

ADJACENT VERTEX DISTINGUISHING TOTAL COLOURING OF A CERTAIN 3-REGULAR GRAPHS

Submitted in partial fulfillment of the requirements for the award of

Bachelor of Science Degree in Mathematics

By

R. SRIKANTH-40060016

K. SUMITHRA-40060017



**DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND HUMANITIES**

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

(DEEMED TO BE UNIVERSITY)

Accredited with Grade “A” by NAAC I 12B Status

by UGC I Approved by AICTE

JEPPIAAR NAGAR, RAJIV GANDHI SALAI,

CHENNAI - 600 119

APRIL-2023



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Jeppiaar Nagar, Rajiv Gandhi Salai, Chennai – 600 119

www.sathyabama.ac.in

DEPARTMENT OF MATHEMATICS

BONAFIDE CERTIFICATE

This is to certify that this project report is the bonafide work of **R. SRIKANTH (40060016)**, **K. SUMITHRA (40060017)** who carried out the project entitled "ADJACENT VERTEX DISTINGUISHING TOTAL COLOURING OF A CERTAIN 3-REGULAR GRAPHS" under our supervision from JANUARY 2023 to APRIL 2023.

Internal Guide

Dr. A. VELANKANNI

Head of the Department (Name in Capital letters with Seal & Signature)

Dr. M. NIRMALA, M.Sc., M.Phil., Ph.D.,
Head of the Department
Dept. of Mathematics
(Deemed to be University), Chennai-600 119.

Submitted for Viva voce Examination held on 09/05/2023

Internal Examiner

External Examiner

DECLARATION

We R. SRIKANTH (40060016), K. SUMITHRA (40060017) hereby declare that the Project Report entitled "ADJACENT VERTEX DISTINGUISHING TOTAL COLOURING OF A CERTAIN 3-REGULAR GRAPHS" done by we under the guidance of Dr. A. VELANKANNI (Internal) and at SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY submitted in partial fulfillment of the requirements for the award of Bachelor of Science Degree in Mathematics.

DATE: 02/05/2023

PLACE: Chennai

R. Srikanth.

K. Sumithra.

Signature of Candidates

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ABSTRACT

Let k be a positive integer. An adjacent vertex distinguishing (for short, AVD) total k -colouring Φ of a graph G is a proper total k -colouring of G such that no pair of adjacent vertices have the same set of colours, where the set of colours at a vertex v is $\{\phi(v)\} \cup \{\phi(e): e \text{ is incident to } v\}$. The minimum number of colours required for an adjacent vertex distinguishing total colouring of G is called the adjacent vertex distinguishing total chromatic number and is denoted by $\chi''_a(G)$. In 2005, Zhang et al. conjectured that every graph with maximum degree Δ has an AVD Total $(\Delta + 3)$ - Colouring. In this project, we have studied an adjacent vertex distinguishing total colouring of a certain 3-regular Graphs. In particular, we have proved that for any graph G , $\chi''_a(G) = 5$, where $G = C_n \equiv C_n$, for $n \geq 4$, the graph $G = C_n \equiv C_n$ is obtained by adding a set of independent n edges between the cycles of length n . Also, we have shown that for any graph G , $\chi''_a(G) = 5$, where $G = C_{2n}^*$ is obtained by joining n chords to the vertices of the cycle of length $2n$ and is called cycle with n chords.

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CHAPTER 1

1. INTRODUCTION

1.1 GRAPH THEORY

Graph theory is the branch of mathematics concerned with the study of graphs which are mathematical structures used to model pairwise relations between objects. In real life, many situations can be conveniently described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. Graph theory was first introduced in the 18th century by the Swiss Mathematician Leonhard Euler. His work on the famous “seven bridges of konigsberg problem”, is considered as the origin of graph theory.

A graph G is an ordered triple $(V(G), E(G), \Phi)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$ of edges and an incidence function that associates with each edge of G an unordered pair of vertices of G . If e is an edge and u and v are vertices such that $\Phi(e) = uv$ then e is said to join u and v . The vertices u and v are called the end vertices of e . Graph provides a helpful tool to quantify and simplify the many moving parts of dynamics systems.

Graphs can be used to model many types of relations and processes in physical, biological and information systems. It is used in social network connection, ranking hyperlinks in search engines and GPS maps to find the shortest route. In computer science, cybernetics uses graphs to represent networks of communication, data organization, computational devices and the flow of computation. Graph theory is useful in biology such as conservation efforts where a vertex can represent regions where certain species exist (or inhabit) and the edges represent migration paths or movement between the regions. This information is important when looking at

breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

1.2 APPLICATION OF GRAPH THEORY

1.2.1 DNA Sequencing:

DNA is the building block of life. It consists of two chains of molecules arranged in a double helix. Each link in the chain is one of four molecules called nucleotides. They are cytosine (C), guanine (G), adenine (A) or thymine (T).

DNA sequencing is the process of trying to work out what order these modules appear in the chain. Since these chains are very long it is necessary to split the chain up into smaller fragments. We can then sequence the fragments and put them back together. This becomes a problem that we don't know what order the fragments come in. Luckily, we can apply graph theory to solve this problem. When we have our fragments of DNA they can overlap. We can use this to help stitch them together. Assume our fragments (often called mers) are three molecules long (3-mers). For example, we might have the fragments: ATG, TGC, GTC etc. and assume they overlap by two molecules. This means the fragment ATG must be followed by a fragment beginning with TG eg. TGC

We can create a graph where every vertex is a fragment and there is an edge going from a vertex to another if they overlap by two letters. So the vertex ATG would have an edge connecting it to the vertex TGC. If we find a path that visits every vertex once (Hamiltonian path) we have found an ordering of the fragment that makes up the whole DNA sequence.

1.2.2 Web Pages:

Websites and web pages are represented in a graph. In web search engines such as Google, Yahoo and Bing, which helps to rank websites and makes it possible for Google to display the best result at the top. In web search engines it searches for web pages, where web pages are linked to each other on the internet via hyperlinks. In the graph, every vertex represents a separate webpage, and every edge represents a link (or hyperlink) between two pages. It is important to note that links only go one way, so this graph is directed and extremely large.

The algorithms PageRank and Googlebot are used to help the connectivity process. The graph connects web pages and sites, which helps to rank pages and enables google to display the best page.

1.3 GRAPH COLOURING

Graph Colouring is one of the most important and active research areas in graph theory. Francis Guthrie postulated the four colour conjecture noting that four colours were sufficient to colour the map so that no regions sharing a common border received the same color. Guthrie's brother passed on the question to his mathematics teacher Augustus De Morgan. Arthur Cayley raised the four colour problem at a meeting of the London Mathematical Society in 1879. In Four Colour Problem, consider a region of the map as a vertex and two regions sharing a common border in the map as an edge connecting the two corresponding vertices form a graph. A colouring problem that arose is referred to as vertex colouring of a graph.

Graph colouring is used in various research areas of computer science such as data mining, image segmentation, clustering, image capturing, networking, mobile radio frequency assignment, register allocation, pattern matching, designing seating plans and also solving sudoku puzzles. Graph colouring enjoys many practical applications as well as theoretical challenges.

1.4 PRELIMINARIES AND BASIC TERMINOLOGY

Graph

A graph is an ordered pair $G = (V, E)$ comprising

- i) V , a set of vertices
- ii) E , a set of edges which are unordered pairs of vertices.

Simple Graph

A graph having no self-loops and no parallel edges in it is called a simple graph.

Undirected Graph

A graph in which all the edges are not directed is called an undirected graph.

Example:



Figure 1.4.1 Undirected graph

Directed Graph

A graph in which all the edges are directed is called a directed graph. In other words, all the edges of a directed graph contain some direction. Directed graphs are also called digraphs.

Example:

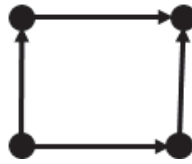


Figure 1.4.2 Directed graph

Degree of a vertex

In a Graph $G = (V, E)$, the number of edges incident to a vertex v of G is called the degree of a vertex and is denoted by $\deg(v)$. The maximum degree of a graph G is denoted by $\Delta(G)$, is the degree of the vertex with the greatest number of edges incident to it. The minimum degree of a graph G denoted by $\delta(G)$, is the degree of the vertex with the least number of edges incident to it.

Complete Graph

A graph in which exactly one edge is present between every pair of vertices is called a complete graph. A complete graph of ‘ n ’ vertices is represented as K_n .

Example:

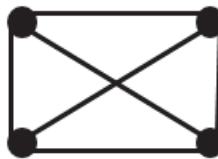


Figure 1.4.3 Complete graph K_4

Regular Graph

A graph in which the degree of all the vertices are the same is called a regular graph. If all the vertices in a graph are of degree ‘ k ’, then it is called a “ k -regular graph”.

Example:

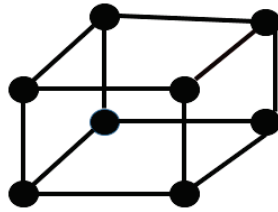


Figure 1.4.4 Hypercube graph

Petersen's Theorem:

Every Cubic, bridgeless graph contains a perfect matching.

Theorem:

Every Cubic graph has an even number of vertices.

Connected Graph

A graph in which at least one path exists between every pair of vertices is called a connected graph.

Example:

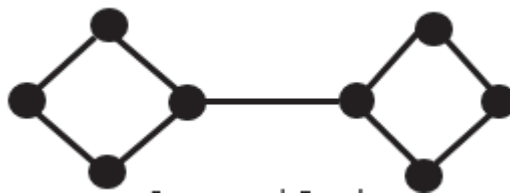


Figure 1.4.5 Connected graph

Walk

A walk of the graph G is an alternating sequence of vertices and edges such that it begins and ends with vertices.

Trail and Path

A walk is called a trail if all edges are distinct. A trail is called a Path if all its vertices are distinct.

Cycle

A closed path of a graph is called a cycle. A cycle of length n is denoted by C_n .

Disconnected Graph

A graph in which there does not exist any path between at least one pair of vertices is called a disconnected graph.

Example:



Figure 1.4.6 Disconnected graph

Acyclic Graph

Acyclic graph is a graph having no cycles.

Example:

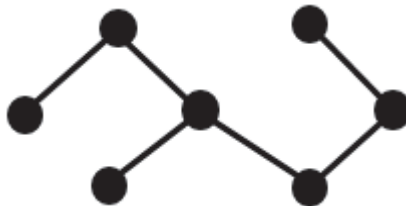


Figure 1.4.7 Acyclic graph

Bipartite Graph

A graph G is called a bigraph or bipartite graph if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a vertex of V_1 to a vertex of V_2 . (V_1, V_2) is called a bipartition of G .

Example:

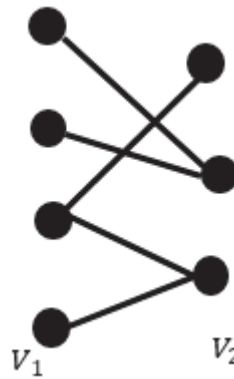


Figure 1.4.8 Bipartite graph with bipartition (V_1, V_2)

Planar Graph

A planar graph is a graph that we can draw in a plane such that no two edges of it cross each other.

Example:

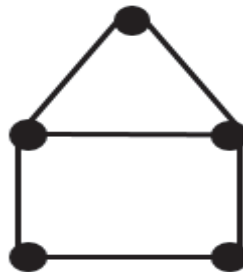


Figure 1.4.9 Planar graph

Independent Set

A subset S of V is called an independent set of G if no two vertices of S are adjacent in G . An independent set S is maximum if G has no independent set S' with $|S'| > |S|$.

Matching

A subset M of E is called a matching in G if no two edges are adjacent in M ; the two end vertices of an edge in M are said to be matched under M . A matching M saturates a vertex v then the vertex v is said to be M - saturated vertex, if some edge of M is incident with v ; otherwise, the vertex v is called M -unsaturated.

Perfect matching

A Perfect Matching in a graph is a matching that covers every vertex of the graph. More formally, given a graph $G = (V, E)$, a perfect matching in G is a subset M of edge set E , such that every vertex in the vertex set V is adjacent to exactly one edge in M .

Vertex Colouring

Vertex colouring is the assignment of colours to each vertex of a graph G such that no adjacent vertices receive the same color. The smallest number of colours required to colour a graph G is called the chromatic number and is denoted by $\chi(G)$.

Example:

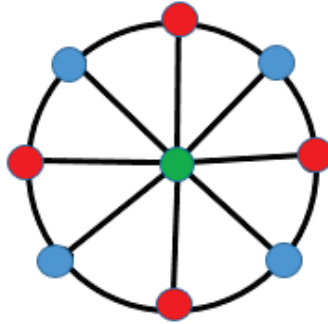


Figure 1.4.10 Vertex colouring of wheel graph

Brooks' theorem:

If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.

Edge Colouring

A k -edge colouring of G is an assignment of k colours to the edges of G so that any two adjacent edges must receive different colours. If G has a k -edge colouring, then G is said to be k -edge colorable. The edge chromatic number of G is denoted by $\chi'(G)$, is the minimum number k for which G is k -edge colorable.

Example:

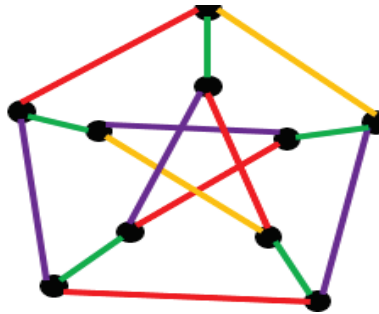


Figure 1.4.11 Edge colouring of Petersen graph

Vizing's Theorem:

If G is any simple graph, then either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$.

Total Colouring

Total colouring is a colouring of both the vertices and edges of a graph such that no adjacent vertices, no adjacent edges and no vertex with its incident edge are assigned the same color. The total chromatic number $\chi''(G)$ of a graph G is the minimum colours needed in any total colouring of G .

Example:

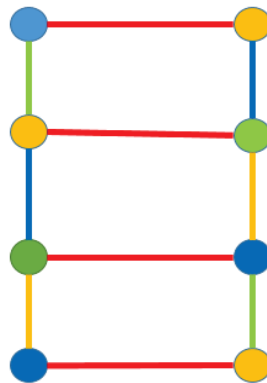


Figure 1.4.12 Total colouring of ladder graph

Total Colouring Conjecture:

If G is any simple graph, then $\chi''(G) = \Delta(G) + 2$.

Adjacent Vertex Distinguishing Total Colouring

An adjacent vertex distinguishing total colouring of a graph G is a proper total colouring of G such that no pair of adjacent vertices have the same set of colours, where the set of colours at a vertex v is $\{\phi(v)\} \cup \{\phi(e) : e \text{ is incident to } v\}$. The minimum number of colours required for an adjacent vertex distinguishing total colouring of G is called the adjacent vertex distinguishing total chromatic number and is denoted by $\chi''_{at}(G)$.

Example:

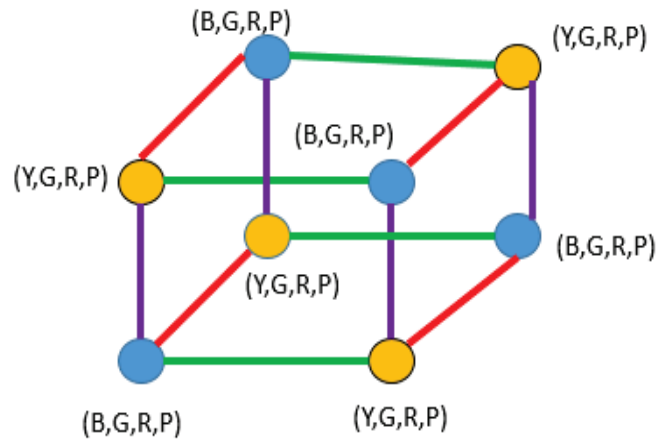


Figure 1.4.13 Adjacent vertex distinguishing total colouring of hypercube graph

CHAPTER 2

2.1 LITERATURE SURVEY

Adjacent vertex distinguishing total colouring is a total colouring with an additional property that for any adjacent vertices u and v , the set of colours used on the edges incident on u including the colour of u is different from the set of colours used on the edges incident on v including the colour of v . The adjacent vertex distinguishing total chromatic number $\chi''_a(G)$ of a graph G is the fewest colours needed in an adjacent vertex distinguishing total colouring of G . In 2005, Zhang et al. conjectured that for any graph G , $\chi''_a(G) \leq \Delta(G) + 3$, where $\Delta(G)$ is the maximum degree of the graph G . The following lower bound for the adjacent vertex distinguishing total chromatic number can be obtained from the definition that if a simple graph G has two adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta(G) + 2$.

Weifan Wang, Danjun Huang [19] have characterized completely the adjacent vertex distinguishing total chromatic number of planar graphs G with large maximum degree Δ by showing that if $\Delta \geq 14$, then $\Delta + 1 \leq \chi''_a(G) \leq \Delta + 2$ and $\chi''_a(G) = \Delta + 2$ if and only if G contains two adjacent vertices of maximum degree. Weifan Wang, Yiqiao Wang [20] have shown that if G is a graph with $mad(G) < 3$ and $\Delta(G) \geq 5$ then $\Delta + 1 \leq \chi''_a(G) \leq \Delta + 2$ and $\chi''_a(G) = \Delta + 2$ if and only if G contains two adjacent vertices of maximum degree. Further, they also proved that (i) if G is a graph with $mad(G) < 3$ and $\Delta(G) \leq 4$ then $\chi''_a(G) \leq 6$, (ii) if G is a graph with $mad(G) < \frac{8}{3}$ and $\Delta(G) \leq 3$ then $\chi''_a(G) \leq 5$, where $mad(G)$ denotes maximum average degree of a graph.

In 2012, Danjun Huang, Weifan Wang and Chengchao Yan [6] have shown that $\chi''_a(G) \leq 2\Delta(G)$, for any graph G with maximum degree $\Delta \geq 3$. Xiaohan Cheng, Guanghui Wang and Jian-Liang Wu [22] have shown that if G is a planar graph with $\Delta \geq 10$, then $\chi''_a(G) \leq \Delta(G) + 3$, using the approach of discharging method and combinatorial nullstellensatz. Yulin Chang, Jie Hu, Guanghui Wang and Xiaowei Yu [26] have proved for every planar graph with $\Delta(G) \geq 8$, then $\chi''_a(G) \leq \Delta(G) + 3$.

In 2005, Zhang et al [28] determined the adjacent vertex distinguishing total chromatic number of some graphs such as cycle, complete graph, complete bipartite graph, fan, wheel and tree. In 2008, Xiang'en Chen [21] proved that for any graph G with maximum degree $\Delta(G) = 3$, the adjacent vertex distinguishing total chromatic number is at most 6. Yiqiao Wang, Weifan Wang [23] have characterized completely the adjacent vertex distinguishing total chromatic number of outerplanar graphs. AVD Total Colouring Conjecture is true for many corona product like generalized, simple and l -coronas of graphs was proved by Hanna Furmanczyk, Rita Zuazua [7]. Papaioannou and Raftopoulou [12] have described an algorithmic procedure that gives an adjacent vertex distinguishing total coloring of any 4-regular graph with seven colors.

Meirun Chen, Xiaofeng Guo [11] have proved that $\chi'_{av}(Q_n) = \Delta(Q_n) + 1$, for $n \geq 3$ and $\chi''_{at}(Q_n) = \Delta(Q_n) + 2$, for $n \geq 2$, where Q_n denotes the hypercube graph of dimension n . Jie Hu, Guanghui Wang, Jianliang Wu, Donglei Yang, Xiaowei Yu [9] have proved that any planar graph with maximum degree at least 10 has an AVD total $(\Delta + 2)$ -colouring and the bound $\Delta + 2$ is sharp. Atilio G. Luiz, C. N. Campos, C. P. de Mello [1] has shown that for any complete equipartite graph G of even order, then $\chi''_a(G) \geq \Delta(G) + 2$. Jonathan Hulan [10] has given concise

proofs for an upper bound on the adjacent vertex distinguishing total chromatic number of graphs of maximum degree three. K. Thirusangu, R. Ezhilarasi [16] have proved the exact value of the AVD total chromatic number of certain families of graphs like quadrilateral snake, double quadrilateral snake, alternate quadrilateral snake and double quadrilateral snake.

Atilio G. Luiz, C. N. Campos, C. P. de Mello, J. Hulgan [2] have verified the Hulgan's conjecture that any simple graph G with maximum degree three has adjacent vertex distinguishing total chromatic number is at most 5. Further, they also determined the exact value of $\chi''_a(G)$ for the following families of snarks, that is, the flower snarks, generalized Blanuša snarks and LP_l snarks. Shaily Verma, Hung - Lin Fu, B. S Panda [14] have studied the adjacent vertex distinguishing total coloring conjecture is true for split graphs and also for the certain classes of split graphs. K.Thirusangu and R.Ezhilarasi [17] have shown that the exact value of the adjacent vertex distinguishing total chromatic number of line graph of Fan graph F_n , Double Star graph $K_{l,n,n}$, Friendship graph F^n and also for the splitting graph of Path P_n , Cycle C_n , Star $K_{l,n}$ and Sunlet graph S^{2n} .

CHAPTER 3

3 MAIN RESULTS

Observation: If G is a graph with at least two adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta(G) + 2$.

3.1 Adjacent vertex distinguishing total colouring of a graph $G = C_n \equiv C_n$

Theorem: For a graph G , $\chi''_a(G) = 5$, where $G = C_n \equiv C_n$, for $n \geq 4$.

Proof: Consider the graph $G = C_n \equiv C_n$, for $n \geq 4$. By definition of the graph $G = C_n \equiv C_n$, it is a 3-regular graph and has a set of independent n edges. The independent edges of the graph G forms perfect matching. By the structural property of the graph G , removal of a perfect matching edges results in two disjoint cycles of length n . We define an adjacent vertex distinguishing total colouring of the G as follows.

Colouring Procedure:

The colouring procedure is divided into two cases depending on whether n is even or odd.

Case 1: n is even

For the case n is even, the graph G is bipartite. It is enough to colour the vertices of the graph G using two colours say c_1 and c_2 alternately such that the adjacent vertices received distinct colours.

Since G has a perfect matching, we assign the colour c_5 to the set of independent 'n' edges. As the removal of perfect matching edges from the graph G results into two disjoint cycles of even length. Assign the colours c_3 and c_4 alternately to the edges of each cycle such that the adjacent edges received distinct colours.

By the above colouring procedure, each vertex in the graph G received the colour either c_1 or c_2 . Suppose the vertex received the colour c_1 then its incident edges have received the colours c_3, c_4 and c_5 . Suppose the vertex received the colour c_2 then its incident edges have received the colours c_3, c_4 and c_5 by the above defined colouring procedure. Thus, the vertex and its incident edges received distinct colours.

Further, suppose any two adjacent vertices say v_i and v_j in the graph G . The colour class of the vertices v_i and v_j are either $\{c_1, c_3, c_4, c_5\}$ or $\{c_2, c_3, c_4, c_5\}$ by the above colouring procedure. Suppose the vertex v_i has the colour class $\{c_1, c_3, c_4, c_5\}$ and its adjacent vertex v_j has the colour class $\{c_2, c_3, c_4, c_5\}$ and vice versa. The adjacent vertices received different colour classes. Hence the colouring procedure is a proper adjacent vertex distinguishing total colouring using 5 colours.

Illustration:

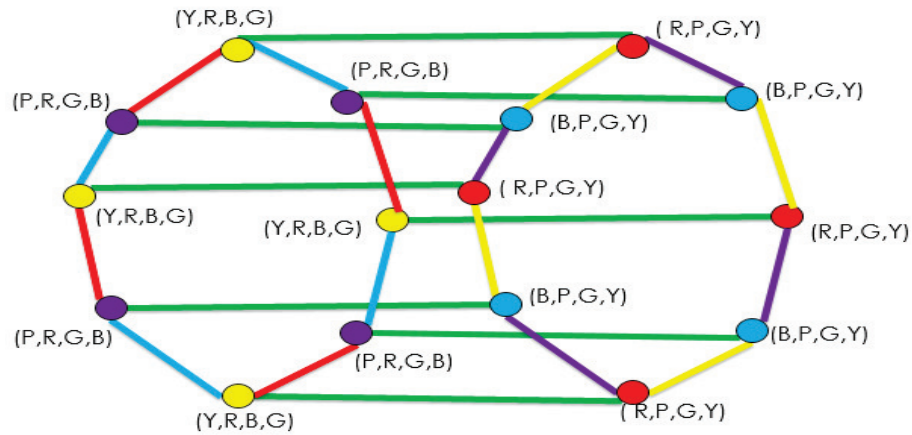


Figure 3.1.1 Adjacent vertex distinguishing total colouring of a graph $G = C_8 \cong C_8$

Case 2: n is odd

For the case n is odd, the graph G is non-bipartite. Assume the label of the vertices of one odd cycle be $u_1, u_2, \dots, u_n$ and label the vertices of another odd cycle in the graph G be $v_1, v_2, \dots, v_n$. Assign the colours say c_1 and c_2 to the vertices $u_1, u_2, \dots, u_{n-1}$ alternately and assign the colours say c_3 and c_4 to the vertices $v_1, v_2, \dots, v_{n-1}$ alternately and also assign the last vertices u_n and v_n the colour c_4 and c_2 respectively in the graph G such that the adjacent vertices received distinct colours. Since G has a perfect matching, we assign the colour c_5 to the set of independent 'n' edges. As the removal of perfect matching edges from the graph G results into two disjoint cycles of odd length. Assign the colours c_1, c_2, c_3 and c_4 to the edges $u_{n-1}u_n, u_1u_n, v_{n-1}v_n$ and v_1v_n respectively in the graph G . Assign the colours c_3 and c_4 alternately to the edges of path $u_1, u_2, \dots, u_{n-1}$ and also assign the colours c_1 and c_2 alternately to the edges of path $v_1, v_2, \dots, v_{n-1}$ in the graph G such that the adjacent edges received distinct colours. Thus the above colouring is a proper total colouring.

Further, the colour class of the vertices u_1 and u_{n-1} are $\{c_1, c_2, c_3, c_5\}$ and the colour class of the vertex u_n is $\{c_1, c_2, c_4, c_5\}$. Similarly, the colour class of the vertices v_1 and v_{n-1} are $\{c_1, c_3, c_4, c_5\}$ and the colour class of the vertex v_n is $\{c_2, c_3, c_4, c_5\}$. The colour class of the vertex u_i of the path $u_2, \dots, u_{n-1}$ is either $\{c_2, c_3, c_4, c_5\}$ or $\{c_1, c_3, c_4, c_5\}$ when i is even or odd. Similarly, The colour class of the vertex v_i of the path $v_2, \dots, v_{n-1}$ is either $\{c_1, c_2, c_4, c_5\}$ or $\{c_1, c_2, c_3, c_5\}$ when i is even or odd. By the above colouring procedure, adjacent vertices received different colour classes. Hence the colouring procedure is a proper adjacent vertex distinguishing total colouring using 5 colours. Hence $\chi''_a(G) \leq 5$, where $G = C_n \equiv C_n$, for $n \geq 4$.

By Observation, it is clear that $\chi''_a(G) \geq 5$, where $G = C_n \equiv C_n$, for $n \geq 4$. Therefore, the adjacent vertex distinguishing total chromatic number of the graph $G = C_n \equiv C_n$ is 5, that is, $\chi''_a(G) = 5$, where $G = C_n \equiv C_n$, for $n \geq 4$.

Illustration:

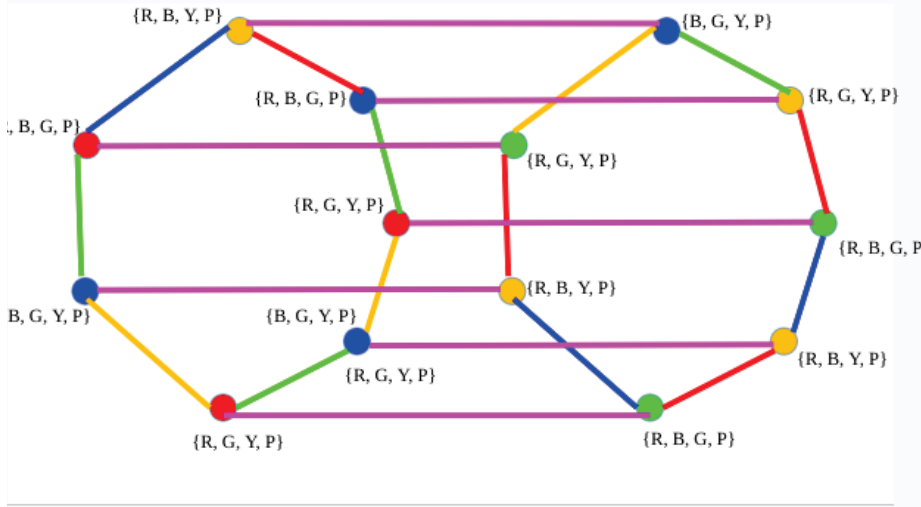


Figure 3.1.2 Adjacent vertex distinguishing total colouring of a graph $G = C_7 \equiv C_7$

3.2 Adjacent vertex distinguishing total colouring of a graph $G = C_{2n}^*$

Theorem 2: For a graph G , $\chi''_a(G) = 5$, where $G = C_{2n}^*$, for $n \geq 3$.

Proof: Consider the graph $G = C_{2n}^*$, for $n \geq 3$. By definition of the graph $G = C_{2n}^*$, it is a 3-regular bipartite graph and has a set of independent n edges. The independent edges of the graph G forms perfect matching. By the structural property of the graph G , removal of a perfect matching edges results into a cycle of length $2n$. We define an adjacent vertex distinguishing total colouring of the G as follows.

Colouring Procedure:

Since the graph G is bipartite. It is enough to colour the vertices of the graph G using two colours say c_1 and c_2 alternately such that the adjacent vertices received distinct colours.

Since G has a perfect matching, we assign the colour c_5 to the set of independent 'n' edges. As the removal of perfect matching edges from the graph G results in a cycle of even length. Assign the colours c_3 and c_4 alternately to the edges of an even cycle such that the adjacent edges received distinct colours.

By the above colouring procedure, each vertex in the graph G received the colour either c_1 or c_2 . Suppose the vertex received the colour c_1 then its incident edges have received the colours c_3, c_4 and c_5 . Suppose the vertex received the colour c_2 then its incident edges have received the colours c_3, c_4 and c_5 by the above defined colouring procedure. Thus, the vertex and its incident edges received distinct colours.

Illustration:

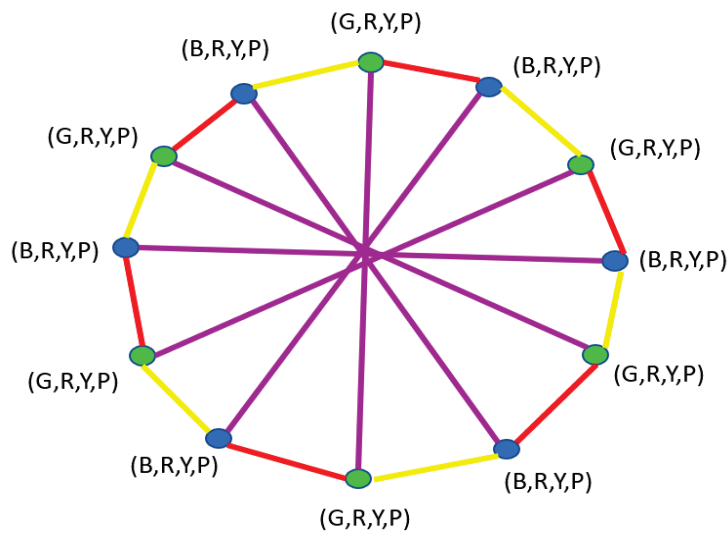


Figure 3.2.1 Adjacent vertex distinguishing total colouring of a graph $G = C_{12}^*$

Further, suppose any two adjacent vertices say v_i and v_j in the graph G . The colour class of the vertices v_i and v_j are either $\{c_1, c_3, c_4, c_5\}$ or $\{c_2, c_3, c_4, c_5\}$ by the above colouring procedure. Suppose the vertex v_i has the colour class $\{c_1, c_3, c_4, c_5\}$ and its adjacent vertex v_j has the colour class $\{c_2, c_3, c_4, c_5\}$ and vice versa. Thus the adjacent vertices received different colour classes. Hence the colouring procedure is a proper adjacent vertex distinguishing total colouring using 5 colours. Hence $\chi''_a(G) \leq 5$, where $G = C_{2n}^*$, for $n \geq 3$.

By Observation, it is clear that $\chi''_a(G) \geq 5$, where $G = C_{2n}^*$, for $n \geq 3$. Therefore, the adjacent vertex distinguishing total chromatic number of the graph $G = C_{2n}^*$ is 5, that is, $\chi''_a(G) = 5$, where $G = C_{2n}^*$, for $n \geq 3$.

CHAPTER 4

4.1 APPLICATION OF GRAPH COLOURING

4.1.1 Network Task Efficiency:

Suppose we have a network of computers, servers, facilities, or any group of stations, which can be called nodes and these nodes are connected to other nodes in the network via physical conductors such as wires or pipelines. A time comes when every facility and every connection must undergo a certain task such as an update, maintenance or test to measure its quality or security. Whenever this task is performed on a particular node or connection, that object must be shut down and remain offline until the task is finished. Each task takes the same amount of time, regardless of the node or connection. The administration wishes to get these tasks completed as soon as possible, while adhering to the following rules,

1. No two adjacent nodes can be offline at the sametime.
2. When a node is operational, at most one of its connections can be offline at any given time.
3. When a node is offline, all of its connections are required to remain operational to support the network as a whole.

The problem is to schedule the tasks for all these nodes and connections so that the process takes the least amount of time. The solution is to represent nodes with vertices and connections with edges, and properly total colour the graph using the fewest colour, where each colour represents the interval of time in which an element will be offline as it undergoes the required task.

4.1.2 Scheduling:

Suppose we want to make an examination schedule for a university. We have listed different subjects and students enrolled in every subject. Many subjects would have common students (of the same batch, some backlog students, etc). How do we schedule the examination so that no two exams with a common student are scheduled at the same time? How many minimum time slots are needed to schedule all examinations? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices means there is a common student. Hence, this is a graph colouring problem where the minimum number of time slots is equal to the chromatic number of the graph.

CHAPTER 5

CONCLUSION

Graph colouring problem is one of the interesting and significant research areas in graph theory. In this project, we studied the adjacent vertex distinguishing total colouring and its literature survey. Inspired by the Zhang et al. conjecture, which states that for any graph G , $\chi''_a(G) \leq \Delta(G) + 3$, where $\Delta(G)$ is the maximum degree of the graph G , we have determined the adjacent vertex distinguishing total chromatic number for certain classes of 3-regular graphs. More precisely, we have proved that for any graph G , $\chi''_a(G) = 5$, where $G = C_n \equiv C_n$, for $n \geq 4$, the graph $G = C_n \equiv C_n$ is obtained by adding a set of independent n edges between the cycles of length n . Also, we have shown that for any graph G , $\chi''_a(G) = 5$, where $G = C_{2n}^*$, for $n \geq 3$, the graph $G = C_{2n}^*$ is obtained by joining n chords to the vertices of the cycle of length $2n$.

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