

**Fixed point results on
 C^* -algebra-valued suprametric spaces**

Submitted in partial fulfillment of the requirements for the award of Master of
Science degree in Mathematics (M.Sc., Mathematics)

By

SHELCIA JHENCI J M
(42630009)



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND HUMANITIES

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)

Accredited with Grade "A++" by NAAC
JEPPIAAR NAGAR, RAJIV GANDHI SALAI,
CHENNAI-600119

APRIL - 2024



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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **SHELZIA JHENCI J M (42630009)** who carried out the project entitled "**Fixed point results on C^* -algebra-valued suprametric spaces**" under our supervision from November 2023 to April 2024.

Internal Guide

Dr. SUMAIYA TASNEEM Z

External Guide

Head of the Department

Dr. NIRMALA M

Submitted for Viva voce Examination held on 20/4/24

Internal Examiner

External Examiner

DECLARATION

I **SHELCIA JHENCI J M (42630009)** hereby declare that the Project Report entitled "**Fixed Point Results on C^* -algebra-valued suprametric Spaces**" done by me under the guidance of **Dr. SUMAIYA TASNEEM Z** (Internal) and _____ (External) at **Sathyabama Institute of Science and Technology (Deemed to be University)** is submitted in partial fulfillment of the requirements for the award of Master of Science degree in Mathematics.

DATE : 20/4/24

PLACE : CHENNAI



SIGNATURE OF THE CANDIDATE

ACKNOWLEDGEMENT

I am pleased to acknowledge my sincere thanks to The Board of Management of **SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY** for their encouragement in doing this project and for completing it successfully. I am grateful to them.

I convey my thanks to **Dr. REKHA CHAKRAVARTHI**, Dean, School of Science and Humanities and **Dr. NIRMALA M**, Head of the department, Department of Mathematics for providing me necessary support during the progressive reviews.

I would like to express my sincere and deep sense of gratitude to my Project Guide **Dr. Z .SUMAIYA TASNEEM** , for her valuable guidance, suggestions and constant encouragement paved way for the successful completion of my project work.

I wish to express my thanks to all Teaching and Non-teaching staff members of the Department of Mathematics who were helpful in many ways for the completion of the project.

ABSTRACT

In the twenty-first-century nonlinear analysis is an intriguing new field that combines operator theory, topology, mathematical modelling, applications, and nonlinear functional analysis in a novel way. In the last sixty years or so, it has become apparent that the theory of fixed points is a very useful and effective method for studying nonlinear phenomena. The existence of a fixed point for a suitable map or operator in a wide range of mathematical, computing, economic, modelling, and engineering problems is comparable to the availability of a solution to a computational or real-world problem.

The first fixed point theorem for contraction mappings in metric spaces was established by the Polish mathematician Banach in [1]. This theorem is known as the Banach's fixed point theorem or the Banach contraction principle. Since then, a number of mathematical fields have found this theorem to be among their most useful and effective tools in tackling practical issues.

In this present work, a new notion of C^* -algebra-valued suprametric space, that extends the concept of suprametric spaces and C^* -algebra-valued metric spaces is presented in accordance with the idea and properties of C^* -algebras. It also provides some fundamental fixed point theorems for self-maps with contractive conditions on such spaces. As applications, existence and uniqueness results for an integral equation are given.

The project is divided into five chapters. The first chapter starts with an introduction. The first chapter is an introduction that covers a thorough explanation of fixed point theorems as well as the work of other researchers in the subject.

Chapter 2 deals with the preliminaries. It incorporates recently found conceptions that have been significant in establishing theorems. Besides establishing the notations and terminologies to be utilized in the following chapters, this chapter provides a summary of the work in the context of fixed point theory.

In Chapter 3, a novel metric space, known as C^* -algebra-valued suprametric spaces is defined. An example is presented for the proposed metric and the notion of convergence, Cauchy sequence and completeness is discussed under the same metric.

Chapter 4 explores the importance of suprametric spaces with C^* -algebra, as well as associated fixed point theorem for self maps involving Banach type contractive mapping. Our approach extends the Theorem 2.1 of [10].

In Chapter 5, we use the proven fixed point theorem to investigate the existence and uniqueness of solutions to an integral equation in the context of complete C^* -algebra-valued suprametric spaces.

The project concludes with a bibliography of the different publications referenced in this research work.

Keywords and Phrases: C^* -algebra; C^* -algebra-valued suprametric spaces; Contractive mapping; Fixed point theorem.

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CHAPTER 1

THE OVERVIEW OF FIXED POINT THEORY

1 Introduction

Fixed point theory continues to be a fascinating subject of research despite having a history of more than hundred years. The attraction of fixed point theory lies in its application which belongs to numerous domain. As is well known, the Banach contraction mapping principle is a very useful, simple and classical tool in modern analysis, and it has many applications in applied mathematics. In particular, it is an important tool for solving existence problems in many branches of mathematics and physics.

The Banach contraction principle is one of the most fundamental and powerful tools of nonlinear analysis. It states that in complete metric space (X, d) if a mapping $f : X \rightarrow X$ is a contraction, that is, there exists a constant $c \in [0, 1)$ such that $d(fx, fy) \leq cd(x, y)$ for all $x, y \in X$, then f has a unique fixed point. Moreover, the sequence $\{f^n x\}$ of n -iterate of f converges to this fixed point for any initial point $x \in X$. Due to its wide range of applications in various fields of mathematics, much research has been devoted to its extension, generalization or improvement.

In some research works, the metric space has been generalized to more general spaces such as the partial metric spaces of Matthew [4], the rectangular metric spaces of Branciari [3] or the b -metric spaces of Czerwik [6]. Some improvements were carried out by relaxing the contraction. For instance, a local contraction type has been investigated by Krasnoselskii [7].

In [8], Edelstein introduce a more general contractive condition and replace the completeness of the metric space by the existence of a nonempty ω -limit set. Further results extending the Banach contraction principle may be found in the following incomplete list of references [2, 5, 9, 18, 14, 16].

Maher Berzig [11] proposed the idea of suprametric space and studied a few key aspects of its topology in the year 2022. Following that, the author demonstrated that specific contraction maps in suprametric spaces possess an unique fixed point if the space is complete or comprises a non-empty $\varkappa$ -limit set.

In [19], Ma established the notion of C^* -algebra-valued metric spaces and proved some fixed point theorems for self maps with contractive or expansive mappings. The main idea consists in using the set of all positive elements of a unital C^* -algebra instead of the set of real numbers. Further in [20], Ma introduced a concept of C^* -algebra-valued b -metric spaces which generalizes the concept of C^* -algebra valued metric spaces.

Motivated by all the above mentioned works, in this work we have initiated the concept of C^* -algebra-valued suprametric spaces that generalize the concept of C^* -algebra-valued metric spaces and suprametric spaces. A Banach fixed point theorem is proved in this metric, and as an application, the existence and uniqueness of the solution of an integral equation are discussed.

CHAPTER 2

LITERATURE SURVEY

2 Introduction

Metric fixed point theory and applications that will be very beneficial to both novices and experts in the field. It is based on late-nineteenth-century methods of establishing the existence and uniqueness of solutions to equations, particularly differential equations, by iterative approximations. Although Picard's work is notably linked with this technique, it was Banach who first introduced the concepts in an abstract form, see [20]. Indeed, Banach's fixed point theorem in metric spaces has become a very prominent technique for solving existence problems in many domains of mathematical analysis and applied fields owing to its efficiency, utility, and applicability.

The technique to broaden the scope of fixed point theory is to extend the class of spaces that are stronger conceptions than metric spaces. Therefore, we begin with a few of them, which will be utilized subsequently.

Czerwik developed the notion of b -metric spaces [6], as follows:

Definition 2.1. *Let X be a nonempty set and $b \geq 1$ be any real number. A function $d : X \times X \rightarrow [0, \infty)$ is a b -metric iff for each $\varpi, \varrho, \varsigma \in X$, the preceding conditions are satisfied:*

1. $0 \leq d(\varpi, \varrho)$ for all $\varpi, \varrho \in X$ and $d(\varpi, \varrho) = 0$ if and only if $\varpi = \varrho$;
2. $d(\varpi, \varrho) = d(\varrho, \varpi)$ for all $\varpi, \varrho \in X$;
3. $d(\varpi, \varrho) \leq b[d(\varpi, \varsigma) + d(\varsigma, \varrho)]$ for all $\varpi, \varrho, \varsigma \in X$;

The pair (X, d) is called a b -metric space.

Remark 2.2. *It is significant to mention that the class of b -metric space are much larger than that of metric spaces. A b -metric is, in fact, a metric if and only if $b = 1$.*

The preceding illustration demonstrates that a b -metric does not have to be a metric.

Example 2.3. Let $d_b(\varpi, \varrho) = (d(\varpi, \varrho))^b$, where $d(\varpi, \varrho)$ is a metric space and $b > 1$ is a real number. The conditions (1) and (2) of Definition 2.1 are met. If $1 < b < \infty$, the function's convexity $g(\varpi) = \varpi^b$ where $\varpi > 0$ then indicates $\left(\frac{u+w}{2}\right)^b \leq \frac{1}{2}(u^b + w^b)$, i.e., $(u + w)^b \leq 2^{b-1}(u^b + w^b)$ is true. Hence for each $\varpi, \varrho, \varsigma \in X$, we have

$$\begin{aligned} d_b(\varpi, \varrho) &= (d(\varpi, \varrho))^b \leq (d(\varpi, \varsigma) + d(\varsigma, \varrho))^b \\ &\leq 2^{b-1}[(d(\varpi, \varsigma))^b + (d(\varsigma, \varrho))^b] \\ &= 2^{b-1}[d(\varpi, \varsigma) + d(\varsigma, \varrho)]. \end{aligned}$$

Therefore, condition (3) of Definition 2.1 is met and d_b is a b -metric with $b = 2^{b-1}$. It is crucial to note that (X, d_b) is not typically a metric space.

The following gives the definition of suprametric space:

Definition 2.4. [11] Let $X \neq \emptyset$. A function $d_s : X \times X \rightarrow [0, \infty)$ is a suprametric if for all $\varpi, \varrho, \varsigma \in X$, the following conditions hold:

1. $d_s(\varpi, \varrho) = 0$ if and only if $\varpi = \varrho$;
2. $d_s(\varpi, \varrho) = d_s(\varrho, \varpi)$ for all $\varpi, \varrho \in X$;
3. $d_s(\varpi, \varrho) \leq d_s(\varpi, \varsigma) + d_s(\varsigma, \varrho) + \rho d_s(\varpi, \varsigma) d_s(\varsigma, \varrho)$

for some constant $\rho \in \mathcal{R}^+$. Then d_s is called a suprametric on X and (X, d_s) is named as suprametric space.

Hereafter, $\mathcal{A}$ denotes an unital C^* -algebra. Set $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. We call an element $a \in \mathcal{A}$ a positive element, denote it by $\theta \preceq a$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where θ is a zero element in $\mathcal{A}$ and $\sigma(a)$ is the spectrum of a . There is a natural partial ordering on $\mathcal{A}_h$ given by $a \preceq b$ if and only if $\theta \preceq b - a$. From now on, $\mathcal{A}_+$ and $\mathcal{A}'$ will denote the set $\{a \in \mathcal{A} : \theta \preceq a\}$ and the set $\{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$ and $|\varpi| = (\varpi^* \varpi)^{\frac{1}{2}}$ respectively.

Lemma 2.5. [19] Suppose that $\mathcal{A}$ is a unital C^* -algebra with a unit I .

1. For any $\varpi \in \mathcal{A}_+$ we have $\varpi \preceq I \Leftrightarrow \|\varpi\| \leq 1$.
2. If $a \in \mathcal{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(1 - a)^{-1}\| < 1$.
3. Suppose that $a, b \in \mathcal{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.
4. Let $a \in \mathcal{A}'$, if $b, c \in \mathcal{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathcal{A}'_+$ is an invertible operator, then $(I - a)^{-1}b \succeq (I - a)^{-1}c$.

The researchers in [19] developed the notion of C^* -algebra-valued metric space using concept of positive components in $\mathcal{A}$, as shown below.

Definition 2.6. [19] Let $X \neq \emptyset$ and the mapping $d_{\mathcal{A}} : X \times X \rightarrow \mathcal{A}$ satisfies:

1. $\theta \preceq d_{\mathcal{A}}(\varpi, \varrho)$ and $d_{\mathcal{A}}(\varpi, \varrho) = \theta$ iff $\varpi = \varrho$;
2. $d_{\mathcal{A}}(\varpi, \varrho) = d_{\mathcal{A}}(\varrho, \varpi)$;
3. $d_{\mathcal{A}}(\varpi, \varrho) \preceq d_{\mathcal{A}}(\varpi, \alpha) + d_{\mathcal{A}}(\alpha, \varrho)$, $\forall \varpi, \varrho, \alpha \in X$.

Then $d_{\mathcal{A}}$ is said to be a C^* -algebra-valued metric on X and $(X, \mathcal{A}, d_{\mathcal{A}})$ is named as C^* -algebra-valued metric space.

Example 2.7. [19] Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$. Define

$$d_{\mathcal{A}}(\varpi, \varrho) = \text{diag}(|\varpi - \varrho|, c|\varpi - \varrho|),$$

where $\varpi, \varrho \in \mathbb{R}$ and $c \geq 0$ is a constant. The completeness of $\mathbb{R}$ makes it simple to demonstrate that $d_{\mathcal{A}}$ is a C^* -algebra valued metric space and $(X, \mathcal{A}, d_{\mathcal{A}})$ is a complete C^* -algebra valued metric space.

It is evident that the class of C^* -algebra valued metric space enlarge the idea of metric spaces by adopting $\mathcal{A}_+$ for the set of real numbers.

CHAPTER 3

OBJECTIVE AND SCOPE OF THE STUDY

3 Introduction

C^* -algebras were initially studied in quantum mechanics to represent the algebras of physical observables. This area of investigation began with Werner Heisenberg's matrix mechanics and was further developed mathematically by Pascual Jordan about 1933. C^* -algebras are widely employed in the theory of unitary representations of locally compact groups, as well as in algebraic quantum mechanics formulations. Segal coined the term C^* -algebra in 1947 to refer to norm-closed subalgebras of $B(H)$, i.e. the space of bounded operators on some Hilbert space H . Segal defined a C^* -algebra as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space" in [15].

Spurred by the previously mentioned research in Chapter 2 in this work, we introduce the concept of C^* -algebra-valued suprametric space which is a valid generalization of suprametric metric spaces and C^* -algebra-valued metric spaces. Further we discuss the conception of convergence, Cauchy sequence and completeness and provide an illustration in the setting of C^* -algebra-valued suprametric space.

Definition 3.1. *Let X be a nonempty set. A function $d : X \times X \rightarrow \mathcal{A}$ is called C^* -algebra-valued suprametric if for all $\varpi, \varrho, \varsigma \in X$ the following properties hold:*

1. $d(\varpi, \varrho) = \theta$ if and only if $\varpi = \varrho$;
2. $d(\varpi, \varrho) = d(\varrho, \varpi)$;
3. $d(\varpi, \varrho) \preceq d(\varpi, \varsigma) + d(\varsigma, \varrho) + Ad(\varpi, \varsigma)d(\varsigma, \varrho)$ for some $A \in \mathcal{A}_+$

Then d is called a C^* -algebra-valued suprametric on X and $(X, \mathcal{A}, d)$ is called a C^* -algebra-valued suprametric space.

Example 3.2. *Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$. Define $d(\varpi, \varrho) = \begin{bmatrix} \delta(\varpi, \varrho) & 0 \\ 0 & \alpha\delta(\varpi, \varrho) \end{bmatrix}$ where $\delta(\varpi, \varrho)$ is a suprametric space with $\rho = \frac{2}{\beta}$, $\varpi, \varrho \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. It is very easy to verify d is a C^* -algebra-valued suprametric and $(\mathbb{R}, M_2(\mathbb{R}), d)$ is a C^* -algebra-valued suprametric space with $A = \begin{bmatrix} \frac{2}{\beta} & 0 \\ 0 & \frac{2}{\beta} \end{bmatrix}$.*

Example 3.3. Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. By $L(H)$ we denote the set of bounded linear operators on Hilbert space H . Clearly $L(H)$ is a C^* -algebra with the usual operator norm. Define $d : X \times X \rightarrow L(H)$ by

$$d(\varpi, \varrho) = \pi_{|\varpi - \varrho|(|\varpi - \varrho| + \alpha)}, \quad (\forall \varpi, \varrho \in X)$$

where $\pi_h : H \rightarrow H$ is the multiplication operator defined by

$$\pi_h(\phi) = h \cdot \eta$$

for $\eta \in H$. Then d is a C^* -algebra-valued suprametric and $(X, L(H), d)$ is a C^* -algebra-valued suprametric space.

Definition 3.4. Suppose that $(X, \mathcal{A}, d)$ is a C^* -algebra-valued suprametric space. Let $\{\varpi_n\}$ be a sequence in X and $\varpi \in X$. If $d(\varpi_n, \varpi) \rightarrow \theta(n \rightarrow \infty)$, then it is said that $\{\varpi_n\}$ converges to ϖ with respect to $\mathcal{A}$, and we denote it by $\lim_{n \rightarrow \infty} \varpi_n = \varpi$.

If for any $p \in \mathbb{N}$, $d(\varpi_{n+p}, \varpi_n) \rightarrow \theta(n \rightarrow \infty)$, then $\{\varpi_n\}$ is called a Cauchy sequence with respect to $\mathcal{A}$.

We say $(X, \mathcal{A}, d)$ is a complete C^* -algebra-valued suprametric space if every Cauchy sequence is convergent in X with respect to $\mathcal{A}$.

Definition 3.5. Suppose that $(X, \mathcal{A}, d)$ is a C^* -algebra-valued suprametric spaces. We call a mapping $T : X \rightarrow X$ a C^* -algebra-valued contractive mapping on X , if there exists an element $B \in \mathcal{A}$ with $\|B\| < 1$ such that

$$d(T\varpi, T\varrho) \preceq B^* d(\varpi, \varrho) B, \quad \forall \varpi, \varrho \in X. \quad (3.1)$$

CHAPTER 4

FIXED POINT THEOREM

4 Main Result

In this chapter, we prove the fixed point theorem in the setting of C^* -algebra valued suprametric space.

Theorem 1. *If $(X, \mathcal{A}, d)$ is a complete C^* -algebra valued suprametric space and $T : X \rightarrow X$ is a contractive mapping there exists a unique fixed point in X .*

Proof. Let $(X, \mathcal{A}, d)$ be a complete C^* -algebra valued suprametric with $A \succeq \theta$ (The case $\mathcal{A} = \theta$ is the Banach contraction Principle). Define the sequence ϖ_n by $\varpi_{n+1} = T\varpi_n, \forall n$. Consider,

$$\begin{aligned} d(\varpi_{n+1}, \varpi_n) &= d(T\varpi_n, T\varpi_{n-1}) \\ &\preceq B^* d(\varpi_n, \varpi_{n-1}) B \\ \Rightarrow \|d(\varpi_n, \varpi_{n+1})\| &\preceq \|B\|^2 \|d(\varpi_{n-1}, \varpi_n)\| \\ &\preceq \|d(\varpi_{n-1}, \varpi_n)\| \end{aligned}$$

Hence the sequence $\{d(\varpi_n, \varpi_{n+1})\}$ is decreasing and for all fixed integer k it satisfies

$$d(\varpi_n, \varpi_{n+1}) \preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}, \forall n > k$$

This implies

$$\lim_{n \rightarrow \infty} \|d(\varpi_n, \varpi_{n+1})\| = 0$$

So there exist $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $\|d(\varpi_n, \varpi_{n+1})\| \leq 1$. We shall prove that $\{\varpi_n\}$ is Cauchy. Consider

$$\begin{aligned} d(\varpi_n, \varpi_m) &\preceq d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_m) + A d(\varpi_n, \varpi_{n+1}) d(\varpi_{n+1}, \varpi_m) \\ &\preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + d(\varpi_{n+1}, \varpi_m) \\ &\quad + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} d(\varpi_{n+1}, \varpi_m) \\ &\preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + \left[1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} \right] \\ &\quad \left[d(\varpi_{n+1}, \varpi_m) \right] \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
d(\varpi_{n+1}, \varpi_m) &\preceq d(\varpi_{n+1}, \varpi_{n+2}) + d(\varpi_{n+2}, \varpi_m) + Ad(\varpi_{n+1}, \varpi_{n+2})d(\varpi_{n+2}, \varpi_m) \\
&\preceq (B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1} + d(\varpi_{n+2}, \varpi_m) \\
&\quad + A(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1}d(\varpi_{n+2}, \varpi_m) \\
&\preceq (B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1} + \left[[1 + A(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1}] \right. \\
&\quad \left. d(\varpi_{n+2}, \varpi_m) \right]
\end{aligned} \tag{4.2}$$

Therefore, inequality (4.1) becomes,

$$\begin{aligned}
d(\varpi_n, \varpi_m) &\preceq (B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k} \\
&\quad + [1 + A(B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k}] \left[(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1} \right. \\
&\quad \left. + [1 + A(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1}]d(\varpi_{n+2}, \varpi_m) \right] \\
&\preceq (B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k} + \left[[1 + A(B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k}] \right. \\
&\quad \left. (B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1} \right] + \left[[1 + A(B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k}] \right. \\
&\quad \left. [1 + A(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1}]d(\varpi_{n+2}, \varpi_m) \right]
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
d(\varpi_{n+2}, \varpi_m) &\preceq d(\varpi_{n+2}, \varpi_{n+3}) + d(\varpi_{n+3}, \varpi_m) + Ad(\varpi_{n+2}, \varpi_{n+3})d(\varpi_{n+3}, \varpi_m) \\
&\preceq (B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2} + d(\varpi_{n+3}, \varpi_m) \\
&\quad + A(B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2}d(\varpi_{n+3}, \varpi_m) \\
&\preceq (B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2} \\
&\quad + \left[[1 + A(B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2}]d(\varpi_{n+3}, \varpi_m) \right]
\end{aligned} \tag{4.4}$$

Thereby inequality (4.3) becomes,

$$\begin{aligned}
d(\varpi_n, \varpi_m) &\preceq B^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k} \\
&\quad + \left[[1 + A(B^*)^{n-k}d(\varpi_k, \varpi_{k+1})B^{n-k}] (B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1} \right] \\
&\quad + [1 + A(B^*)^{n-k+1}d(\varpi_k, \varpi_{k+1})B^{n-k+1}] \left[(B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2} \right. \\
&\quad \left. + [1 + A(B^*)^{n-k+2}d(\varpi_k, \varpi_{k+1})B^{n-k+2}]d(\varpi_{n+3}, \varpi_m) \right]
\end{aligned}$$

$$\begin{aligned}
& \preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} \\
& + \left[[1 + A(B^*)^{n-k} d(x_k, x_{k+1}) B^{n-k}] (B^*)^{n-k+1} d(x_k, x_{k+1}) B^{n-k+1} \right] \\
& + \left[[1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + AA(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) \right. \\
& \left. B^{n-k+1}] (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} \right] \\
& + \left[[1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + A(B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2}] \right. \\
& \left. [1 + A(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] d(\varpi_{n+3}, \varpi_m) \right] \\
& \vdots \\
& \preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} \\
& + \left[[1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} \right] \\
& + \left[[1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + A(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] \right. \\
& \left. (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} \right] \\
& \vdots \\
& + \left[[1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + AA(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] \dots \right. \\
& [1 + A(B^*)^{m-k-3} d(\varpi_k, \varpi_{k+1}) B^{m-k-3}] (B^*)^{m-k-2} d(\varpi_k, \varpi_{k+1}) B^{m-k-2} + \\
& [1 + A(B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + A(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] \\
& \left. \dots [1 + A(B^*)^{m-k-3} d(\varpi_k, \varpi_{k+1}) B^{m-k-3}] (B^*)^{m-k-1} d(\varpi_k, \varpi_{k+1}) B^{m-k-1} \right] \\
& \preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + \\
& \sum_{i=1}^{m-n-1} \prod_{j=1}^i (B^*)^{n-k+j-1} d(\varpi_k, \varpi_{k+1}) B^{n-k+j-1} (B^*)^{n-k+j} d(\varpi_k, \varpi_{k+1}) B^{n-k+j}
\end{aligned} \tag{4.5}$$

Taking $d(\varpi_k, \varpi_{k+1}) = B_k$ in inequality (4.5), we get

$$\begin{aligned}
d(\varpi_n, \varpi_m) & \preceq (B^*)^{n-k} B_k B^{n-k} + \\
& \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[[1 + A(B^*)^{n-k+j-1} B_k B^{n-k+j-1}] (B^*)^{n-k+j} B_k B^{n-k+j} \right] \\
& \preceq (B^*)^{n-k} B_k^{1/2} B_k^{1/2} B^{n-k}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[[1 + A^{1/2} A^{1/2} (B^*)^{n-k+j-1} B_k^{1/2} B_k^{1/2} B^{n-k+j-1}] (B^*)^{n-k+j} B_k^{1/2} B_k^{1/2} B^{n-k+j} \right] \\
& \leq [(B^*)^{n-k} B_k^{1/2}] [B_k^{1/2} B^{n-k}] + \\
& \quad \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [A^{1/2} (B^*)^{n-k+j-1} B_k^{1/2}] [A^{1/2} B^{n-k+j-1} B_k^{1/2}] \right. \\
& \quad \left. [(B^*)^{n-k+j} B_k^{1/2}] [B_k^{1/2} B^{n-k+j}] \right] \\
& \leq [B^{n-k} B_k^{1/2}]^* [B_k^{1/2} B^{n-k}] + \\
& \quad \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [A^{1/2} B^{n-k+j-1} B_k^{1/2}]^* [A^{1/2} B^{n-k+j-1} B_k^{1/2}] \right. \\
& \quad \left. [B^{n-k+j} B_k^{1/2}]^* [B_k^{1/2} B^{n-k+j}] \right] \\
& \leq \|B^{n-k} B_k^{1/2}\|^2 + \\
& \quad \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [\|A^{1/2} B^{n-k+j-1} B_k^{1/2} B_k^{1/2}\|^2 \|B^{n-k+j} B_k^{1/2}\|^2] \right] \\
& \leq \|B^{n-k}\|^2 \|B_k^{1/2}\|^2 + \\
& \quad \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [\|A^{1/2}\|^2 \|B^{n-k+j-1}\|^2 \|B_k^{1/2}\|^2 \|B^{n-k+j}\|^2 \|B_k^{1/2}\|^2] \right] \\
& \leq \|B^{n-k}\|^2 \|B_k\| + \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [\|A\| \|B^{n-k+j-1}\|^2 \|B_k\| \|B^{n-k+j}\|^2 \|B_k\|] \right] \\
& \leq \|B^{n-k}\|^2 \|B_k\| + \sum_{i=1}^{m-n-1} \prod_{j=1}^i \left[1 + [\|A\| \|B^{n-k+j-1}\|^2 \|B_k\| \|B^{n-k+j}\|^2] \right]
\end{aligned} \tag{4.6}$$

Let $\mathcal{U}_i = \prod_{j=1}^i [1 + \|A\| \|B^{n-k+j-1}\|^2 \|B_k\| \|B^{n-k+j}\|^2]$. By Ratio Test, $\sum_{n=1}^{\infty} \mathcal{U}_i$ converges as $\lim_{i \rightarrow \infty} \|\frac{\mathcal{U}_{i+1}}{\mathcal{U}_i}\| < 1$ and $\|B\| < 1$. Inequality (4.6) thus lead to the conclusion that $d(\varpi_n, \varpi_m)$ tends to zero as n, m tends to infinity. Therefore $\{\varpi_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. By the completeness of $\{X, \mathcal{A}, d\}$, there exists an $\varpi \in X$ such that $\lim_{n \rightarrow \infty} \varpi_n = \lim_{n \rightarrow \infty} T\varpi_{n-1} = \varpi$.

Since

$$\begin{aligned}
\theta & \leq d(T\varpi, \varpi) \\
& \leq d(T\varpi, \varpi_n) + d(\varpi_n, \varpi) \\
& \leq B^* d(\varpi, \varpi_n) B + d(\varpi_n, \varpi) \\
& \rightarrow \theta.
\end{aligned} \tag{4.7}$$

Hence $d(T\varpi, \varpi) = 0$ which implies $T\varpi = \varpi$. i.e., ϖ is a fixed point of T . Through the use of inequality (3.1), we can easily show that ϖ is a unique fixed point of T . $\square$

CHAPTER 5

APPLICATION

5 Existence and Uniqueness of solutions for Integral Equation

In this chapter, we study the existence and uniqueness of solutions for an integral equation.

Example 5.1. Consider the integral Equation

$$\varpi(t) = \int_E K(t, s, \varpi(s))ds + g(t), t \in E$$

where E is a Lebesgue measurable set. Suppose that

1. $K : E \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $g \in L^\infty(E)$;
2. there exists a continuous function $\phi : E \times E \rightarrow \mathbb{R}$ and $k \in (0, 1)$ such that $|K(t, s, \varpi(s)) - K(t, s, \varrho(s))| \leq p|\phi(t, s)| |\varpi(s) - \varrho(s)|$ for $t \in E$ and $\varpi, \varrho \in L^\infty(E)$;
3. $\sup_{t \in E} \int_E K(t, s, \varpi(s))ds \leq 1$;
Then the integral equation has a unique solution ϖ^* in $L^\infty(E)$.

Proof. Let $X = L^\infty(E)$ and $H = L^2(E)$. For $\varpi, \varrho \in X$, we set $d : X \times X \rightarrow L(H)$ by

$$d(\varpi, \varrho) = \pi_{|\varpi - \varrho|(|\varpi - \varrho| + \alpha)}$$

where $\pi_h : H \rightarrow H$ is the multiplication operator defined by

$$\pi_\eta(\phi) = h.\eta$$

for $\eta \in H$. Then d is a C^* -algebra-valued suprametric by Example 3.2 and $(X, L(H), d)$ is a complete C^* -algebra-valued suprametric space. Let $T : L^\infty(E) \rightarrow L^\infty(E)$ be

$$T\varpi(t) = \int_E K(t, s, \varpi(s))ds + g(t), \quad t \in E$$

Set $B = kI$, then $B \in L(H)$ and $\|B\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
\|d(T\varpi, T\varrho)\| &= \sup_{\|h\|=1} (\pi_{|T\varpi-T\varrho|(|T\varpi-T\varrho|+\alpha)} h, h) \\
&= \sup_{\|h\|=1} \int_E \left[\left| \int_E K(t, s, \varpi(s)) - K(t, s, \varrho(s)) \right. \right. \\
&\quad \left. \left. \left(K(t, s, \varpi(s)) - K(t, s, \varrho(s)) + \alpha \right) ds \right| h(t) \overline{h(t)} dt \right] \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E |K(t, s, \varpi(s)) - K(t, s, \varrho(s))| \right. \\
&\quad \left. \left(|K(t, s, \varpi(s))| - |K(t, s, \varrho(s))| + \alpha \right) ds \right] \|h(t)\|^2 dt \\
&= \sup_{\|h\|=1} \int_E \left[\int_E \left(|K(t, s, \varpi(s)) - K(t, s, \varrho(s))|^2 \right. \right. \\
&\quad \left. \left. + \alpha |K(t, s, \varpi(s))| - |K(t, s, \varrho(s))| \right) ds \right] \|h(t)\|^2 dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E \left(k |\phi(t, s)| \varpi(s) - \varrho(s) \right)^2 \right. \\
&\quad \left. + \alpha k |\phi(t, s)| \varpi(s) - \varrho(s) \right) ds \right] \|h(t)\|^2 dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E \left(k^2 |\phi(t, s)|^2 |\varpi(s) - \varrho(s)|^2 \right. \right. \\
&\quad \left. \left. + \alpha k |\phi(t, s)| |\varpi(s) - \varrho(s)| \right) ds \right] \|h(t)\|^2 dt \\
&\leq \left[\sup_{\|h\|=1} \int_E \|h(t)\|^2 dt \right] \left[k^2 \left[\sup_{\|h\|=1} \int_E |\phi(t, s)|^2 \right] \|\varpi - \varrho\|_\infty^2 \right. \\
&\quad \left. + \alpha k \left[\sup_{t \in E} \int_E |\phi(t, s)| ds \right] \|\varpi - \varrho\|_\infty \right] \\
&\leq k^2 \|\varpi - \varrho\|_\infty^2 + \alpha k \|\varpi - \varrho\|_\infty \\
&\leq k \|\varpi - \varrho\|_\infty \left[\|\varpi - \varrho\|_\infty + \alpha \right] \\
&= \|B\| \|d(\varpi, \varrho)\|
\end{aligned}$$

Since $\|B\| < 1$, the integral equation has a unique solution ϖ^* in $L^\infty(E)$. □

CHAPTER 6

SUMMARY AND CONCLUSION

6 CONCLUSION

In this study, we examined the existence and uniqueness of fixed points in suprametric spaces, and we attempted to prove the fixed point theorem under the same metric. Thus, the study is based on the fixed point theorems for contractive type mapping in suprametric spaces. In the second chapter, we introduced the concept of C^* -algebra valued suprametric spaces and provided an illustration for the same metric. Moreover, the Banach type contractions in the setting of suprametric spaces with C^* -algebra are investigated, and a fixed point theorem is proved beneath the related contraction. Finally, we used our proven fixed point theorems to verify the existence and uniqueness of solutions to integral equations.

FUTURE WORK:

Several investigations and research studies have been carried out on coupled systems with fractional differential equations. Due to their various uses, coupled systems of differential equations of different orders possess a primary significance for research. The future work is to utilize coupled fixed point theorems in the context of our newly defined metric space, namely C^* -algebra-valued suprametric spaces, to confirm the existence and uniqueness of solutions to coupled systems of fractional differential equations.

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