

A STUDY ON ODD GRACEFUL LABELING FOR SOME FAMILIES OF GRAPHS

Submitted in partial fulfilment of the requirements for the award of
Master of science degree in Mathematics

By

N POOVIZHI (42630005)



**DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND HUMANITIES**

SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

**Accredited with Grade “A++” by NAAC
JEPPIAAR NAGAR, RAJIV GANDHI SALAI,
CHENNAI-600119**

APRIL - 2024



SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

(DEEMED TO BE UNIVERSITY)

Accredited "A++" Grade by NAAC | 12B Status by UGC | Approved by AICTE

www.sathyabama.ac.in

DEPARTMENT OF MATHEMATICS

BONAFIDE CERTIFICATE

This is to certify that this project report is the bonafide work of **N POOVIZHI (42630005)** who carried out the project entitled "**A STUDY ON ODD GRACEFUL LABELING FOR SOME FAMILIES OF GRAPHS**" under my supervision from **JANUARY 2024** to **APRIL 2024**.

Internal Guide

Dr. Franklin Thamil Selvi M S

Head of the Department

Dr. NIRMALA M

Submitted for Viva voce Examination held on 20. 4. 2024

Internal Examiner

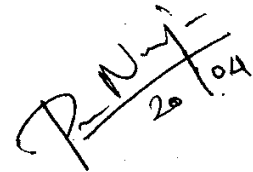
External Examiner

DECLARATION

I **N POOVIZHI (42630005)** hereby declare that the Project Report entitled “**A STUDY ON ODD GRACEFUL LABELING FOR SOME FAMILIES OF GRAPHS**” done by me under the guidance of **Dr. FRANKLIN THAMIL SELVI M S** at **SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY** submitted in partial fulfilment of the requirements for the award of Master of Science degree in Mathematics.

DATE: 20/04/2024

PLACE: CHENNAI



**SIGNATURE OF THE
CANDIDATE**

ACKNOWLEDGEMENT

I delighted to acknowledge my sincere thanks to Board of Management of **SATHYABAMA** for their kind encouragement in doing this project and for completing it successfully. I am oblige to them for providing me such opportunity.

I deliver my thanks to **Dr. Rekha Chakravarthi**, Dean, school of Science and Humanities and **Dr. M. Nirmala**, Head of the Department, Department of **Mathematics** for providing the necessary support and details at the right time during the progressive reviews.

I would like to express my sincere and deep sense of gratitude to my Project Guide **Dr. Franklin Tamil Selvi M S** for her valuable guidance, suggestions and constant encouragement paved way for the successful completion of the project work. And I am grateful that she supported me in every possible way till the end.

I wish to express my thanks to all Teaching and Non-Teaching staff members of the Department of Mathematics who supported me in many ways for the completion of the project.

ABSTRACT

A graph with p vertices and q edges is said to be odd graceful labeling if there exist a mapping $f: p \rightarrow \{0, 1, \dots, 2q - 1\}$ where every edges of V_1V_2 are label from its difference $(|V_1 - V_2|)$ then the line labels are $\{1, 2, 3, \dots, 2q - 1\}$. In this project we have showed that the graph acquired by securing n copies of bistar by an $(n-1)$ edge is odd graceful and a graph acquired by securing n copies of comb by an $(n-1)$ edge is odd graceful.

KEYWORDS: Labeling, Graceful labeling, Odd graceful labeling, Bistar, Comb.

CONTENTS

CHAPTER No.	TITLE	PAGE No.
1	INTRODUCTION	1
	1.1 BASIC CONCEPTS	2
	1.1.1 Graph labeling	2
	1.1.2 Graceful labeling	3
2	LITERATURE SURVEY	4
3	AIM	6
	3.1 AIM	6
4	ODD GRACEFUL GRAPH	8
	4.1 DEFINITION	8
	4.2 APPLICATIONS	8
	4.3 AN OUTLINE OF THE PROJECT	9
5	ODD GRACEFUL LABELING FOR n-COPIES OF BISTAR	10
	5.1 DEFINITIONS	10
	5.1.1 Bistar	10
	5.1.2 n-copies of Bistar	10

	5.2	APPLICATIONS OF BISTAR	11
	5.3	THEOREM 1	12
6		ODD GRACEFUL LABELING FOR n – COPIES OF COMB	16
	6.1	DEFINITIONS	16
		6.1.1 6.1.1 Comb	16
		6.1.2 n-copies of Bomb	16
	6.2	APPLICATIONS OF COMB	16
	6.3	THEOREM 2	18
7		CONCLUSION	24
		REFERENCE	25

LIST OF FIGURES

FIGURE No.	TITLE	PAGE No.
1.1.1	Graph labeling	2
1.1.2	Graceful labeling	3
5.1	Odd graceful labeling	8
5.1.1	Bistar graph	10
5.3.1	3-copies of bistar which is Odd graceful graph ($\overline{B_3}$)	14
6.1.1	Comb graph	16
6.3.1	3-copies of comb (with even a_j) which is Odd graceful graph ($\overline{C_3}$)	20
6.3.2	3-copies of comb (with odd a_j) which is Odd graceful graph ($\overline{C_3}$)	23

CHAPTER – 1

INTRODUCTION

Graph labeling is a fundamental aspect of graph theory that involves assigning labels or values to the vertices, edges, or other components of a graph. These labels can be numerical, alphabetical, or symbolic, and they play a crucial role in various graph-related applications, such as network analysis, communication systems, and data representation. Graph labeling encompasses several different types, each serving different purposes and applications. Here, are some common types of graph labeling.

- Vertex labeling
- Graph labeling
- Total labeling
- Distance labeling
- Graceful labeling
- Edge labeling

Graph labeling plays a crucial role in various graph-related algorithms, optimization problems, and data analysis tasks. By assigning meaningful labels to the vertices, edges, or entire graphs, researchers and practitioners can gain valuable insights into the underlying structures, properties, and relationships within complex networks and systems. Moreover, graph labeling techniques enable the development of efficient algorithms for solving a wide range of problems in diverse domains, including computer science, operations research, telecommunications, and computational biology.

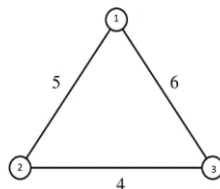
Graceful labeling in graph theory involves assigning distinct integers to the vertices in such a way that the absolute differences of the labels of adjacent vertices result in a set of distinct integers. This labeling is used to create graceful graphs, which have applications in coding theory, communication networks and combinatorial design. The graceful labeling conjecture proposed by Rosa in 1967, remains unsolved: every tree and most other graphs have graceful labelings. Graceful labeling has connections to number theory and continues to be an area of active research, contributing to various fields through its applications and theoretical implications.

1.1 BASIC CONCEPTS

1.1.1 Graph labeling

Graph labeling is the concept of assigning some numbers / integers to the points(p) or lines(e) or for both points(p) and lines(e) of the graph under certain conditions.

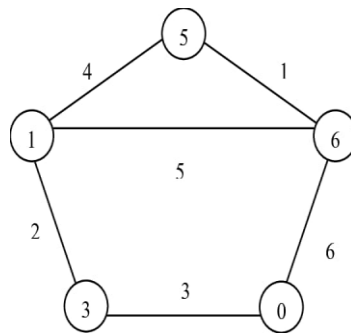
Graph labeling is a process of assigning labels or values to the vertices and/or edges of a graph, typically following certain rules or properties. These labels could be integers, colors, or any other symbols, and they help in representing different attributes or characteristics of the graph's components. The goal of graph labeling is often to provide additional information about the structure or properties of the graph, which can then be utilized in various applications such as graph algorithms, network analysis, or visualization.



1.1.1 Graph Labeling

1.1.2 Graceful labeling

The graph G with p vertices and q edges is said to be graceful labeling if there exist a function f such that $f: p \rightarrow \{0, 1, \dots, 2q - 1\}$ and every lines are labeled by the difference of its points ($|V_1 - V_2|$) then the line labels are $\{1, 2, 3, \dots, 2q - 1\}$.



1.1.2 Graceful Labeling

CHAPTER – 2

LITERATURE SURVEY

R. B. Gnanajothi⁴ (1991) explores various topics within graph theory, likely covering a range of concepts, algorithms, and applications related to graphs. Likely covers fundamental concepts such as graph representation, connectivity, graph traversal algorithms (like DFS and BFS), graph coloring, and maybe some advanced topics like graph isomorphism or planarity. May delve into specific types of graphs such as trees, cycles, bipartite graphs, or directed graphs, discussing their properties and algorithms applicable to them. Might include uses of graph theory in various fields like computer science, operations research, social network analysis, etc.

J. Jeba Jesintha⁶, R. Jaya Glory⁶ and K. Farzana⁶ (1969-1977) a study on odd graceful labeling specifically applied to revised sunflower graphs, focusing on assigning labels to the vertices in a way that certain conditions are satisfied. Introduces the concept of odd graceful labeling and applies it to a specific type of graph, the revised sunflower graph. Likely provides definitions and theorems related to odd graceful labeling and discusses their implications in the context of revised sunflower graphs. Presents methodologies or algorithms for constructing odd graceful labelings on revised sunflower graphs and possibly explores properties of such labelings.

Rosa A¹² (1966) discusses certain estimation of vertices in graph, possibly introducing new valuation methods or exploring properties of existing ones. Introduces novel vertex valuation methods or explores properties of existing ones such as chromatic number, degree sequence, or other graph parameters. May include proofs or discussions on the relationships between different vertex valuations and their implications on graph theory. Likely presents applications of vertex valuations in solving specific graph theory problems or in other fields where graph theory is applied.

Eldergill P² (1997) likely to investigate methods for decomposing complete graphs with an even number of vertices into smaller substructures or components. Probably introduces different decomposition techniques applicable to complete graphs, such as edge decompositions, vertex decompositions, or decomposition into cycles. Likely

explores theoretical results regarding the existence or non-existence of certain decompositions in even-order complete graphs. May discuss practical applications of decomposing complete graphs in various fields like network design, coding theory, or combinatorial optimization.

Gallian J.A.³ (2020) provides a comprehensive overview of graph labeling techniques, covering wide range of labeling schemes and their applications in graph theory. Presents an overview of various types of graph labeling, including graceful labeling, prime labeling, cordial labeling, etc. Likely discusses recent developments and open problems in the field of graph labeling, providing a snapshot of the current research landscape. May include examples and applications of different types of graph labeling in solving practical problems or in other areas of mathematics and computer science.

CHAPTER – 3

AIM

3.1 AIM

Our aim is to exposed two families of graph which is odd graceful labeling. Inspired by the labeling of graphs odd graceful labeling, we have obtained the odd graceful labeling for n -copies of bistar and n -copies of comb which is connected by $n-1$ edges .

CHAPTER – 4

ODD GRACEFUL GRAPH

An odd graceful graph is a type of mathematical graph in which the vertices are labeled with distinct positive integers, and the edges are labeled with positive integers such that the difference between the labels of adjacent vertices is a distinct value. Here's a brief introduction highlighting key points, An odd graceful labeling of a graph is a labeling of its vertices with distinct positive integers such that when each edge is labeled with the absolute difference between the labels of its endpoints, the resulting edge labels form a set of consecutive odd integers.

In an odd graceful labeling, each edge is assigned a distinct positive integer label, ensuring that no two edges share the same label .The edge labels form a set of consecutive odd integers, starting from 1, 3, 5, and so on. This property distinguishes odd graceful graphs from other types of graceful graphs.

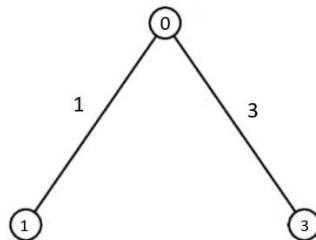
The concept of odd graceful labeling extends the notion of graceful labeling, which typically involves labeling the edges with consecutive integers. Odd graceful labeling adds the constraint of using only odd integers for the edge labels. While odd graceful graphs may seem like purely theoretical constructs, they find applications in various areas such as coding theory, network design, and combinatorics. They provide insights into graph labeling problems and can be used to model real-world scenarios.

Despite their theoretical importance, constructing odd graceful graphs can be challenging. Finding optimal labeling schemes that satisfy the constraints of distinctiveness and consecutive odd integers requires careful consideration and often involves complex computational techniques. In general, odd graceful graphs are an intriguing area of study within graph theory, offering insights into graph labeling problems and their applications in diverse fields. Their unique properties

make them both mathematically fascinating and potentially useful in practical scenarios.

4.1 DEFINITION

The graph G with p points and q lines is said to be odd graceful labeling if there exist a function f such that $f: p \rightarrow \{0, 1, \dots, 2q - 1\}$ and every line is labeled by the difference of its points $(|V_1 - V_2|)$ then the line labels are $\{1, 3, 5, \dots, 2q - 1\}$. Here all the labelings are distinct for both vertices and edges.



4.1 Odd Graceful Labeling

4.2 APPLICATIONS

Odd graceful labeling has applications in various fields, including:

- **Network Communication:** In networking, odd graceful labeling can be used to assign unique identifiers to network nodes or devices, aiding in efficient routing and communication.
- **Graph Theory Algorithms:** Odd graceful labeling can be utilized in algorithms related to graph theory, such as graph colouring or path finding algorithms. The labeling scheme might provide insights into the structure of the graph and help in developing efficient algorithms for various graph-related problems.
- **Error Detection in Data Transmission:** Odd graceful labeling can be applied to detect errors in data transmission. By assigning unique labels to data

packets or chunks of data, and ensuring that the differences between adjacent labels follow the odd graceful property, it becomes possible to detect if any data packets are missing or corrupted during transmission.

- **Cryptography:** Odd graceful labeling might have applications in certain cryptographic schemes where unique labeling or numbering of elements is required. It could be used in creating cryptographic protocols or algorithms for secure communication and data exchange.
- **Parallel and Distributed Computing:** In parallel and distributed computing environments, odd graceful labeling can assist in assigning unique identifiers to computational units or tasks. This labeling scheme can help coordinate and synchronize activities across distributed systems efficiently.
- **Data Structures:** Odd graceful labeling may inspire the design of data structures that exploit the properties of labeled graphs. These data structures could be useful in various applications such as database management systems, where efficient representation and manipulation of graph-like structures are needed.

These are just a few examples of how odd graceful labeling can find applications in various domains. Depending on the specific problem or application, odd graceful labeling may offer benefits in terms of efficiency, organization, or error detection.

4.3 OUTLINE OF THE PROJECT

Chapter 1 presents an introduction and certain basic concepts in graph theory that are necessary for the project. In **Chapter 2** description of odd graceful graph and its application is presented. In **Chapter 3** a brief survey of graceful and odd graceful labeling is provided. In **Chapter 4** we obtain the odd graceful labeling for n -copies of bistar and we have discussed its application. In **Chapter 5** we obtain the odd graceful labeling for n -copies of comb and we have discussed its application. We concluded by summarizing the significance of the study in **chapter 6**. We have compiled a list of references utilized throughout the

completion of this project, which we hereby present in accordance with the sequence outlined in the **Chapter7**.

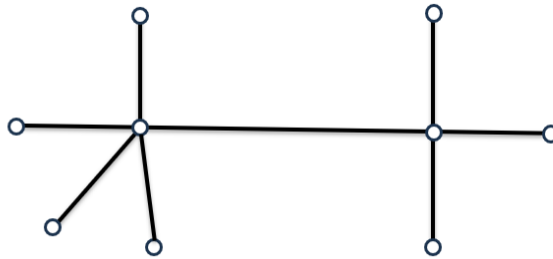
CHAPTER – 5

ODD GRACEFUL LABELING FOR BISTAR

5.1 DEFINITION

5.1.1 Bistar

A graph acquired by connecting the centre of two stars (either same or distinct) $A_{1,n}$ and $A_{1,m}$ with one edge is known as Bistar. In general, Bistar is denoted by $\hat{A}_{n,m}$.



5.1.1 Bistar graph

5.1.2 n-copies of Bistar

The n-Bistar graph is a graph obtained by connecting n copies of bistar say $B_1, B_2, \dots, B_p$ in any sequence such that we connect the point s of B_j with the point t of B_{j+1} ($j = 1, 2, \dots, p$) with $p - 1$ edges such that all the graphs are connected. Its denoted as $\overline{B_p}$.

5.2 APPLICATIONS OF BISTAR

Odd graceful labeling on bistar involves assigning unique integer labels to the edges of a graph in a manner that the totals of labels incident to each vertex form a set of consecutive integers. This labeling scheme finds applications in various areas of computer science and information technology, including network communication, graph algorithms, error detection, and parallel computing.

- **Network Communication:** The labeled graph can represent network connections, with vertex sums indicating the load or capacity of network nodes. This information is crucial for optimizing network traffic, ensuring efficient data transmission, and managing network resources effectively.
- **Graph Algorithms:** Algorithms such as shortest path algorithms or flow algorithms can utilize the labeled graph to optimize resource allocation, routing decisions, or flow control in various applications like transportation networks, telecommunications, or data centres.
- **Error Detection:** Odd graceful labeling of bistar can aid in error detection in data transmission by ensuring the consistency of edge label sums. Any deviation from the expected sum sequence can indicate missing or corrupted data, prompting appropriate error recovery mechanisms.
- **Parallel Computing:** In parallel computing environments, the labeled graph can assist in load balancing or task scheduling by providing insights into the distribution of computational resources. By analyzing the sums of edge labels incident to each vertex, efficient task allocation strategies can be devised to maximize system performance and throughput.

In general, odd graceful labeling of bistar offers a structured approach to assigning labels to graph edges, with the property that the sums of incident edge labels at each vertex form consecutive integers. This labeling scheme finds practical applications in optimizing network communication, graph algorithms, error detection, and parallel computing tasks in computer science and information technology domains.

5.3 THEOREM 1

The bistar graph $\overline{B_p}$ is said to be odd graceful.

Proof:

Let $\hat{A}_{n,m}$ be a bistar and let the graph $B_j = \hat{A}_{n_j,m_j}$ where $j = 1, 2, \dots, p$ be the p (same or distinct) copies of bistar B . Let the centre vertices be s_j and t_j and let n_j and m_j where $j = 1, 2, \dots, p$ be the count of vertices which are adjacent to s_j and t_j . Here s_j be the centre vertex of $s_{j1}, s_{j2}, \dots, s_{jn_j}$ and t_j be the centre vertex of $t_{j1}, t_{j2}, \dots, t_{jm_j}$.

In the end $\overline{B_p}$ is the graph that obtained by connecting p non-similar bistar graphs $B_1, B_2, \dots, B_p$. Let k_{1j} be the no of vertices joining s_j and k_{2j} be the no of vertices joining t_j . By connecting all the s_j with t_j by an edge we get,

the number of edge of the graph $\overline{B_p}$ is

$$(2p - 1) + \sum_{j=1}^p (n_j + m_j)$$

And the number of vertices of the graph $\overline{B_p}$ is

$$\sum_{j=1}^p (n_j + m_j + 2).$$

By the definition odd graceful we know that,

$$h : v(\overline{B_p}) \rightarrow \{0, 1, \dots, 2N - 1\}$$

Where N represents the total number of edges in the graph.

From the above definition,

Define:

For, $j=1$, $k_{1j} = 1, 2, \dots, n_j$.

$$h(s_j) = 2N - 1 \tag{5.1}$$

$$h(s_{1k_{1j}}) = 2(k_{1j} - 1) \tag{5.2}$$

For, $j=2, 3, \dots, p$.

$$h(s_j) = h(t_{(j-1)m_{(j-1)}}) - 2 \tag{5.3}$$

$$h(s_{j k_{1j}}) = h(t_{(j-1)}) + 2 \quad k_{1j} = 1 \quad (5.4)$$

$$h(s_{j k_{1j}}) = h(s_{j(k_{1j}-1)}) + 2 \quad k_{1j} = 2, 3, \dots, n_j \quad (5.5)$$

For, $j=1, 2, \dots, p$.

$$h(t_j) = h(s_{j n_1}) + 21 \quad (5.6)$$

$$h(t_{j k_{2j}}) = h(s_j) - 2 \quad k_{2j} = 1 \quad (5.7)$$

$$h(t_{j k_{2j}}) = h(t_{j(k_{2j}-1)}) - 2 \quad k_{2j} = 2, 3, \dots, m_j \quad (5.8)$$

by proceeding this till p we obtain a graph with odd graceful labeling.

From the definition of h we get the labelling of vertices in some sequence, such as

$$C_1: s_{11}, s_{12}, \dots, s_{1n_1}, t_1, s_{21}, \dots, s_{2n_2}, t_2, s_{31}, \dots, t_{p-1}, s_{(p-1)1}, s_{(p-1)2}, \dots, s_{(p-1)n_{(p-1)}}, t_p.$$

The sequence C_1 is monotonically increasing sequence that is,

$$C_1: 0, 2, \dots, 2(n_1 + n_2 + \dots + n_p + 1)$$

And the labeling of vertices,

$$C_2: s_1, t_{11}, t_{12}, \dots, t_{1m_1}, s_2, t_{21}, \dots, s_{(p-1)}, t_{(p-1)1}, \dots, t_{(p-1)n_{(p-1)}}$$

The sequence C_2 is monotonically decreasing sequence that is,

$$C_2: 2N - 1, 2N - 3, \dots, \left\{ \left[2 \left(\sum_{j=1}^P n_j + 1 \right) \right] + 1 \right\}$$

By combining C_1 and C_2 we get the following sequence,

$$C: 2N - 1, 2N - 3, \dots, \left\{ \left[2 \left(\sum_{j=1}^P n_j + 1 \right) \right] + 1 \right\}, 2 \sum_{j=1}^P n_j + 1, \dots, 2, 0.$$

Finally, by applying the definition of odd graceful labeling to label the edges, we achieve a monotonically increasing sequence.

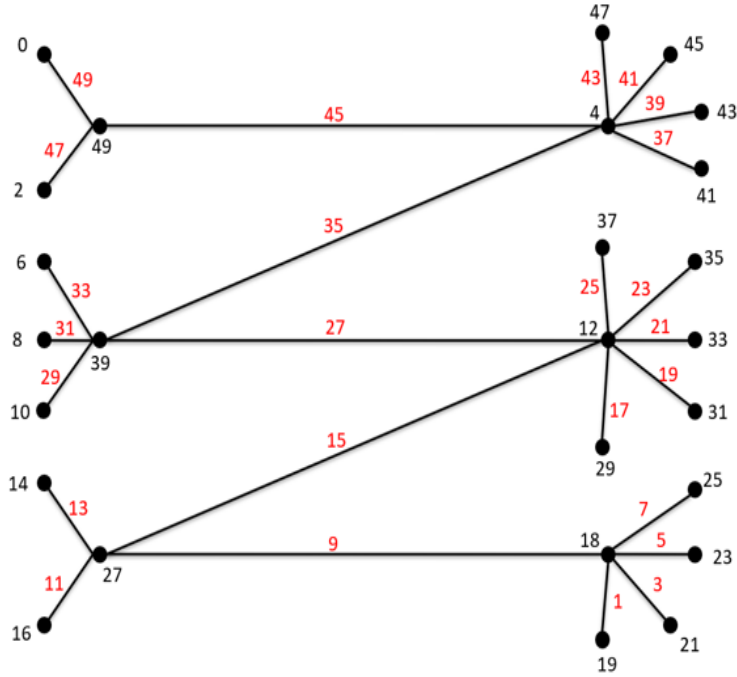


Figure 5.3.1 3-copies of bistar which is Odd graceful graph $(\overline{B_3})$

And the labeling of vertices,

$$C_2: s_1, t_{11}, t_{12}, \dots, t_{1m_1}, s_2, t_{21}, \dots, s_{(p-1)}, t_{(p-1)1}, \dots, t_{(p-1)n_{(p-1)}}$$

The sequence C_2 is monotonically decreasing sequence that is,

$$C_2: 2N - 1, 2N - 3, \dots, \left\{ \left\lceil 2 \left(\sum_{j=1}^P n_j + 1 \right) \right\rceil + 1 \right\}$$

By combining C_1 and C_2 we get the following sequence,

$$C : 2N - 1, 2N - 3, \dots, \left\{ \left\lceil 2 \left(\sum_{j=1}^P n_j + 1 \right) \right\rceil + 1 \right\}, 2 \sum_{j=1}^P n_j + 1, \dots, 2, 0.$$

Finally, by applying the definition of odd graceful labeling to label the edges, we achieve a monotonically increasing sequence,

$t_p t_{pm_p}, t_p t_{p(m_p-1)}, \dots, t_p t_{p1}, t_p s_p, s_p s_{pn_p}, s_p s_{p(n_p-1)}, \dots, t_1 t_{11}, t_1 s_1, s_1 s_{1n_1}, s_1 s_{1(n_1-1)}, \dots, s_1 s_{11}.$

The above sequence is of the form,

$$1, 3, 5, \dots, 2N - 1.$$

Consequently, the graph $\overline{B_p}$ is an odd graceful graph.

Therefore, this concludes the proof.

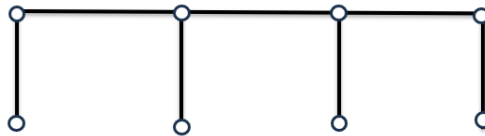
CHAPTER – 6

ODD GRACEFUL LABELING FOR COMB

6.1 DEFINITION

6.1.1 Comb

A graph obtained by joining pendent vertex to all vertex of a path is called as comb.



6.1.1 Comb graph

6.1.2 n-copies of Comb

The n-Comb graph is a graph obtained by connecting n copies of comb say $C_1, C_2, \dots, C_p$ in any sequence such that we connect the point s_{jk_j} of C_j with the point s_{j1} of C_{j+1} ($j = 1, 2, \dots, p$) with $p - 1$ edges such that all the graphs are connected. Its denoted as $\overline{C_p}$.

6.2 APPLICATIONS ON COMB

Odd graceful labeling on comb graphs involves assigning unique integer assigning labels to both vertices and edges of a comb graph to ensure specific properties are met. These labeled comb graphs can be used in computer science and information technology for tasks such as network communication, algorithm design, error detection, and data representation.

- **Graph Representation:** A comb graph is a special type of graph that resembles the structure of a comb, with a central path (the spine) and uniformly spaced teeth branching off from it.
- **Vertex and Edge Labeling:** Each vertex and edge of the comb graph is assigned a unique integer label. The labeling scheme must satisfy specific constraints, such as ensuring that the differences between adjacent labels form a set of consecutive odd integers.
- **Network Communication:** Labeled comb graphs can represent communication networks, with vertices representing network nodes and edges representing communication links. The odd graceful labeling ensures efficient routing and communication between nodes, optimizing network performance.
- **Algorithm Design:** Algorithms in graph theory, such as path finding or network flow algorithms, can leverage the labeled comb graph's structure to develop efficient solutions for various computational problems. The labeling scheme provides additional information about the graph's topology, aiding algorithm design and optimization.
- **Error Detection:** The structured labeling of comb graphs can be utilized in error detection mechanisms for data transmission or storage systems. By ensuring that the differences between adjacent labels follow a specific pattern, deviations from this pattern can indicate errors or inconsistencies in the data.
- **Data Representation:** Labeled comb graphs can serve as a data representation or modelling tool in information technology applications. They can represent hierarchical structures, organizational charts, or other data relationships, with the odd graceful labeling providing a systematic way to assign labels to graph elements.

In brief, odd graceful labeling of comb graphs offers a structured approach to graph labeling, with applications in network communication, algorithm design, error detection, and data representation in computer science and information technology. The labeling scheme provides valuable insights into the graph's structure, facilitating efficient solutions to various computational problems.

6.3 THEOREM 2

The comb graph $\overline{C_p}$ is said to be odd graceful

Proof:

Let the graph C be a comb.

By definition of comb: The graph obtained by connecting pendant vertices to all vertices along a path (P_a) is called as comb.

Let us denote the vertices of the path in the comb as $s_1, s_2, \dots, s_j$ and the pendant attached to the vertices of the path be $t_1, t_2, \dots, t_j$.

Let $\overline{C_p}$ be the graph derived from connecting p copies of comb C of even or odd order of path in comb.

Let $P_{a_j} : s_{j1}, s_{j2}, \dots, s_{ja_j}$ be the path of the graph $\overline{C_p}$ and the single vertex connected to the paths be $t_{j1}, t_{j2}, \dots, t_{ja_j}$ where $j = 1, 2, \dots, p$. Here a_j be the no. of vertices in the path C_j or number of single vertices connected to the path C_j and $k_j = 1, 2, \dots, a_j$.

Here $\overline{C_p}$ be a graph gained by connecting p copies of $C_1, C_2, \dots, C_p$ of C by connecting vertex t_{ja_j} of C_j to the vertex $t_{(j+1)1}$ of C_{j+1} .

the number of edge of the graph $\overline{C_p}$ is

$$2 \sum_{j=1}^p a_j$$

And the number of vertices of the graph $\overline{C_p}$ is

$$\left[2 \sum_{j=1}^p (a_j) \right] - 1.$$

By the definition odd graceful we know that,

$$h: v(\overline{C_p}) \rightarrow \{0, 1, \dots, 2N - 1\}$$

Where N represents the total no. of edges in the graph.

Case 1: if all a_j are even.

For $j = 1, k_j = 1$

$$h(s_{jk_j}) = 2N - 1 \quad (6.1)$$

$$h(t_{jk_j}) = 2(j - 1) \quad (6.2)$$

For $j = 1, 2, \dots, \dots, p, k_j = 2, 4, \dots a_j$.

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) + 2 \quad (6.3)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) - 2 \quad (6.4)$$

For $j = 1, 2, \dots, \dots, p$ & $k_j = 3, 5, \dots a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) - 2 \quad (6.5)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) + 2$$

by proceeding this till p we obtain a graph with odd graceful labeling.

From the definition of h we get the labelling of vertices in some sequence for even a_j , such as

$$S_1: t_{11}, s_{12}, \dots, s_{1a_1}, t_{21}, s_{22}, \dots, s_{2a_2}, t_{31}, s_{32}, \dots, s_{3a_3}, \dots, t_{p1}, s_{p2}, \dots, s_{pa_p}.$$

The sequence g_1 is monotonically increasing sequence that is,

$$S_1: 0, 2, \dots, N - 1$$

And the labeling of vertices for even a_j ,

$$S_2: s_{11}, t_{12}, \dots, t_{1a_1}, s_{21}, t_{22}, \dots, t_{2a_2}, \dots, s_{p1}, t_{p2}, \dots, t_{pa_p}$$

The sequence S_2 is monotonically decreasing sequence that is,

$$S_2: 2N - 1, 2N - 3, 2N - 5, \dots, N$$

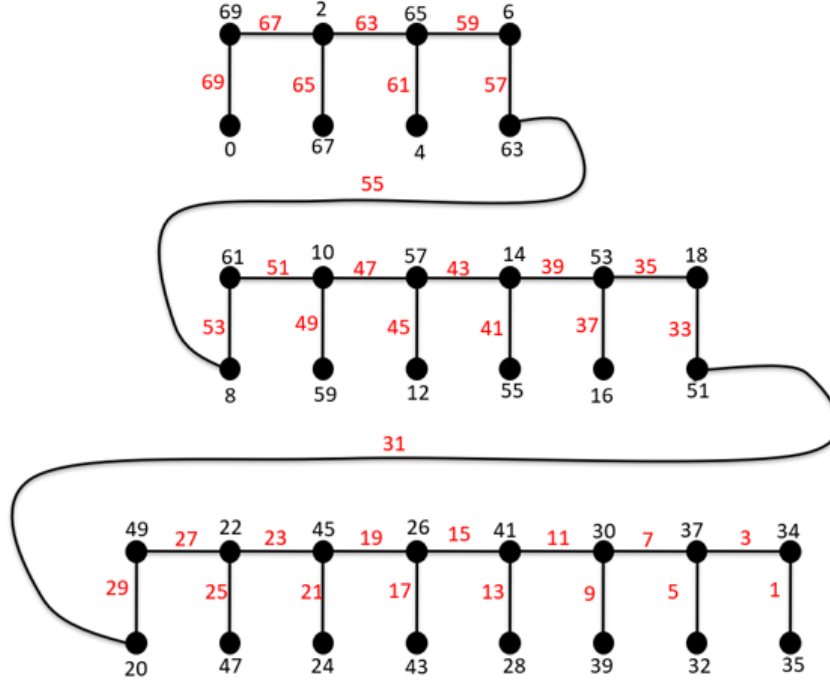


Figure 6.3.1 3-copies of comb (with even a_j) which is Odd graceful graph ($\overline{C_3}$)

By combining S_1 and S_2 we get the following sequence for even a_j ,

$$S : 2N - 1, 2N - 3, \dots, N - 1, N, \dots, 2, 0.$$

Case 2: if all a_j are odd.

For $j = 1$ & $k_j = 1$

$$h(s_{jk_j}) = 2N - 1 \tag{6.6}$$

$$h(t_{jk_j}) = 2(j - 1) \tag{6.7}$$

For $j = 1, 3, 5, \dots, p$ & $k_j = 2, 4, \dots, a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) + 2 \quad (6.8)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) - 2 \quad (6.9)$$

For $j = 1, 3, 5, \dots, p$ & $k_j = 3, 5, \dots, a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) - 2 \quad (6.10)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) + 2 \quad (6.11)$$

For $j = 2, 4, \dots, p$ & $k_j = 1$

$$h(s_{jk_j}) = h(t_{(j-1)k_{(j-1)}}) + 2 \quad (6.12)$$

$$h(t_{jk_j}) = h(s_{(j-1)k_{(j-1)}}) - 2 \quad (6.13)$$

For $j = 2, 4, \dots, p$ & $k_j = 2, 4, \dots, a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) - 2 \quad (6.14)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) + 2 \quad (6.15)$$

For $j = 2, 4, \dots, p$ & $k_j = 3, 5, \dots, a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) + 2 \quad (6.16)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) - 2 \quad (6.17)$$

For $j = 3, 5, \dots, p$ & $k_j = 1$

$$h(s_{ja_j}) = h(t_{(j-1)a_{(j-1)}}) - 2 \quad (6.18)$$

$$h(t_{ja_j}) = h(s_{(j-1)a_{(j-1)}}) + 2 \quad (6.19)$$

For $j = 3, 5, \dots, p$ & $k_j = 2, 4, \dots, a_j$

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) + 2 \quad (6.20)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) - 2 \quad (6.21)$$

For $j = 3, 5, \dots, p$ & $k_j = 3, 5, \dots, a_j$.

$$h(s_{jk_j}) = h(t_{j(k_j-1)}) - 2 \quad (6.22)$$

$$h(t_{jk_j}) = h(s_{j(k_j-1)}) + 2 \quad (6.23)$$

by proceeding this till p we obtain a graph with odd graceful labeling.

From the definition of h we get the labelling of vertices in some sequence for odd a_j , such as

$$S_1: t_{11}, s_{12}, \dots, t_{1a_1}, s_{21}, t_{22}, \dots, s_{3a_2}, t_{31}, s_{32}, \dots, t_{3a_3}, \dots, t_{p1}, s_{p2}, \dots, t_{pa_p}.$$

The sequence S_1 is monotonically increasing sequence that is,

$$S_1: 0, 2, \dots, N - 1$$

And the labeling of vertices for odd a_j ,

$$S_2: s_{11}, t_{12}, \dots, s_{1a_1}, t_{21}, s_{22}, \dots, t_{2a_2}, \dots, s_{p1}, t_{p2}, \dots, s_{pa_p}$$

The sequence S_2 is monotonically decreasing sequence that is,

$$S_2: 2N - 1, 2N - 3, 2N - 5, \dots, N$$

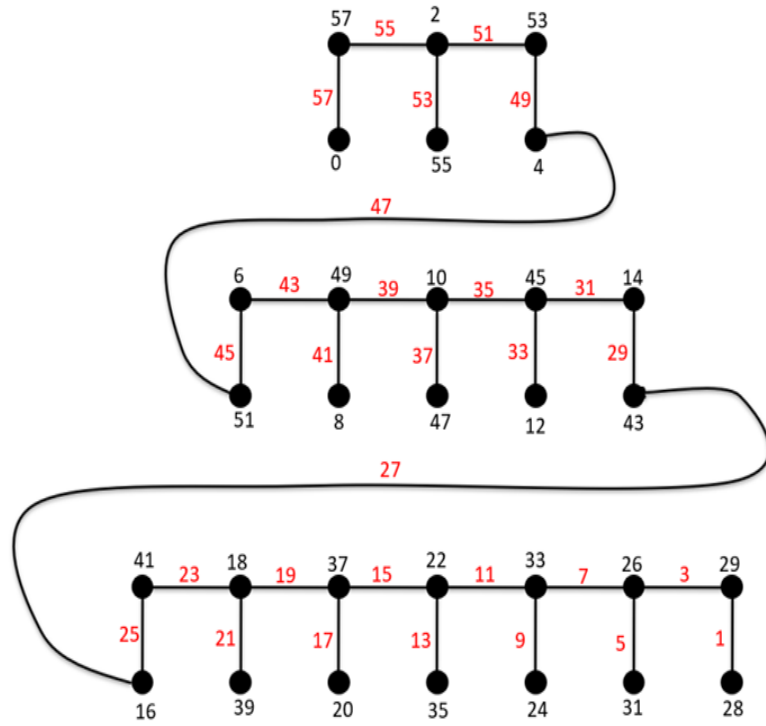


Figure 6.3.2 3-copies of comb (with even a_j) which is Odd graceful graph ($\overline{C_3}$)

By combining S_1 and S_2 we get the following sequence for odd a_j ,

$$S : 2N - 1, 2N - 3, \dots, N - 1, N, \dots, 2, 0.$$

Consequently, from case 1 and case 2 we know that the graph $\overline{C_p}$ is odd graceful graph.

Therefore, this concludes the proof.

CHAPTER – 7

CONCLUSION

In the text of graph, its remarkable to explore the families of odd graceful graph. By Theorem 1 & 2, we have exposed two families of graph which is odd graceful labeling. Inspired by the labeling of graphs odd graceful labeling, we have obtained the odd graceful labeling for n -copies of bistar and n -copies of comb which is connected by $n-1$ edges. This can be extended for arbitrary super subdivision of certain families of graphs in future.

REFERENCE

- [1] Bloom G.S. and S. W. Golomb, Applications of numbered undirected graphs, Proceeding of IEEE, 65(4), (1977), pp. 562-570.
- [2] Eldergill P. (1997). Decomposition of the Complete Graph with an Even Number of Vertices, M. Sc. Thesis, McMaster University.
- [3] Gallian J.A. (2020). A dynamic survey of Graph labeling, The Electronic Journal of Combinatorics, (2020), 77-81.
- [4] Gnanajothi R. B., Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, 1991.
- [5] Govindarajan R. and V. Srividya, Odd Graceful Labeling of Cycle with Parallel P_k Chords, Annals of Pure and Applied Mathematics Vol. 8, No. 2, (2014), 123-129.
- [6] Jeba Jesintha J , R. Jaya Glory and K. Farzana, Odd graceful labeling on revised sunflower graph, International Journal of Pure and Applied Mathematics, vol 120, 8, 2018, 69-77.
- [7] Kaladevi V. and Backiyalakshmi P (2011). Maximum Distance Matrix of Super Subdivision of Star Graph, J. Comp. & Math. Sci. 2(6), 828-835.
- [8] Kathiresan K. M., Some Class of Odd Graceful Graphs, ANJAC J. Sci., 7(1), (2008), pp. 5-10.

- [9] Li W. Z. and Q. T. Yan, Odd graceful labeling of graph $P_{r,(2s-1)}$. Math. Prac. Thero. 41 (61) (2011) 254-258.
- [10] Mahmoud I Moussa, El-Sayed Badr (2009). Odd Graceful labelings of Crown Graphs, 1st Int Conference on Computer Science from Algorithms to Applications (CSAA-2009), Cairo, Egypt, 8-10.
- [11] Neela N and Selvaraj C. (2016). Conjecture on odd graceful graphs, J. Combin Math Combin Comput., 97, 65-82.
- [12] Rosa A (1966). On certain valuations of the vertices of a graph, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris, (1967), 349-355.
- [13] Sundaram M., R. Ponraj and S. Somasundaram Prime Cordial Labeling of Graphs, Journal of Indian Academy of Mathematics, 27 (2005) 373-390.
- [14] Sridevi R., S. Navaneethakrishnan and K. Nagarajan, Odd-Even graceful graphs, J. Appl.Math. & Informatics Vol. 30 (2012), No. 5-6, pp. 913-923.
- [15] Sushant Kumar Rout, Debdas Mishra and Purna Chandra Nayak, Odd Graceful Labeling Some New Types of Graphs, International Journal of Mathematics Trends and Technology Volume 41 Number 1- January 2017.
- [16] Vaidya S. K., Odd Graceful Labeling of Some New Graphs, Vol. 4, NO. 10; October 2010.

- [17] Vaidya S. K. and U. M. Prajapati, Fibonacci and Super Fibonacci Graceful Labelings of some Cycle related Graphs, International J. Math. Combi. Vol. 4, (2011), 59-69.
- [18] Vaidya S. K. and N. H. Shah, Some new Odd harmonious graphs, Int J. Math. Soft. Comput. 1 (1) (2011) 9-16.
- [19] Vaidya S. K. and N. H. Shah, Odd harmonious labeling of some graphs, J. Math. Combin. 3 (2012) 105-102.
- [20] Vasuki R. and A. Nagarajan, Studies in Graph Theory – Labeling Problems in Graphs (Thesis), Manonmaniam Sundaranar University, December 2010.
- [21] Yan Q. T., Odd gracefulness and odd strongly harmoniousness of the product graphs $P_m \times P_n$, J. Systems Sci. Math. Sci., 30, 2010, 341 -348.