

SOME STUDIES ON INDEPENDENT DOMINATION IN GRAPHS

Submitted in partial fulfilment of the requirements for the award of Bachelor of
Science degree in Mathematics

by

P. Surya (38060012)



DEPARTMENT OF MATHEMATICS

SCHOOL OF SCIENCE & HUMANITIES

SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

Accredited with Grade "A" by NAAC

JEPPIAAR NAGAR, RAJIV GANDHI SALAI, CHENNAI - 600 119

April -2021



SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited with "A" grade by NAAC
Jeppiaar Nagar, Rajiv Gandhi Salai, Chennai – 600 119
www.sathyabama.ac.in

DEPARTMENT OF MATHEMATICS

BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **P. Surya (38060012)** who carried out the project entitled "**Some studies on independent domination in graphs**" under my supervision from January 2021 to April 2021

(Dr.T.KAMARAJ)

Internal Guide
(Name in Capital letters
with Signature)

External Guide (if Applicable)
(Name in Capital letters
with signature)

(Dr. M.NIRMALA)

Head of the Department
(Name in Capital letters with Seal & Signature)

Submitted for Viva voce Examination held on _____

Internal Examiner

(Name in Capital letters
letterswith Signature) with signature)

External Examiner

(Name in Capital

DECLARATION

P. Surya (38060012) hereby declare that the Project Report entitled “**Some studies on independent domination in graphs**” done by me under the guidance of **Dr.T.Kamaraj** (Internal) and _____ (External) at Sathyabama institute of Science and Technology is submitted in partial fulfilment of the requirements for the award of Bachelor of Science degree in Mathematics.

DATE: 27-03-2021



PLACE: Chennai

SIGNATURE OF THE CANDIDATE

ACKNOWLEDGEMENT

I am pleased to acknowledge my sincere thanks to Board of Management of SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY for their kind encouragement in doing this project and for completing it successfully. I am grateful to them.

I convey my thanks to Dr. Nirmala, Head of the Department, Dept. of Mathematics for providing me necessary support and details at the right time during the progressive reviews.

I would like to express my sincere and deep sense of gratitude to my Project Guide Dr. T. Kamaraj for his valuable guidance, suggestions and constant encouragement paved way for the successful completion of my project work.

I wish to express my thanks to all Teaching and Non-teaching staff members of the Department of Mathematics who were helpful in many ways for the completion of the project.

ABSTRACT

A dominating set in a Graph G is a set of vertices D such that each vertex of G is either in D or has a neighbour in D . An independent set in a graph G is a set of vertices in G such that no two of which are adjacent. An independent dominating set in a graph G is a set of vertices which is both independent and dominating set. Independent domination number is the minimum cardinality of an independent dominating set.

In this thesis, we offer a survey of recent results on independent domination in some selected graphs like wheel related graphs, snake like graphs, number theory related graphs and product graphs.

TABLE OF CONTENTS

ABSTRACT.....	(i)
LIST OF FIGURES.....	(iv)
LIST OF TABLES.....	(v)
CHAPTERS	
1. INTRODUCTION	1
1.1 Graph Theory	1
1.1.1 Common Graphs	2
1.2 Domination in graphs	4
1.2.1 Dominating sets and domination numbers	5
1.3 Independent domination	6
1.3.1 Values in common graphs	7
2.LITERATURE SURVEY	8
3.AIM AND SCOPE OF THE PRESENTATION	9
4.INDEPENDENT DOMINATION ON VARIOUS TYPES OF GRAPHS	10
4.1 Wheel related graphs	10
4.1.1 Wheel graph	10
4.1.2 Helm graph	10
4.1.3. Closed helm graph	11
4.1.4 Web graph	12
4.1.5 Gear graph	13

4.2 Triangular and snake like graphs	13
4.2.1 Triangular snake	13
4.2.2 Alternate Triangular snake	14
(ii)	
4.2.3 Double Triangular snake	15
4.2.4 Double Alternate Triangular snake	16
4.2.5 Quadrilateral snake	16
4.2.6 Alternate Quadrilateral snake	17
4.3 Independent domination in graph based on number theory	17
4.3.1 Cayley graph	17
4.3.2 Euler totient Cayley graph(G_n, φ)	18
4.3.3 Arithmetic graph	19
4.4 Products graphs	20
4.4.1 Corona product of graphs	20
4.4.2 Lexicographic product of graphs	21
4.4.3 Cartesian product of graphs	21
5. RESULTS AND DISCUSSION	23
6. SUMMARY AND CONCLUSION	31
7. REFERENCES	32

(iii)

LIST OF FIGURES

1.1 Konigsberg bridge problem	1
1.2 Complete graph	2
1.3 Bipartite graph	3
1.4 Queens on chess board	5
1.5 Dominating sets	6
1.6 Domination number	6
1.7 Independent domination	7
4.1 Wheel graph	10
4.2 Helm graph	10
4.3 Independent domination on helm H_5	11
4.4 Independent domination on closed helm CH_9	11
4.5 Web graph $W(m,n)$	12
4.6 Independent domination on $W(3,4)$	12
4.7 Gear graph	13
4.8 Independent domination on gear G_5	13
4.9 Triangular snake T_n	14
4.10 Independent domination of Triangular snake T_6	14
4.11 Independent domination on Triangular snake T_7	14
4.12 Independent domination on $A(T_6)$	15

4.13 Independent domination on $D(T_6)$	15
4.14 Independent domination on $D(T_7)$	16
4.15 Independent domination on $D(A(T_6))$	16
(iv)	
4.16 Independent domination on Q_6	17
4.17 Independent domination on $A(Q_6)$	17
4.18 Independent domination on $A(Q_8)$	17
4.19 Independent domination on Euler totientCayley graph $G(Z_{11}, \varphi)$	19
4.20 Independent domination on Arithmetic V_{60} graph	20
4.21 The corona product two graphs	21
4.22 Independent domination on Corona product of graphs $(G \circ H)$	21
4.23 Lexicographic product of two graphs	22
4.24 Cartesian product of two graphs	22

LIST OF TABLES

4.1 Independent domination on Euler totientCayley graph	19
4.2 Independent domination on Arithmetic V_{60} graph	20

(v)

1. INTRODUCTION

Domination is an area of graph theory with an extensive research activity. The study of domination sets and domination numbers in graphs assumes a lot of significance as a number of its variations are applied to practical problems. A dominating set in a Graph is a set of vertices D such that each vertex of the graph is either in D or has a neighbour in D . An independent set in a graph is a set of vertices such that no two of which are adjacent. An independent dominating set in a graph is a set of vertices which is both independent and dominating set. Independent domination number is the minimum cardinality of an independent dominating set. Mathematical study of domination in graphs began around 1960. In the following sequel we provide some basic definitions about graph theory in general, followed by a discussion of domination and independent domination in graphs.

1.1 GRAPH THEORY

Graph theory, branch of mathematics concerned with networks of points connected by lines. The subject of graph theory had its beginnings in recreational math problems, but it has grown into a significant area of mathematical research, with applications in chemistry, operation research, social sciences, and computer science.

The history of graph theory may be specifically traced to 1735, when the swiss mathematician Leonhard Euler solved the Königsberg bridge problem. The Königsberg bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice. Euler argued that no such path exists. His proof involved only references to the physical arrangement of the bridges, but essentially, he proved the first theorem in graph theory.

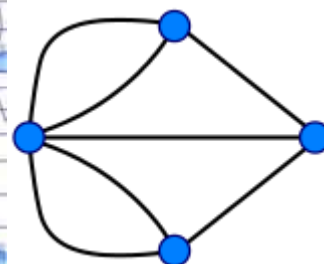
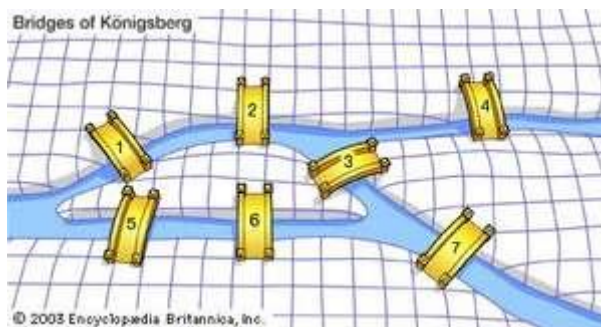


Fig: 1.1 Königsberg bridge problem

A graph $G = (V, E)$ consists of a non-empty set V of vertices and a set E of edges. We shall only consider simple graphs, which contain no loops and no repeated edges. That is, E is a set of unordered pairs $\{u, v\}$ of distinct elements from V . The order of G is $|V(G)| = n$, and the size of G is $|E(G)| = m$. If $e = \{v_i, v_j\} \in E(G)$, then v_i and v_j are adjacent. Vertex v_i and edge e are said to be incident

Degree of a vertex

The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $\deg_G(v)$ or $\deg(v)$ or $d(v)$. The maximum and the minimum degrees of the vertices of G are respectively denoted by $\Delta(G)$ and $\delta(G)$.

1.1.1 Common Graphs

Complete Graph

The complete graph K_n has the maximum possible edges $n(n-1)/2$. See Figure 1.2 for the graphs of K_4 and K_5 . The complement $\bar{G}$ of a graph G has $V(\bar{G}) = V(G)$ and $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$. Thus, the complement of a complete graph is the empty graph

Example

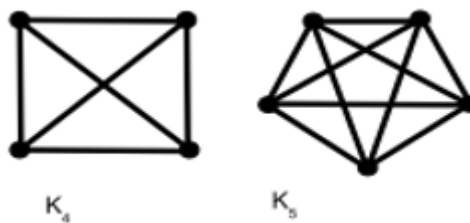


Fig: 1.2 Complete graph

Bipartite graph

A bipartite graph is one that can be partitioned as $V = V_1 \cup V_2$ with no two adjacent vertices in the same V_i . We define the chromatic number of a graph G to be the minimum k such that $V(G)$ can be partitioned into sets $S_1, S_2, \dots, S_k$ and each S_i is independent. That is, for each i , no two vertices in S_i are adjacent. Denote the chromatic number of G by $X(G)$. If $X(G) = k$, then G is k -colorable which means we can color the vertices of G with k colors in such a way that no two adjacent vertices

are the same color. Observe that a graph is 2-colorable if and only if it is a bipartite graph.

Example

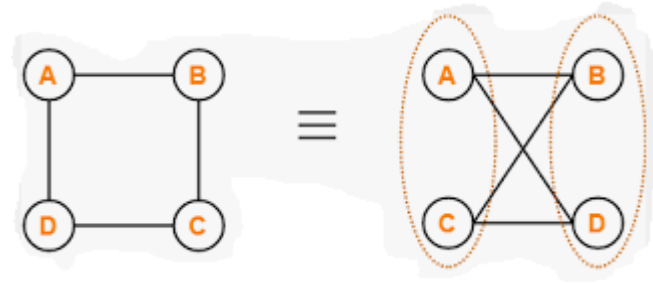


Fig:1.3 Bipartite graph

Path Graph (P_n)

A path graph or linear graph is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \dots, n - 1$. Equivalently, a path with at least two vertices is connected and has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2.

Cycle graph (C_n)

A simple graph with ' n ' vertices ($n \geq 3$) and ' n ' edges is called a cycle graph if all its edges form a cycle of length ' n '.

If the degree of each vertex in the graph is two, then it is called a Cycle Graph. It is denoted by C_n .

Regular graph

A graph G is said to be regular, if all its vertices have the same degree. In a graph, if the degree of each vertex is ' k ', then the graph is called a ' k -regular graph'.

Cubic graph

In the mathematical field of graph theory, a cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3-regular graph. Cubic graphs are also called trivalent graphs.

1.2 DOMINATION IN GRAPHS

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics around 1860. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board. Three basic types of problems that chess players studied during this time. These include the following:

1. **Covering:** Determine the minimum number of chess pieces of a given type that are necessary to cover (attack) every square of an $n \times n$ chess board.
2. **Independent Covering:** Determine the smallest number of mutually nonattacking chess pieces of a given type that are necessary to dominate every square of an $n \times n$ board
3. **Independence:** Determine the maximum number of chess pieces of a given type that can be placed on an $n \times n$ chess board such that no two pieces attack each other. Note that if the chess piece being considered is the queen, this type of problem is commonly known as the N-queens Problem.

A standard 8×8 chessboard, can be seen in Figure 1.4. Each square can be represented by a vertex in a graph G . Consider placing several queens on the board. A queen may move any number of spaces vertically, horizontally, or diagonally. Any square (or vertex) to which a queen is able to move is adjacent to the square containing the queen. Therefore, there is an edge between those two squares, or vertices of the graph G . One is interested in finding minimum number of queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. A graph G may be formed from an 8×8 chessboard by taking the squares as the vertices with two vertices adjacent if a queen situated on one square attacks the other square. The graph G is known as the queens graph. The minimum number of queens that attack all the squares of a chessboard is the domination number $\gamma(G)$. For the queens graph G , we note that $\gamma(G) = 5$

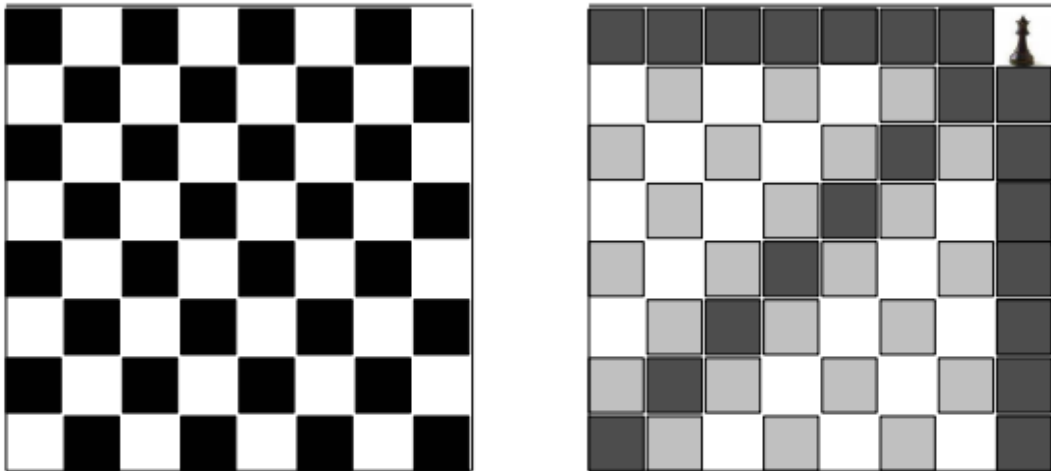


Fig: 1.4 Queens on chess board

1.2.1 Dominating Sets and Domination Numbers

A set $D \subseteq V(G)$ is called a dominating set if every vertex in $V(G) - D$ is adjacent to some vertex of D . A vertex u is said to be dominated by a vertex $v \in G$ if either $u = v$ or $u v \in E(G)$.

Example

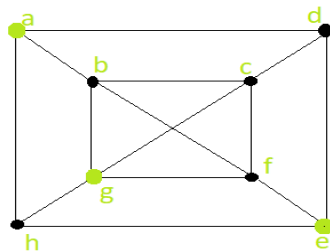


Fig: 1.5 Dominating sets

In this graph of 8 vertices and 14 edges. Every black vertex is adjacent to at least one green vertex, which is then said that the black vertex is dominated by the green vertex. So, dominate set of this graph is $S = \{a, g, e\}$

Domination number

The dominating set with minimum cardinality is called the γ -set and its cardinality is the domination number $\gamma(G)$.

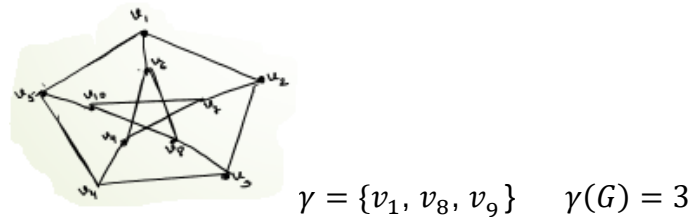


Fig: 1.6 Domination number

1.3 INDEPENDENT DOMINATION

The idea of an independent dominating set arose in chessboard problems. In 1862, De Jaenisch [4] posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. The minimum number of mutually non-attacking queens that attack all the squares of a chessboard is the independent domination number $i(G)$. For the queens graph G , we note that $i(G) = 7$

Independent dominating set

A set $D \subseteq V$ is an Independent set of G if $u, v \in D, N(u) \cap \{v\} = \emptyset$.

A set $D \subseteq V$ is an Independent dominating set of G if D is both an independent and a dominating set. Minimum cardinality of independent dominating set is denoted by $i(G)$. Maximum cardinality of an independent set is known as independence number of graph G , denoted by $\beta(G)$. The following result is simple to visualize,

$$\gamma(G) \leq i(G) \leq \beta(G)$$

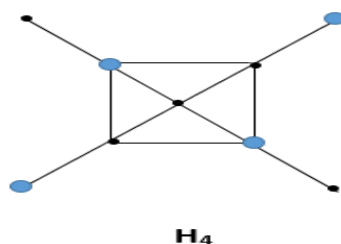


Fig: 1.7 independent domination

The highlighted blue color vertices represent the independent domination set

1.3.1 Values in common graphs

- (a) If G is a connected for the path and cycle, $i(P_n) = i(C_n) = \lceil n/3 \rceil$
- (b) If G is a bipartite graph without isolated vertices on n vertices, then $i(G) \leq n/2$.
- (c) If G is a r -regular graph on n vertices with no isolated vertex, then $i(G) \geq n/(r + 1)$
- (d) If G is a Cubic graph on n vertices other than $K_{3,3}$ then $i(G) \leq 2n/5$.

2. LITERATURE SURVEY

- The study of domination in graphs was developed mathematically in the late 1960's, beginning with Claude Berge [2] in 1962. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability," which is now known as the domination number of a graph.
- Oystein Ore [10] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962.
- The theory of independent domination was formalized by Berge [2] and Ore [10] in 1962.
- A decade later, The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [3].
- Domination and independent domination numbers of some common graphs are discussed by Allan, R. B. and Laskar, R in [1]. Independent dominating sets in regular graphs and in cubic graphs in particular, are studied by Goddard [5]
- A wheel graph is a graph [6] formed by connecting a single universal vertex to all vertices of a cycle. Vaidya and Pandit [15] studied independent domination in some wheel related graphs. The triangular snake [13] T_n is obtained from the path P_n by replacing each edge of the path by a triangle C . In [11] independent domination on triangular and quadrilateral snake graphs are studied
- Nathanson [9] paved the way for the emergence of a new class of graphs, namely, Arithmetic Graphs by introducing the concepts of Number Theory, particularly, the Theory of congruences in Graph Theory.
- The Cayley graph associated with Euler Totient function is called an Euler Totient Cayley graph. Madhavi [8] introduced the concept of Euler totient Cayley graphs and studied some of its properties.

- Independent domination in number theory based graphs is studied in [14]
- The product graphs and their domination sets are studied respectively in [7] and [12]

3.AIM AND SCOPE

The main aim of this thesis work is to survey the study of independent domination insome variant of wheel and path graphs and also in graphs based on number theory.

The concept of independent dominating sets occur in variety of practical problems. A number of these problems are motivated by communication network problems. For example,the communication network includes a set of nodes, where one node can communicate with another if it is directly connected to that node. In order to send a message directly from a set of nodes (senders) to all others (receivers), one needs to choose this set such that all others nodes are connected to at least one node in the (sender) set. Now, such a set is an independent dominating set in a graph which represents the network.

For other applications of independent domination, the job allocation problem, land surveying, and routings can be mentioned. For an example of Job allocation problem, consider a bipartite graph, where one part represents jobs and other part represents skills and edges represent skills of each person. Each person may take more than one job. One is interested to find the minimum number of people such that all jobs are engaged.

4. INDEPENDENT DOMINATION IN VARIOUS TYPES OF GRAPHS

4.1 WHEEL RELATED GRAPHS

4.1.1 Wheel graph

The wheel W_n is defined to be the joint $C_{n-1}+K_1$. The vertex corresponding to K_1 is known as apex vertex and the vertices corresponding to cycle are known as rim vertices. The edges corresponding to cycle are known as rim edges and the edges incident with the apex vertex are known as spoke edges.

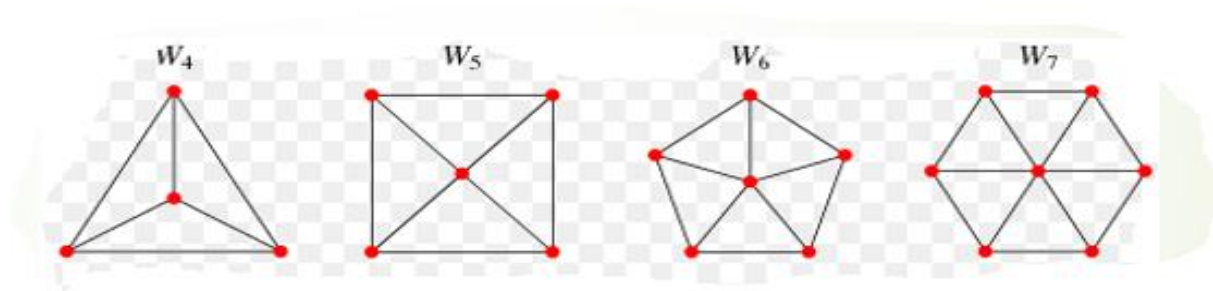


Fig:4.1 wheel graph

4.1.2 Helm graph

The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices.

Example

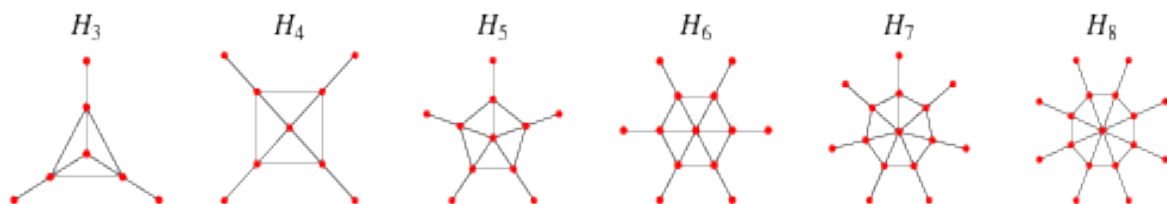


Fig:4.2 Helm graph

Independent domination on helm H_5

$$i(H_5) = 5$$

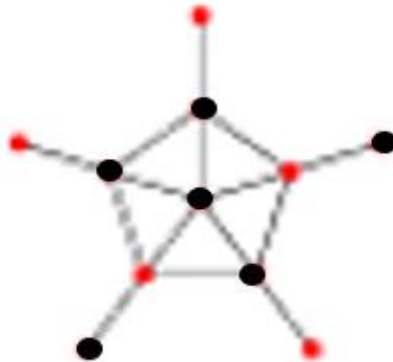


Fig: 4.3 Independent domination on the helm H_5

The highlighted red color vertices represent the independent domination set.

4.1.3 Closed helm graph

The closed helm CH_n is the graph obtained from a $helmH_n$ by joining each pendant vertex to form a cycle.

Independent domination on the closed helm CH_9

$$i(CH_9) = 6$$

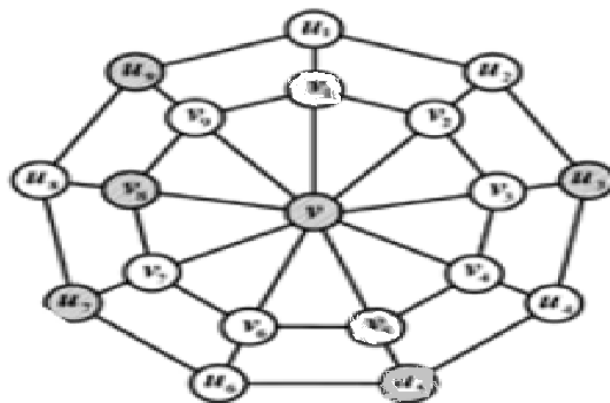


Fig:4.4 Independent domination on closed helm CH_9

The highlighted vertices represent the independent domination set.

4.1.4 Web graph

A web graph is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Generalized Web Graph $W(m,n)$ contains m number of cycles and n number of vertices in each cycle

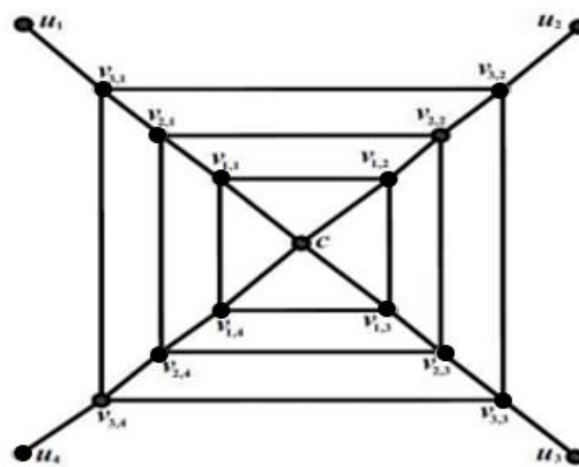


Figure 4.5 web graph $W(m,n)$

Independent domination on $w(3, 4)$

$$i(w(3, 4)) = 6$$

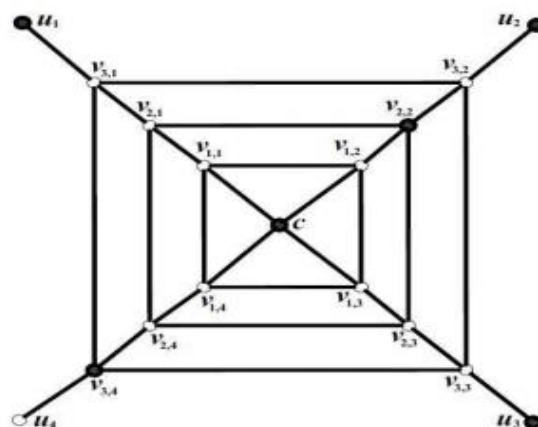


Fig:4.6 Independent domination on $w(3,4)$

The highlighted vertices represent the independent domination set.

4.1.5 Gear graph

A gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the $(n-1)$ -cycle of W_n .

Example

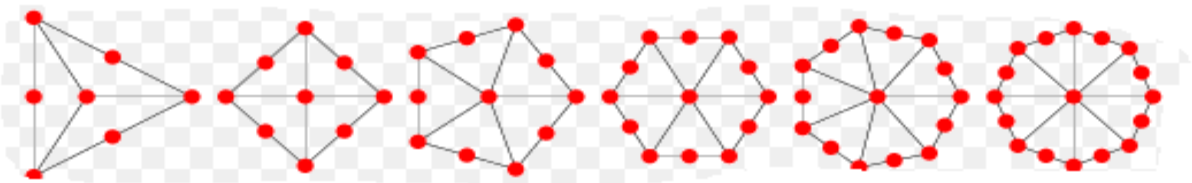


Fig:4.7 Gear graph

Independent domination on Gear G_5

$$i(G_5)=4$$

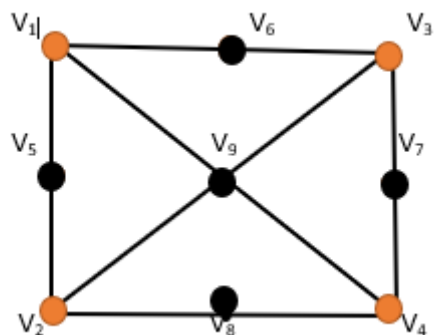


Fig:4 .8 independent domination on Gear G_5

The highlighted vertices represent the independent domination set.

4.2 TRIANGULAR AND SNAKE LIKE GRAPHS

4.2.1 Triangular snake:

The triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3 .

Example

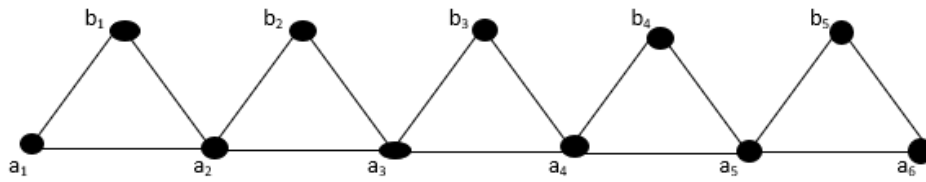


Fig: 4.9 Triangular snake T_n

Independent domination of Triangular snake T_n

$$i(T_6) = 3$$

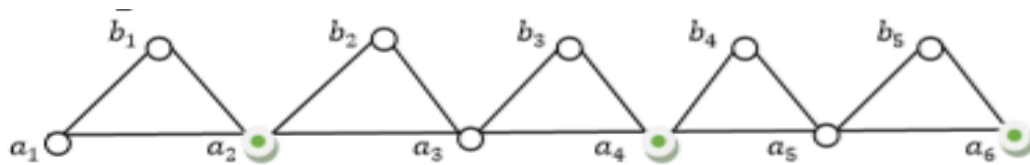


Fig: 4.10 Independent domination of Triangular snake T_6

$$i(T_7) = 3$$

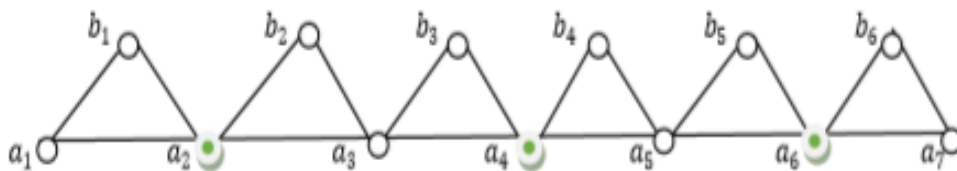


Fig: 4.11 independent domination on Triangular snake T_7

The highlighted vertices represent the independent domination set.

4.2.2 Alternate triangular snake

An alternate triangular snake $A(T_n)$ is obtained from a path $a_1, a_2, \dots, a_n$ by joining a_i and a_{i+1} (alternately) to a new vertex V_i .

Independent domination on $A(T_6)$

$$i(A(T_6)) = 3$$

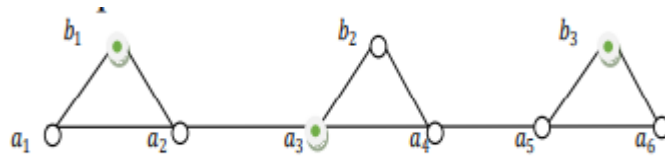


Fig: 4.12 Independent domination on $A(T_6)$

The highlighted vertices represent the independent domination set.

4.2.3 Double Triangular snake:

A double triangular snake $D(T_n)$ consists of two triangular snakes that have a common path

Independent domination on $D(T_n)$

$$i(D(T_6)) = 3$$

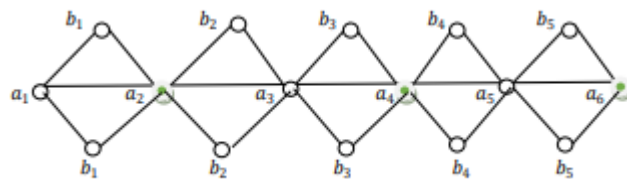


Fig: 4.13 Independent domination on $D(T_6)$

$$i(D(T_7)) = 3$$

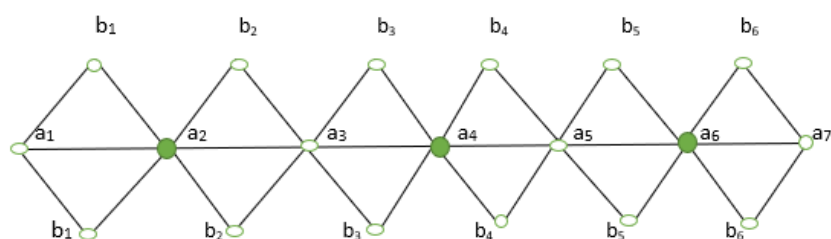


Fig: 4.14 Independent domination on $D(T_7)$

The highlighted vertices represent the independent domination set.

4.2.4 Double Alternate triangular snake:

A double alternate triangular snake $D(A(T_n))$ consists of two alternate triangular snakes that have a common path.

Independent domination on $D(A(T_6))$

$$i(D(A(T_6))) = 3$$

For example

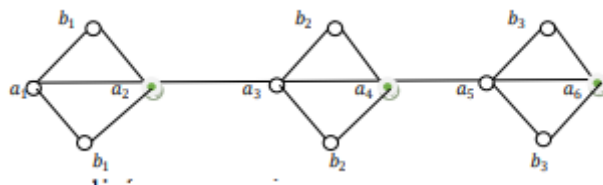


Fig: 4.15 Independent domination on $D(A(T_6))$

The highlighted vertices represent the independent domination set.

4.2.5 Quadrilateral snake:

A quadrilateral snake Q_n is obtained from a path $a_1, a_2, \dots, a_n$ by joining a_{i+1} to new vertices b_i and c_i respectively and joining the vertices b_i and c_i for $i=1, 2, \dots, n-1$. That is every edge of a path is replaced by a cycle C_4 .

Independent nomination on Q_n

$$i(Q_6) = 6$$

Example

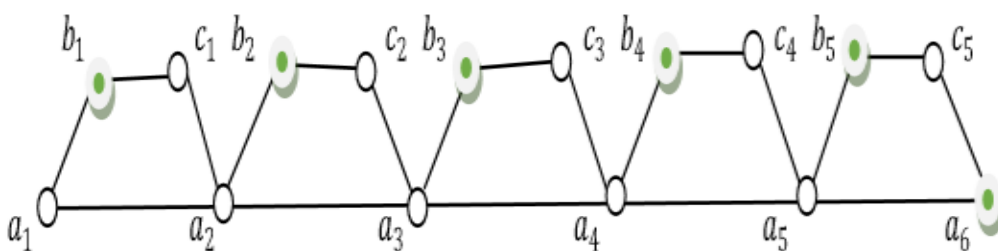


Fig: 4.16 Independent domination on Q_6

The highlighted vertices represent the independent domination set

4.2.6 Alternate quadrilateral snake:

An alternate quadrilateral snake $A(Q_n)$ is obtained from a path $a_1, a_2, \dots, a_n$ by joining a_i and a_{i+1} to new vertices b_i and c_i respectively and joining the vertices b_i and c_i for $i \equiv 1 \pmod{2}$ and $i \leq n-1$ and then joining b_i and c_i . That is every alternate edge of a path is replaced by a cycle C_4 .

Independent domination on $A(Q_n)$

$$i(A(Q_6)) = 5$$

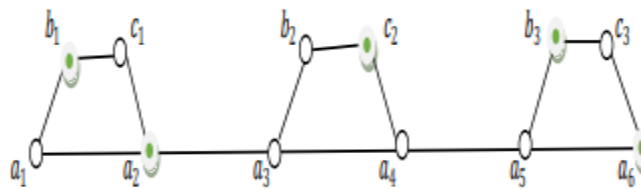


Fig: 4.17 Independent domination on $A(Q_6)$

$$i(A(Q_8)) = 6$$

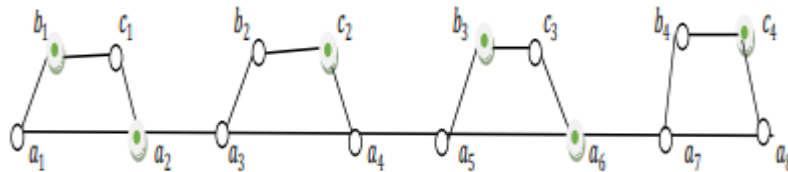


Fig: 4.18 Independent domination on $A(Q_8)$

The highlighted vertices represent the independent domination set

4.3 INDEPENDENT DOMINATION IN GRAPH BASED ON NUMBER THEORY

4.3.1 Cayley graph

Let H be a finite group and let $S \subseteq H$ be a subset. The corresponding Cayley graph $C(H, S)$ has vertex set equal to H . Two vertices $g, h \in H$ are joined by a directed edge from g to h if and only if there exists $s \in S$ such that $g = sh$. Each edge is labeled to

denote that it corresponds to $s \in S$. If G is a graph such that there exists a group H and generating set $S \subseteq H$ with $G \cong C(H, S)$, then G is said to be Cayley.

4.3.2 Euler totientCayley graph $G(Z_n, \varphi)$

The Euler totient Cayley graph, For any positive integer n , let $Z_n = \{0, 1, 2, \dots, n-1\}$ be the residue classes modulo n . Then $(Z_n, \oplus)$, where $\oplus$ addition modulo n , is an abelian group of order n .

The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called an Euler totient function. Let S denote the set of all positive integers less than n and relatively prime to n that is $S = \{r/1 < r < n \text{ and } \text{GCD}(r, n) = 1\}$. Then $|S| = \varphi(n)$

The Euler totientCayley graph is defined as follows.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0, 1, 2, \dots, n-1\}$ and the edge set is $E = \{(x, y) / x - y \in S \text{ or } y - x \in S\}$. Euler totient cayley graph is denoted by $G(Z_n, \varphi)$, where Z_n is set of integer and addition modulo and φ is set of all the co-prime numbers

Example

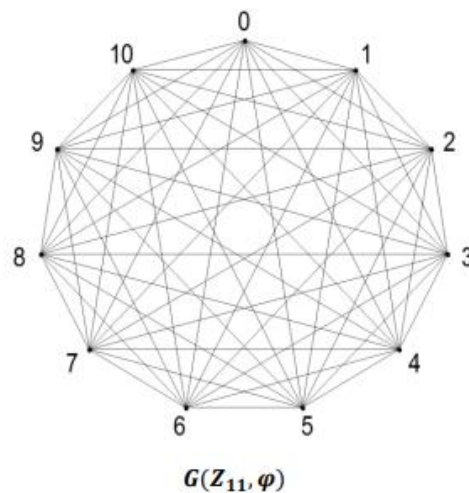


Fig: 4.19 Independent domination on Euler totient Cayley graph $G(Z_{11}, \varphi)$

$G(Z_n, \varphi)$	$n = 11$
Minimum Independent Dominating Set	$\{0\}$
$\gamma_l(G(Z_n, \varphi))$	1

Table 4.1 Independent domination on *Euler totient Cayley graph*

4.3.3 Arithmetic graph

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if $\text{GCD}(u, v) = p_i$ for some prime divisor p_i of n .

Example

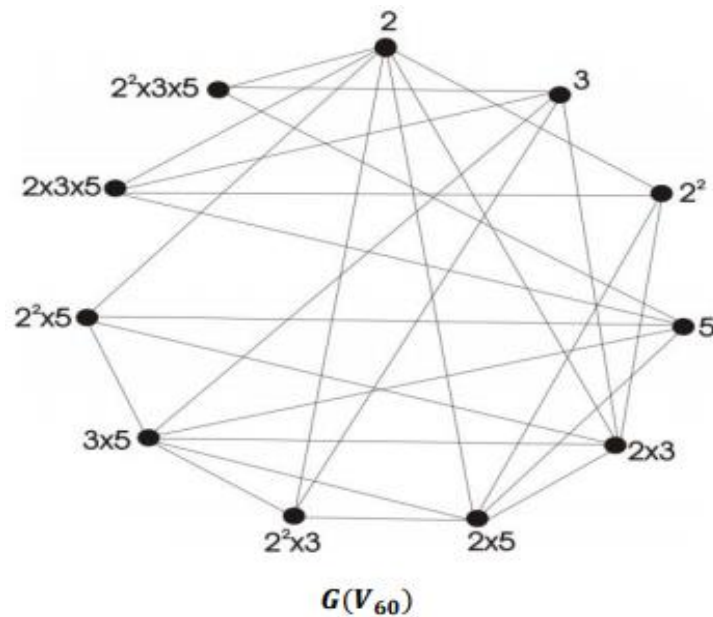


Fig: 4.20 Independent domination on Arithmetic V_{60} graph

$G(V_n)$	$n = 60$ $= 2^2 \times 3 \times 5$
Minimum Independent Dominating Set	$\{2, 15\}$
$\gamma_l(G(V_n))$	2

Table 4. 2 Independent domination on Arithmetic V_{60} graph

4.4 PRODUCT GRAPHS

4.4.1 Corona of graphs

The corona $G \odot H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H , and then joining the i th vertex of G to every vertex in the i th copy of H . We denote by $v + H^v$ a subgraph of $G \odot H$ obtained by joining the vertex $v \in V(G)$ to every vertex of the copy H^v of H .

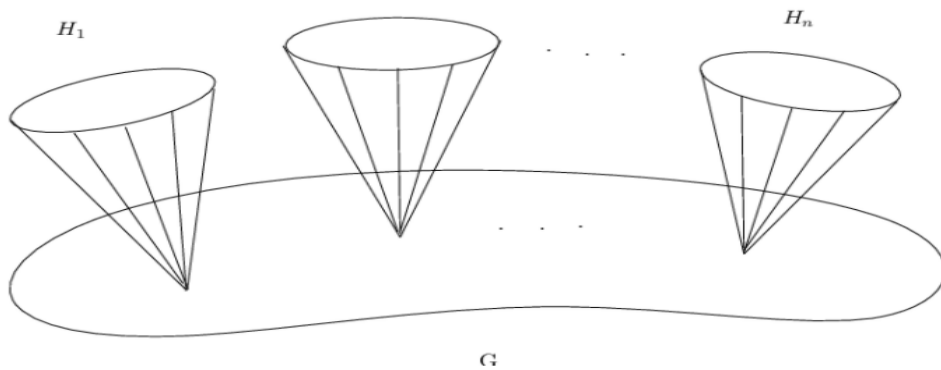


Fig: 4.21 The corona product of two graphs

Independent domination on corona product of graphs ($G \odot H$)

$$i(G \odot H) = 4$$

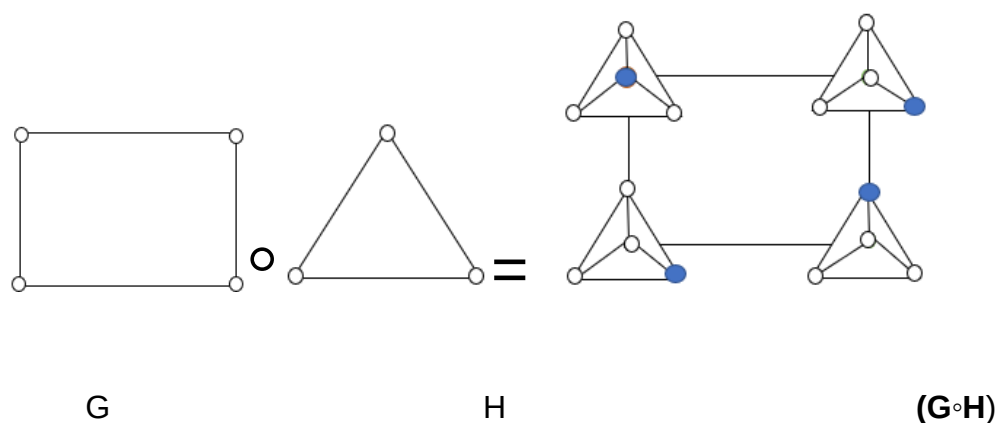


Fig: 4.22 Independent domination on corona product of graphs ($G \circ H$)

The highlighted vertices represent the independent dominating set

4.4.2 Lexicographic product of graphs

The lexicographic product $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $u, v \in E(G)$ or $u = v$ and $u', v' \in E(H)$.

Observe that any non-empty subset C of $V(G[H]) = V(G) \times V(H)$ (in fact, any set of ordered- pairs) can be expressed or written as $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any non-empty subset C of $V(G) \times V(H)$.

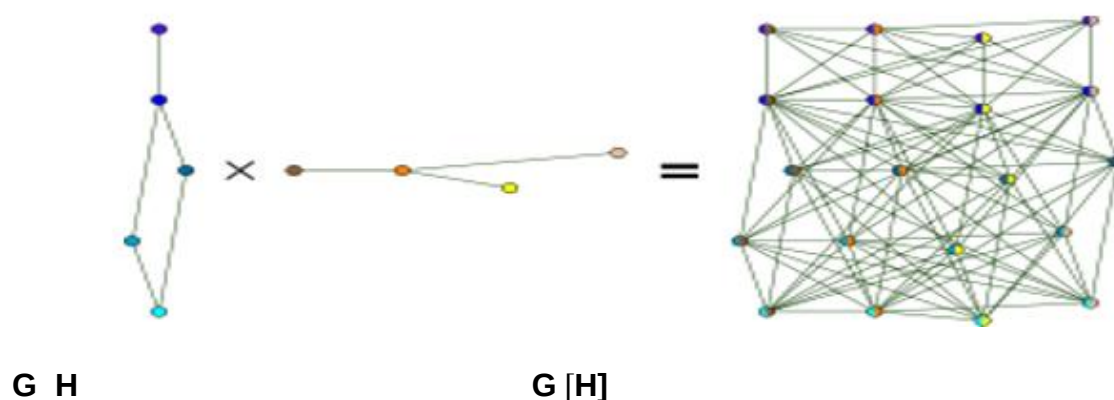


Fig: 4. 23 Lexicographic products of two graphs

4.4.3 Cartesian product of graphs

The cartesian product $G \times H$ of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u') (v, v') \in E(G \times H)$ if and only if either $u v \in E(G)$ and $u' = v'$ or $u = v$ and $u' v' \in E(H)$.

. Let C be a subset of $V(G \times H)$. Then the G -projection C_G and H -projection C_H of C are the following sets:

$$C_G = \{x \in V(G) : (x, a) \in C \text{ for some } a \in V(H)\}$$

$$C_H = \{b \in V(H) : (y, b) \in C \text{ for some } y \in V(G)\}$$

Further, if $x \in C_G$, then $Tx = \{a \in C_H : (x, a) \in C\}$ and if $a \in C_H$, then $Da = \{x \in C_G : (x, a) \in C\}$.

Example

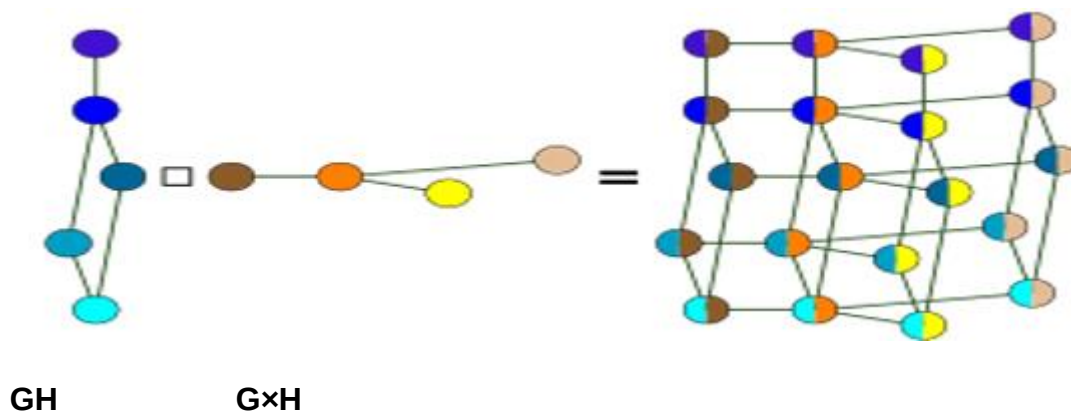


Fig: 4.24 Cartesian product of two graphs

5. RESULTS AND DISCUSSION

In this chapter, we denote the cardinality of a set A by $|A|$. Results given here are mainly taken from [11,12,14,15]

Theorem 5.1

(a) The independent domination number of path and cycle are given as, $i(P_n) = i(C_n) = \lceil n/3 \rceil$

(b) For the complete bipartite graph, $i(K_{r,s}) = \min(r, s)$.

Theorem 5.2

For the helm H_n ($n \geq 4$), $i(H_n) = n$

Proof: The helm H_n has $2n+1$ vertices and $3n$ edges. It contains wheel W_n and $n-1$ pendant vertices.

In order to dominate the pendant vertices of H_n , at least $n-1$ vertices of H_n are required. Moreover, it is also possible to take $n-1$ pairwise non-adjacent vertices which can dominate the pendant vertices as well as the remaining vertices of the helm H_n . Therefore, for any independent dominating set S of the helm H_n , $|S| \geq n$ which implies that $i(H_n) = n$

Theorem 5.3

For the closed helm CH_n ($n \geq 5, n \neq 6$), $i(CH_n) = \lceil (n-3) \rceil$.

Proof: Let c denote the apex vertex of wheel W_n . For the closed helm CH_n ,

$$|V(CH_n)| = 2n + 1 \text{ and } |E(CH_n)| = 4n$$

For $n = 5$, at least two non-adjacent vertices of CH_n , one from W_n and one from the outer cycle C_{n-1} are required to dominate all vertices of CH_n . Hence,

$$i(CH_n) = 2 = \lceil (n-3) \rceil \text{ for } n = 5$$

For $n > 6$, $\deg(c) = n - 1 = \Delta(CH_n)$ and the vertex c dominates W_n . Therefore, in order to attain the minimum cardinality, any independent dominating set of CH_n must contain the vertex c . Now, $i(C_n) = \lceil n/3 \rceil$. Hence, at least $\lceil (n-1)/3 \rceil$ pairwise nonadjacent vertices of C_{n-1} are required to dominate all vertices of outer cycle C_{n-1} of CH_n . Therefore, for any independent dominating set S of CH_n ,

$$|S| \geq \lceil (n-1)/3 \rceil + 1 = \lceil (n-3) \rceil$$

implying that

$$i(CH_n) = \lceil (n-3) \rceil.$$

Theorem 5.4

For the web graph $W(3, n)$ where $n \geq 4$.

$$i(W(3, n)) = \lceil (4n+2)/3 \rceil.$$

Proof: Label the vertices of the web graph $W(3, n)$ as follows:

Denote the vertices of the innermost cycle of $W(3, n)$ successively as $v_{1,1}, v_{1,2}, v_{1,3}, \dots, v_{1,n-1}$. Then denote the vertices adjacent to $v_{1,1}, v_{1,2}, v_{1,3}, \dots, v_{1,n-1}$ on the second cycle (middle cycle) as $v_{2,1}, v_{2,2}, v_{2,3}, \dots, v_{2,n-1}$ respectively, and the vertices adjacent to $v_{2,1}, v_{2,2}, v_{2,3}, \dots, v_{2,n-1}$ on the third cycle (outermost cycle) as $v_{3,1}, v_{3,2}, v_{3,3}, \dots, v_{3,n-1}$. Next denote the pendant vertices of $W(3, n)$ as $u_1, u_2, u_3, \dots, u_{n-1}$ and the apex vertex of W_n as c . Then,

$$|V(W(3,n))| = 4n - 1 \text{ and}$$

$|E(W(3,n))| = 7n$. Now, to attain the minimum cardinality, every independent dominating set of $W(3,n)$ should contain the vertex c because

$$\deg(c) = n - 1 = \Delta W(3,n)$$

and the vertex c dominates the innermost cycle of $W(3,n)$. Moreover, in order to dominate $n - 1$ pendant vertices of $W(3,n)$, at least $n - 1$ vertices are required and these $n - 1$ pendant vertices dominate themselves as well as all vertices of outermost cycle of $W(3,n)$. Now, $i(C_n) = \lceil n/3 \rceil$. Therefore, at least $\lceil (n - 1)/3 \rceil$ pairwise non-adjacent vertices are required to dominate all vertices of middle cycle of $W(3,n)$. Thus, for $n \equiv 0, 1 \pmod{3}$, at least

$$1 + n - 1 + \lceil (n - 1)/3 \rceil = \lceil (4n + 2)/3 \rceil$$

pairwise non-adjacent vertices are required to dominate the web graph $W(3,n)$. But for $n \equiv 2 \pmod{3}$, to retain the minimum cardinality of S , it should contain a vertex $v_{3,n-1}$ because $v_{3,n-1}$ dominates a pendant vertex as well as a vertex of the middle cycle of $W(3,n)$. Hence, at least

$$n - 2 + (n - 2) / 3 = (4n + 8) / 3$$

vertices and the remaining vertices of the outermost as well as the middle cycles of $W(3,n-1)$. Therefore, at pairwise non-adjacent vertices are essential to dominate the remaining pendant least

$$1 + 1 + (4n + 8)/3 = (4n + 12)/3$$

pairwise non-adjacent vertices are required to dominate all vertices of $W(3,n)$ for $n \equiv 2 \pmod{3}$. Thus, every independent dominating set of $W(3,n)$ must contain at least $\lceil (4n + 2)/3 \rceil$ vertices of $W(3,n)$.

Hence, we construct a vertex set $S \subset V W(3,n)$ as follows:

$$S = \begin{cases} \{c, v_{2,2}, v_{2,5}, \dots, v_{2,3i+2}, u_1, u_2, \dots, u_{n-1}\}, & \text{if } n \equiv 0 \text{ or } 1 \pmod{3}, \\ \{c, v_{2,2}, v_{2,5}, \dots, v_{2,3i+2}, u_1, u_2, \dots, u_{n-2}\} \cup \{v_{3,n-1}\}, & \text{otherwise,} \end{cases}$$

where $0 \leq i \leq \lfloor (n-3)/3 \rfloor$ with $|S^*| = \lceil (4n+2)/3 \rceil$ for $n \geq 4$.

Since each vertex in $W(3, n)$ is either in S or is adjacent to a vertex in S , it follows that the set S is a dominating set of $W(3, n)$. Moreover, the set S is also an independent set of $W(3, n)$ because no two vertices in S are adjacent. Therefore, the set S is an independent dominating set of $W(3, n)$. As

$$|S^*| = \lceil (4n+2)/3 \rceil,$$

the set S is of minimum cardinality.

Hence, the set S is an independent dominating set with minimum cardinality implying that

$$i(W(3, n)) = \lceil (4n+2)/3 \rceil.$$

Theorem 5.5

$$i(G_n) = \lceil 2(n+1)/3 \rceil$$

Proof: Let c denote the apex vertex of wheel W_n . The gear graph G_n has $2n-1$ vertices and $3(n-1)$ edges. The gear graph G_n contains the outer cycle $C_{2(n-1)}$ and

For the gear graph G_n ($n \geq 4$),

$$V(G_n) = V(C_{2(n-1)}) \cup \{c\}$$

Now,

$$i(C_{2(n-1)}) = \lceil 2(n+1)/3 \rceil$$

Therefore, at least $\lceil 2(n+1)/3 \rceil$ pairwise non-adjacent vertices are essential to dominate all $i(C_{2(n-1)})$ vertices of G_n . Moreover, these vertices also dominate the vertex c . Hence, for any independent dominating set S of G_n ,

$$|S^*| \geq \lceil 2(n+1)/3 \rceil \text{ Thus, } i(G_n) = \lceil 2(n+1)/3 \rceil.$$

Theorem 5.6

For the triangular snake graph T_n ,

$$i(T_n) = \begin{cases} \lceil (n-1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil n/2 \rceil & \text{when } n \text{ is even} \end{cases}$$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of T_n and b_i be the new vertex joining a_i and a_{i+1} of a path by a triangle C_3 .

Now we select the vertices for the set S (independent domination set) from a graph T_n , in such a way that each vertex in S is adjacent to atleast one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for T_n as (i.e.,)

$$i(T_n) = \begin{cases} \lceil (n-1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil n/2 \rceil & \text{when } n \text{ is even} \end{cases}$$

Theorem 5.7

If $A(T_n)$ is an alternate triangular snake then

$$i(A(T_n)) = \lceil n/2 \rceil$$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of $A(T_n)$ and b_j be the new vertex joining alternatively a_j and a_{j+1} of $A(T_n)$ a path by a triangle C_2 .

Now we select the vertices for the set S (independent domination set) from a graph $A(T_n)$, in such a way that each vertex in S is adjacent to atleast one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for $A(T_n)$ as (i.e.,)

$$i(A(T_n)) = \lceil n/2 \rceil$$

Theorem 5.8

If $D(T_n)$ is a double triangular snake obtained from two triangular snake with a common path then

$$i(D(T_n)) = \begin{cases} \lceil (n-1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil n/2 \rceil & \text{when } n \text{ is even} \end{cases}$$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of $D(T_n)$ and b_i be the new vertex joining a_i and a_{i+1} both above and below the common path of $a_1, a_2, \dots, a_n$.

Now we select the vertices for the set S (independent domination set) from a graph $D(T_n)$, in such a way that each vertex in S is adjacent to at least one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for $D(T_n)$ as (i.e.,)

$$i(D(T_n)) = \begin{cases} \lceil (n-1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil n/2 \rceil & \text{when } n \text{ is even} \end{cases}$$

Theorem 5.9

Let $D(A(T_n))$ be the double alternative triangular snake obtained from alternate double triangular snake then

$$i(D(A(T_n))) = \lceil n/2 \rceil$$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of $D(A(T_n))$ and b_j be the new vertex joining alternatively a_j and a_{j+1} both above and below the common path of $a_1, a_2, \dots, a_n$.

Now we select the vertices for the set S (independent domination set) from a graph $D(A(T_n))$, in such a way that each vertex in S is adjacent to at least one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for as (i.e.,)

$$i(D(A(T_n))) = \lceil n/2 \rceil$$

Theorem 5.10

Let Q_n be the quadrilateral snake obtained from the path P_n by replacing each edge of the path by a cycle C_4 then $i(Q_n) = n + 1$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of Q_n and b_j and c_j be the new vertex joining a_j and a_{j+1} of a path by a cycle C_4 .

Now we select the vertices for the set S (independent domination set) from a graph Q_n , in such a way that each vertex in S is adjacent to at least one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for Q_n as (i.e.,)

$$i(Q_n) = n + 1$$

Theorem 5.11

Let $A(Q_n)$ be the alternate quadrilateral snake obtained from the path by replacing each alternate edge of the path by a cycle C_4 then

$$i(A(Q_n)) = \begin{cases} \lceil (3n + 1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil 3n/2 \rceil & \text{when } n \text{ is even} \end{cases}$$

Proof: Let $a_1, a_2, \dots, a_n$ be the vertices of $A(Q_n)$ and b_j and c_j be the new vertex joining alternatively a_j and a_{j+1} of a path by a cycle C_4 .

Now we select the vertices for the set S (independent domination set) from a graph $A(Q_n)$, in such a way that each vertex in S is adjacent to atleast one vertex in $V-S$ and that each vertex in S is independent.

Choosing such a way we get the independent domination number for $A(Q_n)$ as (i.e.,)

$$i(A(Q_n)) = \begin{cases} \lceil (3n + 1)/2 \rceil & \text{when } n \text{ is odd} \\ \lceil \frac{3n}{2} \rceil & \text{when } n \text{ is even} \end{cases}$$

Theorem 5.12

If n is a prime, then the independent domination number of $G(Z_n, \varphi)$ is 1.

Proof: Let n be a prime. Then $G(Z_n, \varphi)$ is a complete graph. Let $D = \{u\}$ where u is any vertex in V . Then every t in V is adjacent to vertex u . Thus every vertex in $V - D$ is adjacent to u , so that D forms a dominating set in $G(Z_n, \varphi)$. since $|D| = 1$ it is evident that D is a minimum dominating set in $G(Z_n, \varphi)$

In fact every singleton vertex set forms a minimum dominating set and also becomes an independent dominating set of $G(Z_n, \varphi)$

Thus $i(G(Z_n, \varphi)) = 1$

Theorem 5.13

Let G be a connected graph of order n and let H be any graph. Then $i(G \circ H) = \beta(G) + (n - \beta(G))i(H)$.

(Where $\beta(G)$ = Independent number of G)

Proof: Firstly, $i(H) \neq 1$. Let C be a minimum independent dominating set in $G \circ H$. Let $C_1 = C \cap V(G)$ and $C_2 = C \setminus C_1$. Then C_1 is a maximum independent set in G hence $|C_1| = \beta(G)$. Therefore

$$i(G \circ H) \geq \beta(G) + (n - \beta(G))i(H).$$

Next, let C_1 be a maximum independent set in G and D^* be a minimum independent dominating set in H . For each $v \in V(G) \setminus C_1$, let $D^v \subseteq V(H^v)$ be such that $\langle D^v \rangle \cong \langle D^* \rangle$. Let $C_2 = \bigcup \{D^v : v \in V(G) \setminus C_1\}$. Then $C^* = C_1 \cup C_2$ is an independent dominating set in $G \circ H$. Therefore

$$i(G \circ H) \leq |C^*| = \beta(G) + (n - \beta(G))i(H).$$

This establishes the required equality

Suppose now that $i(H) = 1$. For each $v \in V(G)$, let $D^v = \{x_v\}$ be a (independent) dominating set in the copy H_v of H . Then, clearly, $C = \bigcup \{D^v : v \in V(G)\}$ is a minimum independent dominating set in $G \circ H$. Therefore, $i(G \circ H) = |C| = n$. Since $i(H) = 1$, $\beta(G) + (n - \beta(G))i(H) = n$.

6. SUMMARY AND CONCLUSION

In this thesis work, we surveyed results on independent dominating sets in some selected graphs. These results convey the independent domination numbers in those graphs. These results can be utilized to establish key relationships between the independent domination number and other parameters, including the independence number, and the chromatic number

7.REFERENCES

- [1]Allan, R. B. and Laskar, R. (1978). On domination and independent domination numbers of graph, Discrete Math., 23 (1978), 73-76
- [2]Berge. C., Theory of Graphs and its Applications, Methuen, London, 1962
- [3] Cockayne E. J., and Hedetniem S.T., Towards a theory of domination in graphs. Networks, 7 (1977) 247-261
- [4] De Jaenisch C.F., Applications, de l'AnalyseMathematique an Jeu des Echecs(Petrograd,1862)
- [5] Goddard, W., Henning, M., Lyle, J. and Southey, J., On the independent domination number of regular graphs, Ann. Comb., 16 (2012), 719-732
- [6] Harary.F, Graph theory, Addisionwesley, New Delhi
- [7] Harary, F. On the group of the composition of two graphs, Duke Math. J., 26, 29-36 (1959).
- [8] Madhavi, L.-Studies on domination parameters and enumeration of cycles in some Arithmetic Graphs, Ph.D. Thesis submitted to S.V. University, Tirupati, India, 2002
- [9] Nathanson, Melvyn, B. – Connected components of arithmetic graphs, Monat. fur. Math, 29 (1980), 219 – 220
- [10] Ore.O, Theory of graphs. Amer. Math. Soc. Transl., 38 (1962), 206-212

- [11] Senthurpriya. N and Meenakshi. S., Independent domination number in triangular and quadrilateral snake graph, International Journal of recent technology and engineering, 8 (2019) 20-23
- [12] Sergio Canoy Jr.R., Another look at independent domination in graphs, International journal of Mathematical analysis, 8 (2014), 2075-2082
- [13] Sunoj B.S and Mathew Varkey T.K., Square difference prime labeling for some snake graphs, 3(2017) 1083-1089
- [14] Uma Maheswari. S., and Maheswari B., Independent domination number of Euler totient Cayley graph and Arithmetic graph, International Journal of advanced research in engineering and technology, 7 (2016), 56-65
- [15].Vaidya S.K and Pandit R.M, Independent domination in some wheel related graphs, Applications and applied Mathematics, 11 (2016), 397-407