

“FUZZY SUBALGEBRAS OF Z-ALGEBRAS”

Submitted in partial fulfillment of the requirements for the award of
Master of Science Degree in Mathematics

By

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SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

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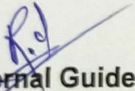
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BONAFIDE CERTIFICATE

This is to certify that this Project Report is the bonafide work of **PRATIKSHA SACHAN (41630008)** who carried out the project entitled "**FUZZY SUBALGEBRAS OF Z-ALGEBRAS**" under my supervision from **JANUARY 2023 to APRIL 2023**.



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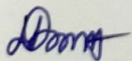
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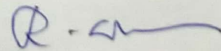
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DECLARATION

I PRATIKSHA SACHAN (41630008) hereby declare that the Project Report entitled "FUZZY SUBALGEBRAS OF Z-ALGEBRAS" done by me under the guidance of "Dr. R. DELHI BABU" (Internal) at department of Mathematics SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY is submitted in partial fulfillment of the requirements for the award of Master of Science Degree in Mathematics.

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ABSTRACT

In this project, we have introduced the notions of Fuzzy Z -Subalgebras and Fuzzy Z -Ideals in Z -Algebras and proved some interesting results. Moreover, we have described how to deal with the Z - homomorphic image and inverse image of fuzzy Z - Subalgebras (fuzzy Z -ideals) in Z -algebras. Further, we have also proved that the Cartesian product of fuzzy Z - Subalgebras (fuzzy Z -ideals) is a fuzzy Z - Subalgebra (fuzzy Z -ideal) in Z -algebras.

KEYWORDS:

Z -algebra, Z -Subalgebra, Z -homomorphism, Level Z -Subalgebras, Fuzzy Z -Subalgebras, Fuzzy Z -Ideals.

TABLE OF CONTENT

CHAPTER NO.	TITLE	PAGE NO
1.	INTRODUCTION	2
	1.1 Z-Algebra	3
	1.2 Z-Subalgebra	4
	1.3 Z-Ideal	5
2	LITERATURE SURVEY	7-8
3	AIM AND SCOPE	
	3.1 Aim	10
	3.2 Scope	10
4	METHODOLOGY	
	4(a) Fuzzy Z- Subalgebra	12
	4.1 – 4.27 (Related Definitions, Theorems and Results)	13-26
	4(b) Fuzzy Z-Ideal	27
	4.28 – 4.44 (Related Definitions, Theorems and Results)	28-38
5	RESULTS AND DISCUSSION	40-43
6	CONCLUSION	45
	REFERENCES	46-47

LIST OF TABLES:

TABLE NO.	TITLE	PAGE NO.
1.1	Example of Z-Algebra.	4
1.2	Example of Z-Subalgebra.	4
1.3	Example of Z-Ideal	5
4.2	Example of Fuzzy Z-Subalgebra.	12
4.29	Example of Fuzzy Z- Ideal	17
4.36 (table1)	Z-homomorphic image of a fuzzy set is a fuzzy Z-ideal.	32
4.36 (table2)	Z-homomorphic image of a fuzzy set is a fuzzy Z-ideal.	32

CHAPTER 1

1. INTRODUCTION:

Z – Algebra:

A proposition is a statement. Propositional logic is a mathematical model that allows us to reason about the truth or falsehood of logical expressions that are simplifications of a conditional expression. With the help of propositional calculus or logic, we represent a statement or proposition by a symbol called propositional variables. The values that can be assigned to a proposition are called truth values — either true or false. To model reasoning, algebra is associated with the propositional logic. There are several such algebras such as BCK-algebras, BCI-algebras, BCH-algebras, Q-algebras, B-algebras, d-algebras and so on.

One such algebra which has been introduced recently, in the year 2017, by Chandramouleeswaran and others is that of the Z-algebras.

Fuzzy Sets:

The real physical world contains many problems of uncertainty, in which one cannot predict the property of “belongingness” of an object with certainty. For example, rose cannot be ascertained as a flower whose color is rose, because we have rose flower with different colors (red, yellow...). In order to deal with the problem of uncertainty in the real physical world, in 1965, Zadeh introduced the notion of fuzzy sets. In studying the theory of fuzzy sets each element of the universe is assigned a membership grading whose values lie in $[0, 1]$.

The study of fuzzy sets and their application to mathematical concepts has reached to what is commonly called as fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld. During 1985, Bhattacharya and Mukherji introduced fuzzy structure in

the form of fuzzy relations and fuzzy groups. In 1991, Xi applied the concept of fuzzy sets to BCK-algebras. For the general development of BCK/BCI-algebras, the ideal theory and its fuzzifications play an important role.

THE PROBLEM OF STUDY:

In this project, we study the fuzzy structures on Z-algebras. In our project work, we combine the theory of fuzzy sets and the theory of Z-algebras. Thus, we have studied the notions of fuzzy Z-Subalgebras, fuzzy Z-Ideals in Z-Algebra with some interesting results.

FUZZY SET:

In fuzzy sets, each element is mapped to $[0, 1]$ by membership function, $\mu_A: X \rightarrow [0, 1]$, Where $[0, 1]$ means real numbers between 0 and 1 (Including 0 and 1).

MEMBERSHIP FUNCTION:

A membership function for a set A on the universe X is defined as $\mu_A: X \rightarrow [0, 1]$, where each element of X is mapped to a value between 0 and 1. This value μ_A is called membership value or degree of membership, quantifies the grade of membership of the element in X to the fuzzy set A.

1.1 Z-ALGEBRA:

A Z-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

- $x * 0 = 0$
- $0 * x = x$
- $x * x = X$
- $x * y = y * x$, when $x \neq 0$ and $y \neq 0$ for all $x, y \in X$.

EXAMPLE:

Let $X = \{0,1,2,3\}$ be a set with constant 0 and a binary operation $*$ defined on X with the following Cayley table, is a Z-algebra.

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

1.2 Z-SUBALGEBRA:

Let S be a nonempty subset of a Z-algebra $(X, *, 0)$. Then, S is called a Z-Subalgebra of $(X, *, 0)$, if $x * y \in S$, for all $x, y \in S$.

Example:

Let $X = \{0,1,2,3\}$ be a set with constant 0 and a binary operation $*$ defined on X with the following Cayley table, is a Z-algebra.

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, the subsets $A = \{1,3\} \subset X$ and $B = \{2,3\} \subset X$ are Z-Subalgebras of X , but the subset $C = \{1,2,3\} \subset X$ is not a Z-Subalgebra of X , for $1 * 2 = 0 \notin C$.

1.3 Z-IDEAL:

Let $(X, *, 0)$ be a Z-algebra and I be a subset of X . Then, I is called an Z-Ideal of X , if it satisfies the following conditions: For all $x, y \in X$,

- $0 \in I$
- $x * y \in I$ and $y \in I$ implies $x \in I$

EXAMPLE:

Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and a binary operation $*$ defined on X with the following Cayley table, is a Z-algebra.

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $I = \{0, 1, 2\} \subset X$ is an Z-Ideal of X .

CHAPTER 2

2. LITERATURE SURVEY:

An important point in the evaluation of the modern concept of uncertainty was the paper by Zadeh. In his paper, Zadeh introduced a theory whose objects - fuzzy sets, are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. This concept is being used and is also found to be more appropriate in solving problems of all disciplines such as Economics, Expert System, Artificial Intelligence, Computer Science, Control Engineering, Psychology, Management Science, Operations Research, Pattern recognition, Robotics, Washing Machines, Topological Spaces, and Algebraic Structures etc.

Christopher Jefferson and Chandramouleeswaran (2015) applied fuzzy algebraic structures in BP-algebras.

Abbass and Neamah (2016) discussed fuzzy implicative ideal of a BH-algebra and gave some properties of this fuzzy ideal and link it with other types of fuzzy ideals of a BH-algebra.

Mostafa and Ola Wageeh Abd El-Baseer (2018) introduced the notion of fuzzy implicative ideals of KU-algebras and several basic properties are investigated.

Poungsumpao et al. (2019) studied the notions of fuzzy UP-Subalgebras and fuzzy UP-ideals of UP-algebras in terms of level subsets of a fuzzy set and investigated some of their properties.

Lee et al. (2009) discussed the concept of fuzzy translations and fuzzy multiplications of fuzzy Subalgebras in BCK/ BCI-algebras.

Chandramouleeswaran et al. (2013) have been discussed similar concept in BF-algebras.

Abu Ayub Ansari and Chandramouleeswaran (2014) discussed the concepts of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy β -Subalgebras and fuzzy β -ideals in β -algebras.

Sithar Selvam and Nagalakshmi (2016) discussed the notions of homomorphism and Cartesian product on fuzzy α -translations and fuzzy α -multiplications of PMS-algebras.

Hameed and Mohammed (2017) discussed the ideas of fuzzy translation and fuzzy multiplication in CI-algebras. In the same year, Hameed and Alkurdi investigated fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy QS-Subalgebras and fuzzy QS-ideals in QS-algebras.

Prasanna et al. (2020) investigated some algebraic properties of ω -fuzzy translations and ω -fuzzy multiplications in BP-algebras.

CHAPTER 3

3. AIM AND SCOPE:

Imai and Iseki (1966) introduced two new classes of abstract algebras: BCK-algebras and BCI-algebras. Chandramouleeswaran et al. (2017) introduced the concept of Z-algebras as a new structure of algebra based on propositional calculus. Zadeh (1965) introduced the fundamental concept of a fuzzy set which is a generalization of an ordinary set. Following the idea of fuzzy groups, Xi (1991) introduced the notion of fuzzy BCK-algebras.

Christopher Jefferson and Chandramouleeswaran (2015) applied fuzzy algebraic structures in BP-algebras.

3.1 AIM:

- Our aim is to define the notion of Fuzzy Z-Subalgebras in Z-algebras and obtain some of their properties.
- Next, we discuss the notion of Fuzzy Z-Ideals in Z-algebras and obtain some of their properties.

3.2 SCOPE:

- It will be useful in soft set theory because fuzzy soft set is an extended model of soft set and a new mathematical tool that has great advantages in dealing with uncertain information and is proposed by combining soft sets and fuzzy sets.
- It has a great scope in graph theory because Fuzzy graphs have numerous applications in diverse parts of Science and Engineering like Social Network, data hypothesis, producing and so on.

CHAPTER 4

4. METHODOLOGY:

4(a) FUZZY Z-SUBALGEBRA:

DEFINITION 4.1:

Let $(X, *, 0)$ be a Z-algebra. A fuzzy set A in X with membership function, is said to be a fuzzy Z-Subalgebra of a Z-algebra X if for all $x, y \in X$ the following condition is satisfied:

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

Example 4.2:

Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Then $(X, *, 0)$ is a Z-algebra.

Define a fuzzy set A_1 in X whose membership function is given by

$$\mu_{A_1}(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2, 3 \end{cases}$$

Then A_1 is a fuzzy Z-Subalgebra of a Z-algebra X .

THEOREM 4.3:

Intersection of any two fuzzy Z-Subalgebras of a Z-algebra X is again a fuzzy Z-Subalgebra.

Proof:

Let A_1 and A_2 be fuzzy Z-Subalgebras of a Z-algebra X. Then, for all $x, y \in X$,

$$\begin{aligned}\mu_{A_1 \cap A_2}(x * y) &= \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \\ &\geq \min\{\min\{\mu_{A_1}(x), \mu_{A_1}(y)\}, \min\{\mu_{A_2}(x), \mu_{A_2}(y)\}\} \\ &= \min\{\min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \\ &= \min\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\}\end{aligned}$$

That is

$$\mu_{A_1 \cap A_2}(x * y) \geq \min\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\}$$

Hence $A_1 \cap A_2$ is a fuzzy Z-Subalgebra of a Z-algebra X.

The following example illustrates Theorem 4.3

EXAMPLE 4.4:

Consider Z-algebra X and fuzzy Z-Subalgebra A, as in Example 4.2. The

fuzzy set A_2 in X defined by $\mu_{A_2}(x) = \begin{cases} 0.7 & \text{if } x = 0 \\ 0.5 & \text{if } x = 1 \\ 0.2 & \text{if } x = 2, 3 \end{cases}$

is also a fuzzy Z-Subalgebra of X.

Now, $\mu_{A_1 \cap A_2}(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2, 3 \end{cases}$

Then, $A_1 \cap A_2$ is a fuzzy Z-Subalgebra of a Z-algebra X.

We can generalize the theorem 4.3 as follows.

Corollary 4.5:

Let $\{A_i \mid i \in \Omega\}$ where $i \in \Omega$ an index set, be a family of fuzzy Z-Subalgebras of a Z-algebra X. Then $\bigcap_{i \in \Omega} A_i$ is also a fuzzy Z-Subalgebra of X.

THEOREM 4.6:

Let A be a fuzzy Z-Subalgebras of a Z-algebra X such that $\mu_A(0) \neq 0$. Let $\mu_B: X \rightarrow [0,1]$ be a fuzzy set defined by $\mu_B(x) = \frac{\mu_A(x)}{\mu_A(0)}$ for all $x \in X$. Then B is a fuzzy Z-Subalgebra of X.

Proof:

$$\begin{aligned} \text{Let } x, y \in X \text{ Then } \mu_B(x * y) &= \frac{\mu_A(x * y)}{\mu_A(0)} \\ &\geq \frac{1}{\mu_A(0)} \min\{\mu_A(x), \mu_A(y)\} \\ &= \min\left\{\frac{\mu_A(x)}{\mu_A(0)}, \frac{\mu_A(y)}{\mu_A(0)}\right\} \\ &= \min\{\mu_B(x), \mu_B(y)\} \end{aligned}$$

Therefore B is a fuzzy Z-Subalgebra of a Z-algebra X.

THEOREM 4.7:

If A be a fuzzy Z -Subalgebra of a Z -algebra X , then the characteristic function χ_A defined by $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is a fuzzy Z -Subalgebra of X .

Proof: Let $x, y \in X$

Case (i) If $x, y \in A$ then $x * y \in A$. Hence $\chi_A(x) = 1, \chi_A(y) = 1$ and

$\chi_A(x * y) = 1$. Thus,

$$\chi_A(x * y) \geq 1 = \min\{1, 1\} = \min\{\chi_A(x), \chi_A(y)\}, \text{ for all } x, y \in X.$$

Case (ii) If $x, y \notin A$ then $\chi_A(x) = 0, \chi_A(y) = 0$ and

$$\chi_A(x * y) \geq 0 = \min\{0, 0\} = \min\{\chi_A(x), \chi_A(y)\}, \text{ for all } x, y \in X.$$

Case (iii) If $x \in A, y \notin A$ then $\chi_A(x) = 1, \chi_A(y) = 0$ and

$$\chi_A(x * y) \geq 0 = \min\{1, 0\} = \min\{\chi_A(x), \chi_A(y)\}, \text{ for all } x, y \in X.$$

Case (iv) Interchanging the roles of x and y in case (iii) we can prove that A is a fuzzy Z -Subalgebra of X when $x \notin A, y \in A$.

Thus the theorem is completely proved.

The converse of the above theorem is also true.

Corollary 4.8:

Let A be any subset of a Z -algebra X . If the characteristic function χ_A is a fuzzy Z -Subalgebra of X , then A is a Z -Subalgebra of X .

Proof:

For any $x, y \in X$, $\chi_A(x) = 1 = \chi_A(y)$

Therefore, $\chi_A(x * y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1, 1\} = 1$.

Hence $\chi_A(x * y) = 1 \Rightarrow x * y \in A$.

The theorem 4.7 can be generalized as follows.

THEOREM 4.9:

Let A be any Z -Subalgebra of a Z -algebra X and $\mu_A: X \rightarrow [0,1]$ be a fuzzy set defined by $\mu_A(x) = \begin{cases} t_0 & \text{if } x \in A \\ t_1 & \text{if } x \notin A \end{cases}$ with $t_0 > t_1$. Then A is a fuzzy Z -Subalgebra of X .

THEOREM 4.10:

A fuzzy set A of a Z -algebra X is a fuzzy Z -Subalgebra if and only if for every $t \in [0,1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is either empty or a Z -Subalgebra of X .

Proof:

If A is a fuzzy Z -Subalgebra of X , for any $t \in [0,1]$, assume that $U(\mu_A; t) \neq \emptyset$

For any $x, y \in U(\mu_A; t)$, we have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$.

Then $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{t, t\} = t$

This implies $x * y \in U(\mu_A; t)$

That is, $U(\mu_A; t)$ is a Z -subalgebra of a Z -algebra X .

Conversely, assume that $U(\mu_A; t)$ is a Z -Subalgebra of a Z -algebra X .

Let $x, y \in A$ and let $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$.

Then $x \in U(\mu_A; t_1)$ and $y \in U(\mu_A; t_2)$.

If $t_1 \leq t_2$, then $U(\mu_A; t_2) \subseteq U(\mu_A; t_1)$ and so $y \in U(\mu_A; t_1)$.

Since $U(\mu_A; t_1)$ is a Z -Subalgebra of X , $x * y \in U(\mu_A; t_1)$.

Thus $\mu_A(x * y) \geq t_1 = \min\{\mu_A(x), \mu_A(y)\}$, proving that A is a fuzzy Z -Subalgebra of a Z -algebra X .

DEFINITION 4.11:

Let A be a fuzzy Z -Subalgebra of a Z -algebra X . For any $t \in [0,1]$, Z -Subalgebras $U(\mu_A; t)$ are called upper level Z -Subalgebras of A .

REMARK 4.12:

Henceforth, in this section, the upper level Z-Subalgebras of A will be referred as level Z-Subalgebras of A.

THEOREM 4.13:

Any Z-Subalgebra of a Z-algebra X can be realized as a level Z-Subalgebra of some fuzzy Z-Subalgebra of X.

Proof:

Let S be a Z-Subalgebra of a Z-algebra X and A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{Where } t \in [0,1] \text{ is fixed.}$$

Clearly $U(\mu_A; t) = S$. We consider the following cases:

Case (i) If $x, y \in S$ then $x * y \in S$.

Hence $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t$ and

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = t$$

Case (ii) if $x, y \notin S$ then $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = 0$.

Then $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0$.

Case (iii) if at most one of $x, y \in S$

then at least one of $\mu_A(x)$ and $\mu_A(y)$ is equal to 0.

Hence $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0$.

This shows that S is a level Z-Subalgebra of a Z-algebra X corresponding to the fuzzy Z-Subalgebra A of X.

THEOREM 4.14:

Let X be a Z-algebra. Then given any chain of Z-Subalgebras $S_0 \subset S_1 \subset \dots \subset S_r = X$, There exists a fuzzy Z-Subalgebra A of X whose upper level Z-Subalgebras are exactly the Z-Subalgebras of this chain.

Proof:

Consider a set of numbers $t_0 > t_1 > t_2 > \dots > t_r$, where each $t_i \in [0,1]$.

Let A be a fuzzy set defined by $\mu_A(S_0) = t_0$ and $\mu_A(S_i - S_{i-1}) = t_i, i = 1, 2, \dots, r$.

Claim: A is a fuzzy Z -Subalgebra of a Z -algebra X .

Let $x, y \in X$. Then we classify it into two cases as follows:

Case (i): Let $x, y \in S_i - S_{i-1}$. then by the definition of A , $\mu_A(x) = t_i = \mu_A(y)$.

Since S_i is a Z -Subalgebra of X , it follows that $x, y \in S_i$ and so either

$x * y \in S_i - S_{i-1}$ (Or) $x * y \in S_{i-1}$. In any case, we conclude that

$$\mu_A(x * y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$$

Case (ii): For $i > j$, Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$.

Then $\mu_A(x) = t_i$; $\mu_A(y) = t_j$ and $x * y \in S_i$ since S_i is a Z -Subalgebra of X and $S_j \subset S_i$.

Hence $\mu_A(x * y) \geq t_j = \min\{\mu_A(x), \mu_A(y)\}$.

Thus A is a fuzzy Z -Subalgebra of a Z -algebra X .

From the definition of A , it follows that $\text{Im}(A) = \{t_0, t_1, \dots, t_r\}$

Hence the upper level Z -Subalgebras of A are given by the chain of Z -Subalgebras.

$$U(\mu_A; t_0) \subset U(\mu_A; t_1) \subset U(\mu_A; t_2) \subset \dots \subset U(\mu_A; t_r) = X.$$

$$\text{Now } U(\mu_A; t_0) = \{x \in X \mid \mu_A(x) \geq t_0\} = S_0.$$

Finally, we prove that $U(\mu_A; t_i) = S_i$ for $i = 1, 2, \dots, r$.

Clearly $S_i \subseteq U(\mu_A; t_i)$.

If $x \in U(\mu_A; t_i)$, then $\mu_A(x) \geq t_i$ which implies that $x \notin S_j$ for $j > i$.

Hence $\mu_A(x) \in \{t_1, t_2, \dots, t_i\}$ and so $x \in S_k$ for some $k \leq i$.

As $S_k \subseteq S_i$, it follows that $x \in S_i \Rightarrow U(\mu_A; t_i) = S_i$ for $i = 1, 2, \dots, r$.

This completes the proof.

NOTE:

If X is a finite Z —algebra, then the number of Z -Subalgebras of X is finite where as the number of level Z - Subalgebras of a fuzzy Z -Subalgebra A appears to be infinite. But since every level Z -Subalgebra is indeed Z - Subalgebra of X , not all these Z -Subalgebras are distinct. The next theorem characterizes this aspect.

THEOREM 4.15:

Let A be a fuzzy Z -Subalgebra of a Z -algebra X . Two level Z -Subalgebras $U(\mu_A; t)$ and $U(\mu_A; s)$ (with $t < s$) of A are equal if and only if there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Proof:

Assume that $U(\mu_A; t) = U(\mu_A; s)$ for some $t < s$ and there exists $x \in X$ such that $t \leq \mu_A(x) < s$.

Then $U(\mu_A; s)$ is a proper subset of $U(\mu_A; t)$ which is a contradiction.

Hence there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Conversely, suppose that there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Since $t < s$, we get $U(\mu_A; s) \subseteq U(\mu_A; t)$ (1)

If $x \in U(\mu_A; t)$ then $\mu_A(x) \geq t$ and so $\mu_A(x) > s$, because $\mu_A(x)$ does not lie between t and s .

Hence $x \in U(\mu_A; s)$.

Hence $U(\mu_A; t) \subseteq U(\mu_A; s)$ (2)

From (1) and (2) we get $U(\mu_A; t) = U(\mu_A; s)$.

REMARK 4.16:

As a consequence of Theorem 4.15, the level Z -Subalgebras of a fuzzy Z -Subalgebra A of a finite Z -algebra X form a chain $U(\mu_A; t_0) \subset U(\mu_A; t_1) \subset U(\mu_A; t_2) \subset \dots \subset U(\mu_A; t_r) = X$, where $t_0 > t_1 > t_2 > \dots > t_r$.

Corollary 4.17:

Let X be a finite Z -algebra and A be a fuzzy Z -Subalgebra of X . If $\text{Im}(A) = \{t_1, \dots, t_n\}$, then the family of Z -Subalgebras $U(\mu_A; t_i)$, $i = 1, 2, \dots, n$, constitutes all the level Z -Subalgebras of A .

Proof:

Let $t \in [0, 1]$ and $t \notin \text{Im}(A)$.

Suppose $t_1 < t_2 < \dots < t_n$, without loss of generality.

If $t \leq t_1$, then $U(\mu_A; t_1) = X = U(\mu_A; t)$,

If $t > t_n$, then $U(\mu_A; t) = \emptyset$ obviously.

If $t_{i-1} < t < t_i$, then $U(\mu_A; t) = U(\mu_A; t_i)$ by Theorem 4.15.

Thus for any $t \in [0, 1]$, the level Z -Subalgebra is one of $\{U(\mu_A; t_i) \mid i = 1, 2, \dots, n\}$.

Lemma 4.18:

Let X be a Z -algebra and A be a fuzzy Z -Subalgebra of X . If $\text{Im}(A)$ is finite, say $\{t_1, t_2, \dots, t_n\}$, then for any $t_i, t_j \in \text{Im}(A)$, $U(\mu_A; t_i) = U(\mu_A; t_j) \Rightarrow t_i = t_j$.

Proof: Assume that $t_i \neq t_j$ and $t_i < t_j$.

If $x \in U(\mu_A; t_j)$ then $\mu_A(x) \geq t_j > t_i$.

Hence $x \in U(\mu_A; t_i)$

Let $x \in X$ such that $t_i < \mu_A(x) < t_j$.

Then $x \in U(\mu_A; t_i)$ but $x \notin U(\mu_A; t_j)$

Hence $U(\mu_A; t_j) \subset U(\mu_A; t_i)$ and

$U(\mu_A; t_j) \neq U(\mu_A; t_i)$, a contradiction.

Therefore $U(\mu_A; t_i) = U(\mu_A; t_j) \Rightarrow t_i = t_j$.

THEOREM 4.19:

Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras. If $\text{Im}(A) = \{t_1, t_2, \dots, t_r\}$ and $\text{Im}(B) = \{q_1, q_2, \dots, q_k\}$ where $t_1 \geq t_2 \geq \dots \geq t_r$ and $q_1 \geq q_2 \geq \dots \geq q_k$. Then

- i. $k = r$
- ii. $U(\mu_A; t_i) = U(\mu_B; q_i), i = 1, 2, \dots, r.$
- iii. If $x \in X$ such that $\mu_A(x) = t_i$ then $\mu_B(x) = q_i, i = 1, 2, \dots, r.$

Proof:

Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras with $F(A) = F(B)$ where

$$F(A) = \{U(\mu_A; t_i) \mid i = 1, 2, \dots, r\} \text{ And } F(B) = \{U(\mu_B; q_i) \mid i = 1, 2, \dots, k\} .$$

$$\text{Let } \text{Im}(A) = \{t_1, t_2, \dots, t_r\} \text{ where } t_1 \geq t_2 \geq \dots \geq t_r \quad (1)$$

$$\text{and let } \text{Im}(B) = \{q_1, q_2, \dots, q_k\} \text{ where } q_1 \geq q_2 \geq \dots \geq q_k \quad (2)$$

$$\text{From (1) we get } U(\mu_A; t_1) \subseteq U(\mu_A; t_2) \subseteq \dots \subseteq U(\mu_A; t_r) = X \quad (3)$$

$$\text{From (2) we get } U(\mu_B; q_1) \subseteq U(\mu_B; q_2) \subseteq \dots \subseteq U(\mu_B; q_k) = X \quad (4)$$

To prove (i): $k = r$

Suppose $k \neq r$, then we consider the following cases:

Case (i): $k > r$

Let $k > r$ then $U(\mu_A; t_i) = U(\mu_B; q_i); i = 1, 2, \dots, r.$

This shows that both t_i and $q_i \in \text{Im}(A)$

For $i > r$ we observe that $t_i \notin \text{Im}(A)$ and hence,

$$U(\mu_A; t_i) \neq U(\mu_B; q_i), i = r + 1, r + 2, \dots, k.$$

Case (ii): $r > k$

Let $r > k$ then $U(\mu_A; t_i) = U(\mu_B; q_i); i = 1, 2, \dots, k.$

This shows that both t_i and $q_i \in \text{Im}(B).$

For $i > k$ we observe that $q_i \notin \text{Im}(B)$ and hence

$$U(\mu_A; t_i) \neq U(\mu_B; q_i); i = k + 1, k + 2, \dots, r.$$

From (3) and (4) we get $t_i \neq q_i$ for all $i = 1, 2, \dots, r$.

Hence we can find some i such that $U(\mu_A; t_i) \neq U(\mu_B; q_i)$.

This contradicts that $F(A) = F(B)$.

Hence we conclude that $k = r$.

To prove (ii): By part (i), we have proved that $k = r$.

Since A and B have identical family of level Z -Subalgebras, we have

$$U(\mu_A; t_i) = U(\mu_B; q_i); i = 1, 2, \dots, r.$$

To prove (iii): Let $x \in X$ such that $\mu_A(x) = t_i$ and $\mu_A(x) = q_j$ where $0 \leq i \leq k$ and $0 \leq j \leq k$.

From (ii) follows that $x \in U(\mu_B; q_i)$ thus $\mu_B(x) \geq q_i$ and $q_j \geq q_i$

Therefore $U(\mu_B; q_j) \subseteq U(\mu_B; q_i)$

Since $x \in U(\mu_B; q_j) = U(\mu_A; t_j)$, we get $t_i = \mu_A(x) \geq t_j$

This gives $U(\mu_B; q_i) = U(\mu_A; t_i) \subseteq U(\mu_A; t_j) = U(\mu_B; q_j)$

Thus $U(\mu_B; q_j) = U(\mu_B; q_i)$ and by above lemma: 4.18 we get $q_j = q_i$

Hence $\mu_B(x) = q_i$

Hence the proof.

Corollary 4.20:

Let A and B be two fuzzy Z -Subalgebras of a Z -algebra X with identical family of level Z -Subalgebras. Then $\text{Im}(A) = \text{Im}(B) \Rightarrow A = B$.

Proof:

Let $\text{Im}(A) = \text{Im}(B) = \{q_1, q_2, \dots, q_r\}$ where $q_1 \geq q_2 \geq \dots \geq q_r$

By Theorem 4.19, for any $x \in X$ there exists q_i such that $\mu_A(x) = q_i = \mu_B(x)$

Thus $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

This implies $A = B$.

THEOREM 4.21:

Let h be a Z -homomorphism from a Z -algebra $(X, *, 0)$ onto a Z -algebra $(Y, *, 0')$ and let A be a fuzzy Z -Subalgebra of X with the sup property. Then the image of A denoted by $h(A)$ is a fuzzy Z -Subalgebra of Y .

Proof:

Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that

$$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t)$$

$$\mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$$

Now,

$$\begin{aligned} \mu_{h(A)}(a * 'b) &= \sup_{t \in h^{-1}(a * 'b)} \mu_A(t) \geq \mu_A(x_0 * y_0) \\ &\geq \min\{\mu_A(x_0), \mu_A(y_0)\} \\ &= \min\{\sup_{t \in h^{-1}(a)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t)\} \\ &= \min\{\mu_{h(A)}(a), \mu_{h(A)}(b)\} \end{aligned}$$

Hence $h(A)$ is a fuzzy Z -Subalgebra of a Z -algebra Y .

THEOREM 4.22:

Let $h: (X, *, 0) \rightarrow (Y, *, 0')$ be a Z -homomorphism of Z -algebras. If A is a fuzzy Z -Subalgebra of Y then the pre-image of A denoted by $h^{-1}(A)$ is a fuzzy Z -Subalgebra of X . Converse is true if h is an Z -epimorphism.

Proof:

Let $x, y \in X$. Then,

$$\begin{aligned}\mu_{h^{-1}(A)}(x * y) &= \mu_A(h(x * y)) = \mu_A(h(x) * 'h(y)) \\ &\geq \min\{\mu_A(h(x)), \mu_A(h(y))\} \\ &= \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}\end{aligned}$$

Hence $\mu_{h^{-1}(A)}(x * y) \geq \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}$

Therefore, $h^{-1}(A)$ is a fuzzy Z -Subalgebra of a Z -algebra X .

On the other hand,

Assume that h is an Z -epimorphism and $h^{-1}(A)$ is a fuzzy Z -Subalgebra of X .

Let $y_1, y_2 \in Y$.

Since h is an Z -epimorphism, there exists $x_1, x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

This implies $x_1 = h^{-1}(y_1)$ and $x_2 = h^{-1}(y_2)$.

$$\begin{aligned}\text{Now, } \mu_A(y_1 * 'y_2) &= \mu_A(h(x_1) * 'h(x_2)) \\ &= \mu_A(h(x_1 * x_2)) \\ &= \mu_{h^{-1}(A)}(x_1 * x_2) \\ &\geq \min\{\mu_{h^{-1}(A)}(x_1), \mu_{h^{-1}(A)}(x_2)\} \\ &= \min\{\mu_A(h(x_1)), \mu_A(h(x_2))\} \\ &= \min\{\mu_A(y_1), \mu_A(y_2)\}\end{aligned}$$

Hence A is a fuzzy Z -Subalgebra of a Z -algebra Y .

Definition 4.23:

Let h be an Z -endomorphism of Z -algebras and A be a fuzzy set in a Z -algebra X . We define a new fuzzy set A^h in X as $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$.

THEOREM 4.24:

Let h be an Z -endomorphism of a Z -algebra $(X, *, 0)$. If A be a fuzzy Z -Subalgebra of X . Then A^h is also a fuzzy Z -Subalgebra of X .

Proof:

Let $x, y \in X$. Then,

$$\begin{aligned}\mu_{A^h}(x * y) &= \mu_A(h(x * y)) = \mu_A(h(x) * h(y)) \\ &\geq \min\{\mu_A(h(x)), \mu_A(h(y))\} \\ &= \min\{\mu_{A^h}(x), \mu_{A^h}(y)\}\end{aligned}$$

$$\Rightarrow \mu_{A^h}(x * y) \geq \min\{\mu_{A^h}(x), \mu_{A^h}(y)\}$$

Hence A^h is a fuzzy Z -Subalgebra of a Z -algebra X .

THEOREM 4.25:

If A and B are fuzzy Z -Subalgebras of a Z -algebra X then $A \times B$ is also a fuzzy Z -Subalgebra of $X \times X$.

Proof:

For any $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$\begin{aligned}\mu_{A \times B}((x_1, x_2) * (y_1, y_2)) &= \mu_{A \times B}(x_1 * y_1, x_2 * y_2) \\ &= \min\{\mu_A(x_1 * y_1), \mu_B(x_2 * y_2)\} \\ &\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_A(x_1), \mu_B(x_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A \times B}(x_1, x_2), \mu_{A \times B}(y_1, y_2)\}\end{aligned}$$

Hence $A \times B$ is also a fuzzy Z -Subalgebra of $X \times X$.

We can generalize the above theorem as follows.

THEOREM 4.26:

Let $\{X_i | i = 1, 2, \dots, n\}$ be a finite collection of Z -algebras and $X = \prod_{i=1}^n X_i$. Let $A_i, i = 1, 2, \dots, n$ be fuzzy Z -Subalgebras of X , respectively. Then $A = \prod_{i=1}^n A_i$ is also a fuzzy Z -Subalgebra of X .

THEOREM 4.27:

If B is a fuzzy Z -Subalgebra of a Z -algebra X then the strongest fuzzy relation A_B is a fuzzy Z -Subalgebra of $X \times X$.

Proof:

For all $(x_1, x_2), (y_1, y_2) \in X \times X$, then

$$\begin{aligned} \mu_{A_B}((x_1, y_1) * (x_2, y_2)) &= \mu_{A_B}(x_1 * x_2, y_1 * y_2) \\ &= \min\{\mu_B(x_1 * x_2), \mu_B(y_1 * y_2)\} \\ &\geq \min\{\min\{\mu_B(x_1), \mu_B(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_B(x_1), \mu_B(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A_B}(x_1, y_1), \mu_{A_B}(x_2, y_2)\} \end{aligned}$$

Therefore A_B is a fuzzy Z -Subalgebra of $X \times X$.

4(b) FUZZY Z-IDEAL:

DEFINITION 4.28:

Let $(X, *, 0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A , is said to be a fuzzy Z-Ideal of a Z-algebra X if it satisfies the following conditions;
For all x, y in X ,

- i. $\mu_A(0) \geq \mu_A(x)$
- ii. $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$

Example 4.29:

Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

Then $(X, *, 0)$ is a Z-algebra.

Define a fuzzy set A_1 and A_2 in X whose membership functions μ_{A_1} and μ_{A_2} by

$$\mu_{A_1}(x) = 0.9 \text{ for all } x = 0, 1, 2, 3 \text{ and } \mu_{A_2}(x) = \begin{cases} 0.8 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2, 3 \end{cases}.$$

Then A_1 is a fuzzy Z-Ideal of X , while A_2 is not. For,

$$\begin{aligned} \mu_{A_2}(2) &= 0.5 \not\geq 0.8 = \min\{0.8, 0.8\} = \min\{\mu_{A_2}(0), \mu_{A_2}(0)\} \\ &= \min\{\mu_{A_2}(2 * 0), \mu_{A_2}(0)\} \end{aligned}$$

THEOREM 4.30:

Arbitrary intersection of fuzzy Z-ideals of Z-algebra X is also a fuzzy Z-ideal.

Proof:

Let $\{A_i | i \in \Omega\}$ be a family of fuzzy Z-ideals of a Z-algebra X.

For any $x, y \in X$,

$$\begin{aligned}\mu_{\bigcap_{i \in \Omega} A_i}(0) &= \inf_{i \in \Omega}(\mu_{A_i}(0)) \geq \inf_{i \in \Omega}(\mu_{A_i}(x)) = \mu_{\bigcap_{i \in \Omega} A_i}(x) \\ \mu_{\bigcap_{i \in \Omega} A_i}(x) &= \inf_{i \in \Omega}(\mu_{A_i}(x)) \geq \inf_{i \in \Omega}(\min\{\mu_{A_i}(x * y), \mu_{A_i}(y)\}) \\ &= \min\{\inf_{i \in \Omega}(\mu_{A_i}(x * y)), \inf_{i \in \Omega}(\mu_{A_i}(y))\} \\ &= \min\{\mu_{\bigcap_{i \in \Omega} A_i}(x * y), \mu_{\bigcap_{i \in \Omega} A_i}(y)\}\end{aligned}$$

Hence $\bigcap_{i \in \Omega} A_i$ is a fuzzy Z-ideal of a Z-algebra X.

Proposition 4.31:

Let $\{A_i | i \in \Omega\}$ be a chain of fuzzy Z-ideals of a Z-algebra X. Then $\bigcup_{i \in \Omega} A_i$ is a fuzzy Z-ideal of X.

Proof:

Let $x \in X$. Then

$$\mu_{\bigcup_{i \in \Omega} A_i}(0) = \sup_{i \in \Omega}(\mu_{A_i}(0)) \geq \sup_{i \in \Omega}(\mu_{A_i}(x)) = \mu_{\bigcup_{i \in \Omega} A_i}(x)$$

Let $x, y \in X$. Then, we have

$$\begin{aligned}\mu_{\bigcup_{i \in \Omega} A_i}(x) &= \sup_{i \in \Omega}(\mu_{A_i}(x)) \geq \sup_{i \in \Omega}(\min\{\mu_{A_i}(x * y), \mu_{A_i}(y)\}) \\ &= \min\{\sup_{i \in \Omega}(\mu_{A_i}(x * y)), \sup_{i \in \Omega}(\mu_{A_i}(y))\} \\ &= \min\{\mu_{\bigcup_{i \in \Omega} A_i}(x * y), \mu_{\bigcup_{i \in \Omega} A_i}(y)\}\end{aligned}$$

Hence $\bigcup_{i \in \Omega} A_i$ is a fuzzy Z-ideal of a Z-algebra X.

THEOREM 4.32:

A fuzzy set A of a Z -algebra $(x, *, 0)$ is a fuzzy Z -ideal if and only if for any $t \in [0,1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is a Z -ideal of X where $U(\mu_A; t) \neq \emptyset$.

Proof:

Suppose A is a fuzzy Z -ideal of a Z -algebra X and $U(\mu_A; t) \neq \emptyset$ for any $t \in [0,1]$.

Let $x \in U(\mu_A; t)$, then $\mu_A(x) \geq t$.

By definition of fuzzy Z -ideal, we have $\mu_A(0) \geq \mu_A(x) \geq t$.

If $x * y \in U(\mu_A; t)$ and $y \in U(\mu_A; t)$, then $\mu_A(x * y) \geq t$ and $\mu_A(y) \geq t$.

By definition, we have $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \geq \min\{t, t\} = t$

Therefore $x \in U(\mu_A; t)$. hence $U(\mu_A; t)$ is a Z -ideal of X .

Conversely, suppose that for each $t \in [0,1]$, $U(\mu_A; t)$ is either empty or a Z -ideal of a Z -algebra X .

For any $x \in X$, let $\mu_A = t$. Then $x \in U(\mu_A; t)$

Since $U(\mu_A; t) \neq \emptyset$ is a Z -ideal of X , we have $0 \in U(\mu_A; t)$ and

Hence $\mu_A(0) \geq t = \mu_A(x)$.

Thus $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.

Assume $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$ is not true.

Then There exists $x_0, y_0 \in X$ such that $\mu_A(x_0) < \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$

Let $t_0 = \frac{1}{2}[\mu_A(x_0) + \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}]$

Then $\mu_A(x_0) < t_0 < \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\}$

This implies $x_0 * y_0, y_0 \in U(\mu_A; t_0)$ and $x_0 \notin U(\mu_A; t_0)$

But $U(\mu_A; t_0)$ is a Z -ideal of X . So $x_0 \in U(\mu_A; t_0)$ by the definition of Z -ideal.

This implies $\mu_A(x_0) \geq t_0$.

This is a contradiction.

Therefore $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$, for all $x, y \in X$.

Hence A is a fuzzy Z -ideal of a Z -algebra X .

THEOREM 4.33:

A fuzzy set A of a Z -algebra $(X, *, 0)$ is a fuzzy Z -ideal if and only if every nonempty upper level subset $U(\mu_A; q)$ of A , $q \in \text{Im}(A)$ is an Z -ideal.

Proof:

Let A be a fuzzy Z -ideal of a Z -algebra X .

Claim: $(\mu_A; q)$, $q \in \text{Im}(A)$ is an Z -ideal.

Since $U(\mu_A; q) \neq \emptyset$ there exists $x \in U(\mu_A; q)$, such that $\mu_A(x) \geq q$.

Since A is a fuzzy Z -ideal of X , $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.

Hence for this $x \in U(\mu_A; q)$, $\mu_A(0) \geq q$, which shows that $0 \in U(\mu_A; q)$.

Now, for any $x, y \in X$, assume that $x * y \in U(\mu_A; q)$ and $y \in U(\mu_A; q)$

Then $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$.

This shows that, $\min\{\mu_A(x * y), \mu_A(y)\} \geq q$.

Since A is a fuzzy Z -ideal of X , $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \geq q$,

Thus $x \in U(\mu_A; q)$, this proves that $U(\mu_A; q)$ is an Z -ideal of a Z -algebra X .

Conversely, let $U(\mu_A; q)$, $q \in \text{Im}(A)$ be an Z -ideal of a Z -algebra X .

Claim: A is a fuzzy Z -ideal of a Z -algebra X .

Let $x, y \in X$. For any $q \in \text{Im}(A)$, let $q = \min\{\mu_A(x * y), \mu_A(y)\}$.

Therefore, $\mu_A(x * y) \geq q$ and $\mu_A(y) \geq q$.

This shows that $x * y, y \in U(\mu_A; q)$.

Since $U(\mu_A; q)$ is an Z -ideal, we have $x \in U(\mu_A; q)$.

This proves that $\mu_A(x) \geq q = \min\{\mu_A(x * y), \mu_A(y)\}$

This shows that A is a fuzzy Z -ideal of a Z -algebra X .

THEOREM 4.34:

Let A be a fuzzy Z -ideal of Z -algebra X and let $x \in X$. Then $\mu_A(x) \geq t$ if and only if $x \in U(\mu_A; t)$ but $x \notin U(\mu_A; q)$ for all $q > t$.

Proof: Assume $\mu_A(x) = t$, so that $x \in U(\mu_A; t)$.

If possible, let $x \in U(\mu_A; q)$ for $q > t$.

Then $\mu_A(x) \geq q > t$.

This contradicts the fact that $\mu_A(x) = t$. Hence $x \notin U(\mu_A; q)$ for all $q > t$.

Conversely, let $x \in U(\mu_A; t)$, but $x \notin U(\mu_A; q)$ for all $q > t$.

$x \in U(\mu_A; t) \Rightarrow \mu_A(x) \geq t$.

Since $x \notin U(\mu_A; q)$ for all $q > t$, $\mu_A(x) = t$.

THEOREM 4.35:

Let h be a Z -homomorphism from a Z -algebra $(X, *, 0)$ onto a Z -algebra $(Y, *, 0')$ and A be a fuzzy Z -ideal of X with the sup property. Then image of A denoted by $h(A)$ is a fuzzy Z -ideal of Y .

Proof:

Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that

$$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t); \mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t).$$

$$\mu_{h(A)}(0') = \sup_{t \in h^{-1}(0')} \mu_A(t) \geq \mu_A(0) \geq \mu_A(x_0) = \sup_{t \in h^{-1}(0')} \mu_A(t) = \mu_{h(A)}(a)$$

$$\begin{aligned} \min\{\mu_{h(A)}(a * 'b), \mu_{h(A)}(b)\} &= \min\{\sup_{t \in h^{-1}(a * 'b)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t)\} \\ &\leq \min\{\mu_A(x_0 * y_0), \mu_A(y_0)\} \\ &\leq \mu_A(x_0) \\ &= \sup_{t \in h^{-1}(a)} \mu_A(t) \\ &= \mu_{h(A)}(a) \end{aligned}$$

This implies, $\mu_{h(A)}(a) \geq \min\{\mu_{h(A)}(a * 'b), \mu_{h(A)}(b)\}$

Hence $h(A)$ is a fuzzy Z -ideal of a Z -algebra Y .

Example 4.36:

Consider the Z -algebras $(X, *, 0)$ and $(Y, *', 0')$ with the following Cayley tables 1 and 2 respectively.

Table 1

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Table 2

*'	0'	1	2	3
0'	0'	1	2	3
1	0'	1	2	3
2	0'	2	2	3
3	0'	3	3	3

Now, the function $h: (X, *, 0) \rightarrow (Y, *', 0')$ such that

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 3 & \text{if } x = 1, 2, 3 \end{cases} \text{ is a } Z\text{-homomorphism.}$$

And a fuzzy set A in X defined by $\mu_A(x) = 0.4$ for all $x \in X$ is a fuzzy Z -ideal of X

Then, the Z -homomorphic image of A , $h(A)$ is a fuzzy Z -ideal of Y .

THEOREM 4.37:

Let $h: (X, *, 0) \rightarrow (Y, *, 0')$ be a Z -homomorphism of Z -algebras. If B is a fuzzy Z -ideal of Y , then $h^{-1}(B)$ is a fuzzy Z -ideal of X .

Proof:

For any $x \in X$, we have

$$\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$$

Let $x, y \in X$. Then

$$\begin{aligned} \min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\} &= \min\{\mu_B(h(x * y)), \mu_B(h(y))\} \\ &= \min\{\mu_B(h(x) * 'h(y)), \mu_B(h(y))\} \\ &\leq \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \end{aligned}$$

$$\Rightarrow \mu_{h^{-1}(B)}(x) \geq \min\{\mu_{h^{-1}(B)}(x * y), \mu_{h^{-1}(B)}(y)\}$$

Hence $h^{-1}(B)$ is a fuzzy Z -ideal of a Z -algebra X .

THEOREM 4.38:

Let $h: (X, *, 0) \rightarrow (Y, *, 0')$ be an Z -epimorphism of Z -algebras. Let B be a fuzzy set of Y . If $h^{-1}(B)$ is a fuzzy Z -ideal of X then B is a fuzzy Z -ideal of Y .

Proof:

Let $y \in Y$, there exists $x \in X$ such that $h(x) = y$. Then

$$\mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0')$$

This implies, $\mu_B(0') \geq \mu_B(y)$. Let $x, y \in Y$. Then there exists $a, b \in X$ such that $h(a) = x$ and $h(b) = y$. It follows that

$$\begin{aligned} \mu_B(x) = \mu_B(h(a)) &= \mu_{h^{-1}(B)}(a) \geq \min\{\mu_{h^{-1}(B)}(a * b), \mu_{h^{-1}(B)}(b)\} \\ &= \min\{\mu_B(h(a * b)), \mu_B(h(b))\} \\ &= \min\{\mu_B(h(a) * 'h(b)), \mu_B(h(b))\} \\ &= \min\{\mu_B(x * 'y), \mu_B(y)\} \end{aligned}$$

Hence B is a fuzzy Z -ideal of a Z -algebra Y .

THEOREM 4.39:

Let h be an Z -endomorphism of a Z -algebra X and A be a fuzzy set in X . Then $A^h: X \rightarrow [0,1]$ defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$, is a fuzzy Z -ideal of X if A is a fuzzy Z -ideal.

Proof:

Obvious.

THEOREM 4.40:

If A and B be fuzzy Z -ideals in a Z -algebra X then $A \times B$ is a fuzzy Z -ideal in $X \times X$.

Proof:

Let $(x_1, x_2) \in X \times X$,

$$\mu_{A \times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A \times B}(x_1, x_2),$$

$$\text{Hence } \mu_{A \times B}(0,0) \geq \mu_{A \times B}(x_1, x_2) \quad (1)$$

Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then,

$$\begin{aligned} \mu_{A \times B}(x_1, x_2) &= \min\{\mu_A(x_1), \mu_B(x_2)\} \\ &\geq \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_A(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A \times B}((x_1 * y_1), (x_2 * y_2)), \mu_{A \times B}(y_1, y_2)\} \\ &= \min\{\mu_{A \times B}((x_1, x_2) * (y_1, y_2)), \mu_{A \times B}(y_1, y_2)\} \end{aligned} \quad (2)$$

$$\text{Hence } \mu_{A \times B}(x_1, x_2) \geq \min\{\mu_{A \times B}((x_1, x_2) * (y_1, y_2)), \mu_{A \times B}(y_1, y_2)\}$$

By (1) and (2) we get, $A \times B$ is a fuzzy Z -ideal in $X \times X$.

THEOREM 4.41:

Let A and B be fuzzy sets in a Z -algebra X such that $A \times B$ is a fuzzy Z -ideal of $X \times X$. Then,

- i. Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.
- ii. If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_B(x)$ (or) $\mu_B(0) \geq \mu_B(x)$
- iii. If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$, then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_A(x)$

Proof:

(i) If $\mu_A(0) < \mu_A(x_1)$ and $\mu_B(0) \geq \mu_B(x_2)$ for some $x_1, x_2 \in X$.

Then, $\mu_{A \times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \min\{\mu_A(0), \mu_B(0)\}$
 $= \mu_{A \times B}(0, 0)$, which is a contradiction.

Hence, either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(ii) Let $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Assume that there exists $x_1, x_2 \in X$ such that $\mu_B(0) < \mu_A(x_1)$

and $\mu_B(0) < \mu_B(x_2)$.

Then, $\mu_{A \times B}(0, 0) = \min\{\mu_A(0), \mu_B(0)\} = \mu_B(0)$

$\mu_{A \times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \mu_B(0) = \mu_{A \times B}(0, 0)$

$\Rightarrow \mu_{A \times B}(x_1, x_2) > \mu_{A \times B}(0, 0)$, which is a contradiction.

Hence either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$

(iii) Will obtain by interchanging the roles of A and B in part (ii).

THEOREM 4.42:

Let A and B be fuzzy sets in a Z-algebra X and AB is a fuzzy Z-ideal of $X \times X$ then either A or B is a fuzzy Z-ideal of X.

Proof:

By Theorem 4.41 (i), we can assume that $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$. Then, by Theorem 4.41 (iii), either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$

Let $\mu_A(0) \geq \mu_B(x)$ for any $x \in X$, then

$$\begin{aligned}
 \mu_B(x) &= \min\{\mu_A(0), \mu_B(x)\} = \mu_{A \times B}(0, x) \\
 &\geq \min\{\mu_{A \times B}((0, x) * (0, y)), \mu_{A \times B}(0, y)\} \\
 &= \min\{\mu_{A \times B}((0 * 0), (x * y)), \mu_{A \times B}(0, y)\} \\
 &= \min\{\mu_{A \times B}(0, x * y), \mu_{A \times B}(0, y)\} \\
 &= \min\{\min\{\mu_A(0), \mu_B(x * y)\}, \min\{\mu_A(0), \mu_B(y)\}\} \\
 &= \min\{(\mu_B(x * y), \mu_B(y))\}
 \end{aligned}$$

Therefore, $\mu_B(x) \geq \min\{(\mu_B(x * y), \mu_B(y))\}$, for all $x, y \in X$

Hence B is a fuzzy Z-ideal of a Z-algebra X.

By Theorem 4.41 (i) and (ii), assume that $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ and $\mu_B(0) \geq \mu_A(x)$ for any $x \in X$.

Then A is a fuzzy Z-ideal of a Z-algebra X.

This completes the proof.

THEOREM 4.43:

Let A be a fuzzy relation on a Z -algebra X and A_g be the strongest fuzzy relation on X , where B is a fuzzy set of X . If B is a fuzzy Z -ideal of a Z -algebra X , then A_B is a fuzzy Z -ideal of $X \times X$.

Proof:

Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\mu_{A_B}(0,0) = \min\{\mu_B(0), \mu_B(0)\} \geq \min\{\mu_B(x_1), \mu_B(x_2)\} = \mu_{A_B}(x_1, x_2)$$

$$\text{And also } \mu_{A_B}(x_1, x_2) = \min\{\mu_B(x_1), \mu_B(x_2)\}$$

$$\begin{aligned} &\geq \min\{\min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_B(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A_B}((x_1 * y_1), (x_2 * y_2)), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\} \end{aligned}$$

Therefore A_B is a fuzzy Z -ideal of $X \times X$.

THEOREM 4.44:

Let A be a fuzzy relation on a Z -algebra X and B be a fuzzy set of X . If the strongest fuzzy relation A_B is a fuzzy Z -ideal of $X \times X$, then B is a fuzzy Z -ideal of a Z -algebra X .

Proof:

For all $(x_1, x_2), (y_1, y_2) \in X \times X$,

$$\min\{\mu_B(0), \mu_B(0)\} = \mu_{A_B}(0,0) \geq \mu_{A_B}(x_1, x_2) = \min\{\mu_B(x_1), \mu_B(x_2)\}$$

$$\text{Then, } \mu_B(0) \geq \min\{\mu_B(x_1), \mu_B(x_2)\}$$

$$\Rightarrow \mu_B(0) \geq \mu_B(x_1) \text{ (Or) } \mu_B(0) \geq \mu_B(x_2) \text{ for all } x_1, x_2 \in X.$$

Also,

$$\begin{aligned} \min\{\mu_B(x_1), \mu_B(x_2)\} &= \mu_{A_B}(x_1, x_2) \geq \min\{\mu_{A_B}((x_1, x_2) * (y_1, y_2)), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\mu_{A_B}((x_1 * y_1), (x_2 * y_2)), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \end{aligned}$$

$$\text{Put } x_2 = y_2 = 0, \text{ we get } \mu_B(x_1) \geq \min\{\mu_B(x_1 * y_1), \mu_B(y_1)\}$$

Hence B is a fuzzy Z -ideal of a Z -algebra X .

CHAPTER 5

5. RESULTS AND DISCUSSION:

5.1. FUZZY Z-SUBALGEBRA:

1. Intersection of any two fuzzy Z-Subalgebras of a Z-algebra X is again a fuzzy Z-Subalgebra.
2. Let A be a fuzzy Z-Subalgebras of a Z-algebra X such that $\mu_A(0) \neq 0$. Let $\mu_B: X \rightarrow [0,1]$ be a fuzzy set defined by $\mu_B(x) = \frac{\mu_A(x)}{\mu_A(0)}$ for all $x \in X$. Then B is a fuzzy Z-Subalgebra of X.
3. If A be a fuzzy Z-Subalgebra of a Z-algebra X, then the characteristic function χ_A defined by $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is a fuzzy Z-Subalgebra of X.
4. Let A be any Z-Subalgebra of a Z-algebra X and $\mu_A: X \rightarrow [0,1]$ be a fuzzy set defined by $\mu_A(x) = \begin{cases} t_0 & \text{if } x \in A \\ t_1 & \text{if } x \notin A \end{cases}$ with $t_0 > t_1$. Then A is a fuzzy Z-Subalgebra of X.
5. A fuzzy set A of a Z-algebra X is a fuzzy Z-Subalgebra if and only if for every $t \in [0,1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is either empty or a Z-Subalgebra of X.

5.2. UPPER LEVEL Z-SUBALGEBRA:

6. Any Z-Subalgebra of a Z-algebra X can be realized as a level Z-Subalgebra of some fuzzy Z-Subalgebra of X.
7. Let X be a Z-algebra. Then given any chain of Z-Subalgebras $S_0 \subset S_1 \subset \dots \subset S_r = X$, There exists a fuzzy Z-Subalgebra A of X whose upper level Z-Subalgebras are exactly the Z-Subalgebras of this chain.

8. Let A be a fuzzy Z -Subalgebra of a Z -algebra X . Two level Z -Subalgebras $U(\mu_A; t)$ and $U(\mu_A; s)$ (with $t < s$) of A are equal if and only if there is no $x \in X$ such that $t \leq \mu_A(x) < s$.
9. Let A and B be two fuzzy Z -Subalgebras of a Z -algebra X with identical family of level Z -Subalgebras. If $\text{Im}(A) = \{t_1, t_2, \dots, t_r\}$ and $\text{Im}(B) = \{q_1, q_2, \dots, q_k\}$ where $t_1 \geq t_2 \geq \dots \geq t_r$ and $q_1 \geq q_2 \geq \dots \geq q_k$. Then
 - (i) $k = r$
 - (ii) $U(\mu_A; t_i) = U(\mu_B; q_i), i = 1, 2, \dots, r.$
 - (iii) If $x \in X$ such that $\mu_A(x) = t_i$ then $\mu_B(x) = q_i, i = 1, 2, \dots, r.$
10. Let h be a Z -homomorphism from a Z -algebra $(X, *, \mathbf{0})$ onto a Z -algebra $(Y, *', \mathbf{0}')$ and let A be a fuzzy Z -Subalgebra of X with the sup property. Then the image of A denoted by $h(A)$ is a fuzzy Z -Subalgebra of Y .
11. Let $h: (X, *, \mathbf{0}) \rightarrow (Y, *', \mathbf{0}')$ be a Z -homomorphism of Z -algebras. If A is a fuzzy Z -Subalgebra of Y then the pre-image of A denoted by $h^{-1}(A)$ is a fuzzy Z -Subalgebra of X . Converse is true if h is an Z -epimorphism.

5.3. Z-ENDOMORPHISM:

12. Let h be an Z -endomorphism of a Z -algebra $(X, *, \mathbf{0})$. If A be a fuzzy Z -Subalgebra of X . Then A^h is also a fuzzy Z -Subalgebra of X .
13. If A and B are fuzzy Z -Subalgebras of a Z -algebra X then $A \times B$ is also a fuzzy Z -Subalgebra of $X \times X$.
14. Let $\{X_i | i = 1, 2, \dots, n\}$ be a finite collection of Z -algebras and $X = \prod_{i=1}^n X_i$. Let $A_i, i = 1, 2, \dots, n$ be fuzzy Z -Subalgebras of X , respectively. Then $A = \prod_{i=1}^n A_i$ is also a fuzzy Z -Subalgebra of X .

15. If B is a fuzzy Z -Subalgebra of a Z -algebra X then the strongest fuzzy relation A_B is a fuzzy Z -Subalgebra of $X \times X$.

5.4. FUZZY Z -IDEAL:

16. Arbitrary intersection of fuzzy Z -ideals of Z -algebra X is also a fuzzy Z -ideal.
17. A fuzzy set A of a Z -algebra $(x, *, 0)$ is a fuzzy Z -ideal if and only if for any $t \in [0, 1]$, $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is a Z -ideal of X ,
Where $(\mu_A; t) \neq \emptyset$
18. A fuzzy set A of a Z -algebra $(X, *, 0)$ is a fuzzy Z -ideal if and only if every nonempty upper level subset $U(\mu_A; q)$ of A , $q \in \text{Im}(A)$ is a Z -ideal.
19. Let A be a fuzzy Z -ideal of Z -algebra X and let $x \in X$. Then $\mu_A(x) \geq t$ if and only if $x \in U(\mu_A; t)$ but $x \notin U(\mu_A; q)$ for all $q > t$.
20. Let h be a Z -homomorphism from a Z -algebra $(x, *, 0)$ onto a Z -algebra $(Y, *, ', 0')$ and A be a fuzzy Z -ideal of X with the sup property. Then image of A denoted by $h(A)$ is a fuzzy Z -ideal of Y .
21. Let $h: (X, *, 0) \rightarrow (Y, *, ', 0')$ be a Z -homomorphism of Z -algebras. If B is a fuzzy Z -ideal of Y , then $h^{-1}(B)$ is a fuzzy Z -ideal of X .
22. Let $h: (X, *, 0) \rightarrow (Y, *, ', 0')$ be a Z -epimorphism of Z -algebras. Let B be a fuzzy set of Y . If $h^{-1}(B)$ is a fuzzy Z -ideal of X then B is a fuzzy Z -ideal of Y .
23. Let h be a Z -endomorphism of a Z -algebra X and A be a fuzzy set in X . Then $A^h: X \rightarrow [0, 1]$ defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$, is a fuzzy Z -ideal of X if A is a fuzzy Z -ideal.

24. If A and B be fuzzy Z -ideals in a Z -algebra X then $A \times B$ is a fuzzy Z -ideal in $X \times X$.
25. Let A and B be fuzzy sets in a Z -algebra X such that $A \times B$ is a fuzzy Z -ideal of $X \times X$. Then,
- (i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.
 - (ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$
 - (iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$, then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$
26. Let A and B be fuzzy sets in a Z -algebra X and AB is a fuzzy Z -ideal of $X \times X$ then either A or B is a fuzzy Z -ideal of X .
27. Let A be a fuzzy relation on a Z -algebra X and A_B be the strongest fuzzy relation on X , where B is a fuzzy set of X . If B is a fuzzy Z -ideal of a Z -algebra X , then A_B is a fuzzy Z -ideal of $X \times X$.
28. Let A be a fuzzy relation on a Z -algebra X and B be a fuzzy set of X . If the strongest fuzzy relation A_B is a fuzzy Z -ideal of $X \times X$, then B is a fuzzy Z -ideal of a Z -algebra X .

CHAPTER 6

6. CONCLUSION:

We have initiated fuzzy Z - Subalgebras and fuzzy Z -ideals in Z -algebras and obtained some characterization theorems using level subsets. We have also proved some theorems using the concepts of strongest fuzzy relation, Z -homomorphism and Cartesian product on fuzzy Z - Subalgebras and fuzzy Z -ideals in Z -algebras.

For future study, one can apply these results to various algebraic structures and also in the fields of graph theory and soft set theory.

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