

**ALGEBRAIC PROPERTIES OF
 ϑ –TRANSLATION AND ϑ –MULTIPLICATION
IN FUZZY Z– ALGEBRA**

Submitted in partial fulfillment of the requirements for the award of Master of
Science degree in Mathematics

By

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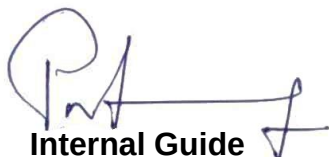
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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonfide work of **JOEL JAIKUMAR MADHURAM G (42630002)** who carried out the project entitled "**Algebraic Properties of ϑ – Translation and ϑ – Multiplication in Fuzzy Z- Algebra**" under our supervision from November **2023** to **April 2024**.



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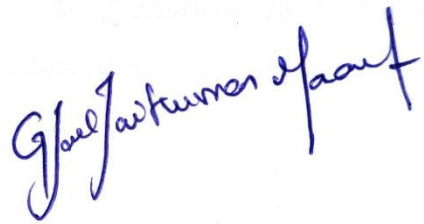
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DECLARATION

I **JOEL JAIKUMAR MADHURAM G (42630002)** hereby declare that the Project Report entitled “**Algebraic Properties of ϑ –Translation and ϑ –Multiplication in Fuzzy Z- Algebra** ” done by me under the guidance of **Dr. PREMKUMAR M (Internal)** at **Satyabhama Institute of Science and Technology (Deemed to be University)** is submitted in partial fulfillment of the requirements for the award of Master of Science degree in Mathematics.

DATE : 20.04.2024

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A handwritten signature in blue ink, reading "Joel Jaikumar Madhuram G", is written over a faint, circular official stamp of Satyabhama Institute of Science and Technology.

SIGNATURE OF THE CANDIDATE

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Abstract:

In this project, we define the new notation of Algebraic structures of ϑ –Translations and ϑ –Multiplication in Z –Sub Algebra of Z –Algebras and Z –Ideal of Z –Algebra and discussed some of their properties in detail by Z –Algebras. Also, we define the new notation of on characteristics of ϑ –Translations and ϑ –Multiplication in Q -Fuzzy Z –Sub Algebras and also explored the Q -Fuzzy Z –Ideal on Z –Algebra. We also define the new notation of Doubt Fuzzy Z –Sub Algebra and Z –Ideal of Z –Algebras under the ϑ –translations and ϑ –Multiplications.

Keywords:

Fuzzy Set, Fuzzy Subset, Z -Algebra, Z -Ideal, Z –Sub Algebra , Fuzzy Z -Ideal, Fuzzy Z –Algebra, Fuzzy Z –Sub Algebra, Fuzzy ϑ – Translation, Fuzzy ϑ – Multiplication, Q –fuzzy Z -Algebra, Q -fuzzy Z -Ideal, Q –fuzzy Z –Sub Algebra , Q -Fuzzy ϑ – Translation, Q -Fuzzy ϑ –Multiplication, Doubt Fuzzy Z -Ideal, Doubt Fuzzy Z –Sub Algebra, Doubt Fuzzy ϑ – Translation, Doubt Fuzzy ϑ – Multiplication.

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CHAPTER- 1

INTRODUCTION

1.1 INTRODUCTION

In 1965, Zadeh L A [20], initiated by the concept of fuzzy sets. Several researchers explored on the generalization of the notion of fuzzy subset. The study of fuzzy subsets and its applications to various mathematical contents has given rise to what is now commonly called fuzzy mathematics. In 1971, Rosenfield [15] introduced the concept of Fuzzy groups. Iseki K and Tanaka S [4], introduced the concept of an introduction to the theory of BCK-algebras in 1978. In 1980, Iseki K [5], first introduced the notation on BCI-algebras. Das P S [3] formulated the concept of Fuzzy groups and level subgroups in 1981. In 1984, Mukherjee N P & Bhattacharya P [8] introduced Fuzzy normal subgroups and fuzzy cosets. Mordeson J N, Bhutani K R & Rosenfield A [7] gave the concept of Fuzzy group theory in 2005. In 2008, Solairaju A and Nagarajan R [16]&[17] introduced Q -fuzzy left R-subgroups of near rings with respect to T -norms and they also gave a new structure and construction of Q -fuzzy Groups in 2009. Kyoung Ja Lee, Young Bae Jun and Myung Im Doh [6], introduced the concept of fuzzy translations and fuzzy multiplication of BCK/BCI-algebras in 2009. Abu Ayub Ansari and Chandramouleeswaran M [1], introduced the concept of fuzzy translation of fuzzy β – ideals of β – algebras in 2014. In 2014, Priya and Ramachandran T [14], introduced the new notation of fuzzy translation and multiplication on PS-algebras. Prasanna A, Premkumar M and Ismail Mohideen S [9] & [10], introduced the concept of fuzzy translation and multiplication on B-algebras in 2018 and also derived from Fuzzy Translation and Fuzzy Multiplication in BG – Algebras in 2019. In 2021, Premkumar [11] derived the new notation of Algebraic Properties on Fuzzy Translation and Multiplication in BP– Algebras. Premkumar [12] & [13], introduced the new concept of Algebraic Properties on ω – Fuzzy Translation and Multiplication in BH – Algebras in 2020 and also derived from the concept of Characteristics of κ – Q – Fuzzy Translation and Fuzzy Multiplication in T-Ideals in T-Algebra in 2022. Sowmiya [18] & [19] initiated by the concept on Fuzzy Z-ideals in Z-algebras and also Fuzzy Algebraic Structure in Z-Algebras in 2019.

1.2 PRELIMINARIES

1.2.1 Definition

A Z -algebra $(\tilde{\omega}, *, 0)$ is a non-empty set $\tilde{\omega}$ with constant 0 and a binary operation $*$ satisfying the following conditions:

- (i) $\acute{r} * 0 = 0$
- (ii) $0 * \acute{r} = \acute{r}$
- (iii) $\acute{r} * \acute{r} = \acute{r}$
- (iv) $\acute{r} * \acute{s} = \acute{s} * \acute{r}$ when $\acute{r} \neq 0$ and $\acute{s} \neq 0$ for all $\acute{r}, \acute{s} \in \tilde{\omega}$

1.2.2 Definition

Let $\tilde{\omega}$ be a Z -algebra and I be a subset of $\tilde{\omega}$. Then, I is called a Z -ideal of $\tilde{\omega}$, if it satisfies the following conditions

- (i) $0 \in I$
- (ii) $\acute{r} * \acute{s} \in I$ and $\acute{s} * \acute{r} \in I$ implies $\acute{r} \in I$, For all $\acute{r}, \acute{s}$ in $\tilde{\omega}$,

1.2.3 Definition

A Z -algebra $(\tilde{\omega}, *, 0)$ be a Z -algebra. A fuzzy set A in $\tilde{\omega}$ with a membership function $\check{\mathfrak{Z}}_A$ is said to be a fuzzy- Z -sub algebra of a Z -algebra $\tilde{\omega}$ if, for all $\acute{r}, \acute{s}$ in $\tilde{\omega}$ the following condition is satisfied

$$\check{\mathfrak{Z}}(\acute{r} * \acute{s}) \geq \{\check{\mathfrak{Z}}(\acute{r}) \wedge \check{\mathfrak{Z}}(\acute{s})\}$$

1.2.4 Definition

A Z -algebra $(\tilde{\omega}, *, 0)$ be a Z -algebra. A fuzzy set V in $\tilde{\omega}$ with a membership function $\check{\mathfrak{Z}}_A$ is said to be a fuzzy- Z -sub algebra of a Z -algebra $\tilde{\omega}$ if, for all $\acute{r}, \acute{s}$ in $\tilde{\omega}$ the following condition is satisfied

- (i) $\check{\mathfrak{Z}}(0) \geq \check{\mathfrak{Z}}(\acute{r})$
- (ii) $\check{\mathfrak{Z}}(\acute{r}) \geq \{\check{\mathfrak{Z}}(\acute{r} * \acute{s}) \wedge \check{\mathfrak{Z}}(\acute{s})\}$

1.2.5 Definition

Let Q and $\check{G}$ be a set and group respectively. The mapping $\mu : \check{\omega} \times Q \rightarrow [0,1]$ is called Q -Fuzzy Set in $\check{G}$. For any Q -Fuzzy Set μ in $\check{G}$ and $\mathfrak{t} \in [0,1]$ we define the set $U(\mu; \mathfrak{t}) = \{\mathfrak{p} \in \check{\omega} / \mu(\mathfrak{p}, q) \geq \mathfrak{t}, q \in Q\}$ which is called an upper cut of " μ " and can be used to the characterization of μ .

CHAPTER- 2

LITERATURE SURVEY

In 1965, Zadeh L A [20], initiated by the concept of fuzzy sets. Several researchers explored on the generalization of the notion of fuzzy subset. The study of fuzzy subsets and its applications to various mathematical contents has given rise to what is now commonly called fuzzy mathematics. In 1971, Rosenfield [15] introduced the concept of Fuzzy groups. Iseki K and Tanaka S [4], introduced the concept of an introduction to the theory of BCK-algebras in 1978. In 1980, Iseki K [5], first introduced the notation on BCI-algebras. Das P S [3] formulated the concept of Fuzzy groups and level subgroups in 1981. In 1984, Mukherjee N P & Bhattacharya P [8] introduced Fuzzy normal subgroups and fuzzy cosets. Mordeson J N, Bhutani K R & Rosenfield A [7] gave the concept of Fuzzy group theory in 2005. In 2008, Solairaju A and Nagarajan R [16]&[17] introduced Q -fuzzy left R-subgroups of near rings with respect to T -norms and they also gave a new structure and construction of Q -fuzzy Groups in 2009. Kyoung Ja Lee, Young Bae Jun and Myung Im Doh [6], introduced the concept of fuzzy translations and fuzzy multiplication of BCK/BCI-algebras in 2009. Abu Ayub Ansari and Chandramouleeswaran M [1], introduced the concept of fuzzy translation of fuzzy β – ideals of β – algebras in 2014. In 2014, Priya and Ramachandran T [14], introduced the new notation of fuzzy translation and multiplication on PS-algebras. Prasanna A, Premkumar M and Ismail Mohideen S [9] & [10], introduced the concept of fuzzy translation and multiplication on B-algebras in 2018 and also derived from Fuzzy Translation and Fuzzy Multiplication in BG – Algebras in 2019. In 2021, Premkumar [11] derived the new notation of Algebraic Properties on Fuzzy Translation and Multiplication in BP– Algebras. Premkumar [12] & [13], introduced the new concept of Algebraic Properties on ω – Fuzzy Translation and Multiplication in BH – Algebras in 2020 and also derived from the concept of Characteristics of κ – Q – Fuzzy Translation and Fuzzy Multiplication in T-Ideals in T-Algebra in 2022. Sowmiya [18] & [19] initiated by the concept on Fuzzy Z-ideals in Z-algebras and also Fuzzy Algebraic Structure in Z-Algebras in 2019.

This project holds our work of implying ϑ –Translations and ϑ – Multiplications on Z –Algebra and also defined Q- Fuzzy Z –Sub Algebra and Doubt Fuzzy Z –Sub Algebra

CHAPTER- 3

AIM AND SCOPE

In this project, we define the new notation of Algebraic structures of ϑ –Translations and ϑ –Multiplication in Z –Sub Algebra of Z –Algebras and Z –Ideal of Z –Algebra and discussed some of their properties in detail by Z –Algebras. Also, we define the new notation of Algebraic structures of ϑ –Translations and ϑ –Multiplication in Z –Sub Algebra of Q fuzzy Z –Algebras. Moreover we define the ϑ –translations and ϑ –Multiplications in Z –Ideal of Q fuzzy Z –Algebra and discussed some of their properties in detail. In this project, we also define the new notation of Algebraic structures of ϑ –translations and ϑ –Multiplications in Doubt Fuzzy Z –Sub Algebra of Z –Algebras. And also defined the ϑ –translations and ϑ –Multiplications in Doubt Fuzzy Z –Ideal of Z –Algebra and discussed some of their properties in detail by Z –Algebras.

CHAPTER- 4

ON CHARACTERISTICS OF ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN FUZZY Z –SUB ALGEBRA AND Z –IDEAL ON Z –ALGEBRA

In this chapter, attempts have made to explain ϑ –Translations and ϑ –Multiplications in Fuzzy Z –Sub Algebra and Z –Ideal on Z –Algebra with adequate illustration comprising of necessary theorems and examples.

4.1 ON CHARACTERISTICS OF ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN FUZZY Z –SUB ALGEBRA ON Z –ALGEBRA

Let $\tilde{\omega}$ be a Z -algebra. For any fuzzy set $\check{z}$ of $\tilde{\omega}$, we defined $T=1-\sup\{\check{z}(r)/r \in \tilde{\omega}\}$, unless otherwise we specified.

4.1.1 Definition

Let $\check{z}$ be a fuzzy subset of $\tilde{\omega}$ and $\vartheta \in [0, T]$. A mapping $\check{z}_{\vartheta}^T: \tilde{\omega} \rightarrow [0,1]$ is said to be a fuzzy ϑ – translation of $\check{z}$ if it satisfies $\check{z}_{\vartheta}^T = \check{z}(r) + \vartheta, \forall r \in \tilde{\omega}$.

4.1.2 Definition

Let $\check{z}$ be a fuzzy subset of $\tilde{\omega}$ and $\vartheta \in [0,1]$. A mapping $\check{z}_{\vartheta}^M: \tilde{\omega} \rightarrow [0,1]$ is said to be a fuzzy ϑ – multiplication of $\check{z}$ if it satisfies $\check{z}_{\vartheta}^M = \vartheta \check{z}(r), \forall r \in \tilde{\omega}$.

4.1.2.1 Example

Let $\tilde{\omega} = \{0,1,2,3\}$ be the set with the following table.

*	0	1	2	3
3	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(\tilde{\omega}, *, 0)$ is a Z – algebra.

Define Fuzzy Set $\check{Z}$ is of $\tilde{\omega}$ by $\check{Z}(\acute{r}) = \begin{cases} 0.4 & \text{if } \acute{r} \neq 1 \\ 0.3 & \text{if } \acute{r} = 1 \end{cases}$

Thus $\check{Z}$ is a *Fuzzy Z – Subalgebra* of X .

Hence $T = 1 - \sup\{\check{Z}(\acute{r})/\acute{r} \in \tilde{\omega}\} = 1 - 0.4 = 0.6$, Choose $\vartheta = 0.2 \in [0,1]$ and $\vartheta = 0.3 \in [0,1]$.

Then the mapping $\check{Z}_{0.2^T} : \tilde{\omega} \rightarrow [0,1]$ is defined by

$$\check{Z}_{0.2^T} = \begin{cases} 0.2 + 0.4 = 0.6 & \text{if } \acute{r} \neq 1 \\ 0.2 + 0.3 = 0.5 & \text{if } \acute{r} = 1 \end{cases}$$

Which satisfies $\check{Z}_{0.2^T}(\acute{r}) = \check{Z}(\acute{r}) + 0.2, \forall \acute{r} \in \tilde{\omega}$, is a Fuzzy 0.2-Translation.

The mapping $\gamma_{0.3^M} : \tilde{\omega} \rightarrow [0,1]$ is defined by

$$\check{Z}_{0.3^M} = \begin{cases} 0.3 * 0.4 = 0.12 & \text{if } \acute{r} \neq 1 \\ 0.3 * 0.3 = 0.09 & \text{if } \acute{r} = 1 \end{cases}$$

Which satisfies $\check{Z}_{0.3^M}(\acute{r}) = \check{Z}(\acute{r})(0.3), \forall \acute{r} \in \tilde{\omega}$, is a Fuzzy 0.3-Multiplication.

4.1.3 Theorem

If $\check{Z}$ of $\tilde{\omega}$ is a fuzzy Z -sub-algebra and $\vartheta \in [0,1]$, then the fuzzy ϑ –translation.

$\check{Z}_{\vartheta}^T(\acute{r})$ of $\check{Z}$ is also a fuzzy Z - sub algebra of $\tilde{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \tilde{\omega}$ and $\vartheta \in [0, T]$

Then, $\check{Z}(\acute{r} * \acute{s}) \geq \check{Z}(\acute{r}) \wedge \check{Z}(\acute{s})$

$$\begin{aligned} \text{Now, } \check{Z}_{\vartheta}^T(\acute{r} * \acute{s}) &= \check{Z}(\acute{r} * \acute{s}) + \vartheta \\ &\geq [\check{Z}(\acute{r}) \wedge \check{Z}(\acute{s})] + \vartheta \end{aligned}$$

$$\begin{aligned}
&= [(\check{Z}(\acute{r}) + \vartheta) \wedge (\check{Z}(\check{s}) + \vartheta)] \\
&= [\check{Z}_{\vartheta}^T(\acute{r}) \wedge \check{Z}_{\vartheta}^T(\check{s})]. \forall \acute{r}, \check{s} \in \check{\omega}
\end{aligned}$$

4.1.4 Theorem

Let $\check{Z}$ be a fuzzy subset of $\check{\omega}$ such that the fuzzy ϑ – translation $\check{Z}_{\vartheta}^T(\acute{r})$ of $\check{Z}$ is a fuzzy sub algebra of $\check{\omega}$, for some $\vartheta \in [0, T]$, then $\check{Z}$ is a fuzzy Z -sub algebra of $\check{\omega}$.

Proof:

Assume that $\check{Z}_{\vartheta}^T(\acute{r})$ is a fuzzy sub algebra of $\check{\omega}$ for some $\vartheta \in [0, T]$

Let $\acute{r}, \check{s} \in \check{\omega}$, we have

$$\begin{aligned}
\check{Z}(\acute{r} * \check{s}) + \vartheta &= \check{Z}_{\vartheta}^T(\acute{r} * \check{s}) \\
&\geq [\check{Z}_{\vartheta}^T(\acute{r}) \wedge \check{Z}_{\vartheta}^T(\check{s})] \\
&= [(\check{Z}(\acute{r}) + \vartheta) \wedge (\check{Z}(\check{s}) + \vartheta)] \\
&= [\check{Z}(\acute{r}) \wedge \check{Z}(\check{s})] + \vartheta \\
\Rightarrow \check{Z}(\acute{r} * \check{s}) &\geq [\check{Z}(\acute{r}) \wedge \check{Z}(\check{s})], \forall \acute{r}, \check{s} \in \check{\omega}
\end{aligned}$$

Hence, $\check{Z}$ is fuzzy sub algebra of $\check{\omega}$.

4.1.5 Theorem

For any fuzzy Z - sub algebra $\check{Z}$ of $\check{\omega}$ and $\vartheta \in [0, 1]$, if the fuzzy ϑ – multiplication $\check{Z}_{\vartheta}^M(\acute{r})$ of $\check{Z}$ is a fuzzy Z -sub algebra of $\check{\omega}$.

Proof:

Let $\acute{r}, \check{s} \in \check{\omega}$ and $\vartheta \in [0, T]$

Then $\check{Z}(\acute{r} * \check{s}) \geq \check{Z}(\acute{r}) \wedge \check{Z}(\check{s})$

$$\begin{aligned}
\text{Now, } \check{Z}_{\vartheta}^M(\acute{r} * \check{s}) &= \vartheta \check{Z}(\acute{r} * \check{s}) \\
&\geq \vartheta [\check{Z}(\acute{r}) \wedge \check{Z}(\check{s})]
\end{aligned}$$

$$\begin{aligned}
&\geq [\vartheta \check{\mathfrak{Z}}(\acute{r}) \wedge \vartheta \check{\mathfrak{Z}}(\check{s})] \\
&= [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s})] \\
\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) &\geq [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s})]
\end{aligned}$$

Therefore, $\check{\mathfrak{Z}}_{\vartheta}^M$ is a fuzzy Z - sub algebra of $\check{\mathfrak{O}}$.

4.1.6 Theorem

For any fuzzy subset $\check{\mathfrak{Z}}$ of $\check{\mathfrak{O}}$ and $\vartheta \in [0,1]$, if the fuzzy ϑ – multiplication $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -sub algebra of $\check{\mathfrak{O}}$, then so in $\check{\mathfrak{Z}}$.

Proof:

Assume that $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a fuzzy Z - sub algebra of $\check{\mathfrak{O}}$ for some $\vartheta \in [0, T]$

Let $\acute{r}, \check{s} \in \check{\mathfrak{O}}$, we have

$$\begin{aligned}
\vartheta \check{\mathfrak{Z}}(\acute{r} * \check{s}) &= \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) \\
&\geq [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s})] \\
&= [\vartheta \check{\mathfrak{Z}}(\acute{r}) \wedge \vartheta \check{\mathfrak{Z}}(\check{s})] \\
&= \vartheta [\check{\mathfrak{Z}}(\acute{r}) \wedge \check{\mathfrak{Z}}(\check{s})] \\
\Rightarrow \check{\mathfrak{Z}}(\acute{r} * \check{s}) &\geq \vartheta [\check{\mathfrak{Z}}(\acute{r}) \wedge \check{\mathfrak{Z}}(\check{s})]
\end{aligned}$$

Hence, $\check{\mathfrak{Z}}$ is a fuzzy Z - sub algebra of $\check{\mathfrak{O}}$.

4.2 ON CHARACTERISTICS OF ϑ –TRANSLATIONS AND ϑ – MULTIPLICATIONS IN FUZZY Z –IDEAL ON Z –ALGEBRA

4.2.1 Theorem

If the fuzzy ϑ – translation $\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r})$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -Ideal, then it satisfies the condition $\check{\mathfrak{Z}}_{\vartheta}^T(\check{s} * (\acute{r} * \check{s})) \geq \check{\mathfrak{Z}}_{\vartheta}^T(\acute{r})$.

Proof:

$$\begin{aligned}
\check{\mathfrak{Z}}_{\vartheta}^T(\check{s} * (\check{r} * \check{s})) &= \check{\mathfrak{Z}}(\check{s} * (\check{r} * \check{s})) + \vartheta \\
&\geq \{\check{\mathfrak{Z}}(0 * (\check{s} * (\check{r} * \check{s}))) + \vartheta \wedge \check{\mathfrak{Z}}(0) + \vartheta\} \\
&\geq \{\check{\mathfrak{Z}}(0 * (\check{s} * (\check{s} * \check{r}))) + \vartheta \wedge \check{\mathfrak{Z}}(0) + \vartheta\} \\
&= \{\check{\mathfrak{Z}}(0 * ((\check{s} * \check{s}) * \check{r})) + \vartheta \wedge \check{\mathfrak{Z}}(0) + \vartheta\} \\
&= \{\check{\mathfrak{Z}}(0 * (\check{s} * \check{r})) + \vartheta \wedge \check{\mathfrak{Z}}(0) + \vartheta\} \\
&= \{\check{\mathfrak{Z}}((\check{s} * \check{r}) * 0) + \vartheta \wedge \check{\mathfrak{Z}}(0) + \vartheta\} \\
&\geq \{\check{\mathfrak{Z}}((\check{s} * \check{r}) * 0) + \vartheta \wedge \check{\mathfrak{Z}}(\check{r}) + \vartheta\} \\
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T(0) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{r})\} \\
&= \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}). \\
\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^T(\check{s} * (\check{r} * \check{s})) &\geq \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \quad \forall \check{r}, \check{s} \in \check{\mathfrak{O}}
\end{aligned}$$

4.2.2 Theorem

If $\check{\mathfrak{Z}}$ is a fuzzy Z-ideal of $\check{\mathfrak{O}}$, then the fuzzy ϑ -translation $\check{\mathfrak{Z}}_{\vartheta}^T(\check{r})$ of $\check{\mathfrak{Z}}$ is a fuzzy Z-ideal of $\check{\mathfrak{O}}$, for all $\vartheta \in [0, T]$.

Proof:

Let $\check{\mathfrak{Z}}$ be a fuzzy Z-ideal of $\check{\mathfrak{O}}$ and let $\vartheta \in [0, T]$. Then,

$$\begin{aligned}
\text{(i) } \check{\mathfrak{Z}}_{\vartheta}^T(0) &= \check{\mathfrak{Z}}(0) + \vartheta \\
&\geq \check{\mathfrak{Z}}(\check{r}) + \vartheta = \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \\
\text{(ii) } \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) &= \check{\mathfrak{Z}}(\check{r}) + \vartheta \\
&\geq \{\check{\mathfrak{Z}}(\check{r} * \check{s}) \wedge \check{\mathfrak{Z}}(\check{s})\} + \vartheta \\
&= \{(\check{\mathfrak{Z}}(\check{r} * \check{s}) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{s}) + \vartheta)\} \\
&= \{\check{\mathfrak{Z}}_{\vartheta}^T(\check{r} * \check{s}) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s})\} \\
&\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \geq \{\check{\mathfrak{Z}}_{\vartheta}^T(\check{r} * \check{s}) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s})\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T(\check{r})$ of $\check{\mathfrak{Z}}$ is a fuzzy Z-ideal of $\check{\mathfrak{O}}$, $\forall \vartheta \in [0, T]$

4.2.3 Theorem

Let $\check{Z}$ is a fuzzy subset of $\check{\omega}$ such that the fuzzy ϑ – translation $\check{Z}_\vartheta^T(\acute{r})$ of $\check{Z}$ is a fuzzy Z - ideal of $\check{\omega}$, for some $\vartheta \in [0, T]$, then $\check{Z}$ is a fuzzy Z - ideal of $\check{\omega}$.

Proof:

Assume that $\check{Z}_\vartheta^T$ is a fuzzy Z - ideal of $\check{\omega}$ for some $\vartheta \in [0, T]$.

Let $\acute{r}, \acute{s} \in \check{\omega}$. Then,

$$\begin{aligned} \text{(i)} \quad \check{Z}(0) + \vartheta &= \check{Z}_\vartheta^T(0) \\ &\geq \check{Z}_\vartheta^T(\acute{r}) \\ &= \check{Z}(\acute{r}) + \vartheta \end{aligned}$$

$$\text{And so} \quad \Rightarrow \check{Z}(0) \geq \check{Z}(\acute{r})$$

$$\begin{aligned} \text{(ii)} \quad \check{Z}(\acute{r}) + \vartheta &= \check{Z}_\vartheta^T(\acute{r}) \\ &\geq \{\check{Z}_\vartheta^T(\acute{r} * \acute{s}) \wedge \check{Z}_\vartheta^T(\acute{s})\} \\ &= \{(\check{Z}(\acute{r} * \acute{s}) + \vartheta) \wedge (\check{Z}(\acute{s}) + \vartheta)\} \\ &= \{\check{Z}(\acute{r} * \acute{s}) \wedge \check{Z}(\acute{s})\} + \vartheta \end{aligned}$$

$$\text{and so} \quad \check{Z}(\acute{r}) \geq \{\check{Z}(\acute{r} * \acute{s}) \wedge \check{Z}(\acute{s})\}$$

Hence $\check{Z}$ is a fuzzy Z -ideal of $\check{\omega}$.

4.2.4 Theorem

Let $\vartheta \in [0, T]$ and let $\check{Z}$ be a fuzzy Z -ideal of $\check{\omega}$. If $\check{\omega}$ is a Z -algebra, then the fuzzy ϑ –translation $\check{Z}_\vartheta^T$ of $\check{Z}$ is a fuzzy Z -sub-algebra of $\check{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \check{\omega}$. Now, we have

$$\begin{aligned} \check{Z}_\vartheta^T(\acute{r} * \acute{s}) &= \check{Z}(\acute{r} * \acute{s}) + \vartheta \\ &\geq \{\check{Z}((\acute{r} * \acute{s}) * \acute{s}) \wedge \check{Z}(\acute{s})\} + \vartheta \end{aligned}$$

$$\begin{aligned}
&= \{\check{\mathfrak{Z}}(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s})) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \text{ by Theorem 3.2.1} \\
&\geq \{\check{\mathfrak{Z}}(0) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\
&\geq \{\check{\mathfrak{Z}}(\mathfrak{r}) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\
&\geq \{(\check{\mathfrak{Z}}(\mathfrak{r}) + \vartheta) \wedge (\check{\mathfrak{Z}}(\mathfrak{s}) + \vartheta)\} \\
&= \{\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T$ is a fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

4.2.5 Theorem

If the fuzzy ϑ -translation $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$, $\vartheta \in [0, T]$, then $\check{\mathfrak{Z}}$ is a fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

Proof:

Let us assume that $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -ideal of $\check{\mathfrak{O}}$.

Then

$$\begin{aligned}
\check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) + \vartheta &= \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r} * \mathfrak{s}) \\
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T((\mathfrak{r} * \mathfrak{s}) * \mathfrak{s}) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})\} \\
&= \{\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s})) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})\} \text{ by Theorem 3.2.1} \\
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})\} \\
&= \{(\check{\mathfrak{Z}}(\mathfrak{r}) + \vartheta) \wedge (\check{\mathfrak{Z}}(\mathfrak{s}) + \vartheta)\} \\
&= \{\check{\mathfrak{Z}}(\mathfrak{r}) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\
&\Rightarrow \check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) \geq \{\check{\mathfrak{Z}}(\mathfrak{r}) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}$ is a fuzzy Z -sub algebra of $\check{\mathfrak{O}}$.

4.2.6 Theorem

Let $\check{Z}$ is a fuzzy subset of $\check{\omega}$ such that the fuzzy ϑ – Multiplication $\check{Z}_\vartheta^M(r)$ of $\check{Z}$ is a fuzzy Z- ideal of $\check{\omega}$, for some $\vartheta \in (0,1]$, then $\check{Z}$ is a fuzzy Z- ideal of $\check{\omega}$.

Proof:

Assume that $\check{Z}_\vartheta^M$ is a fuzzy Z- ideal of $\check{\omega}$ for some $\vartheta \in [0, T]$.

Let $r, s \in \check{\omega}$

$$\begin{aligned} \text{(i)} \quad \vartheta \check{Z}(r) &= \check{Z}_\vartheta^M(0) \\ &\geq \check{Z}_\vartheta^M(r) \\ &= \vartheta \check{Z}(r) \end{aligned}$$

And so $\Rightarrow \check{Z}(0) \geq \check{Z}(r)$

$$\begin{aligned} \text{(ii)} \quad \vartheta \check{Z}(r) &= \check{Z}_\vartheta^M(r) \\ &\geq \{ \check{Z}_\vartheta^M(r * s) \wedge \check{Z}_\vartheta^M(s) \} \\ &= \{ (\vartheta \check{Z}(r * s)) \wedge (\vartheta \check{Z}(s)) \} \\ &= \vartheta \{ \check{Z}(r * s) \wedge \check{Z}(s) \} \end{aligned}$$

and so $\check{Z}(r) \geq \{ (r * s) \wedge \check{Z}(s) \}$

Hence $\check{Z}$ is a fuzzy Z-ideal of $\check{\omega}$.

4.2.7 Theorem

If $\check{Z}$ is a fuzzy Z- ideal of $\check{\omega}$, then the fuzzy ϑ – multiplication $\check{Z}_\vartheta^M(r)$ of $\check{Z}$ is a fuzzy Z- ideal of $\check{\omega}$, for all $\vartheta \in (0,1]$.

Proof:

Let $\check{Z}$ be a fuzzy Z-ideal of $\check{\omega}$ and let $\vartheta \in (0,1]$. Then

$$\begin{aligned}
 \text{(i)} \quad \check{Z}_{\vartheta}^M(0) &= \vartheta \check{Z}(\acute{r}) \\
 &\geq \vartheta \check{Z}(\acute{r}) \\
 &= \check{Z}_{\vartheta}^M(\acute{r}) \\
 &\Rightarrow \check{Z}_{\vartheta}^M(0) \geq \check{Z}_{\vartheta}^M(\acute{r}) \\
 \text{(ii)} \quad \check{Z}_{\vartheta}^M(\acute{r}) &= \vartheta \check{Z}(\acute{r}) \\
 &\geq \vartheta \{ \check{Z}(\acute{r} * \acute{s}) \wedge \check{Z}(\acute{s}) \} \\
 &= \vartheta \{ \check{Z}(\acute{r} * \acute{s}) \wedge \check{Z}(\acute{s}) \} \\
 &= \{ (\vartheta \check{Z}(\acute{r} * \acute{s})) \wedge (\vartheta \check{Z}(\acute{s})) \} \\
 &\geq \{ \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}) \wedge \check{Z}_{\vartheta}^M(\acute{s}) \} \\
 &\Rightarrow \check{Z}_{\vartheta}^M(\acute{r}) \geq \{ \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}) \wedge \check{Z}_{\vartheta}^M(\acute{s}) \}
 \end{aligned}$$

Hence $\check{Z}_{\vartheta}^M$ of $\check{Z}$ is a fuzzy Z-ideal of $\check{\omega}$, $\forall \acute{r}, \acute{s} \in (0,1]$.

4.2.8 Theorem

Let $\vartheta \in (0,1]$ and let $\check{Z}$ be a fuzzy Z-ideal of a Z-algebra $\check{\omega}$. Then the fuzzy multiplication $\check{Z}_{\vartheta}^M(\acute{r})$ of $\check{Z}$ is a fuzzy Z- sub algebra of $\check{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \check{\omega}$.

Now, we have

$$\begin{aligned}
 \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}) &= \vartheta \check{Z}(\acute{r} * \acute{s}) \\
 &\geq \vartheta \{ \check{Z}((\acute{r} * \acute{s}) * \acute{s}) \wedge \check{Z}(\acute{s}) \} \\
 &= \{ \vartheta \check{Z}(\acute{s} * (\acute{r} * \acute{s})) \wedge \vartheta \check{Z}(\acute{s}) \}
 \end{aligned}$$

$$\begin{aligned}
&= \vartheta\{\check{\mathfrak{Z}}(0) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} \\
&\geq \vartheta\{\check{\mathfrak{Z}}(\mathfrak{r}) \wedge \check{\mathfrak{Z}}(\mathfrak{s})\} \\
&\geq \left\{ \left(\vartheta\check{\mathfrak{Z}}(\mathfrak{r}) \right) \wedge \left(\vartheta\check{\mathfrak{Z}}(\mathfrak{s}) \right) \right\} \\
&= \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}) \right\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^M$ is a fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$, $\forall \mathfrak{r}, \mathfrak{s} \in (0,1]$.

4.2.9 Theorem

If the fuzzy ϑ -translation $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -ideal of $\check{\mathfrak{O}}$, $\vartheta \in (0,1]$, then $\check{\mathfrak{Z}}$ is a fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

Proof:

Let us assume that $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a fuzzy Z -ideal of $\check{\mathfrak{O}}$.

$$\begin{aligned}
\text{Then} \quad \vartheta\check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) &= \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r} * \mathfrak{s}) \\
&\geq \left\{ \check{\mathfrak{Z}}_{\vartheta}^M((\mathfrak{r} * \mathfrak{s}) * \mathfrak{s}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}) \right\} \\
&= \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s})) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}) \right\} \\
&= \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(0) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}) \right\} \\
&\geq \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}) \right\} \\
&= \left\{ \left(\vartheta\check{\mathfrak{Z}}(\mathfrak{r}) \right) \wedge \left(\vartheta\check{\mathfrak{Z}}(\mathfrak{s}) \right) \right\} \\
&\Rightarrow \check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) \geq \left\{ \check{\mathfrak{Z}}(\mathfrak{r}) \wedge \check{\mathfrak{Z}}(\mathfrak{s}) \right\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}$ is a fuzzy Z -sub algebra of $\check{\mathfrak{O}}$.

4.2.10 Theorem

Intersection and union of any two ϑ – translation of a fuzzy Z-ideal of $\check{\mathfrak{Z}}$ of $\check{\mathfrak{O}}$ is also a fuzzy Z-ideal of $\check{\mathfrak{O}}$.

Proof:

Let $\check{\mathfrak{Z}}_{\vartheta}^T$ and $\check{\mathfrak{Z}}_{\delta}^T$ be two fuzzy translations of a fuzzy Z-ideal of $\check{\mathfrak{Z}}$ of $\check{\mathfrak{O}}$, where $\vartheta, \delta \in [0,1]$.

Assume that $\vartheta \leq \delta$.

Then by theorem 3.2.8, $\check{\mathfrak{Z}}_{\vartheta}^T$ and $\check{\mathfrak{Z}}_{\delta}^T$ are fuzzy Z-ideals of $\check{\mathfrak{O}}$.

$$\begin{aligned} \text{Now, } (\check{\mathfrak{Z}}_{\vartheta}^T \cap \check{\mathfrak{Z}}_{\delta}^T)(\check{r}) &= \{\check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \wedge \check{\mathfrak{Z}}_{\delta}^T(\check{r})\} \\ &= \{(\check{\mathfrak{Z}}(\check{r}) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{r}) + \delta)\} \\ &= \check{\mathfrak{Z}}(\check{r}) + \vartheta \\ &= \check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \end{aligned}$$

And

$$\begin{aligned} (\check{\mathfrak{Z}}_{\vartheta}^T \cup \check{\mathfrak{Z}}_{\delta}^T)(\check{r}) &= \{\check{\mathfrak{Z}}_{\vartheta}^T(\check{r}) \vee \check{\mathfrak{Z}}_{\delta}^T(\check{r})\} \\ &= \{(\check{\mathfrak{Z}}(\check{r}) + \vartheta) \vee (\check{\mathfrak{Z}}(\check{r}) + \delta)\} \\ &= \check{\mathfrak{Z}}(\check{r}) + \delta \\ &= \check{\mathfrak{Z}}_{\delta}^T(\check{r}) \end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T \cap \check{\mathfrak{Z}}_{\delta}^T$ and $\check{\mathfrak{Z}}_{\vartheta}^T \cup \check{\mathfrak{Z}}_{\delta}^T$ are fuzzy Z-ideals of $\check{\mathfrak{O}}$

CHAPTER- 5

ON FUNDAMENTAL ATTRIBUTES OF ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN Q- FUZZY Z –SUB ALGEBRA AND IDEAL ON Z –ALGEBRA

In this chapter, attempts have made to explain ϑ –Translations and ϑ –Multiplications in Q- Fuzzy Z –Sub Algebra and Q- Fuzzy Z –Ideal on Z –Algebra with adequate illustration comprising of necessary theorems and examples.

5.1 ON FUNDAMENTAL ATTRIBUTES OF ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN Q- FUZZY Z –SUB ALGEBRA IN Z –ALGEBRA

Let $\tilde{\omega}$ be a Z-algebra. For any fuzzy set $\check{\mathfrak{Z}}$ of $\tilde{\omega}$, we define $T=1-\sup\{\check{\mathfrak{Z}}(\acute{r}, q) / \acute{r} \in \tilde{\omega} \text{ and } q \in Q\}$, unless otherwise we specified.

5.1.1 Definition

Let $\check{\mathfrak{Z}}$ be a fuzzy subset of $\tilde{\omega}$ and $\vartheta \in [0, T]$. A mapping $\check{\mathfrak{Z}}_{\vartheta}^T: \tilde{\omega} \times Q \rightarrow [0,1]$ is said to be a fuzzy ϑ – translation of $\check{\mathfrak{Z}}$ if it satisfies $\check{\mathfrak{Z}}_{\vartheta}^T = \check{\mathfrak{Z}}(\acute{r}, q) + \vartheta, \forall \acute{r} \in \tilde{\omega} \text{ and } q \in Q$.

5.1.2 Definition

Let $\check{\mathfrak{Z}}$ be a fuzzy subset of $\tilde{\omega}$ and $\vartheta \in [0,1]$. A mapping $\check{\mathfrak{Z}}_{\vartheta}^M: \tilde{\omega} \times Q \rightarrow [0,1]$ is said to be a fuzzy ϑ – multiplication of $\check{\mathfrak{Z}}$ if it satisfies $\check{\mathfrak{Z}}_{\vartheta}^M = \vartheta \check{\mathfrak{Z}}(\acute{r}, q), \forall \acute{r} \in \tilde{\omega} \text{ and } q \in Q$.

5.1.2.1 Example

Let $\tilde{\omega} = \{0,1,2,3\}$ be the set with the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(\tilde{\omega}, *, 0)$ is a Z – algebra.

Define Q -Fuzzy Set $\check{z}$ is of $\tilde{\omega}$ by $\check{z}(\acute{r}) = \begin{cases} 0.4 & \text{if } \acute{r} \neq 1 \\ 0.3 & \text{if } \acute{r} = 1 \end{cases}$

Thus $\check{z}$ is a Q – Fuzzy Subalgebra of X .

Hence $T = 1 - \sup\{\check{z}(\acute{r}, q) / \acute{r} \in \tilde{\omega} \text{ and } q \in Q\} = 1 - 0.4 = 0.6$,

Choose $\vartheta = 0.2 \in [0, 1]$ and $\vartheta = 0.3 \in [0, 1]$.

Then the mapping $\check{z}_{0.2^T} : \tilde{\omega} \rightarrow [0, 1]$ is defined by

$$\check{z}_{0.2^T} = \begin{cases} 0.2 + 0.4 = 0.6 & \text{if } \acute{r} \neq 1 \\ 0.2 + 0.3 = 0.5 & \text{if } \acute{r} = 1 \end{cases}$$

Which satisfies $\check{z}_{0.2^T}(\acute{r}) = \check{z}(\acute{r}) + 0.2, \forall \acute{r} \in \tilde{\omega}$, is a Q -Fuzzy 0.2-Translation

The mapping $\gamma_{0.3^M} : \tilde{\omega} \rightarrow [0, 1]$ is defined by

$$\check{z}_{0.3^M} = \begin{cases} 0.3 * 0.4 = 0.12 & \text{if } \acute{r} \neq 1 \\ 0.3 * 0.3 = 0.09 & \text{if } \acute{r} = 1 \end{cases}$$

Which satisfies $\check{z}_{0.3^M}(\acute{r}) = \check{z}(\acute{r})(0.3), \forall \acute{r} \in \tilde{\omega}$, is a Q -Fuzzy 0.3-Multiplication

5.1.3 Theorem

If $\check{z}$ of $\tilde{\omega}$ is a Q -fuzzy Z -sub-algebra and $\vartheta \in [0, 1]$, then the fuzzy ϑ – translation.

$\check{z}_{\vartheta}^T(\acute{r}, q)$ of $\check{z}$ is also a fuzzy Z - sub algebra of $\tilde{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \tilde{\omega}$ and $\vartheta \in [0, T]$

Then, $\check{z}(\acute{r} * \acute{s}, q) \geq \check{z}(\acute{r}, q) \wedge \check{z}(\acute{s}, q)$

Now, $\check{z}_{\vartheta}^T(\acute{r} * \acute{s}, q) = \check{z}(\acute{r} * \acute{s}, q) + \vartheta$

$$\geq [\check{z}(\acute{r}, q) \wedge \check{z}(\acute{s}, q)] + \vartheta$$

$$\begin{aligned}
&= [\check{\mathfrak{Z}}(\acute{r}, q) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{s}, q) + \vartheta)] \\
&= [\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)] . \quad \forall \acute{r}, \check{s} \in \check{\mathfrak{W}}
\end{aligned}$$

5.1.4 Theorem

Let $\check{\mathfrak{Z}}$ be a fuzzy subset of $\check{\mathfrak{W}}$ such that the fuzzy ϑ -translation $\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q)$ of $\check{\mathfrak{Z}}$ is a fuzzy sub algebra of $\check{\mathfrak{W}}$, for some $\vartheta \in [0, T]$, then $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{W}}$.

Proof:

Assume that $\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q)$ is a fuzzy sub algebra of $\check{\mathfrak{W}}$ for some $\vartheta \in [0, T]$

Let $\acute{r}, \check{s} \in \check{\mathfrak{W}}$, we have

$$\begin{aligned}
\check{\mathfrak{Z}}(\acute{r} * \check{s}, q) + \vartheta &= \check{\mathfrak{Z}}_{\vartheta}^T(\acute{r} * \check{s}, q) \\
&\geq [\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)] \\
&= [(\check{\mathfrak{Z}}(\acute{r}, q) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{s}, q) + \vartheta)] \\
&= [\check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)] + \vartheta \\
&\Rightarrow \check{\mathfrak{Z}}(\acute{r} * \check{s}, q) \geq [\check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)] , \quad \forall \acute{r}, \check{s} \in \check{\mathfrak{W}}
\end{aligned}$$

Hence, $\check{\mathfrak{Z}}$ is fuzzy sub algebra of $\check{\mathfrak{W}}$.

5.1.5 Theorem

For any Q -fuzzy Z -sub algebra $\check{\mathfrak{Z}}$ of $\check{\mathfrak{W}}$ and $\vartheta \in [0, 1]$, if the fuzzy ϑ -multiplication $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}, q)$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{W}}$.

Proof:

Let $\acute{r}, \check{s} \in \check{\mathfrak{W}}$ and $\vartheta \in [0, T]$

Then $\check{\mathfrak{Z}}(\acute{r} * \check{s}, q) \geq \check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)$

$$\begin{aligned}
\text{Now, } \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}, q) &= \vartheta \check{\mathfrak{Z}}(\acute{r} * \check{s}, q) \\
&\geq \vartheta [\check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)]
\end{aligned}$$

$$\begin{aligned}
&\geq [\vartheta \check{\mathfrak{Z}}(\mathfrak{r}, q) \wedge \vartheta \check{\mathfrak{Z}}(\mathfrak{s}, q)] \\
&= [\check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}, q)] \\
\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r} * \mathfrak{s}, q) &\geq [\check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}, q)]
\end{aligned}$$

Therefore, $\check{\mathfrak{Z}}_{\vartheta}^M$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{O}}$.

5.1.6 Theorem

For any fuzzy subset $\check{\mathfrak{Z}}$ of $\check{\mathfrak{O}}$ and $\vartheta \in [0, 1]$, if the fuzzy ϑ –multiplication $\check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}, q)$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{O}}$, then so is $\check{\mathfrak{Z}}$.

Proof:

Assume that $\check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}, q)$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{O}}$ for some $\vartheta \in [0, T]$

Let $\mathfrak{r}, \mathfrak{s} \in \check{\mathfrak{O}}$, we have

$$\begin{aligned}
\vartheta \check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}, q) &= \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r} * \mathfrak{s}, q) \\
&\geq [\check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\mathfrak{s}, q)] \\
&= [\vartheta \check{\mathfrak{Z}}(\mathfrak{r}, q) \wedge \vartheta \check{\mathfrak{Z}}(\mathfrak{s}, q)] \\
&= \vartheta [\check{\mathfrak{Z}}(\mathfrak{r}, q) \wedge \check{\mathfrak{Z}}(\mathfrak{s}, q)] \\
\Rightarrow \check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}, q) &\geq \vartheta [\check{\mathfrak{Z}}(\mathfrak{r}, q) \wedge \check{\mathfrak{Z}}(\mathfrak{s}, q)]
\end{aligned}$$

Hence, $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{O}}$.

5.2 ON FUNDAMENTAL ATTRIBUTES OF ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN Q -FUZZY Z –IDEAL IN Z –ALGEBRA

5.2.1 Theorem

If the fuzzy ϑ – translation $\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r})$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -Ideal, then it satisfies the condition $\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s}), q) \geq \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r}, q)$.

Proof:

$$\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s}), q) = \check{\mathfrak{Z}}(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s}), q) + \vartheta$$

$$\begin{aligned}
&\geq \{\check{Z}(0 * (\check{s} * (\acute{r} * \check{s})), q) + \vartheta \wedge \check{Z}(0, q) + \vartheta\} \\
&\geq \{\check{Z}(0 * (\check{s} * (\check{s} * \acute{r})), q) + \vartheta \wedge \check{Z}(0, q) + \vartheta\} \\
&= \{\check{Z}(0 * ((\check{s} * \check{s}) * \acute{r}), q) + \vartheta \wedge \check{Z}(0, q) + \vartheta\} \\
&= \{\check{Z}(0 * (\check{s} * \acute{r}), q) + \vartheta \wedge \check{Z}(0, q) + \vartheta\} \\
&= \{\check{Z}((\check{s} * \acute{r}) * 0, q) + \vartheta \wedge \check{Z}(0, q) + \vartheta\} \\
&\geq \{\check{Z}((\check{s} * \acute{r}) * 0, q) + \vartheta \wedge \check{Z}(\acute{r}, q) + \vartheta\} \\
&\geq \{\check{Z}_\vartheta^T(0, q) \wedge \check{Z}_\vartheta^T(\acute{r}, q)\} \\
&= \check{Z}_\vartheta^T(\acute{r}, q). \\
&\Rightarrow \check{Z}_\vartheta^T(\check{s} * (\acute{r} * \check{s}), q) \geq \check{Z}_\vartheta^T(\acute{r}, q) \quad \forall \acute{r}, \check{s} \in \check{\omega}
\end{aligned}$$

5.2.2 Theorem

If $\check{Z}$ is a fuzzy Z -ideal of $\check{\omega}$, then the fuzzy ϑ -translation $\check{Z}_\vartheta^T(\acute{r}, q)$ of $\check{Z}$ is a Q -fuzzy Z -ideal of $\check{\omega}$, for all $\vartheta \in [0, T]$.

Proof:

Let $\check{Z}$ be a Q -fuzzy Z -ideal of $\check{\omega}$ and let $\vartheta \in [0, T]$

Then,

$$\begin{aligned}
\text{(i)} \quad \check{Z}_\vartheta^T(0, q) &= \check{Z}(0, q) + \vartheta \\
&\geq \check{Z}(\acute{r}, q) + \vartheta \\
&= \check{Z}_\vartheta^T(\acute{r}, q) \\
\text{(ii)} \quad \check{Z}_\vartheta^T(\acute{r}, q) &= \check{Z}(\acute{r}, q) + \vartheta \\
&\geq \{\check{Z}(\acute{r} * \check{s}, q) \wedge \check{Z}(\check{s}, q)\} + \vartheta \\
&= \{(\check{Z}(\acute{r} * \check{s}, q) + \vartheta) \wedge (\check{Z}(\check{s}, q) + \vartheta)\} \\
&= \{\check{Z}_\vartheta^T(\acute{r} * \check{s}, q) \wedge \check{Z}_\vartheta^T(\check{s}, q)\}
\end{aligned}$$

$$\Rightarrow \check{Z}_\vartheta^T(\acute{r}, q) \geq \{\check{Z}_\vartheta^T(\acute{r} * \check{s}, q) \wedge \check{Z}_\vartheta^T(\check{s}, q)\}$$

Hence $\check{Z}_\vartheta^T(\acute{r}, q)$ of $\check{Z}$ is a Q -fuzzy Z -ideal of $\check{\omega}$, $\forall \vartheta \in [0, T]$

5.2.3 Theorem

Let $\check{Z}$ is a fuzzy subset of $\check{\omega}$ such that the fuzzy ϑ –translation $\check{Z}_{\vartheta}^T(\acute{r}, q)$ of $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$, for some $\vartheta \in [0, T]$, then $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$.

Proof:

Assume that $\check{Z}_{\vartheta}^T$ is a Q -fuzzy Z - ideal of $\check{\omega}$ for some $\vartheta \in [0, T]$. Let $\acute{r}, \acute{s} \in \check{\omega}$. Then,

$$\begin{aligned} \text{(i)} \quad \check{Z}(0, q) + \vartheta &= \check{Z}_{\vartheta}^T(0, q) \\ &\geq \check{Z}_{\vartheta}^T(\acute{r}, q) \\ &= \check{Z}(\acute{r}, q) + \vartheta \end{aligned}$$

$$\text{And so } \Rightarrow \check{Z}(0, q) \geq \check{Z}(\acute{r}, q)$$

$$\begin{aligned} \text{(ii)} \quad \check{Z}(\acute{r}, q) + \vartheta &= \check{Z}_{\vartheta}^T(\acute{r}, q) \\ &\geq \{ \check{Z}_{\vartheta}^T(\acute{r} * \acute{s}, q) \wedge \check{Z}_{\vartheta}^T(\acute{s}, q) \} \\ &= \{ (\check{Z}(\acute{r} * \acute{s}, q) + \vartheta) \wedge (\check{Z}(\acute{s}, q) + \vartheta) \} \\ &= \{ \check{Z}(\acute{r} * \acute{s}, q) \wedge \check{Z}(\acute{s}, q) \} + \vartheta \end{aligned}$$

$$\text{and so } \check{Z}(\acute{r}, q) \geq \{ (\acute{r} * \acute{s}, q) \wedge \check{Z}(\acute{s}, q) \}$$

Hence $\check{Z}$ is a Q -fuzzy Z -ideal of $\check{\omega}$.

5.2.4 Theorem

Let $\vartheta \in [0, T]$ and let $\check{Z}$ be a Q -fuzzy Z -ideal of $\check{\omega}$. If $\check{\omega}$ is a Q -fuzzy Z -algebra, then the fuzzy ϑ –translation $\check{Z}_{\vartheta}^T$ of $\check{Z}$ is a Q -fuzzy Z -sub-algebra of $\check{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \check{\omega}$. Now, we have

$$\check{Z}_{\vartheta}^T(\acute{r} * \acute{s}, q) = \check{Z}(\acute{r} * \acute{s}, q) + \vartheta$$

$$\begin{aligned}
&\geq \{\check{\mathfrak{Z}}((\acute{r} * \check{s},) * \check{s}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)\} + \vartheta \\
&= \{\check{\mathfrak{Z}}(\check{s} * (\acute{r} * \check{s}), q) \wedge \check{\mathfrak{Z}}(\check{s}, q)\} + \vartheta \text{ by Theorem 4.2.1} \\
&\geq \{\check{\mathfrak{Z}}(0, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)\} + \vartheta \\
&\geq \{\check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)\} + \vartheta \\
&\geq \{(\check{\mathfrak{Z}}(\acute{r}, q) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{s}, q) + \vartheta)\} \\
&= \{\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T$ is a Q -fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

5.2.5 Theorem

If the fuzzy ϑ -translation $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$, $\vartheta \in [0, T]$, then $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

Proof:

Let us assume that $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -ideal of $\check{\mathfrak{O}}$.

Then $\check{\mathfrak{Z}}(\acute{r} * \check{s}, q) + \vartheta = \check{\mathfrak{Z}}_{\vartheta}^T(\acute{r} * \check{s}, q)$

$$\begin{aligned}
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T((\acute{r} * \check{s}) * \check{s}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)\} \\
&= \{\check{\mathfrak{Z}}_{\vartheta}^T(\check{s} * (\acute{r} * \check{s}), q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)\} \\
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T(0, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)\} \\
&\geq \{\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^T(\check{s}, q)\} \\
&= \{(\check{\mathfrak{Z}}(\acute{r}, q) + \vartheta) \wedge (\check{\mathfrak{Z}}(\check{s}, q) + \vartheta)\} \\
&= \{\check{\mathfrak{Z}}(\acute{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q)\} + \vartheta
\end{aligned}$$

$$\Rightarrow \check{Z}(\acute{r} * \check{s}, q) \geq \{\check{Z}(\acute{r}, q) \wedge \check{Z}(\check{s}, q)\}$$

Hence $\check{Z}$ is a Q -fuzzy Z -sub algebra of $\check{\omega}$.

5.2.6 Theorem

Let $\check{Z}$ is a fuzzy subset of $\check{\omega}$ such that the fuzzy ϑ – Multiplication $\check{Z}_\vartheta^M(\acute{r}, q)$ of $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$, for some $\vartheta \in (0, 1]$, then $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$.

Proof:

Assume that $\check{Z}_\vartheta^M$ is a Q -fuzzy Z - ideal of $\check{\omega}$ for some $\vartheta \in [0, T]$.

Let $\acute{r}, \check{s} \in \check{\omega}$

$$\begin{aligned} \text{(i)} \quad \vartheta \check{Z}(\acute{r}, q) &= \check{Z}_\vartheta^M(0, q) \\ &\geq \check{Z}_\vartheta^M(\acute{r}, q) \\ &= \vartheta \check{Z}(\acute{r}, q) \end{aligned}$$

$$\text{And so} \quad \Rightarrow \check{Z}(0, q) \geq \check{Z}(\acute{r}, q)$$

$$\begin{aligned} \text{(ii)} \quad \vartheta \check{Z}(\acute{r}, q) &= \check{Z}_\vartheta^M(\acute{r}, q) \\ &\geq \{\check{Z}_\vartheta^M(\acute{r} * \check{s}, q) \wedge \check{Z}_\vartheta^M(\check{s}, q)\} \\ &= \{\left(\vartheta \check{Z}(\acute{r} * \check{s}, q)\right) \wedge (\vartheta \check{Z}(\check{s}, q))\} \\ &= \vartheta \{\check{Z}(\acute{r} * \check{s}, q) \wedge \check{Z}(\check{s}, q)\} \end{aligned}$$

$$\text{And so} \quad \Rightarrow \check{Z}(\acute{r}, q) \geq \{\check{Z}(\acute{r} * \check{s}, q) \wedge \check{Z}(\check{s}, q)\}$$

Hence $\check{Z}$ is a Q -fuzzy Z -ideal of $\check{\omega}$.

5.2.7 Theorem

If $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$, then the fuzzy ϑ –multiplication $\check{Z}_\vartheta^M(\acute{r}, q)$ of $\check{Z}$ is a Q -fuzzy Z - ideal of $\check{\omega}$, for all $\vartheta \in (0, 1]$.

Proof:

Let $\check{Z}$ be a Q -fuzzy Z -ideal of $\check{\omega}$ and let $\vartheta \in (0,1]$. Then

$$\begin{aligned}
 \text{(i)} \quad \check{Z}_{\vartheta}^M(0, q) &= \vartheta \check{Z}(\acute{r}, q) \\
 &\geq \vartheta \check{Z}(\acute{r}, q) \\
 &= \check{Z}_{\vartheta}^M(\acute{r}, q) \\
 &\Rightarrow \check{Z}_{\vartheta}^M(0, q) \geq \check{Z}_{\vartheta}^M(\acute{r}, q) \\
 \text{(ii)} \quad \check{Z}_{\vartheta}^M(\acute{r}, q) &= \vartheta \check{Z}(\acute{r}, q) \\
 &\geq \vartheta \{ \check{Z}(\acute{r} * \acute{s}, q) \wedge \check{Z}(\acute{s}, q) \} \\
 &= \vartheta \{ \check{Z}(\acute{r} * \acute{s}, q) \wedge \check{Z}(\acute{s}, q) \} \\
 &= \{ (\vartheta \check{Z}(\acute{r} * \acute{s}, q)) \wedge (\vartheta \check{Z}(\acute{s}, q)) \} \\
 &\geq \{ \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}, q) \wedge \check{Z}_{\vartheta}^M(\acute{s}, q) \} \\
 &\Rightarrow \check{Z}_{\vartheta}^M(\acute{r}, q) \geq \{ \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}, q) \wedge \check{Z}_{\vartheta}^M(\acute{s}, q) \}
 \end{aligned}$$

Hence $\check{Z}_{\vartheta}^M$ of $\check{Z}$ is a Q -fuzzy Z -ideal of $\check{\omega}$, $\forall \acute{r}, \acute{s} \in (0,1]$.

5.2.8 Theorem

Let $\vartheta \in (0,1]$ and let $\check{Z}$ be a Q -fuzzy Z -ideal of a Z -algebra $\check{\omega}$. Then the fuzzy ϑ -multiplication $\check{Z}_{\vartheta}^M(\acute{r})$ of $\check{Z}$ is a Q -fuzzy Z -sub algebra of $\check{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \check{\omega}$. Now, we have

$$\begin{aligned}
 \check{Z}_{\vartheta}^M(\acute{r} * \acute{s}, q) &= \vartheta \check{Z}(\acute{r} * \acute{s}, q) \\
 &\geq \vartheta \{ \check{Z}((\acute{r} * \acute{s}) * \acute{s}), q \} \wedge \check{Z}(\acute{s}, q)
 \end{aligned}$$

$$\begin{aligned}
&= \{ \vartheta \check{\mathfrak{Z}}((\check{s} * (\check{r} * \check{s})), q) \wedge \vartheta \check{\mathfrak{Z}}(\check{s}, q) \} \\
&= \vartheta \{ \check{\mathfrak{Z}}(0, q) \wedge \check{\mathfrak{Z}}(\check{s}, q) \} \\
&\geq \vartheta \{ \check{\mathfrak{Z}}(\check{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q) \} \\
&\geq \{ (\vartheta \check{\mathfrak{Z}}(\check{r}, q)) \wedge (\vartheta \check{\mathfrak{Z}}(\check{s}, q)) \} \\
&= \{ \check{\mathfrak{Z}}_{\vartheta}^M(\check{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}, q) \}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^M$ is a Q -fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$, $\forall \check{r}, \check{s} \in (0, 1]$.

5.2.9 Theorem

If the fuzzy ϑ –translation $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -ideal of $\check{\mathfrak{O}}$, $\vartheta \in (0, 1]$, then $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub-algebra of $\check{\mathfrak{O}}$.

Proof:

Let us assume that $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -ideal of $\check{\mathfrak{O}}$. Then,

$$\begin{aligned}
\vartheta \check{\mathfrak{Z}}(\check{r} * \check{s}, q) &= \check{\mathfrak{Z}}_{\vartheta}^M(\check{r} * \check{s}, q) \\
&\geq \{ \check{\mathfrak{Z}}_{\vartheta}^M((\check{r} * \check{s}) * \check{s}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}, q) \} \\
&= \{ \check{\mathfrak{Z}}_{\vartheta}^M(\check{s} * (\check{r} * \check{s}), q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}, q) \} \\
&= \{ \check{\mathfrak{Z}}_{\vartheta}^M(0, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}, q) \} \\
&\geq \{ \check{\mathfrak{Z}}_{\vartheta}^M(\check{r}, q) \wedge \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}, q) \} \\
&= \{ (\vartheta \check{\mathfrak{Z}}(\check{r}, q)) \wedge (\vartheta \check{\mathfrak{Z}}(\check{s}, q)) \}
\end{aligned}$$

$$\Rightarrow \check{\mathfrak{Z}}(\check{r} * \check{s}, q) \geq \{ \check{\mathfrak{Z}}(\check{r}, q) \wedge \check{\mathfrak{Z}}(\check{s}, q) \}$$

Hence $\check{\mathfrak{Z}}$ is a Q -fuzzy Z -sub algebra of $\check{\mathfrak{O}}$.

5.2.10 Theorem

Intersection and union of any two ϑ –translation of a Q -fuzzy Z -ideal of $\check{Z}$ of $\check{\omega}$ is also a Q -fuzzy Z -ideal of $\check{\omega}$.

Proof:

Let $\check{Z}_\vartheta^T$ and $\check{Z}_\delta^T$ be two fuzzy translations of a Q -fuzzy Z -ideal of $\check{Z}$ of $\check{\omega}$, where $\vartheta, \delta \in [0,1]$.

Assume that $\vartheta \leq \delta$.

Then by theorem 4.2.8, $\check{Z}_\vartheta^T$ and $\check{Z}_\delta^T$ are Q -fuzzy Z -ideals of $\check{\omega}$.

$$\begin{aligned} \text{Now, } (\check{Z}_\vartheta^T \cap \check{Z}_\delta^T)(r, q) &= \{\check{Z}_\vartheta^T(r, q) \wedge \check{Z}_\delta^T(r, q)\} \\ &= \{(\check{Z}(r, q) + \vartheta) \wedge (\check{Z}(r, q) + \delta)\} \\ &= \check{Z}(r, q) + \vartheta \\ &= \check{Z}_\vartheta^T(r, q) \end{aligned}$$

$$\begin{aligned} \text{And } (\check{Z}_\vartheta^T \cup \check{Z}_\delta^T)(r, q) &= \{\check{Z}_\vartheta^T(r, q) \vee \check{Z}_\delta^T(r, q)\} \\ &= \{(\check{Z}(r, q) + \vartheta) \vee (\check{Z}(r, q) + \delta)\} \\ &= \check{Z}(r, q) + \delta \\ &= \check{Z}_\delta^T(r, q) \end{aligned}$$

Hence $\check{Z}_\vartheta^T \cap \check{Z}_\delta^T$ and $\check{Z}_\vartheta^T \cup \check{Z}_\delta^T$ are Q -fuzzy Z -ideals of $\check{\omega}$.

CHAPTER- 6

ALGEBRAIC STRUCTURES OF DOUBT FUZZY Z –SUB ALGEBRA AND IDEAL UNDER THE ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN Z –ALGEBRA

In this chapter, attempts have made to explain ϑ –Translations and ϑ –Multiplications in Doubt Fuzzy Z –Sub Algebra and Ideal on Z –Algebra with adequate illustration comprising of necessary theorems and examples.

6.1 ALGEBRAIC STRUCTURES OF DOUBT FUZZY Z –SUB ALGEBRA UNDER ϑ –TRANSLATIONS AND ϑ –MULTIPLICATIONS IN Z –ALGEBRA

Let, $\tilde{\omega}$ be a Z -algebra. For any *Doubt Fuzzy Set* $\check{\xi}$ of $\tilde{\omega}$, we define $T=1-\sup\{\check{\xi}(r)/r \in \tilde{\omega}\}$, unless otherwise we specified.

6.1.1 Definition

Let, $\check{\xi}$ be a Doubt Fuzzy Subset of $\tilde{\omega}$ and $\vartheta \in [0, T]$. A mapping $\check{\xi}_{\vartheta}^T: \tilde{\omega} \rightarrow [0, 1]$ is said to be *Doubt Fuzzy ϑ – Translation* of $\check{\xi}$ if it satisfies $\check{\xi}_{\vartheta}^T = \check{\xi}(r) + \vartheta, \forall r \in \tilde{\omega}$.

6.1.2 Definition

Let, $\check{\xi}$ be a Doubt Fuzzy Set of $\tilde{\omega}$ and $\vartheta \in [0, 1]$. A mapping $\check{\xi}_{\vartheta}^M: \tilde{\omega} \rightarrow [0, 1]$ is said to be a *Doubt Fuzzy ϑ – Multiplication* of $\check{\xi}$ if it satisfies $\check{\xi}_{\vartheta}^M = \vartheta \check{\xi}(r), \forall r \in \tilde{\omega}$.

6.1.2.1 Example

Let $\tilde{\omega} = \{0, 1, 2, 3\}$ be the set with the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(\tilde{\omega}, *, 0)$ is a Z – algebra.

Define Doubt Fuzzy Set $\check{z}$ is of $\tilde{\omega}$ by $\check{z}(\acute{r}) = \begin{cases} 0.4 & \text{if } \acute{r} \neq 1 \\ 0.3 & \text{if } \acute{r} = 1 \end{cases}$

Thus $\check{z}$ is a *Doubt Fuzzy – Z – Sub Algebra* of X .

Hence $T = 1 - \sup\{\check{z}(\acute{r})/\acute{r} \in \tilde{\omega}\} = 1 - 0.4 = 0.6$,

Choose $\vartheta = 0.2 \in [0,1]$ and $\vartheta = 0.3 \in [0,1]$.

Then the mapping $\check{z}_{0.2^T} : \tilde{\omega} \rightarrow [0,1]$ is defined by

$$\check{z}_{0.2^T} = \begin{cases} 0.2 + 0.4 = 0.6 & \text{if } \acute{r} \neq 1 \\ 0.2 + 0.3 = 0.5 & \text{if } \acute{r} = 1 \end{cases}$$

satisfies $\check{z}_{0.2^T}(\acute{r}) = \check{z}(\acute{r}) + 0.2, \forall \acute{r} \in \tilde{\omega}$, is a Doubt Fuzzy 0.2-Translation.

The mapping $\gamma_{0.3^M} : \tilde{\omega} \rightarrow [0,1]$ is defined by

$$\check{z}_{0.3^M} = \begin{cases} 0.3 * 0.4 = 0.12 & \text{if } \acute{r} \neq 1 \\ 0.3 * 0.3 = 0.09 & \text{if } \acute{r} = 1 \end{cases}$$

satisfies $\check{z}_{0.3^M}(\acute{r}) = \check{z}(\acute{r})(0.3), \forall \acute{r} \in \tilde{\omega}$, is a Doubt Fuzzy 0.3-Multiplication.

6.1.3 Theorem

If $\check{z}$ of $\tilde{\omega}$ is a *Doubt Fuzzy – Z – Sub Algebra* and $\vartheta \in [0,1]$, then the *Doubt Fuzzy ϑ – Translational* $\check{z}_{\vartheta}^T(\acute{r})$ of $\check{z}$ is also a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$.

Proof:

Let, $\acute{r}, \acute{s} \in \tilde{\omega}$ and $\vartheta \in [0, T]$

Then, $\check{z}(\acute{r} * \acute{s}) \leq \check{z}(\acute{r}) \vee \check{z}(\acute{s})$

Now,

$$\check{z}_{\vartheta}^T(\acute{r} * \acute{s}) = \check{z}(\acute{r} * \acute{s}) + \vartheta$$

$$\begin{aligned}
&\leq [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\mathfrak{s})] + \vartheta \\
&= [(\check{\mathfrak{Z}}(\acute{r}) + \vartheta) \vee (\check{\mathfrak{Z}}(\mathfrak{s}) + \vartheta)] \\
&= [\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})]. \forall \acute{r}, \mathfrak{s} \in \tilde{\omega}
\end{aligned}$$

6.1.4 Theorem

Let, $\check{\mathfrak{Z}}$ be a *Doubt Fuzzy Subset* of $\tilde{\omega}$ such that the *Doubt Fuzzy ϑ – Translation* $\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r})$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$, for some $\vartheta \in [0, T]$, then $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$.

Proof:

Assume that $\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r})$ is a *Doubt Fuzzy Z – Sub Algebra* of $\tilde{\omega}$ for some $\vartheta \in [0, T]$

Let $\acute{r}, \mathfrak{s} \in \tilde{\omega}$, we have

$$\begin{aligned}
\check{\mathfrak{Z}}(\acute{r} * \mathfrak{s}) + \vartheta &= \check{\mathfrak{Z}}_{\vartheta}^T(\acute{r} * \mathfrak{s}) \\
&\leq [\check{\mathfrak{Z}}_{\vartheta}^T(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})] \\
&= [(\check{\mathfrak{Z}}(\acute{r}) + \vartheta) \vee (\check{\mathfrak{Z}}(\mathfrak{s}) + \vartheta)] \\
&= [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\mathfrak{s})] + \vartheta \\
\Rightarrow \check{\mathfrak{Z}}(\acute{r} * \mathfrak{s}) &\geq [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\mathfrak{s})], \forall \acute{r}, \mathfrak{s} \in \tilde{\omega}
\end{aligned}$$

Hence, $\check{\mathfrak{Z}}$ is *Doubt Fuzzy Z – Sub Algebra* of $\tilde{\omega}$.

6.1.5 Theorem

For any *Doubt Fuzzy – Z – Sub Algebra* $\check{\mathfrak{Z}}$ of $\tilde{\omega}$ and $\vartheta \in [0, 1]$, if the *Doubt Fuzzy ϑ – Multiplication* $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$.

Proof:

Let $\acute{r}, \acute{s} \in \tilde{\omega}$ and $\vartheta \in [0, T]$

Then $\check{\mathfrak{Z}}(\acute{r} * \acute{s}) \geq \check{\mathfrak{Z}}(\acute{r}) \wedge \check{\mathfrak{Z}}(\acute{s})$

Now,

$$\begin{aligned} \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \acute{s}) &= \vartheta \check{\mathfrak{Z}}(\acute{r} * \acute{s}) \\ &\leq \vartheta [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\acute{s})] \\ &\leq [\vartheta \check{\mathfrak{Z}}(\acute{r}) \vee \vartheta \check{\mathfrak{Z}}(\acute{s})] \\ &= [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\acute{s})] \\ \Rightarrow \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \acute{s}) &\leq [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\acute{s})] \end{aligned}$$

Therefore, $\check{\mathfrak{Z}}_{\vartheta}^M$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$.

6.1.6 Theorem

For any *Doubt Fuzzy* Subset, $\check{\mathfrak{Z}}$ of $\tilde{\omega}$ and $\vartheta \in [0, 1]$, if the *Doubt Fuzzy* ϑ – *Multiplication* $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$, then so in $\check{\mathfrak{Z}}$.

Proof:

Assume that, $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$ for some $\vartheta \in [0, T]$

Let $\acute{r}, \acute{s} \in \tilde{\omega}$, we have

$$\begin{aligned} \vartheta \check{\mathfrak{Z}}(\acute{r} * \acute{s}) &= \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \acute{s}) \\ &\leq [\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\acute{s})] \\ &= [\vartheta \check{\mathfrak{Z}}(\acute{r}) \vee \vartheta \check{\mathfrak{Z}}(\acute{s})] \\ &= \vartheta [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\acute{s})] \\ \Rightarrow \check{\mathfrak{Z}}(\acute{r} * \acute{s}) &\leq \vartheta [\check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\acute{s})] \end{aligned}$$

Hence, $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\tilde{\omega}$.

6.2 ALGEBRAIC STRUCTURES OF DOUBT FUZZY Z – IDEAL UNDER ϑ – TRANSLATIONS AND ϑ – MULTIPLICATIONS IN Z – ALGEBRA

6.2.1 Theorem

If the *Doubt Fuzzy ϑ – Translation* $\check{Z}_{\vartheta}^T(\check{r})$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal*, then it satisfies the condition $\check{Z}_{\vartheta}^T(\check{s} * (\check{r} * \check{s})) \leq \check{Z}_{\vartheta}^T(\check{r})$.

Proof:

$$\begin{aligned}
 \check{Z}_{\vartheta}^T(\check{s} * (\check{r} * \check{s})) &= \check{Z}(\check{s} * (\check{r} * \check{s})) + \vartheta \\
 &\leq \{\check{Z}(0 * (\check{s} * (\check{r} * \check{s}))) + \vartheta \vee \check{Z}(0) + \vartheta\} \\
 &\leq \{\check{Z}(0 * (\check{s} * (\check{s} * \check{r}))) + \vartheta \vee \check{Z}(0) + \vartheta\} \\
 &= \{\check{Z}(0 * ((\check{s} * \check{s}) * \check{r})) + \vartheta \vee \check{Z}(0) + \vartheta\} \\
 &= \{\check{Z}(0 * (\check{s} * \check{r})) + \vartheta \vee \check{Z}(0) + \vartheta\} \\
 &= \{\check{Z}((\check{s} * \check{r}) * 0) + \vartheta \vee \check{Z}(0) + \vartheta\} \\
 &\leq \{\check{Z}((\check{s} * \check{r}) * 0) + \vartheta \vee \check{Z}(\check{r}) + \vartheta\} \\
 &\leq \{\check{Z}_{\vartheta}^T(0) \vee \check{Z}_{\vartheta}^T(\check{r})\} \\
 &= \check{Z}_{\vartheta}^T(\check{r}). \\
 \Rightarrow \check{Z}_{\vartheta}^T(\check{s} * (\check{r} * \check{s})) &\leq \check{Z}_{\vartheta}^T(\check{r}) \quad \forall \check{r}, \check{s} \in \check{\omega}
 \end{aligned}$$

6.2.2 Theorem

If, $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$, then the *Doubt Fuzzy ϑ – Translation* $\check{Z}_{\vartheta}^T(\check{r})$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$, for all $\vartheta \in [0, T]$.

Proof:

Let, $\check{Z}$ be a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$ and let $\vartheta \in [0, T]$

Then,

$$\begin{aligned}
 \text{(i)} \quad \check{Z}_{\vartheta}^T(0) &= \check{Z}(0) + \vartheta \\
 &\leq \check{Z}(\check{r}) + \vartheta \\
 &= \check{Z}_{\vartheta}^T(\check{r})
 \end{aligned}$$

$$\text{(ii)} \quad \check{Z}_{\vartheta}^T(\check{r}) = \check{Z}(\check{r}) + \vartheta$$

$$\begin{aligned}
&\leq \{\check{Z}(\acute{r} * \check{s}) \vee \check{Z}(\check{s})\} + \vartheta \\
&= \{(\check{Z}(\acute{r} * \check{s}) + \vartheta) \vee (\check{Z}(\check{s}) + \vartheta)\} \\
&= \{\check{Z}_{\vartheta}^T(\acute{r} * \check{s}) \vee \check{Z}_{\vartheta}^T(\check{s})\} \\
&\Rightarrow \check{Z}_{\vartheta}^T(\acute{r}) \leq \{\check{Z}_{\vartheta}^T(\acute{r} * \check{s}) \vee \check{Z}_{\vartheta}^T(\check{s})\}
\end{aligned}$$

Hence $\check{Z}_{\vartheta}^T(\acute{r})$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\tilde{\omega}$, $\forall \vartheta \in [0, T]$

6.2.3 Theorem

Let, $\check{Z}$ is a *Doubt Fuzzy Subset* of $\tilde{\omega}$ such that the *F ϑ – Translation* $\check{Z}_{\vartheta}^T(\acute{r})$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\tilde{\omega}$, for some $\vartheta \in [0, T]$, then $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\tilde{\omega}$.

Proof:

Assume that, $\check{Z}_{\vartheta}^T$ is a *Doubt Fuzzy – Z – Ideal* of $\tilde{\omega}$ for some $\vartheta \in [0, T]$.

Let $\acute{r}, \check{s} \in \tilde{\omega}$

Then,

$$\begin{aligned}
\text{(i)} \quad \check{Z}(0) + \vartheta &= \check{Z}_{\vartheta}^T(0) \\
&\leq \check{Z}_{\vartheta}^T(\acute{r}) \\
&= \check{Z}(\acute{r}) + \vartheta
\end{aligned}$$

$$\text{And so} \quad \Rightarrow \check{Z}(0) \leq \check{Z}(\acute{r})$$

$$\begin{aligned}
\text{(ii)} \quad \check{Z}(\acute{r}) + \vartheta &= \check{Z}_{\vartheta}^T(\acute{r}) \\
&\leq \{\check{Z}_{\vartheta}^T(\acute{r} * \check{s}) \vee \check{Z}_{\vartheta}^T(\check{s})\} \\
&= \{(\check{Z}(\acute{r} * \check{s}) + \vartheta) \vee (\check{Z}(\check{s}) + \vartheta)\}
\end{aligned}$$

$$= \{\check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) \vee \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta$$

$$\text{and so} \quad \Rightarrow \check{\mathfrak{Z}}(\mathfrak{r}) \leq \{(\mathfrak{r} * \mathfrak{s}) \vee \check{\mathfrak{Z}}(\mathfrak{s})\}$$

Hence, $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$.

6.2.4 Theorem

Let, $\vartheta \in [0, T]$ and let $\check{\mathfrak{Z}}$ be a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$. If $\check{\mathfrak{O}}$ is a Z-algebra, then the fuzzy *Doubt Fuzzy ϑ – Translation* $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$.

Proof:

Let, $\mathfrak{r}, \mathfrak{s} \in \check{\mathfrak{O}}$.

Now, we have

$$\begin{aligned} \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r} * \mathfrak{s}) &= \check{\mathfrak{Z}}(\mathfrak{r} * \mathfrak{s}) + \vartheta \\ &\leq \{\check{\mathfrak{Z}}((\mathfrak{r} * \mathfrak{s}) * \mathfrak{s}) \vee \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\ &= \{\check{\mathfrak{Z}}(\mathfrak{s} * (\mathfrak{r} * \mathfrak{s})) \vee \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \text{ by Theorem 5.2.1} \\ &\geq \{\check{\mathfrak{Z}}(0) \vee \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\ &\leq \{\check{\mathfrak{Z}}(\mathfrak{r}) \vee \check{\mathfrak{Z}}(\mathfrak{s})\} + \vartheta \\ &\leq \{(\check{\mathfrak{Z}}(\mathfrak{r}) + \vartheta) \vee (\check{\mathfrak{Z}}(\mathfrak{s}) + \vartheta)\} \\ &= \{\check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^T(\mathfrak{s})\} \end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$.

6.2.5 Theorem

If the *Doubt Fuzzy ϑ – Translation* $\check{\mathfrak{Z}}_{\vartheta}^T$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$, $\vartheta \in [0, T]$, then $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$.

Proof:

Let us assume that, $\check{Z}_\vartheta^T$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$.

Then

$$\begin{aligned}
\check{Z}(\acute{r} * \check{s}) + \vartheta &= \check{Z}_\vartheta^T(\acute{r} * \check{s}) \\
&\leq \left\{ \check{Z}_\vartheta^T((\acute{r} * \check{s}) * \check{s}) \vee \check{Z}_\vartheta^T(\check{s}) \right\} \\
&= \left\{ \check{Z}_\vartheta^T(\check{s} * (\acute{r} * \check{s})) \vee \check{Z}_\vartheta^T(\check{s}) \right\} \\
&\leq \left\{ \check{Z}_\vartheta^T(0) \vee \check{Z}_\vartheta^T(\check{s}) \right\} \\
&\leq \left\{ \check{Z}_\vartheta^T(\acute{r}) \vee \check{Z}_\vartheta^T(\check{s}) \right\} \\
&= \{(\check{Z}(\acute{r}) + \vartheta) \vee (\check{Z}(\check{s}) + \vartheta)\} \\
&= \{\check{Z}(\acute{r}) \vee \check{Z}(\check{s})\} + \vartheta \\
&\Rightarrow \check{Z}(\acute{r} * \check{s}) \leq \{\check{Z}(\acute{r}) \vee \check{Z}(\check{s})\}
\end{aligned}$$

Hence $\check{Z}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\omega}$.

6.2.6 Theorem

Let, $\check{Z}$ is a *Doubt Fuzzy Subset* of $\check{\omega}$ such that the *Doubt Fuzzy ϑ – Multiplication* $\check{Z}_\vartheta^M(\acute{r})$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$, for some $\vartheta \in (0,1]$, then $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$.

Proof:

Assume that, $\check{Z}_\vartheta^M$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$ for some $\vartheta \in [0, T]$.

Let $\acute{r}, \check{s} \in \check{\omega}$.

$$(i) \quad \vartheta \check{Z}(\acute{r}) = \check{Z}_\vartheta^M(0)$$

$$\leq \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$$

$$= \vartheta \check{\mathfrak{Z}}(\acute{r})$$

$$\text{And so } \Rightarrow \check{\mathfrak{Z}}(0) \leq \check{\mathfrak{Z}}(\acute{r})$$

$$\begin{aligned} \text{(ii) } \vartheta \check{\mathfrak{Z}}(\acute{r}) &= \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \\ &\leq \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}) \right\} \\ &= \left\{ \left(\vartheta \check{\mathfrak{Z}}(\acute{r} * \check{s}) \right) \vee \left(\vartheta \check{\mathfrak{Z}}(\check{s}) \right) \right\} \\ &= \vartheta \{ \check{\mathfrak{Z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}(\check{s}) \} \end{aligned}$$

$$\text{and so } \check{\mathfrak{Z}}(\acute{r}) \leq \{ (\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}(\check{s}) \}$$

Hence $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$.

6.2.7 Theorem

If, $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$, then the *Doubt Fuzzy ϑ – Multiplication $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$* of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$, for all $\vartheta \in (0,1]$.

Proof:

Let, $\check{\mathfrak{Z}}$ be a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$ and let $\vartheta \in (0,1]$. Then

$$\begin{aligned} \text{(i) } \check{\mathfrak{Z}}_{\vartheta}^M(0) &= \vartheta \check{\mathfrak{Z}}(\acute{r}) \\ &\leq \vartheta \check{\mathfrak{Z}}(\acute{r}) \\ &= \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \\ &\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^M(0) \leq \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \end{aligned}$$

$$\begin{aligned} \text{(ii) } \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) &= \vartheta \check{\mathfrak{Z}}(\acute{r}) \\ &\leq \vartheta \{ \check{\mathfrak{Z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}(\check{s}) \} \\ &= \vartheta \{ \check{\mathfrak{Z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}(\check{s}) \} \end{aligned}$$

$$\begin{aligned}
&= \left\{ \left(\vartheta \check{\mathfrak{Z}}(\acute{r} * \check{s}) \right) \vee \left(\vartheta \check{\mathfrak{Z}}(\check{s}) \right) \right\} \\
&\leq \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}) \right\} \\
&\Rightarrow \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \leq \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}) \right\}
\end{aligned}$$

Hence, $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\mathfrak{O}}$, $\forall \acute{r}, \check{s} \in (0,1]$.

6.2.8 Theorem

Let, $\vartheta \in (0,1]$ and let, $\check{\mathfrak{Z}}$ be a *Doubt Fuzzy – Z – Ideal* of a Z-algebra $\check{\mathfrak{O}}$. Then the *Doubt Fuzzy ϑ – Multiplication* $\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r})$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$.

Proof:

Let, $\acute{r}, \check{s} \in \check{\mathfrak{O}}$.

Now, we have

$$\begin{aligned}
\check{\mathfrak{Z}}_{\vartheta}^M(\acute{r} * \check{s}) &= \vartheta \check{\mathfrak{Z}}(\acute{r} * \check{s}) \\
&\leq \vartheta \{ \check{\mathfrak{Z}}((\acute{r} * \check{s}) * \check{s}) \vee \check{\mathfrak{Z}}(\check{s}) \} \\
&= \{ \vartheta \check{\mathfrak{Z}}(\check{s} * (\acute{r} * \check{s})) \vee \vartheta \check{\mathfrak{Z}}(\check{s}) \} \\
&= \vartheta \{ \check{\mathfrak{Z}}(0) \vee \check{\mathfrak{Z}}(\check{s}) \} \\
&\leq \vartheta \{ \check{\mathfrak{Z}}(\acute{r}) \vee \check{\mathfrak{Z}}(\check{s}) \} \\
&\leq \left\{ \left(\vartheta \check{\mathfrak{Z}}(\acute{r}) \right) \vee \left(\vartheta \check{\mathfrak{Z}}(\check{s}) \right) \right\} \\
&= \left\{ \check{\mathfrak{Z}}_{\vartheta}^M(\acute{r}) \vee \check{\mathfrak{Z}}_{\vartheta}^M(\check{s}) \right\}
\end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^M$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$, $\forall \acute{r}, \check{s} \in (0,1]$.

6.2.9 Theorem

If the *Doubt Fuzzy ϑ – Translation* $\check{\mathfrak{Z}}_{\vartheta}^M$ of $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$, $\vartheta \in (0,1]$, then $\check{\mathfrak{Z}}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\mathfrak{O}}$.

Proof:

Let us assume that, $\check{Z}_\vartheta^M$ of $\check{Z}$ is a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$. Then

$$\begin{aligned}
 \vartheta \check{Z}(\check{r} * \check{s}) &= \check{Z}_\vartheta^M(\check{r} * \check{s}) \\
 &\leq \left\{ \check{Z}_\vartheta^M((\check{r} * \check{s}) * \check{s}) \vee \check{Z}_\vartheta^M(\check{s}) \right\} \\
 &= \left\{ \check{Z}_\vartheta^M(\check{s} * (\check{r} * \check{s})) \vee \check{Z}_\vartheta^M(\check{s}) \right\} \\
 &= \left\{ \check{Z}_\vartheta^M(0) \vee \check{Z}_\vartheta^M(\check{s}) \right\} \\
 &\leq \left\{ \check{Z}_\vartheta^M(\check{r}) \vee \check{Z}_\vartheta^M(\check{s}) \right\} \\
 &= \left\{ (\vartheta \check{Z}(\check{r})) \vee (\vartheta \check{Z}(\check{s})) \right\} \\
 &\Rightarrow \check{Z}(\check{r} * \check{s}) \leq \{ \check{Z}(\check{r}) \vee \check{Z}(\check{s}) \}
 \end{aligned}$$

Hence $\check{Z}$ is a *Doubt Fuzzy – Z – Sub Algebra* of $\check{\omega}$.

6.2.10 Theorem

Intersection and union of any two ϑ – Translation of a *Doubt Fuzzy – Z – Ideal* of $\check{Z}$ of $\check{\omega}$ is also a *Doubt Fuzzy – Z – Ideal* of $\check{\omega}$.

Proof:

Let $\check{Z}_\vartheta^T$ and $\check{Z}_\delta^T$ be two *Doubt Fuzzy ϑ – Translation of a Doubt Fuzzy – Z – Ideal* of $\check{Z}$ of $\check{\omega}$, where $\vartheta, \delta \in [0,1]$.

Assume that $\vartheta \leq \delta$.

Then by theorem 5.2.8, $\check{Z}_\vartheta^T$ and $\check{Z}_\delta^T$ are *Doubt Fuzzy – Z – Ideals* of $\check{\omega}$.

$$\begin{aligned}
 \text{Now, } (\check{Z}_\vartheta^T \cap \check{Z}_\delta^T)(\check{r}) &= \left\{ \check{Z}_\vartheta^T(\check{r}) \vee \check{Z}_\delta^T(\check{r}) \right\} \\
 &= \{ (\check{Z}(\check{r}) + \vartheta) \vee (\check{Z}(\check{r}) + \delta) \} \\
 &= \check{Z}(\check{r}) + \vartheta \\
 &= \check{Z}_\vartheta^T(\check{r})
 \end{aligned}$$

And

$$\begin{aligned}
 (\check{\mathfrak{Z}}_{\vartheta}^T \cup \check{\mathfrak{Z}}_{\delta}^T)(\dot{r}) &= \{\check{\mathfrak{Z}}_{\vartheta}^T(\dot{r}) \wedge \check{\mathfrak{Z}}_{\delta}^T(\dot{r})\} \\
 &= \{(\check{\mathfrak{Z}}(\dot{r}) + \vartheta) \wedge (\check{\mathfrak{Z}}(\dot{r}) + \delta)\} \\
 &= \check{\mathfrak{Z}}(\dot{r}) + \delta \\
 &= \check{\mathfrak{Z}}_{\delta}^T(\dot{r})
 \end{aligned}$$

Hence $\check{\mathfrak{Z}}_{\vartheta}^T \cap \check{\mathfrak{Z}}_{\delta}^T$ and $\check{\mathfrak{Z}}_{\vartheta}^T \cup \check{\mathfrak{Z}}_{\delta}^T$ are *Doubt Fuzzy – Z – Ideals* of $\check{\mathfrak{W}}$.

CHAPTER- 7

CONCLUSION

By this project, we defined the new notation of Algebraic structures of ϑ –Translations and ϑ –Multiplication in Z –Sub Algebra of Z –Algebras. And also defined the ϑ –translations and ϑ – Multiplications in Z –Ideal of Z –Algebra and discussed some of their properties in detail by Z –Algebras. Also we defined the new notation of Algebraic structures of ϑ –Translations and ϑ –Multiplication in Z –Sub Algebra of Q -fuzzy Z –Algebras. And also defined the ϑ –translations and ϑ –Multiplications in Z –Ideal of Q -fuzzy Z –Algebra and discussed some of their properties in detail. we also defined the new notation of Algebraic structures of ϑ –translations and ϑ –Multiplications in Doubt Fuzzy Z –Sub Algebra of Z –Algebras. And also defined the ϑ –translations and ϑ –Multiplications in Doubt Fuzzy Z –Ideal of Z –Algebra and discussed some of their properties in detail by Z –Algebras.

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