

NEW IDEAL TOPOLOGIES ON TREES OF ORDER LESS THAN OR EQUAL TO FOUR

Submitted in partial fulfillment of the requirements for the award of
Master of Science Degree in Mathematics

By

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SATHYABAMA

**INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)**

Accredited with Grade “A” by NAAC | 12B Status

by UGC | Approved by AICTE

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APRIL 2023



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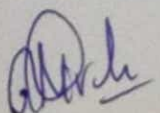
BONAFIDE CERTIFICATE

This is to certify that the Project Report is the bonafide work of SARUMATHY M (41630005) who carried out the project entitled "NEW IDEAL TOPOLOGIES ON TREES OF ORDER LESS THAN OR EQUAL TO FOUR" under our supervision from JANUARY 2023 to APRIL 2023.

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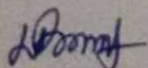
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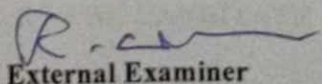

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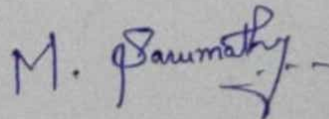

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DECLARATION

I SARUMATHY M (41630005) hereby declare that the Project Report entitled "NEW IDEAL TOPOLOGIES ON TREES OF ORDER LESS THAN OR EQUAL TO FOUR" done by me under the guidance of DR. R. MANOHARAN (Internal) at SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY submitted in partial fulfillment of the requirements for the award of Master of Science Degree in Mathematics.

DATE : 26-04-2023

PLACE : CHENNAI



SIGNATURE OF THE CANDIDATE

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ABSTRACT

In our project work, we study various topologies generated by certain simple undirected graphs without isolated vertices. At first, we will generate a topology from a simple undirected graph having no isolated vertices. Then try to investigate some properties of the topologies generated by those graphs. Later we introduce the concept of minimal adjacency of vertices of the graph G and analyze how topologies are generated in various types of graphs like cyclic graphs, bipartite graphs and so on. Also, we try to prove the continuity condition of a function via two graphs and the generated topologies

And the second part of our project work evolves around the ideals and properties of various topological spaces. In a simple finite undirected graph, a topology is generated using minimal adjacency of each vertex of the graph. It is possible to define different ideals on a graph based on the collection of vertices of the graph. Using the ideals as well as the topology, we can study some of the new topological spaces on a graph. In this, we introduce and study some of the ideal topological spaces on a simple finite undirected graph using minimal adjacency of each vertex of a graph. Adding to it, we approach some of the topological properties in graphs via ideals.

KEYWORDS

Minimal adjacency, ideals, topological spaces, isolated vertices, bipartite graph, non-isomorphic graphs, power set, discrete topology.

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CHAPTER - 1

1.INTRODUCTION

In mathematics, it is a sub-field which deals with the study of graphs which are the mathematical structures widely used to model pairwise relation between objects. It is the study of relationship between the nodes (vertices) and lines (edges). Formally, a graph is a pair (V, E) where V is the finite set of vertices and E is the finite set of edges.

Topology is relatively a new branch of topology where most of the proofs and theories were discussed in 1990's. It is a significant branch of mathematics because of its contributing properties to other branches of mathematics. In this project work, we are aiming to obtain a topological space by using a simple undirected graph without isolated vertices. Later study the topological properties and graph properties and investigate the relationship between them.

1.1 GRAPH THEORY

Graph theory is promptly moving into the main stream of Mathematics as it has various applications in all fields which includes genomics, coding theory, operations research, algorithms and computations, so on.

Swiss Mathematician Leonhard Euler transfer the Königsberg bridge problem in to a graph and wrote a paper and issued in 1736. He is well thought-out as the originator of graph theory. In detailed, the word “graph” was introduced in a paper by Sylvester which was issued in 1878. The first book on graph theory was written by Denes König and was issued in 1936.

In computer science, Graph theory is widely used in representing the network of communication. Also in Google Maps, the concept of graph theory is helpful in finding the shortest path in a road or network. Also, in the branch of science the 3D structure of complicated atomic structures can be studied quantitatively by gathering the statistics on graph-theoretic properties related to the topology of the atoms.

1.1.1 ADJACENT AND INCIDENT VERTICES

A graph is an ordered pair $(V(G), E(G))$, where $V(G)$ is set of vertices, $E(G)$ is set of edges linking to any unordered pair of vertices of G . A graph is called **simple graph**, if there is at most one edge linking to arbitrary two vertices of the graph and it has not a loop.

- If two vertices in a graph are connected by an edge, then we say the vertices are **adjacent**.

- A vertex is said to be **incident** with an edge if the vertex is one of the endpoints of that edge.

1.1.2 BIPARTITE GRAPHS

- Let $G = (V, E)$ be a graph. If vertices set V can be divided into two subsets A and B so that each edge of G has one end in A and one end in B , then we say G is a bipartite graph.
- In other words, a graph G is bipartite if and only if vertices set V of G can be divided into two independent sets of the graph.

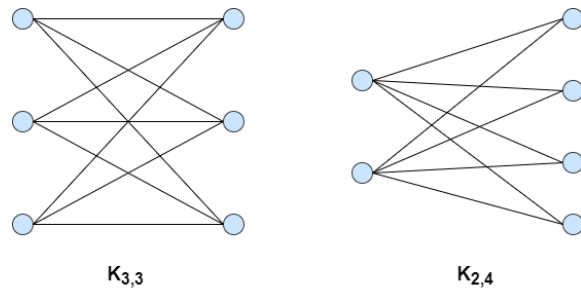


Fig 1.1 Bipartite graph

1.1.3 COMPLETE GRAPH AND COMPLETE BIPARTITE GRAPHS

COMPLETE GRAPH:

In the mathematical field of graph theory, a complete graph is a simple undirected graph in which each pair of distinct vertices are connected by a unique edge. Complete graph is denoted as K_n .

COMPLETE BIPARTITE GRAPH:

- In the field of graph theory, a complete bipartite graph (biclique) is a kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.
- A **complete bipartite graph** is a graph whose vertices are partitioned into two subsets namely V_1 , and V_2 , such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in the different subsets is also a part of the graph.
- That is, it is a bipartite graph (V_1, V_2, E) such that for every two vertices $v_1, \in V_1$, and $v_2, \in V_2$, v_1v_2 is an edge.

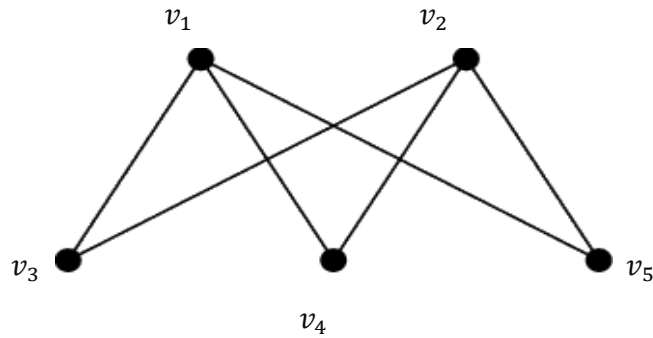


Fig 1.2 Complete bipartite graph

1.1.4 CYCLIC GRAPH

A graph containing at least one graph cycle is said to be cyclic graph. If the graph is not cyclic then it is said to be acyclic graph. Cyclic graphs are usually denoted as C_n . Some examples for cyclic graphs are mentioned below,

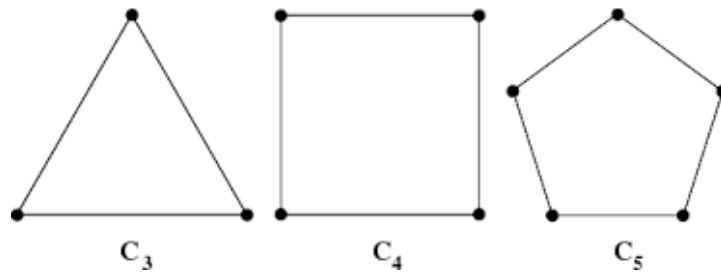


Fig 1.3 Cyclic Graphs

1.2 INTRODUCTION TO TOPOLOGY

Topology is the area of mathematics which investigates continuity and related concepts. The term topology actually refers to a specific mathematical idea central to the area of mathematics known as topology. A topology shows how elements of a set relate spatially to each other. i.e., the same set can have different topologies. For any instance, the real line, the complex plane, and the Cantor set can have same set with different topologies. There are different types of topologies such as

- General Topology
- Algebraic Topology
- Geometric Topology

- Differential Topology

The basic part of study is topological spaces, which are the sets equipped with a topology, that is, a family of subsets, called as open sets, which is closed under finite intersections and (finite or infinite) unions. The fundamental concepts of topology, such as continuity, compactness, and connectedness, can be defined in the terms of open sets. And, continuous functions take any nearby points to nearby points. Compact sets are the ones that can be covered by finitely many sets of arbitrarily small size. Connected sets are the sets that usually cannot be divided into two pieces that are far apart. The words nearby, arbitrarily small, and far apart can all be made precise by using the concept of open sets. Several topologies can be defined on any given space. Changing a topology consists of changing the collection of any open sets. This changes which the functions are continuous and which subsets are compact or connected.

1.2.1 CONDITIONS FOR TOPOLOGICAL SPACE

Let X be a non-empty set. A collection of subsets of x namely τ is said to be topology on X if it satisfies the following conditions:

- (i) $X, \emptyset$ are members of τ
- (ii) Arbitrary union of members of τ is again a member of τ
- (iii) Finite intersection of members of τ is again a member of τ

Then the pair (X, τ) is called **topological space**.

CHAPTER – 2

2.LITERATURE SURVEY

In 2013, M. Amiri et. al. has created a topology using vertices of an undirected graph. In 2018, K.A. Abdu and A. Kilic,man have investigated the topologies generated by directed graphs.

S. Somasundaram and R. Ponraj introduced mean labelling of graphs in (2003a) and investigate mean labelling for some standard graphs in (2003b) where it is useful to study the mean labelling of graphs and their applications.

C.Paul Bonnington published a book “The foundations of Topological graph theory” on 20 October 1995 which the author clearly explained about the fundamental concepts of topological graph theory and also discussed the major topics of topology.

John L. Kelley published “General Topology” on 7 March 2017 which is a systematic exposition of the part of general topology that has proven useful in different branches of mathematics, this volume is especially intended as background for modern analysis.

D.V. Gowda and K.S. Shashidara have proposed an article naming “Recent Advances in Graph theory and its applications”, this provides a description of implementations of graphical theory in a number of heterogeneous areas but focuses mostly on information science, electrical engineering, linguistics, physics and chemistry, computer network science, biotechnology and graphical theoretical applications.

Soon-Mo Jung and Doyun Nam proposed an article on July 2019 “Some properties of Interior and Closure in Topology” in July 2019. They introduced necessary and sufficient condition for an open subset of a closed subspace of a topological space to be open. As its duality, they also gave discussion about necessary and sufficient condition for a closed subset of an open subspace to be closed.

M. Shokri proposed “Generating Topology on Graphs by operation on graphs”, Applied Mathematical Sciences Vol-9 in the year 2015 which explains about the relationship between the topology and the operation on graphs.

Antoine Vella published a thesis on “A fundamentally topological perspective on Graph theory”. In this work he discussed the model of “classical” Topologized graphs that translates the graph isomorphism into topological homeomorphism, so that all these combinatorial concepts are expressed in a purely topological language.

Jonathan L. Gross, Thomas W. Tucker published a book named “Topological Graph theory” that gives comprehensive introduction which emphasizes graph imbedding but also covers the connections between topological graph theory and also the other areas of Mathematics. In this book the discussion of imbedding’s into surfaces is combined with a proof of classification of the closed surfaces.

R Manoharan and P. Thangavelu proposed an article on the topic “Some New sets and Topologies in Ideal Topological spaces” in January 2013 which is an eye-opening subject towards Ideal topological spaces and also, they investigated some theorems and conditions in the concept of Ideals and ideal topological space.

CHAPTER - 3

3. AIM AND SCOPE

3.1 AIM

In this project, we study graph properties and topological properties using various concepts like ideals, minimal adjacencies and some theorems explaining how topologies were generated in certain types of graphs. Adding to it, our aim is to find the relationship between the graph properties and topological properties on the non-isomorphic trees of order less than or equal to 4 (i.e., $n \leq 4$)

3.2 SCOPE

Listed below are some of the scopes from this project work:

- So many topological properties can be studied by defining a topology in graphs.
- New results can be found by analysing topologies defining on a graph.
- Studying new concepts of graph properties as well as topological properties.
- In future, many results and conclusion can be derived by this concept of ideals for the vertices of order $n > 4$.
- The ideal topology generated by the ideal of odd degree vertices can be analyzed.

CHAPTER - 4

4.METHODOLOGY AND THEOREMS

4.1 TOPOLOGIES GENERATED BY VERTEX SET OF GRAPH G

For any simple undirected graph, we can generate topologies with the help of minimal adjacency of the vertex of the graph and the base of the topology. At first, we consider any vertex 'v' and find out the adjacent vertices of v. Then using minimal adjacency concept, we write down the minimal adjacency of vertex 'v', then for the base of the topology we combinedly write all the possible minimal adjacency of every vertex v. Later from this we generate the topology.

4.1.1 τ TOPOLOGICAL SPACE

Let X be a non-empty set. A collection of subsets of x namely τ is said to be topology on X if it satisfies the following conditions:

- (i) $X, \emptyset$ are members of τ
- (ii) Arbitrary union of members of τ is again a member of τ
- (iii) Finite intersection of members of τ is again a member of τ

Then the pair (X, τ) is called **topological space**.

4.1.2 MINIMAL ADJACENCY OF THE VERTEX OF GRAPH G

Definition:

Let $G = (V, E)$ be a graph. Then the set of vertices becoming adjacent to a vertex 'u' is called **adjacency of u** and it is denoted $A_G(u)$. Minimal adjacency of u is defined as,

$$[u]_G = \bigcap_{u \in A_G(v)} A_G(v).$$

4.1.3 BASE CLASS OF THE TOPOLOGY (β)

Let $G = (V, E)$ be a simple undirected graph without any isolated vertices. Then the class $\beta = \{[u]_G : u \in V\}$ is a **base** for a topology on V.

4.1.4 τ_G GENERATION

Let $G = (V, E)$ be a simple undirected graph without any isolated vertices. Then the topology generated by $\beta = \{[u]_G : u \in V\}$ is called the topology generated by the graph G. It is denoted as τ_G .

This topology is of the form,

$$\tau_G = \{G \subseteq V: G = \bigcup_{[u]_G \in \beta} [u]_G, u \in U \subseteq V\}.$$

The class of closed sets of the above-mentioned topology is of the form:

$$C_G = \{G^c: G \in \tau_G\}$$

4.1.5 EXAMPLE FOR GENERATING TOPOLOGY

The below given graph is the graph of 6 vertices.

Let the vertex set be $V = \{x, y, z, t, u, v\}$

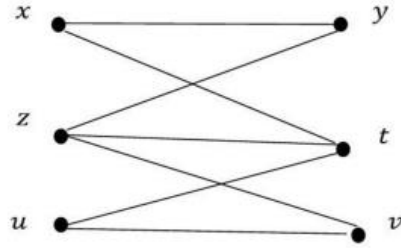


Fig 4.1 Example graph G (bipartite)

The adjacency of each vertex is,

$$\begin{aligned} A_G(x) &= \{y, t\} & A_G(u) &= \{t, v\} \\ A_G(y) &= \{x, z\} & A_G(v) &= \{u, z\} \\ A_G(z) &= \{y, t, v\} & A_G(t) &= \{u, z, x\} \end{aligned}$$

Minimal Adjacencies of the above-mentioned vertices are,

$$\begin{aligned} [x]_G &= \{z, x\}, [y]_G = \{t, y\}, [z]_G = \{z\}, \\ [u]_G &= \{u, z\}, [v]_G = \{v, t\}, [t]_G = \{t\} \end{aligned}$$

Therefore, the base class of the topology is given by,

$$\beta = \{\{z, x\}, \{t, y\}, \{z\}, \{t\}, \{u, z\}, \{v, t\}\}$$

and the topology τ_G generated by the given graph G is,

$$\begin{aligned}\tau_G = \{ & V, \emptyset, \{t\}, \{z\}, \{z, x\}, \{t, y\}, \{u, z\}, \{v, t\}, \{t, z\}, \{z, y, t\}, \{z, v, t\}, \\ & \{x, t, z\}, \{z, u, t\}, \{x, y, t, z\}, \{x, u, z\}, \{x, z, v, t\}, \{y, z, u, t\}, \{y, v, t\}, \\ & \{x, z, t, u\}, \{x, y, z, u, t\}, \{x, y, z, v, t\}, \{y, z, v, t\}, \{t, z, u, v\}, \\ & \{x, z, u, t, v\}, \{y, z, t, v, u\} \}\end{aligned}$$

τ_G is the topology generated by the graph G.

4.2 TOPOLOGIES ON CERTAIN GRAPHS

Different topologies can be generated by different types of graphs.

In this part, we will discuss about certain theorems which involves in obtaining topologies from graphs like complete graph, complete bipartite graphs and cyclic graphs.

4.2.1 TOPOLOGY ON A COMPLETE GRAPH

THEOREM 1:

Let $K_n = \{V, E\}$ be a complete graph, where $V = \{v_1, v_2, \dots, v_n\}$. Then the topology generated by K_n is a discrete topology on V.

PROOF:

The minimal neighbourhoods of vertices set $V = \{v_1, v_2, \dots, v_n\}$ were mentioned below,

$$[v_1]_G = \{v_1\}$$

$$[v_2]_G = \{v_2\}$$

.....

.....

$$[v_n]_G = \{v_n\}$$

Thus,

$$\beta K_n = \{\{v_n\} : v_n \in V\}$$

The topology generated by K_n is as follows:

$$\tau K_n = P(V).$$

From this proof we observe that τK_n is **discrete topology on V**.

HENCE PROVED

4.2.2 TOPOLOGY ON A CYCLIC GRAPH

THEOREM 2:

Let $C_n = (V, E)$ be a cycle whose vertices set is $V = \{v_1, v_2, \dots, v_n\}$ where $n \geq 3$, then the topological space generated by the cycle $C_n = (V, E)$ is a discrete topological space.

PROOF:

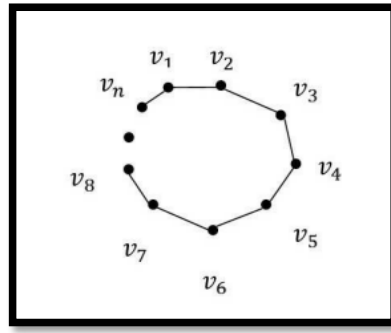


Fig 4.2 The cycle C_n graph

- The adjacencies of the vertices in the above figure are,

$$A_G(v_1) = \{v_n, v_2\}, A_G(v_2) = \{v_1, v_3\}, A_G(v_3) = \{v_2, v_4\},$$

$$\dots\dots\dots, A_G(v_{n-1}) = \{v_{n-2}, v_n\} \text{ and } A_G(v_n) = \{v_{n-1}, v_1\}.$$

- The minimal adjacency of each vertex is given as,

$$[v_1]_G = \bigcap_{v_1 \in A_G(v_i)} A_G(v_i) = \{v_1\} \text{ and } [v_2]_G = \bigcap_{v_2 \in A_G(v_i)} A_G(v_i) = \{v_2\}$$

$$\dots\dots \text{ and } [v_{n-1}]_G = \bigcap_{v_{n-1} \in A_G(v_i)} A_G(v_i) = \{v_{n-1}\}$$

$$\text{and } [v_n]_G = \bigcap_{v_n \in A_G(v_i)} A_G(v_i) = \{v_n\}$$

- $\beta = \{\{v_n\} : v_n \in V\}$, where β is a base for discrete topology on V. Also, we observe that the topological space generated by the graph C_n is the **discrete topological** space on U.

- Assuming $n = 4$, the graph C_4 is a complete bipartite graph. The topological space generated the graph C_4 whose vertex set is $V = \{v_1, v_2, v_3, v_4\}$ is τC_n , where $\tau C_n = \{V, \emptyset, \{v_1, v_4\}, \{v_2, v_3\}\}$. This topology is not a discrete topology, but it is a **quasi-discrete topology**.

4.2.3 TOPOLOGY ON A COMPLETE BIPARTITE GRAPH

THEOREM 3:

Let $K_{A,B} = (V, E)$ be a complete bipartite graph. Then the topology

PROOF:

Since $K_{A,B}$ is a bipartite graph, for every $x \in V$, $x \in A$ or $x \in B$

Implies that, $A \cap B = \emptyset$ and $A \cup B = V$

Let $x \in A$. implies $A_{K_{A,B}}(x) = B$ and $[x] K_{A,B} = A$

Let $x \in B$ implies $A_{K_{A,B}}(x) = A$ and $[x] K_{A,B} = B$

Thus, the base of the topology generated by the graph $K_{A,B}$ is given by,

$$\beta K_{A,B} = \{A, B\}$$

- The topology generated by $K_{A,B}$ is as follows:

$$\tau K_{A,B} = \{V, \emptyset, A, B\}$$

From this we conclude that, $\tau K_{A,B}$ is the quasi – discrete topology on U

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4.2.4 CONTINUITY PROPERTY OF A FUNCTION VIA TWO GRAPHS

In this part we will discuss about the continuity property of a function via two graphs and the generated topologies with a help of theorem that consists of necessary and sufficient condition.

THEOREM 4:

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs without isolated vertices. Let τ_G and τ_G' be the topologies generated by the two graphs G and G' respectively and let $f: (V, \tau_G) \rightarrow (V', \tau_G')$ be a function. Then f is continuous if and only if for every $v \in V$, $f([v]_G) \subseteq [f(v)]_{G'}''$.

PROOF:

PROOF FOR FIRST PART:

Let the function f be defined $\mathbf{f: (V, \tau_G) \rightarrow (V', \tau_{G'})}$ be a continuous function. We need to prove that $f([v]_G) \subseteq [f(v)]_{G'}$.

Now $\beta = \{[v]_G : v \in V\}$ be the base of the topology τ_G .

And $\beta' = \{[v']_{G'} : v' \in V'\}$ be the base of the topology $\tau_{G'}$.

Since f is continuous, there is $B \in \beta$ such that $f(B) \subseteq [f(v)]_{G'}$ for every $v \in V$. $[v]_G$ is the minimal element containing v of β .

Therefore, we can say that

$$f([v]_G) \subseteq [f(v)]_{G'} \text{ ----- (1)}$$

Hence proved the first part.

PROOF FOR SECOND PART:

Conversely, let us consider $f([v]_G) \subseteq [f(v)]_{G'}$ for every $v \in V$.

To prove: f is a continuous function

Since it's been observed that $[f(v)]_{G'} \in \beta'$ for every $v \in V$

Also $f([v]_G) \subseteq [f(v)]_{G'}$ and $[v]_G \in \beta_{G(v)}$

We can conclude that $\mathbf{f: (V, \tau_G) \rightarrow (V', \tau_{G'})}$ is a **continuous function** ----- (2)

Hence proved the second part.

Therefore, from equations (1) and (2), the if and only if condition is satisfied.

HENCE PROVED.

CHAPTER - 5

5.RESULTS AND FURTHER PROOFS

5.1 INTRODUCTION TO IDEAL TOPOLOGICAL SPACES

A triplet $(X, \tau, \mathfrak{I})$ is called as an ideal topological space where X is non-empty set, $\mathfrak{I}$ is an ideal of subsets of X and τ is a topology on X . In this chapter, we will generate a topology using the minimal adjacency of each vertex of a simple finite undirected graph. Later we study some of the new topological spaces on a graph using ideals and topology. Also, we examine some relationship between the topological properties and graph properties via ideals.

5.1.1 IDEALS AND TOPOLOGICAL SPACES

- A triplet $(X, \tau, \mathfrak{I})$ is called an ideal topological space where X is non-empty set, $\mathfrak{I}$ is an ideal of subsets of X and τ is a topology on X .
- A nonempty collection of subsets of a set X is said to be an ideal on set X , when it satisfies the two conditions listed below,
 - i. If $A \in \mathfrak{I}$ and $B \subseteq A$, then $B \in \mathfrak{I}$
 - ii. If $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$, then $A \cup B \in \mathfrak{I}$

$\Rightarrow$ An ideal topological space $(X, \tau, \mathfrak{I})$ (also known as ideal space) is a topological space (X, τ) with an ideal $\mathfrak{I}$ defined on the set X . Also let (X, τ) be a topological space with an ideal $\mathfrak{I}$, then for any subset A of X ,

$$A^*(\mathfrak{I}, \tau) = \{x \in X / U_x \cap A \notin \mathfrak{I} \text{ for every neighborhood } U_x \text{ of } x\}$$

$$\text{and, } Cl^*(A) = A \cup A^*$$

5.1.2 TOPOLOGY GENERATED BY CONCEPT OF MINIMAL ADJACENCY OF VERTEX

Let $G(V, E)$ be a simple finite undirected graph, then β is the class which is generated as,

$$\beta = \{ [v]_G : v \in V \}$$

And the topology generated by the base class β is denoted as,

$$\tau_G = \{ \bigcup_{[v]_G \in \beta} [v]_G : v \in V \}$$

Let us discuss this concept with a simple finite undirected graph which consists of 3 vertices v_1, v_2 and v_3 .

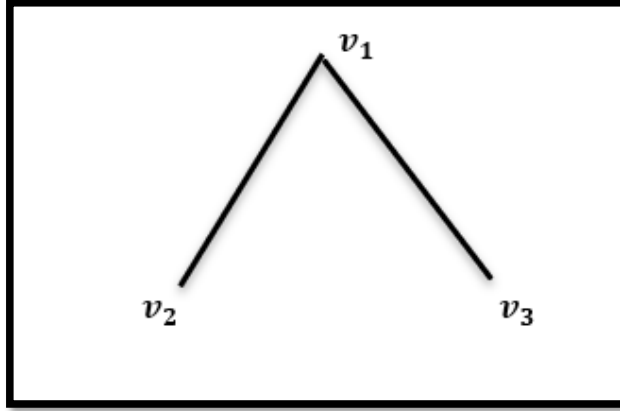


Fig 5.1 Graph G with 3 vertices

By the concept of minimal adjacency of graphs, for the above graph consisting of 3 vertices v_1 , v_2 and v_3 the adjacency for each vertex is denoted as,

$$A_G(v_1) = \{ v_2, v_3 \}$$

$$A_G(v_2) = \{ v_1 \}$$

$$A_G(v_3) = \{ v_1 \}$$

Thus, the minimal adjacency is given by,

$$[v_1]_G = \{v_1\} \cap \{v_1\} = \{v_1\}$$

$$[v_2]_G = \{ v_2, v_3 \}$$

$$[v_3]_G = \{ v_2, v_3 \}$$

The base class is given by,

$$\beta = \{ \{v_1\}, \{ v_2, v_3 \} \}$$

So, the topology τ is generated as follows,

$$\tau_G = \{V, \phi, \{v_1\}, \{ v_2, v_3 \} \}$$

$$O = \{ v_2, v_3 \} \text{ (Set of vertices of odd degree)}$$

The ideal $\mathfrak{I}$ on the topological space is defined as,

$$\mathfrak{I} = \wp(o) = \{ \phi, \{v_2\}, \{v_3\}, \{v_2, v_3\} \}$$

Then $A^* = \{x \in V / U_x \cap A \notin \mathfrak{I}, \text{ for every neighbourhood } U_x \text{ of } x\}$ for all $A \subseteq V$ is given by,

$$\{v_1\}^* = \{v_1\} \quad \{v_1, v_2\}^* = \{v_1\}$$

$$\{v_2\}^* = \emptyset \quad \{v_2, v_3\}^* = \emptyset$$

$$\{v_3\}^* = \emptyset \quad \{v_3, v_1\}^* = \{v_1\}$$

Then $Cl^*(A) = A \cup A^*$,

$$Cl^*\{v_1\} = \{v_1\} \cup \{v_1\} = \{v_1\} \quad Cl^*\{v_1, v_2\} = \{v_1, v_2\} \cup \{v_1\} = \{v_1, v_2\}$$

$$Cl^*\{v_2\} = \{v_2\} \cup \emptyset = \{v_2\} \quad Cl^*\{v_2, v_3\} = \{v_2, v_3\} \cup \emptyset = \{v_2, v_3\}$$

$$Cl^*\{v_3\} = \{v_3\} \cup \emptyset = \{v_3\} \quad Cl^*\{v_3, v_1\} = \{v_3, v_1\} \cup \{v_1\} = \{v_3, v_1\}$$

If $Cl^*(A) = A$, then A is closed. So $A^C \in \tau_G^*$.

Therefore, $\tau_G^* = \{V, \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$

Hence in this graph, τ_G^* is generated using the property of ideal topological spaces.

Also we observe that $\tau_G \subset \tau_G^*$.

5.2 RELATION BETWEEN NON-ISOMORPHIC TREES AND IDEAL TOPOLOGICAL SPACES

Let us now study the relationship between non – isomorphic trees of order less than or equal to 4 and the topological properties via ideals.

Using ideals, we study the topological and graph properties with the help of the below table:

Table 5.1 Comparison for ORDER 1 (n = 1)

ORDER OF TREE n	NON-ISOMORPHIC TREE	τ_G	τ_G^*	OBSERVATION
n=1 (v ₁)	 v ₁	$\tau_G = \{V, \emptyset\}$	$\tau_G^* = \{V, \emptyset\}$	$\tau_G = \tau_G^*$

Table 5.2 Comparison for ORDER 2 (n = 2)

ORDER OF TREE n	NON-ISOMORPHIC TREE	τ_G	τ_G^*	OBSERVATION
n=2 (v ₁ ,v ₂)		$\tau_G = \{V, \emptyset, \{v_1\}, \{v_2\}\}$	$\tau_G^* = \{V, \emptyset, \{v_1\}, \{v_2\}\}$	$\tau_G = \tau_G^* = \wp(V)$ Discrete topology

Table 5.3 Comparison for ORDER 3 (n = 3)

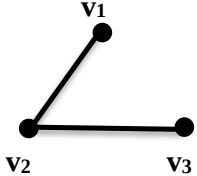
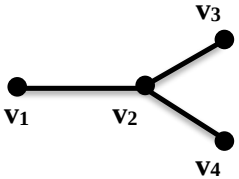
ORDER OF TREE n	NON-ISOMORPHIC TREE	τ_G	τ_G^*	OBSERVATION
n=3 (v ₁ ,v ₂ ,v ₃)		$\tau_G = \{V, \emptyset, \{v_2\}, \{v_3, v_1\}\}$	$\tau_G^* = \{V, \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$	$\tau_G \subset \tau_G^*$

Table 5.4 Comparison for ORDER 4 (n = 4)

ORDER OF TREE n	NON-ISOMORPHIC TREE	τ_G	τ_G^*	OBSERVATION
n=4 (v ₁ ,v ₂ ,v ₃ ,v ₄)	(a) 	$\tau_G = \{V, \emptyset, \{v_2\}, \{v_1, v_3, v_4\}\}$	$\tau_G^* = \wp(V)$	$\tau_G \subset \tau_G^*$
	(b) 	$\tau_G = \{V, \emptyset, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3\}\}$	$\tau_G^* = \{V, \emptyset, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$	$\tau_G = \tau_G^*$

Considering the non-isomorphic trees for the vertex set V_n where $n \leq 4$ we have found the relationship between the graph properties and topological properties. Also, the observations from the tabular column are noted down.

Similarly for higher order of n , many results and properties can also be found and discussed.

5.3 RESULT AND CONCLUSION

From the above tables 5.1, 5.2, 5.3 and 5.4, it is evident that if a tree is of order 1 and 2 (i.e) complete graph, we get discrete topology. We have approached new topological properties via ideals. Also, different ideals can result in getting different topologies. From this paper, topological properties with respect to trees can be analyzed for the vertex set V_n where $n=1, 2, 3$ and 4 . Also for higher order n , topologies and closures $cl^*(A)$ can be generated using minimal adjacency and ideals of the graph.

CHAPTER - 6

6.CONCLUSION

This project work helps people to understand the properties of the Topologies and certain graphs. We discussed some theorems and obtained results from it. Using the concept of ideal topological space, we can observe so many relationships between the graph properties as well as topological properties. For different types of non-isomorphic trees of order $n \leq 4$, we analysed how τ_G and τ_G^* are generated and the observations are noted down. Adding to it, we approached some of the topological properties in graphs via ideals.

In future, the research regarding this concept can be proceeded with different types of graphs having higher order vertices and results can be obtained. Also, we can generate many τ_G and τ_G^* from which new relations and results can be found.

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