

SPECTRAL GRAPH THEORY ON VARIOUS FIELDS

Submitted in partial fulfillment of the requirements for the award of

BACHELOR OF SCIENCE degree in **MATHEMATICS**

BY

PHILOMINA VASANTHA KUMARI. A (41060010)

PRIYA. G (41060011)



DEPARTMENT OF MATHEMATICS

SCHOOL OF SCIENCE AND HUMANITIES

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

(DEEMED TO BE UNIVERSITY)

Accredited with Grade “A++” by NAAC

JEPPIAAR NAGAR, RAJIV GANDHI SALAI, CHENNAI - 600 119

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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **PHILOMINA VASANTHA KUMARI . A (41060010), PRIYA. G (41060011)** who carried out the project entitled **"SPECTRAL GRAPH THEORY ON VARIOUS FIELDS"** under our supervision from **December 2023 to April 2024**


Internal Guide
Dr. VINITHA.V


Head of the department
Dr.NIRMALA.M

Submitted for Viva voce Examination held on _____

Internal Examiner

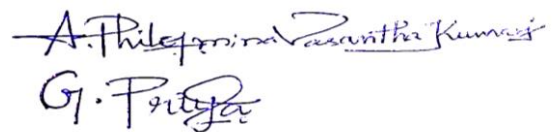
External Examiner

DECLARATION

We **PHILOMINA VASANTHA KUMARA.A (41060010)**, **PRIYA.G (41060011)** hereby declare that the Project Report entitled “**SPECTRAL GRAPH THEOREY ON VARIOUS FIELDS**” done by us under the guidance of **Dr. VINITHA. V** at **SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY** is submitted in partial fulfilment of the requirements for the award of Bachelor of Science degree in Mathematics.

DATE : 26.04.2024

PLACE: CHENNAI



SIGNATURE OF THE CANDIDATES

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ABSTRACT

This study uses the positive definite of Laplacian equation and energy minimization in spectral graph theory. This energy plays a fundamental role in optimizing various system across multiple discipline and also it optimizes molecular structure and calculate the most stable configuration of molecules by comparing with other energy and find the equilibrium configuration of a mechanical system. It helps to optimize the design of Structure to minimize stresses and strains. The advantage of this approach is that it is quite easy to achieve convergence for the minimize.

KEYWORDS: Laplacian equation, Dirichlet energy, Harmonic function,potential energy

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CHAPTER 1

INTRODUCTION

Spectral graph theory is a powerful framework that studies graphs using the spectral properties of their associated matrices, particularly the Laplacian matrix. In recent years, it has found wide applications in various fields, including computer science, machine learning, and optimization. One fascinating application of spectral graph theory is in energy minimization, where the laplacian matrix plays a pivotal role. The laplacian matrix captures essential structural information about the graph, representing connections and interactions between vertices. Specifically, it encodes the graphs topology and allows us to analyze its spectral properties.

In the context of energy minimization, spectral graph theory provides a rigorous mathematical foundation for optimizing functions defined on graphs. By leveraging the eigenvalues and eigenvectors of the laplacian matrix, we can devise efficient algorithms to minimize energy functions over the graph. Key to this approach is the concept of positive definiteness. A matrix is positive definite if all its eigenvalues are positive indicating a well-behaved convex optimization landscape. In energy minimization ensuring the laplacian matrix (possibly modified with additional terms) remains positive definite guarantees convergence to a global optimum.

Through the lens of Spectral Graph Theory, energy minimization becomes not just an optimization task but a deeper exploration of the underlying graph structure. By exploiting spectral properties, we can uncover hidden patterns, detect communities, and reveal intricate relationships within complex networks. In summary, Spectral Graph Theory provides a powerful framework for energy minimization on graphs. By harnessing the spectral properties of the laplacian matrix, we can develop efficient algorithms that ensure convergence to optimal solutions while gaining insights into the structural properties of the underlying.

1.1 SPECTRAL GRAPH THEORY

- The study of the properties of a graph in relationship to the characteristic polynomial, eigen values, and eigen vectors of matrices associated with the graph, such, as its adjacency matrix or Laplacian matrix. The adjacency matrix of a simple undirected graph is a real symmetric matrix and is therefore orthogonally diagonalizable. Its eigenvalues are real algebraic integers.
- While the adjacency matrix depends on the vertex labelling, its spectrum is a graph invariant, although not a complete one.
- Spectral graph theory is also concerned with graph parameters that are defined via multiplicities of eigenvalues of matrices associated to the graph, such as the Colin de Verdière number.

1.2 PRELIMINARIES

1.2.1 GRAPH

A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it called its end points. An edge is said to connect its endpoints.

EXAMPLE:

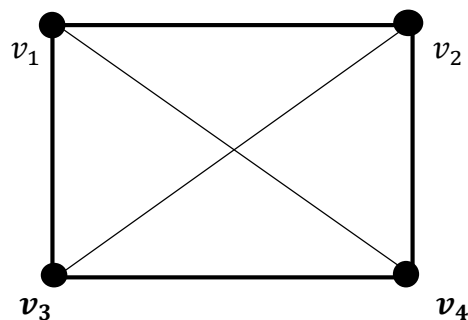


Fig 1.3.1 Graph

1.2.2 ADJACENCY MATRIX

The adjacency matrix, also called the connection matrix, the matrix containing rows and columns which is used to represent a simple labelled graph, with 0 or 1 the position of (v_1, v_2) according to the condition whether V_i and V_j are adjacent or not.

EXAMPLE:

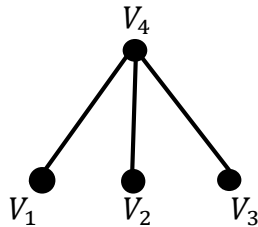


Fig 1.2.2 Adjacency matrix

$$A = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

1.3.2 DEGREE MATRIX

Given a graph $G = (V, E)$ with $|V| = n$, the degree matrix D for G is a $n \times n$ diagonal matrix defined as

$$D_{ij} = \begin{cases} \deg(v_i) & , \text{ if } i = j \\ 0 & , \text{ otherwise} \end{cases} \quad (1.1)$$

Where the degree $\deg(v_i)$ of a vertex counts the number of times an edge terminates at that vertex. In an undirected graph, this means that each loop increases the degree of a vertex by two. In a directed graph, the term degree may refer either to in degree (the number of incoming edges at each vertex) or out degree (the number of outgoing edges at each vertex).

EXAMPLE:

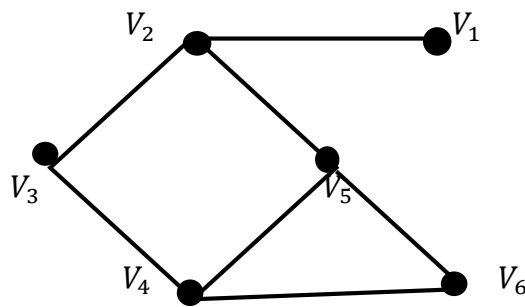


Fig 1.2.3 Degree matrix

Degree matrix of D =

$$\begin{array}{c}
 v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \\
 \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 3 & 0 & 0 & 0 & 0 \\
 0 & 0 & 2 & 0 & 0 & 0 \\
 0 & 0 & 0 & 3 & 0 & 0 \\
 0 & 0 & 0 & 0 & 3 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2
 \end{bmatrix}
 \end{array}$$

1.2.4 LAPLACIAN MATRIX

❖ The Laplacian matrix of a graph is a mathematical representation that encapsulates certain structural properties of the graph. It is defined as the difference between the degree matrix and the adjacency matrix of the graph.

❖ The Laplacian matrix L is defined as:

$$L = A - D$$

Where,

❖ D is the degree matrix, a diagonal matrix where D_{ii} represents the degree of vertices i.

❖ A is the adjacency matrix, where A_{ij} is the weight of the edge connecting vertices i and j.

1.2.5 WEIGHTED GRAPH

A graph in which each edge has an associated numerical value is called a weighted Graph. The numerical value associated to an edge is the weight of the edge. The weight of an edge can represents a distance, a cost... etc.

EXAMPLE:

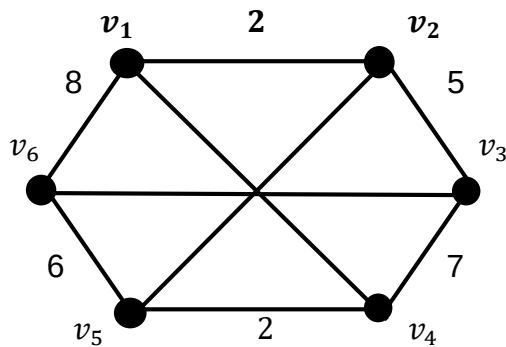


Fig 1.2.5 Weighted graph

Weighted graph of $w =$

v_1	v_2	v_3	v_4	v_5	v_6
0	2	0	4	0	8
2	0	5	0	3	0
0	5	0	7	0	3
4	0	7	0	2	0
0	3	0	2	0	6
8	0	3	0	6	0

1.2.6 CONNECTED GRAPH

- ❖ A graph is said to be connected if every pair of vertices in the graph is connected. This means that there is a path between every pair of vertices.

- ❖ A connected graph can be simple or have loops and multiple edges, as long as it maintains the connectivity property.

EXAMPLE

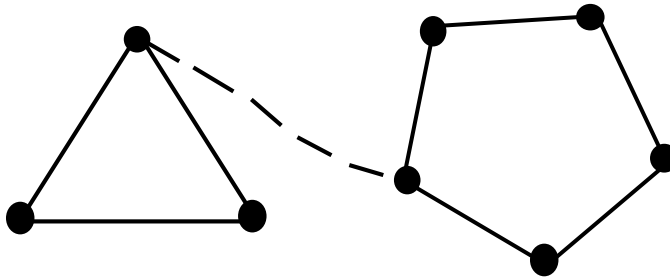


Fig 1.2.6 Connected graph

1.2.7 BLOCK MATRIX

- ❖ A block matrix is the result of performing some vertical and horizontal cuts on a matrix so as to subdivide it into blocks.
- ❖ A matrix interpreted as a block matrix can be visualized as the original matrix with a collection of horizontal and vertical lines, which break it up, or partition it, into a collection of smaller matrices.

EXAMPLE:

The matrix

$$P = \begin{bmatrix} 1 & 2 & 2 & 7 \\ 1 & 5 & 6 & 2 \\ 3 & 3 & 4 & 5 \\ 3 & 3 & 6 & 7 \end{bmatrix}$$

Can be visualized as divided into four blocks, as

$$P = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 7 \\ 1 & 5 & 6 & 2 \\ \hline 3 & 3 & 4 & 5 \\ 3 & 3 & 6 & 7 \end{array} \right]$$

The horizontal and vertical lines have no special mathematical meaning but are a common way to visualize a partition. By this partition is partitioned into four 2×2 blocks, as

$$P_{11} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \quad P_{12} = \begin{bmatrix} 2 & 7 \\ 6 & 2 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \quad P_{22} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

The partitioned matrix can then be written as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

1.2.8 HARMONIC FUNCTION

Let $G = (V, E)$ be a graph with vertex set V and edge set E . A function $f: V \rightarrow R$ is said to be harmonic on a subset $S \subseteq V$ if for each vertex $v \in S$ the value of $f(v)$ is equal to the average of the values of f over all neighboring vertices of V in S .

Formally, for each $v \in S$, the condition for f to be harmonic on S is given by

$$f(v) = \frac{1}{deg(v)} \sum_{U \in N(v)} f(U) \quad (1.2)$$

Where,

$N(v)$ represents the set of neighbors of V in S .

$deg(v)$ is the degree of vertex V . (i.e.) the number of edges incident to V .

EXAMPLE:

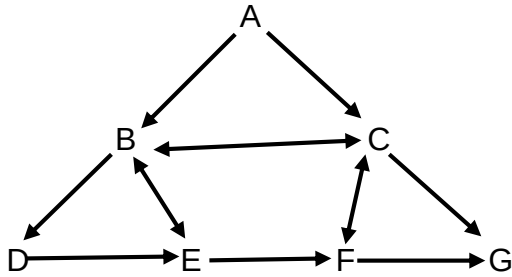


Fig 1.2.8 Harmonic function

Here the Vertices A, B, C, D, E, F and G represent cities. Edges represent connection between cities.

Where, The weight of each edge represents the distance between of each city. The population of the city,

$$f(A)=100, f(B)=80, f(C)=120, f(D)=60, f(E)=70, f(F)=90, f(G)=110$$

To check if f is harmonic on subset S of vertices Let's consider $S = \{B, C, E, F\}$, the subset of the vertices containing cities B, C, E and F.

By using harmonic function;

Let's check this condition for each vertex V in S ,

$$S = \{B, C, E, F\}$$

$$\begin{aligned}\text{For cities B: } f(B) &= \frac{f(A)+f(C)+f(D)+f(E)}{4} \\ &= \frac{100+120+60+70}{4} \\ &= 87.5\end{aligned}$$

$$\begin{aligned}\text{For cities C: } f(C) &= \frac{f(A)+f(B)+f(F)+f(G)}{4} \\ &= \frac{100+80+90+110}{4} \\ &= 95\end{aligned}$$

$$\begin{aligned}\text{For cities E: } f(E) &= \frac{f(B)+f(D)+f(G)}{3} \\ &= \frac{80+60+90}{3} \\ &= 76.67\end{aligned}$$

$$\begin{aligned}\text{For cities F: } f(F) &= \frac{f(E)+f(G)+f(C)}{3} \\ &= \frac{70+110+120}{3} \\ &= 100\end{aligned}$$

So ,In this example, we can see that the function f is harmonic on the subset $S=\{B,C,E,F\}$ of vertices each city's population density is equal to the average population density of its neighboring cities with in S . satisfying the condition for f to be harmonic on S .

1.2.9 DIRICHLET ENERGY

Given a function $x: V \rightarrow \mathbb{R}$ where V is the set of vertices of a graph G . The Dirichlet energy $\Sigma(f)$ is defined as:

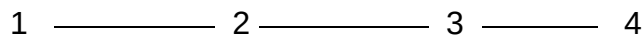
$$\Sigma(x) = \frac{1}{2} \sum_{(u,v) \in E} w(u,v) (x(u) - x(v))^2 \quad (1.3)$$

Where;

- ❖ E is the set of edges of the graph G .
- ❖ $w(u, v)$ is the weight of the edge between vertices u and v .
- ❖ $x(u)$ and $x(v)$ are the function values at vertices u and v respectively.

EXAMPLE

Consider a one-dimensional chain graph G with vertices $V = \{1,2,3,4\}$ and edges between adjacent vertices. The weights of the edges represent the distance between the vertices.



suppose we have a function f defined on the vertices of this chain graph. Such that $f(1) = 3, f(2) = 5, f(3) = 4$ and $f(4) = 2$

Now, let's calculate the Dirichlet energy $\sum(f)$ for this function f :

$$\begin{aligned}\sum(x) &= \frac{1}{2}w(u,v)[(x(1) - x(2))^2 + (x(2) - x(3))^2 + (x(3) - x(4))^2] \\ &= \frac{1}{2}[(3 - 5)^2 + (5 - 4)^2 + (4 - 2)^2] \\ \sum(x) &= 4.5\end{aligned}$$

So, for this function f , the Dirichlet energy $\sum(f)$ is 4.5

CHAPTER 2

LITERATURE SURVEY

S Li, W Xia [1] consider a undirected signed weighted graphs and prove that the signed Laplacian matrix has at most negative eigenvalues for a graph with negative edges, and for the signed Laplacian corresponding to a tree, it has exactly negative eigenvalues.

Qin Ni, Liqun Qi, and Fei Wang [2] consider an eigenvalue method for testing positive definiteness of a multivariate form. An advantage of the E-eigenvalues and the E-characteristic polynomial is that the degree of the E-characteristic polynomial is much lower than the degree of the characteristic polynomial.

Daniel A. Spielman [3] the design of provably fast algorithms for solving linear equations in the Laplacian matrices of graphs. These algorithms motivate and rely upon fascinating primitives in graph theory, including low-stretch spanning trees, graph sparsifiers, ultra-sparsifiers, and local graph clustering. These are all connected by a definition of what it means for one graph to approximate another.

Daniel Zelazo and Mathias Burger [4] consider a definiteness of the weighted graph Laplacian matrix with negative edge weights. The definiteness of the weighted Laplacian is studied in terms of certain matrices that are related via congruent and similarity transformations. The utility of these results are demonstrated in a weighted consensus network where appropriately placed negative weight edges can induce a clustering behavior for the protocol.

Xiao-Dong Zhang [5] consider a The Laplacian matrix of a simple graph is the difference of the diagonal matrix of vertex degree and the (0,1) adjacency matrix. it has been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning.

DH Tran, TX Vu, S Chatzinotas [6] we design the UAV trajectory to minimize the total energy consumption while satisfying the requested timeout (RT) requirement and energy budget, which is accomplished via jointly optimizing the path and UAV's velocities along subsequent hops. It is also shown that the proposed solutions outperform the state-of-the-art benchmarks in terms of both energy consumption and outage performance.

D Zhang, M Hebert [7] This representation is based on the well-studied theory of Harmonic Maps which studies the mapping between different metric manifolds from the energy minimization point of view and they preserve both the shape and the continuity of the underlying surfaces, they are robust to occlusion and they are independent of surface sampling strategy. The proposed representation is applied to solve the surface-registration problem.

Braxton Osting, Chris D. White and Edouard Oudet [8] introduced a new nonconvex graph partitioning objective where the optimality criterion is given by the sum of the Dirichlet eigenvalues of the partition components. A relaxed formulation is identified and a novel rearrangement algorithm is proposed, which we show is strictly decreasing and converges in a finite number of iterations to a local minimum of the relaxed objective function.

Amie Albrecht, Phil Howlett, Peter Pudney, Xuan Vu, Peng Zhou [9] discussed the problem of finding an energy-efficient driving strategy for a train journey on an undulating track with steep grades subject to a maximum prescribed journey time. We examined the characteristic optimal control modes, studied allowable control transitions and established the existence of optimal switching points.

sA Abrams, R Kenyon [10] discussed the map from conductances to edge energies for harmonic functions on finite graphs with Dirichlet boundary conditions and proved that for any compatible acyclic orientation and choice of energies there is a unique choice of conductances such that the associated harmonic function realizes those orientations and energies.

CHAPTER 3

AIM AND SCOPE

3.1 AIM

To prove the positive definiteness of the Laplacian matrix $L(S, S)$ corresponding to the vertex set S of a connected graph G , where B is a nonempty, proper subset of V and $S=V-B$ and also to prove energy minimization by comparing harmonic and nonharmonic function where $\mathcal{E}(x)$ is minimized by setting $x(S)$ so that x is harmonic on S .

3.2 SCOPE

The scope of the positive definiteness of the Laplacian matrix is to provide a clear explanation of the proof while addressing potential challenges and extensions of the result. It should cater to an audience with a background in graph theory, linear algebra, and possibly spectral graph theory. The scope of the energy minimization techniques in spectral graph theory have a broad scope encompassing graph partitioning, clustering, dimensionality reduction, graph embedding, regularization, and various applications in fields such as computer science, physics, chemistry, and engineering.

CHAPTER 4

THEOREM OF LAPLACIAN OF POSITIVE DEFINITE AND ENERGY MINIMIZATION

4.1 LAPLACIAN OF POSITIVE DEFINITE

Let L be the Laplacian matrix of a connected graph $G = (V, E, w)$, let B be a nonempty, proper subset of V , and let $S = V - B$. Then, $L(S, S)$ is positive definite.

PROOF:

Let $S_1, \dots, S_k$ be the connected components of vertices of $G(S)$.

We can use these to write $L(S, S)$ as a block matrix with blocks equal to $L(S_i, S_i)$. Each of these blocks can be written

$$L(S_i, S_i) = L_{G_{S_i}} + X_{S_i} \quad (4.1)$$

As G is connected, there must be some vertex in S_i with an edge to a vertex not in S_i

This implies that X_{S_i} is not the zero matrix.

By this lemma,

Let L be the Laplacian matrix of a connected graph and let X be a nonnegative, diagonal matrix with at least one nonzero entry. Then $L + X$ is positive definite.

Given that X is a nonnegative, diagonal matrix with at least one nonzero entry, we can write X as:

$$X = \text{diag}(x_1, x_2, \dots, x_n) \quad (4.2)$$

Where $x_i \geq 0$ and at least one x_i is nonzero.

Now, consider the Laplacian matrix L of a connected graph. The matrix $L+X$ is given by:

$$L + S = \begin{bmatrix} d_1 + x_1 & -w_{12} & \cdots & -w_{1n} \\ -w_{12} & d_2 + x_2 & \cdots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \cdots & d_n + x_n \end{bmatrix}$$

Where d_i is the degree of vertex i and w_{ij} is the weight of the edge between vertices i and j .

To show that $L+X$ is positive definite, we need to show that all its eigenvalues are positive.

Let λ be an eigenvalue of $L+X$ and let V be the corresponding eigenvector. Then, we have: (Laplacian linear equation)

$$(L+X) V = \lambda V$$

$$LV + XV = \lambda V$$

$$LV = \lambda V - XV$$

$$LV = (\lambda - X) V$$

Since X is a diagonal matrix with nonnegative entries, X is positive semidefinite.

Therefore, $\lambda - X$ is an eigenvalue of L , which is known to have all positive eigenvalues.

Since $\lambda - X$ is positive and X has at least one nonzero entry, λ must be positive.

Since $\lambda - X$ is positive and X has at least one nonzero entry, λ must be positive.

Therefore, all eigenvalues of $L+X$ are positive which implies that $L+X$ is positive definite.

To prove that $L(S_i, S_i)$ is invertible, we can use the fact that $L(S_i, S_i)$ is positive definite.

Since $L(S_i, S_i)$ is positive definite, all its eigenvalues are positive.

Let's denote the eigenvalues of $L(S_i, S_i)$ as $\lambda_1, \lambda_2, \dots, \lambda_k$ Where k is the dimension of $L(S_i, S_i)$.

Since $L(S_i, S_i)$ is positive definite, all its eigenvalues are positive.

(i, e) $\lambda_i > 0$ for all i

Now, for a matrix to be invertible, its eigenvalues must be non-zero.

Since all eigenvalues of $L(S_i, S_i)$ are positive, they are also non-zero.

Therefore, $L(S_i, S_i)$ is invertible.

As the matrix $L(S, S)$ is invertible, the equations have a solution, and it must be unique.

$L(S, S)$ is invertible equal to $L(S_i, S_i)$ is positive definite.

4.2 ENERGY MINIMIZATION

Let $G = (V, E, W)$ be a connected, weighted graph, let $B \subset V$, and let $S = V - B$. Given $x(b)$, show that $\varepsilon(x)$ is minimized by setting $x(s)$ so that x is harmonic on S .

PROOF

Let G be a given graph for a function x on a subset S of a vertices in a graph is harmonic function every $V \in S$ the value of $x(v)$ is equal to average value of its neighbors in S . That is

$$X(v) = \frac{1}{|N(V) \cap S|} \sum_{U \in N(V) \cap S} X(u) \quad (4.3)$$

where $N(V)$ denotes the neighbors of vertex.

Now, Let's consider $E(x)$, the Dirichlet energy of x , defined as:-

$$E(x) = \frac{1}{2} \sum_{\{U, V\} \in E} W(U, V) (x(U) - x(V))^2 \quad (4.4)$$

Where $W(U, V)$ is the weight of the edge between vertices U and V .

To minimize $E(x)$ with respect to $x(s)$ given $x(b)$ where Dirichlet energy can be compared with potential energy by setting $y(s)$ on s where $y(s)$ is non harmonic function and it is not average value of its neighbors in s .

Let's assume a simple connected graph with boundary nodes have fixed potential energy in harmonic function.

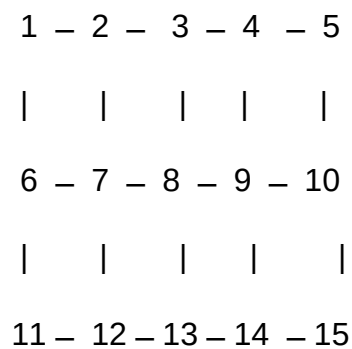


Fig 4.2.1 Harmonic function with fixed potential energy

For the above fig 4.2.1, the nodes 2,7,8 and 12 all have a potential of 2 and nodes 3,8,9 and 13 all have a potential of 3. Thus, neighboring nodes are evenly distributed.

Now, Let's assume one more simple connected graph with boundary nodes have fixed potential energy in nonharmonic function.

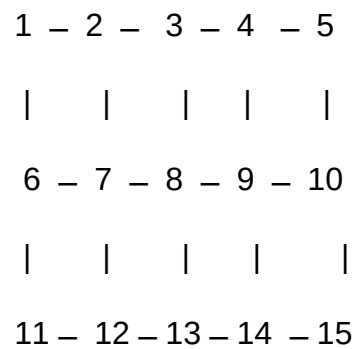


Fig 4.2.2 Non harmonic function with fixed potential energy

For the above fig4.2.2, the nodes 2 have potential of 1 while its neighboring nodes 1,3,7 and 8 hence potential of 0, 2, 3. Thus neighboring nodes are not evenly distributed

From the 4.2.1 and 4. 2.2 graph of harmonic function and non harmonic function $E(x)$ is minimized because potential energy are evenly distributed in harmonic function within S . Thus resulting in a smooth and evenly distributed potential energy. Finally it leads to a lower energy $E(x)$ compared to non harmonic function.

Therefore $\varepsilon(x)$ is minimized by setting $x(s)$ so that x is harmonic on s .

CHAPTER 5

RESULTS AND DISCUSSIONS

5.1 RESULTS

In this chapter, we proved that the positive definiteness of the Laplacian matrix $L(S, S)$ corresponding to the vertex set S of a connected graph G , where B is a non empty, proper subset of V and $S=V-B$. Secondly we proved that energy minimization by comparing harmonic and non harmonic function where $\varepsilon(x)$ is minimized by setting $x(S)$ so that x is harmonic on S .

5.2 DISCUSSIONS

- ❖ We can prove $L(S, S)$ is positive definite is equal to $L(S_i, S_i)$. Is invertible. Let using $L(S, S)$ block matrix with blocks matrix to $L(S_i, S_i)$ since x is a diagonal matrix with non negative entries, x is positive semidefinite. In this also using degree matrix and weighted matrix.
- ❖ We can solve energy minimization by setting harmonic function and by comparing with non harmonic. Using Dirichlet energy we can compare with other energy source to find average of its neighbors. Graph of harmonic function and non harmonic function are compared. Then the final result gives that by only using harmonic function the energy can be minimized.

CHAPTER 6

CONCLUSIONS

In this project, we found that Laplacian $L(S, S)$ is positive definite. Then we are prove to $L(S_i, S_i)$ invertible. The Laplacian matrix is connected graph and X be a non negative, diagonal matrix with at least one non zero entry. Then we have applying a Laplacian liner equation in diagonal matrix. The final result of $L(S, S)$ is equal to $L(S_i, S_i)$ are positive. They are also non zero. We describe minimizing $E(x)$ with respect to $x(S)$ by setting $x(S)$ to be harmonic on ensures that the Dirichlet energy is minimized while $x(B)$ remains fixed and also by comparing non harmonic energy. Therefore, to achieve the minimum $E(x)$, we should choose $x(S)$ such that it is harmonic on S .

FUTURE WORK

In future, we try to minimize the energy in connected weighted graph using the application of spectral graph theory.

REFERENCES

- [1] Abrams, Aaron, and Richard Kenyon. "Fixed-energy harmonic functions." arXiv preprint arXiv:1505.05785 (2015).
- [2] Albrecht, Amie, Phil Howlett, Peter Pudney, Xuan Vu, and Peng Zhou. "The key principles of optimal train control—Part 2: Existence of an optimal strategy, the local energy minimization principle, uniqueness, computational techniques." *Transportation Research Part B: Methodological* 94 (2016): 509-538.
- [3] Lai, Rongjie, Zaiwen Wen, Wotao Yin, Xianfeng Gu, and Lok Ming Lui. "Folding-free global conformal mapping for genus-0 surfaces by harmonic energy minimization." *Journal of Scientific Computing* 58 (2014): 705-725.
- [4] Li, Shuang, and Weiguo Xia. "On the definiteness and the second smallest eigenvalue of signed laplacian matrices." *IEEE Control Systems Letters* 6 (2022): 2347-2352.
- [5] Ni, Qin, Liqun Qi, and Fei Wang. "An eigenvalue method for testing positive definiteness of a multivariate form." *IEEE Transactions on Automatic Control* 53, no.5 (2008): 1096-1107.
- [6] Osting, Braxton, Chris D. White, and Édouard Oudet. "Minimal Dirichlet energy partitions for graphs." *SIAM Journal on Scientific Computing* 36, no. 4 (2014): A1635A1651.
- [7] Spielman, Daniel A. "Algorithms, graph theory, and linear equations in Laplacian matrices." In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pp. 2698-2722. (2010).

- [8] Tran, Dinh-Hieu, Thang X. Vu, Symeon Chatzinotas, Shahram ShahbazPanahi, and Björn Ottersten. "Trajectory design for energy minimization in uav-enabled wireless communications with latency constraints." arXiv preprint arXiv:1910.08612 (2019).
- [9] Zelazo, Daniel, and Mathias Bürger. "On the definiteness of the weighted Laplacian and its connection to effective resistance." In *53rd IEEE Conference on Decision and Control*, pp. 2895-2900. IEEE,(2014).
- [10] Zhang, Xiao-Dong. "The Laplacian eigenvalues of graphs: a survey." arXiv preprint arXiv:1111.2897 (2011).