

REAL ANALYSIS THROUGH TOPOLOGICAL VIEW POINT

Submitted in partial fulfillment of the requirements for the award of
Bachelor of Science Degree in Mathematics

By

V. JANS1(38060005)



**DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND HUMANITIES**

SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

**Accredited with Grade "A" by NAAC I12B Status by UGC I approved by AICTE
JEPPIAAR NAGAR, RAJIV GANDHI SALAI, CHENNAI - 600 119.**

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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **V. JANSI (38060005)** who carried out the project entitled "**REAL ANALYSIS THROUGH TOPOLOGICAL VIEW POINT**" under our supervision from **DECEMBER 2020 to MARCH 2021**.

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Head of the Department
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Submitted for Viva voce Examination held on 09/04/2021.

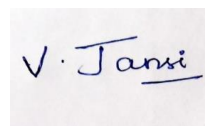
Internal Examiner
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DECLARATION

I **JANSI.V** hereby declare that the Project Report entitled “**REAL ANALYSIS THROUGH TOPOLOGICAL VIEW POINT**” done by me under the guidance of **Dr. L. PRASANNA VENAKATESH** (Internal) at **SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY** is submitted in partial fulfillment of the requirements for the award of Bachelor of Science degree in Mathematics.

DATE: 28/03/2021

A handwritten signature in blue ink that reads "V. Jansi". The signature is written in a cursive style with a horizontal line under the name.

PLACE: CHENNAI SIGNATURE OF THE CANDIDATE.

ACKNOWLEDGEMENT

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I convey my thanks to **Dr. M. NIRMALA**, Head of the Department, Department of Mathematics for providing me necessary support and details at the right time during the progressive reviews.

I would like to express my sincere and deep sense of gratitude to my Project Guide **Dr. L. PRASANNA VENKATESH** for his valuable guidance, suggestions and constant encouragement paved way for the successful completion of my project work.

I wish to express my thanks to all Teaching and Non-teaching staff members of the Department of Mathematics who were helpful in many ways for the completion.

ABSTRACT:

In this project work we have introduced the concept of metric space, discussed the motivation behind studying this concept. We also discussed how this concept is being used in real life scenario. Being closely related to the notion of distance, it has many applications in the field of navigation. One such instance where it is being associated with the navigation of aeroplanes is discussed. We have discussed various metrics on the set of reals, on $\mathbb{R}^n$, on the set of functions etc. to illustrate the abstract nature of the distance. Later the notions of metric subspace and super space were introduced and a method of finding a metric super space from a given metric space is discussed. In fact, we observed that we can have uncountable extensions of a given metric to some superset of that space. We also touched upon the concept of isometry i.e., distance preserving functions and discussed some standard examples associated with this concept. Finally we discussed special types of metrics determined by a norm on a vector space, which relates the algebraic properties of the vector space with its geometric properties. We hope that the content presented in the chapter would be of immense help to the students who are beginners in the subject. Hence, we can conclude that many proofs in real analysis is much smoother and easier to understand once we stop thinking about δ and ϵ and consider everything in terms of neighborhood in alternate. This substantiates the role of topology and metric space in real analysis.

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1. INTRODUCTION TO METRIC SPACES

1.1. INTRODUCTION

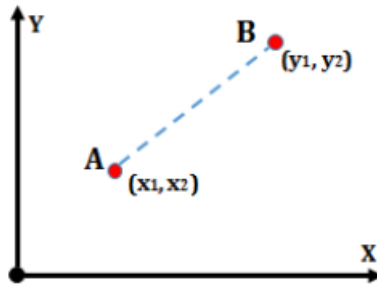
A metric space is a non-empty set equipped with structure determined by a well-defined notion of distance. The term 'metric' is derived from the word metro (measure). Natural and immediate questions that come to mind are what do we mean by measure, what can be measured and how it can be measured? In the search of answers to these questions, let us consider the following example:

Suppose a person wants to go from New Delhi to Mumbai. The adjoining figure gives possible routes from New Delhi to Mumbai. Depending on the situation, he may travel by taking any of the given possible options. We note that there are two different ways to interpret his journey.

- (i) Navigational distance (in km) from New Delhi to Mumbai.
- (ii) Navigational time (in hrs) to reach from New Delhi to Mumbai

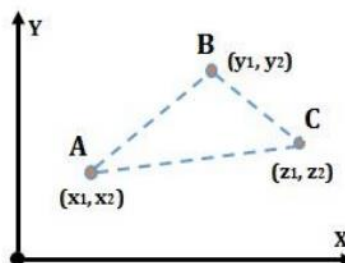
Suppose he travels via NH48, then the distance travelled is 1402 km and time taken is 21h 31min. So in this example, Time and Distance represent two different modes of measurement. Here, we shall discuss and learn about a very special class of functions that 'measure difference' which mathematicians were able to identify in the beginning of the 20th century. In the mathematical literature, this special class is represented as "distance". In the plane, distance between two points is measured along the straight line joining them. Our objective in this chapter is to illustrate through examples the different ways of measuring difference (distance) between objects besides straight line measurements, so that students can grasp the abstract nature of the subject. To begin with, let us observe the fundamental properties of straight line distance measured between two points in $\mathbb{R}^2$. From high school geometry, we know that straight line distance between points A and B is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



1.2 Properties of straight-line distance

1. Measurement between distinct points is a positive real number.
2. Two points in a space are identical if and only if measurement between them is zero
3. Measurement is symmetric in nature i.e., distance measured along A to B is same as measured along B to A.
4. Measurement between two points is less than or equal to the total distance taken when we travel via some other point.



$$AC \leq AB + BC$$

From the first two properties, we observe that straight line distance is non-negative real number. How to generalize all these ideas under one notion so that the properties remain intact? The solution is provided by real valued functions which measures difference. Such functions are known as metric in the mathematical literature. Further since the prototype for such functions is straight line distance, these functions are often regarded as distance functions. These functions were first considered in 1905, by the French mathematician Maurice Frechet who thought of generalizing the notion of distances and extending them to arbitrary sets. In his doctoral dissertation “Less Espaces Abstrait”, he introduced the concept of a metric on a set.

2.THE ROLE OF TOPOLOGY IN ANALYSIS

It is possible to proceed with Real analysis without mentioning the word to topology at all.

Indeed, one of the most popular texts of the previous generation by G.H. Hardy entitled “A course of pure mathematics” has no mention of the word “topology”.

However to study analysis in this way is to study analysis with one eye closed! Indeed many of the major theorems of real analysis including the intermediate value theorem and the extreme value theorem can be easily proven using topology.

In fact, a strong argument can be made that these theorems are really a consequence of topology and studying them without invoking topology does not give a complete picture. So we will spend some time studying topology.

One of the central aims of real analysis is to formalise the definition of a continuous function in a mathematically precise way. Several eminent mathematicians like Newton, Leibniz, Euler, Cauchy and others have given definitions of continuity. Let's summarise some of these “definitions” before we move on to the modern definition which is attributed to Bolzano and Weierstrass.

1. A function is continuous if its graphs has no “gaps” or “jumps”. An example of function with a jump in the famous Heaviside function that takes the value zero when $x \leq 0$ and 1 when $x > 0$. This very simple function crops up repeatedly in many branches of science and engineering.
2. A function is continuous if its graph can be freely traced using a pencil without lifting.
3. A function is said to be continuous if a small change in input creates only a small change in the output.

None of the above three “definitions” are mathematically precise. Specifically, number of intuitive notions like “gaps” and “freely traced” are used. Since all of this began with the intermediate value theorem, let us give a mock definition inspired by the second one above.

2.1. MOCK DEFINITION OF CONTINUITY:

Let $f:(a, b) \rightarrow \mathbb{R}$ be a function. We say f has the intermediate value property if for $\forall a_1, a_2 \in (a, b)$, $a_1 < a_2$, and $d_1 \in [f(a_1), f(a_2)]$ ($d_1 \in [f(a_2), f(a_1)]$ if $f(a_1) > f(a_2)$), we can find $c_1 \in [a_1, a_2]$ such that $f(c_1) = d_1$.

The above definition seeks to make precise the notion that the graph of a continuous function can be fully traced without lifting the pencil. The following example demonstrates why the above definition will not work. Consider the function called the topologists sine curve:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

The problem with this function is that as we get closer and closer to the origin, the function starts oscillating more and more rapidly. We would not want to consider such functions as continuous. Near the origin, a small change in the input produces a spectacularly large change in the output.

PROBLEM:

Note that the above function has the intermediate value property on any interval. Before we venture further, it might be a good idea to take a closer look at one more weird functions. After all it is our objective to give a formal definition of continuity that guarantees that continuous functions are not too weird!

Example:

This example deals with the famous Dirichlet function also known colloquially as the salt-and-pepper function. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Note that if $r \in \mathbb{R}$ then in any interval $(r - \varepsilon, r + \varepsilon)$ there are both rational and irrational numbers irrespective of how small $\varepsilon > 0$ is. In fact, there are infinitely many rationals and irrationals in any non-empty open interval. This means that this function jumps infinitely often in any given open set! Let us analyse a bit more.

We can construct sequences $Q \ni x_n \rightarrow r$ and $\mathbb{R} \setminus Q \ni y_n \rightarrow r$. consequently,

- $f(x_n)$ is the constant function 1,
- $f(y_n)$ is the constant function 0.

We note that $f(x_n) \rightarrow f(r)$ if r is rational. This suggests that to capture intuitive notion of continuity using a precise mathematical definition, we have to consider the behaviour of the function along all sequences converging to a particular point. But before we get to this, we will introduce notions from topology which provides the machinery that allows to accomplish this in an elegant manner.

Loosely speaking, topology is the abstract study of “space”.

We might wonder what is so different about topology compared to geometry which you are very familiar with from school. Well, topology studies only those aspects of space that do not change under a “continuous” transformation of the space. Concepts like area and angle are in the domain of geometry and not topology. Informally, it is often remarked that to a topologist a coffee cup and a doughnut. In some sense, the idea of continuity is sort of built-in the core of topology. From our perspective, Topology is relevant to our study because it deals with “closeness”.

2.2. DEFINITION:

A set X , whose element we shall call points, is said to be a metric spaces if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q , such that,

- (a) $d(p, q) > 0$ if $p \neq q$; $d(p, q) = 0$;
- (b) $d(p, q) = d(q, p)$;
- (c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a distance functions, or metric.

EXAMPLES:

The most important examples of metric spaces, from one standpoint are the Euclidean spaces R^K , especially R^1 (real line) and R^2 (the complex plane); the distance in R^K is defined by

$$d(x, y) = |x - y|$$

By the condition of a metric space. It is important to observe that every subset Y of a metric space X is a metric space in its own right, with same distance functions. For it is clear that if conditions (a) to (c) to metric space hold for $p, q, r \in X$, they also hold if we restrict p, q, r to lie in Y .

Thus every subset of a Euclidean space is a metric space.

3. METRIC TOPOLOGICAL TERMINOLOGIES AND THEOREMS

SEGMENT:

By the segment (a, b) we mean the set of all real numbers x such that $a < x < b$.

INTERVAL:

By the interval $[a, b]$ we mean the set of all numbers x such that $a \leq x \leq b$.

NOTE:

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consist of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

K- CELL:

If $a_i < b_i$ for $i = 1, \dots, k$, the set of all points $x = (x_1, \dots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ ($1 \leq i \leq k$) is called a k -cell. Thus a 1-cell is an interval, a 2-cell is a rectangle etc.

OPEN OR CLOSED BALLS:

If $x \in R^k$ and $r > 0$, the open (or closed) ball B with centre at x and radius r is defined to be the set of all $y \in R^k$ such that $|y - x| < r$ (or $|y - x| \leq r$).

CONVEX BALLS:

We call a set $E \subset R^k$ convex if

$$\lambda x + (1 - \lambda) y \in E$$

Whenever $x \in E$, $y \in E$, and $0 < \lambda < 1$.

For example, balls are convex. For if $|y - x| < r$, $|z - x| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda y + (1 - \lambda) z - x| &= |\lambda (y - x) + (1 - \lambda) (z - x)| \\ &\leq \lambda |y - x| + (1 - \lambda) |z - x| < \lambda r + (1 - \lambda) r \\ &= r. \end{aligned}$$

The same proof applies to closed balls. It is also easy to see that k -cells are convex.

DEFINITIONS:

Let X be a metric space. All points and sets mentioned below are understood to be elements and subset of X .

(a) *Neighborhood:*

A neighborhood of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the radius of $N_r(p)$.

(b) *Limit point:*

A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) *Isolated point:*

If $p \in E$ and p is not a limit point of E , the p is called an Isolated point.

(d) *Closed set:*

E is closed if every limit point of E is a point of E .

(e) *Interior point:*

A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.

(f) *Open set:*

E is open if every point of E is an interior point of E .

(g) *Complement of a set:*

The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) *Perfect set:*

E is perfect if E is closed and if every point of E is a limit point of E .

(i) *Bounded set:*

E is bounded if there is a real number M and a point $q \in E$.

(j) *Dense set:*

E is dense in X if every point of X is a limit point of E , or a point of E (or both).

Let us note that in R^1 neighborhood are segments, whereas in R^2 neighborhoods are interior of circles.

Theorem 3.1:

Every neighborhood is an open set.

Proof:

Consider a neighborhood $E = N_r(p)$, and let q be any point of E .

Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(p, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E .

Theorem 3.2:

If p is a limit point of a set E , then every neighborhood of p contains infinitely many points of E .

Proof:

Suppose there is a neighborhood N of p which contains only a finite number of points of E . Let $q_1, \dots, q_n$ be those points of $N \cap E$, which are distinct from p , and put

$$r = \min d(p, q_m) \quad 1 \leq m \leq n.$$

[we use this notation to denote the smallest of the number $d(p, q_1), \dots, d(p, q_n)$]. The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E . This contradiction establishes the theorem.

Corollary: A finite point set has no limit points.

Theorem 3.3:

Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α . Then

$$(\cup_\alpha E_\alpha)^c = (\cap_\alpha E_\alpha^c)$$

Proof:

Let A and B be the left and right members of limit point of E . If $x \in A$, then $x \notin \bigcup_{\alpha} E_{\alpha}$, hence $x \notin E_{\alpha}$ for any α , hence $x \in E_{\alpha}^c$ for every α , so that $x \in \bigcap_{\alpha} E_{\alpha}^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_{\alpha}^c$ for every α , $x \notin E_{\alpha}$ for any α , hence $x \notin (\bigcup_{\alpha} E_{\alpha})^c$ so that $x \in (\bigcup_{\alpha} E_{\alpha})^c$. Thus $B \subset A$.

It follows that $A = B$.

Theorem 3.4:

A set E is open if and only if its complement is closed.

Proof:

First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c . Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E , and E is open.

Next, suppose E is open. Let x be a limit point of E^c . Then every neighborhood of x contains a point of E^c , so that x is not an interior point of E . Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary: A set F is closed if and only if its complement is open.

Theorem 3.5:

(a) For any collection $\{G_{\alpha}\}$ of open sets, $\bigcup_{\alpha} G_{\alpha}$ is open.

(b) For any collection $\{F_{\alpha}\}$ of closed sets, $\bigcap_{\alpha} F_{\alpha}$ is closed.

(c) For any finite collection $G_1, \dots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, \dots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Proof:

Put $G = \bigcup_{\alpha} G_{\alpha}$. If $x \in G$, then $x \in G_{\alpha}$ for some α . Since x is an interior point of G_{α} , x is also an interior point of G , and G is open. This proves (a).

By Theorem finite or infinite collection of sets E_{α} ,

$$(\bigcap_{\alpha} F_{\alpha})^c = \bigcup_{\alpha} (F_{\alpha}^c)$$

and F_{α}^c is open, by the above theorem. Hence (a) implies that the function is open so $\bigcap_{\alpha} F_{\alpha}$ that is closed.

Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x with radii r_i , such that $N_i \subset G_i$ ($i = 1, \dots, n$). Put

$$r = \min(r_1, \dots, r_n),$$

and let N be the neighborhood of x of radius r . Then $N \subset G_i$ for $i = 1, \dots, n$, so that $N \subset H$, and H is open. By taking complements, (d) follows from (c):

$$(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n (F_i^c).$$

Definition:

If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X , then the closure of E is the set $\bar{E} = E \cup E'$.

Theorem 3.6:

If X is a metric space and $E \subset X$, then

- (a) E is closed,
- (b) $E = \bar{E}$ if and only if E is closed,
- (c) $\bar{E}$ for every closed set $F \subset X$ such that $E \subset F$.

By (a) and (c), $\bar{E}$ is the smallest closed subset of X that contains E .

Proof:

- (a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E .

Hence p has a neighborhood which does not intersect E . The complement of E is therefore open. Hence $\bar{E}$ is closed.

(b) If $E = \bar{E}$ implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions], hence $\bar{E} = E$

(c) If F is closed and $F \supset E$ then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

Definition:

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition:

A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K , then there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

The notion of compactness is of great importance in analysis, especially in connection with continuity.

It is clear that every finite set is compact. The existence of a large class of infinite compact sets will be following theorem.

We observed earlier that if $E \subset Y \subset X$, then E may be open relative to Y without being open relative to X . The property of being open thus depends on the space in which E is embedded. The same is true of the property of being closed.

Compactness, however, behaves better, as we shall now see. To formulate the next theorem, let us say, temporarily, that K is compact relative to X if the requirements of the definition.

Theorem 3.7:

Compact subsets of metric spaces are closed.

Proof:

Let K be a compact subset of a metric space X . We shall prove that the complement of K is an open subset of X .

Suppose $p \in X$, $p \notin K$. If $q \in K$, let V_q and W_q be neighborhoods of p and q respectively, of radius less than $\frac{1}{2}d(p, q)$. Since K is compact, there are finitely many points, $q_1, \dots, q_n$ in K such that

$$K \subset W_{q_1} \cup \dots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \dots \cap V_{q_n}$ then V is a neighborhood of p which does not intersect W . Hence $V \subset K^c$, so that p is an interior point of K^c . The theorem follows.

Theorem 3.8:

Closed subsets of compact sets are compact.

Proof:

Suppose $F \subset K \subset X$, F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F . If F^c is adjoined to $\{V_\alpha\}$ we obtain an open cover Ω of K . Since K is compact, there is a finite subcollection ϕ of Ω which covers K and hence F . If F^c is a member of ϕ , we may remove it from ϕ and still retain an open cover of F . We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F .

Corollary: If F is closed and K is compact, then $F \cap K$ is compact.

Definition:

Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A .

A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

Theorem 3.9:

A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Proof:

If there exist $x \in E$, $y \in E$, and some $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where

$$A_z = E \cap (-\infty, z), \quad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence E is not connected.

To prove the converse, suppose E is not connected. Then there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup (A \cap [x, y]).$$

By Theorem of a nonempty set of real numbers $z \in \bar{A}$; hence $z \notin B$. In particular, $x < z < y$.

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \in \bar{B}$, hence there exists such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

Definition:

Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y , and p is a limit point of E . We write $f(x) \rightarrow q$ as $x \rightarrow p$, or

$$\lim_{x \rightarrow p} f(x) = p$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y , respectively.

If X and /or Y are replaced by the real line, the complex plane, or by some euclidean space R^k , the distances d_X , d_Y are of course replaced by absolute values, or by norms of differences.

It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \rightarrow p} f(x)$.

We can recast this definition in terms of limits of sequences:

Theorem 3.10:

Let X, Y, E, f , and p be as in above definition. Then

$$\lim_{x \rightarrow p} f(x) = p$$

if and only if $\lim_{n \rightarrow \infty} f(p_n) = q$.

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \quad \lim_{n \rightarrow \infty} p_n = p.$$

Proof:

Suppose $\lim_{x \rightarrow p} f(x) = p$ holds. Choose $\{p_n\}$ in E satisfying $\lim_{n \rightarrow \infty} p_n = p$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that $\lim_{n \rightarrow \infty} f(p_n) = q$ holds.

Conversely, suppose $\lim_{x \rightarrow p} f(x) = p$ is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$, for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ ($n = 1, 2, 3, \dots$), we thus find a sequence in E satisfying $\lim_{n \rightarrow \infty} p_n = p$ for which $\lim_{n \rightarrow \infty} f(p_n) = q$ is false.

Corollary: If f has a limit at p , this limit is unique.

Theorem 3.11:

Suppose $E \subset X$, a metric space, p is a limit point of E , f and g are complex functions on E , and

$$\lim_{x \rightarrow p} f(x) = A \quad \quad \quad \lim_{x \rightarrow p} g(x) = B.$$

Then

- (a) $\lim_{x \rightarrow p} (f + g)(x) = A + B$;
- (b) $\lim_{x \rightarrow p} (fg)(x) = AB$;
- (c) $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Definition:

Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y . Then

f is said to be continuous at p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E , then f is said to be continuous on E .

It should be noted that f has to be defined at the point p in order to be continuous at p .

If p is an isolated point of E , then our definition implies that every function f which has E as its domain of definition is continuous at p . For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$; then

$$d_Y(f(x), f(p)) = 0 < \varepsilon.$$

Theorem 3.12:

A mapping f of a metric space X into a metric space Y is continuous on X if and only iff $f^{-1}(V)$ is open in X for every open set V in Y .

Proof:

Suppose f is continuous on X and V is an open set in Y . We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if $d_Y(f(p), y) < \varepsilon$; and since f is continuous at p , there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y . Fix $p \in X$ and $\varepsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \varepsilon$. Then V is open; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(p, x) < \delta$. But if $x \in f^{-1}(V)$ then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$.

This completes the proof.

Corollary:

A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y .

This follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.

We now turn to complex-valued and vector-valued functions, and to functions defined on subsets of $\mathbb{R}^k$.

Definition:

A mapping f of a set E into $\mathbb{R}^k$ is said to be bounded if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Statement:

Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Statement:

If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $f(X)$ is closed and bounded. Thus, f is bounded.

Definition:

Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$, form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x}$$

and define,

$$f'(x) = \lim_{t \rightarrow x} \phi(t),$$

provided this limit exists in accordance with continuity definitions.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit exists; f' is called the derivative of f .

If f' is defined at a point x , we say that f is differentiable at x . If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E . It is possible to consider right-hand and left-hand limits in above equation; this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b , the derivative, if it exists, is a right-hand or left-hand derivative, respectively. We shall not, however, discuss one-sided derivatives in any detail.

If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined. But $f'(a)$ and $f'(b)$ are not defined in this case.

Theorem 3.13:

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x .

Proof:

As $t \rightarrow x$, we have,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0.$$

The converse of this theorem is not true. It is easy to construct continuous functions which fail to be differentiable at isolated points. We shall even become acquainted with a function which is continuous on the whole line without being differentiable at any point!

Note: Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg , and f/g are differentiable at x , and

$$(a) (f + g)'(x) = f'(x) + g'(x);$$

$$(b) (fg)'(x) = f'(x)g(x) + f(x)g'(x);$$

$$(c) \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

(In (c), we assume of course that $g(x) \neq 0$.)

Theorem 3.14:

The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of (b) and (c) then shows that x^n is differentiable, and that its derivative is nx^{n-1} , for any integer n (if $n < 0$, we have to restrict ourselves to $x \neq 0$). Thus every polynomial is differentiable, and so is every rational function, except at the points where the denominator is zero.

The following theorem is known as the “chain rule” for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives.

Note:

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f , and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x , and

$$h'(x) = g'(f(x)) f'(x).$$

Examples:

Let f be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Taking for granted that the derivative of $\sin x$ is $\cos x$ (we shall discuss the trigonometric functions).

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad (x \neq 0).$$

At $x = 0$, these theorems do not apply any longer, since $1/x$ is not defined there, and we appeal directly to the definition: for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \neq 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

(a) Let f be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

As above, we obtain

$$f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad (x \neq 0).$$

At $x = 0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t| \quad (t \neq 0);$$

letting $t \rightarrow 0$, we see that

$$f'(0) = 0.$$

Thus f is differentiable at all points x , but f' is not a continuous function, since $\cos(1/x)$ in above does not tend to a limit as $x \rightarrow 0$.

Definition:

Let f be a real function defined on a metric space X . We say that f has a local maximum at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

4. MEAN VALUE THEOREM AND TAYLORS THEOREM

GENERALISED MEAN VALUE THEOREM:

If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b) , then there is a point $x \in (a, b)$ at which

$$[f(a) - f(b)] g'(x) = [g(b) - g(a)] f'(x)$$

Note that differentiability is not required at the endpoints.

Proof:

Put

$$h(t) = [f(b) - f(a)] g(t) - [g(b) - g(a)] f(t) \quad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b) , and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$. If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum theorem shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a generalized mean value theorem; the following special case is usually referred to as “the” mean value theorem:

MEAN VALUE THEOREM:

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) , then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(x).$$

Proof:

Take $g(x) = x$ in the above theorem and hence the proof.

Definition:

If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f . Continuing in this manner, we obtain functions

$$f, f', f'', f^{(3)}, \dots, f^{(n)}.$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the n th derivative, or the derivative of order n , of f .

In order for $f^{(n)}(x)$ to exist at a point x , $f^{(n-1)}(t)$ must exist in a neighborhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x . Since $f^{(n-1)}$ must exist in a neighborhood of x , $f^{(n-2)}$ must be differentiable in that neighborhood.

TAYLOR'S THEOREM**Theorem:**

Suppose f is a real function on $[a, b]$, n is a positive integer, f is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n.$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$, and that allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof:

Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n.$$

and put

$$g(t) = f(t) - P(t) - M(t - \alpha)^n \quad (a \leq t \leq b).$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β .

$$g^{(n)}(t) = f^{(n)}(t) - n!M \quad (a < t < b).$$

Hence the proof will be complete if we can show that $g^{(n)} = 0$ for some x between α and β .

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \dots, n-1$, we have

$$G(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β , by the mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1 . After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1} that is, between α and β .

5. CONCLUSION

From a general perspective, the most important way to construct a topology on a set is to define the topology in terms of a metric on the set. This approach underlies our intuitive understanding of open and closed sets on the real line. Generally speaking, a metric captures the notion of a distance between two elements of a set. Topologies that are defined through metrics possess a number of properties that make them suitable for analysis. Identifying these common properties permits the unified treatment of different spaces that are useful in solving engineering problems. To gain better insight into metric spaces, we need to review the notion of a metric and to introduce a definition for topology.

There are three important theorems that uses the concept of topology in real analysis which are:

1. Extreme Value Theorem: A continuous function f on a closed bounded interval attains its maximum and minimum value on that interval.
2. A continuous function on a closed bounded interval is uniformly continuous.
3. Intermediate Value Theorem: Let f be a continuous function on $\mathbb{R}$ such that there are real numbers a, b for which $f(a) < 0$ and $f(b) > 0$. Then there exist $c \in (a, b)$ such that $f(c) = 0$.

The first two theorems make use of a topological property known as compactness. The Heine – Borel Theorem is what tells you that in $\mathbb{R}$ with Euclidean metric, being closed and bounded is equivalent to being compact. Number 3 makes use of the topological property that $\mathbb{R}$ with the national metric is connected.

Number 2 is especially important to know that why a continuous functions on a closed bounded interval is Riemann Integrable; this should be enough motivation for you to study topology.

The following are the example of where things get generalised in topology. In a metric space (in fact in a topological space that is at least T_2) this cannot happen.

Put the trivial on $\mathbb{R}$, the family $A_n = (0, 1/n)$ is a countable collection of non – empty compact sets such that the intersection of any finite sub-collection is non-empty, but clearly

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

The part of topology that is useful in real analysis is called “point set topology”. The three relevant topics of that part of topology are

- 1) Continuity, which you’ve already had in “regular” real analysis,
- 2) Connectivity and
- 3) Compactness.

In this work, we have introduced the concept of metric space, discussed the motivation behind studying this concept. We also discussed how this concept is being used in real life scenario. Being closely related to the notion of distance, it has many applications in the field of navigation. One such instance where it is being associated with the navigation of aeroplanes is discussed. We have discussed various metrics on the set of reals, on $\mathbb{R}^n$, on the set of functions etc. to illustrate the abstract nature of the distance. Later in the chapter, the notions of metric subspace and superspace were introduced and a method of finding a metric superspace from a given metric space is discussed. In fact, we observed that we can have uncountable extensions of a given metric to some superset of that space. We also touched upon the concept of isometry i.e., distance preserving functions and discussed some standard examples associated with this concept. Finally we discussed special types of metrics determined by a norm on a vector space, which relates the algebraic properties of the vector space with its geometric properties. We hope that the content presented in the chapter would be of immense help to the students who are beginners in the subject. Hence, we can conclude that many proofs in real analysis is much smoother and easier to understand once we stop thinking about δ and ϵ and consider everything in terms of neighborhood in alternate. This substantiates the role of topology and metric space in real analysis.

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