

INVESTIGATION ON STRONG CHROMATIC NUMBERS

Submitted in partial fulfilment of the requirements for the award of
Master of Science degree in Mathematics

by

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SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

Accredited with Grade "A" by NAAC

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APRIL-2021



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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **V.Sivaranjani (39630012)** who carried out the project entitled "**INVESTIGATION ON STRONG CHROMATIC NUMBERS**" under my supervision from January 2021 to April 2021.

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DECLARATION

I **V.Sivaranjani (39630012)** hereby declare that project Report entitled **“INVESTIGATION ON STRONG CHROMATIC NUMBERS”** done by me under the guidance of **Dr.R.MARY JEYA JOTHI**(Internal) at **Sathyabama Institute of Science and Technology** is submitted in partial fulfillment of the requirements for the award of Master of Science degree in Mathematics.

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ACKNOWLEDGEMENT

I am pleased to acknowledge my sincere thanks to Board of Management of **SATHYABAMA** for their kind encouragement in doing this project and for completing it successfully. I am grateful to them.

I convey my thanks to **Dr.M.Nirmala**, Head of the Department, Dept. of Mathematics for providing necessary support and details at the right time during the progressive reviews.

I would like to express my sincere and deep sense of gratitude to my Project Guide **Dr.R.MARYJEYA JOTHI** for her valuable guidance, suggestions and constant encouragement paved way for the successful completion of my project work.

I wish to express my thanks to all Teaching and Non-teaching staff members of the Department of Mathematics who were helpful in many ways for the completion of the project.

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ABSTRACT

A graph is said to be strongly k -colourable if, each partite vertex subset of size k admits a strong colouring (A strong colouring, in reference to a partition of the vertices into subsets (equal size with disjoint), may be a vertex colouring (proper) wherein all colour performs accurately once in each part. If there exists a least k such that (The strong chromatic number) $s\chi(G) = k$, then G is strongly k -colourable. A graph is called strongly k -chromatic if it consumes strong chromatic number k . In this paper, it is discussed the strong coloring of some graphs.

KEYWORDS

Colouring, strong colouring, chromatic number, strong chromatic number.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS		ABBREVIATIONS
C_n	-	Cycle Graph
P_n	-	Path Graph
$K_{m,n}$	-	Complete Bipartite Graph
K_n	-	Complete Graph
$P_{n,k}$	-	Peterson Graph
$W_{n,r}$	-	Web Graph
L_n	-	Ladder Graph
CL_n	-	Circular Ladder Graph

CHAPTER-1

INTRODUCTION

Graph-theoretical ideas date back to at least the 1730's, when a researcher published his paper on the problem of Seven Bridges of Königsberg [1]. This puzzle asks whether there is a continuous walk that crosses each of the seven bridges of Königsberg only once and if so, whether a closed walk can be found. Furthermore, the large part of graph theory has been motivated by the study of games and recreational mathematics. Graphs are very convenient tools for representing the relationships among objects, which are represented by points. In their turn, relationships among points are represented by connections [3]. In general, any mathematical object involving points and connections among them can be called a graph. For a great diversity of problems such pictorial representations may lead to a solution. Examples of such applications include databases, physical networks, organic molecules, map colourings, signal flow graphs, web graphs, tracing mazes as well as less tangible interactions occurring in social networks, ecosystems and in a flow of a computer program.

Graph theory research from the middle of the twentieth century has been motivated by problems in combinatorial optimization, computer science, operations research, game theory, decision theory, urban planning, molecular biology, social science, geometry and so on. These problems have led to the introduction and study of various concepts and classes of graphs. In fact, many of these concepts or theoretical questions arise from practical reasons and not just from the mathematician's imaginations for solving real problems modelled on graphs [2]. Moreover, researchers in graph theory try to find efficient algorithms for solving these problems.

The graph models can be further classified into different categories. For instance, two atoms in an organic molecule may have multiple connections between them, an electronic circuit may use a model in which each line represents a direction, or a computer program may consist of loop structures. Therefore, for these examples we need multi graphs, directed graphs or graphs that allow loops [4]. Hence graphs can serve as mathematical models to solve an appropriate graph-theoretic

problem, and then interpret the solution in terms of the original problem. Thus, graph theory has now grown to touch almost every mathematical discipline, in one form or the other, and it serves as a tool to solve problems in other disciplines, particularly in computer science.

Some conjectures in graph theory had a fundamental impact. For example, the development of graph theory over the last four decades has been strongly influenced by the Strong Perfect Graph Conjecture (SPGC) and perfect graphs introduced by a researcher [8]. Also, scientific conjectures are another important research area of graph theory. Scientific conjectures express the most interesting theoretic statements, which are neither proved nor disproved. They are basically posed in order to draw attention of scientific community and advance progress in the corresponding field.

1.1 Graph Theory

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices which are connected by edges. A distinction is made between undirected graphs, where edges link two vertices symmetrically, and directed graphs, where edges link two vertices asymmetrically; Graph (discrete mathematics) for more detailed definitions and for other variations in the types of graph that are commonly considered. Graphs are one of the prime objects of study in discrete mathematics.

V and E are usually taken to be finite, and many of the well-known results are not true (or are rather different) for infinite graphs because many of the arguments fail in the infinite case. Moreover, V is often assumed to be non-empty, but E is allowed to be the empty set. The order of a graph is $|V|$, its number of vertices. The size of a graph is $|E|$, its number of edges. The degree or valency of a vertex is the number of edges that are incident to it, where a loop is counted twice.

In the edge (x, y) directed from x to y , the vertices x and y are called the endpoints of the edge, x the tail of the edge and y the head of the edge. The edge (y, x) is

called the inverted edge of (x, y) . The edge is said to join x and y and to be incident on x and on y . A vertex may exist in a graph and not belong to an edge. A loop is an edge that joins a vertex to itself. Multiple edges are two or more edges that join the same two vertices.

1.2 Applications of Graph Theory

Graph Theory is used in vast area of science and technologies. Some of them are given below:

In computer science graph theory is used for the study of algorithms like:

- Dijkstra's Algorithm
- Prim's's Algorithm
- Kruskal's Algorithm
- Graphs are used to define the flow of computation.
- Graphs are used to represent networks of communication.
- Graphs are used to represent data organization.
- Graph transformation systems work on rule-based in-memory manipulation of graphs. Graph databases ensure transaction-safe, persistent storing and querying of graph structured data.
- Graph theory is used to find shortest path in road or a network.
- In Google Maps, various locations are represented as vertices or nodes and the roads are represented as edges and graph theory is used to find the shortest path between two nodes.

In Electrical Engineering, graph theory is used in designing of circuit connections. These circuit connections are named as topologies. Some topologies are series, bridge, star and parallel topologies.

In linguistics, graphs are mostly used for parsing of a language tree and grammar of a language tree. Semantics networks are used within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words. Methods in phonology (e.g. theory of optimality, which uses lattice graphs) and morphology (e.g. morphology

of finite - state, using finite-state transducers) are common in the analysis of language as a graph.

In physics and chemistry, graph theory is used to study molecules. The 3D structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. Statistical physics also uses graphs. In this field graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems. Graphs are also used to express the micro-scale channels of porous media, in which the vertices represent the pores and the edges represent the smaller channels connecting the pores. Graph is also helpful in constructing the molecular structure as well as lattice of the molecule. It also helps us to show the bond relation in between atoms and molecules, also help in comparing structure of one molecule to other [8].

In computer network, the relationships among interconnected computers within the network, follow the principles of graph theory. Graph theory is also used in network security. We can use the vertex coloring algorithm to find a proper coloring of the map with four colors. Vertex coloring algorithm may be used for assigning at most four different frequencies for any GSM (Grouped Special Mobile) mobile phone networks.

Graph theory is also used in sociology. For example, to explore rumor spreading, or to measure actors' prestige notably through the use of social network analysis software. Acquaintanceship and friendship graphs describe whether people know each other or not. In influence graphs model, certain people can influence the behavior of others. In collaboration graphs model to check whether two people work together in a particular way, such as acting in a movie together.

Nodes in biological networks represent biomolecular such as genes, proteins or metabolites, and edges connecting these nodes indicate functional, physical or chemical interactions between the corresponding biomolecular. Graph theory is used in transcriptional regulation networks. It is also used in Metabolic networks. In PPI (Protein - Protein interaction) networks graph theory is also useful. Characterizing drug - drug target relationships.

In mathematics, operational research is the important field. Graph theory provides many useful applications in operational research. Like: Minimum cost path. A scheduling problem.

Generally, Graphs are used to represent the routes between the cities. With the help of tree that is a type of graph, we can create hierarchical ordered information such as family tree.

1.3 BASIC CONCEPTS

In this section, we present many of the graph theoretic terms used in the thesis. A graph G consists of a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set whose elements are called Points or Vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called Lines or Edges of the graph G . If $e = \{u, v\} \in E(G)$, the edge e is said to join u and v . We write $e = uv$ and we say that the vertices u and v are adjacent. We also say that the vertex u and the edge e are incident with each other. If two distinct edges e_1 and e_2 are incident with a common vertex then they are called adjacent edges. A graph with p vertices and q edges is called a (p, q) graph.

The number of vertices of a graph G is denoted by $|V|$. The number of edges of a graph G is denoted by $|E|$. The graph G is said to be finite or infinite according as the number of vertices is finite or infinite. A graph with only one vertex and no edge is called a trivial graph. Let $e = uv$ be an edge of a graph G . The vertex u and v are called the end vertices or terminal vertices of the edge. An edge joining a vertex to itself is called a loop. A graph G is said to be a complete graph if every vertex is adjacent to every other vertex. The complete graph with n vertices is denoted by K_n . A graph G is called a bigraph or bipartite graph, if V can be partitioned into two distinct subsets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2 . (V_1, V_2) is called a bipartition of G . If G contains every edge joining the vertices of V_1 to the vertices of V_2 , then G is called a complete bigraph or complete bipartite graph. If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph G is denoted by $K_{m,n}$. The number of edges incident on the vertex v of a graph G is called the degree of the vertex v in G . It is

denoted by $d_G(v)$ or $d(v)$ or $\deg(v)$. A vertex v of degree 0 is called an isolated vertex. A vertex v of degree 1 is called a pendant vertex or end vertex.

For any graph G , $\delta(G) = \min \{\deg(v) / v \in V\}$ and $\Delta(G) = \max \{\deg(v) / v \in V\}$. If all the vertices of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r . A graph $H = (V_1, E_1)$ is called a subgraph of $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$. If H is a subgraph of G we say that G is a supergraph of H . H is called a spanning subgraph of G if $V_1 = V$. H is called an induced subgraph of G if H is the maximal subgraph of G with vertex set V_1 and it is denoted by $G[V_1]$. Thus if H is an induced subgraph of G , two vertices are adjacent in H if and only if they are adjacent in G . A clique of G is a set $X \subseteq V$ such that $G[X]$ is complete. The cardinality of the largest clique is called the clique number and it is denoted by $\omega(G)$. The removal of a vertex v from a graph G results in that subgraph $G - v$ of G consisting of all vertices of G except v and all the edges not incident with v . The removal of an edge e from G yields $G - e$ containing all edges of G except e .

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \rightarrow V_2$ such that u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2 . If G_1 is isomorphic to G_2 , we write $G_1 \cong G_2$. Isomorphic graphs are clearly different drawings of the configuration of the same graph. For this reason, isomorphic graphs are also called equivalent graphs or equal graphs. For a given graph G , the complement $\overline{G}$ of G is defined to be the graph which has V as its set of vertices and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G . A graph G is said to be self-complementary if G is isomorphic to $\overline{G}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$,

- (i) The sum $G_1 + G_2$ is $G_1 \cup G_2$ together with all the edges joining vertices of V_1 to the vertices of V_2 .
- (ii) The product $G_1 \times G_2$ has $V(G_1 \times G_2) = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

If G_1 and G_2 are (p_1, q_1) and (p_2, q_2) graphs, then (1) $G_1 + G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2 + p_1 p_2$ edges (2) $G_1 \times G_2$ has $p_1 p_2$ vertices and $q_1 p_2 + q_2 p_1$ edges. A walk of a graph is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$ beginning and ending with vertices such that each edge e_i is incident with v_{i-1} and v_i . We say the walk joins v_0 and v_n and it is called $v_0 - v_n$ walk, v_0 is called the initial vertex and v_n is called the terminal vertex of the walk. The above walk is also denoted by $v_0 v_1 v_2 \dots v_n$, the edges of a walk being self evident and n is called the length of this walk. Single vertex is considered as a walk of length 0. A walk is called a trail if all its edges are distinct. A walk is called a path if all the vertices are distinct. A $v_0 - v_n$ walk is closed if $v_0 = v_n$. A closed walk $v_0 v_1 v_2 \dots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, \dots, v_n$ are distinct is called a cycle or circuit of length n . The graph consisting of a cycle of length n is denoted by C_n . An odd cycle is a cycle with odd number of vertices. An even cycle is a cycle with even number of vertices. The number of vertices in a cycle equals the number of edges. A *hole* is a chordless cycle of length at least four and an *antihole* is the complement of such a cycle. Holes and antiholes are *odddoreven* according to the parity of their number of vertices. If a graph contains loops then every loop is a closed walk. A graph G is said to be connected if any two of its vertices can be joined by a path. Two vertices u and v are said to be connected in a graph G if there exists a $u - v$ path. A graph that is not connected is said to be disconnected. A maximal connected subgraph of G is called a component of G . Let G be a graph and u, v be any two vertices. There may be a number of paths connecting u and v . The length of the shortest path is called the distance between the points u and v and it is denoted by $d(u, v)$. The diameter of a graph G is the maximum distance between any two vertices in the graph and is denoted by $\text{diam}(G)$.

A graph that contains no cycle is called an acyclic graph. A connected acyclic graph is called a tree. A collection of trees is called a forest. So a forest is a disconnected graph whose connected components are trees. A subset M of E is called a matching of G if no two edges are adjacent in G . If $uv \in M$, we say that u and v are matched under M . We also say that the vertices u and v are M -saturated. If every vertex of G is M -saturated, then the matching is called a perfect matching. M is called a maximum matching of G if there is no matching M_1 with $|M_1| > |M|$. Clearly every perfect matching is a maximum matching. A subset S of

V is called an independent set or stable set of G if no two vertices of S are adjacent in G . An independent set S is called maximum if G has no independent set S_1 with $|S_1| > |S|$.

By vertex colouring or colouring of a graph G , we mean assigning different colours to the vertices of the graph such that no two adjacent vertices get the same colour. The chromatic number of a graph G is the minimum number of colours that is required to colour G . It is denoted by $\chi(G)$. If $\chi(G) = k$ then G is said to be k -chromatic. A subset D of $V(G)$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D . A subset S of V is said to be a minimal dominating set if $S - \{u\}$ is not a dominating set for any $u \in S$. A dominating set of minimum cardinality is called a minimum dominating set. The domination number $\gamma(G)$ of a graph G equals the minimum cardinality of a minimal dominating set. The domination number of its complement $\bar{G}$ is called the co - domination number of G and is denoted by $\gamma(\bar{G})$ or simply $\bar{\gamma}$.

1.4 Graph coloring

In graph theory, graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color; this is called a vertex coloring [9].

Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges are of the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color [7].

Vertex coloring is usually used to introduce graph coloring problems, since other coloring problems can be transformed into a vertex coloring instance. For example, an edge coloring of a graph is just a vertex coloring of its line graph, and a face coloring of a plane graph is just a vertex coloring of its dual. However, non-vertex coloring problems are often stated and studied as-is. This is

partly pedagogical, and partly because some problems are best studied in their non-vertex form, as in the case of edge coloring [5].

The convention of using colors originates from coloring the countries of a map, where each face is literally colored. This was generalized to coloring the faces of a graph embedded in the plane. By planar duality it became coloring the vertices, and in this form it generalizes to all graphs [6].

In mathematical and computer representations, it is typical to use the first few positive or non-negative integers as the "colors". In general, one can use any finite set as the "color set". The nature of the coloring problem depends on the number of colors but not on what they are [10].

1.5. Application of Graph Coloring

The Graph coloring problem has huge number of applications.

1) Making Schedule or Time Table

Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph [18].

2) Mobile Radio Frequency

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed? This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

3) Sudoku

Sudoku is also a variation of Graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block [12].

4) Registration Allocation

In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem [17].

5) Bipartite Graphs

We can check if a graph is Bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not.

6) Map Coloring

Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map [13].

CHAPTER-2

LITERATURE SURVEY

Most Strong chromatic number methods trace their origin to one introduced by Alon Noga [14] in 1988 and Fellows Michael in 1990. In the paper of “The Strong Chromatic Number of a graph”, Alon states that, It is shown that there is an absolute constant c with the following property: For any two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices, where G_1 has maximum degree at most d and G_2 is a vertex disjoint union of cliques of size cd each, the chromatic number of the graph $G = (V, E_1 \cup E_2)$ is precisely cd . The proof is based on probabilistic arguments. And on this another paper “the linear arboricity of graphs”,

Alon states that, a linear forest is a forest in which each connected component is a path. The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests whose union is the set of all edges of G . The linear arboricity conjecture asserts that for every simple graph G with maximum degree $\Delta = \Delta(G)$, $la(G) \leq \lceil \Delta/2 \rceil$. Although this conjecture received a considerable amount of attention, it has been proved only for $\Delta \leq 6$, $\Delta = 8$ and $\Delta = 10$, and the best known general upper bound for $la(G)$ is $la(G) \leq \lceil 3\Delta/5 \rceil$ for even Δ and $la(G) \leq \lceil (3\Delta + 2)/5 \rceil$ for odd Δ . Here we prove that for every $\varepsilon > 0$ there is a $\Delta_0 = \Delta_0(\varepsilon)$ so that $la(G) \leq (\frac{1}{2} + \varepsilon) \Delta$ for every G with maximum degree $\Delta \geq \Delta_0$. To do this, we first prove the conjecture for every G with an even maximum degree Δ and with girth $g \geq 50\Delta$. In the paper “Transversals of Vertex Partitions in Graphs” Fellow states that for every partition of the vertex set that satisfies an upper(or lower) bound on the number of elements in each partition class, there is a transversal of the partition that is an independent or (dominating) set [15].

A possible application to fault-tolerant data storage is discussed, and bounds for the parameters that are functions of minimum and maximum degree are established. In the book of Jensen [11], it Contains a wealth of information previously scattered in research journals, conference proceedings and technical reports. Identifies more than 200 unsolved problems. Every problem is stated in a

self-contained, extremely accessible format, followed by comments on its history, related results and literature [19]. The book will stimulate research and help avoid efforts on solving already settled problems. Each chapter concludes with a comprehensive list of references which will lead readers to original sources, important contributions and other surveys. In the paper “On the Strong chromatic number of graphs” Haxell [16] called the Strong chromatic number to G and considering partition of the vertices of the resulting graph into disjoint subsets $V_1, \dots, V_{\lceil n/k \rceil}$ of size k each, one can find a proper k -vertex-coloring of the graph such that each part V_i , $i=1,2,\dots,\lceil n/k \rceil$, contains exactly one vertex of each color. For any graph with maximum degree Δ , it is easy to see that $\chi_s(G) \geq \Delta + 1$. Recently, Haxell proved that $\chi_s(G) \leq 3\Delta - 1$.

In 2008 Haxell [15] found “An improved bound for the strong chromatic number”, Let $\eta > 0$ be given. Then there exists $d_0 = d_0(\eta)$ such that the following holds. Let G be a finite graph with maximum degree at most $d \geq d_0$ whose vertex set is partitioned into classes of size αd , where $\alpha \geq 11/4 + \eta$. Then there exists a proper coloring of G with αd colors in which each class receives all αd colors.

CHAPTER-3

DEFINITIONS AND RESULTS

3.1 DEFINITIONS

3.1.1 Graph Coloring

Graph coloring is a special case of graph labeling; it is traditionally called "colors" to elements of a graph. It is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color; this is called a vertex coloring.

3.1.2 Chromatic Number

The chromatic number of a graph is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color.

3.1.3 Cycle graph

A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3, if the graph is simple) connected in a closed chain. The cycle graph with n vertices is called C_n .

3.1.4 Path graph

A path in a graph is a finite or infinite sequence of edges which joins a sequence of vertices which, by most definitions, are all distinct (and since the vertices are distinct, so are the edges).

3.1.5 Complete Bipartite graph

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and

every possible edge that could connect vertices in different subsets is part of the graph.

3.1.6 Complete graph

A Complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges (one in each direction).

3.1.7 Web graph

The webgraph describes the directed links between pages of the World Wide Web. A graph, in general, consists of several vertices, some pairs connected by edges. In a directed graph, edges are directed lines or arcs. The webgraph is a directed graph, whose vertices correspond to the pages of the WWW, and a directed edge connects page X to page Y if there exists a hyperlink on page X, referring to page Y.

3.1.8 Peterson graph

The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.

3.1.9 Ladder graph

In the mathematical field of graph theory, the ladder graph L_n is a planar undirected graph with $2n$ vertices and $3n-2$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_{n,1} = P_n \times P_2$.

3.1.10 Circular ladder graph

The circular ladder graph CL_n is constructible by connecting the four 2-degree vertices in a straight way, or by the Cartesian product of a cycle of length $n \geq 3$ and an edge. In symbols, $CL_n = C_n \times P_2$.

3.1.11 Strong coloring

In graph theory, a strong coloring with respect to a partition of the vertices into (disjoint) subsets of equal sizes, is a (proper) vertex coloring in which every color appears exactly once in every part.

3.1.12 Strong Chromatic Number

The strong chromatic number $s\chi(G)$ of a graph G is the least k such that G is strongly k -colorable. A graph is strongly k -chromatic if it has strong chromatic number k .

3.2 MAIN RESULTS

3.2.1 Theorem 1

Show that the Strong Chromatic number of cycle graph is 2. (i.e)

$$s\chi(C_n) = 2$$

Proof:

Consider a cycle graph C_n .

Let the vertices $v_1, v_2, v_3, \dots, v_n$ be coloured by colours 1 and 2 alternatively.

To prove $s\chi(C_n) = 2$.

Suppose $s\chi(C_n) \neq 2$.

Then there exists some partitions such that all partitions have the same number (three or more) of colours.

Without loss of generality, assume that there exist 3 colours in each partition.

$\Rightarrow$ Each vertex partition set has colouring number 3.

Which is a contradiction to $\chi(C_n) = 2$.

Hence, $s\chi(C_n) = 2$.

The above theorem is illustrated in fig 3.1.

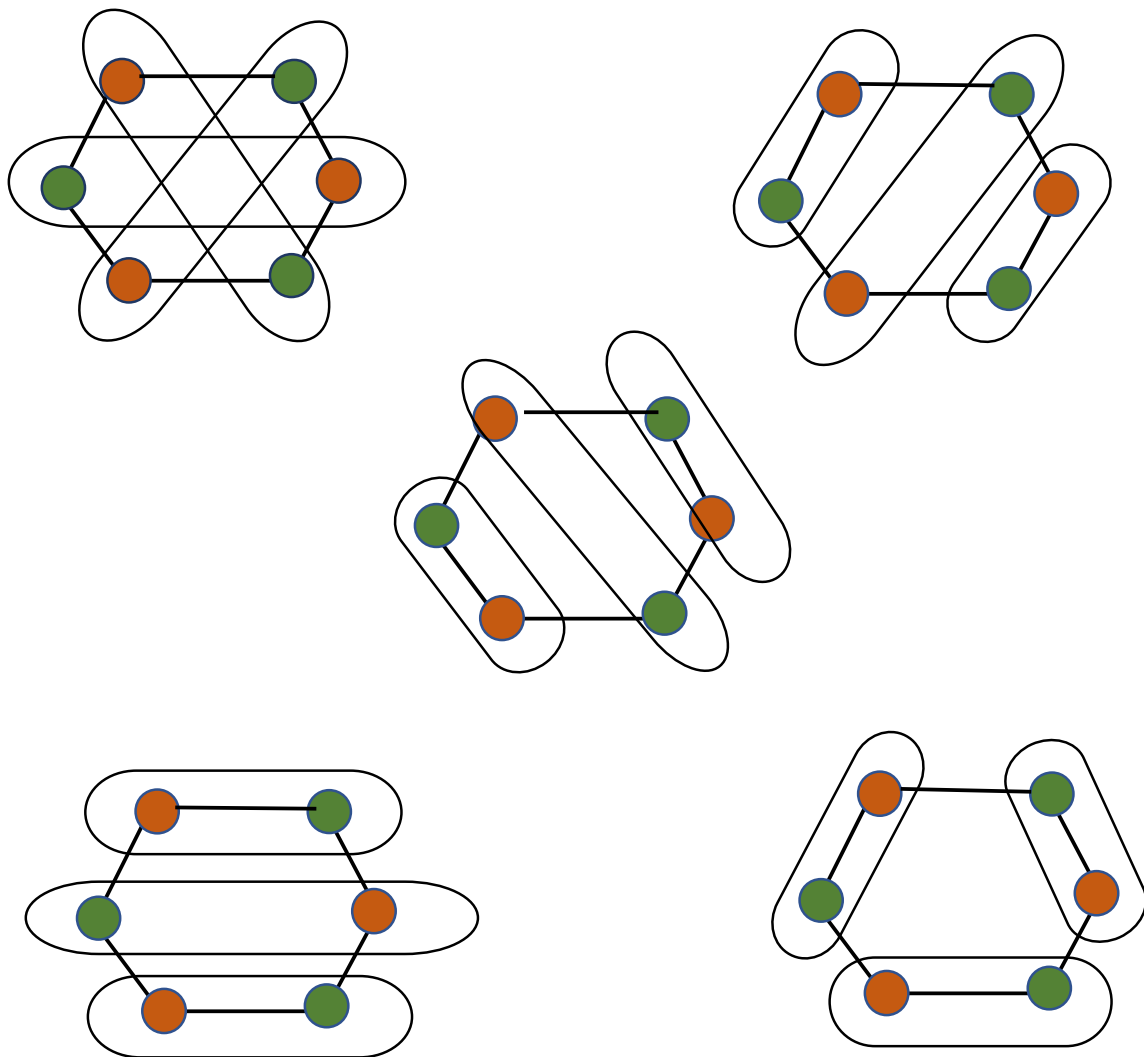


Fig 3.1 Graph C_n

3.2.2 Theorem 2

Show that the strong Chromatic number of Path Graph is 2.(i.e)

$$s\chi(P_n) = 2$$

Proof:

Consider a path graph P_n .

Let the vertices be $u_1, u_2, u_3, \dots, u_n$.

As $\chi(P_n) = 2$, there exists $\frac{n}{2}$ partitions of the vertex set such that all partitions

have 2 colours.

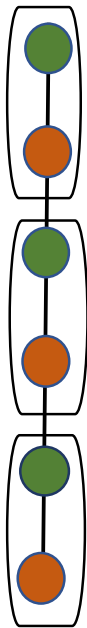
$$\Rightarrow s\chi(P_n) = 2.$$

The above theorem is illustrated in fig 3.2

P₄



P₆



P₈



Fig 3.2 Graph P_4 , P_6 , P_8

3.2.3 Theorem 3

Show that the strong Chromatic Number of Complete Bipartite Graph is 2.

$$(i.e) s_{\chi}(K_{m,n}) = 2$$

(where m, n are both even or odd).

Proof:

Consider a complete bipartite graph $K_{m,n}$.

Let the vertices be partitioned into $U = \{u_1, u_2, u_3, \dots, u_m\}$ and $V = \{v_1, v_2, v_3, \dots, v_n\}$ such that $|U \cup V| = m+n$.

Let the vertices of U and V be coloured by colours 1 and 2.

$\Rightarrow$ There exists $\frac{m+n}{2}$ vertex partitions such that each partition has same number of colours.

$$\Rightarrow s_{\chi}(K_{m,n}) = 2.$$

The above theorem is illustrated in fig 3.3

$K_{2,2}K_{4,4}$

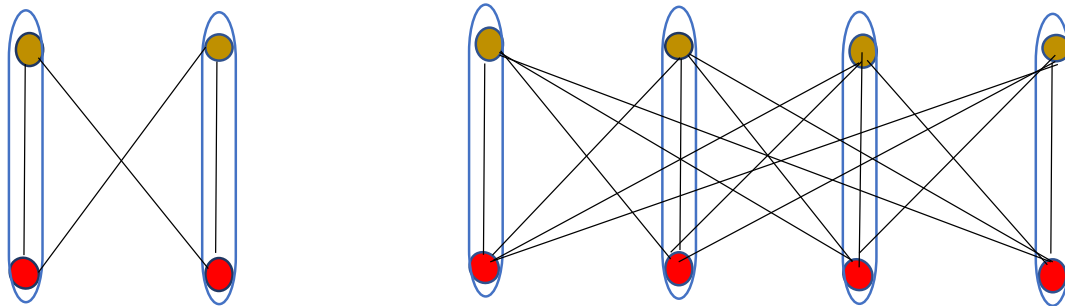


Fig 3.3 Graph $K_{2,2}$ and Graph $K_{4,4}$

3.2.4 Theorem 4

Show that the strong Chromatic Number of Complete Graph is n (i.e)

$$s\chi(K_n) = n$$

(when n is both even and odd).

Proof:

Consider a complete graph K_n .

Let the vertices be $u_1, u_2, u_3, \dots, u_n$.

To prove $s\chi(K_n) = n$.

Suppose $s\chi(K_n) \neq n$.

Then there exists some partitions such that all partitions have the same number
(but less than n) of colours.

Without loss of generality, assume that there exist 2 partitions.

$\Rightarrow$ Vertex set can be partitioned into two equal sets and each having colouring
number less than n

Which is a contradiction to $\chi(K_n) = n$.

$\Rightarrow$ there exists only one partition with n colours.

Hence, $s\chi(K_n) = n$.

The above theorem is illustrated in fig 3.4

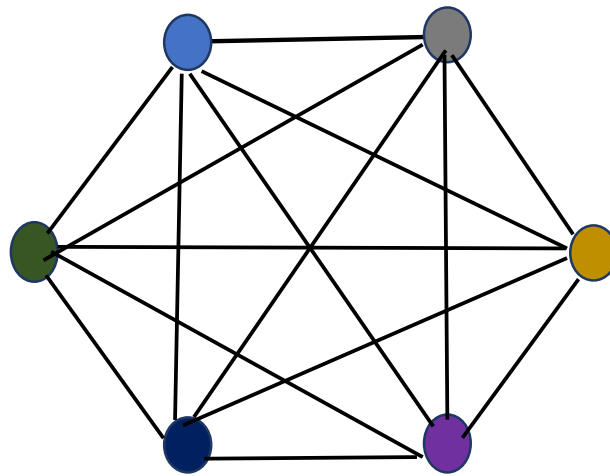


Fig 3.4 Graph K_n

3.2.5 Theorem 5

Show that the strong Chromatic Number of Peterson Graph is 5. (i.e)

$$s\chi(P_{5,2}) = 5.$$

Proof:

Consider the above Petersen graph $G = P(5, 2)$, with 10 vertices.

As $\chi(G) = 3$, if the vertex set is divided into some partitions such that each partition has 3 colours then each partition does not have equal size.

If $s\chi(G) = 4$, then also, each partition does not have equal size with same colours.

Hence, to divide the vertex into two equal partitions such that each partition has same number (all distinct), its needed 5 colours.

Hence, $s\chi(G) = 5$.

The above theorem is illustrated in fig 3.5

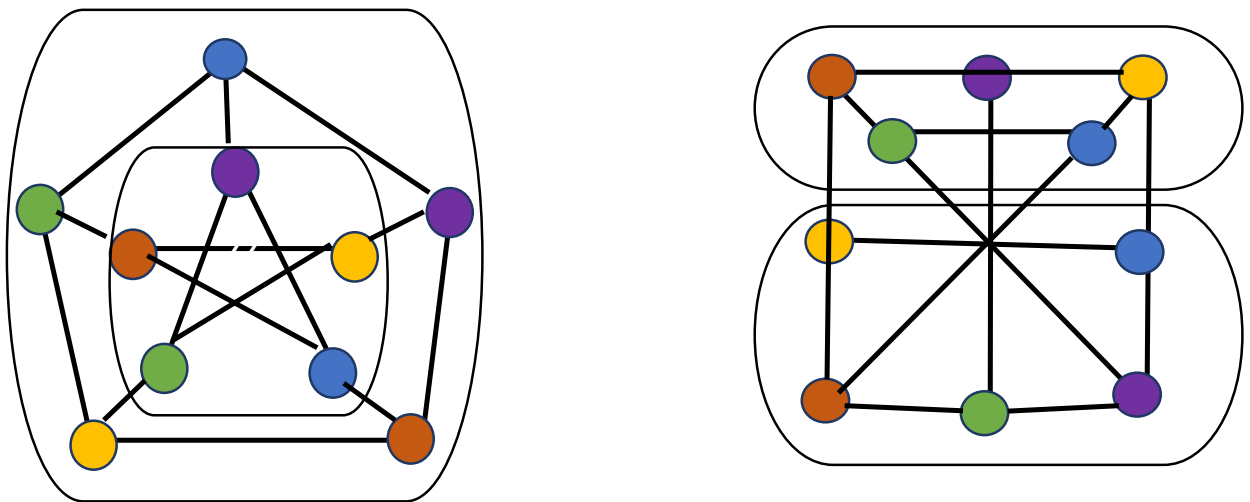


Fig 3.5 Graph $P_{5,2}$

3.2.6 Theorem 6

If r is even, then

$$s_{\chi}(W_{n,r}) = 2$$

where $n \geq 4$, n is even and

$$s_{\chi}(W_{n,r}) = n$$

where $n \geq 3$, n is odd.

Proof:

For $W_{n,r}$, where n is even, Inner and outer even cycles are colored with 2 colors and partitioning the vertex set into $\frac{nr}{2}$ disjoint subsets such that each partition has 2 colours.

$$\Rightarrow s_{\chi}(W_{n,r}) = 2.$$

For $W_{3,2}$, inner and outer cycles C_3 are coloured with 3 colors and partitioning the vertex set into 2 distinct subsets such that each partition has 3 colours.

$$\Rightarrow s_{\chi}(W_{3,2}) = 3.$$

For $W_{5,2}$, If the inner and outer odd cycles are coloured with 3 or 4 colors then partitioning the vertex set into equal distinct subsets is not possible. Hence it is required minimum 5 colours such that each there exists two disjoint partition such that each partition has 5 colours.

$$\Rightarrow s_{\chi}(W_{5,2}) = 5 = \text{length of the cycle } C_5.$$

For $W_{7,2}$, If the inner and outer odd cycles are coloured with 3, 4, 5 or 6 colors then partitioning the vertex set into equal distinct subsets is not possible. Hence it is required minimum 7 colours such that each there exists two disjoint partition such that each partition has 7 colours.

$$\Rightarrow s_{\chi}(W_{7,2}) = 7 = \text{length of the cycle } C_7.$$

Proceeding like this, $s_\chi(W_{n,2}) = n$ where $n \geq 3$, n is odd.

The above theorem is illustrated in fig 3.6

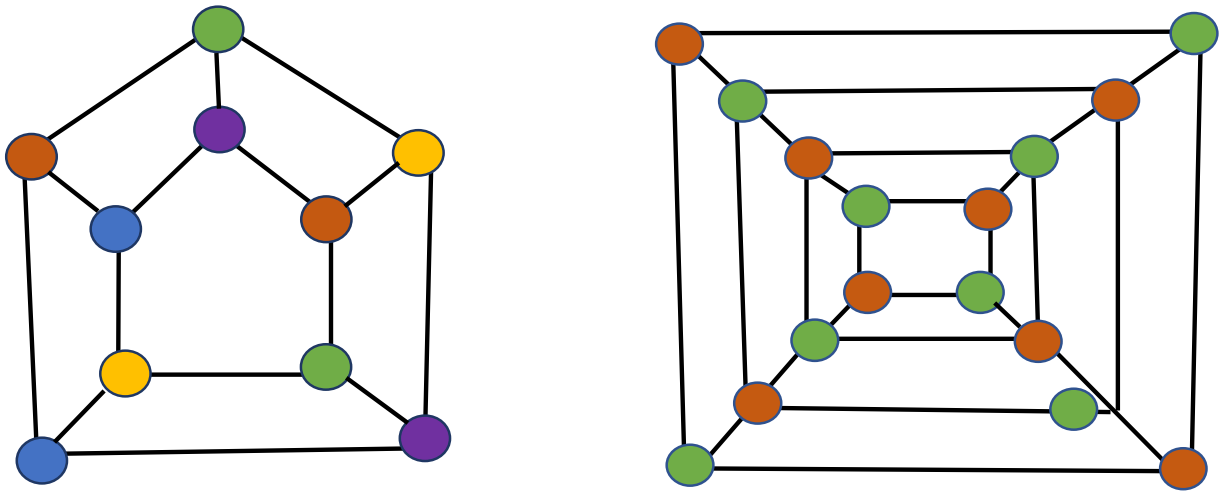


Fig 3.6 Graph $W_{n,r}$

3.2.7 Theorem 7

If n is even then

$$s\chi(CL_n) = 2,$$

if n is odd then

$$s\chi(CL_n) = n.$$

Proof:

Consider a circular ladder graph CL_n .

As CL_n has $2n$ vertices, CL_n has a structure of two concentric even cycles with the corresponding vertices are connected.

Case (i) if n is even

As $\chi(C_n) = 2$ for even n , in CL_n there exists $\frac{2n}{2} = n$ partitions of the vertex set such that all partitions have 2 colours.

$$\Rightarrow s\chi(CL_n) = 2.$$

Case (ii) if n is odd

As CL_n has $2n$ vertices and $\chi(C_n) = 3$ for odd n , partitioning (equal partite with three colours) $2n$ vertices with three colours is not possible

$\Rightarrow$ Each partite set should have some colours k such that k would be the divisor of $2n$.

As n is odd, the divisor of n are also the divisor of $2n$.

So the divisors of n can be $1, 2, 3, \dots, 2n$

Since $\chi(CL_n) = 3$ for odd n , the divisors would be three or more.

Let us prove this by induction on ' n ', the length of the cycle $n \geq 3$ (n is odd).

To prove $s\chi(CL_3) = 3$.

As CL_3 has 6 vertices, the divisors of 6 are 1, 2, 3, 6.

Since $\chi(CL_3) = 3$, $s_\chi(CL_3) \neq 1$ and $s_\chi(CL_3) \neq 2$.

$\Rightarrow CL_3$ may be coloured using 3 (or) 6 colours with respect to 2 (or) 1 partition.

But 3 is the least number, hence $s_\chi(CL_3) = 3$, where 3 is the length of the cycle C_3 .

Now consider CL_5 ,

It has 10 vertices and its divisors are 1,2,5,10.

By recalling the above case, $s_\chi(CL_5) = 5$, where 5 is the length of the cycle C_5 .

When it proceeded like this $s_\chi(CL_n) = n$, where n is the length of the cycle C_n .

The above theorem is illustrated in fig 3.7

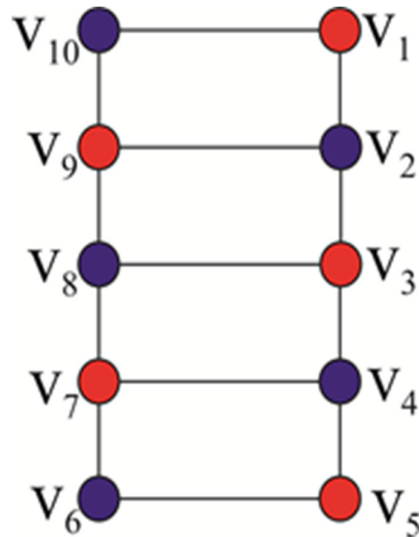


Fig 3.7 Graph L_6

3.2.8 Theorem 8

Show that the Strong Chromatic Number of Circular Ladder Graph is 5. (i.e)

$$s_{\chi}(CL_n)=2.$$

Proof:

Consider a ladder graph L_n .

Let the vertices $v_1, v_2, v_3, \dots, v_n$ be coloured by colours 1 and 2 alternatively.

To prove $s_{\chi}(L_n) = 2$.

Suppose $s_{\chi}(L_n) \neq 2$.

Then there exists some partitions such that all partitions have the same number (three or more) of colours.

Without loss of generality, assume that there exist 3 colours in each partition.

$\Rightarrow$ Each vertex partition set has colouring number 3.

Which is a contradiction to $\chi(L_n) = 2$.

Hence, $s_{\chi}(L_n) = 2$.

The above theorem is illustrated in fig 3.8

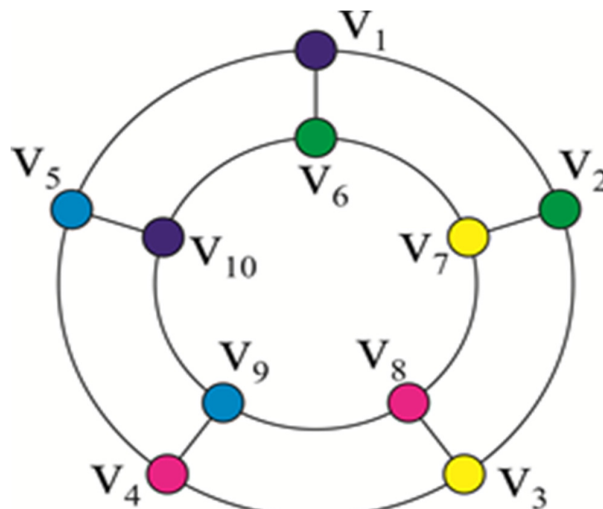


Fig 3.8 Graph CL₅

CHAPTER 4

CONCLUSION

This projects presents the strongly k-chromatic number of cycle, path, complete, complete bipartite, Peterson, web, Ladder and circular ladder graphs with even number of vertices.

As we cannot partition odd number of vertices into equal partitions, for the above graph classes with odd number of vertices, their corresponding strongly k-chromatic number will be obtained by adding some condition in partitioning.

Paper Presented

Presented the paper “Strong Chromatic Number of Some Graphs” in the 2nd International Conference on Mathematical Techniques and Applications (e-ICMTA-2021).

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