

A STUDY ON NEW FAMILIES OF ODD GRACEFUL GRAPHS

Submitted in partial fulfillment of the requirements for the award of

BACHELOR OF SCIENCE degree in **MATHEMATICS**

BY

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SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)

Accredited with Grade "A++" by NAAC

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APRIL - 2024



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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **R. ABINAYA** (41060001), **V. AGATHIALAKSHMI** (41060002) & **S. SOWMIYA** (41060012) who carried out the project entitled "**A STUDY ON NEW FAMILIES OF ODD GRACEFUL GRAPHS**" under our supervision from **JANUARY 2024** to **APRIL 2024**.


Internal Guide

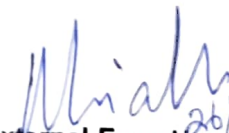
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DECLARATION

We **R. ABINAYA (41060001)**, **V. AGATHIYALAKSHMI (41060002)**,
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“**A STUDY ON NEW FAMILIES OF ODD GRACEFUL GRAPHS**” done
by us under the guidance of **Dr. A. VELANKANNI** at **SATHYABAMA**
INSTITUTE OF SCIENCE AND TECHNOLOGY is submitted in partial
fulfillment of the requirements for the award of Bachelor of Science
degree in Mathematics.

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ABSTRACT

Let $G = (V, E)$ be a finite, simple and undirected graph. A graph G with q edges admit odd graceful labeling if there is an injection $f: V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \dots, 2q - 1\}$. A graph which admits an odd graceful labeling is called an odd graceful graph. In this project, we prove the odd graceful labeling of some new families of graphs such as arbitrary super subdivided shell graph fused with star graph, the jewel graph without the prime edge fused with star graph and arbitrary super subdivision of bistar graph. Also, we show that the splitting graph of comb and the shadow graph of comb admits odd graceful labeling.

Keywords: Odd graceful labeling, Arbitrary super subdivision, Fusing, Splitting, Shadow graph, Shell graph, Jewel graph without the prime edge, Star, Bistar, Comb.

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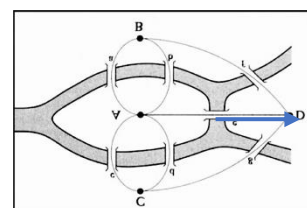
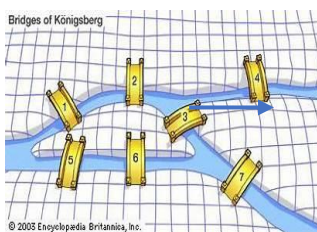
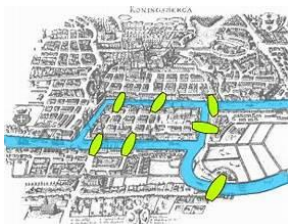
CHAPTER 1

INTRODUCTION

1.1 GRAPH THEORY

Graph theory is a branch of mathematics concerned with networks of points connected by lines. The subject of graph theory had its beginnings in recreational math problems, but it has grown into a significant area of mathematical research, with applications in chemistry, operations research, social sciences and computer science.

The history of graph theory may be specifically traced to 1735, when the Swiss Mathematician Leonhard Euler solved the konigsberg bridge problem. The konigsberg bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice. Euler proved that no such path exists. His work on the famous seven bridges of the konigsberg problem is considered the origin of graph theory.



1.2 APPLICATIONS OF GRAPH THEORY

Graph theory is like the ultimate puzzle solver in mathematics and computer science, helping us to understand complex connections in everything from internet networks to how living cells function. It breaks down big, complicated systems into nodes (as dots) and edges (the lines connecting the dots), making it easier to see how things interact. Also, it's figuring out the quickest route for your road trip, analyzing friendships on social media, improving how data travels across the internet.

1.2.1 Biological Network

In biology and bioinformatics, graph theory is used to model and analyze biological networks such as gene regulatory networks, protein-protein interaction networks, and metabolic pathways. It aids in understanding biological processes, predicting gene functions and identifying potential drug targets.

Graphs can capture the associations between any type of biological entity such as proteins, genes, small molecules, metabolites, ligands, diseases, drugs, or even database records. In Protein-Protein Interaction Networks (PPI), proteins rarely work alone; they often interact with other proteins to carry out cellular functions. Graph theory helps in studying PPI networks to identify protein complexes, pathways, and functional modules. This information is crucial for understanding diseases and developing targeted therapies.

Graph theory is employed to identify potential drug targets by analyzing network properties and identifying nodes that, when targeted, could disrupt disease-associated pathways or protein interactions. This approach aids in drug discovery and development. Graph-based computational methods can predict the functions of unknown genes by analyzing their interactions within biological networks. This helps prioritize experimental validation efforts and accelerate gene function discovery.

1.2.2 Social Network

Social network analysis involves studying the structure and dynamics of social networks, such as friendships, interactions, and information flow among individuals or entities. Graph theory provides tools and techniques to analyze network properties, identify influential nodes and understand social phenomena.

Graph theory allows us to represent social networks as graphs, where each person is a node and their relationships are edges. This makes it easier to visualize and analyze who is connected to whom. By applying graph theory algorithms, we can identify influential individuals or groups within a social network. These could be people who connect different groups together or those with a lot of connections, often called “hubs” or “influencers”.

1.3 GRAPH LABELING

Graph labeling is an important research area in graph theory and its unity in application issues like crystallography, networks and decomposition of graphs. A graph labeling is an assignment of integers to edges or vertices or both. The origin of graph labeling arose from the problem of graph decomposition. In 1967 Alexander Rosa introduced a labeling technique to investigate the decomposition of graphs. One of the labelings, called the β -labelings was later called graceful labeling by Solomon W. Golomb. A function f is called graceful labeling of a graph G with m edges if f is an injective function from $V(G)$ to $\{0, 1, 2, \dots, m\}$ such that when every edge uv is assigned the edge label $|f(u) - f(v)|$, then the resulting edge labels are distinct.

1.4 APPLICATIONS FOR GRACEFUL LABELING

1.4.1 The Coding Theory:

The design of certain important classes of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the complete graph in such a way that all the edge labels are distinct. The node labels then determine the time positions at which pulses are transmitted.

1.4.2 The X-ray Crystallography:

X-ray diffraction is one of the most powerful techniques for characterizing the structural properties of crystalline solids, in which a beam of X-rays strikes a crystal and diffracts into many specific directions. In some cases, more than one structure has the same diffraction information. This problem is mathematically equivalent to determining all labeling of the appropriate graphs which produce a pre specified set of edge labels.

CHAPTER 2

LITERATURE SURVEY

In 1991, Gnanajothi [15] introduced another type of labeling called an odd graceful labeling. A graph G with q edges to be odd graceful if there is an injection f from $V(G)$ to $\{0, 1, 2, \dots, 2q-1\}$ such that when each edge xy is assigned a label $|f(x) - f(y)|$, then the resulting edge labels are $\{1, 3, 5, \dots, 2q-1\}$. The graph which admits odd graceful labeling is called an odd graceful graph. She proved the following graphs admits odd graceful labeling which includes the path graph P_n , the cycle graph C_{2n} , the complete bipartite graph $K_{m,n}$, books, crowns $C_n \odot K_1$, caterpillar graph, product graph $P_n \times P_2$.

She proved that the class of odd-graceful graphs lies between the class of graphs with α -labeling and the class of bipartite graphs by showing that every graph with an α -labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. Vaidya and Lekha [36] have shown that the shadow graph of path and star are odd graceful. They also proved that the graph obtained by fusing all the n vertices of C_n of even order with the apex vertices of n copies of $K_{1,m}$ is odd graceful.

Yan [39] proved that $P_m \times P_n$ is odd graceful. Manisha and Veena Shinde-Deare [22] have shown that the corona graph $C_{2n} \times K_1$ for $n \geq 2$ is odd graceful. The m -splitting graphs for paths, stars and symmetric product between paths and null graphs are odd graceful proved by M. E. Abdel-Aal [1]. Moussa and Badr [23] proved the ladders and subdivisions of ladders with pendant edges are odd graceful. Liu, Wang, and Lu [21] proved that a class of bicyclic graphs with a common edge is odd graceful. Zhen-Bin Gao, Zhang, and Xu [40] proved that $P_n \times P_m$ ($m = 2, 3$ or 4) generalized crown graphs $C_n \odot K_{1,t}$, and gears are odd graceful.

Ezhilarasi Hilda and Jeba Jesintha [10] proved that subdivided shell flower graphs admit odd graceful labeling. They also proved that the subdivided uniform shell bow graphs are odd graceful. Gao [13] has proved the following graphs are odd graceful: the union of any number of paths; the union of any number of stars; the union of any number of stars and paths; $C_m \cup P_n$; $C_m \cup C_n$ and the union of any number of cycles each of which has order divisible by 4. Seoud, Diab, and Elsakhawi [27] have shown that a connected complete r -partite graph is odd graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd graceful.

Velankanni. A, Bernick Raj. A and Sujasree. M [37] proved that the odd graceful labeling of arbitrary super subdivision of certain graphs. Vaidya and Bijukumar [35] proved the following are odd- graceful: graphs obtained by joining two copies of C_n by a path; graphs that are two copies of an even cycle that share a common edge; graphs that are the splitting graph of a star; and graphs that are the tensor product of a star and P_2 . Kathiresan [20] proved that ladders and graphs obtained from them by subdividing each step exactly once are odd graceful graphs.

CHAPTER 3

AIM AND SCOPE

The main aim of the project is to study the literature survey of an odd graceful labeling in graph labeling and to investigate a new family of odd graceful graphs using operations. The scope of graph labeling is widely used in many applications like routing of data packets in network design, error correction and detection, optical character recognition, circuit design and coding theory.

CHAPTER 4

DEFINITIONS

4.1 PRELIMINARIES AND BASIC TERMINOLOGY

Graph

A graph G consists of a pair $(\mathcal{V}(G), E(G))$ where $\mathcal{V}(G)$ is a non-empty finite set whose elements are called points or vertices and $E(G)$ is a set of unordered pairs of distinct elements of $\mathcal{V}(G)$. The elements of $E(G)$ are called lines or edges of the graph G .

Example:

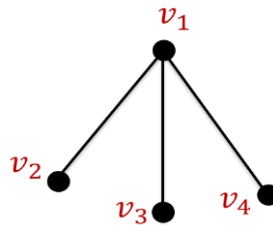


Figure 4.1.1 Graph

Simple Graph

A Graph that has neither self-loops nor parallel edges is called a simple graph.

Example:

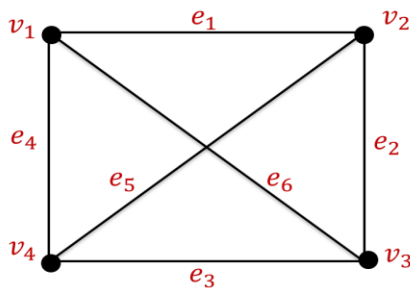


Figure 4.1.2 Simple graph

Regular Graph

A graph in which the degree of all the vertices is the same is called a regular graph. If all the vertices in a graph are of degree ' k ', then it is called a " k -regular graph".

Example:

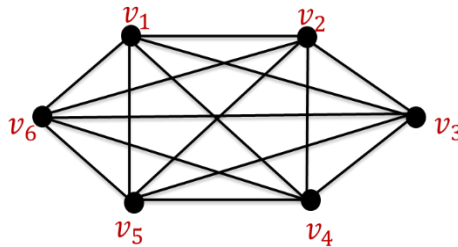


Figure 4.1.3 Regular graph

Complete Graph

A complete graph is a graph in which each pair of distinct vertices is connected by an edge. A complete graph of ' n ' vertices is represented as K_n .

Example:

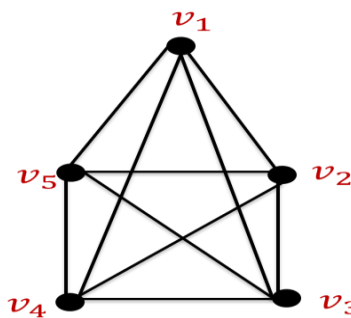


Figure 4.1.4 Complete graph

Connected Graph

A graph G is said to be a connected graph if there is at least one path between every pair of vertices in G .

Example:

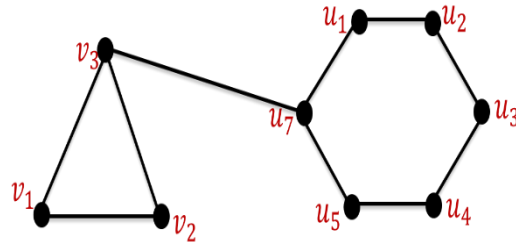


Figure 4.1.5 Connected graph

Disconnected Graph

A graph which is not connected is said to be disconnected.

Example:

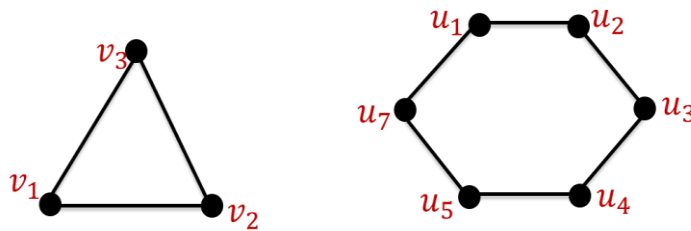


Figure 4.1.6 Disconnected graph

Bipartite Graph

A bipartite graph is a graph where the vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in another set.

Example:

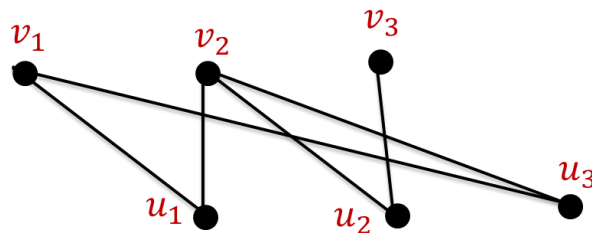


Figure 4.1.7 Bipartite graph

Complete Bipartite Graph

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset and every possible edge that could connect vertices in different subsets is part of the graph.

Example:

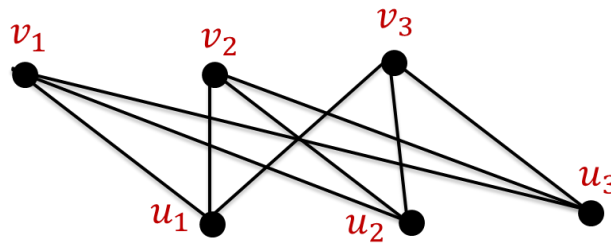


Figure 4.1.8 Complete bipartite graph

Odd Graceful Labeling

A graph G with q edges to be odd graceful if there is an injection f from $V(G)$ to $\{0, 1, 2, \dots, 2q-1\}$ such that when each edge xy is assigned a label $|f(x) - f(y)|$, then the resulting edge labels are $\{1, 3, 5, \dots, 2q-1\}$.

Example:

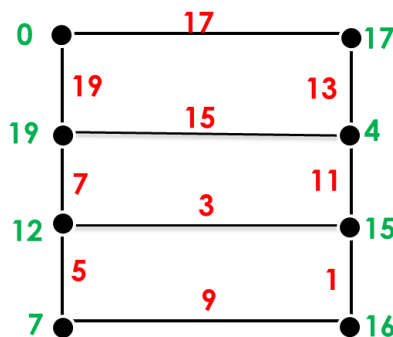


Figure 4.1.9 Odd graceful labeling of ladder graph

Star Graph

A star graph is defined as the tree which has one vertex and is adjacent to all the other vertices. The star graph is denoted as $K_{1,r}$ where r is the number of pendant vertices.

Example:

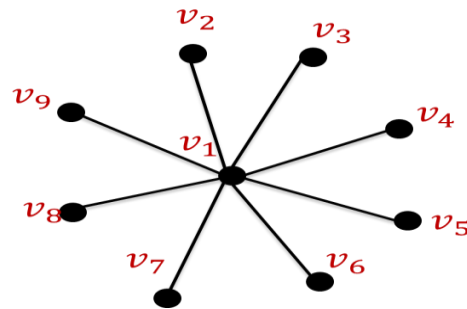


Figure 4.1.10 Star graph, $K_{1,8}$

Bistar Graph

Bistar is the graph obtained by joining the centre of two stars $K_{1,m}$ and $K_{1,n}$ with an edge and is denoted by $B_{m,n}$.

Example:

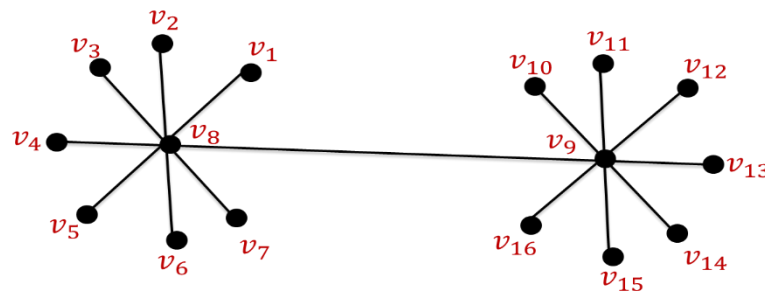


Figure 4.1.11 bistar graph, $K_{7,7}$

Comb Graph

Comb is a graph which is obtained by attaching a single pendant edge to every vertex of a path.

Example:

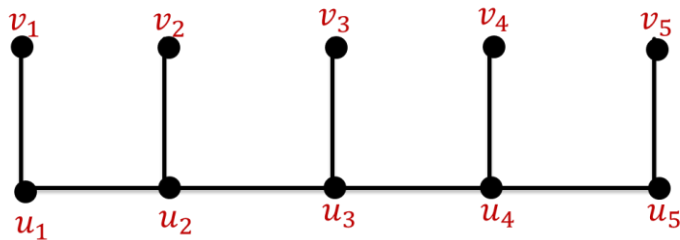


Figure 4.1.12 Comb graph

Shell Graph

Shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common endpoint called the apex. Shell graphs are denoted as $\mathcal{C}(n, n-3)$.

Example:

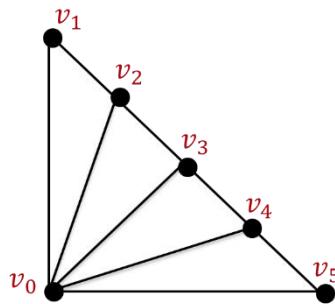


Figure 4.1.13 Shell graph, $\mathcal{C}(6,3)$

Jewel Graph

The jewel graph J_n is the graph with the vertex set $V(J_n) = \{ v_x, v_y, u_x, u_y, u_i: 1 \leq i \leq n \}$ and the edge set $E(J_n) = \{ v_x u_x, v_x v_y, v_y u_x, u_x u_y, v_y u_y, v_x u_i, u_y u_i: 1 \leq i \leq n \}$. The prime edge in a jewel graph is defined to be the edge joining the vertices v_y and u_x .

Example:

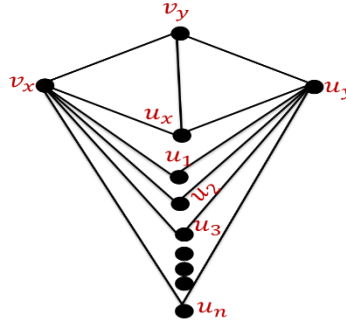


Figure 4.1.14 Jewel graph

The Jewel Graph Without Prime Edge

The jewel graph without the prime edge J_n^* is defined as the jewel graph in which the prime edge joining the vertices v_y and u_x is removed.

Example:

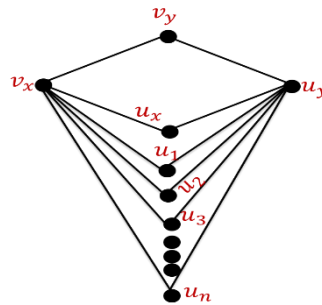


Figure 4.1.15 Jewel graph without prime edge

Subdivided Graph

The subdivision of a graph G denoted by $S(G)$ is obtained from G by replacing every edge by a path of length two.

Example:

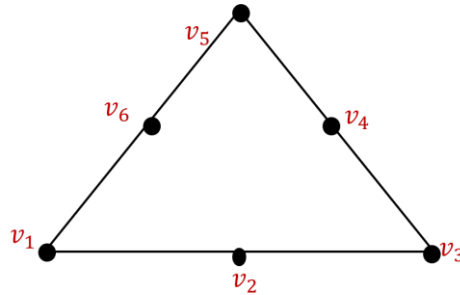


Figure 4.1.16 Subdivided graph, $S(C_3)$

Arbitrary Super Subdivision

An arbitrary super subdivision of a graph G is obtained from G by replacing every edge uv of G by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the vertices x and y respectively, where $\{x, y\}$ is a partition of $K_{2,m}$.

Example:

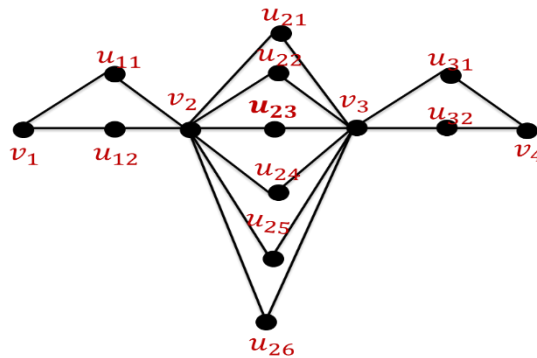


Figure 4.1.17 Arbitrary super subdivision of path graph

Fusing

Let u and v be two distinct vertices of a graph G . A new graph G_1 is constructed by fusing two vertices u and v by a single vertex x in G_1 such that every edge which was incident with either u (or) v in G is now incident with x in G_1 .

Splitting Graph

The splitting graph $SP(G)$ of a graph G is the graph obtained from G by taking a new vertex u' for each $u \in V(G)$ and joining u' to all vertices of G adjacent to u .

Example:

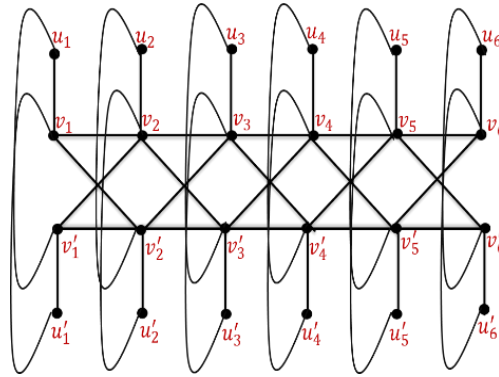


Figure 4.1.18 The Splitting graph of a comb

Shadow Graph

Shadow graph $D(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'' , join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G'' .

Example:

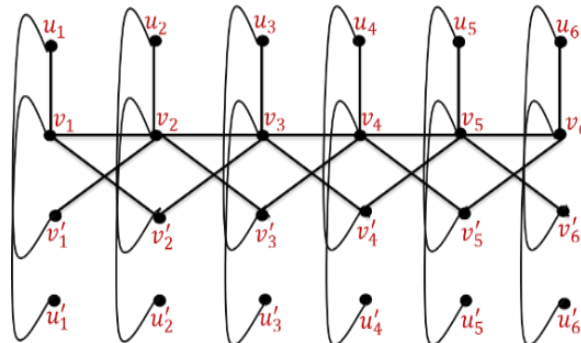


Figure 4.1.19 The Shadow graph of a comb

CHAPTER 4

MAIN RESULTS

5.1 ODD GRACEFUL GRAPHS USING OPERATIONS

Theorem 5.1.1:

The graph obtained by fusing star graph $K_{1,r}$ to its apex of an arbitrary super subdivided shell graph G^* admits odd graceful labeling.

Proof:

Let G be the graph obtained by fusing the star graph $K_{1,r}$ to the apex of the arbitrary super subdivided shell graph G^* with $n + \sum_{i=1}^{n-2} m_i$ vertices. Let $\mathcal{V}(G)$ and $E(G)$ be the number of vertices and edges of the graph G respectively, such that $|\mathcal{V}(G)| = p$ and $|E(G)| = q$.

The graph G is described as follows:

Let $w_1, w_2, w_3, \dots, w_r$ denote the pendant vertices of $k_{1,r}$. Denote the apex of the G^* as v_0 and the remaining vertices of G^* in G by v_i , for $1 \leq i \leq n-1$. Denote the vertices in the m_i – vertex part of the complete bipartite graph K_{2,m_i} , $1 \leq i \leq n-2$ as u_{ij} , $1 \leq j \leq m_i$.

Thus, the graph G has

$p = r + n + \sum_{i=1}^{n-2} m_i$ vertices and $q = r + (n-1) + 2 \sum_{i=1}^{n-2} m_i$ edges.

Define $f: \mathcal{V}(G) \rightarrow \{0, 1, 2, \dots, 2q-1\}$ in the following manner.

$$f(w_i) = 2q - (2i - 1)$$

$$= 2q - 2i + 1 \quad \forall 1 \leq i \leq r$$

$$f(v_0) = 0$$

$$f(v_1) = 2q - 2r - 2n + 3$$

$$f(v_i) = f(v_{i-1}) + 2 \quad \text{for } 2 \leq i \leq n-1$$

$$f(v_{11}) = 2q - 2r - 2n + 2$$

For $2 \leq j \leq m$,

$$\text{Define } f(u_{1j}) = f(u_{1(j-1)}) - 4$$

For $2 \leq i \leq n-2$,

$$f(u_{ij}) = \begin{cases} f(u_{(i-1)m_{i-1}}) - 2 & , j = 1 \\ f(u_{i(j-1)}) - 4 & , 2 \leq j \leq m_i \end{cases}$$

The labels of the vertices of the star $K_{1,r}$ and the labels of the base vertices of G^* in G are all odd and form a monotonically decreasing sequence $f(w_1), f(w_2), \dots, f(w_r), f(v_{n-1}), f(v_{n-2}), \dots, f(v_1)$. The labels of the non-base vertices are all even and form a monotonically decreasing sequence $[f(u_{i1}), f(u_{i2}), \dots, f(u_{im_i})]_{i=1}^{n-2}$. Hence the vertex labels of the graph G are distinct.

The labels of the edges

$$v_1u_{11}, v_2u_{11}, v_1u_{12}, v_2u_{12}, \dots, v_0v_1, v_0v_2, \dots, v_0v_{n-2}, v_0v_{n-1}, v_0w_r, v_0w_{r-1}, \dots, v_0w_1$$

Forms a monotonically increasing sequence $1, 3, 5, \dots, 2q-1$. The edge labels of the graph G are distinct. Hence the graph G admits odd graceful labeling.

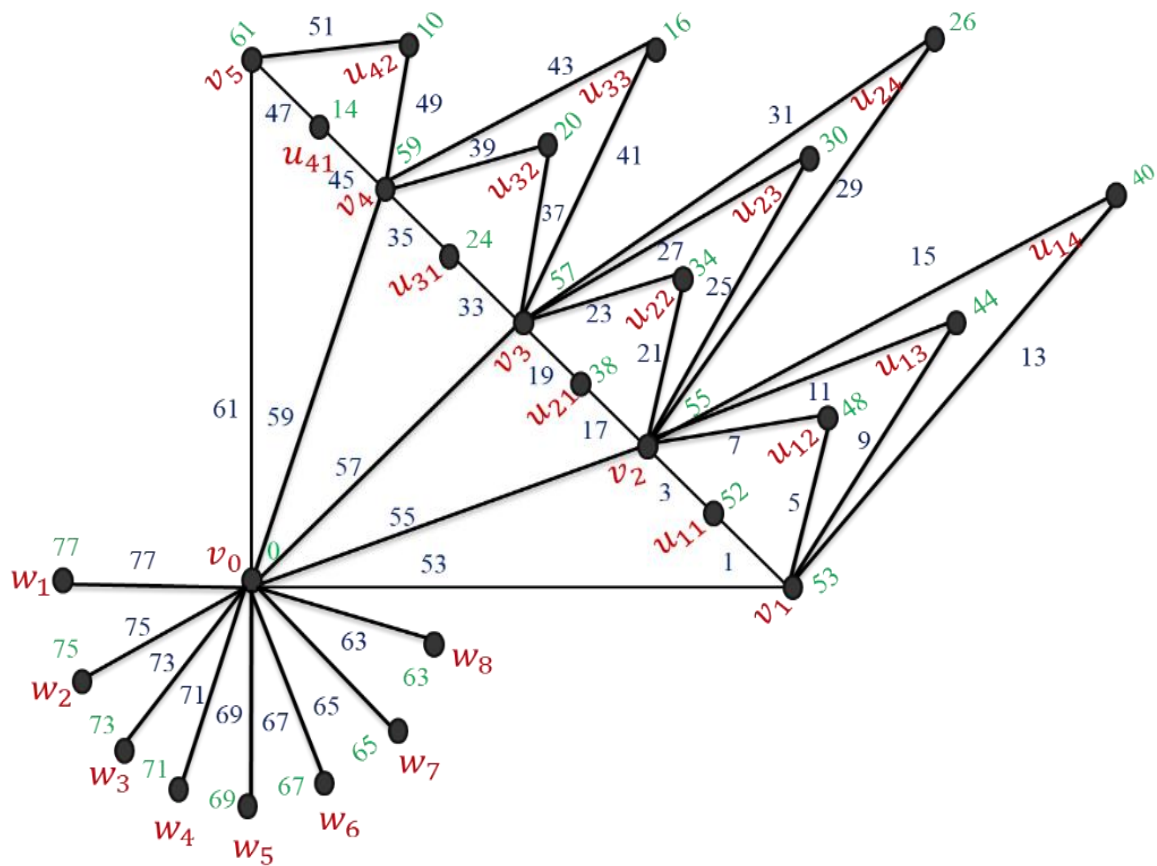


Figure 5.1.1.1 Odd graceful labeling of an arbitrary super subdivided shell graph G^* fused with star graph $K_{1,8}$.

Theorem 5.1.2:

The graph obtained by fusing star graph $K_{1,r}$ with the jewel graph without the prime edge J_n^* admits odd graceful labeling.

Proof:

Let G be the graph obtained by fusing the star graph $K_{1,r}$ to the vertex v_y of the Jewel graph J_n^* without the prime edge. Let $V(G)$ and $E(G)$ be the number of vertices and edges of the graph G respectively. Thus, $|V(G)| = p = r + n + 4$ and $|E(G)| = q = r + 2n + 4$. The graph G is described as follows:

Let $w_1, w_2, \dots, w_r$ denote the pendant vertices of $K_{1,r}$. Denote the jewel vertices in J_n^* as $u_1, u_2, \dots, u_n$ and the remaining vertices u_x, v_x, u_y and v_y such that $v_x u_x, v_y v_x, v_y u_y, u_x u_y, v_x u_i$ and $u_y u_i, 1 \leq i \leq n$, are the edges of J_n^* in G .

Define $f: V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ in the following manner.

$$f(w_i) = 2q - (2i - 1) \quad \forall 1 \leq i \leq r$$

$$= 2q - 2i + 1$$

$$f(v_x) = 2q - 2r - 1, \quad f(v_y) = 0$$

$$f(u_y) = 2q - 2r - 3$$

$$f(u_x) = 4$$

$$f(u_i) = (i + 1)f(u_x)$$

The labels of the vertices of G form a monotonically decreasing sequence $f(w_1), f(w_2), \dots, f(w_r), f(v_x), f(u_y)$ and also form increasing sequence $f(v_y), f(u_x), f(u_1), \dots, f(u_n)$. Thus the labels of edges are distinct odd numbers. Hence the graph G admits odd graceful labelings.

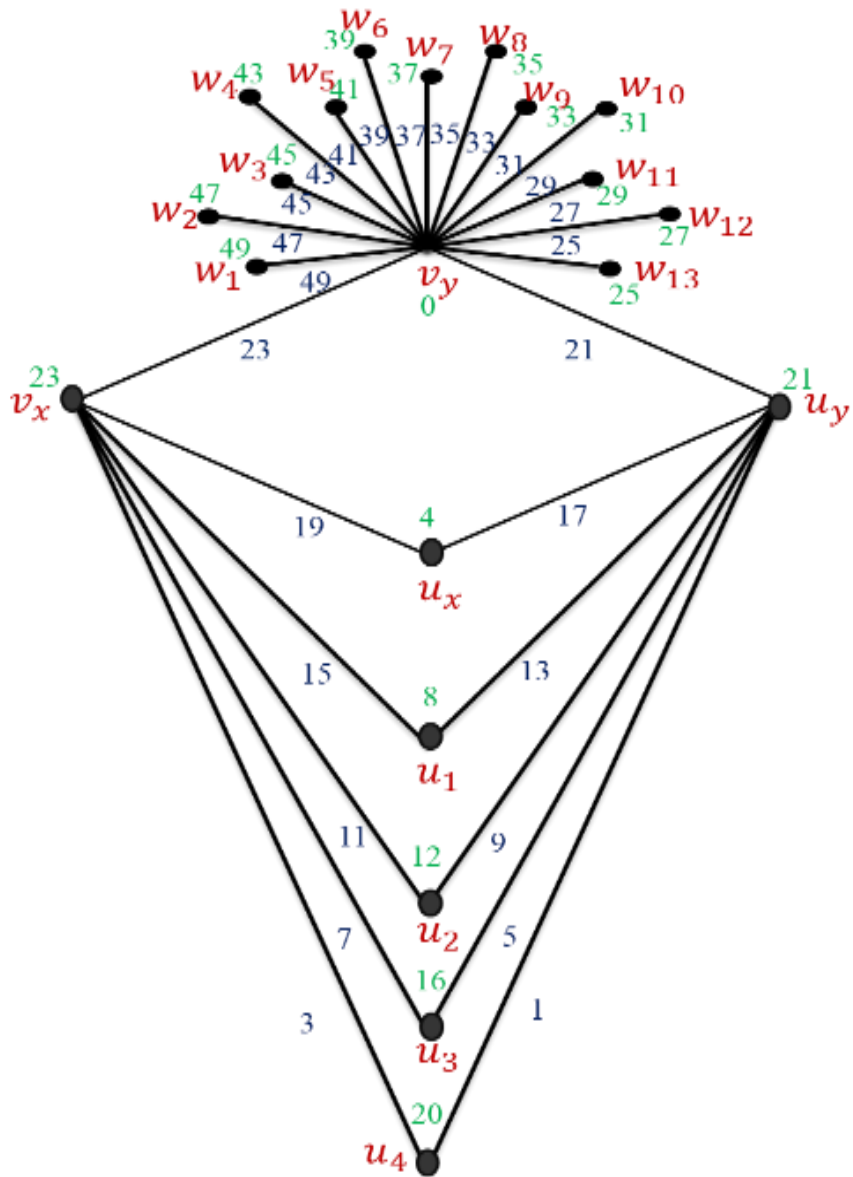


Figure 5.1.2.1 Odd graceful labeling of jewel graph without the prime edge J_4^* fused with star graph $K_{1,13}$

Theorem 5.1.3:

Arbitrary super subdivision of the Bistar graph is odd graceful.

Proof:

Let $G = B_{m,n}$ be a bistar with central vertices v_1 and v_2 such that the vertices v_{1i} , $1 \leq i \leq m$ and v_{2j} , $1 \leq j \leq n$ are the pendant vertices of the bistar that are adjacent to the central vertices v_1 and v_2 respectively. Replace the edge v_1v_2 by the complete bipartite graph $K_{2,r}$ (where r is any positive integer) and merge the 2-vertex part of $K_{2,r}$ with the end vertices of the edge v_1v_2 . Denote the vertices in the r -vertex part of $K_{2,r}$ as u_j , $1 \leq j \leq r$. Thus the arbitrary super subdivision graph of a bistar graph is denoted as $SSD(G)$. The graph $SSD(G)$ has $p = m + n + r + 2$ vertices and $q = m + n + 2r$ edges.

Define $f: V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ in the following manner.

$$f(v_1) = 0 ; \quad f(v_2) = 2$$

$$f(v_{1i}) = 2q - 2i + 1 \quad \forall 1 \leq i \leq m$$

$$f(u_1) = 2q - 2m - 1$$

$$f(u_j) = f(u_{j-1}) - 4 \quad \forall 2 \leq j \leq r$$

$$f(v_{2i}) = 2i + 1 \quad \forall 1 \leq i \leq n.$$

The labels of the vertices of $SSD(G)$ (except the central vertices) are all odd and form a monotonically decreasing sequence $f(v_{11}), f(v_{12}), \dots, f(v_{1m}), f(u_1), \dots, f(u_r), f(v_{2n}), f(v_{2n-1}), \dots, f(v_{21})$.

The labels of the central vertices $f(v_1)$ and $f(v_2)$ are 0 and 2 respectively. Thus, the labels of the vertices and edges of the graph $SSD(G)$ are distinct.

Thus, the graph $SSD(G)$ admits odd graceful labeling.

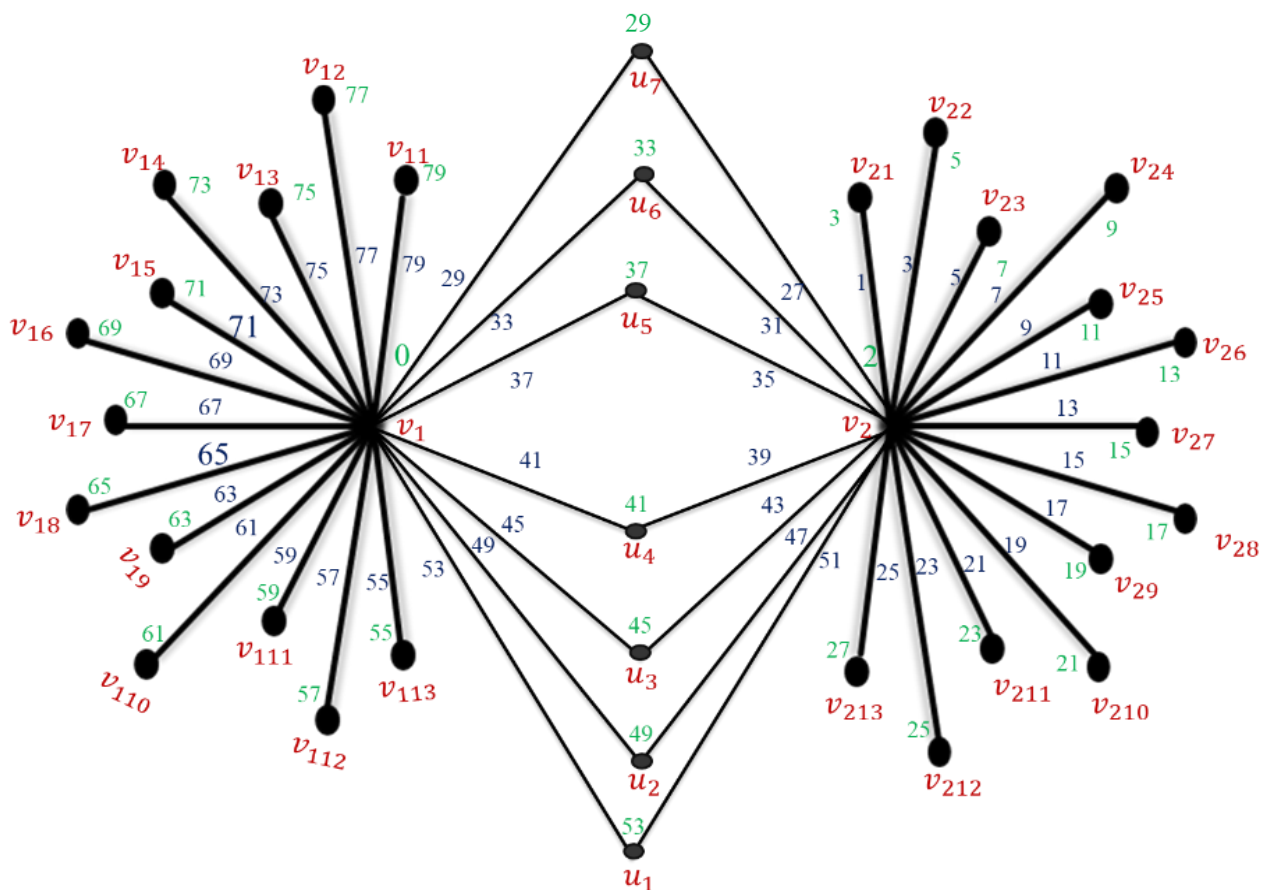


Figure 5.1.3.1 Odd graceful labeling of arbitrary super subdivision
of bistar graph $B_{13,13}$

Theorem 5.1.4:

Let G be a comb graph with $2n$ vertices, $n \geq 2$ then the shadow graph of comb $D(G)$ admits odd graceful labeling.

Proof:

Let G be a comb graph having the vertices $v_1, v_2, \dots, v_n$ of the path and also the pendant vertices of the path as $u_1, u_2, \dots, u_n$ respectively. Let the duplicate vertices of the vertices of path be $v'_1, v'_2, \dots, v'_n$ and also the duplicate vertices of the pendant vertices of path as $u'_1, u'_2, \dots, u'_n$. Hence the shadow graph is denoted by $D(G)$ with

$$\begin{aligned} |V(D(G))| &= p = 4n \text{ and } |E(D(G))| = q = 6n - 3 + (2n - 1) \\ &= 8n - 4 \end{aligned}$$

Define $f: V(D(G)) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ as follows.

$$f(v_1) = 0$$

$$f(v'_1) = f(v_1) + 2$$

$$= 2$$

$$f(v_i) = f(v_{i-2}) + 16 \quad i = 3, 5, \dots, n$$

$$f(v'_i) = f(v'_{i-2}) + 16 \quad i = 3, 5, \dots, n$$

$$f(v_2) = 2q - 9$$

$$f(v'_2) = f(v_2) - 4$$

$$f(v_i) = f(v_{i-2}) - 16 \quad i = 4, 6, \dots, n$$

$$f(v'_i) = f(v'_{i-2}) - 16 \quad i = 4, 6, \dots, n$$

$$f(u_1) = 2q - 1$$

$$f(u_2) = 8$$

$$f(u_i) = f(u_{i-2}) - 16 \quad i = 3, 5, \dots, n$$

$$f(u_i) = f(u_{i-2}) + 16 \quad i = 4, 6, \dots, n$$

$$f(u'_i) = f(u'_i) - 4 \quad i = 1, 3, \dots, n$$

$$f(u'_i) = f(u'_i) + 2 \quad i = 2, 4, \dots, n$$

From the above labeling it is clear that the labels of the vertices and the edges are distinct. Hence the graph $D(G)$ admits odd graceful labeling. Thus, the labels of the edges of $D(G)$ are distinct odd numbers. Hence the graph $D(G)$ is an odd graceful graph.

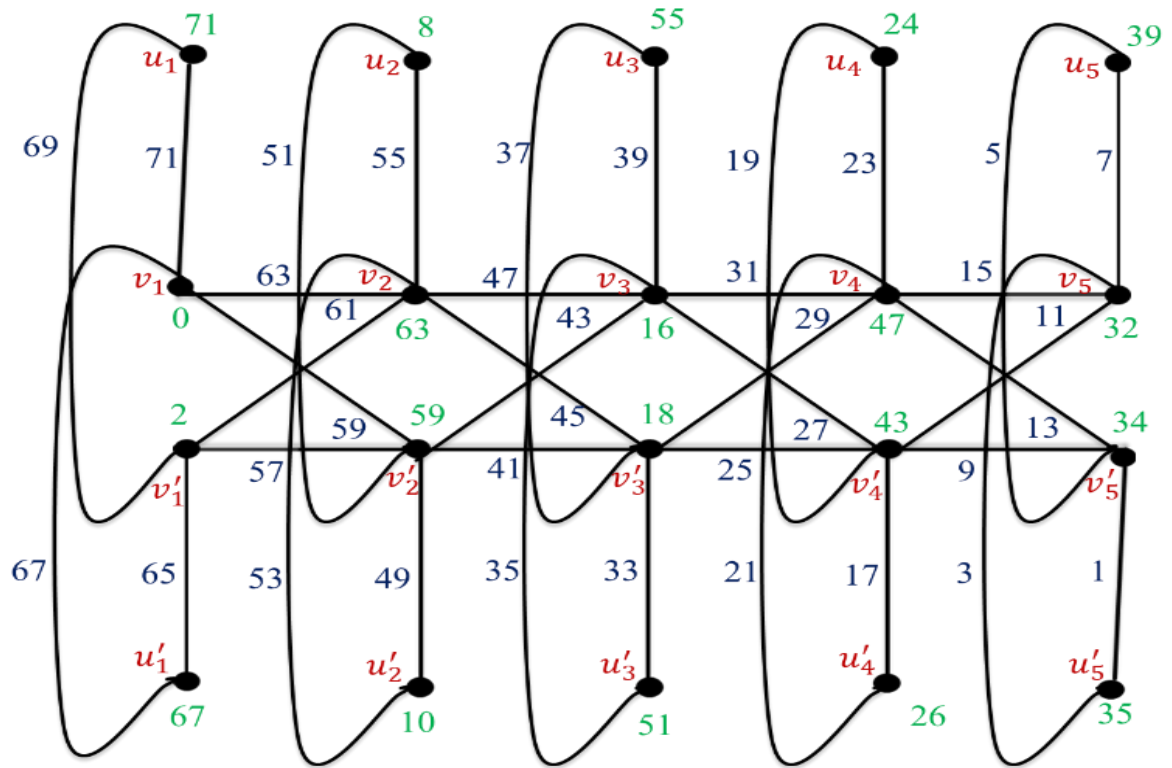


Figure 5.1.4.1 Odd graceful labeling of shadow of comb graph

Theorem 5.1.5:

Let G be a comb graph with $2n$ vertices, $n \geq 2$ then the splitting graph of comb $SP(G)$ admits odd graceful labeling.

Proof:

Let G be a comb graph having the vertices $v_1, v_2, \dots, v_n$ of the path and also the pendant vertices of the path as $u_1, u_2, \dots, u_n$. Let $v'_1, v'_2, \dots, v'_n$ denote the duplicate vertices of the path vertices $v_1, v_2, \dots, v_n$ and let $u'_1, u'_2, \dots, u'_n$ denote the duplicate vertices of the pendant vertices $u_1, u_2, \dots, u_n$ respectively. Thus, the splitting graph of comb, G is denoted as $SP(G)$ such that

$$\begin{aligned} |V[SP(G)]| = p = 4n \quad \text{and} \quad |E[SP(G)]| = q &= (2n - 1) + 2(2n - 1) \\ &= 2n - 1 + 4n - 2 \\ &= 6n - 3 \end{aligned}$$

Define $f: V[D(G)] \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ as follows

$$f(v_1) = 2q - 1$$

$$f(v_i) = f(v_{i-2}) - 12 \quad i = 3, 5, \dots, n$$

$$f(v'_1) = 2q - 3$$

$$f(v'_i) = f(v_i) - 2 \quad i = 3, 5, \dots, n$$

$$f(v_2) = 8$$

$$f(v'_2) = f(v_2) - 2$$

$$f(v_i) = f(v_{i-2}) + 12 \quad i = 4, 6, \dots, n$$

$$f(v'_i) = f(v_i) - 2 \quad i = 4, 6, \dots, n$$

$$f(u'_1) = 0$$

$$f(u'_i) = f(u'_{i-2}) + 12 \quad i = 3, 5, \dots, n$$

$$f(u'_2) = 2q - 9$$

$$f(u'_i) = f(u'_{i-2}) - 12 \quad i = 4, 6, \dots, n$$

$$f(u_i) = f(u'_i) + 2 \quad i = 1, 3, 5, \dots, n$$

$$f(u_i) = f(u'_i) + 2 \quad i = 1, 2, \dots, n$$

From the above labeling it is clear that the labels of the vertices and the edges are distinct. Hence the graph $SP(G)$ admits odd graceful labeling. Thus, the labels of the edges of $SP(G)$ are distinct odd numbers. Thus, the graph $SP(G)$ is an odd graceful graph.

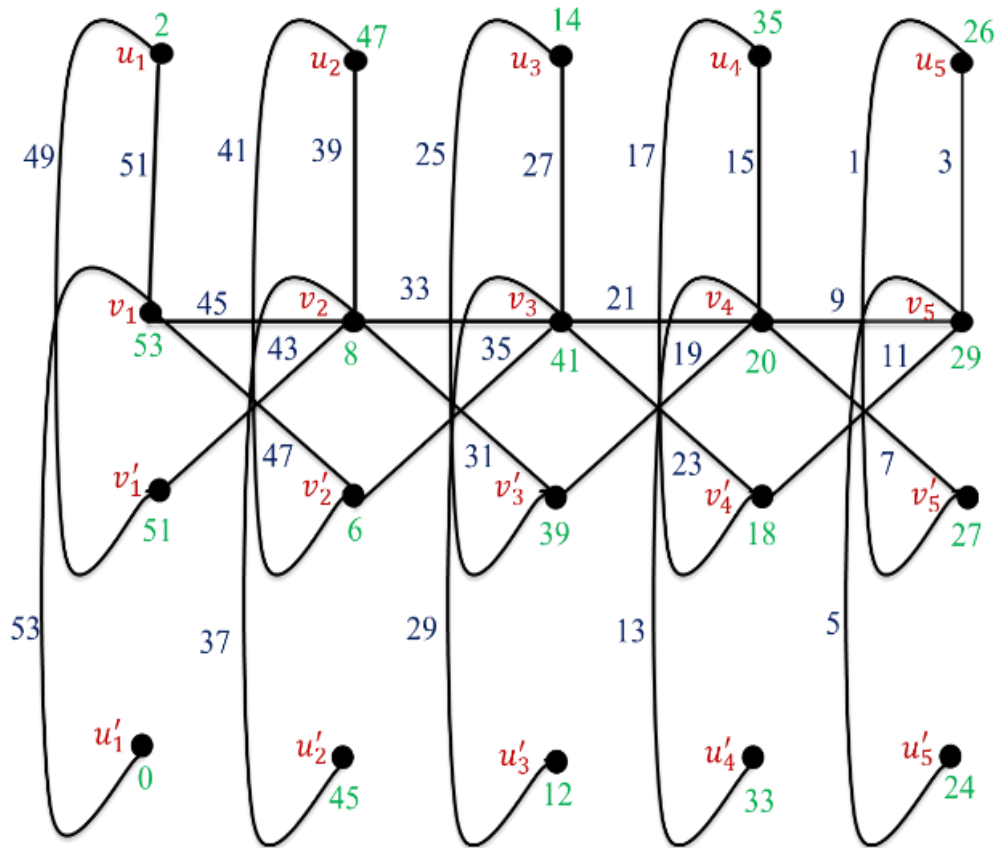


Figure 5.1.5.1 Odd graceful labeling of splitting of comb graph

CHAPTER 6

CONCLUSIONS

Graph labeling is one of the interesting and fascinating research areas in graph theory. Gnanajothi introduced a type of labeling is called an odd graceful labeling in 1991. She proved some families of graphs admit odd graceful labeling. She also observed that the class of odd graceful graphs are bipartite by showing a every graph with an odd cycle is not odd graceful. As every graph does not admit odd graceful labeling it is very interesting to identify the families of graph that admit odd graceful labeling. Inspired by the investigation of odd graceful graph, we studied a detailed literature survey on odd graceful graphs and investigated a new family of graphs using an operation which admits odd graceful labeling.

In this project work, we prove some new families of graphs such as an arbitrary super subdivided shell graph fused with star graph, the jewel graph without the prime edge fused with star graph and arbitrary super subdivision of bistar graph odd graceful labeling. Also, we show that the splitting graph and the shadow graph of comb admits odd graceful labeling. Investigating odd graceful graphs by using different operations techniques is an interesting problem that can be carried out as an open area of future research.

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