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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – I – ELECTOSTATICS AND MAGNETOSTATICS – SPH5105

ELECTROMAGNETIC THEORY

UNIT-I

SPH 5105

UNIT 1 ELECTROSTATICS AND MAGNETOSTATICS **12 Hrs.**

Dielectric and its Polarization – External Field of a Dielectric Medium – Electric Field Inside a Dielectric – Dielectric Constant and Displacement Vector – Relation Between **D**, **P** and **E**-Polarization of Non-Polar Molecules (Clausius-Mossotti Relation) – Polarization of Polar Molecules – Electrostatic Energy. **Magnetostatics:** Ampere's Circuital Law – Magnetic Scalar Potential – Magnetic Vector Potential – Magnetisation and Magnetisation Current – Magnetic Intensity – Magnetic Susceptibility and Permeability.

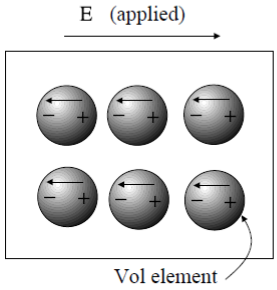
1.0 Aims and Objectives

This lesson deals with potential and fields due to electric dipole. The relation between electric susceptibility, polarization, displacement will be obtained. The molecular field, derivation of claussion mossotti relation for non-polar molecules, Debge formula for polar molecules are explained in detail. The derivation of electrostatic energy and energy density who has has discussed.

Polarization of Dielectrics

Dielectrics are materials which have no free charges; all electrons are bound and associated with the nearest atoms. An external electric field causes a small separation of the centres of the electron cloud and the positive ion core so that each infinitessimal element of volume behaves as an electric dipole. Dielectrics may be subdivided into two groups :

Non-Polar	which behave as above
Polar	in which the molecules or atoms possess a permanent dipole moment which is ordinarily randomly oriented, but which become more or less oriented by the application of an external electric field.



The induced dipole field opposes the applied field. In the diagram shown opposite the volume element indicated could represent an atom, a molecule, or a small region.

The type of polarization on a microscopic scale is determined by the material. Most materials exhibit polarization only in the presence of an external field. A few however show permanent polarization:

Ferroelectric	crystals exhibit spontaneous permanent polarization.
Electrets	become permanently polarized if allowed to solidify in the presence of a strong electric field.

The type of polarization may be additionally subdivided into the following categories :

Electronic	a displacement of the electronic cloud w.r.t the nucleus.
Ionic	separation of +ve and -ve ions in the crystal.
Orientational	alignment of permanent dipoles (molecules).
Space-charge	free electrons are present, but are prevented from moving by barriers such as grain boundaries - the electrons "pile up".

The Electric Polarization P is the dipole moment per unit volume at a given point**.**

$\mathbf{P} = N\mathbf{p}$ where $\mathbf{p}$ is the average dipole moment per molecule 1.
N is the number of molecules per unit volume.

Any molecule develops a dipole moment which is proportional to the applied field

$\mathbf{p} = \alpha\mathbf{E}$ where α is the polarizabilty. 2.

Example The electronic polarizability of a simple atom.
With an applied field the electron cloud is displaced until the mutual attractive force between it and the ion core is is just balanced by that produced by the field **E**

$$F = eE = e \cdot \frac{1}{4\pi\varepsilon_0 r^2} \cdot \frac{r^3}{R^3} e$$

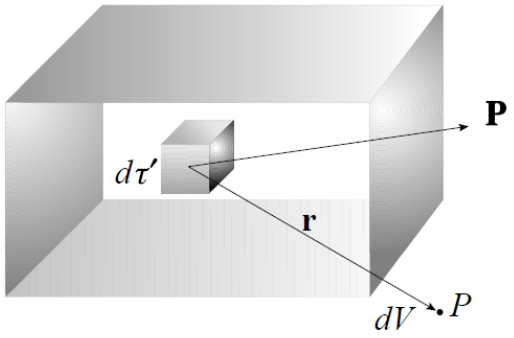
$$eE = \frac{e^2 r}{4\pi\varepsilon_0 R^3}$$

$$gives\ p = er = 4\pi\varepsilon_0 R^3 E = \alpha_e E \qquad 3.$$

where α_e = electronic polarizabilty = $4\pi\varepsilon_0 R^3$

The Electric Field at an Exterior point

Consider a block of dielectric, **P** = dipole moment per unit volume.
We need to calculate the potential V at an exterior point.



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2}\,d\tau' \qquad 4.$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_\tau \mathbf{P}\cdot\nabla'\left(\frac{1}{r}\right)d\tau' \qquad 5.$$

integrating over the block

Now $\nabla\cdot f\mathbf{A} = f\left(\nabla\cdot\mathbf{A}\right) + \mathbf{A}\cdot\nabla f$ 6.

so, putting $f = \left(\frac{1}{r}\right)$ and $\mathbf{P} = \mathbf{A}$

$$V = \frac{1}{4\pi\varepsilon_0}\int_\tau\left(\nabla'\cdot\frac{\mathbf{P}}{r}\right)d\tau' - \frac{1}{4\pi\varepsilon_0}\frac{(\nabla'\cdot\mathbf{P})}{r}\,d\tau' \qquad 7.$$

By the divergence theorem this becomes

$$V = \frac{1}{4\pi\varepsilon_0}\int_{s'}\frac{\mathbf{P}\cdot\mathbf{da}'}{r} - \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\nabla'\cdot\mathbf{P}}{r}\,d\tau' \qquad 8.$$

for the potential due to the dipoles at the external point.

Both of the terms in equation 8 have the form of potentials produced by charge distributions ;

i.e. a surface charge density $\sigma_b = \mathbf{P}\bullet\mathbf{n}$ where n is the outward unit normal vector
and a volume charge density $\rho_b = -\nabla'\bullet\mathbf{P}$

Thus $$V = \frac{1}{4\pi\varepsilon_0}\left[\int_s\frac{\sigma_b\,da}{r} + \int_\tau\frac{\rho_b\,d\tau}{r}\right]$$ due to the dielectric only 9.

and

$$\mathbf{E} = -\nabla V = \frac{1}{4\pi\varepsilon_0}\left[\int_{s'}\frac{\sigma_b\,da'\,\hat{\mathbf{r}}}{r^2} + \int_{\tau'}\frac{\rho_b\,d\tau'\,\hat{\mathbf{r}}}{r^2}\right] \qquad 10.$$

The Macroscopic Field

To the above must be added the effects of the external charge distributions that are responsible for the polarization. These are simply additive.

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_s\frac{(\sigma_f+\sigma_b)}{r}\,da + \int_\tau\frac{(\rho_f+\rho_b)}{r}\,d\tau\right] \qquad 11.$$

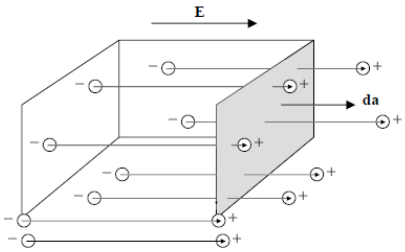
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\int_s\frac{(\sigma_f+\sigma_b)\,\hat{\mathbf{r}}}{r^2}\,da + \int_\tau\frac{(\rho_f+\rho_b)\,\hat{\mathbf{r}}}{r^2}\,d\tau\right] \qquad 12.$$

This expression is completely general.
The effect of these expressions is that for the purposes of calculation the dielectric may be replaced by the equivalent bound charge distributions σ_b and ρ_b without affecting the field outside the dielectric.
Example cylindrical electret.

The Bound Charge Densities ρ_b and σ_b

We now demonstrate how the displacement of charges within the dielectric results in net volume and surface charge densities.

Consider a small volume inside the dielectric, where the electric field **E** is the resultant of an external field and the field due to the dipoles. The positive and negative charges are separated by an average distance s due to the influence of **E**. Consider the element of surface da and the charge which has crossed it. If we fix the origin in the negative charges we need only consider the movement of the positive charges.



Then, the amount of charge dQ crossing da is just the amount of positive charge within the volume $d\tau = \mathbf{s} \cdot \mathbf{da}$

i.e. $dQ = NQ\,\mathbf{s} \cdot \mathbf{da}$ $\quad Q\mathbf{s} = \mathbf{p}$, $\quad NQ\mathbf{s} = \mathbf{P}$

i.e. $dQ = \mathbf{P} \cdot \mathbf{da}$ 13.

If da is on the surface of the material, this charge accumulates there in a layer of thickness $\mathbf{s} \cdot \mathbf{n}$ (which is small, of molecular dimensions) and the charge can be treated as a surface layer with density

$$\sigma_b = dQ/da = \mathbf{P} \cdot \mathbf{n} \qquad 14.$$

We can similarly show that $-\nabla \cdot \mathbf{P}$ represents a real volume density of charge as follows.

The net charge flowing out of a volume τ across the elementary area da of its surface is $\mathbf{P} \cdot \mathbf{da}$ as found above in (13). Thus the total charge flowing **out** of the surface bounding τ is the integral of this over the surface, *i.e.*

$$Q = \int_s \mathbf{P} \cdot \mathbf{da}$$

and the net charge remaining within is $-Q$.

If the density of this remaining charge is ρ_b then

$$\int_\tau \rho_b d\tau = -Q = -\int_s \mathbf{P} \cdot \mathbf{da} \qquad 15.$$

$$= -\int_\tau (\nabla \cdot \mathbf{P})\, d\tau$$

hence $$\rho_b = -\nabla \cdot \mathbf{P} \qquad 16.$$

i.e. the bound charge density is numerically equal to minus the divergence of the polarization. An important consequence of this is that if the polarization is uniform within a region and its divergence is zero, then so is the bound charge density ρ_b.

div E in Dielectrics

Gauss's law states

$$\int_s \mathbf{E} \cdot \mathbf{da} = \int_\tau \nabla \cdot \mathbf{E}\, d\tau = \frac{Q}{\varepsilon_0} \qquad 17.$$

Q here is the total charge enclosed, which for dielectrics must include free as well as bound charges. *i.e.*

$$Q = \int_\tau (\rho_f + \rho_b)\, d\tau \qquad 18.$$

where the integration is intended to cover both volume and surface distributions. Hence substituting (18) into (17) and equating the integrands of the volume integrals then

$$\nabla \cdot \mathbf{E} = \frac{(\rho_f + \rho_b)}{\varepsilon_0} \qquad 19.$$

or

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho_t}{\varepsilon_0}} \qquad 20.$$

This is Gauss's law in its more general form and is one of the four **Maxwell's Equations.**. In obtaining it we have implicitly assumed that space derivatives of **E** exist. This is not the case at the interface between media and where this applies we must use the integral form i.e we must integrate $\mathbf{E} \cdot \mathbf{da}$ over a closed surface.

N.B. Since $\mathbf{E} = -\nabla V$ this leads to $\nabla^2 V = -\rho_t/\varepsilon_0$ Poisson's equation.

The Electric Displacement

Now

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad 21.$$

$$\therefore \quad \nabla \cdot E = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right) \qquad 22.$$

or $$\nabla \cdot \left(\varepsilon_0 \mathbf{E} + \mathbf{P}\right) = \rho_f \qquad 23.$$

i.e. the vector $(\varepsilon_o \mathbf{E} + \mathbf{P})$ has a divergence that depends only on ρ_f.
This is called the *Displacement*, $\mathbf{D} = (\varepsilon_o \mathbf{E} + \mathbf{P})$ 24.

i.e. $$\nabla \cdot \mathbf{D} = \rho_f \qquad 25.$$

This is the equivalent Maxwell's equation for a dielectric. Note it does not contain the permittivity ε and is thus **independent of the medium**. It can be regarded as Gauss's law for **D**. In its integral form it becomes

$$\int_s \mathbf{D} \cdot \mathbf{da} = \int_\tau \rho_f \, d\tau \qquad 26.$$

Note that both div**D** and $\int_s \mathbf{D} \cdot \mathbf{da}$ are unaffected by bound charges.

From the definition of displacement $(\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P})$ we have

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} - \frac{\mathbf{P}}{\varepsilon_0} \qquad 27.$$

i.e. the electric field inside the dielectric is the resultant of two fields,
$\mathbf{D}/\varepsilon_0$ associated with free charges, since $\nabla \cdot (\mathbf{D}/\varepsilon_0) = \rho_f / \varepsilon_0$ 28.

and $-\mathbf{P}/\varepsilon_0$ associated with bound charges, since $\nabla \cdot (-\mathbf{P}/\varepsilon_0) = \rho_b / \varepsilon_0$ 29.

NB Lines of **D** begin and end only on free charges.
Lines of **E** begin and end on either free or bound charges.

In writing down expressions for the divergence of **E** and **P** we have implicitly assumed their existence. It should be noted that the space derivatives do not exist at a point charge or at the interface between two media. In such cases the integral form of Gauss's law must be used.

The Susceptibility

Provided that $\mathbf{P} \propto \mathbf{E}$, which it is in practice for moderate fields

since $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ 30.

$= \varepsilon_0\mathbf{E}(1 + \chi_e)$ $\qquad \chi_e = \mathbf{P}/\varepsilon_0\,\mathbf{E}$ 31.

χ_e is the *electric susceptibility*

i.e. $\mathbf{P} = \varepsilon_0\,\chi_e\,\mathbf{E}$ indicates the relative ease of polarization. 32.

The Permittivity

$\mathbf{D} = \varepsilon_0(1 + \chi_e)\mathbf{E}$ 33.

$= \varepsilon_0\varepsilon_r\mathbf{E}$ $\quad \varepsilon_r$ is the relative permittivity or dielectric constant 34.

$\mathbf{D} = \varepsilon\mathbf{E}$ $\qquad$ in vacuum $\chi = 0$, ε_r or $k = 1$ 35.

where $\varepsilon = \varepsilon_0\varepsilon_r$ $\qquad$ is the permittivity. 36.

ε_r or k are usually in the range 1 - 7, but some non-linear materials have k as high as 10^5.
Pure water has k ~ 80. For all materials k is a function of frequency.

In class A dielectrics $D \propto E$, and then since ε_r is a constant for class A, then

$\nabla \cdot \mathbf{D} = \rho_f$ 37.

and if $\varepsilon_r \neq f(x,y,z)$ $\quad \nabla \cdot \mathbf{E} = \rho_f/\varepsilon$ 38.

$\therefore \quad \nabla^2 V = -\rho_f/\varepsilon$ Poisson's equation for class A dielectrics 39.

If the material is not class A then

$$\nabla^2 V = -\rho_t/\varepsilon_0 = (\rho_f + \rho_b)/\varepsilon_0 \qquad 40.$$

The Relationship between ρ_f and ρ_b

In a class A dielectric

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E} = \frac{\varepsilon_r - 1}{\varepsilon_r}\mathbf{D} \qquad 41.$$

$$\therefore \quad \nabla \cdot \mathbf{P} = \left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)\nabla \cdot \mathbf{D} = \left(1 - \frac{1}{\varepsilon_r}\right)\rho_f \qquad 42.$$

$$\text{or} \quad -\rho_b = \left(1 - \frac{1}{\varepsilon_r}\right)\rho_f \qquad 43.$$

The total charge density is $\rho_t = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P} = \dfrac{\rho_f}{\varepsilon_r}$ 44.

This is smaller than ρ_f. ρ_f and ρ_b have opposite signs.
Note also that if ρ_f is zeri (in class A dielectrics) then so is ρ_b. This is nearly always the case. Thus the bound charges will nearly always be located only on the surface of the dielectric.

The Surface Charge Densities σ_b and σ_f

At the interface between dielectric and conductor there is :

a bound charge density σ_b on the dielectric

a free charge density σ_f on the conductor.

In the steady state $\mathbf{E} = 0$ inside the conductor, and inside the dielectric, from Gauss's Law

$$\varepsilon_0 E = \sigma_f + \sigma_b \qquad 45.$$

and $\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_0\,\mathbf{E} = \sigma_f$ from Gauss 46.

Hence $\sigma_f + \sigma_b = \sigma_f/\varepsilon_r$ (cf 44.) 47.

Boundary Conditions

The calculation of electric field variation across a boundary between two media needs a knowledge of the boundary conditions. These are :
At the boundary between media

- V is continuous $V_1 = V_2$

  (else the field is infinite)

- The normal component of D is continuous
  or discontinuous by the free charge density at the interface $D_{n1} = D_{n2}\ (+\sigma_f)$

This can be shown by applying Gauss's Law to a small pillbox at the boundary. The net outward displacement = free charge enclosed. Note that between dielectrics the free charge density σ_f is usually zero. At a dielectric / conductor interface, if E is constant, then D = 0 inside the conductor, and $D_n = \sigma_f$ in the dielectric.

- The tangential component of E is continuous. $E_{t1} = E_{t2}$

This can be shown by evaluating the line integral $\oint E \cdot dl$ around a thin rectangular loop lying parallel to the boundary. If the boundary is between a dielectric and a conductor, then E = 0 in the conductor, and hence $E_t = 0$ in both media. Therefor E must be normal at the surface of a conductor.

Forces on Dielectrics

A dipole in a uniform field experiences a *torque*, but no net force

$$\tau = p \times E$$

A net force is experienced only in a non-uniform field

$$F = (p \cdot \nabla)E$$

or force per unit volume $\bar{F} = N(p \cdot \nabla)E = (P \cdot \nabla)E$

and $P = (\varepsilon - \varepsilon_0)E$

$$\bar{F} = (\varepsilon - \varepsilon_0)(E \cdot \nabla)E = \frac{1}{2}(\varepsilon - \varepsilon_0)\nabla E^2$$

$$= \frac{\varepsilon_r - 1}{\varepsilon_r}\nabla\left(\frac{1}{2}\varepsilon E^2\right)$$

Ex. Calculate the force in the dielectric of a coaxial cable, radii R_1, R_2 permitivitty ε with charge per unit length λ.

By Gauss $E = \dfrac{\lambda}{2\pi\varepsilon\rho}$, $V = \displaystyle\int_{R_1}^{R_2} E\,d\rho = \frac{\lambda}{2\pi\varepsilon}\ln\frac{R_2}{R_1}$

$$E = \frac{\lambda}{2\pi\varepsilon\rho} = \frac{V}{\rho\ln(R_2/R_1)}$$

$$\therefore \frac{F}{\tau} = \frac{1}{2}\varepsilon_0(\varepsilon_r - 1)\frac{V^2}{\ln^2(R_2/R_1)}\left|\nabla\frac{1}{\rho^2}\right| = -\frac{\varepsilon_0(\varepsilon_r - 1)V^2}{\ln^2(R_2/R_1)\,\rho^3}$$, radial, inwards.

1.6 MOLECULAR FIELD IN A DIELECTRIC (CLAUSIUS-MOSSOTTI RELATION)

The aim of this section is to examine the molecular nature of the dielectric and to consider how the electric field & responsible for polarizing the molecule is related to the macroscopic electric filed. *The electric field which is responsible for polarizing a molecule of*

the dielectric is called the molecular field. This is the electric field at a molecular field or local field or local field is produced by all external sources and by all polarized molecules in the dielectric, except one molecule under consideration.

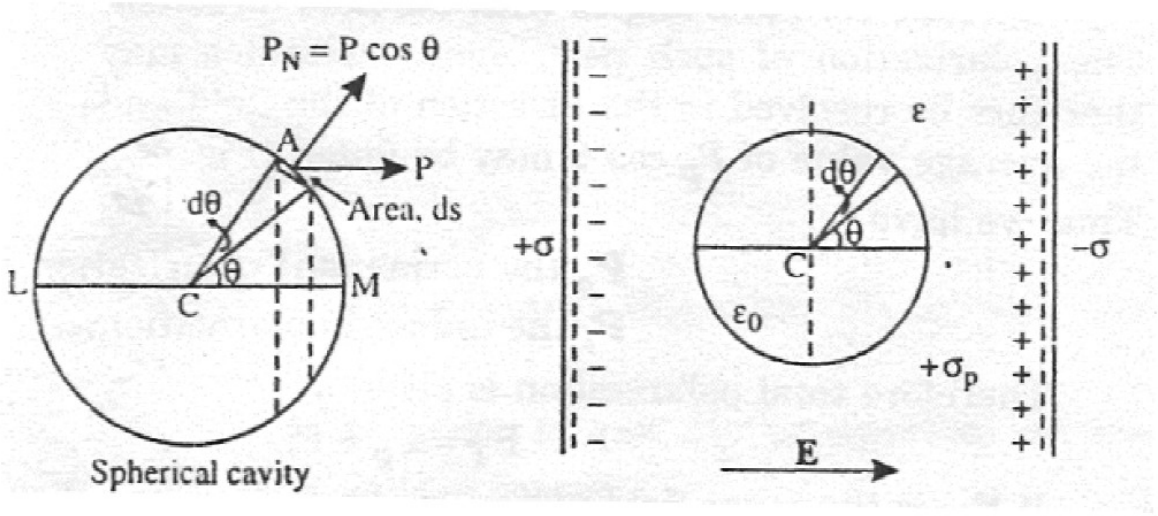


Fig.1.6(a) Fig.1.6(b)

The molecular field, E_{in}, may be calculated in the following way. Let us cut a spherical cavity of radius r (such that its dimensions are very great as compared to the molecular dimensions and very small as compared to the volume of the dielectric) surrounding the point at which the molecular field is to be computed. The dielectric which is now left, will be treated as a continuum. The cavity is put in its original position in the dielectric (molecule by molecule) except the molecule where the molecular field is to be computed. The molecules which are just replaced in the cavity are treated as individual dipoles and not as a continuum.

Let the dielectric be placed in the uniform electric field between two parallel plates (of a condenser) as shown in fig (1.6b). The dotted lines show the boundary of the dielectric. Let the surface density of real charges on the capacitor plates be σ. Again let the surface of cavity has polarized charges of surface density σ_p.

The field experienced by the molecule of the dielectric at the centre of the cavity **C**, $\mathbf{E_{in}}$, given by

$$\mathbf{E_{in}} = \mathbf{E_1} + \mathbf{E_2} + \mathbf{E_3} + \mathbf{E_4}$$

Where

(i) $\mathbf{E_1}$ is the field between two plates with no dielectric, so that $\mathbf{E_1} = \sigma / \varepsilon_0$,

(ii) $\mathbf{E_2}$ is the field at C due to polarized charges on the plane surfaces of the dielectric facing the capacitor plates and is given by $\mathbf{E_2} = \sigma_p / \varepsilon_0$.

(iii) $\mathbf{E_3}$ is the field at C due to polarized charges on the surface of cavity, to be calculated.

(iv) $\mathbf{E_4}$ is the field at C due to permanent dipoles. But in present case for non-polar isotropic dielectrics $\mathbf{E_4} = 0$.

Thus,
$$E_{in} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0} + E_3 \qquad (1)$$

Evaluation of E_3: Consider a small elemental area *ds* on the surface of cavity of an angular width $d\theta$ and an angle θ with the direction field **E**. The vector **P** shows the direction of displacement at the centre of **ds**, Fig (1.6a). The normal component of displacement is

$$\mathbf{P_N} = \boldsymbol{P} \cos\theta.$$

By definition of polarization, it is the surface charge per unit area. Such a charge on *ds* to provide flux normal to *ds* is,

$$\mathbf{P_N} = \boldsymbol{P} \cos\theta \, ds$$

and the electric intensity at C due to this charge is given by,

$$= \frac{P \cos\theta ds}{4\pi\varepsilon_0 r^2}$$

where r is the radius of cavity. The field is directed along the radius CA. Resolving the intensity along and perpendicular to the applied field, we have the components

$$= \frac{P\cos\theta ds}{4\pi\varepsilon_0 r^2}\cos\theta \qquad \text{along the field}$$

$$= \frac{P\cos\theta ds}{4\pi\varepsilon_0 r^2}\sin\theta \qquad \text{perpendicular to field}$$

If the area *ds* be taken round through 2π radians about *LM*, it will describe a ring, the surface area of which is given by,

$$= 2\pi r \sin\theta .r.d\theta$$

$$= 2\pi r^2 \sin\theta d\theta$$

Therefore intensity at C due to ring in the field direction is

$$= \frac{P\cos^2\theta ds}{4\pi\varepsilon_0 r^2} 2\pi r^2 \sin\theta d\theta$$

$$= \frac{P}{2\varepsilon_0 r^2}\cos^2\theta \sin\theta d\theta$$

while the normal components of intensity due to the ring cancel each other.

Integrating the intensity at C due to charges on the surface of cavity, we have

$$E_3 = \frac{P}{2\varepsilon_0}\int_0^{\pi} \cos^2\theta \sin\theta d\theta$$

$$= \frac{P}{2\varepsilon_0} \times \frac{2}{3}$$

$$= \frac{P}{3\varepsilon_0}$$

therefore the total intensity or E_{in} at C is given by eq.(1) as

$$E_{in} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0} + \frac{P}{3\varepsilon_0} \qquad (2)$$

The resultant field, E between the plates is

$$= \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0}$$

Therefore $$E_{in} = E + \frac{P}{3\varepsilon_0}$$

Further we know that $$D = \varepsilon E = \varepsilon_0 E + P$$

So that $$E = \frac{P}{\varepsilon - \varepsilon_0}$$

Putting this value in eq.(2) for E_{in}, we get

$$E_{in} = \frac{P}{\varepsilon - \varepsilon_0} + \frac{P}{3\varepsilon_0}$$

$$= \frac{P}{3\varepsilon_0}\left(\frac{\varepsilon + 2\varepsilon_0}{\varepsilon - \varepsilon_0}\right) \qquad (3)$$

if the number of the molecules per unit volume , α molecular polarizability then polarization, P, is defined as electric moment per unit volume . That is

$$P = n\alpha E_{in}$$

Putting this value in eq,(3), we get

$$\frac{P}{n\alpha} = \frac{P}{3\varepsilon_0}\left(\frac{\varepsilon + 2\varepsilon_0}{\varepsilon - \varepsilon_0}\right)$$

$$\frac{n\alpha}{3\varepsilon_0} = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$$

where ε_r is the relative permittivity or dielectric constant. Equation (6) is well known *Claussius-Mossotti relation*.

We know that $\varepsilon_r = \mu_g^2$ where μ_g is the refractive index of gas. Hence equation (6) can be expressed in terms of refractive index. The relation in that case is

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\mu_g^2 - 1}{\mu_g^2 - 2} \qquad (4)$$

called *Lorentz* formula and is valid only as long as ε_r is frequency independent.

Validity of Claussius-Mossotti Relation:

The number of molecules per unit volume is proportional to the density of gas. If molecular polarizability α is taken to be a constant, the number n is proportional to $\frac{\varepsilon_r - 1}{\varepsilon_{r+2}}$.

Also the constant α is proportional to the cube of the radius of molecule. Hence finding ε_r experimentally and calculating n from the density at definite temperature and pressure, the value of the radius of the molecule of dielectric may be reckoned. The values, so obtained, fairly agree with the values obtained by other methods and proves the validity of the relation. It is true only in case of monatomic gases and weak solutions. The experimental and theoretical values disagree in case of strong solutions and solids. It is due to the fact that in these cases the interaction forces among molecules are sufficiently great, the account of which has not been taken.

$$= \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}$$

$$\frac{n\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

The Magnetic Field

let us consider a +ve test chare q_0, with a velocity v through a point P. If this charge experiences a side-way deflecting force F, then a magnetic field is said to exist at that point. This field is defined by means of a vector quantity B and is called magnetic Induction, shown in Y axis.

If a charge moving through a point P in a magnetic field experiences no side way deflecting force then the direction of motion of the charge is defined as the direction of B. Conditions are

(1) when V is parallel to B (in the same direction) F is minimum. (2) when V is perpendicular to B, F is maximum.

Definition: If a +ve test charge q_0 moving with velocity v through point P in a magnetic field experiences a deflecting force F, then magnetic induction B at P is defined by

$$F = q_0 V \times B \text{ ---------(1)}$$

The above relation defines both the direction and magnitude of B is

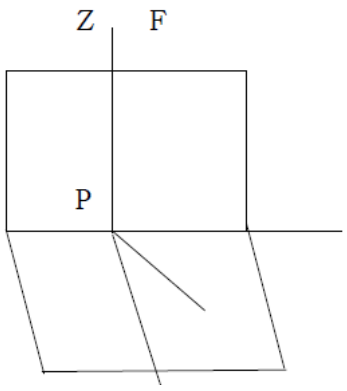
$$F = q_0 V B \sin\theta$$

$$B = F / q_0 V \sin\theta \ ; \theta \text{ is angle between V x B.}$$

From the above V is parallel to B, $\theta = 0$; F = 0 (min) v is perpendicular

$$\theta = 90 \ ; F = Bq_0 V \text{ (max)}$$

Units; Newton / amp –turn



Ampere's force law (Force on current element)

The concept of magnetic field is introduced by considering a test charge q moving in a region of space with velocity V. Suppose the charge experiences a force F, then the region is said to be having magnetic field B, we write

$$F = qV \times B \text{ ----------(1)}$$

The above equation in terms of current i.e. the current crossing a surface is defined as the rate at which charge flows across the surface,

$$\text{ie} \qquad I = dq / dt \text{ ---------(2)}$$

ie, the force experienced (dF) by the charge dq moving with velocity V then Eq. (1) becomes

$$dF = dq\ VxB$$
$$= I\ dt\ VxB.$$

Suppose in the time dt, charge dq travels along the length 'dl' of the conductor then

$$V = dl / dt \text{ ie., } dF = I\ dt\ (dl / dt) \times B$$

$$dF = I\ dl \times B \text{ --------(3)}$$

this is Ampere's force law

The total force experienced by the total volume containing the charge can be calculated by integrating the above equation

$$F = \int_{vol} (J \times B)\ dv \qquad [\text{ie, } I = J.ds \text{ and } ds.dl = V]$$

Ampere Circuital law

(4) It depends upon the nature of the medium

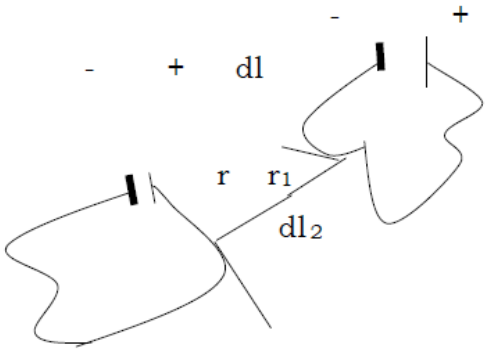


Fig. 2.4 Two currents I_1 and I_2

The force exerted on current element dl_2 by current element dl_1 is given by

$$dF_{21} = (\mu_0 / 4) (I_1, I_2) (1 / r^2_{21}) [dl_2 x (dl_1 x r_{21} / r_{21})]$$

$(\mu_0 / 4)$ ---------- arises due to (4) -------nature of the media

I_1, I_2 ---------- due to (1)

$(1 / r^2_{21})$ ----------due to (2)

$[dl_2 x ((dl_1 x r_{21} / r_{21})]$ arise due to (3) r_{21} / r_{21}

represents unit vector along r_{21}

The above equation dF_{21} can be written for whole lengths of the conductors as

$$F_{21} = (\mu_0 / 4) (I_1, I_2) \quad 1 \quad 2 \quad [dl_1 x (dl_1 x r_2)] / r_3 \, 21 \quad \text{-------(2)}$$

The above equations not does have much practical value because of r^3_{21} The force cannot expressed as the interaction of current I_1 with field current I_2 . However the above equation. can be written as,

$$F_{21} = I_1 \; 1 \; dl_1 x [\mu_0 / 4 \quad I_2 \; 2 \; (dl_2 \; x \; r_{2\,1}) / \; r^3_{2\,1} \quad \text{------(3)}$$

$$F_{21} \quad I_2 \, dl_2 \quad x \, B_1$$

$$\text{Where } B_1 = (\mu_0 / 4) \; 2 \, dl_2 \; x \, r_{2\,1} / r^3_{2\,1} \text{ --------(4)}$$

B_1 is called the magnetic induction, magnetic flux density or magnetic field current. Unit is web / m^2 or Tesla. In general the magnetic induction B at a position r due to a current carrying circuit of element I dl will be

$$B = (\mu_0 / 4) \quad I \, dl \, x \, r / r^3$$

This is Biot savart law

Point (1) If the current I is distributed in space with a current density J then I dl = Jd

$$\text{Here } B = \mu_0 / 4 \quad J \times r / r^3 \, d$$

Point (2) If a single charge q moving with velocity v then

$$B = \mu_0 / 4 \; q (v \times r / r^3)$$

or

$$B = (\mu_0 \; _0 \, v \times q \, r) / 4 \quad _0 r^3$$

$$\text{But } \mu_0 \; _0 = 1 / c^2 \text{ and } E = qr/4 \quad _0 r^3$$

$$B = v \times E / c^2$$

The above gives the relation between electric (E) and magnetic fields (B) of a uniformally moving charge as v<<c.

THE DIVERAGENCE OF THE MAGNETIC INDUCTION B

We had demonstrated that the magnetic field of moving charges were such that $\nabla \cdot B = 0$. It is also possible to arrive at this same result for steady currents starting from the Biot – Savart law. We know that,

$$B = (\mu_0 / 4) \int_{r'} (J_1 \times r_1) / r^3 \, d' \quad \text{---------(1)}$$

$$\nabla \cdot B = (\mu_0 / 4) \int_{r'} (J_f \, r_1) / r^2 * dr' = (\mu_0 / \nabla) \int_{r'} . (J_f \times r_1/r^2) \, d'$$

$$\nabla . (J_f \times r_1 \nabla r^2) \quad r_1 / r^2 . \nabla \quad \times J_f) - J_f . (\quad \times r_1/r^2)$$

Where the first term on the right is zero because J_f is a function of the source point x' , y' , z' while the del operator involves derivatives with respect to the field point x , y , z. The second term on the right is also zero because

$$(\nabla \times r_1)/r^2 = (\nabla \times r)/ r^3 = \begin{vmatrix} i & j & k \\ / x & / y & / z \\ (x - x')/r^3 & (y - y')/r^3 & (z - z')/r^3 \end{vmatrix} \quad 0$$

Then

$$\nabla .B = 0 \quad \text{----------(2)}$$

This equation follows form the definition of B given in Eq.(1) we also know that it is a consequence of Coulomb's law and of the Lorentz transformation. The fact that $\nabla . \quad . B$ is zero means that these cannot be sources of B.

The net flux of magnetic induction through any closed surfaces is equal to zero since

$$B.da = \int_r \nabla B \, dr = 0 \;\rightarrow\; B.da = \int_r \nabla . B dr = 0 \quad \text{----------(3)}$$

The Vector Potential A (Magnetic Vector Potential)

The calculation of electric fields was much simplified by the introduction of the electrostatic potential. For an electrostatic field, the relation between electrostatic field E and electrostatic potential V is given by

$$E = \nabla V$$

Here V is a scalar quantity

In the case of a magnetic field,

$$\text{div } B = 0$$

Since the divergence of any curl is zero, it is reasonable to assume that the magnetic induction may be written as,

$$B = \text{Curl } A = \nabla \times A$$

'A' refers to magnetic potential and is called the magnetic vector potential. Therefore the magnetic vector potential A can be defined as the vector, whose curl at any point gives the vale of the magnetic fields B at that point.

The only other requirement placed on A is that

$$\nabla \times B = \nabla \times [\nabla \times A] = \mu_0 J$$

The unit of A is Wb / m

Derivation of the magnetic vector potential of a current loop:-

According to the Biot-Savart law, the magnetic induction at a distance r from the element of length l carrying a current I (Fig. 2.6) is given by

$B = \mu_0 I / 4 \ . \ l \times r / r^3$ -------(1)

We have $\nabla(1 / r) = - r / r^3$ ----(2)

$$B = (\mu_0 I / 4) \; 1 \; x \{ -\nabla \; (1/r)\}$$

$$= (\mu_0 I / 4) \; \{\nabla \; x1/r) \; x \; 1 \} \text{ ------(3)}$$

We have the vector identity.

$$\nabla \; x \; (A) \; = (\nabla \;) \; x \; A + \; (\; \nabla \; x \; A)$$

Where is a scalar and A is a vector

Or $(\nabla \;) \; x \; A \; = \nabla \; x \; (A) \; - \; (\nabla \; x \; A)$

So $\nabla \; (1/r) \; x \; 1 = \nabla \; x \; (1 / r) - 1/r \; (\nabla \; x \; 1) \; \text{------(4)}$

Using Eq. (4) in Eq.(3) we get

$$B = (\mu_0 I / 4) [\; \nabla \; x \; (1 / r) - 1/r \; (\nabla \; x \; 1) \; \text{-----(5)}$$

In this equation, $\nabla \; x \; 1 = 0$ because the operator ∇ is a function of (x,y,z) and the current element is not a function of (x,y,z) as shown in the Fig. 2.6

Then Eq. (5) reduces to

$$B = \; (\mu_0 I / 4 \;) \; \nabla \; x \; (1 / r) \text{ ----- (6)}$$

Therefore the total magnetic induction at the given point by a closed loop carrying current is given by

$$B = (\mu_0 I / 4) \; \nabla \; x \; (dI/r) \text{ -------(7)}$$

The operation $\nabla \; x$ is independent of the integration of dI /r around the closed loop.

The Eq.(7) can be rewritten as

$$B = \nabla \; x \; [\mu_0 I / 4 \quad dI/r \;] = \; (\text{curl } \mu_0 I / 4 \;) \quad dI/r \text{ ------(8)}$$

Thus, we conclude that a vector exists such that by taking its curl, the magnetic induction produced at any point by a closed loop carrying current may be obtained. This vector is known as magnetic vector potential A. Thus

$$B = \text{curl } A \text{ ------(9)}$$

Where magnetic vector potential $A = (\mu_0 I / 4) \quad dI/r$ ---------(10)

Eq. (9) is frequently used to derive magnetic induction B at any point from the magnetic vector potential A at that point.

If the current is flowing through the length element is distributed over a cross-sectional area a, we writel = Ja. Eq (10) is thus written as

$$A = \mu_0 / 4 \quad {}_v (J /r) \text{ x } dV \text{ ----------(11)}$$

The vector potential defined by Eq. (11) is not uniquely defined. We find that we can add any term, whose curl is zero to the vector potential and it still gives the same magnetic field. Unlike V, A does not have a physical significance. It serves as a convenient intermediate step for the computation of B.

2.7 THE CURL OF THE MAGNETIC INDUCTION B

We have shown that the magnetic induction is always equal to the curl of the vector potential : $B = \nabla \text{ x } A$. We shall now show that

$$\nabla \text{ x } B = \mu_0 J_f \quad \text{---------(1)}$$

assuming a steady state and the absence of magnetic materials. In terms of A,

$$\nabla \text{ x } B = \nabla \text{ x} \nabla \text{ x } A = \nabla (\nabla .A) - \nabla^2 A \text{ ----(2)}$$

We have already shown that $\nabla . A$ is proportional to the time derivative of the electric potential V, then, with the above assumptions.

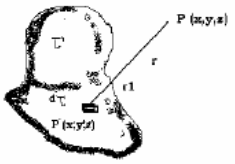
$$\nabla . A = 0 \text{ ----------(3)}$$

For the second term we have from the definition of A that

$$\nabla^2 A = (\mu_0 / 4\pi) \int \nabla^2 (J_f / r)\, d\tau' \text{ ---------(4)}$$

Where we have interchanged the order of differentiation and integration. Let us imagine At the field point P (x,y,z) where we wish to compute $\nabla^2 A$, we form the vector $J\, d\tau' / r$, where J_f of and $d\tau'$ are respectively the current density and the volume element at the source point P', and where r is the distance form P' to P. We compute the Laplacian of this vector at P by taking the appropriate derivatives with respect to the coordinates x, y, z of P. We then sum the contributions from all such sources in the volume τ' which includes all points at which J_f exists. The volume τ' may include the field point P, where r = 0.

Figure 2.7 Source point P' and field point P for the calculation of $\nabla^2 A$



Since J_f is not a function of the coordinates of P, we can write the integral as

$$\nabla^2 A = (\mu_0 / 4\pi) \int J_f \nabla^2 (1/r)\, d\tau' \text{ -------(5)}$$

Now, by differentiation of

$$(1/r) = 1/[\,(x-x')^2 + (y - y')^2 + (z - z')^2\,]^{1/2} \text{ -----(6)}$$

we find that $\nabla^2 (1/r) = 0$ if $r \neq 0$. There can thus be no contribution to the integral from any element $d\tau'$ except possibly if P and P' coincide and r is zero

To investigate the integral at r = 0 we consider a small volume enclosing the point P, where we wish to calculate $\nabla^2 A$, situated inside the current distribution.

We take the volume so small that J_f does not change appreciably within it; J_f may then be removed from the integral:

$$\nabla^2 A = (\mu_0 J_f / 4) \int_{V'} \nabla^2 (1/r)\, dV' \quad \text{-------(7)}$$

The meaning of this integral is as follows. For each element of volume dV' centered at the point P' within V' we calculate

$$\nabla^2 (1/r) = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2) * 1 / [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} \quad \text{-----(8)}$$

Multiply by dV' and sum the results. Since $\nabla^2 (1/r) = \nabla'^2 (1/r)$

$$\nabla^2 A = (\mu_0 J_f / 4) \int_{V'} \nabla^2 (1/r)\, dV' \quad \text{---------(9)}$$

$$= (\mu_0 J_f / 4) \int_{V'} \nabla' \cdot \nabla' (1/r)\, dV' \quad \text{---------(10)}$$

$$(\mu_0 J_f / 4) \int_{S'} \nabla' (1/r) \cdot da \quad \text{--------(11)}$$

from the divergence theorem,

$$\nabla^2 A = -(\mu_0 J_f / 4) \int_{S'} (r_1 \cdot da) / r^2 \quad \text{-----(12)}$$

where r_1 is the unit vector from the source point to the field point. In this case r_1 points inward toward the point P, Thus

$$\nabla^2 A = -(\mu_0 J_f / 4) \int_{S'} d\Omega \quad \text{-----(13)}$$

where $d\Omega$ is the element of solid angle subtended at the point P by the element of area da. Since the surface S' completely surrounds P,

$$\nabla^2 A = -\mu_0 J_f \quad \text{-------(14)}$$

and

$$\nabla \times B = \mu_0 J_f \quad \text{------(15)}$$

This result is again valid only for static fields and in the absence of magnetic materials.

Magnetic Scalar Potential

Consider a closed current loop carrying current I (Fig 2.9). Consider a point P (r) having position vector r relative to current element I dI. From Biot - Savart law, the magnetic induction B at P due to whole loop is

$$B = (\mu_0 / 4)\ I\ dI \times r / r^3$$

Let the point of observation P(r) be moved through an infinitesimal distance dr say from P (r) to Q (r+dr). then,

$$B.dr = (\mu_0 / 4)\ (I\ dI) \times r / r^3 .dr$$

$$(\mu_0 I / 4)\ dr.\ (dI \times r) / r^3$$

$$\mu_0 I / 4\ (dr \times dI) . r / r^3$$

When the point P is shifted to Q, the solid angle subtended by the loop at P changes by d . But we can also get the same change in solid angle d keeping P fixed and giving every point of the loop the same but opposite displacement (-dx). Then, the above equation becomes

$$B.dr = -\ \mu_0 I / 4\ (-\ dr \times dI\ ) . r / r^3$$

But, -dx * dI = dS = area traced out by current element dI during the displacement (-dx)

Therefore B. dr = $-\mu_0 I/4$ dS . r / r^3

But, (dS . r) / r^3 = d = Change in solid angle subtended by current loop when point P is displaced to Q

Therefore B.dr = - ($\mu_0 I/4$) d ---------(1)

Since is a scalar function of (x,y,z)

d = ∇ .dr

Hence Eq. (1) becomes

B.dr = - ($\mu_0 I/4$)∇ . d

Or

B = - ($\mu_0 I/4$)∇ = ∇ [$\mu_0 I$ / 4] ----(2)

The direction of B is that of - ∇ , so that B points away from the loop along its positive normal.

Comparing Eq.(2) with B = - ∇V_m we get

Magnetic Scalar Potential V_m = ($\mu_0 I$)/ 4

= μ_0 /4 x current x solid angle ------(3)

Negative gradient of V_m gives the magnetic induction B.

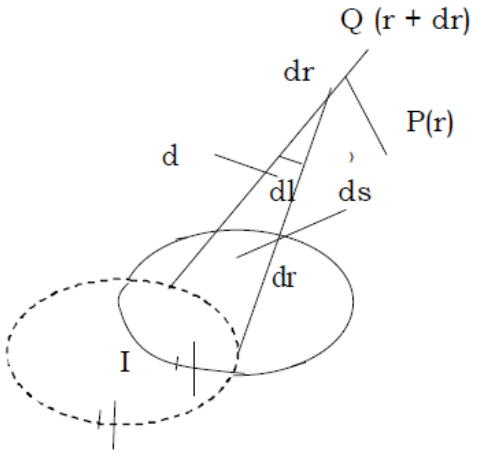


Fig. 2.9

Magnetization and Susceptibility

The H-field inside a long solenoid is nI. If there is a vacuum inside the solenoid, the B-field is $\mu_0 H = \mu_0 nI$. If we now place an iron rod of permeability μ inside the solenoid, this doesn't change H, which remains nI. The B-field, however, is now $B = \mu H$. This is greater than $\mu_0 H$, and we can write

$$B = \mu_0 (H + M).$$

The quantity M is called the *magnetization* of the material. In SI units it is expressed in A m^{-1}. We see that there are two components to B. There is the $\mu_0 H = \mu_0 nI$, which is the externally imposed field, and the component $\mu_0 M$, originating as a result of something that has happened within the material.

It might have occurred to you that you would have preferred to define the magnetization from $B = \mu_0 H + M$, so that the magnetization would be the excess of B over $\mu_0 H$. The equation $B = \mu_0 H + M$, would be analogous to the familiar $D = \varepsilon_0 E + P$, and the magnetization would then be expressed in tesla rather than in A m^{-1}. This viewpoint does indeed have much to commend it, but so does $B = \mu_0 (H + M)$. The latter is the recommended definition in the SI approach, and that is what we shall use here.

The ratio of the magnetization M ("the result") to H ("the cause"), which is obviously a measure of how susceptible the material is to becoming magnetized, is called the *magnetic susceptibility* χ_m of the material:

$$M = \chi_m H.$$

On combining this with equation 12.3.1 and $B = \mu H$, we readily see that the magnetic susceptibility (which is dimensionless) is related to the relative permeability $\mu_r = \mu / \mu_0$ by

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DEPARTMENT OF PHYSICS

UNIT – 2 – FIELD EQUATIOS AND CONSERVATION LAWS– SPH5105

ELECTROMAGNETIC THEORY

Sub code: SPH 5105

UNIT 2

FIELD EQUATION AND CONSERVATION LAWS

Equation of Continuity – Displacement Current – Maxwell's Equations – Derivations and Physical Significance – Energy in electromagnetic fields (Poynting's theorem) – Poynting Vector – Electromagnetic Potentials **A** and Φ - Maxwell's Equations in terms of Electromagnetic Potentials –Concept of Guage – Lorentz Guage – Coulomb Guage.

Aim and Objectives

continuity equation, displacement current and also the derivation of Maxwell's equations in differential and integral form also you study about the scalar (Φ) and vector potential (A) and its relation to Lornetz and Coulomb gauge.

Equation of Continuity

According to the principle of conservation of charge the net amount of charge in an isolated system remains constant. The principle can be stated as follows.

In the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in a manner that the time rate of decrease of charge within the volume equals the net rate of flow of charge out of the volume. This statement can be expressed by the equation of continuity.

Derivation: Let us consider that charge density ρ, is a function of time. The transport of charge constitutes the current i.e.,

$$I = dq/dt = d/dt \int_v \rho . dV \text{ ---------- (1)}$$

Here we have considered that the current is extended in space of volume V closed by the surface 'S'. The net amount of charge which crosses a unit area (normal to the direction of charge flow) of a surface in unit time is defined as the current density J. We know, if a net amount of current is flowing outward closed surface the charge contained within that volume should decrease at the rate

$$-dq / dt = I \text{ ---------- (2)}$$

Where I is the total current flowing through surface S. if J is the current density, then by definition, total current I will le

$$I = \int_s J . ds \text{------- (3)}$$

From equations (2) and (3) we get

$$\int_s J.\,ds = -\,dq/dt$$

$$= -d/dt \int_v \rho.\,dV \text{ ---------- (4)}$$

{Using equ. (1) }

Because it is ρ which is changing with time, we can write

$$d/dt \int_v \rho.dV = \int_v \partial\rho/\partial t.dV$$

so that equ. (4) becomes

$$\int_s J.ds = -\int_v (\partial\rho/\partial t).\,dV \text{ ----------(5)}$$

From divergence theorem, we have

$$\int_s J.ds = -\int_v (\text{div } J)\,dV$$

so that equ (5) becomes

$$\int_v (\text{div } J)\,dv = -\int_v (\partial\rho/\partial t).\,dV$$

(or)

$$\int_v (\text{div } J + \partial\rho/\partial t)dV = 0 \text{ ------(6)}$$

Since Eq. (6) holds for any arbitrary volume, we can put integral equal to zero. i.e.,

$$\text{div } J + \partial\rho/\partial t = 0 \text{ --------(7)}$$

It is referred to as the equation of continuity. It is the mathematical expression for the conservation of charge. It states that the " total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that charge cannot the created or destroyed. i.e., no sources and sinks are present in that volume". In case of stationary currents, charge density at any point within the region remain constant

$$\text{i.e.., } \partial\rho/\partial t = 0$$

So that div J=0 or $\nabla . J = 0$ which express the fact – that there is no net outward flux of current density J.

Displacement current (D)

Maxwell changed the definition of total current density to adapt the equation of continuity to time dependent fields

Ampere's circuital law is

$$\int_s B.\,dl = \mu_0 I$$

$$\int_s H.\,d\rho = I = \int_s J.ds$$

Changing line integral into surface integral, by stoke`s theorem,

$$\int_s \text{curl } H.\,ds = \int_s J.ds$$

(or)

$$\text{curl } H = J \text{ ---------- (1)}$$

Let us substitute it in equation of continuity, then

$$\text{div } J = -\ \rho\ /\ t$$

we get $\text{div } (\text{curl } H) = -\ \rho /\ t$

$$0 = -\ \rho\ /\ t$$

Here equ (1) leads to steady state condition in which charge density is not changing. Therefore for time dependent (changing) fields, Eq (1) should be modified. Maxwell suggested that the definition of total current density is incomplete and advised to add something to it. Let it be J' then Eq (1) becomes

$$\text{Curl } H = (J + J') \text{ ---------- (2)}$$

In order to identify J', we take divergence of Eq (2) That is

$$\text{div } (\text{curl } H) = \text{div}(J + J')$$

$$0 = \text{div } J + \text{div } J'$$

(or)

$$\text{div } J' = -\text{div } J = \rho /\ t \text{ ----------(3)}$$

We know that

$$\rho_v = \nabla \cdot D$$

So that Eq. (3) becomes

$$\text{div } J' = \partial / \partial t = (\nabla \cdot D)$$

$$= \nabla \cdot \partial D / \partial t$$

$$= \text{div } (\partial D / \partial t)$$

(or)

$$\text{div } [J' - (\partial D / \partial t)] = 0 \text{ --------- (4)}$$

Eq. (4) is true for any arbitrary volume, we can have

$$J' = (\partial D / \partial t) \text{ -------(5)}$$

Therefore the modified form of the ampere's law is

$$\text{Curl } H = J + (\partial D / \partial t) \text{ --------- (6)}$$

Note: 1. Since J' arises due to the variation of electric displacement D with time, it is termed as displacement current density . According to Maxwell it is just as effective as J, the conduction current density in producing magnetic field. (2) The important inference that we get from Eq. (6) is that, since displacement current J' is related to the electrtic field vector D (as $D = \varepsilon E$) it is not possible in case of time varying fields to deal separately with electric and magnetic fields but, instead the two fields are interlinked giving rise to electromagnetic fields. Thus J' results into unification of electric and magnetic phenomenon.

The Maxwell's equations (Differential form)

The four equations of Maxwell's are.,

(i) $\nabla \cdot D = \rho$ ---- obtained by the application of Gauss theorem in electrostatics. D is the electric displacement in coulomb / meter^2 and ρ is the free charge density in coulomb/meter^3.

(ii) $\nabla \cdot B = 0$; obtained by the application of Gauss theorem to magnetic field and B is the magnetic induction in Weber/mt^2

(iii) $\nabla \cdot E = -B/t$; obtained by Faraday's and Lenz`s law in electromagnetic induction and E is the electric intensity in volt / meter

(iv) $\nabla \times H = J + D/t$: obtained by Maxwell's modification of Ampere's law in a circuital form for magnetic field accompanying an electric current and H is magnetic field intensity in ampere/meter and. J is the current density in ampere/meter^2 .

(A) Derivation of Maxwell's equations

(i) div D= ρ

Consider a surface S bounding a volume V in a dielectric medium. From Gauss theorem the integral E.ds of the normal component of E over any closed surface is equal to the total charge enclosed within the surface. Also we know that the total charge must include both the free and the

polarisation charges or the total charge density ρ_p = -div p and ρ is the free charge density at a point in a small volume element dV. Thus total charge density at that point will be, ρ - (divp) then Gauss law can be expressed as

$$\int_s E.ds = \int_v \text{div } E.dV = 1/\varepsilon_0 (\rho - \text{div } p)\, dV$$

(or)

$$\text{div } (\varepsilon_0 E + P)dV = \rho dV$$

the quantity (ε_0E+P) is D called electric displacement, so that

$$\text{div } D\, dV = \rho dV$$

(or)

$$(\text{div } D - \rho)dV = 0$$

Since this equation is true for all volume, the integrand in this equation must vanish i.e., Div D= ρ

when the medium is isotropic the three vectors D,E, P are in the same direction and for small field, D is proportional to E, that is

$$D = \varepsilon E$$

Where ε is called dielectric constant of the medium.

(ii) $\text{div } B = 0$

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic line of force entering any arbitrary closed surface is exactly the same as leaving it. It means that the flux of magnetic induction B across any closed surface is always zero i.e..,

$$B.ds = 0$$

Transforming the surface integral into volume integral, we have

$$\text{div } B \, dV = 0$$

The integrand should vanish for the surface boundary as the volume is arbitrary, i.e.,

$$\text{div } B = 0$$

(iii) curl E - B/ t

By Faraday law we know that emf induced in a closed loop is given by

$$e = - \quad / t = \quad B/ t \, . \, ds$$

Since the flux = $_s$ B.ds where S is any surface having the loop as boundary. E.m.f `e` can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus e = E. dl where E is the intensity of the electric field associated with induced em f

Therefore, equating above two equations, we get

$$E.dl = -_s \; B/ t \, . \, ds$$

Applying stokes` theorem, the line integral can be transformed into surface integral i.e.,

$$_s \; (\nabla \times E) \, . \, ds = -_s \; (B/ t) \, . \, ds$$

This equation must be true for any surface whether small or large in the field. Therefore the two vectors in the integrands must be equal at every point, i.e.,

$$\nabla \times E = - B/ t$$

$$\text{Crul } E = - B/ t$$

(IV) --- Curl H = J+ D/ t

Ampere's law in the circuital form gives this equation. According to this law, the work done in carrying a unit magnetic pole once round closed arbitrary path linked with the current is expressed as

$$H.\, dl = I \text{ (or) } = J.\, ds$$

where the integral on the right is taken over the surface through which the charge flow corresponding to the current I take place. Now changing the line integral into surface integral by stoke`s theorem

$$_s \text{ curl } H.ds = J.ds$$

$$\text{Curl } H = J$$

The above relation, derived on the basis of Amperes` law, stands only for steady closed current. But for the changing electric fields, the current density should be modified. The divergence of the above equation is

$$\text{div (curl H)} = 0 \text{ (or)}$$

div J=0 which conflicts with the equation of continuity div J = (- ρ/ t.) Adding J we get curl H= (J+J'). Taking divergence of the above equation, we get

$$\text{div (curl H)} = (\text{divJ}+\text{divJ}') \text{ (or)}$$

$$0 = \text{div J}+\text{divJ}' \text{ (or)}$$

$$\text{div J}' = -\text{divJ} = + \rho / t.$$

$$\text{we know that } \rho = \nabla . D$$

Substituting this value in the expression for Div J', we get

$$\text{Div J}' = / t (\nabla . D)$$

$$\nabla . J` = (\nabla . D/ t)$$

therefore the Maxwell's fourth relation can be written as

$$\text{Curl H} = J + (D/ t)$$

3.4 Maxwell's equation in free space

In the free space, where the current density J and volume charge density ρ are zero, Maxwell's equations reduce to

$$\nabla .D = 0$$

$$\nabla .B = 0$$

$$\nabla \times E = - B/ t$$

$$\nabla \times H = D/ t$$

Maxwell's equations in linear Isotropic Media

In linear isotropic media

$$D = E$$

$$\text{and } H = B/\mu$$

Where is the dielectric constant, μ permeability of the medium.

The Maxwell's equations become

$$\nabla . E = \rho /$$

$$\nabla \cdot H = 0$$

$$\nabla \times E + \mu(\partial H/\partial t) = 0$$

$$\nabla \times H - (\partial E/\partial t) = J.$$

Energy in electromagnet fields: Poynting vector – (Poynting theorem)

Energy may be transported through space by means of e.m. waves. Let the material inside S be isotropic homogenous and characterised by permeability μ, permittivity and conductivity . For derivation, consider a volume V bounded by a closed surface S.

Maxwell's third and fourth relations are

$$\text{Curl } E = -\ B/\ t$$

$$\text{Curl } H = J + D/\ t$$

Taking scalar product of both sides of the above equations with H and E respectively and subtracting, we get

$$E.\ \text{curl } H\ .\ H \text{ curl } E = J.\ E + [(E.(\ D/\ t) + H\ (\ B/\ t)] \quad ---(1)$$

But we know that

$$H.\ \text{curl } E - E.\ \text{curl } H = \text{div } (E \text{x} H)$$

So that equ. (1) becomes

$$-\text{div } (E \text{ x } H) = J.\ E[E.(\ /\ t)\ (\ .\ E) + H(\ /\ t)\ (\mu H)]$$

$$\text{(or) } J.E + (1/2.\ /\ t\ (\ E^2) + 1/2.\ /\ t(\mu H^2) + \text{div } (E \text{x} H) = 0$$

$$\text{(or) } J.E + (1/2.\ /\ t\ (E.\ \ E) + 1/2.\ /\ t\ (H.\ \mu H) + \text{div } (E \text{x} H) = 0$$

$$\text{(or) } J.E + (1/2.\ /\ t\ (E.D) + 1/2.\ /\ t(H.B) + \text{div } (E \text{x} H) = 0 \quad --(2)$$

Integrating over the volume V bounded by the surface S, we get

$$_v\ (J.E)\ dv + _v\ 1/2.\ /\ t(E.D + H.\ B) + _v \text{div } (E \text{x} H)\ dv = 0 \quad ----(3)$$

But as $_v$ div (ExH) dv = (ExH).ds

We write equation (3) as

$$_v\ (J.B)\ dv + _v\ 1/2\ /\ t\ (E.D + H.B) = -\ _s\ (E \text{x} H).ds \quad ----\ (4)$$

Integrating the second term of Equ. (4) we get

$$1/2\ /\ t(E.D + H.B)dv = 1/2\ .\ /\ t(E.\ E + H\ .\mu H\ )dv$$

$$= /\ t\ (1/2\ .\ E^2 + 1/2\ \mu H^2)\ dv \quad -----(5)$$

The first and second term on right hand side represent the time rate of increase of energy stored in the electric and magnetic fields respectively in the volume V. Considering the Eq.(4), LHS of this represents the sum of the

power expended by the fields due to the motion of charge and the time rate of increase of stored energy in the fields. On the other hand RHS of Eq. (4) must represent the power flow into the volume V across the surface S, or the power flow out of the volume V across the surface S

$$= {}_s(ExH) \cdot ds$$

$$= {}_s P.ds$$

where P= (ExH) ----(6)

It then follows that the vector P has the meaning of power density associated with the electromagnetic filed at that point. The statement represented by Eq.(6) is known as poynting theorem and the vector P is known as the poynting vector.

3.6 Electromagnetic potentials – Maxwell's equations is terms of electromagnetic Potentials

Consider the Maxwell's equations

$$\mu \text{Curl } H = \mu J + \mu \ D/\ t \text{ -------(1)}$$

$$\text{Curl } B = \mu J + \mu \ E/\ t \text{ -----(2)}$$

Where and μ are permittivity and permeability, Substituting for B (i.e. B = Curl A) and E(-grad φ- A/ t) (where A and φ are electromagnetic potentials), we get

$$\text{Curl (CurlA)} = \mu J + \mu \quad / \ t \ (-\text{grad } \varphi - A/\ t)$$

$$\text{i.e.., grad div A-} \nabla^2 A = \mu J - \mu \quad / \ t \ (\text{grad } \varphi) - \mu \quad {}^2A/\ t^2)$$

$$\text{i.e., } \nabla^2 A - \mu \quad {}^2A/\ t^2 - \text{grad (divA} + \mu \quad \varphi/\ A) = -\mu J \text{ ----(3)}$$

Considering other Maxwell's equations, namely

$$\text{Div } D = \rho$$

$$\text{div } E = \rho$$

$$\text{i.e., div (- grad } \varphi - A/\ t) = \rho /$$

$$\text{i.e., } \nabla^2 \varphi = \ / \ t(\text{div } A) = - \rho /$$

Adding and subtracting μ $\varphi^2/\ t^2$ it becomes

$$\nabla^2\varphi - \mu \quad {}^2\varphi/\ t^2 - \ /\ t(\text{div } A+\mu \quad \varphi/\ t) = -\rho/\ \ --(4)$$

Equations (3) and (4) are field equations in terms of electromagnetic potentials. Here Maxwell's equations are reduced from four to two by electromagnetic potentials,

Note: Electromagnetic potential define the field vectors uniquely though they themselves are non-unique. We get the same field vectors when we use the set (A, φ) or (A', φ'). These transformations are called gauge transformations.

3.7 Lorentz Gauge

Maxwell's field equations in terms of electromagnetic potentials are

$$\nabla^2 A - \mu \quad {}^2A/\ t^2 - \text{grad}\ (\text{div}A+\mu \quad \varphi/\ t) = -\mu J \ ---(1)$$

$$\nabla^2\ \varphi - \mu \quad {}^2\varphi/\ t^2 + \ /\ t(\text{div } A+\mu \quad \varphi/\ t) = -\rho/\ \ -(2)$$

The above equations may be simplified as

$$\text{div } A + \mu \quad \varphi/\ t = 0 \ ----(3)$$

This requirement is called the Lorentz condition and when the vector and scalar potential satisfy it, the gauge is known as Lorentz gauge.

So with Lorentz condition field equations reduce to

$$\nabla^2 A - \mu \quad ({}^2A/\ t^2) = -\mu J \ ----- \ (4)$$

and

$$\nabla^2\varphi - \mu \quad ({}^2\varphi/\ t^2) = -\rho/\ \ -----(5)$$

But we know $\mu = 1/v^2$ (i.e., $V = 1\ \overline{\mu}$)

Hence Equations (4) and (5) can be written as

$${}^2A = -\mu J \ ------(6)$$

$${}^2\ \varphi = -\rho/\ \ -----(7)$$

and $\quad {}^2 = \nabla^2 - 1/v^2 * ({}^2\Lambda\ /\ t^2)$

Equations (6) and (7) are inhomogenous wave equations and are known as D' Alembertian equations and can be solved. The potentials obtained by solving these equations are called retarded potentials.

To determine the requirement that Lorentz condition Λ, we substitute A' and φ' from equations already given earlier.

$$\text{Div } (A' - \text{grad}\ \Lambda) + \mu \quad /\ t\ (\varphi' + \Lambda\ /\ t) = 0$$

$$\text{i.e., div } A' = \mu\ (\quad \varphi'/\ t) = \nabla^2\ \Lambda - \mu\ (\quad {}^2\Lambda \quad /\ t^2)$$

Hence A' and φ' will satisfy the equations (3) i.e.., Lorentz condition provides

$$\nabla^2 A - \mu (\ ^2 \Lambda \ / \ t^2) = 0 \text{ ----- (8)}$$

$$\text{i.e., } \ ^2 A = 0$$

Lorentz condition is invariant under those gauge transformations for which the gauge functions are solutions of the homogeneous wave equations.

Advantages:

(1) It makes equations for A and φ independent of each other.

(2) It leads to the wave equation.

(3) It is a concept which is independent of Co – ordinate system.

3.8 Coulomb Gauge

Consider the field equations in terms of electromagnetic potentials we get,

$$\nabla \ ^2 \varphi + \ / \ t \ (\text{div } A) = - \rho / \quad \text{-----(1)}$$

If we assume div A = 0

The above equation (1) reduces to Poisson's equation

$$\nabla^2 \ \varphi_{(r,t)} = \rho(r' \ t) / \quad \text{---(2)}$$

Whose solution is

$$\varphi_{(r,t)} = 1/4 \quad (\rho \ (r' \ t)/R) \ d \ ' \text{----(3)}$$

i.e., the scalar potential is just the instantaneour Colombian potential due to charge ρ (x', y', z',t). This is the origin of the name coulomb gauge.

From equations (2) (3) we get

$$\nabla^2 \{ 1/4 \quad (\rho \ (r' \ t)/R) \ d \ ' \} = -\rho \ (\ r' \ , t) / \quad \text{---(4)}$$

As Poisson's equations holds good for both scalar and vectors replacing ρ (r' t) by J' we get

$$\nabla^2 \ [1/4 \quad (J' \ /R) \ d \ ' \ ] = -J' \ / \quad \text{--------(5)}$$

As J' is confined to the volume ' the surface contributions will vanish, so

$$\nabla \ (J' \ /R) \ d \ ' = \ (\nabla' J' \ /R) \ d \ ' \text{ --------(6)}$$

(Since J' = -(1/4) ∇ ∇ . J' /R) d ' +1/4 ∇ x ∇ x (J' /R*) d ']

We know ∇ x (J' /R) = 1/R ∇ ' x J' x ∇ ' (1/R)

= (' x J') /R) d ² + (J' /R) x ds

(as ∇ x V d ² = - $_s$V x ds)

As J' confined to volume ' surface contribution will vanish so

$$\nabla \ x \ (J' \ /R) \ d \ ' = (\nabla' \ x \ J' \ /R) \ ^* d \ ' \text{ --------(7)}$$

Now as $\nabla' \times J' = \nabla \times [-1/4 \nabla (\nabla' . J'/R) d\tau'$ ---------- (8)

i.e., $\nabla' \times J' = 0$ (as curl grad $\varphi = 0$)

and $\nabla . J' = \nabla [\nabla \times \nabla' \times J' /R) d\tau'$

i.e. $\nabla . J = 0$ (as div curl V = 0)

In terms of vector potential

$$\nabla^2 A - (1/v^2) \partial^2 A/ \partial t^2 = -\mu J$$

$$\nabla^2 A = -\mu J$$

i.e.., the equation for A can be expressed entirely interms of the transverse current.

The Coulomb gauge has a certain advantage. In it the scalar potential is exactly the electrostatic potential and electric field is given by

$$E = - \text{grad } \varphi - \partial A/ \partial t$$

It is separable into an electrostatic field $V = \varphi$ and a wave field given by $-\partial A/ \partial t$. This gauge is used when no sources are present. If $\varphi = 0$ and A satisfies the homogeneous wave equation, the field is given by

$$E = - \partial A/ \partial t \text{ and } B = \nabla \times A$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – 3 – PROPAGATION OF PLANE ELECTROMAGNETIC WAVES
SPH5105

SPH5005	Electromagnetic Theory	L	T	P	Credits	Total Marks
		4	1	0	5	100

Unit 3

Propagation of Plane Electromagnetic Waves

Electromagnetic Waves in Free Space – Propagation of Electromagnetic Waves in Isotropic Dielectrics Anisotropic Dielectric – In Conducting Media – In Ionized Gases. Interaction of EMW with Matter o Macroscopic Scale: Boundary Condition of inter faces – Reflection and Refraction – Fresnell's Formula Brewster's Law and Polarization of EMW – Total Internal Reflection and Critical Angle – Reflection from Metallic Surface – Wave Guides – Rectangular Wave Guide.

In this chapter, our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

Electromagnetic Waves in Free Space

We start with the source-free, instantaneous Maxwell's equations writte in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ only. Note that conduction current in the source-fre region is accounted for in the $\sigma\boldsymbol{E}$ term.

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \qquad ①$$

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad ②$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad ③$$

$$\nabla \cdot \boldsymbol{H} = 0 \qquad ④$$

Taking the curl of ①

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H})$$

and inserting ② gives

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$= -\mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad ⑤$$

Taking the curl of ②

$$\nabla \times \nabla \times \boldsymbol{H} = \sigma(\nabla \times \boldsymbol{E}) + \epsilon \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E})$$

and inserting ① yields

$$\nabla \times \nabla \times \boldsymbol{H} = \sigma\left(-\mu \frac{\partial \boldsymbol{H}}{\partial t}\right) + \epsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \boldsymbol{H}}{\partial t}\right)$$

$$= -\mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \qquad ⑥$$

Using the vector identity

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F} \qquad \text{(for any vector } \boldsymbol{F}\text{)}$$

in ⑤ and ⑥ gives

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\overbrace{\nabla \cdot \boldsymbol{E}}^{0 \text{ (from ③)}}) - \nabla^2 \boldsymbol{E} = -\mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \boldsymbol{H} = \nabla(\overbrace{\nabla \cdot \boldsymbol{H}}^{0 \text{ (from ④)}}) - \nabla^2 \boldsymbol{H} = -\mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{E} = \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{H} = \mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

Instantaneous vector wave equations (Helmholtz equations)

For time-harmonic fields, the instantaneous (time-domain) vector $\boldsymbol{F}$ is related to the phasor (frequency-domain) vector $\boldsymbol{F}_s$ by

$$\boldsymbol{F} \quad \Leftrightarrow \quad \boldsymbol{F}_s$$

$$\frac{\partial \boldsymbol{F}}{\partial t} \quad \Leftrightarrow \quad j\omega \boldsymbol{F}_s$$

$$\frac{\partial^2 \boldsymbol{F}}{\partial t^2} \quad \Leftrightarrow \quad (j\omega)^2 \boldsymbol{F}_s$$

Using these relationships, the instantaneous vector wave equations are transformed into the phasor vector wave equations:

$$\nabla^2 \boldsymbol{E}_s = \mu\sigma(j\omega)\boldsymbol{E}_s + \mu\epsilon(j\omega)^2\boldsymbol{E}_s = j\omega\mu(\sigma + j\omega\epsilon)\boldsymbol{E}_s$$

$$\nabla^2 \boldsymbol{H}_s = \mu\sigma(j\omega)\boldsymbol{H}_s + \mu\epsilon(j\omega)^2\boldsymbol{H}_s = j\omega\mu(\sigma + j\omega\epsilon)\boldsymbol{H}_s$$

If we let

$$j\omega\mu(\sigma + j\omega\epsilon) = \gamma^2$$

the phasor vector wave equations reduce to

$$\nabla^2 \boldsymbol{E}_s - \gamma^2 \boldsymbol{E}_s = 0$$

$$\nabla^2 \boldsymbol{H}_s - \gamma^2 \boldsymbol{H}_s = 0$$

Phasor vector wave equations (Helmholtz equations)

The complex constant γ is defined as the *propagation constant*.

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$$

The real part of the propagation constant (α) is defined as the *attenuation constant* while the imaginary part (β) is defined as the *phase constant*. The attenuation constant defines the rate at which the fields of the wave are attenuated as the wave propagates. An electromagnetic wave propagates in an ideal (lossless) media without attenuation ($\alpha = 0$). The phase constant defines the rate at which the phase changes as the wave propagates.

Separate but equivalent units are defined for the propagation, attenuation and phase constants in order to identify each quantity by its units [similar to complex power, with units of VA (complex power), W (real power) and VAR (reactive power)].

γ	**propagation constant (m^{-1})**
α	**attenuation constant (Np/m)**
β	**phase constant (rad/m)**

Given the properties of the medium (μ, ϵ, σ), we may determine equations for the attenuation and phase constants.

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = (\alpha + j\beta)^2 = \alpha^2 + j2\alpha\beta - \beta^2$$

$$\left.\begin{aligned} Re\,\gamma^2 &= \alpha^2 - \beta^2 = -\omega^2\mu\epsilon \\ Im\,\gamma^2 &= 2\alpha\beta = \omega\mu\sigma \end{aligned}\right\} \text{Solve for } \alpha, \beta$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]}$$

Summary of Wave Characteristics – Lossy Media (General case)

Lossy media $\Rightarrow$ $(\sigma>0,\ \mu=\mu_r\mu_o,\ \epsilon=\epsilon_o\epsilon_r)$

$$\gamma=\sqrt{j\omega\mu(\sigma+j\omega\epsilon)}=\alpha+j\beta \qquad \text{(complex)}$$

$$\alpha=\omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]} \qquad \beta=\omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]}$$

$$u=\frac{\omega}{\beta} \qquad \lambda=\frac{2\pi}{\beta} \qquad \eta=\sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}} \qquad \text{(complex)}$$

Summary of Wave Characteristics – Lossless Media

Lossless media $\Rightarrow$ $(\sigma=0,\ \mu=\mu_r\mu_o,\ \epsilon=\epsilon_o\epsilon_r)$

$$\gamma=\sqrt{-\omega^2\mu\epsilon}=j\omega\sqrt{\mu\epsilon}=\alpha+j\beta \qquad \text{(imaginary)}$$

$$\alpha=0 \qquad \beta=\omega\sqrt{\mu\epsilon}=\omega\sqrt{\mu_r\mu_o\epsilon_r\epsilon_o}=\frac{\omega}{c}\sqrt{\mu_r\epsilon_r}$$

$$u=\frac{\omega}{\beta}=\frac{c}{\sqrt{\mu_r\epsilon_r}} \qquad \lambda=\frac{2\pi}{\beta} \qquad \eta=\sqrt{\frac{\mu}{\epsilon}} \qquad \text{(real)}$$

Wave Propagation in Free Space

Air is typically very low loss (negligible attenuation) with little polarization or magnetization. Thus, we may model air as free space (vacuum) with $\sigma=0$, $\epsilon=\epsilon_o$, and $\mu=\mu_o$ ($\epsilon_r=1$, $\mu_r=1$). We may specialize the lossless medium equations for the case of free space.

$$\alpha = 0 \qquad \beta = \frac{\omega}{c}\sqrt{\mu_r \epsilon_r} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r \epsilon_r}} = c \qquad \lambda = \frac{2\pi}{\beta} = \frac{c}{f}$$

$$\eta = \eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 377\ \Omega$$

Wave Propagation in Good Conductors ($\sigma \gg \omega\epsilon$)

In a good conductor, displacement current is negligible in comparison to conduction current.

$$\boldsymbol{J}_{total} = \boldsymbol{J}_{conduction} + \boldsymbol{J}_{displacement} = \sigma \boldsymbol{E} + j\omega\epsilon \boldsymbol{E}$$

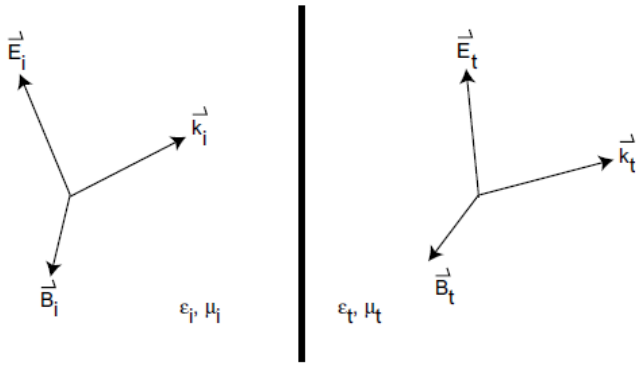
$$|\boldsymbol{J}_{conduction}| \gg |\boldsymbol{J}_{displacement}| \qquad \text{if} \qquad (\sigma \gg \omega\epsilon)$$

Although this inequality is frequency dependent, most good conductors (such as copper and aluminum) have conductivities on the order of 10^7 Ʊ/m and negligible polarization ($\epsilon_r=1$, $\epsilon=\epsilon_o=8.854\times10^{-12}$ F/m) such that we never encounter the frequencies at which the displacement current becomes comparable to the displacement current. Given $\sigma \gg \omega\epsilon$, the propagation constant within a good conductor may be approximated by

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \approx \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma\angle 90^o} = \sqrt{\omega\mu\sigma}\angle 45^o$$

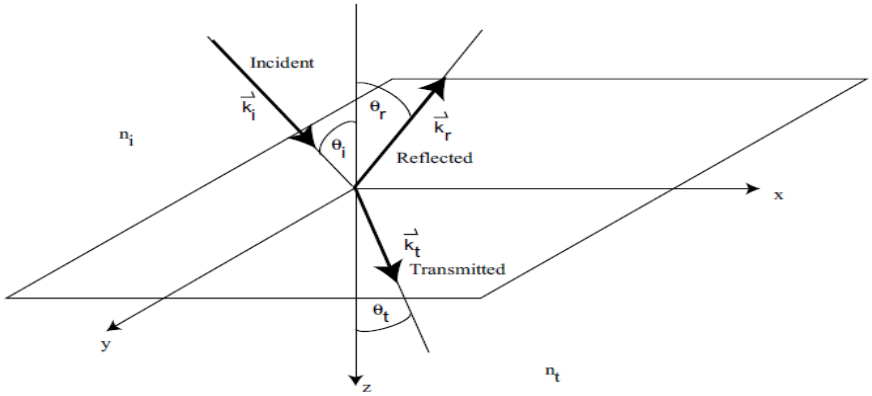
Reflection and Refraction (transmission) at Boundaries

We have seen how plane wave propagate in vacuum or in isotropic homogeneous linear media. The next thing Maxwell equation can give us is a description of the propagation of light across a boundary between two media described by different values of ϵ and μ.



First we will derive the well-known Snell's law, which determines are the reflected and transmitted angles of a incident waves at the boundary. Afterwards, the boundary conditions of an electromagnetic waves at the boundaries between two different materials will be discussed. Finally, the amplitude of the reflected and transmitted waves can be estimated by the Fresnel's formulae.

Snell's law



Consider a plane harmonic wave incident upon a plane boundary separating two different optical media. There will be a reflected wave and a transmitted wave. The space-time dependence of these three waves, aside from constant amplitude factors, is given by the following complex expressions:

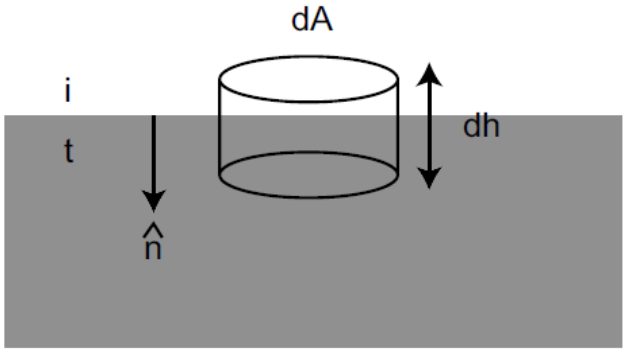
Boundary conditions at the interface

We consider the case where no charges or current are present near the boundary, which is generally true for most optical problems, so the source-free Maxwell equation

may be used, and in integral form

$$\oint_s \vec{D} \cdot d\vec{s} = 0$$
$$\oint_s \vec{B} \cdot d\vec{s} = 0$$
$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_s \vec{B} \cdot d\vec{s}$$
$$\oint \vec{H} \cdot d\vec{l} = \frac{\partial}{\partial t} \int_s \vec{D} \cdot d\vec{s}$$

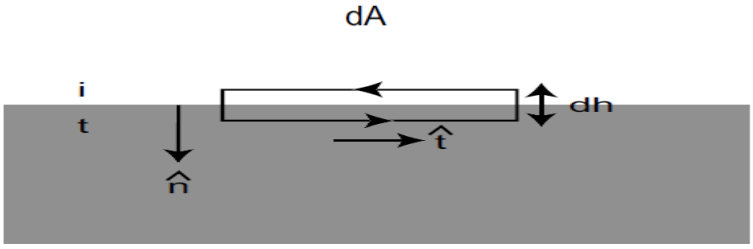
First we consider the first two closed surface integrals. Assume the media interface is surrounded by a cylinder whose faces lie parallel to and each on one side of the interface, as shown in the graph below.



$\hat{n}$ is the unit vector from medium i to t; $d\vec{s} = -\hat{n}dA$ in region i and $d\vec{s} = \hat{n}dA$ in region t. Therefore, the first two Maxwell equation gives us,

$$\vec{D}_i \cdot \hat{n} = \vec{D}_t \cdot \hat{n}$$
$$\vec{B}_i \cdot \hat{n} = \vec{B}_t \cdot \hat{n}$$

which means that "the normal components of $\vec{D}$ and $\vec{B}$ are continuous across the interface". Note that $\vec{D}_i$, $\vec{B}_i$ and $\vec{D}_t$, $\vec{B}_t$ are the total electric displacement and total magnetic field intensity at media i and n, respectively.
Now consider the line integrals of the latter two Maxwell equation by constructing a infinitesimal loop across the interface,



$\hat{t}$ is the unit vector parallel to the interface. As $dh \to 0$, the fluxes $\phi_B = \int_s \vec{B} \cdot d\vec{s} = 0$ and $\phi_D = \int_s \vec{D} \cdot d\vec{s} = 0$; therefore, we have

$$\vec{E}_i \cdot \hat{t} = \vec{E}_t \cdot \hat{t}$$
$$\vec{H}_i \cdot \hat{t} = \vec{H}_t \cdot \hat{t}$$

which further implies that "the tangential components of $\vec{E}$ and $\vec{H}$ are continuous across the interface". Same here that $\vec{E}_i$, $\vec{H}_i$ and $\vec{E}_t$, $\vec{H}_t$ are the total electric field and total magnetic field at media i and n, respectively.

Brewster's angle

In the above graphs, there is a particular angle at which the reflectivity is zero for p polarization light wave, and this angle is known as the "Brewster's angle". From the equation of reflection coefficient of p polarized light,

$$r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \to 0 \qquad \text{when } \theta_i + \theta_t = 90^\circ$$

This situation will happen, according to Snell's law

$$\frac{n_t}{n_i} = \frac{\sin\theta_i}{\sin(90^\circ - \theta_i)} = \frac{\sin\theta_i}{\cos\theta_i}$$

By defining this angle as Brewster's angle θ_B,

$$\boxed{\theta_B = \tan^{-1}(\tfrac{n_t}{n_i})}$$

In other words, at Brewster's angle, all P polarized light is transmitted through the interface, and this happens for both $n_i > n_t$ and $n_t > n_i$.

Total internal reflection

Total internal reflection only happens when light propagates from an optically "denser" medium into a "less dense" medium, i.e. $n_i > n_t$. (For example, light going from water into air). From Snell's law,

$$\theta_t = \sin^{-1}(\frac{n_i}{n_t}\sin\theta_i)$$

When $n_i > n_t$, it is possible to have $\frac{n_i}{n_t}\sin\theta_i > 1$. Lets define the *critical angle* θ_c by requiring that $\frac{n_i}{n_t}\sin\theta_c = 1$, we have

$$\boxed{\theta_c = \sin^{-1}(\tfrac{n_t}{n_i})}$$

At the critical angle of incidence, the transmitted beam has $\theta_t = 90°$, i.e. the beam runs exactly along the interface.
Writing down the general expression for the transmitted wave

$$E_t \propto e^{-i\vec{k}\cdot\vec{r}} = e^{-ik_t(x\sin\theta_t + z\cos\theta_t)}$$

We can rewrite Snell's law as

$$\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i = \frac{\sin\theta_i}{\sin\theta_c}$$

and also

$$\cos\theta_t = \pm\sqrt{1-\sin^2\theta_t} = \pm\sqrt{1-(\frac{\sin\theta_i}{\sin\theta_c})^2}$$

Now when $\theta_i > \theta_c$, the cosine becomes purely imaginary. For notational convenience, select the negative root and

$$\cos\theta_t = -i\sqrt{(\frac{\sin\theta_i}{\sin\theta_c})^2 - 1} \equiv -i\alpha \qquad \theta_c < \theta < \frac{\pi}{2}$$

where

$$\alpha = \sqrt{(\frac{\sin\theta_i}{\sin\theta_c})^2 - 1}$$

The transmitted wave can be written in this notation as

$$E_t \propto e^{-ik_t(x\sin\theta_t + z\cos\theta_t)}$$

$$E_t = e^{-k_t\alpha z}e^{-ik_t x\sqrt{1+\alpha^2}}$$

The transmitted wave propagates parallel to the surface (i.e., along the x axis) and is attenuated exponentially in the z direction (i.e. normal to the propagation direction). Such a damped wave is called an evanescent wave. Except when the incident wave

Reflection from a conductor

When deriving the Fresnel equations for reflection, there are only two assumptions have been made, which are 1) the material response is linear, and 2) ϵ and μ are constant on each side of the interface. Therefore, the Fresnel equations for reflection and transmission will also hold for a complex dielectric constants.

Thus the Fresnel field reflectivity is the same for a conductor, but the transmission angle θ_t is now complex. Therefore, the Snell's law becomes

$$\sin\tilde{\theta}_t = \frac{n_i}{\tilde{n}_t}\sin\theta_i$$

The fact that the reflection coefficient becomes complex means there is a phase shift between incident and reflected waves. Thus linearly polarized light can become elliptically polarized on reflection under certain circumstances. We will not concern ourselves with the details of the general case, but restrict our attention solely to power reflection at normal incidence.

$$\begin{aligned} R &= |r|^2 = \tfrac{\tilde{n}-1}{\tilde{n}+1}\cdot\tfrac{\tilde{n}^*-1}{\tilde{n}^*+1} = \tfrac{(n-1)^2+(n\kappa)^2}{(n+1)^2+(n\kappa)^2} \\ &= 1 - \tfrac{4n}{(n+1)^2+(n\kappa)^2} \end{aligned}$$

Note that if the index were purely imaginary, i.e. $\tilde{n} = in\kappa$, then we would have

$$R = \frac{(in\kappa-1)(-in\kappa-1)}{(in\kappa+1)(-in\kappa+1)} = 1$$

It becomes a "perfect reflector". Indeed, the imaginary part of the refractive index of most metals dominate over the real part, due to the high conductivities of most metals ($\tilde{k} = \omega\sqrt{\mu(\epsilon - \frac{i\sigma}{\psi})}$). And very high reflectivities (90 to 95%) are observed for most metals. That is the reason why metal coatings are often used to made mirrors.

Waveguides

Waveguides, like transmission lines, are structures used to guide electromagnetic waves from point to point. However, the fundamental characteristics of waveguide and transmission line waves (*modes*) are quite different. The differences in these modes result from the basic differences in geometry for a transmission line and a waveguide.

Transmission line

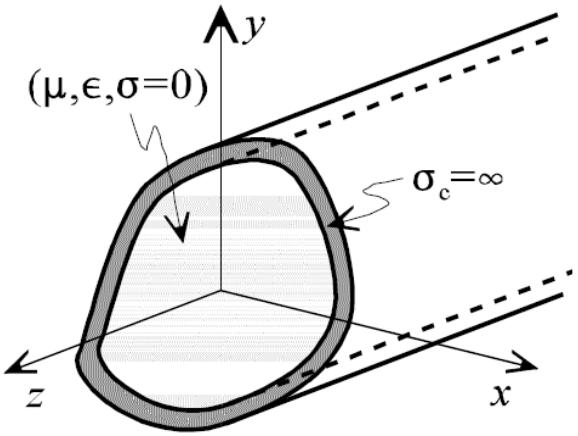
- Two or more conductors separated by some insulating medium (two-wire, coaxial, microstrip, etc.).
- Normal operating mode is the TEM or quasi-TEM mode (can support TE and TM modes but these modes are typically undesirable).
- No cutoff frequency for the TEM mode.
- Significant signal attenuation at high frequencies due to conductor and dielectric losses.

Waveguide

- Typically one enclosed conductor filled with an insulating medium (rectangular, circular, etc.).
- Operating modes are TE or TM modes (cannot support a TEM mode).
- Must operate the waveguide at a frequency above the respective TE or TM mode cutoff frequency for that mode to propagate.
- Lower signal attenuation at high frequencies than transmission lines.

Ideal Waveguide
(PEC tube, perfect insulator inside)

Waves propagate along the waveguide (+z-direction) within the waveguide through the lossless dielectric. The electric and magnetic fields of the guided waves must satisfy



the source-free Maxwell's equations.

Assumptions:

(1) the waveguide is infinitely long, oriented along the z-axis, and uniform along its length.

(2) the waveguide is constructed from ideal materials (enclosing PEC conductor is filled with a perfect insulator).

(3) fields are time-harmonic.

The electric and magnetic fields associated with the waves propagating inside the waveguide must satisfy the source free Maxwell's equations given by

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu \boldsymbol{H}_s$$

$$\nabla \times \boldsymbol{H}_s = j\omega\epsilon \boldsymbol{E}_s$$

These equations can be manipulated into wave equations for the electric and magnetic fields as was shown in the case of unguided waves. These wave equations are

$$\nabla^2 \boldsymbol{E}_s + k^2 \boldsymbol{E}_s = 0$$

$$\nabla^2 \boldsymbol{H}_s + k^2 \boldsymbol{H}_s = 0$$

For certain waveguide geometries, the individual components of the fields can be determined using the separation of variables technique. For a wave propagating along the waveguide in the z-direction, the electric and magnetic fields may be written in rectangular coordinates as

$$\boldsymbol{E}_s(x,y,z) = \boldsymbol{e}_s(x,y)\, e^{-\gamma z}$$

$$\boldsymbol{H}_s(x,y,z) = \boldsymbol{h}_s(x,y)\, e^{-\gamma z}$$

The constant γ is the waveguide propagation constant defined by

$$\gamma = \alpha + j\beta$$

where α is the waveguide attenuation constant and β is the waveguide phase constant. In general, the waveguide propagation constant has very different characteristics than the transmission line propagation constant.

The vectors $\boldsymbol{e}_s(x,y)$ and $\boldsymbol{h}_s(x,y)$ in the waveguide field expressions may contain both transverse field components ($\boldsymbol{a}_x$, $\boldsymbol{a}_y$) and longitudinal field components ($\boldsymbol{a}_z$). By expanding the curl operator of Maxwell's equations in rectangular coordinates, and noting that the derivatives of the transverse components with respect to z can be evaluated as

$$\frac{\partial E_{xs}}{\partial z} = -\gamma E_{xs} \qquad \frac{\partial E_{ys}}{\partial z} = -\gamma E_{ys}$$

$$\frac{\partial H_{xs}}{\partial z} = -\gamma H_{xs} \qquad \frac{\partial H_{ys}}{\partial z} = -\gamma H_{ys}$$

If we equate the vector components on each side of the two Maxwell curl equations, we find

$$j\omega\epsilon E_{xs} = \frac{\partial H_{zs}}{\partial y} + \gamma H_{ys} \qquad (1a)$$

$$j\omega\epsilon E_{ys} = -\gamma H_{xs} - \frac{\partial H_{zs}}{\partial x} \qquad (1b)$$

$$j\omega\epsilon E_{zs} = \frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y} \qquad (1c)$$

$$-j\omega\mu H_{xs} = \frac{\partial E_{zs}}{\partial y} + \gamma E_{ys} \qquad (2a)$$

$$-j\omega\mu H_{ys} = -\gamma E_{xs} - \frac{\partial E_{zs}}{\partial x} \qquad (2b)$$

$$-j\omega\mu H_{zs} = \frac{\partial E_{ys}}{\partial x} - \frac{\partial E_{xs}}{\partial y} \qquad (2c)$$

Equations (1) and (2) are valid for any wave (guided or unguided) propagating in the z-direction in a source-free region with a propagation constant of γ. We may use Equations (1) and (2) to solve for the longitudinal field components in terms of the transverse field components.

solve (1a) and (2b) for H_{ys}

$$E_{xs} = \frac{1}{h^2}\left(-\gamma \frac{\partial E_{zs}}{\partial x} - j\omega\mu \frac{\partial H_{zs}}{\partial y} \right)$$

solve (1b) and (2a) for H_{xs}

$$E_{ys} = \frac{1}{h^2}\left(-\gamma \frac{\partial E_{zs}}{\partial y} + j\omega\mu \frac{\partial H_{zs}}{\partial x} \right)$$

solve (1b) and (2a) for E_{ys}

$$H_{xs} = \frac{1}{h^2}\left(j\omega\epsilon \frac{\partial E_{zs}}{\partial y} - \gamma \frac{\partial H_{zs}}{\partial x} \right)$$

solve (1a) and (2b) for E_{xs}

$$H_{ys} = \frac{1}{h^2}\left(-j\omega\epsilon \frac{\partial E_{zs}}{\partial x} - \gamma \frac{\partial H_{zs}}{\partial y} \right)$$

where the constant h is defined by

$$h^2 = \gamma^2 + \omega^2\mu\epsilon = \gamma^2 + k^2 \qquad \Rightarrow \qquad \gamma = \sqrt{h^2 - k^2}$$

The equations for the transverse fields in terms of the longitudinal fields describe the different types of possible modes for guided and unguided waves.

Transverse electromagnetic (TEM) modes	$E_{zs} = H_{zs} = 0$	plane waves, transmission lines
Transverse electric (TE) modes	$E_{zs} \neq 0, H_{zs} = 0$	waveguide modes
Transverse magnetic (TM) modes	$E_{zs} = 0, H_{zs} \neq 0$	waveguide modes
Hybrid (EH or HE) modes	$E_{zs} \neq 0, H_{zs} \neq 0$	waveguide modes

For TEM modes, the only way for the transverse fields to be non-zero with $E_{zs} = H_{zs} = 0$ is for $h = 0$, which yields

$$\gamma = \sqrt{-k^2} = jk = \alpha + j\beta \qquad \Rightarrow \qquad \beta = k \qquad \textbf{(TEM modes)}$$

For the waveguide modes, h cannot be zero since this would yield unbounded results for the transverse fields. Thus, $\beta \neq k$ and the waveguide propagation constant can be written as

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{-k^2\left(1 - \frac{h^2}{k^2}\right)} = jk\sqrt{1 - \left(\frac{h}{k}\right)^2}$$

The ratio of h/k can be written in terms of the *cutoff frequency* f_c for the given waveguide mode.

$$\frac{h}{k} = \frac{h}{\omega\sqrt{\mu\epsilon}} = \frac{h}{2\pi f\sqrt{\mu\epsilon}} = \frac{f_c}{f}$$

$$f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}} \qquad \textbf{(waveguide cutoff frequency)}$$

The waveguide propagation constant in terms of the waveguide cutoff frequency is

$$\gamma = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

If $f < f_c$, $\gamma = \alpha$ (purely real) $\quad e^{-\gamma z} = e^{-\alpha z}$ waves are attenuated (*evanescent* modes).

If $f > f_c$, $\gamma = j\beta$ (purely imaginary) $\quad e^{-\gamma z} = e^{-j\beta z}$ waves are unattenuated (propagating modes).

TE and TM Modes in Waveguides

If the single non-zero longitudinal field component associated with a given waveguide mode (H_{zs} for a TE mode or E_{zs} for a TM mode) can be determined, the remaining transverse field components can be found using the general wave equations for the transverse fields in terms of the longitudinal fields. The longitudinal magnetic field of the TE mode and the longitudinal electric field of the TM mode are determined by solving the appropriate boundary value problem for the given waveguide geometry.

General Waves

$$E_{xs} = \frac{1}{h^2}\left(-\gamma\frac{\partial E_{zs}}{\partial x} - j\omega\mu\frac{\partial H_{zs}}{\partial y}\right) \qquad H_{xs} = \frac{1}{h^2}\left(j\omega\epsilon\frac{\partial E_{zs}}{\partial y} - \gamma\frac{\partial H_{zs}}{\partial x}\right)$$

$$E_{ys} = \frac{1}{h^2}\left(-\gamma\frac{\partial E_{zs}}{\partial y} + j\omega\mu\frac{\partial H_{zs}}{\partial x}\right) \qquad H_{ys} = \frac{1}{h^2}\left(-j\omega\epsilon\frac{\partial E_{zs}}{\partial x} - \gamma\frac{\partial H_{zs}}{\partial y}\right)$$

TE Modes ($E_{zs} = 0$)

$$E_{xs} = -\frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} \qquad H_{xs} = -\frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x}$$

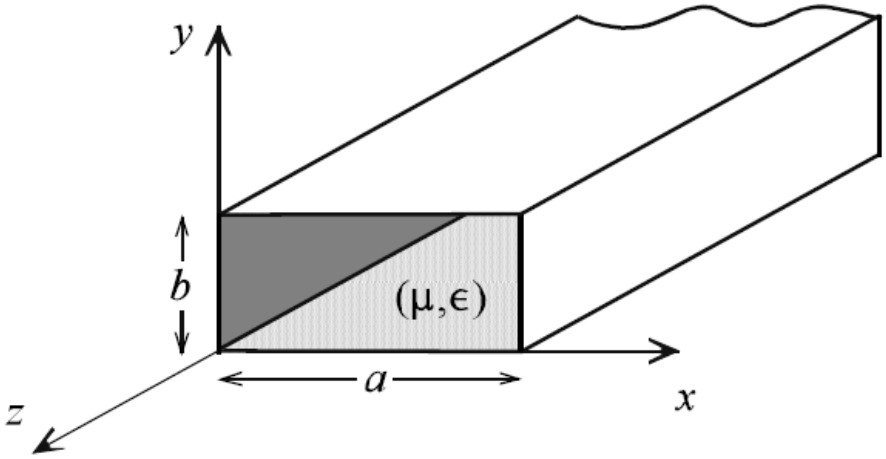
$$E_{ys} = \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} \qquad H_{ys} = -\frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y}$$

TM Modes ($H_{zs} = 0$)

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} \qquad H_{xs} = \frac{j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial y}$$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} \qquad H_{ys} = -\frac{j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial x}$$

Rectangular Waveguide



The rectangular waveguide can support either TE or TM modes. The rectangular cross-section ($a > b$) allows for *single-mode operation*. Single -mode operation means that only one mode propagates in the waveguide over a given frequency range. A square waveguide cross-section does not allow for single-mode operation.

Rectangular Waveguide TM modes

The longitudinal electric field of the TM modes within the rectangular waveguide must satisfy the wave equation

$$\nabla^2 E_{zs} + k^2 E_{zs} = 0$$

which expanded in rectangular coordinates is

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0$$

The electric field function may be determined using the separation of variables technique by assuming a solution of the form

$$E_{zs} = X(x)Y(y)e^{-\gamma z} \qquad (+\boldsymbol{a}_z \text{ traveling waves})$$

Inserting the assumed solution into the governing differential equation

gives

$$Y(y)\frac{d^2X(x)}{dx^2}e^{-\gamma z}+X(x)\frac{d^2Y(y)}{dy^2}e^{-\gamma z}+(\gamma^2+k^2)X(x)Y(y)e^{-\gamma z}=0$$

$$Y(y)\frac{d^2X(x)}{dx^2}e^{-\gamma z}+X(x)\frac{d^2Y(y)}{dy^2}e^{-\gamma z}+h^2X(x)Y(y)e^{-\gamma z}=0$$

where $h^2 = \gamma^2 + k^2$. Dividing this equation by the assumed solution gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+h^2=0 \qquad (1)$$

Note that the first term in (1) is a function of x only while the second term is a function of y only. In order for (1) to be satisfied for every x and y within the waveguide, each of the first two terms in the equation must be constants.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-k_x^2 \quad \Rightarrow \quad \frac{d^2X(x)}{dx^2}+k_x^2X(x)=0$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-k_y^2 \quad \Rightarrow \quad \frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0$$

$$h^2=k_x^2+k_y^2 \qquad \textbf{(separation equation)}$$

The original second order partial differential equation dependent on two variables has been separated into two second order pure differential equations each dependent on only one variable. The general solutions to the two separate differential equations are

$$X(x)=A\sin k_x x+B\cos k_x x$$

$$Y(y)=C\sin k_y y+D\cos k_y y$$

The resulting longitudinal electric field for the rectangular waveguide TM modes is

$$E_{zs}(x,y,z) = (A\sin k_x x + B\cos k_x x)(C\sin k_y y + D\cos k_y y)\, e^{-\gamma z}$$

The TM boundary conditions for the rectangular waveguide are

$$E_{zs}(0,y,z) = E_{zs}(a,y,z) = 0 \qquad \text{(vertical walls)}$$

$$E_{zs}(x,0,z) = E_{zs}(x,b,z) = 0 \qquad \text{(horizontal walls)}$$

The application of the boundary conditions yields

$$E_{zs}(0,y,z) = 0 \quad \Rightarrow \quad B = 0$$

$$E_{zs}(a,y,z) = 0 \quad \Rightarrow \quad k_x a = m\pi \quad (\mathrm{m} = 1,2,3,\ldots) \quad \Rightarrow \quad k_x = \frac{m\pi}{a}$$

$$E_{zs}(x,0,z) = 0 \quad \Rightarrow \quad D = 0$$

$$E_{zs}(x,b,z) = 0 \quad \Rightarrow \quad k_y b = n\pi \quad (\mathrm{n} = 1,2,3,\ldots) \quad \Rightarrow \quad k_y = \frac{n\pi}{b}$$

The resulting product of the constants A and C into combined into one constant (E_o).

$$E_z(x,y,z) = E_o \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-\gamma z}$$

The transverse field components of the TM modes are

$$E_{xs}(x,y,z) = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} = -\frac{\gamma}{h_2}\left(\frac{m\pi}{a}\right) E_o \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-\gamma z}$$

$$E_{ys}(x,y,z) = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} = -\frac{\gamma}{h_2}\left(\frac{n\pi}{b}\right) E_o \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, e^{-\gamma z}$$

$$H_{xs}(x,y,z) = \frac{j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial y} = \frac{j\omega\epsilon}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} e^{-\gamma z}$$

$$H_{ys}(x,y,z) = -\frac{j\omega\epsilon}{h_2}\frac{\partial E_{zs}}{\partial x} = -\frac{j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-\gamma z}$$

The index designation for the discrete TM modes is TM_{mn} where (m = 1, 2, 3, ...) and (n = 1, 2, 3, ...).

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – 4 – PINTERACTION OF EMW WITH MATTER IN MICROSCOPIC SCALE - SPH5105

UNIT-IV

ELECTROMAGNETIC THEORY

INTERACTION OF EMW WITH MATTER ON MICROSCOPIC SCALE

Scattering and Scattering Parameters – Scattering by a Free Electron (Thomson Scattering) – Scattering by a Bound Electron (Rayleigh Scattering) – State of Polarization and Scattered Radiation – Coherence and Incoherence in scattered Light – Dispersion Normal and Anomalous – Dispersion in Gases (Lorentz Theory) – Dispersion in Liquids and Solids.

X-Ray Scattering by a Free Electron

X-rays are electromagnetic radiation, and as such manifest both wave and particle properties: X-ray waves have wavelengths and frequencies of the order of $\lambda \approx 1\,\text{Å} = 10^{-10}$ m and $\omega = 2\pi\nu = 2\pi c/\lambda \approx 2\times10^{19}$ s^{-1}; X-ray photons have Planck-Einstein energies $E_\gamma = h\nu = hc/\lambda$ and de Broglie momenta $p_\gamma = h/\lambda$, where $h = 6.626\times10^{-34}$ J·s is Planck's constant. X-rays with $\lambda = 1$ Å have $E_\gamma = 1.988\times10^{-25}$ J or $V = E_\gamma/q_e = 12.4$ keV, where $q_e = 1.602\times10^{-19}$ C is the electron charge. Corresponding to the dualistic wave/particle nature of X-rays, there are two types of X-ray scattering by electrons: Thomson scattering via electromagnetic wave-electron interactions, and Compton scattering via photon-electron collisions.

Thomson Scattering. If a linearly polarized, monochromatic, plane wave X-ray beam is incident on a stationary free electron with charge e and mass m, the oscillating electric field $E = E_0\,e^{i(\omega t-\delta)}$ of the incident X-ray wave exerts a Coulombic driving force ($F = qE$) on the electron that causes it to oscillate at the same frequency as the incident radiation. By Newton's second law of motion ($F = ma$) the equation of motion for the driven harmonic oscillation of the electron is then

$$ma = q_e E = e\,E_0\,e^{i(\omega t-\delta)}\ ,$$

and the electron experiences the oscillating acceleration

$$a = \frac{e}{m}E_0\,e^{i(\omega t-\delta)} = a_0\,e^{i(\omega t-\delta)}\ .$$

Since an accelerated charge emits electromagnetic radiation, the oscillating electron becomes a new source of X-rays that radiate spherically in all directions and have the same frequency as the incident X-rays. Effectively, the electron scatters the incident X-ray beam in all directions.

According to classical electrodynamical theory, the electromagnetic radiation at a point at $\vec{r}$ far from a charge q that experiences an acceleration $\vec{a}$ has an electric field component

$$\mathcal{E} = -\frac{qa\sin\alpha}{c^2 r}\ ,$$

where c is the speed of light and α is the angle between $\vec{a}$ and $\vec{r}$. So, if a linearly-polarized, monochromatic X-ray beam is incident on a free stationary electron, and the scattered radiation is observed at a position $\vec{r}$, far from the electron, in the equatorial plane through the electron's original at-rest position and perpendicular to its oscillation direction, *i.e.*, perpendicular to the incident beam polarization direction, then $\alpha = 90°$, $\sin\alpha = 1$, and the electric field component of the scattered radiation has amplitude

$$\mathcal{E}_0 = -\frac{qa_0}{c^2 r} = -\frac{q}{c^2 r}\frac{e}{m}E_0 = -\left(\frac{e^2}{mc^2}\right)\frac{E_0}{r}\ .$$

In the present case of forced oscillation of a free electron, although $q = e$ is a negative charge, the product $qe = e^2$ is positive, and the negative sign means that the scattered radiation is 180° or π out of phase with the incident radiation, i.e., there is a 180° phase change – a phase flip or phase reversal – upon scattering.

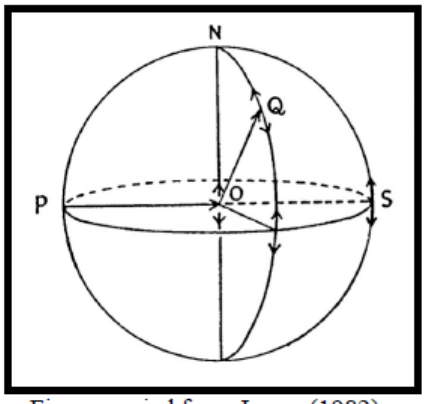


Figure copied from James (1982).

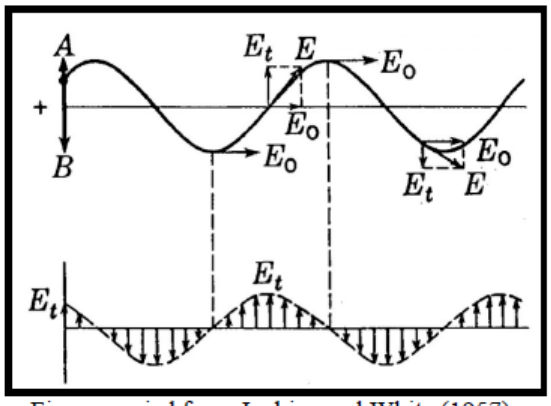


Figure copied from Jenkins and White (1957).

X-rays incident from point P on an electron at the origin O are scattered spherically in all directions. The amplitude of the scattered wave at any observation point such as S or Q is perpendicular to the radius from O to the point, and the phase of the scattered wave with respect to the meridian through the observation point is opposite the phase of the incident wave with respect to the polar axis through the origin. The incident wave and the forced oscillation of the electron have the same frequency, but the phase of the electron oscillation lags $\pi/2$ behind the phase of the incident wave, and the phase of the scattered wave lags $\pi/2$ behind the phase of the electron oscillation. Hence, the phase of the scattered wave lags π behind, i.e., is opposite the phase of the incident wave.

The physics of X-ray scattering by a free electron was first analyzed in terms of classical electromagnetic theory by J.J. Thomson (1856-1940), the discoverer of the electron, and the constant factor in the scattered-to-incident amplitude ratio, $\dfrac{\mathcal{E}_o}{E_0} = -\left(\dfrac{e^2}{mc^2}\right)\dfrac{1}{r} = -\dfrac{r_e}{r}$, is called the *Thomson scattering length of the classical electron*. By dimensional analysis,

$$r_e = \frac{e^2}{mc^2} = \frac{\text{charge}^2}{\text{mass}\cdot\text{velocity}^2} = \frac{\text{charge}^2}{\text{mass}\cdot\text{distance}^2\cdot\text{time}^{-2}} = \frac{\text{force}}{\text{mass}\cdot\text{time}^{-2}} = \frac{\text{acceleration}}{\text{time}^{-2}} = \text{distance},$$

and the constant has dimension length (as it must, because multiplying it by $1/r$ gives the dimensionless ratio $-\mathcal{E}_0/E_0$). For historical reasons, the Thomson scattering length is also called the *classical electron radius*, since for a uniform distribution of charge equal to the electron charge, q_e, confined to a sphere with mass equal to the electron rest mass, m_e, the sphere radius, r_e, results from equating the electrostatic potential energy of the charge distribution, $q_e V = q_e^2/r_e$, and the relativistic energy, $m_e c^2$, equivalent to the electron rest mass.

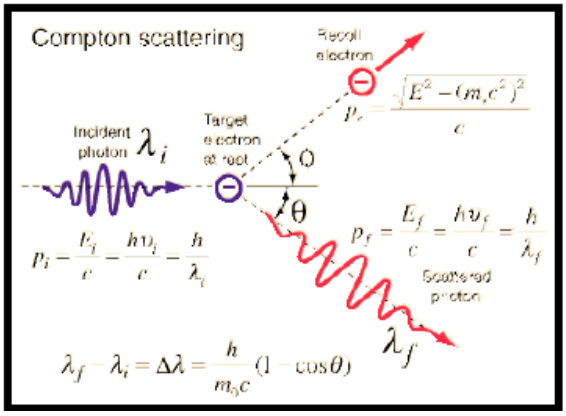
Converting from Gaussian cgs units to SI mks units (in which $1\,\text{C} = 1\,\text{A}\cdot\text{s}$), the value of the Thomson scattering length or classical electron radius is

$$r_e = \frac{e^2}{mc^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_e^2}{m_e c^2} = 2.818\times10^{-15}\,\text{m},$$

where $q_e = 1.602 \times 10^{-19}\,\mathrm{C}$ and $m_e = 9.107 \times 10^{-31}\,\mathrm{kg}$ are, respectively, the charge and rest mass of the electron, $c = 2.998 \times 10^{8}\ \mathrm{m \cdot s^{-1}}$ is the speed of light in free space, and the quantit $\left(4\pi\varepsilon_0\right)^{-1} = 8.988 \times 10^{9}\,\mathrm{C^{-2} \cdot N \cdot m^2}$ is the constant of proportionality in Coulomb's law, $F = \left(4\pi\varepsilon_0\right)^{-1} q_1 q_2 / r_{12}^2$, in which $\varepsilon_0 = 1/\left(\mu_0 c^2\right) = 8.854 \times 10^{-12}\ \mathrm{C^2 \cdot N^{-1} \cdot m^{-2}}$ is the electric permittivity of free space, derived from the defined magnetic permeability of free space, $\mu_0 \equiv 4\pi \times 10^{-7}\,\mathrm{C^{-2} \cdot N \cdot s^2}$.

Because r_e is so small, the scattered-to-incident amplitude ratio $-\mathcal{E}_0 / E_0 = r_e / r$ implies that electron scatters only a tiny fraction of X-radiation incident on it. At a distance from the elect of only a nanometer, the amplitude ratio is only $r_e / r = 2.8 \times 10^{-15}\,\mathrm{m} / 10^{-9}\,\mathrm{m} \approx 10^{-6}$, yet we observe and measure X-ray scattering at sample to detector distances of tens of centimeters because even a tiny material sample contains a huge number of electrons, e.g., in just 1 µg of carbon there are $10^{-6}\,\mathrm{g} \times \frac{1\ \mathrm{mol}}{12\ \mathrm{g}} \times \frac{6.023 \times 10^{23}\ \mathrm{atoms}}{1\ \mathrm{mol}} \times \frac{6\ \mathrm{electrons}}{\mathrm{atom}} \approx 3 \times 10^{17}\ \mathrm{electrons}$.

Compton scattering. Thomson scattering, which accounts for most X-ray scattering by electrons, is elastic, i.e., the incident and scattered X-rays have the same wavelength; and it is Thomson scattering of electromagnetic X-ray waves without change in wavelength that produ the interference effects responsible for diffraction of X-rays by crystals. A small part, howeve of the total scattering of X-rays by matter is inelastic in that some of the scattered X-rays have longer wavelengths that the incident X-rays, because, in the manner of a billiard ball collision, with total energy and total momentum are conserved, energy is transferred from the incident X-ray photon to the electron, and both the photon and the electron are scattered at angles that depend on the size of the photon wavelength change, i.e., on the size of the photon-to-electron energy transfer. In crystallographic X-ray diffraction measurements of Bragg reflection peaks due to coherent, elastic Thomson scattering, the incoherent, inelastic Compton scattering appe as diffuse, uniform background under the peaks.



X-Ray Scattering by a Bound Atomic Electron

If a linearly polarized, monochromatic X-ray beam is incident on an atom, the oscillating electric field of the electromagnetic radiation forces both the nucleus and the electrons surrounding it to oscillate with the same frequency as the incident X-rays. The atom becomes an assembly of oscillating dipoles in which the relatively massive nucleus ($m_p/m_e = 1836$) can be regarded as being stationary and each oscillating electron experiences a Hooke's law restoring force ($F = -kx$) equal to the net resultant of the force of attraction to the nucleus screened by the forces of repulsion from the other electrons.

The equation of motion for the forced harmonic oscillation of an electron in an atom is then

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx + eE_0\,\mathrm{e}^{i(\omega t-\delta)}.$$

This differential equation has a solution of the form

$$x = x_0\,\mathrm{e}^{i(\omega t-\delta)}$$

for the oscillatory displacement, and differentiation gives the oscillatory velocity and acceleration,

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = i\omega x_0\,\mathrm{e}^{i(\omega t-\delta)}$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x_0\,\mathrm{e}^{i(\omega t-\delta)}.$$

Substitution into the equation of motion gives

$$-m\omega^2 x_0\,\mathrm{e}^{i(\omega t-\delta)} = -kx_0\,\mathrm{e}^{i(\omega t-\delta)} + eE_0\,\mathrm{e}^{i(\omega t-\delta)}$$

$$x_0\left(k - m\omega^2\right) = eE_0\,,$$

which yields the electron's oscillation amplitude,

$$x_0 = \frac{eE_0}{k - m\omega^2} = \frac{eE_0}{m\left(\dfrac{k}{m} - \omega^2\right)} = \frac{eE_0}{m\left(\omega_0^2 - \omega^2\right)},$$

or, after changing signs in the numerator and denominator,

$$x_0 = -\frac{eE_0}{m\left(\omega^2 - \omega_0^2\right)},$$

where the quantity introduced as

$$\omega_0 = \sqrt{\frac{k}{m}}$$

is the natural vibration frequency of the dipole oscillator corresponding to the net binding force ($F = -kx = -m\omega_0^2 x$) exerted on the electron by the nucleus as screened by the other electrons.

The oscillating electron's acceleration is then

$$a = -\omega^2 x_0 \, e^{i(\omega t - \delta)} = \frac{eE_0\omega^2}{m\left(\omega^2 - \omega_0^2\right)} e^{i(\omega t - \delta)} = a_0 \, e^{i(\omega t - \delta)},$$

and, from the classical theory of electrodynamics, the radiation scattered to a position $\vec{r}$ in the equatorial plane of the dipole oscillator , which is perpendicular to the X-ray polarization direction, has amplitude

$$\mathcal{E}_0 = -\frac{ea_0}{c^2 r} = -\left(\frac{e^2}{mc^2}\right)\frac{E_0}{r}\frac{\omega^2}{\left(\omega^2 - \omega_0^2\right)}.$$

Therefore, at unit distance, in units of minus the Thomson scattering length of a free classical electron, the scattering factor for a bound atomic electron is

$$f_e = \frac{\mathcal{E}_0(\text{bound})}{\mathcal{E}_0(\text{free})} = \frac{\omega^2}{\omega^2 - \omega_0^2} = \frac{1}{1 - \left(\frac{\omega_0}{\omega}\right)^2},$$

where, again, ω is the frequency of the incident and scattered X-rays and ω_0 is the natural vibration frequency of the nucleus-electron dipole oscillator

Note that in the high-frequency limit with $\omega \gg \omega_0$, which is commonly case, the scattering factor is independent of frequency, and *the scattering by a bound atomic electron is to a good approximation the same as the scattering by a free electron.*

Since atomic electrons move around the nucleus with speeds much less than of the speed of light, they appear, from the point of view of incident X-rays, to be stationary. For example, in a Bohr atom, the condition for a stable electron orbit is that the centripetal force of nuclear attraction balances the tangential centrifugal force of orbital revolution, i.e.,

$$\frac{Ze^2}{r^2} = \frac{mv^2}{r},$$

$$Ze^2 = (mvr)v,$$

and the orbital angular momentum is quantized,

$$L = mvr = \frac{nh}{2\pi}, \qquad n = 1, 2, 3, \ldots .$$

Thus, the speed of an electron in the ground-state, first Bohr orbit in a hydrogen atom is,

$$v = \frac{Ze^2}{L} = Ze^2 \frac{2\pi}{nh}\Bigg]_{Z=1,\, n=1} = \left(\frac{1}{4\pi\varepsilon_0}\right) q_e^2 \frac{2\pi}{h}$$

$$= \frac{\left(8.988 \times 10^9 \, \text{C}^{-2} \cdot \text{N} \cdot \text{m}^2\right) \times \left(1.602 \times 10^{-19} \, \text{C}\right)^2 \times 2\pi}{6.626 \times 10^{-34} \, \text{J} \cdot \text{s}} = 2.2 \times 10^6 \, \text{m} \cdot \text{s}^{-1} = 0.007\, c .$$

less than 1% of the speed of light. Even in the extreme case of a uranium atom with $Z = 90$, the speed of the lowest energy electron, in the first Bohr orbit, is only ~0.6 c.

Not only are atomic electrons generally moving much more slowly than electromagnetic waves, but the oscillatory displacement of an electron imparted by the oscillating electric field of an incident X-ray wave is very small. In the high frequency limit with $\omega \gg \omega_0$ the amplitude of electron oscillation per unit amplitude of incident X-rays is

$$x_0 = \frac{eE_0}{m\left(\omega^2 - \omega_0^2\right)} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_e E_0}{m_e \omega_X^2}$$
$$= \frac{\left(8.988\times10^{9}\,\mathrm{C^{-2}\cdot N\cdot m^2}\right)\left(1.602\times10^{-19}\,\mathrm{C}\right)\left(1\,\mathrm{C\cdot m^{-2}}\right)}{\left(9.107\times10^{-31}\,\mathrm{kg}\right)\left(\sim 20\times10^{18}\,\mathrm{s^{-1}}\right)^2} \approx 4\times10^{-14}\,\mathrm{m} = 4\times10^{-4}\,\text{Å},$$

which is a tiny fraction of an atomic diameter.

Thus all-in-all, *in the high-frequency limit with* $\omega \gg \omega_0$, *for atoms with* $Z \lesssim 20$, *normal scattering of X-rays by a bound atomic electron is independent of frequency or wavelength, and is, to a fair approximation, the same as the scattering by a stationary free electron.*

Dispersion Effects in X-Ray Scattering

In the special case that the X-ray frequency ω happens to be, or is tuned to be, close to the dipole oscillator's natural vibration frequency ω_0, *resonant absorption due to the frequency matching* causes the electron oscillation amplitude $x_0 = \left(e/m\right) E_0 \omega^2 \Big/ \left(\omega^2 - \omega_0^2\right)$ to become very large. The amplitude does not, however, become infinite at $\omega = \omega_0$ because there is a small damping effect due to the energy loss to classical radiation by the oscillating electron.

Taking the damping force to be proportional the electron's velocity ($F = -k' v$), the electron's equation of motion in the damped, driven harmonic oscillator is

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k'\frac{\mathrm{d}x}{\mathrm{d}t} - kx + eE_0 \mathrm{e}^{i(\omega t - \delta)}.$$

Recalling that the restoring force constant was defined as

$$k = m\omega_0^2,$$

and defining the damping force constant as

$$k' = m\gamma,$$

the equation of motion can be rewritten as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\gamma\frac{\mathrm{d}x}{\mathrm{d}t} - \omega_0^2 x + \frac{e}{m}E_0 \mathrm{e}^{i(\omega t - \delta)}.$$

This differential equation too has a solution of the form

$$x = x_0 \mathrm{e}^{i(\omega t + \delta)},$$

and differentiation and substitution gives

$$-\omega^2 x_0 \mathrm{e}^{i(\omega t - \delta)} = -i\gamma\omega x_0 \mathrm{e}^{i(\omega t - \delta)} - \omega_0^2 x_0 \mathrm{e}^{i(\omega t - \delta)} + \frac{e}{m}E_0 \mathrm{e}^{i(\omega t - \delta)}$$

$$x_0\left(\omega^2 - \omega_0^2 - i\gamma\omega\right) = -\frac{e}{m}E_0$$

$$x_0 = -\frac{eE_0}{m\left(\omega^2 - \omega_0^2 - i\gamma\omega\right)}.$$

With this oscillation amplitude, the oscillating acceleration of the electron is

$$a = -\omega^2 x_0 \mathrm{e}^{i(\omega t - \delta)} = a_0 \mathrm{e}^{i(\omega t - \delta)} = \frac{eE_0\omega^2}{m\left(\omega^2 - \omega_0^2 - i\gamma\omega\right)}\mathrm{e}^{i(\omega t - \delta)};$$

the amplitude of radiation scattered to a point at $\vec{r}$ in the equatorial plane of the oscillator is

$$\mathcal{E}_0 = -\frac{qa_0}{c^2 r} = -\left(\frac{e^2}{mc^2}\right)\frac{E_0}{r}\frac{\omega^2}{\left(\omega^2 - \omega_0^2 - i\gamma\omega\right)};$$

and, with the resonance damping correction included, the X-ray scattering factor for a bound atomic electron is

$$f_e = \frac{\mathcal{E}_0(\text{bound})}{\mathcal{E}_0(\text{free})} = \frac{\omega^2}{\omega^2 - \omega_0^2 - i\gamma\omega} = \frac{1}{1 - \left(\frac{\omega_0}{\omega}\right)^2 - \frac{i\gamma}{\omega}} \ .$$

For a free electron, $\omega_0 = \gamma = 0$ and $f_e = 1$ in units of minus the Thomson scattering length.

In order to emphasize that the frequency dependence in the resonant scattering factor is a correction to the frequency-independent normal scattering factor, the resonant scattering factor may be rewritten as

$$f_e = \frac{\omega^2}{\omega^2 - \omega_0^2 - i\gamma\omega} = 1 + \left(\frac{\omega_0^2 + i\gamma\omega}{\omega^2 - \omega_0^2 - i\gamma\omega}\right) \approx 1 + \left(\frac{\omega_0^2}{\omega^2 - \omega_0^2 - i\gamma\omega}\right) ,$$

where to obtain the rightmost expression the imaginary term could be neglected in the numerator because $\gamma \ll \omega$; but, if by accident or design $\omega \approx \omega_0$, the imaginary term cannot be neglected in the denominator. For classical radiation-loss damping it can be shown that $\gamma = \frac{2}{3}\left(\frac{e^2}{mc^2}\right)\frac{\omega^2}{c}$, so that if $\lambda \approx 1\ \text{Å}$, $\omega = 2\pi c/\lambda \approx 2\times 10^{19}\ \text{s}^{-1}$, then $\gamma \approx 2.5\times 10^{15}\ \text{s}^{-1}$ and $\gamma/\omega \approx 10^{-4}$.

Multiplying numerator and denominator of f_e by the complex conjugate of the denominator in order to put the imaginary term in the numerator gives

$$\begin{cases} f_e = f_e' + if_e'' = \dfrac{\omega^2\left(\omega^2 - \omega_0^2\right) + i\gamma\omega^3}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \approx 1 + \left[\dfrac{\omega_0^2\left(\omega^2 - \omega_0^2\right) + i\gamma\omega\omega_0^2}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2}\right] \\ \begin{cases} f_e' = \dfrac{\omega^2\left(\omega^2 - \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \approx 1 + \left[\dfrac{\omega_0^2\left(\omega^2 - \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2}\right] \\ f_e'' = \dfrac{\gamma\omega^3}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \approx \dfrac{\gamma\omega\omega_0^2}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \end{cases} \end{cases} \qquad \begin{cases} f_e = f_e' + if_e'' = |f_e| e^{i\delta} \\ \begin{cases} |f_e| = \sqrt{\left(f_e'\right)^2 + \left(f_e''\right)^2} \\ \delta = \tan^{-1}\left(\dfrac{f_e''}{f_e'}\right) . \end{cases} \end{cases}$$

Note that the real part of the complex-valued scattering factor $f_e = f_e' + if_e''$ might be negative or positive according to

$$\begin{cases} f_e' < 0 & \text{if} \quad \omega < \omega_0 , \\ f_e' > 0 & \text{if} \quad \omega > \omega_0 ; \end{cases}$$

however, the coefficient of the imaginary part is intrinsically positive

$$f_e'' > 0 \ .$$

Thus, it is as if the damped resonant scattering phenomenon that occurs if $\omega \approx \omega_0$ corresponds to, in effect, an elastic X-ray absorption and re-emission that translate to, in effect, a time lag that produces a phase advance $\delta = \tan^{-1}(f''/f')$ with $0 \le \delta < \pi$.

It is in the sense of their frequency or wavelength dependence that the *damped resonant scattering effects* observable when $\omega \approx \omega_0$ are referred to as *dispersion effects*, *i.e.*, refraction effects that are different for different wavelengths, like the refractive bending of visible light by a glass prism that disperses white light into its spectrum of colors.

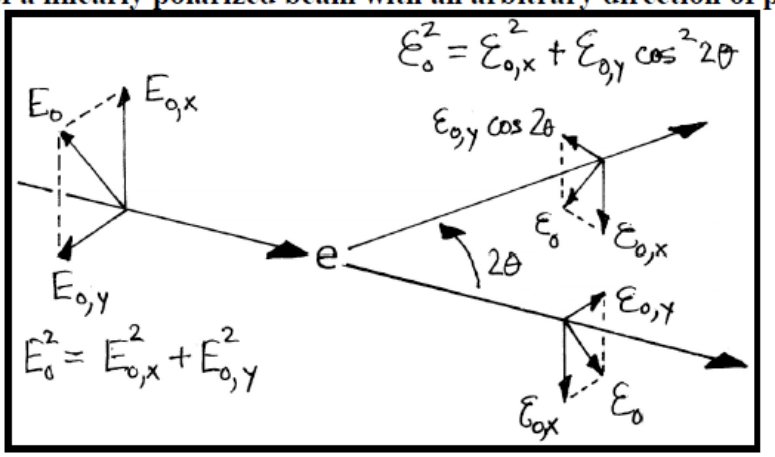
The dispersion effects due to damped resonant scattering are commonly referred to as *anomalous dispersion*, or *anomalous diffraction*, or *anomalous scattering*, since they are not normally observed, but rather are observed only when the incident radiation frequency is close to a scattering electron's natural absorption frequency. In this sense, the dispersion effects are not "normal", but there is nothing "abnormal" or "anomalous" about them; they are entirely natural resonance, or resonant absorption, effects.

Polarization Effects in X-Ray Scattering

To this point, the discussion of X-ray scattering by an electron has assumed a linearly polarized incident beam, and has considered only the scattering in an equatorial plane perpendicular to the incident beam polarization direction.

Scattering by an electron of an unpolarized beam was analyzed by J.J. Thomson by first considering a single, but arbitrary, direction of linear polarization resolved into its components perpendicular and parallel to the plane of scattering as in the figure below, and then averaging over all polarization directions $0 \le \varphi < 2\pi$ around the beam direction.

Scattering of a linearly polarized beam with an arbitrary direction of polarization



In a linearly polarized beam incident along the z axis, the electric field amplitude in any particular direction and its x and y components obey

$$E_0^2 = E_{0,x}^2 + E_{0,y}^2 \ .$$

and, in the beam forward scattered at zero scattering angle along the z axis,

$$\mathcal{E}_0^2 = \mathcal{E}_{0,x}^2 + \mathcal{E}_{0,y}^2 \ .$$

At a point at distance R from the electron,

$$\mathcal{E}_0 = -\left(\frac{e^2}{mc^2}\right)\frac{1}{R}E_0 \ ,$$

and similarly for its x and y components, where the negative sign is present because $\vec{\mathcal{E}}_0$ and its x and y components are antiparallel to $\vec{E}_0$ and its x and y components due to the phase flip that occurs on scattering,.

For consistency with notation to be introduced later, the scattering angle between the incident beam and a scattered beam is denoted 2θ where θ is the angle that the incident and scattered beams each make with the bisector of the scattering angle. Then with respect to the forward scattered beam at zero scattering angle, the beam scattered at an angle 2θ to a point at a distance R from the electron has amplitude components $\mathcal{E}_{0,x}$ and $\mathcal{E}_{0,y}\cos 2\theta$, and the resultant scattered amplitude for the given, particular polarization direction is

$$\mathcal{E}_0^2 = \mathcal{E}_{0,x}^2 + \mathcal{E}_{0,y}^2 \cos^2 2\theta = \left(\frac{e^2}{mc^2} \right)^2 \frac{1}{R^2} \left(E_{0,x}^2 + E_{0,y}^2 \cos^2 2\theta \right) .$$

Finally, averaging the squared x- and y-amplitude components over $0 \le \varphi < 2\pi$ polarization directions around the scattered beam direction gives

$$\left\langle E_{0,x}^2 \right\rangle = E_0^2 \left\langle \cos^2 \varphi \right\rangle = \left\langle E_{0,y}^2 \right\rangle = E_0^2 \left\langle \sin^2 \varphi \right\rangle = \frac{1}{2} E_0^2 ,$$

and thence the relative intensity of Thomson scattering of an unpolarized X-ray beam by a classical electron is

$$\frac{I_{2\theta}}{I_0} = \frac{\mathcal{E}_0^2}{E_0^2} = \left(\frac{e^2}{mc^2} \right)^2 \frac{1}{R^2} \left(\frac{1}{2} + \frac{1}{2} \cos^2 2\theta \right) ,$$

in which the factor $\left(\frac{1}{2} + \frac{1}{2} \cos^2 2\theta \right)$ is called the *polarization factor*.

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DEPARTMENT OF PHYSICS

UNIT – 5 – RELATIVISTIC ELECTRODYNAMICS - SPH5105

RELATIVISTIC ELECTRODYNAMICS

Four Vectors and Tensors – Transformation Equations for Charge and Current Densities – For the Electromagnetic Potentials – The Electromagnetic Field Tensor – Transformation Equations for Electric and Magnetic field Vectors – Covariance of Maxwell Equations in four Vector form – In four Tensor form – Covariance and Transformation Law of Lorentz Force.

Dot products

The Pythagorean theorem says that distances are given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{1}$$

With time as a fourth direction, we find

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2 \tag{2}$$

We can rewrite this by defining two kinds of objects. One is a vector with components

$$(\Delta x, \Delta y, \Delta z, \Delta t) \tag{3}$$

It does not matter at the end whether we write this as a row or a column vector, since we will write all matrix index summations explicitly. All we care about is that it has 4 elements. We write these components with a superscript, i.e. an 'up' index:

$$\Delta x^\mu, \quad \mu = 1, 2, 3, 4 \tag{4}$$

We define another 4-component object

$$\Delta x_\mu : \quad (\Delta x, \Delta y, \Delta z, -c^2\Delta t) \tag{5}$$

Thus the set with index written as a subscript differs from the case with the superscript in that the last term is multiplied by $-c^2$.

This looks like a lot of notation, but it makes our task of computing dot products easier. Each dot product will involve one quantity with an 'up' index and one quantity with a 'down' index. Thus we will write

$$(\Delta s)^2 = \sum_{\mu=1}^{4} \Delta x_\mu \Delta x^\mu \tag{6}$$

One can check that this reproduces (2). The advantage of setting up all this notation is that now we do not see factors of c in the expression for the dot product. We will try to use up and down indices carefully so that we never have to write c explicitly if at all possible. To summarize, our general rule is the following. If we have a vector with up indices

$$V^\mu = (V^1, V^2, V^3, V^4) \tag{7}$$

then we get a version with lower indices as follows

$$V_\mu = (V^1, V^2, V^3, -c^2 V^4) \tag{8}$$

Conversely, if we have a vector with indices down, like

$$W_\mu = (W_1, W_2, W_3, W_4) \tag{9}$$

then we get a version with up indices as follows

$$W^\mu = (W_1, W_2, W_3, -\frac{1}{c^2} W_4) \tag{10}$$

Thus multiplying the last component by $-c^2$ make an up index go down, while dividing by $-c^2$ makes a down index go up.

2 Joining quantities into 4-vectors

Many of the familiar quantities can now be expressed in this new notation. The current and charge density make a 4-vector with an up index

$$J^\mu = (J_x, J_y, J_z, \rho) \tag{11}$$

On the right hand side it does not matter whether we write the indices x, y, z up or down, since these just stand for the three components of the usual current, and when we were dealing with normal 3-component vectors there is no notion of up or down indices. The up and down difference affects only the fourth component.

The derivatives can be grouped as

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \tag{12}$$

Our convention is that the basic coordinates (x, y, z, t) have an up index. In taking a derivative, we see that these coordinates are in the denominator, so the 4-vector of derivatives is a quantity with indices down.

Now we recall the continuity equation:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{13}$$

In full this is

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{14}$$

We see that this is a natural dot product of the derivative operation with the current vector

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} J^\mu = 0 \tag{15}$$

Note that as expected, the J^μ is a quantity with an up index, and the derivatives $\frac{\partial}{\partial x^\mu}$ are quantities with a lower index. so we have a natural dot product. (All dot products should involve a summation sign, an up index, and a down index.) The goal of writing (15) is that it is a lot simpler than (13).

3 The gauge potential

We have one scalar potential V, and three components of a vector potential $\vec{A} = A_x, A_y, A_z$. It turns out that these can be grouped into a 4-vector, which has indices *down*. Its components are

$$A_\mu = (A_x, A_y, A_z, A_t) \tag{16}$$

where the fourth component is

$$A_t = -V \tag{17}$$

How do we know that there should be a negative sign here? We have to get the electric and magnetic fields out of these potentials. Recall the expressions

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{18}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{19}$$

Our goal is to get these electric and magnetic fields from the potentials in a natural way. But as it stands, the equation for $\vec{E}$ looks quite different from the equation for $\vec{B}$. We will now see that in 4-component notation, these equations are actually completely similar.

Note that

$$B_x = \partial_y A_z - \partial_z A_y \tag{20}$$

Thus B_x can be called the $y-z$ component of the curl. Similarly we have

$$B_y = \partial_z A_x - \partial_x A_z \tag{21}$$

$$B_z = \partial_x A_y - \partial_y A_x \tag{22}$$

Now that we have a fourth variable t, we should ask about similar things we can make with t. If we make the $t-x$ component of the curl we find

$$\partial_x A_t - \partial_t A_x = -\partial_x V - \partial_t A_x \tag{23}$$

But we see from (18) that this is just E_x. Thus we have

$$\partial_x A_t - \partial_t A_x = -\partial_x V - \partial_t A_x = E_x \tag{24}$$

$$\partial_y A_t - \partial_t A_y = -\partial_y V - \partial_t A_y = E_y \tag{25}$$

$$\partial_z A_t - \partial_t A_z = -\partial_z V - \partial_t A_z = E_z \tag{26}$$

Thus we can now define a new quantity with *two* indices, both of which are written as 'down' indices. This quantity is called $F_{\mu\nu}$, and is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{27}$$

We then find

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{28}$$

4 Maxwell's equations

In this section we will look at Maxwell's equations and see that they take a nice form in the 4-vector language.

4.1 The Gauss law equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Let us look at the first equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{29}$$

In full, this is

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z = \frac{\rho}{\epsilon_0} \tag{30}$$

In terms of the $F_{\mu\nu}$ quantity that we have defined, this is

$$\frac{\partial}{\partial x}F_{xt} + \frac{\partial}{\partial x}F_{yt} + \frac{\partial}{\partial x}F_{zt} = \frac{\rho}{\epsilon_0} \tag{31}$$

Let us make two more changes. First, since $F_{tt} = 0$, we can add a term involving F_{tt} to the LHS to make it look like a sum over all of x, y, z, t

$$\frac{\partial}{\partial x}F_{xt} + \frac{\partial}{\partial x}F_{yt} + \frac{\partial}{\partial x}F_{zt} - \frac{1}{c^2}\frac{\partial}{\partial t}F_{tt} = \frac{\rho}{\epsilon_0} \tag{32}$$

Let us see why we added this last term in this particular way. Note that on the LHS we now have a sum of terms involving in turn x, y, z, t. We would like to write this sum as a dot product in the usual way. We see that the dot product involves the derivative operator index and the first index of F. Recall that a dot product must involve one up and one down index. Recall also that the derivative operator has a down index. But both indices of F are also down. So we must write the equation a little differently, so that F appears in the equation with the first index as an 'up' index. In the three terms on the LHS of (31) the first index is 1,2 and 3 respectively, so we can think of this index as 'up' if we want. The fourth term which we added is zero anyway, but as written now it is

$$-\frac{1}{c^2}\frac{\partial}{\partial t}F_{tt} = \frac{\partial}{\partial t}F^t{}_t \tag{33}$$

Thus the equation becomes

$$\sum_{\mu=1}^{4}\frac{\partial}{\partial x^\mu}F^\mu{}_t = \frac{\rho}{\epsilon_0} \tag{34}$$

The next thing we do is write the RHS as

$$\frac{\rho}{\epsilon_0} = \mu_0 c^2 \rho \tag{35}$$

Recall from (11) that $\rho = J^t$, the fourth component of the current J^μ. In (34) on the LHS we have an index t that has not been summed over, but that index t is *down*. Thus on the RHS we should also write something with a 'down' index. We note that from our rule for indices

$$-c^2 J^t = J_t \tag{36}$$

Thus we can write the RHS as $-\mu_0 J_t$, and the full equation becomes

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_t = -\mu_0 J_t \tag{37}$$

4.2 The equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Given the equation (37), it is natural to expect the equations

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_x = -\mu_0 J_x \tag{38}$$

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_y = -\mu_0 J_y \tag{39}$$

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_z = -\mu_0 J_z \tag{40}$$

What are these equations? Let us look at the first one, eq. (38). This is

$$\partial_x F^x{}_x + \partial_y F^y{}_x + \partial_z F^z{}_x + \partial_t F^t{}_x = -\mu_0 J_x \tag{41}$$

To make sense of these terms, let us write them in terms of F with both indices down, which is the way we had defined F. For the first term on the LHS, the up index is x, and as we have seen before, there is no change if we write it as a down index. Similarly for the next two terms. Thus we have

$$\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} + \partial_t F^t{}_x = -\mu_0 J_x \tag{42}$$

Now note that $F^t{}_x = -\frac{1}{c^2} F_{tx}$. Thus we have

$$\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} - \frac{1}{c^2}\partial_t F_{tx} = -\mu_0 J_x \tag{43}$$

Now note that $F_{xx} = 0$. Next, note that

$$\partial_y F_{yx} + \partial_z F_{zx} = -\partial_y B_z + \partial_z B_y = -(\vec{\nabla} \times \vec{B})_x \tag{44}$$

So our equation is

$$-(\vec{\nabla} \times \vec{B})_x - \frac{1}{c^2}\partial_t F_{tx} = -\mu_0 J_x \tag{45}$$

Finally, note that $F_{tx} = -E_x$, and the equation is

$$-(\vec{\nabla} \times \vec{B})_x + \frac{1}{c^2}\partial_t E_x = -\mu_0 J_x \tag{46}$$

which is just one component of the last Maxwell equation:

$$(\vec{\nabla} \times \vec{B})_x = \mu_0 J_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{47}$$

Similarly, (39),(40) give the other components of the last Maxwell equation, i.e. we get all three components of

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{48}$$

4.3 The equation $\vec{\nabla} \cdot \vec{B} = 0$

This equation is

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \tag{49}$$

In terms of F, this is

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = 0 \tag{50}$$

Note that $F_{zx} = -F_{xz}$, and we have written the terms so that each comes with a positive sign. In each term the indices x, y, z appear. How does the second term differ from the first? The indices x, y, z have been permuted. If we interchange xy, then the order xyz goes to yxz. If we further permute xz, we get yzx. This is the order in the second term, so it comes from *two* simple permutations put together. We call this an *even* permutation since an even number, 2, of simple permutations were needed. By contrast, if we wanted to just go to the order yxz then one permutation would be needed, and we call yxz an *odd* permutation of the starting set xyz. Note that there are 6 permutations of xyz in all. Note that

$$\partial_x F_{yz} = -\partial_x F_{zy} \tag{51}$$

etc. Thus we can write (50) as

$$-\partial_x F_{zy} - \partial_y F_{xz} - \partial_z F_{yx} = 0 \tag{52}$$

or, adding (50) and (52), we can write the same equation as

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} - \partial_x F_{zy} - \partial_y F_{xz} - \partial_z F_{yx} = 0 \tag{53}$$

Now we have a simple structure. We have taken three indices , y, z. We have written the terms with all 6 possible permutations. The even permutations come with a positive sign, and the odd terms come with a negative sign. The equation just says that the sum of all these terms is zero.

All this may look like a lot of work for rewriting the simple equation $\vec{\nabla} \cdot \vec{B} = 0$. What is the advantage? We will now see that the next Maxwell equation, which looks quite different on the face of it, really has the same structure!

4.4 The equation $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

In the above equation we permuted xyz. But we have 4 indices now to choose from since t is also there, so we can choose some other set of three indices. Let us choose x, y, t. Then the analog of (50) is

$$\partial_x F_{yt} + \partial_y F_{tx} + \partial_t F_{xy} = 0 \tag{54}$$

This is

$$\partial_x E_y - \partial_y E_x + \partial_t B_z = 0 \tag{55}$$

which is

$$(\vec{\nabla} \times \vec{E})_z = -\frac{\partial}{\partial t} B_z \tag{56}$$

This is just the z component of the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \tag{57}$$

We get the other components by taking other sets of indices xzt or yzt.

4.5 Summary of Maxwell equations

We see that the two Maxwell's equations with 'source' can be unified into one form:

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_\nu = -\mu_0 J_\nu \tag{58}$$

where we can set the index ν to 1,2,3 or 4 to get different equations.

We also see that the two 'source free' equations are unified into just one type of equation, which we can write as

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{59}$$

Here we can choose μ, ν, λ as any three different indices from the set x, y, z, t. Different choices give different equations. (Question: What happens if we choose two of the indices to be the same; e.g. $\mu = x,\ \nu = x,\ \lambda = y$?)

5 Lorentz transformations

First consider motion in only one dimension x. Suppose one observer uses coordinates x, t. Suppose another observer is moving in the positive x direction with constant velocity v. The coordinates for the moving observer will be denoted x', t'.

In Newtonian mechanics, time is unchanged, so

$$t' = t \tag{60}$$

while

$$x' = x - vt \tag{61}$$

Thus we can write

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{62}$$

In relativistic mechanics we have a more symmetric transformation

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{63}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Note that for $v \ll c$, the relativistic case reduce to the Newtonian case. With motion in the x direction, the y, z coordinates are not affected. Thus we can write the full Lorentz transformation

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{64}$$

5.1 Tensors

We are now ready to learn all about tensors. Recall that we had written the position coordinates (and their differences) with an 'up' index. Thus we have

$$x^\mu = (x, y, z, t) \tag{65}$$

In the new frame we have

$$x'^\mu = (x', y', z', t') \tag{66}$$

We have the Lorentz transformation matrix as a 4×4 matrix. We call this matrix Λ. Any matrix has two indices, with the first denoting row number

and the second column number. In our present notation, we will write the first index 'up' and the second 'down'. Thus we write

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \tag{67}$$

We can now write (64) as

$$x'^{\mu} = \sum_{\nu=1}^{4} \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{68}$$

Note that the index which is summed over appears once as an up index and once as a down index, just like in any dot product. The other index, μ, is up on the left and so up on the right, as it should be.

What is all this notation good for? The key point comes now:

Any vector with an up index will change the same way when we go to a moving frame as any other vector with an up index.

Thus take the 4-vector for the current J^{μ}. If we go to a moving frame, the new components will be

$$J'^{\mu} = \sum_{\nu=1}^{4} \Lambda^{\mu}{}_{\nu} J^{\nu} \tag{69}$$

Thus suppose in our initial frame we have only a charge density ρ, and no current density. Thus we have

$$J^{\mu} = (0, 0, 0, \rho) \tag{70}$$

Now we go to a frame moving to the right with a velocity v. What will we see? First, the charge will appear to be moving in the negative x direction, so we should see a negative current J_x. But we should also see a length contraction, which will make the charge distribution appear 'compressed'; thus the charge density will appear *higher* than in the original frame.

How can we get all these effects quantitatively? All we have to do is use (69). This will give

$$J'^{1} = -\gamma v \rho, \quad J'^{2} = 0, \quad J'^{3} = 0, \quad J'^{4} = \gamma \rho \tag{71}$$

Thus we see that the charge density in the moving frame is indeed higher

$$J'^{4} = \rho' = \gamma \rho = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rho \tag{72}$$

while we have also found a current in the negative x direction

$$J'^1 = -\gamma v \rho = -v\rho' \tag{73}$$

Thus we see that the current in the moving frame is indeed the charge density in the moving frame times the velocity of that charge in the moving frame, so our old equation $\vec{J} = \rho\vec{v}$ is still true.

5.2 Transformation of fields

So much for the transformations of vectors with an 'up' index. What about vectors with a 'down' index? When we go to a moving frame, the components change again, but the change is slightly different. To illustrate the rule, take the vector A_μ given in (16). Then in the moving frame we will have components given by A'_μ, with

$$A'_\mu = \sum_{\nu=1}^{4} \tilde{\Lambda}_\mu{}^\nu A_\nu \tag{74}$$

Where

$$\tilde{\Lambda}_\mu{}^\nu = \begin{pmatrix} \gamma & 0 & 0 & \frac{\gamma v}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \tag{75}$$

One thing which can be checked is that if we have a dot product

$$\sum_{\mu=1}^{4} W_\mu V^\mu \tag{76}$$

then it does not change if we change V^μ by its rule for vectors with an up index and change W_μ by its rule for vectors with a down index:

$$\sum_{\mu=1}^{4} W'_\mu V'^\mu = \sum_{\mu=1}^{4} W_\mu V^\mu \tag{77}$$

Checking this takes a little work, but is worth doing.

Now we can say what happens to $F_{\mu\nu}$, which has all the electric and magnetic fields in it. There are two 'down' indices. First look at one index, and ignore the other one. Thinking of this as a single down vector index,

write the change expected in the moving frame. Now do the same for the other index. The result is

$$F'_{\mu\lambda} = \sum_{\nu=1}^{4} \sum_{\kappa=1}^{4} \tilde{\Lambda}_{\mu}{}^{\nu} \tilde{\Lambda}_{\lambda}{}^{\kappa} F_{\nu\kappa} \tag{78}$$

Let us now apply this to various cases.