

A STUDY ON 1-HARMONIOUS COLORING OF CERTAIN CLASSES OF GRAPHS

Submitted in partial fulfillment of the requirements for the award of
Bachelor of science degree in Mathematics

By

J SHANMUGAPRIYA (40060014) and R SIVAGAMA SUNDARI (40060015)



**DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND HUMANITIES**

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

(DEEMED TO BE UNIVERSITY)

Accredited with Grade "A" by NAAC | 12B Status

by UGC | Approved by AICTE

**JEPPIAAR NAGAR, RAJIV GANDHI SALAI,
CHENNAI - 600 119**

APRIL – 2023



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Jeppiaar Nagar, Rajiv Gandhi Salai, Chennai – 600 119

www.sathyabama.ac.in

DEPARTMENT OF MATHEMATICS

BONAFIDE CERTIFICATE

This is to certify that this project report is the bonafide work of **J SHANMUGAPRIYA (40060014)** and **R SIVAGAMASUNDARI (40060015)** who carried out the project entitled "A STUDY ON 1-HARMONIOUS COLORING OF CERTAIN CLASSES OF GRAPHS" under our supervision from **JANUARY 2023 to APRIL 2023**

Internal Guide

Dr. Franklin Tamil Selvi M S

Head of the Department (Name in Capital letters with Seal & Signature)

Dr. M. NIRMALA, M.Sc., M.Phil., Ph.D.,

Head of the Department

Dept. of Mathematics

Sathyabama Institute of Science and Technology

(Deemed to be University), Chennai-600 119.

Dr. M. Nirmala

Submitted for Viva voce Examination held on 09/05/2023

Internal Examiner

External Examiner

DECLARATION

We **J SHANMUGAPRIYA (40060014)** and **R SIVAGAMA SUNDARI (40060015)** hereby declare that the Project Report entitled "**A STUDY ON 1-HARMONIOUS COLORING OF CERTAIN CLASSES OF GRAPHS**" done by us under the guidance of **Dr. FRANKLIN THAMIL SELVI M S** (Internal) at **SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY** submitted in partial fulfillment of the requirements for the award of Bachelor of Science degree in Mathematics.

J. Shanmugapriya
R. Sivagama Sundari

DATE: 29/04/2023

PLACE: CHENNAI

SIGNATURE OF THE
CANDIDATES

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ABSTRACT

Graph theory is generally regarded as utmost pleasurable branch of Discrete Mathematics. This is because it has binary nature; it has the brilliant attestations in all the abstract logic, and it has a lot of operations in every field. Graph coloring is a major content of graph proposition with numerous operations and unsolved problems. One such coloring is 1-Harmonious coloring and it is defined as a vertex coloring such that the color pairs of end vertices of every edge are different only for adjacent edges. In this project we have obtained the 1-Harmonious chromatic number of certain classes of graphs like Pan graph, Lollipop graph, Central graph of Cycle graph, Star graph, Pan graph and Lollipop graph.

KEYWORDS

Vertex coloring, Chromatic number, Harmonious coloring, 1-Harmonious coloring, Central graph.

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CHAPTER – 1

INTRODUCTION

In discrete mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices which are connected by edges. Graph theory is considered to have begun 1736 with the publication of Euler's solution of the Königsberg bridge problem. Graph theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science, statistics, operations research, management sciences and the life sciences.

In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation etc. In google maps, various locations are represented as vertices or nodes and the roads are represented as edges and graph theory is used to find the shortest path between two nodes. Graphs, especially trees and binary trees, are used in the representation of data structures. The link structure of a website could be represented by a directed graph: the vertices are the web pages available at the website and a directed edge from page A to page B exists if and only if A contains a link to B. A similar approach can be taken to problems in travel, computer chip design and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. In Electrical Engineering, graph theory is used in designing of circuit connections. Graph theory is also used in sociology. Graph theory provides many useful applications in operational research like: Minimum cost path, A Scheduling problem. Today using graph labeling, the numbered graphs (the labeled graphs) serve as useful models for a broad range of applications such as coding theory, X-ray, crystallography, radar, robotics, astronomy, frequency assignment in mobile and radio network etc.

1.1 BASIC CONCEPTS

An ordered pair of vertices $V(G)$ and edges $E(G)$ is a graph. A sub-graph H of G consists of $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced sub-graph of G consists of few vertices $S \subseteq V(G)$, which contains all the edges with both end points in S . The number of neighbors incident with v is the degree of a vertex in G , which denotes $d(v)$. If $d(v) = r$, then G is r -regular for each $v \in V(G)$. A pendent vertex v is a vertex with $d(v) = 1$ whereas pendent edge e is an edge which is associated with pendent vertex. A single vertex v is an isolated vertex. $\Delta(G)$ and $\delta(G)$ are symbolized as the largest and the smallest degree of a vertex in G , respectively.

An edge with identical end is a loop and if two or more edges of G have the same end vertices, then these edges are parallel edges. A graph with no loops and parallel edges is called a simple graph.

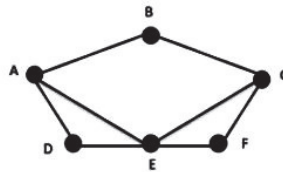


Figure 1.1 Simple Graph

The graphs are basically of two types, directed and undirected. In graph theory, a directed graph is a graph made up of a set of vertices connected by edges, in which the edges have a direction associated with them. The undirected graph is defined as a graph where the set of node are connected together, in which all the edges are bidirectional.

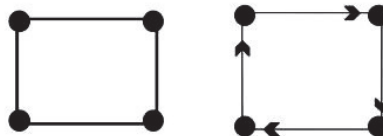


Figure 1.2 Undirected and Directed graph

A path is a set of different vertices $u_0 u_1 \dots u_n$, where u_k and u_{k+1} are adjacent $\forall k = 0, 1, 2, \dots, n$. The walk is referred as path, when the vertices of a walk are distinct. It is a cycle if the path is closed. A trial is a walk that does not have repeated edge. A complete graph is a graph in which each pair of vertices is connected by an edge. The complete graph with n vertices is symbolized by K_n .



Figure 1.3 Complete graph K_5 and K_6

A cycle is a path that originates at one vertex and terminates at the same vertex. cycle with n vertices is denoted as C_n . Total number of edges in a cycle graph is n .



Figure 1.4 Cycle graph C_3 and C_4

A simple graph G is said to be bipartite if its vertex set V can be divided into two disjoint sets such that every edge in G has its initial vertex in the first set and the terminal vertex in the second set. A graph G is connected if any two vertices are joined by a path. A cut vertex is a single vertex whose removal disconnects a graph. An edge e is called a cut edge of G if $G - e$ has more components than G . That is, a single edge whose removal disconnects a graph.

A connected graph with no cycles is called a tree. Tree is a discrete structure that represents hierarchical relationships between individual elements or nodes. A tree in which a parent has no more than children is called a binary tree. A tree is a connected

acyclic undirected graph. There is a unique path between every pair of vertices in G . A tree with N number of vertices contains $(N - 1)$ number of edges. The vertex which is of 0 degree is called root of the tree. The vertex which is of 1 degree is called leaf node of the tree and the degree of an internal node is at least 2. (Bondy J.A (1976), Douglas B (2006), Harary F (2001)) are referred for graph theoretic terms.

CHAPTER – 2

GRAPH COLORING

In graph theory, graph coloring is a special case of graph labelling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color; this is called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges are of the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

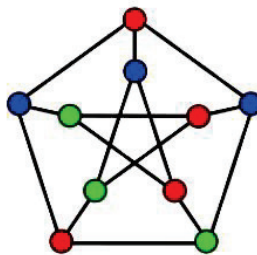


Figure 2.1 Vertex coloring of Petersen graph $G(10, 15)$

A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices. The most common type of vertex coloring seeks to minimize the number of colors for a given graph. Such a coloring is known as a minimum vertex coloring, and the minimum number of colors which the vertices of a graph G may be colored is called the chromatic number, denoted $\chi(G)$. A k -coloring is an assignment of k colors to the vertices of a graph. A k -coloring is said to be proper if adjacent vertices do not receive the same color. The chromatic number of G is the minimal number of colors necessary to appropriately color the vertices of a graph G , and it is indicated by $\chi(G)$. For example, $\chi(G) = 3$ for the graph in Figure 2.2.

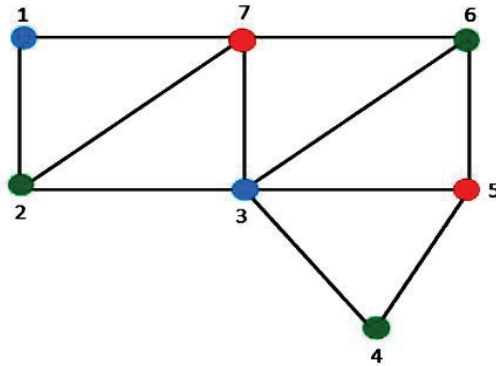


Figure 2.2 A 3-chromatic graph

There are various interesting types of vertex coloring other than proper coloring. A type of graph coloring where colors are assigned to the vertices such that each pair of colors appears together on at most one edge is called Harmonious coloring. The minimum number of coloring used in such a coloring is known as harmonious chromatic number and it is denoted by $\chi_h(G)$.

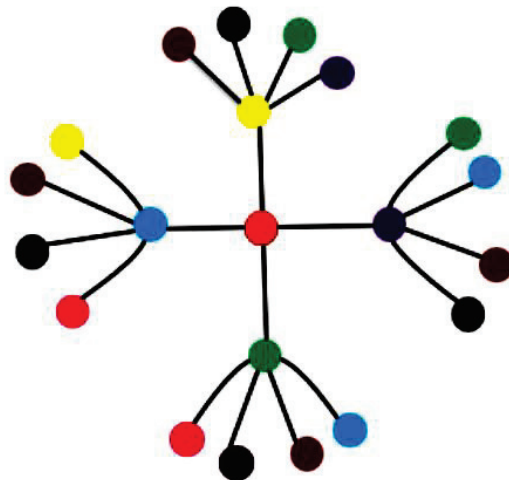


Figure 2.3 Harmonious Coloring $\chi_h(G_{21}) = 8$

The 1-Harmonious coloring is a kind of vertex coloring such that the color pairs of end vertices of every edge are different only for adjacent edges. The minimum number of

coloring used in such a coloring is known as harmonious chromatic number and it is denoted by $\chi_{h_1}(G)$.

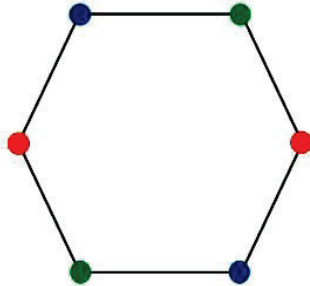


Figure 2.4 1-Harmonious Coloring $\chi_{h_1}(G_6) = 3$

2.1 Application

Coloring problems arise naturally in many practical situations where it is required to partition a set of objects into groups in such a way that the members of each group are mutually compatible according to some criterion. We give below examples of such problems.

Making Schedule - Suppose we want to make an exam schedule for a university. We have list different subjects and students enrolled in every subject. Many subjects would have common students (of same batch, some backlog students, etc). How do we schedule the exam so that no two exams with a common student are scheduled at same time? How many minimum time slots are needed to schedule all exams? This problem can be represented as a graph where every vertex is a subject and an edge between two vertices mean there is a common student. So this is a graph coloring problem where minimum number of time slots is equal to the chromatic number of the graph.

Mobile Radio Frequency Assignment - When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraint? What is the minimum number of frequencies needed?

This problem is also an instance of graph coloring problem where every tower represents a vertex and an edge between two towers represents that they are in range of each other.

Sudoku - Sudoku is also a variation of graph coloring problem where every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

Register Allocation- In compiler optimization, register allocation is the process of assigning a large number of target program variables onto a small number of CPU registers. This problem is also a graph coloring problem.

Bipartite Graphs - We can check if a graph is Bipartite or not by coloring the graph using two colors. If a given graph is 2-colorable, then it is Bipartite, otherwise not.

Map Coloring - Geographical maps of countries or states where no two adjacent cities cannot be assigned same color. Four colors are sufficient to color any map.

Harmonious Coloring problem – Used in Transportation networks, Computer network, Radio navigation, Data compression and Clustering and Designing of minimal hash function

2.2 An outline of the project

In **Chapter 1** presents an introduction and certain basic concepts in graph theory that are necessary for the project. In **Chapter 2** description of graph coloring and its application is presented. In **Chapter 3** a brief survey of Harmonious and 1-Harmonious coloring is provided. In **Chapter 4** we obtain the 1-Harmonious coloring of certain classes graph like Pan graph, Lollipop graph. In **Chapter 5** we obtain the 1-Harmonious coloring of Central graph of certain classes graphs like Cycle graph, Star graph, Pan graph and Lollipop graph. In **chapter 6** it is concluded by summarizing the significance of the study and provides some open problems for future research.

CHAPTER – 3

LITERATURE SURVEY

In this chapter a brief survey of work on Harmonious coloring and 1-Harmonious coloring is presented. Miller and Pritikin were the first to implement this parameter. Since it is sufficient to give every vertex a different color, every graph has a Harmonious coloring; hence $\chi_h(G) \leq |V(G)|$. In literature there are graphs G where $\chi_h(G) > \chi(G)$, in which χ denotes those chromatic numbers that occurs infrequently. P_2 is an example of a path that can be two-colored but does not have a harmonious coloring of two colors.

Hopcroft and Krishnamoorthy investigated that calculating the HCN of a graph is NP-hard and given proof for NP-hard (Hopcroft J., 1983). In 1987, Lee et al. established an upper limit for the HCN of graphs (Lee S., 1987). Lu has calculated the HCN of various graph groups (Zhikang Lu., 1993). He has also found bounds on the HCN of a complete binary and trinary tree, as well as a complete 4-ary tree (Zhikang Lu., 1993, 1995). Lu later gave the accurate value of the HCN of a complete binary tree and trinary tree (Zhikang Lu., 1997, 1998). In 1994, Casse et al. presented minimal colorable designs that were harmonious (Casse L. R., 1994). Havel et al. determined the HCN of quasistars (Havel I., 1996).

Edwards studied the topic of HCN for nearly all trees, bounded degree trees, bounded degree graphs, he also published articles on the relationship between HCN and achromatic number (Edward K. J., 1995-1998). He stressed a new upper limit for the HCN, as well as the HCN of complete r – ary trees (Edward K. J., 1998, 2003).

Mitchem has proved a variety of theorems about HCN and explored a number of open issues (Mitchem J., 1989). The HCN for deep and wide complete n -ary trees was investigated by Mitchem and Schmeichel (Mitchem J., 2002). Mammana determines the HCN of $P(\alpha, K_n)$, $P(\alpha, K_{1,n})$ and $P(\alpha, K_{m,n})$ (Mammana M. F., 2003). Campbell et al. established a new lower limit for the HCN in terms of the independence number (Campbell D., 2004). Thilagavathi et al. have done more research on the HCN (Thilagavathi K., 2003-2009). Vivin et al. found findings on the HCN with central graphs of odd cycles and

complete graphs, as well as the middle graph of $C(C_n)$, $C(K_{1,n})$ and $C(P_n)$ (Vernold Vivin J., 2007). They have also looked at the same issue for line graphs of central graphs of bipartite graphs. The NP-completeness of the HCNP for disconnected interval graphs and cographs is established by Bodlaender (Bodlaender H. L., 1989). Asdre et al. (2007) built on Bodlaender's findings by demonstrating that the problem for connected interval graphs is still NP-complete. Trees and disconnected bipartite permutation graphs, connected bipartite permutation graphs and disconnected quasi-threshold graphs have also been shown to be NP-complete (Edward K. J., 1997).

Wang et al. generalized the harmonious coloring problem and called it the local harmonious coloring problem, which is defined as follows (Wang Y.L, 2010). The Local Harmonious coloring is a d -Harmonious coloring if every color-pair is present at most once for every edge within distance d of any vertex in that graph, and the d -harmonious chromatic number of G , denoted by $h_d(G)$, is the minimum k such that G has a d -Harmonious coloring. Clearly, the original Harmonious coloring problem is the $\lceil x/2 \rceil$ -Harmonious coloring problem if x is the diameter of G . In addition, it was proven that solving the 1-Harmonious coloring problem is NP-complete, and tight bounds were obtained for the path and Cycle graphs, whereas the tight bounds for other graphs have been stated as open problems. Recently, Gao (2015) presented three algorithms to give the coloring procedure for the 1-Harmonious coloring problem, but these algorithms were based on an exhaustive search of vertices and branching rules. Jia-Bao Liu et.al have investigate the sharp bound on 1-Harmonious coloring. Their investigation consists of a collection of basic graphs like a complete graph, wheel, star, tree, fan, and interconnection networks such as a mesh-derived network, generalized honeycomb network, complete multipartite graph, butterfly, and Benes networks (Jia-Bao Liu, 2019).

The above study of the literature justifies the need for systematic mathematical expressions for the 1-Harmonious coloring problem. In the present study, we obtained the 1-Harmonious coloring of certain classes of graphs like Pan graph and Lollipop graph also the 1-Harmonious coloring of certain classes of central graphs like Pan graph, Lollipop graph, Cycle graph and Star graph.

CHAPTER – 4

1-HARMONIOUS COLORING OF CERTAIN CLASSES OF GRAPHS

In this session definitions and 1-Harmonious chromatic number of certain well known graph classes like Pan graph and Lollipop graph are provided.

4.1 Pan Graph

The graph P_n be a pan graph obtained by connecting a Cycle graph with a singleton graph with a bridge.

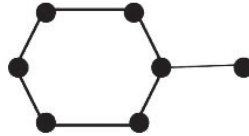


Figure 4.1 Pan graph P_6

Theorem 4.1.1

If $G = P_n$ then $\chi_{h_1}(P_n) = \Delta(P_n) + 1, n \neq 5$, for $n \geq 3$

Proof

Let G be a Pan graph consisting of cycle and bridge. First we color the vertices of cycle using color class (C_1, C_2, C_3, C_4) repeatedly. Then for the end vertex of bridge, choose any one of these color (C_1, C_2, C_3, C_4) which should not adjacently colored to the another vertex of bridge.

This method results in

$$\chi_{h_1}(P_n) = 4 \quad \text{where} \quad n \neq 5$$

We can denote it as

$$\chi_{h_1}(P_n) = \Delta(P_n) + 1, \quad n \neq 5$$

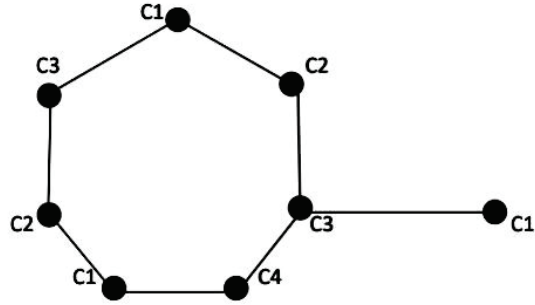


Figure 4.1.1 Illustration of theorem 4.1.1 $\chi_{h_1}(P_7) = 4$

4.2 Lollipop graph

Let $L_{(r,s)}$ be a Lollipop graph obtained by connecting a complete graph of 'r' vertices and a path of 's' vertices as a bridge.

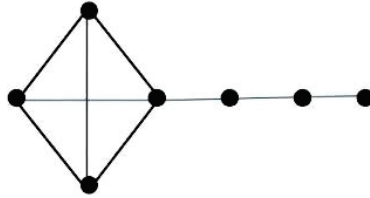


Figure 4.2 Lollipop graph $L_{(4,3)}$

Theorem 4.2.1

If $G = L_{(r,s)}$ then $\chi_{h_1}(L_{(r,s)}) = \Delta(L_{(r,s)}) + 1$, for $s \geq 1$ and $r \geq 3$

Proof

Let $L_{(r,s)}$ be the Lollipop graph obtained by connecting complete graph K_r with the path P_s as a bridge.

Considering the complete graph in $L_{(r,s)}$ can be colored with r colors

i.e $\Delta[L_{(r,s)}]$

Now for the path P_s , one different color is needed to the starting vertex of the path since it is adjacent to one of the vertex in cycle which is adjacent to every vertices in the cycle.

For the remaining path vertices, use any of the already used colors according to the 1-Harmonious definition.

In conclusion, the 1-Harmonious chromatic number for $L_{(r,s)}$ is

$$\Delta(L_{(r,s)}) + 1 \quad \text{where } r \geq 3, s \geq 1.$$

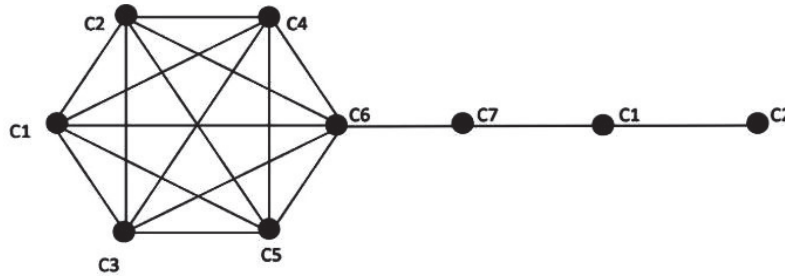


Figure 4.2.1 Illustration of theorem 4.2.1 $\chi_{h_1}(L_{(6,3)}) = 7$

CHAPTER – 5

1-HARMONIOUS COLORING OF CENTRAL GRAPH OF CERTAIN CLASSES OF GRAPHS

We begin this session by giving the definition of Central graph with an example. Then we have given the definition of Star graph and the 1-Harmonious chromatic number of Central graph of certain classes of graphs like Cycle graph, Lollipop graph, Pan graph, and Star graph.

5.1 Central Graph

The Central graph is acquired by splitting each edge of G once and combining all of G 's non-adjacent vertices. $C(G)$ is the abbreviation for it. Central graph of Lollipop graph $L_{(6,2)}$ is sketched in Figure 5.1.

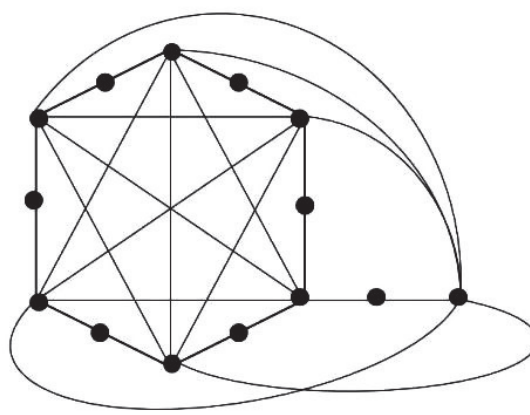


Figure 5.1(a) Central graph of Lollipop graph $L_{(6,2)}$.

Let G_n be graph and $C(G_n)$ be the central graph of G_n . Divide the vertices of $C(G_n)$ into two, original vertices $(V_1, V_2, V_3, \dots, V_n)$ and central vertices $(U_1, U_2, U_3, \dots, U_n)$

to make the coloring easy. The original vertices and the central vertices differ for each graph, given in the below table.

Graph	Original vertices	Central vertices
Central graph of Cycle graph (G_n)	n	n
Central graph of Star graph (S_p)	p	$p - 1$
Central graph of Pan graph	$n + 1$	$n + 1$
Central graph of Lollipop graph ($L_{(r,s)}$)	$r + s$	$r + s$

Figure 5.1(b) Table showing the number of original and central vertices

Theorem 5.1.1

Let G_n be a Cycle graph and $C(G_n)$ be Central graph of G_n , then for $n \geq 3$ $C(G_n)$ confess 1-Harmonius coloring with the chromatic number

$$\chi_{h_1}[C(G_n)] = \begin{cases} \Delta[C(G_n)] + \Delta(G_n) + 1, & \text{when } n \text{ is even} \\ \Delta[C(G_n)] + \Delta(G_n) + 2, & \text{when } n \text{ is odd} \end{cases}$$

Proof

Let G_n be a Cycle graph having $V_1, V_2, V_3, \dots, V_n$ vertices and $C(G_n)$ be the central graph of G_n having $V_1, V_2, V_3, \dots, V_n$ original vertices and $U_1, U_2, U_3, \dots, U_n$ central vertices.

Since the original vertices of $C(G_n)$ are adjacent with themselves, we need n colors to color the original vertices which can be denoted as

$$\Delta[C(G_n)] + 1 \quad \dots\dots(i)$$

Case (i) When n is even

In this case, we have even number of central vertices having degree '2', we can color all the central vertices with 2 colors alternatively which is denoted as

$$\Delta(G_n) \quad \dots\dots(ii)$$

In conclusion, from (i) & (ii)

$$\chi_{h_1}[C(G_n)] = \Delta[C(G_n)] + \Delta(G_n) + 1, \quad n > 3$$

Case (ii), When n is odd,

In this case, we have odd number of central vertices,

So here we need three colors , let ii be (C_x, C_y, C_z)

(C_x, C_y) is to be used as per the previous case and C_z is to color the last odd vertex.

Otherwise it will contradict the 1- Harmonious Coloring.

$$\text{i.e. } \Delta(G_n) + 1 \quad \dots\dots(iii)$$

from (i) & (iii), we get

$$\chi_{h_1}[C(G_n)] = \Delta[C(G_n)] + \Delta(G_n) + 2, \quad n > 3$$

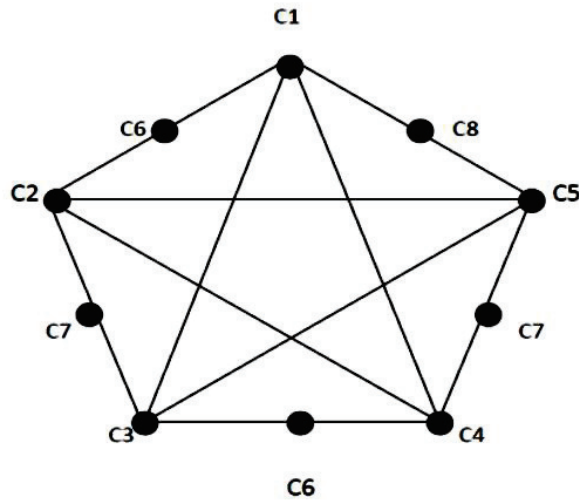


Figure 5.1.1 Illustration of theorem 5.1.1 $\chi_{h_1}[C(G_5)] = 8$

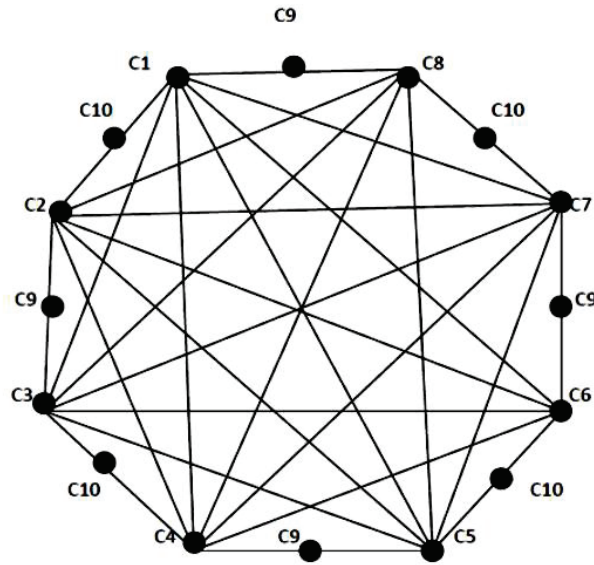


Figure 5.1.2 Illustration of theorem 5.1.1 $\chi_{h_1}[C(G_8)] = 10$

5.2 Star Graph

Let S_p be a Star graph is a tree of order p with one internal node and $(p - 1)$ leaves.

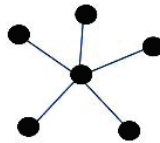


Figure 5.2 Star graph S_6

Theorem 5.2.1

Let S_p be a Star graph and $C(S_p)$ be Central graph of S_p , then

$$\chi_{h_1}[C(S_p)] = n, p > 3$$

Proof

Let S_p be the Star graph with P vertices and $C(S_p)$ be the central graph of Star graph with $(P - 1)$ central vertices adding to the vertices of S_p .

i.e $C(S_p) = (P - 1 + P, E)$

$$V = P - 1 + P = 2P - 1$$

$$\text{Let } 2P - 1 = n$$

Then the total number of vertices in $C(S_p)$ is n .

1) First considering the original vertex which is on the center can be colored with 1 color

2) Then to color the remaining original vertices of $C(S_n)$ except the center vertex i.e $(P - 1)$ vertices, we need $(P - 1)$ unique colors because the remaining vertices are adjacent among them and also to the central vertices, which is on the centre adjacent to the original vertex. Which is on the center

$$\text{i.e } (P - 1)$$

3) Now to the central vertices, we need $(P - 1)$ distinct colors since the central vertices are adjacent to the original vertex, on the center of the graph.

From (1), (2), (3),

We need $(1 + P - 1 + P - 1)$ Colors

$$\chi_{h_1}[C(S_p)] = 2P - 1 = n$$

i.e The total number of vertices in the Central graph of Star graph.

C_x to the central vertex which is on the bridge and (C_y, C_z) to the central vertices on the cycle alternatively .

$$\text{i.e } \Delta(G_n) \dots\dots(ii)$$

Suppose , the cycle having odd number of vertices , again use color C_x to the last odd vertex in the cycle.

In conclusion, we need

$$\Delta[C(G_n)] + \Delta(G_n) + 1 \text{ colors}$$

$$\text{i.e } \chi_{h_1}[C(G_n)] = \Delta[C(G_n)] + \Delta(G_n) + 1, \quad n \geq 3$$

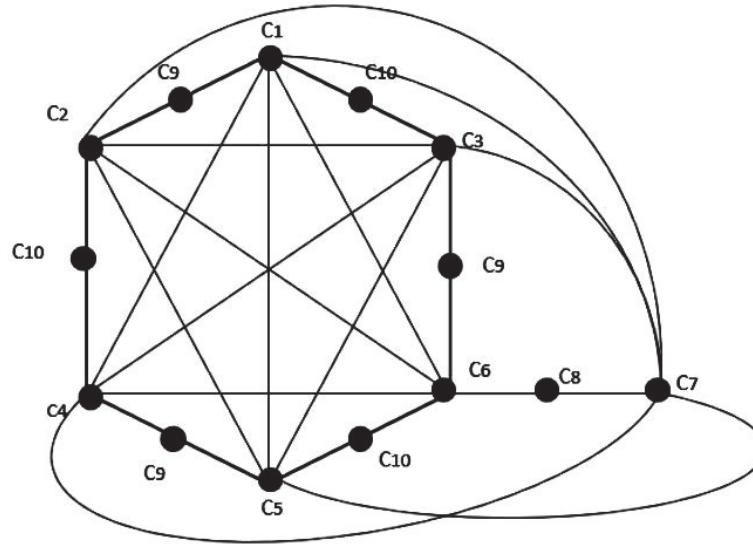


Figure5.3.1: Illustration of theorem 5.3 $\chi_{h_1}[C(G_6)] = 10$

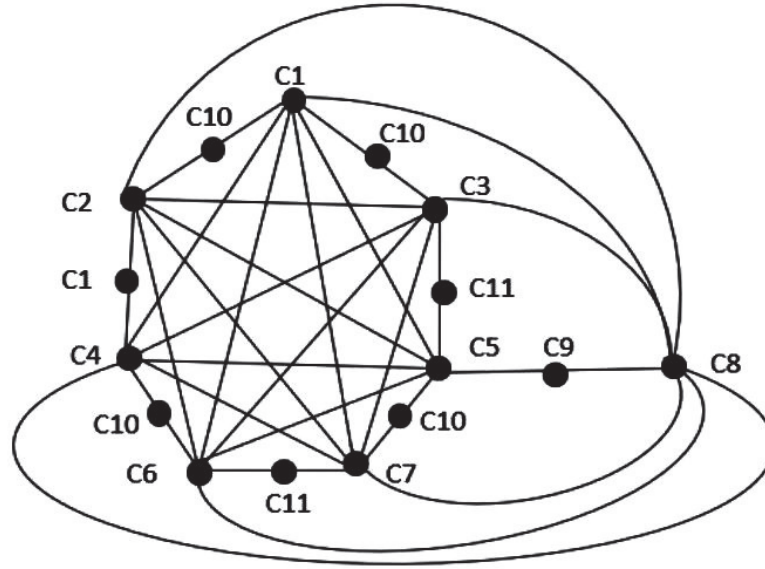


Figure 5.3.2 Illustration of theorem 5.3 $\chi_{h_1}[C(G_7)] = 11$

Theorem 5.4

Let $L_{(r,s)}$ be a Lollipop graph, obtained by connecting a complete graph K_r with the path P_s as a bridge, then the lollipop central graph $C[L_{(r,s)}]$ confesses a 1-Harmonious chromatic number

$$\chi_{h_1}[C(L_{(r,s)})] = \Delta[C(L_{(r,s)})] + 3, \quad r > 3, \quad s > 1$$

Proof

Let $L_{(r,s)}$ be a Lollipop graph obtained by connecting a Complete graph K_r with the path P_s as a bridge, and $C[L_{(r,s)}]$ be a central graph of Lollipop graph having $(r + s)$ original vertices.

Since the original vertices are adjacent with themselves, we can color the original vertices with colors $(1, 2, 3, \dots, r + s)$ which can be expressed as

$$\Delta[C(L_{(r,s)})] + 1$$

Considering the central vertices on the path P_s which are having degree 2, color these vertices with 2 colors alternatively.

And for the remaining central vertices, use any of the already used colors according to the 1-Harmonious definition. In conclusion, we need

$$\Delta[C(L_{(r,s)})] + 3 \text{ colors to color the graph } C(L_{(r,s)})$$

$$\text{i.e } \chi_{h_1}[C(L_{(r,s)})] = \Delta[C(L_{(r,s)})] + 3, \text{ where } r > 3, s > 1$$

Result

If any two Lollipop central graphs are of the form $C(L_{(r,s)})$ & $C(L_{(s,r)})$, then

$$\chi_{h_1}[C(L_{(r,s)})] = \chi_{h_1}[C(L_{(s,r)})], \text{ where } r, s \geq 3$$

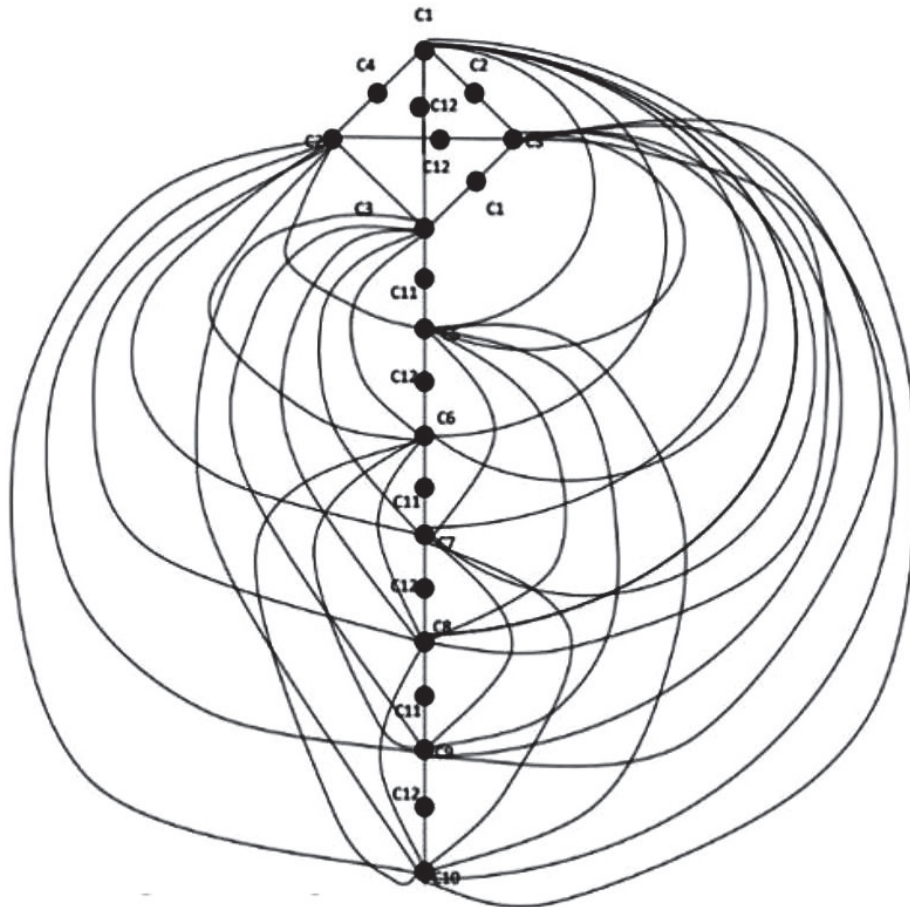


Figure 5.4.1 Illustration of theorem 5.4 $\chi_{h_1}[C(L_{(4,6)})] = 12$

CHAPTER – 6

CONCLUSION

The compilation and investigation of coloring graphs aids network security since these sets are valuable in monitoring a network's edges. Though some research papers dealt with approximation algorithms for the 1-Harmonious coloring of graph, very few analyze the exact values of minimum 1-Harmonious coloring. Our contribution in the study of the 1-Harmonious coloring problem has been devoted in determining the exact solution of this problems for complex structures. In this project we have obtained the 1-Harmonious chromatic number of certain classes of graphs like pan graph, Lollipop graph, Central graph of Cycle graph, Star graph, Pan graph, Lollipop graph. This can be extended for directed graphs in future.

CHAPTER – 7

REFERENCES

- [1] Asdre K., Ioannidou K. and Nikolopoulos S. D. (2007), "The harmonious coloring problem is NP -complete for interval and permutation graphs", *Discrete Applied Math*, Vol. 155, pp. 2377-2382.
- [2] Bondy J. A. and Murty U. S. R. (1976), "Graph theory with applications", London: MacMillan.
- [3] Campbell D. and Edwards K. J. (2004), "A new lower bound for the harmonious chromatic number", *Australasian Journal of Combinatorics*, Vol. 29, pp. 99-102.
- [4] Casse L. R., O'Keefe C. M. and Wilson B. J. (1994), "Minimal harmoniously colorable designs", *Journal of Combinatorial Designs*, Vol. 2, pp. 61-69.
- [5] Douglas B. (2006), "West, Introduction to graph theory", Prentice Hall of India Pvt. Ltd.
- [6] Edwards K. J. (1995), "The harmonious chromatic number of almost all trees", *Combinatorics, Probability and Computing*, Vol. 4, pp. 31-44.
- [7] Edwards K. J. (1996), "The harmonious chromatic number of bounded degree trees", *Combinatorics, Probability and Computing*, Vol. 5, pp. 15-28.
- [8] Edwards K. J. (1997), "The harmonious chromatic number and the achromatic number", *Surveys in Combinatorics*, London, pp. 13-47.
- [9] Edwards K. J. (1998), "A new upper bound for the harmonious chromatic number", *Journal of Graph Theory*, Vol. 29, pp. 257-261.

- [10] Edwards K. J. (2003), "The Harmonious chromatic number of complete p -ary trees", Discrete Mathematics, Vol. 203, pp. 83-99.
- [11] Edwards K. J. (1997), "The harmonious chromatic number of bounded degree graphs", Journal of the London Mathematical Society, Vol. 55, pp. 435-447.
- [12] Edwards K. J. (2000), "Achromatic number versus pseudo achromatic number: A counter example to a conjecture of Hedetniemi", Discrete Mathematics, Vol. 219, pp. 271-274.
- [13] Edwards K. J. and McDiarmid C. J. H. (1995), "The complexity of harmonious coloring for trees", Discrete Applied Math, Vol. 57, pp. 133-144.
- [14] Gao, W. "Three algorithms for graph locally harmonious colouring". J. Differ. Equ. Appl. 2015, 23, 8–20.
- [15] Harary F. (2001), "Graph theory", Narosa Publishing House, New Delhi.
- [16] Havel I. and Laborde J. M. (1996), "On the harmonious chromatic number of quasistars", Manuscript, Prague and Grenoble.
- [17] Hopcroft J. and Krishnamoorthy M. S. (1983), "On the harmonious coloring of graphs", SIAM J. Alg. Disc. Meth, Vol. 4, pp. 306-311.
- [18] Jia-Bao Liu. "Tight bounds on 1-Harmonious coloring of certain graphs". 2019, 917.
- [19] Lee S. and John Mitchum (1987), "An upper bound for the harmonious chromatic number of graphs", Journal of Graph Theory, Vol. 11, pp. 565-567.
- [20] Mammana M. F. (2003), "On the harmonious chromatic number of $P(\alpha, K_n)$ $P(\alpha, K_{1,n})$ and $P(\alpha, K_{m,n})$ ", Utilitas Mathematica, Vol. 64, pp. 25-32.

- [21] McDiarmid, C.; Xinhua, L. Upper bounds for harmonious colourings. J. Graph Theory 1991, 15, 629–636.
- [21] Miller Z. and Pritikin D. (1991), “The harmonious coloring number of a graph”, Discrete Mathematics, Vol. 93, pp. 211-228.
- [22] Mitchem J. (1989), “On the harmonious chromatic number of a graph”, Discrete Mathematics, Vol. 74, pp. 151-157.
- [23] Mitchem J. and Schmeichel E. (2002), “The harmonious chromatic number of deep and wide complete n -ary trees”, Electronic Notes in Discrete Mathematics, Vol. 11, pp. 491-502.
- [24] Thilagavathi K. and Vernold Vivin J. (2006), “Harmonious coloring of graphs”, Far East J. Math. Sci, Vol. 20, pp. 189-197.
- [25] Thilagavathi K. and Vernold Vivin J. (2007), “Harmonious coloring of cycles, regular graphs, star leaf, festoon trees and bigraphs”, Far East J. Math. Sci, Vol. 26, pp. 779-788.
- [26] Thilagavathi K., Nidha D. and Roopesh N. (2009), “On harmonious coloring of $C(W_n)$ and $C(F_{m,n})$ ”, Electronic notes in Discrete Mathematics, Vol. 33, pp. 95-99.
- [27] Thilagavathi K., Vernold Vivin J. and Anitha B. (2007), “On harmonious coloring of $L[C(K_{1,n})]$ and $L[C(C_n)]$ ”, Proc. International Conference on Mathematics and Computer Science, ICMCS 2007, pp. 189-194.
- [28] Zhikang Lu (1993), “Estimates of the harmonious chromatic numbers of some classes of graphs”, Journal of Systems Science and Mathematical Sciences, Vol. 13, 218-223.
- [29] Zhikang Lu (1993), “The harmonious chromatic number of a complete binary and trinary tree”, Discrete Mathematics, Vol. 118, pp. 165-172.

- [30] Zhikang Lu (1995), "The harmonious chromatic number of a complete 4-ary tree", Journal of Mathematical Research and Exposition, Vol. 15, pp. 51-56.
- [31] Zhikang Lu (1997), "The exact value of the harmonious chromatic number of a complete binary tree", Discrete Mathematics, Vol. 172, pp. 93-101.
- [32] Zhikang Lu (1998), "Exact value of the harmonious chromatic number of a complete trinary tree", Systems Science and Mathematical Sciences, Vol. 11, pp. 26- 31.