SATHYABAMA
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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – I - Calculus of functions of more than one variable – SPH1215

Unit 1

Calculus of functions of more than one variable

Find the solution of $\underbrace{(xe^{xy}+2y)dy}_{N} + \underbrace{ye^{xy}dx}_{M}$

The above eqn is in the form of $M dx + N dy$

$M = ye^{xy}$ $N = xe^{xy} + 2y$

To check whether the given eqn is exact (or) in exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = ye^{xy}$$
$$= ye^{xy}x + e^{xy}$$
$$= xye^{xy} + e^{xy} \quad (1)$$

$$\frac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$$
$$= xye^{xy} + e^{xy} \quad (2)$$

$(1) = (2)$

$$\int_{y\,\text{constant}} M\,dx + \int_{\text{free from } x} N\,dy = 0$$

$$\int \cancel{(xe^{xy}+2y)\,dx} + \int y\,\cancel{dx}\,e^{xy}dx$$

$$\int_{y\,\text{constant}} ye^{xy}dx + \frac{y^2}{2} \int_{\text{free } x} xe^{xy}dy$$

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$$\cancel{y\frac{e^{xy}}{y}} + x\frac{e^{xy}}{}$$

$$\int\limits_{y\text{ constant}} y e^{xy} dx + \int\limits_{x\text{ }} \cancel{x e^{xy} dy} = \cancel{c}$$

$\cancel{y\frac{e^{xy}}{y} + \frac{e^{xy}}{} + \frac{x e^{xy}}{x}}$

$$e^{xy} + y^2 = c$$

$(x^4 - a x^2y^2 + y^4)dx + (bx^3y - bxy^3)dy = 0$ is exact then find the value of a and b.

if the given eqn is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

then

$$\int\limits_{y\text{ constant}} M dx + \int\limits_{\text{free from } x} N dy = c$$

$$= \frac{2y^2 - y^4}{4}$$

$$\int (x^4 - ax^2y^2 + y^4)dx + \int (by - by^3)dy = c$$

$$\frac{x^5}{5} - ay^2\frac{x^3}{3} + y^4x + b\left[\frac{y^2}{2} - \frac{y^4}{4}\right] = 0$$

$$x = \left[\frac{x^4}{5} - \frac{ay^2x^2}{3} + y^4\right] + \cancel{\frac{by^2}{2} - by^4}$$

$$\frac{2y^2b - by^4}{4}$$

$$M = x^4 - ax^2y^2 + y^4 \quad N = bx^3y - bxy^3$$

$$\frac{\partial M}{\partial y} = 0 - 2ax^2y + 4y^3 \quad \frac{\partial N}{\partial x} = 3bx^2y - 3by^3$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$-2ax^2y + 4y^3 = 3bx^2y - 3bxy^3$$

$$\cancel{(2ax^2y + 3b}$$

$$-x^2y(+2a+3b) + y^3(4+b) = 0$$

$$2a+3b=0 \qquad 4+b=0$$

$$4-b=0 \qquad \boxed{b=-4}$$

$$2a = 3b \qquad 2a+3\times-4=0$$

$$2a = 12$$

$$\boxed{a=6}$$

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Case) Finding the integrating factor

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$(x^3+y^3)dx = xy^2dy$$

$$(x^3+y^3)dx - xy^2dy = 0$$

To check exact (or) Non exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$M = x^3+y^3$ $N = -xy^2$

$$\frac{\partial M}{\partial y} = 0+3y^2 \quad \frac{\partial N}{\partial x} = -y^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad \frac{\partial N}{\partial x} = -y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ (Non Exact)}$$

$x = kx$ $y = ky$

$$\text{IF} = \frac{1}{Mx+Ny}$$

$$= \frac{1}{(x^4+xy^3)+(-xy^3)}$$

$$= \frac{1}{x^4+xy^3-xy^3}$$

$$\boxed{\text{IF} = \frac{1}{x^4}}$$

If IF is multiplied with both the sides of the eqn

$$M\,dx + N\,dy = c$$

$$\int \frac{x^3+y^3}{x^4} + \int \frac{-xy^2dy}{x^4} = c$$

y const

$\to 0$ (neglect)

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Case 2

Finding the integrating factor

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$(x^3+y^3)dx = xy^2dy$$

$$(x^3+y^3)dx - xy^2dy = 0$$

To check exact (or) Non exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$M = x^3+y^3 \quad N = -xy^2$$

$$\frac{\partial M}{\partial y} = 0+3y^2 \quad \frac{\partial N}{\partial x} = -y^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad \frac{\partial N}{\partial x} = -y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ (Non Exact)}$$

$$x = kx \quad y = ky$$

$$\text{IF} = \frac{1}{Mx+Ny}$$

$$= \frac{1}{(x^4+xy^3)+(-xy^3)}$$

$$= \frac{1}{x^4+xy^3-xy^3}$$

$$\boxed{\text{IF} = \frac{1}{x^4}}$$

If IF is multiplied with both the sides of the eqn

$$M\,dx + N\,dy = c$$

$$\int \frac{x^3+y^3}{x^4} + \int \frac{-xy^2dy}{x^4} = c$$

y const

$\to 0$ (neglect)

Case II

$$(x^3y^2+x)dy+(x^2y^3-y)dx=0$$

$$M=x^2y^3-y \qquad N=x^3y^2+x$$

$$\frac{\partial M}{\partial y}=3x^2y^2-1 \qquad \frac{\partial N}{\partial x}=3x^2y^2+1$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

$$x(x^2y^2+x)\,dy+y(x^2y^2-1)dx=0$$

$$\boxed{x\,g(xy)\,dy+y\,f(xy)\,dx=0}$$

$$IF=\frac{1}{Mx-Ny}$$

$$Mx-Ny\neq 0$$

$$=\frac{1}{x(x^2y^3-y)-(x^3y^2+x)y}$$

$$=\frac{1}{(x^3y^3-yx)-(x^3y^3+xy)}$$

$$=\frac{1}{x^3y^3-yx-x^3y^3+xy}$$

$= \frac{1}{-2xy}$

Multiply the above eqn with M and N

$M = \frac{x^2y^3 - y}{-2xy}$

$= -\frac{x^2y^3}{2xy} + \frac{y}{2xy}$

$= -\frac{xy^2}{2} + \frac{1}{2x}$

$N = \frac{x^3y^2 + x}{-2xy}$

$= \frac{-x^3y^2}{2xy} - \frac{x}{2xy}$

$= -\frac{x^2y}{2} - \frac{1}{2y}$

$$\underset{y\ \text{constant}}{\int} M\,dx + \underset{\text{free from } x}{\int} N\,dy = c$$

$$= \underset{y\ \text{constant}}{\int} \left(-\frac{xy^2}{2} + \frac{1}{2x}\right) dx + \underset{\text{free } (x)}{\int} \left(-\frac{x^2y}{2} - \frac{1}{2y}\right) dy = c$$

$$\Rightarrow -\frac{1}{2} \int \left(xy^2 - \frac{1}{x}\right) dx - \frac{1}{2} \underset{\text{free}(x)}{\int} \left(\frac{1}{y}\right) dy = c$$

$$\Rightarrow -\frac{1}{2}\left[\left(\frac{x^2y^2}{2} - \log x\right) + \log y\right] = c$$

Inexact equation with case 3

$$y(1+xy^2)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$$

$$M = y(1+xy^2) \qquad N = 2(x^2y^2+x+y^4)$$

$$\cancel{\frac{\partial M}{\partial y} = (1+2xy)}$$

$$M = y + xy^3$$

$$\frac{\partial M}{\partial y} = 1+3xy^2 \qquad \frac{\partial N}{\partial x} = 2(2xy^2+1) = 4xy^2+2$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2+2-1-3xy^2}{y+xy^3} = \frac{4xy^2+1-3xy^2}{y+xy^3}$$

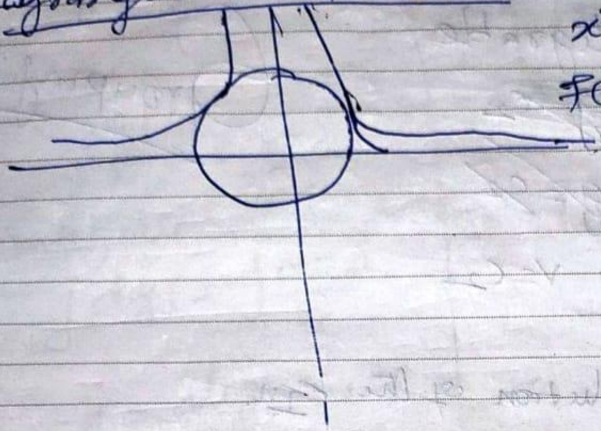
$$= \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y}$$

$$= \cancel{\frac{y(1+xy^2)}{y}\,dx + \frac{2(x^2y^2+x+y^4)}{y}\,dy = 0}$$

$$= (1+xy^2)\,dx$$

$$IF = e^{\int f(y)dy} = e^{\int(\frac{1}{y})dy} = e^{\log y} = y$$

Lagrange's Multipliers



Lagrange's Linear equation.

PDE of the form $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$

where P, Q, R are x, y, z terms

$Pp + Qq = R$

Solutions

For a give PDE $Pp+Qq = R$

① Form Auxilary Equation $\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

② Consider the pair $\frac{dx}{P} = \frac{dy}{Q}$ (or) $\frac{dy}{Q} = \frac{dz}{R}$ (or) $\frac{dz}{R} = \frac{dx}{P}$

Check

(3) The above ~~integra~~ eqns are Straightly integrable

If it is integrable

~~$\phi = c_1$ / $\psi = c_2$~~

~~$f = c_1$~~ ~~$g = c_2$~~

$u = c_1 \quad v = c_2$

Grouping

(4) Complete solution of the eqn.

$f(u, v) = 0$ (or) $\phi(u, v) = 0$

(1) Solve $p + q = \frac{z}{a}$

Part I

$P = 1 \quad Q = 1 \quad R = \frac{z}{a}$

$$A.E = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

Consider $\frac{dx}{1} = \frac{dy}{1}$

Integrate

$$\int dx = \int dy$$

$$x = y + c$$

$x - y = c$ — ①

<u>part II</u>

$$\frac{dy}{1} = \frac{dz}{z/a}$$

Integrate

$$\int dy = \int \frac{dz\, a}{z}$$

$$y = a\log z + c_2$$

$$y - a\log z = c_2$$

Complete solution

$$\phi(u, v) = 0$$

$$\phi(x-y,\ y - a\log z) = 0$$

(OR)

$\cancel{\phi(x-y) =}$

$$(x-y) = \phi(y - a\log z)$$

$$\phi(x, y, z) = 0$$

$$\boxed{x + y + z = c}$$

$$\boxed{x + y + z = 0}$$

Solve $(z-y)p + (x-z)q = y-x$.

$P = z-y \quad Q = x-z \quad R = y-x$

Auxilary Eqn

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \quad —①$$

Then use lagrange Multiplier,

Multiply 1 on numerator and denominator of eqn ①

and add

$$\Rightarrow \frac{dx+dy+dz}{\cancel{z}-\cancel{y}+\cancel{x}-\cancel{z}+\cancel{y}-\cancel{x}} = 0$$

$\Downarrow$ 0

Integrate.

$$\Rightarrow \int dx + \int dy + \int dz = 0$$

$$\boxed{x+y+z = c_1}$$

From eqn ①

$$\Rightarrow \frac{xdx}{x(z-y)} = \frac{ydy}{y(x-z)} = \frac{zdz}{z(y-x)}$$

$$\Rightarrow \frac{xdx}{xz-xy} = \frac{ydy}{yx-yz} = \frac{zdz}{zy-zx}$$

① Add

$$\Rightarrow \frac{xdx+ydy+zdz}{\cancel{xz}-\cancel{xy}+\cancel{yx}-\cancel{yz}+\cancel{zy}-\cancel{zx}} = 0$$

$xdx+ydy+zdz = 0$

Integrate

$$\Rightarrow \int xdx + \int ydy + \int zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2+y^2+z^2 = 2c_1 = c_2$$

Complete solution is

$$\phi(u,v) = 0$$

$$\phi(x+y+z\,;\, x^2+y^2+z^2) = 0$$

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$$\boxed{Pp + Qq = R}$$

Grouping Multipliers.

A.E $\underbrace{\frac{dx}{P} = \frac{dy}{Q}}, \underbrace{\frac{dy}{Q} = \frac{dz}{R}}$

Method of Multiplier

Denominator to be zero.

Solve $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

$$\frac{x \times dx}{x \times x(y^2-z^2)} = \frac{y \times dy}{y \times y(z^2-x^2)} = \frac{z \times dz}{z \times z(x^2-y^2)}$$

$$\frac{x\,dx}{x^2y^2-x^2z^2} = \frac{y\,dy}{y^2z^2-y^2x^2} = \frac{z\,dz}{z^2x^2-z^2y^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2}} = 0$$

$$\boxed{x\,dx + y\,dy + z\,dz = 0}$$

Integrating

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = 2c_0$$

$$\boxed{x^2 + y^2 + z^2 = c_1}$$

II case

$\frac{1}{x}$ is the factor

$$\frac{\frac{1}{x}dx}{\frac{1}{x}x(y^2-z^2)} = \frac{\frac{1}{y}dy}{\frac{1}{y}y(z^2-x^2)} = \frac{\frac{1}{z}dz}{\frac{1}{z}\cdot z(x^2-y^2)}$$

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\cancel{y^2} - \cancel{z^2} + \cancel{z^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2} \Rightarrow 0}$$

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = c_2$$

$\cancel{y^2}p - xyq = x(z - \cancel{2}y)$

$Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\int \frac{dx}{y^2} = \int \frac{dy}{-xy} = \int \frac{dz}{x(z-2y)}$$

$\int$

$\cancel{xy}$ xz $\quad$ xy

$y^2 \quad -xy \quad xz - 2xy$

$x\ 2x$

$y^2 \quad - xyz \quad xyz - 2xy^2$

$2x\cancel{y^2} - \cancel{x}yz + x\cancel{y}z - 2x\cancel{y^2} = 0$

$$\int \frac{2x\,dx}{2xy^2} = \frac{z\,dy}{-xyz} = \frac{y\,dz}{xyz - 2xy^2}$$

$$\frac{2x\,dx + z\,dy + y\,dz}{2xy^2 - xyz + xyz - 2xy^2}$$

$\Rightarrow 0$

$$\int 2x\,dx + \int z\,dy + \int y\,dz = 0$$

$uv = u\,dv + v\,du$

$uv.$

$$\cancel{\int} \frac{\cancel{2}x^2}{\cancel{2}} + zy + yz = 0$$

$\cancel{x^2}$

$$\boxed{y^2 - xy \quad az - 2xy}$$

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(C) $\frac{\cancel{dx}}{\cancel{x}}$ $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$$\boxed{Pp + Qq = R}$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}}{(y-z) \quad (z-x) \quad z-x} =$$

Case 3 $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only (or) constant (or) function of x alone

$$IF = e^{\int f(x)}$$

Case 4 $\dfrac{\dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial x}}{M} = g(y)$ only (or) Constant (or) function of only y.

$$IF = e^{\int f(y)\,dy}$$

$$= \cancel{\frac{(y + xy^3)}{y}\,dx + 2\frac{(x^2y^2 + x + y^4)}{y}\,dy = 0}$$

$$= \cancel{(1 + xy^2)\,dx + 2\frac{(x^2y^2 + x + y^4)}{y}\,dy}$$

$$= (y + xy^3)\,y\,dx + 2y(x^2y^2 + x + y^4)\,dy = 0$$

$$= \int (y^2 + xy^4)\,dx + \int (2x^2y^3 + 2yx + 2y^5)\,dy = c$$

free from x: 0, 0

$$\Rightarrow xy^2 + \frac{x^2y^4}{2} + \cancel{\frac{x^2y^4}{2}}\ \frac{2y^6}{6\,3} = 0$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – II - Dirac Delta function and its properties – SPH1215

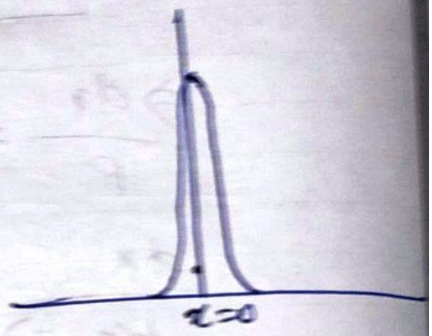
Unit 2

Dirac Delta function and its properties

1 Dimensional Dirac Delta function

$$\delta(x) = \infty \quad x = 0$$
$$= 0 \quad x \neq 0$$

$$\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1}$$



$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx$$

$$\left[\int \delta(x)\,dx = 1\right]$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

Shifting Spike to some other point

$$\delta(x-a) = \infty \text{ at } x = a$$
$$= 0 \text{ at } x \neq a$$

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

$\int_{-\alpha}^{\alpha} \delta(x)\, dx = 1$

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$= f(a) \int_{-\alpha}^{\alpha} \delta(x-a)\, dx$

$\int_{-\alpha}^{\alpha} \delta(a-a)\, dx = 1$

$= f(a)$

$\Rightarrow \int_{0}^{3} x^3 \delta(x-2)\, dx = f(2) = 2^3 = 8$

$=$ Here $f(x) = x^3$ x is constant

Then $f(x) =$ is also constant

$\Rightarrow f(2) \int_{0}^{3} \delta(x-2)\, dx = 8 \int_{0}^{3} 1$

$\left[\cancel{f(2) = 2^3} \quad f(x) = x^3 \quad f(2) = 2^3 = 8 \right]$.

$\int_{0}^{3} x^3 \delta(x-2)\, dx = 8$

$\nabla \cdot V = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta) V_r \right]$

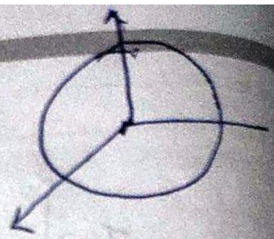
$V = \frac{\hat{r}}{r^2}$

$\boxed{V_r = \frac{1}{r^2}}$ — Magnitude

$\cancel{V_\theta = 0}$ $\quad V_\theta = 0$

$\cancel{V_\phi = 0}$ $\quad V_\phi = 0$

Spherical Coordinates



$$\nabla\cdot V = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta)\cdot\frac{1}{r^2}\right)$$

$$= \frac{\sin\theta}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right)\right]$$

$\sin\theta$ can be taken outside. Because it is not related r.

$$= \frac{\cancel{\sin\theta}}{\cancel{r}}\ \frac{1}{r^2}\frac{\partial}{\partial \theta}(1)$$

$$\nabla\cdot V = 0$$

$$\int_V \nabla\cdot V\, d\tau = \int \vec{V}\, d\vec{a}$$

$$V = \frac{\hat{r}}{r^2}\cdot r^2\sin\theta\, d\theta$$

$$\int \vec{V}\cdot d\vec{a} = 0$$

But

$$\int_{\partial S} \frac{\hat{r}}{r^2}\cdot r^2\sin\theta\, d\theta\, d\phi\ \hat{r}$$

$$\left[\hat{r}\cdot\hat{r} = 1\right]$$

$$\int_S \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi.$$

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$$= \int_0^{\pi} \sin\theta \, d\theta \left[\theta\right]_0^{2\pi}$$

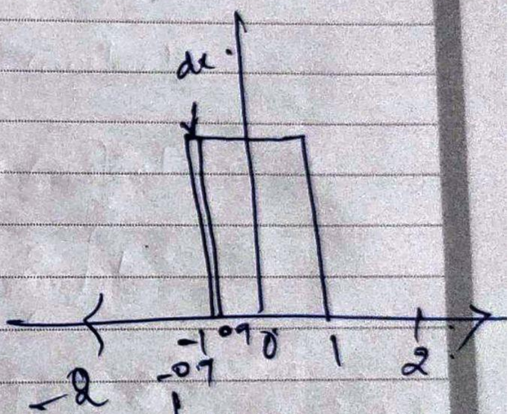
$$= \left[-\cos\theta\right]_0^{\pi} \left[\theta\right]_0^{2\pi}$$

$$= -\left[(-1) - 1\right]\left[2\pi\right]$$

$$= (+1+1)(2\pi)$$

$$= 2(2\pi)$$

$$= 4\pi$$



$$\int_{-1}^{-0.9} dx$$

$$\left[x\right]_{-1}^{-0.9}$$

$$-0.9+1 = 0.1$$

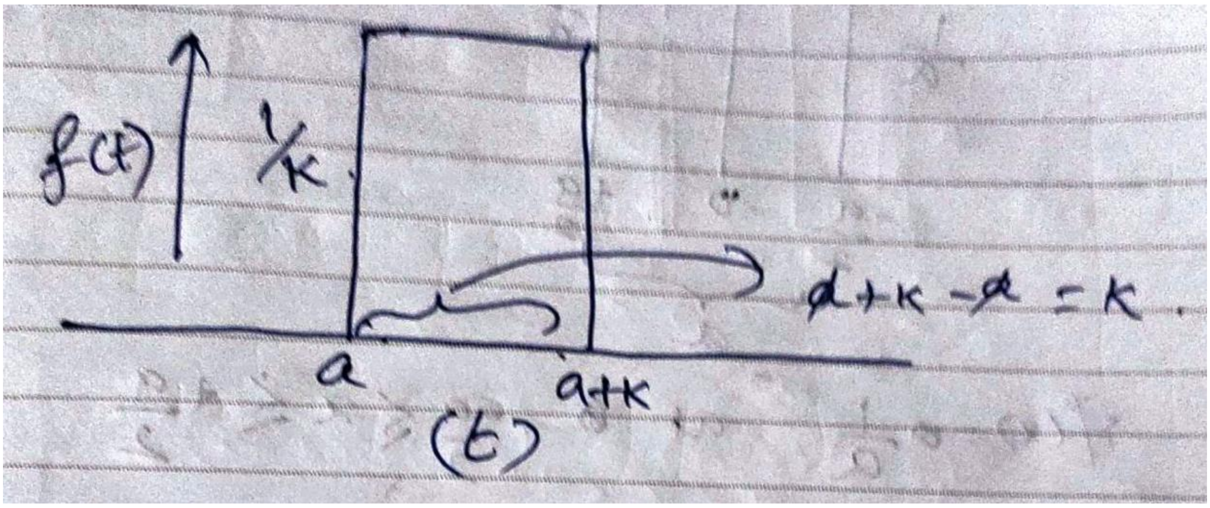
$$\int_{-1} 0.1$$

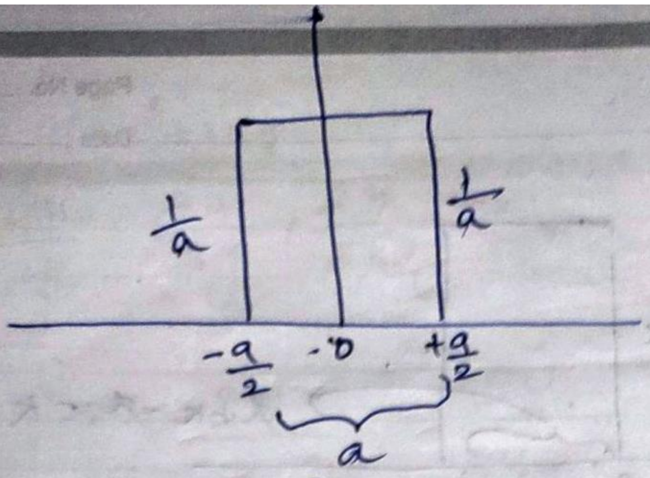
$u = x^3 + 3x^2y$

$\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial x \partial y}$

Prove $\frac{\partial u}{\partial x \partial y} = \frac{\partial u}{\partial y \partial x}$.

$u = x^3 + y^3 + 3x^2y + 3xy^2$





Breadthe = a

Length = $\frac{1}{a}$.

$f(x) = \frac{1}{a}$ at $-\frac{a}{2} \leq x \leq +\frac{a}{2}$

$= 0$ at $x < -\frac{a}{2}$

$= 0$ at $x > +\frac{a}{2}$

$\frac{2}{a}$

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$\frac{2}{a} - \frac{2}{a}$

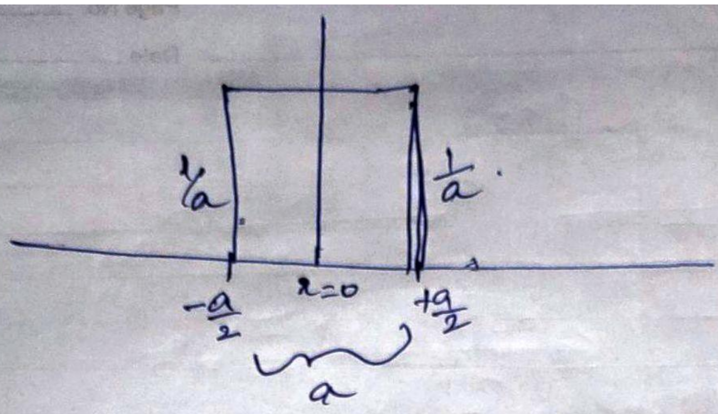
$\frac{2-2=0}{a}$

$\boxed{\begin{array}{ll} \text{If } x = +\frac{a}{2} & f(x) = \frac{1}{2a} \\ (\text{from } 0 \text{ to } +\frac{a}{2}) & \\ x = -\frac{a}{2} & f(x) = \frac{1}{2a}. \\ \text{No need.} & \end{array}}$

~~(from $\frac{a}{2}$ to 0)~~

$= \frac{1}{2a} + \frac{1}{2a}$

$\frac{1+1}{2a} = \frac{2}{2a}$



$$f(x) = \frac{1}{a} \quad -\frac{a}{2} \leq x \leq +\frac{a}{2}$$

$$= 0 \quad x < -\frac{a}{2}$$

$$= 0 \quad x > +\frac{a}{2}$$

If $a \to 0$ [a tends to zero: this is not of zero, becoming zero (reduced)]

Then

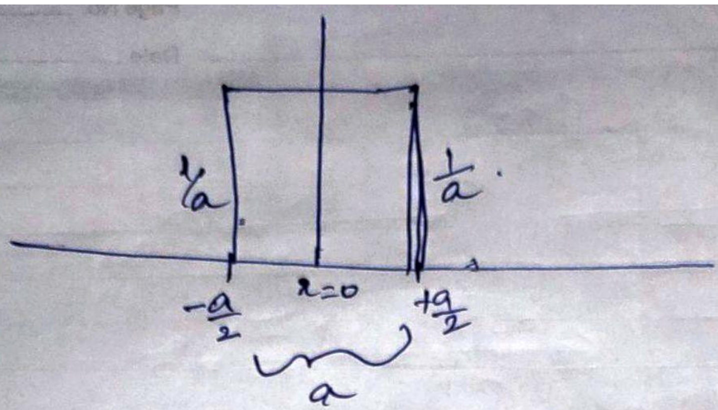
$x = 0$

$$f(x) = \frac{1}{a} = \frac{1}{0} = \infty.$$

This means the gap inbetween $-\frac{a}{2}$ and $+\frac{a}{2}$ becomes zero. that means reduced.

$$\lim_{a \to 0} f(x) \to \delta(x)$$

Here $-\frac{a}{2}$ and $+\frac{a}{2}$ moving towards the centre and combined together. Then becomes infinity.



$$f(x) = \frac{1}{a} \qquad -\frac{a}{2} \leq x \leq +\frac{a}{2}$$

$$= 0 \qquad x < -\frac{a}{2}$$

$$= 0 \qquad x > +\frac{a}{2}$$

If $a \to 0$ [a tends to zero: this is not of zero, becoming zero (reduced)]

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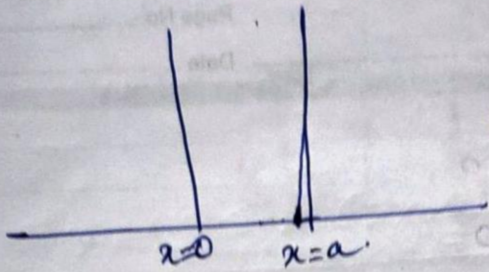
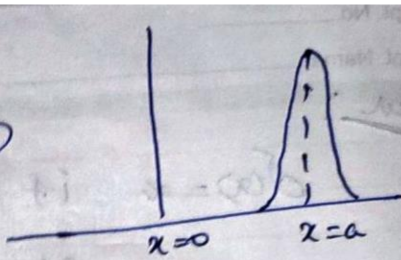


(or)



$\cancel{f(x)\delta(x)\,dx = f(a)}$

$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$. $\left[\because x-a=0,\ \boxed{x=a}\right]$

$\Rightarrow \int_{-\alpha}^{\alpha} f(x)\,\delta(x-a)\,dx$

$x-a=0 \quad x=a$

$\Rightarrow f(a)\int_{-\alpha}^{\alpha}\delta(x-a)\,dx \Rightarrow 1$

$\int_{\alpha}^{\alpha} f(x)\,\delta(x-a)\,dx = f(x)$

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x p} \, dp$$

x - position and p - momentum.

If momentum is applicable then change in position is possible.

Thats what we have integrated the above eqn w.r.to p.

Here x is dependent variable.

x depends on p.

The integration limits can be changed -L to L. because it can be taken as lengths.

$$f(x) = \int_{-L}^{L} e^{2\pi i x p} \, dp$$

$$= \left[\frac{e^{2\pi i x p}}{2\pi i x} \right]_{-L}^{L}$$

$$= \left[\frac{e^{2\pi i x L} - e^{-2\pi i x L}}{2\pi i x} \right]$$

$$= \frac{\sin(2\pi x L)}{\pi x}.$$

$$\left[\frac{e^{ix} - e^{-i}}{2 i x} \right.$$

here x =

Multiphy 2L in numerator as well as denominator

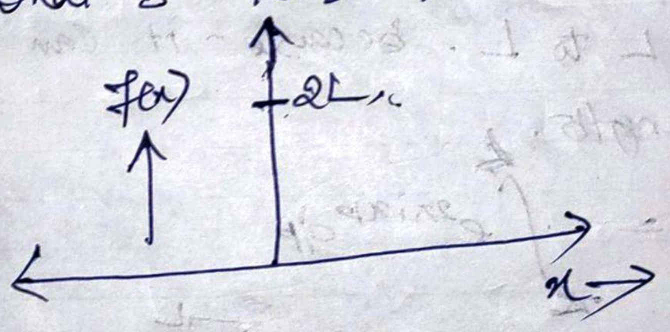
$$f(x) = \frac{\sin(2\pi x L)}{\pi x} \times \frac{2L}{2L}$$

$$= \frac{\sin(2\pi x L)}{2\pi x L} \times 2L \quad —(1)$$

$$\left[\because \frac{\sin x}{x} = 1 \right]$$

$$f(x) = 2L$$

What is f(x). Perpendicular to x



~~If~~ f(x) is 0x. Then what is x

Take θ from the above eqn (1)

$$\theta = 2\pi x L = n\pi \quad —(2)$$

~~Sol Lorr. to~~

~~Corresponding angle is 90°~~

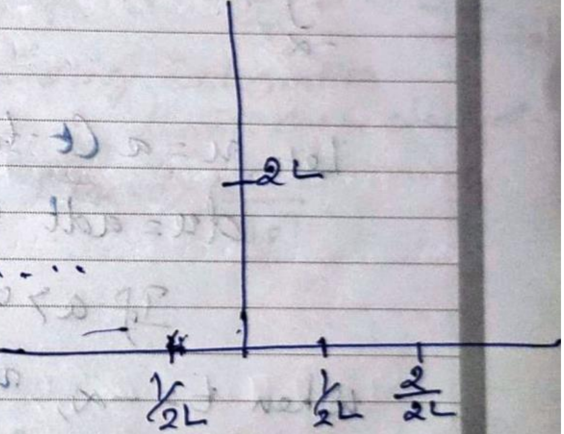
~~It is definitely maximum at 180~~

The difference between $\frac{1}{2L}$ interval 180°c

From (2)

$$2\pi x L = n\pi$$

$$x = \frac{n}{2L}$$



if $n = 1, 2, 3 \ldots$

$$x = \frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L} \ldots$$

Here $\frac{1}{2L} = 0.1$

$2L =$

0.5 0.4

$\frac{1}{0.1}$ $2\times\frac{1}{20}$

$\frac{1}{0.2}$ $\frac{1}{2} = 0.1$

$\frac{1}{0.}$

$$\int_{-\alpha}^{\alpha} \delta(t-t_0)\,dt = 1 \quad \underline{\text{Normalization.}}$$

$$\int_{-\alpha}^{\alpha} f(t)\,\delta(t-t_0)\,dt = f(t_0) \quad \left[\, t = t_0 \,\right]$$

$$\Downarrow$$

<u>localization</u>

$$\int_{-\alpha}^{\infty} f(t)\,\delta(a(t-t_0))\,d\tau = ?\ \cancel{f(at_0)}$$

Let $u = a(t-t_0)$ $\cancel{dt}$

$du = a\,dt$ (or) $dt = \frac{du}{a}$

$\left[\ \because \frac{u}{a} = t - t_0,\quad t = t_0 + \frac{u}{a}\ \right]$

If $a > 0$

When $t = -\alpha$, $u = -\alpha$ when $t = \infty$ $u = \alpha$

$$t = t_0 + \frac{u}{a}$$

$$\cancel{\int_{-\alpha}^{\alpha} f(t_0)} = \int_{-\alpha}^{\infty} f\left(t_0 + \frac{u}{a}\right) \delta(u)\,\frac{du}{a}$$

$$= \frac{1}{a}\int_{-\alpha}^{\alpha} f\left(t_0 + \frac{u}{a}\right) \delta(u-0)\,du \qquad \boxed{u=0}$$

$$= \frac{1}{a} f(t_0)$$

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If $a<0$ [when a is $-ve$)

$u = a(t-t_0) \Rightarrow du = a\,dt \Rightarrow$

$$\boxed{dt = \frac{du}{a}}$$

$$t = t_0 + \frac{u}{a}.$$

When $t = -\alpha$ $u + \alpha$ when $t = \alpha$ $u - \alpha$

$$\int_{-\alpha}^{\alpha} f(t)\,\delta(a(t-t_0))\,dt = \int_{\alpha}^{-\alpha} f\left(t_0 + \frac{u}{a}\right) \delta(u)\,\frac{du}{a = -|a|}$$

for cancelling - sing integration limits can be changed.

$$= \int_{-\alpha}^{\alpha} f\left(t_0 - \frac{u}{|a|}\right) \delta(u-0)\,\frac{du}{|a|}$$

$$\boxed{u=0}$$

$$= \int_{-\alpha}^{\alpha} f\left(t_0 - \frac{u}{a}\right)$$

If $a < 0$

$u = a(t - t_0)$ | $du = a\,dt$

$\frac{u}{a} = t - t_0$ | $dt = \frac{du}{a}$

$$\boxed{t = \frac{u}{a} + t_0}$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(a(t-t_0))\,dt = \int_{+\infty}^{-\infty} f\left(t_0 + \frac{u}{a}\right)\delta(u)\frac{du}{a}$$

If $a = -|a|$ $(a<0)$

$$= \int_{-\infty}^{\infty} f\left(t_0 + \frac{u}{|a|}\right)\delta(u)\frac{du}{|a|}$$

Simply u is given for δ then u is zero

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f(t_0)\,\delta$$

$$= \frac{1}{|a|} f(t_0) \int \delta(u)\,du$$

uncertainty principle

1.34.
3.

$$\Delta x \cdot \Delta p \geqslant \frac{h}{4\pi}$$

$$\Delta x \cdot \Delta p \geqslant \frac{h}{2\pi}\left[\frac{1}{2} \text{ Probability}\right]$$

Dirac Delta function

① $\delta(x) = 0, \; x \neq 0$

$\neq 0, \; x = 0$

② $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

Properties of Dirac Delta function

① $\delta(-x) = \delta(x)$ — even function

$x \rightarrow -x$

$$\int_{\alpha}^{-\alpha} \delta(-x)\, d(-x)$$ — will chang

Won't change. (It is too small)

$\delta(x) = \alpha \quad x = 0$

$= 0 \quad x \neq 0$

② $x\delta(x) = 0$

$x = 0 \quad \delta(x) \neq 0$

$x \neq 0 \quad \delta(x) = 0$

③ (iii) $F(x)\,\delta x = f(0)\,\delta x$

④ (iv) $$\int_{-\alpha}^{\alpha} f(x)\,\delta(x)\,dx = \int_{-\alpha}^{\alpha} f(0)\,\delta x\,dx$$

$$= f(0)\int_{-\alpha}^{\alpha} \delta(x)\,dx$$

$$= f(0)$$

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(5) $\delta(kx) = \frac{1}{k}\delta(x)$

$kx = y$

$x = \frac{y}{k}$

$dx = \frac{dy}{k}$

~~sub~~. above eqn

~~$\delta(y) =$~~

for ~~example~~

$$\int_{-\alpha}^{\alpha} f(x)\,\delta(kx)\,dx \quad (LHS)$$

$$\int_{-\alpha}^{\alpha} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} \quad y \to 0$$

$$\Rightarrow \frac{1}{k}\int_{-\alpha}^{\alpha} f(0)\,\delta y\,dy$$

$$\Rightarrow \frac{f(0)}{k}\int_{-\alpha}^{\alpha}\delta y\,dy \to 1$$

$$\Rightarrow \frac{f(0)}{k} \to \text{What is } f(0)$$

$$= \frac{1}{k}\int_{-\alpha}^{\alpha} f(x)\,\delta(x)\,dx \quad (RHS)$$

LHS = RHS

$$\boxed{\delta(kx) = \frac{1}{k}\delta(x)}$$

(6)

$\cancel{\delta(x)}$.

$\cancel{(vi)\ \delta(-x) = -\delta'(x)}$

$\cancel{\boxed{\therefore\ \delta(x) = \delta(x)}}$

$$\cancel{\frac{\partial\, \delta(-x)}{\partial(-x)} = -\frac{\partial(\delta x)}{\partial x}}$$

$$\cancel{\frac{\partial}{\partial(-x)}\ \ \delta'(x)}$$

$$\cancel{\Rightarrow \frac{\partial\, \partial(-x)}{\partial(-x} = \partial(-x) = \partial(x)}$$

$$\cancel{\Rightarrow \frac{\partial\, \partial(x)}{}}$$

(vi) $\delta'(-x) = -\delta'(x)$

$$\frac{\partial\,(\delta(-x))}{\partial(-x)} = -\frac{\partial\,(\delta(x))}{\partial x}$$

Can be written as $\partial(x)$

$$\frac{\partial\, \delta(-x)}{\partial x} = -\frac{\partial\, \delta x}{\partial x}$$

$$\delta'(-x) = -\delta'(x)$$

(vii) $x \frac{d}{dx} \delta x = -\delta x$

$$\cancel{\int_{-\infty}^{\infty} x \frac{d}{dx} (\delta x) dx}$$

$$\left[\int u\,dv = uv - \int v\,du \right.$$

$$\int_{-\infty}^{\infty} x \frac{d}{dx} \delta(x) dx = x\delta(x) - \int_{-\infty}^{\infty} \delta(x) dx .$$

According to Dirac Delta function 2nd properties

$$x\,\delta(x) = 0$$

$$\int_{-\infty}^{\infty} x \frac{d}{dx} \delta(x) dx = -\int_{-\infty}^{\infty} \delta(x) dx .$$

$$x \frac{d}{dx} \delta(x) = -\delta(x) .$$

(viii) Shifting properties

$\cancel{\delta(x) = 0}$

$$\delta(x) = 0 \quad x \neq 0$$
$$\neq 0 \quad x = 0$$

We can change the above as

Here x can be written as $x-0$

$$\delta(x-0) = 0 \quad x \neq 0$$
$$\delta(x-0) \neq 0 \quad x = 0$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)}$$ Standard equation

Then the above can can be written as.

$$\cancel{\int_{-\infty}^{\infty} f(x-0)\,\delta x}$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-0)\,dx.$$

$x=0$

$$= \int_{-\infty}^{\infty} f(0)\,\delta x\,dx$$

$$= f(0)\int_{-\infty}^{\infty} \delta x\,dx \rightarrow 1$$

$$= f(0)$$

lly

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx = f(x')$$

① $\int_0^3 x^2 \delta(x-2)\,dx = f(2) = 2^2 = 4$

$x = 2$

② $\int_0^2 3x^2 \delta(3x-1)\,dx$.

scaling property

$$\boxed{\delta(kx) = \frac{1}{k}\delta(x)}$$

$\frac{1}{|3|}\delta\left(x - \frac{1}{3}\right)$

$= \frac{1}{\cancel{|3|}}\int_0^2 \cancel{3}x^2 \delta(x - 1/3)\,dx$

$= \int_0^2 x^2 \delta\left(x - \frac{1}{3}\right)dx$

$= \int_0 \cdot \cancel{f\left(\frac{1}{3}\right)}$

$= \int f(\cancel{x}) = x^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

(5) $\int_0^3 (x^2+1)\,\delta(x-2)(x-1)\,dx$

$= \frac{1}{|x-1|} \{$

$(x-2)(x-1)$

$x^2-x-2x+2$

x^2-3x+2

$(x-2)(x-1)$

$x=2 \quad x=1$

(1) $\int_{-\alpha}^{\alpha} 2\,\delta(x)\,dx = 2$

(2) $\int (x^2-1)\,\delta(x)\,dx$

$x=0$

$\cancel{(x^2-1)\big|_{x=0}}$

$\int (-1)\,\delta(x)\,dx$

$= -1$

(3) $\int_{-\alpha}^{\alpha} e^x\,\delta(x)\,dx$

(4) $\int_{-\alpha}^{\alpha} \sin x\,\delta(x-2)\,dx = \int_{-\alpha}^{\alpha} \sin(2)\,\delta(x)\,dx.$

$= \sin(2)$

(5) $\displaystyle\int_{-\alpha}^{4} g(x)\,\delta(x-3)\,dx$

$x = 3$

$\displaystyle\int_{-\alpha}^{4} g(3)\,\delta(x)\,dx$

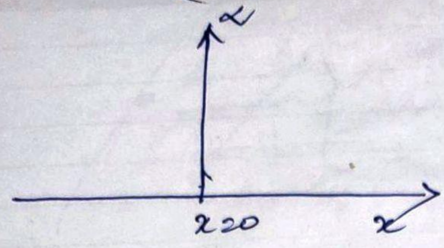
(6) $\displaystyle\int_{-\alpha}^{4} g(x)\,\delta(x-5)\,dx$

$\cancel{\displaystyle\int_{-\alpha}^{4} g(5)}$

0

$g(x)\displaystyle\int_{-\alpha}^{4} \delta(x-5)\,dx = 0$

~~Dis~~ Normal Distribution (or) Gaussian



$$f(x) = \begin{cases} \infty & \text{at } x=0 \\ 0 & \text{at } x \neq 0 \text{ (any where other than 0)} \end{cases}$$

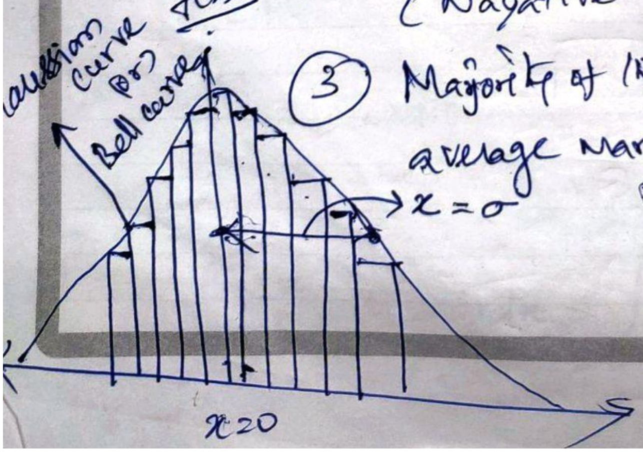
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Normal Distribution

This is the normal distributio curve to sort out our day to day problem.

For ~~example~~ Marks distribution for a class and for a subject.

1. Some students having more marks (positive curve)
2. Some students having less marks (Nagative curve)
3. Majority of the students having average marks



To find the distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

σ - Standard deviatio

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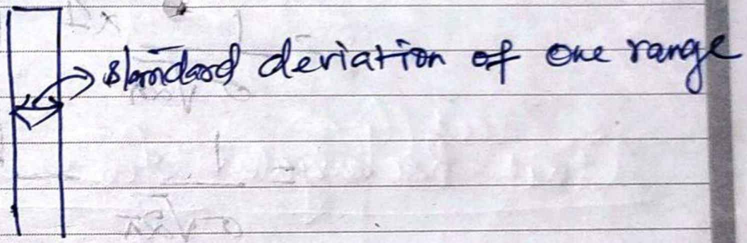
at $x=0$ the Majority of the Students got average Marks

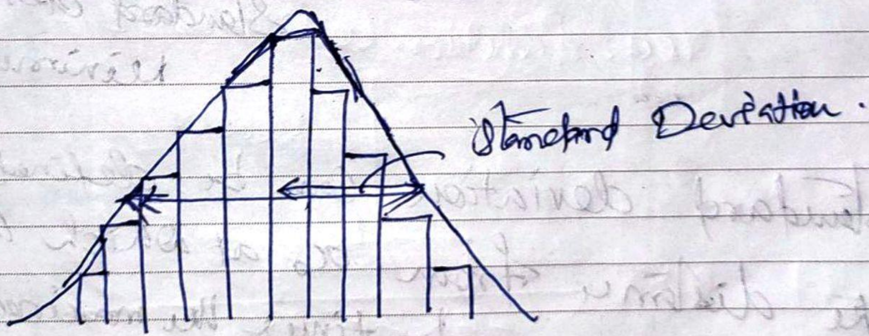
To find the height of the distribution.

The maximum at $x=x_0$ (More students got average Marks (or) pass marks)

Height: $f(x=x_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{0}{2\sigma}} = \frac{1}{\sigma\sqrt{2\pi}}$ Standard Deviation ~~maximum~~ minimum (or) at the centre

Standard deviation is ~~maximum~~ minimum at the centre





Standard Deviation: How the data points are moving towards the average.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

Height

Width.

$\boxed{\text{if } x = x_0}$.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{0}{2\sigma^2}\right)}$$

$$= \frac{1}{\sigma\sqrt{2\pi}\, e^0}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \times 1 \quad [\because e^0 = 1]$$

$$= \frac{1}{\sigma\sqrt{2\pi}}$$

Maximum Height (or) Standard deviation Minimum.

Standard deviation can be defined as the distance from x_0 at which the function has the value of $\frac{1}{\sqrt{e}}$ times the maximum height of the function.

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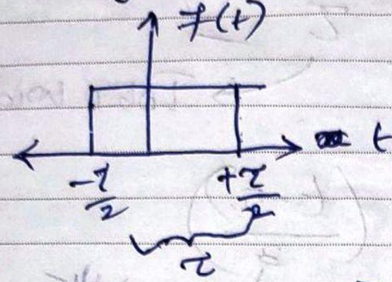
Rectangular Function



$f(t) = \text{rect}\left(\frac{t}{\tau}\right)$

$A\,\text{rect}\left(\frac{t}{\tau}\right) = A \quad -\frac{\tau}{2} < t > \frac{\tau}{2}$

$= \frac{A}{2} \quad |t| = \pm\frac{\tau}{2}$

$= 0$ otherwise

$$\boxed{A\,\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} A & |t| < \frac{\tau}{2} \\ A/2 & |t| = \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}}$$

Using this definition

$|t| = +t, \; t > 0$

$|t| = -t \quad t < 0$

$|t| < \frac{\tau}{2} \xrightarrow{t>0} t < \tau/2$

$\xrightarrow{t<0} -t < \frac{\tau}{2}$

$-\frac{\tau}{2} < t < \frac{\tau}{2}$

Unit Rectangle $\boxed{k \times \frac{1}{k} = \text{Rect}}$

Rect C

if you take Rect as a function we can take

~~$A\,\text{rect}\left(\frac{t}{\tau}\right)$~~ Height (or) Amplitude

$= \text{rect}\left(k \times \frac{t}{k}\right)$

If it is not unit

$= \text{rect}\left(\frac{\tau}{\tau}\right)$

$f(t) = A\,\text{rect}\left(\frac{t}{\tau}\right)$

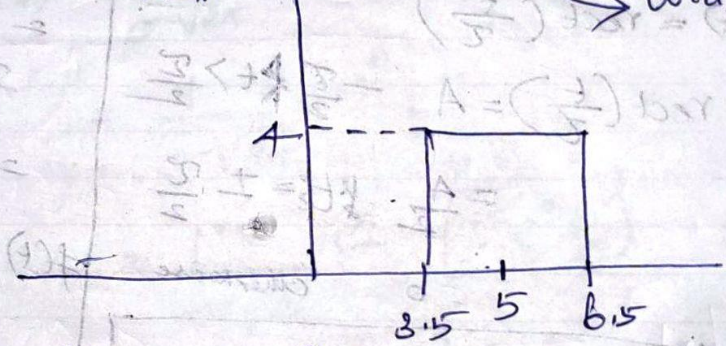
$\frac{2}{\tau}$

General Representation of rect. function.

$$f(t) = \underset{\text{Amplitude}}{A}\ \text{rect}\left[\frac{t - \text{Centre}}{\underset{\text{Total width}}{\tau}}\right]$$

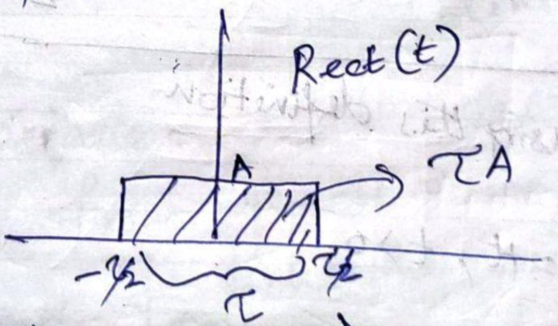
$$f(t) = \underset{\text{Amplitude}}{4}\ \text{rect}\left(\frac{t-5}{\underset{\text{width}}{3}}\right)$$

$\boxed{t = 5}$



Unit Rectangular function

rect. function with Area = 1.



$\tau = 1 \quad (K)$ | Area $= A\tau = 1$

$A = \frac{1}{\tau} \quad (= \frac{1}{K})$.

$A = 1$

Rectangular function is an even function

$$f(t) = f(-t)$$

$$A\,\mathrm{rect}\left(\frac{-t}{\tau}\right) = A\,\mathrm{rect}\left(\frac{t}{\tau}\right)$$

$$A\,\mathrm{rect}\left(-\frac{t \pm t_0}{\tau}\right) = A\,\mathrm{rect}\left(\frac{t \pm t_0}{\tau}\right)$$

$$A\,\mathrm{rect}(b-at) = A\,\mathrm{rect}\left[-a\left(t-\frac{b}{a}\right)\right]$$

$$= A\,\mathrm{rect}\left[a\left(t-\frac{b}{a}\right)\right] \quad A\,\mathrm{rect}\left(-\frac{t}{\tau}\right) = A\,\mathrm{rect}\left(\frac{t}{\tau}\right)$$

$$= A\,\mathrm{rect}\left[at-b\right].$$

① $x(t) = 4\,\mathrm{rect}\left(\frac{-t+4}{2}\right)$

(or) $x(t) = x(-t)$

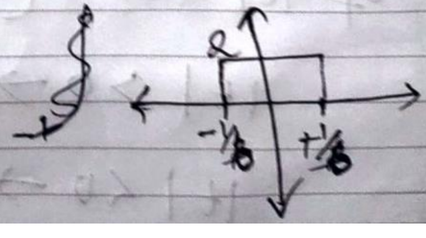
$x(t) = 4\,\mathrm{rect}\left(\frac{t}{2}-2\right)$

$x(t) = 2\,\mathrm{rect}\,(3t)$ Point on x axis.

$= 2\,\mathrm{rect}\left(\frac{t}{1/3}\right)$ → width

Whole width $= \frac{1}{3}$

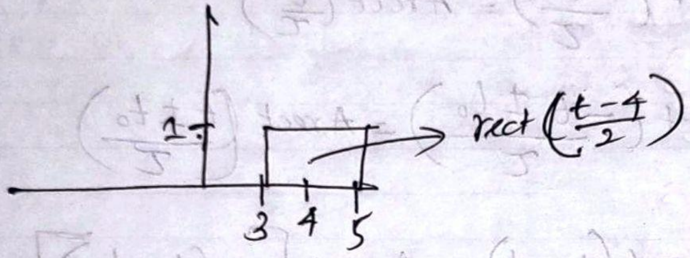
$-\frac{1}{6}$ to $+\frac{1}{6}$



$rect\left(\frac{t-4}{2}\right)$

$\frac{t}{2} - 2 \qquad \frac{t}{2} = 2$

$\boxed{t = 4}$



$$rect(t) = \begin{cases} 1, & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & |t| > 0.5 \end{cases}$$

$$rect\left(\frac{t-4}{2}\right) = \begin{cases} 1, & \left|\frac{t-4}{2}\right| < 0.5 \\ 0.5, & |t| = 0.5 \\ 0, & |t| > 0.5 \end{cases} \longrightarrow \begin{cases} 1, & \left|\frac{t-4}{2}\right| < 0.5 \\ 0.5, & \left|\frac{t-4}{2}\right| = 0.5 \\ 0, & \left|\frac{t-4}{2}\right| > 0.5 \end{cases}$$

If you consider the probability 1 for the value of $|t| < 0.5$ (inbetween -0.5 and $+0.5$)

$|t| > 0 \rightarrow \frac{t-4}{2} > 0 \rightarrow \frac{t-4}{2} < 0.5 \rightarrow t-4 < 1$ $\boxed{t < 5}$

$|t| < 0 \rightarrow \frac{t-4}{2} < 0 \rightarrow -\left(\frac{t-4}{2}\right) < 0.5 \rightarrow$

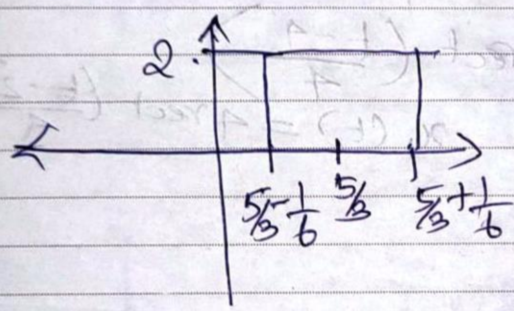
$\rightarrow -(t-4) < 1$

$4 - 1 < t$ $\boxed{3 = t}$

③ $x(t) = 2\,\text{rect}(3t-5)$

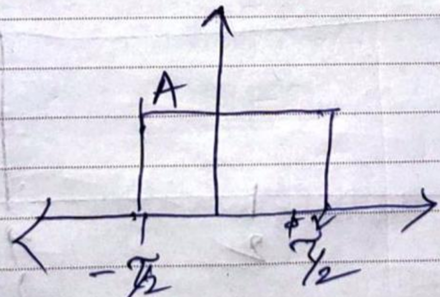
$$= 2\,\text{rect}\left[3\left(t - \frac{5}{3}\right)\right]$$ → Centre

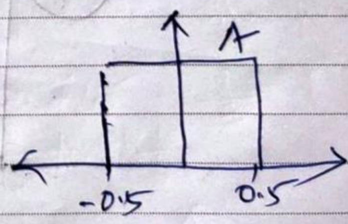
$$= 2\,\text{rect}\left[\frac{\left(t-\frac{5}{3}\right)}{\frac{1}{3}}\right]$$ → width (τ)



$-\frac{1}{6}$ $+\frac{1}{6}$

$$x(t) = A\,\text{rect}\left(\frac{t - \text{Centre}}{\tau}\right)$$





$$x(t) = \underbrace{4\,\mathrm{rect}\left(\frac{-2t+4}{8}\right)}_{x_1(t)} + \underbrace{2\,\mathrm{rect}\left(\frac{4t-16}{16}\right)}_{x_2(t)}$$

Centre

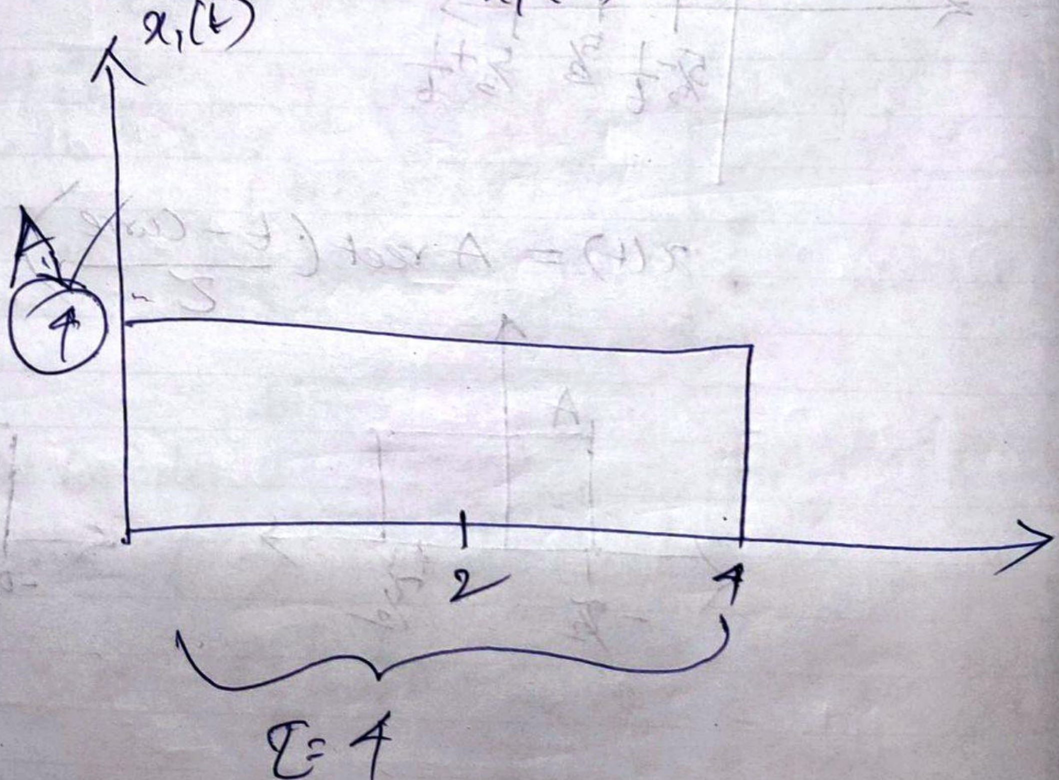
$$x_1(t) = 4\,\mathrm{rect}\left(\frac{\cancel{-2}\left(\cancel{2-t}\ t-2\right)}{\cancel{8}\,{}^{2}\,4}\right)$$

removed

$$= 4\,\mathrm{rect}\left(\frac{t-2}{4}\right)$$

$$x_2(t) = 2\,\mathrm{rect}\left(\frac{t-4}{4}\right)$$

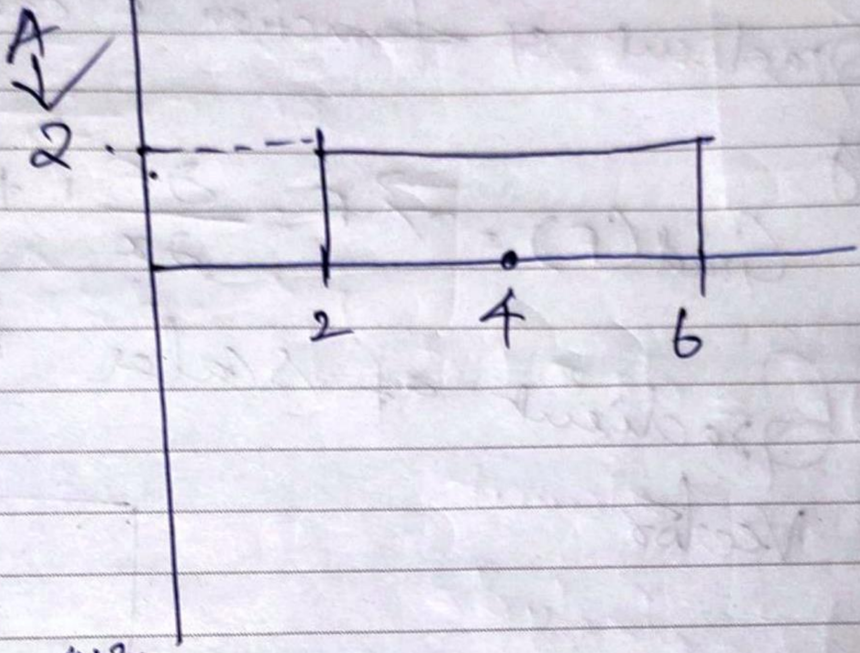
$$x_1(t) = 4\,\mathrm{rect}\left(\frac{t-2}{4}\right)$$

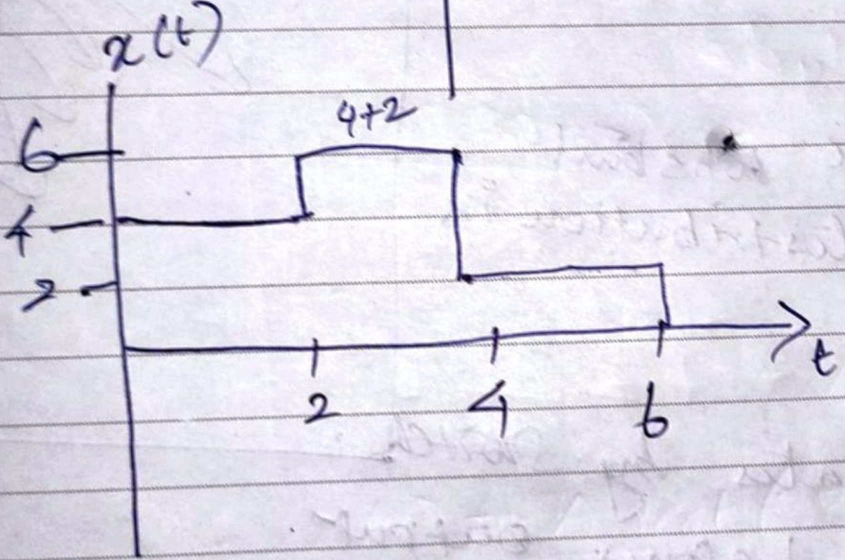


$\tau = 4$

xpt. Name. Date :

$x_2(t) = 2\,\mathrm{rect}\left(\frac{t-4}{4}\right)$





SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – III - Orthogonal Curvilinear Coordinates – SPH1215

Unit 3

Orthogonal Curvilinear Coordinates

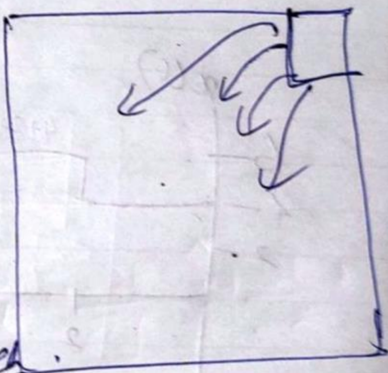
Gradient

(1) Gradient is applied on scalar quantity

(2) Gradient of function F can be calculated by

$$\text{Grad}(F) = \vec{\nabla} F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

(3) Gradient of scalar becomes Vector



[Position of AC constant but its energy distribution is variable]

if AC operates by switch, the flow becomes output $(\vec{\nabla})$

N 09315 36924 – Chemencherry

Divergence

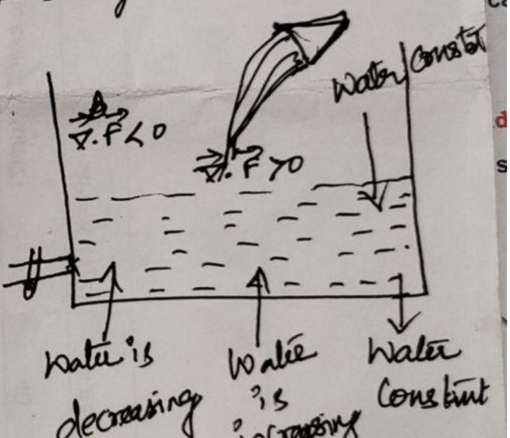
1. Divergence is scalar quantity
2. Divergence is applied on vector quantity
3. Divergence of function $\vec{F}$ can be calculated by

$$Div(\vec{F}) = \vec{\nabla}\cdot\vec{F} = \frac{\partial \vec{F}}{\partial x}\vec{i} + \frac{\partial \vec{F}}{\partial y}\vec{j} + \frac{\partial \vec{F}}{\partial z}\vec{k}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$Div(\vec{F}) = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

4. It explains overall variation of function in x, y and z directions.
5. It explains overall rate of change with respect to co-ordinates.



Find the divergence of function F at point (1,2,1)

$$\vec{F} = xy^2\vec{i} + y\vec{j} + xz\vec{k}$$

$$Div(\vec{F}) = \vec{\nabla}\cdot\vec{F} = \frac{\partial \vec{F}}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial \vec{F}}{\partial z}\vec{k}$$

$$Div(\vec{F}) = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial (xy^2)}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial}{\partial z}(xz)$$

$$y^2 + 1 + x$$

At point (1,2,1)

$$Div(\vec{F}) = \nabla\cdot\vec{F} = 2^2 + 1 + 1$$

~~Gauss divergence theorem~~

$$= 4 + 1 + 1$$
$$= 6$$

$$\vec{\nabla}\cdot\vec{F} = \lim_{\Delta V \to 0} \frac{\oint \vec{F}\cdot ds}{\Delta V}$$

$$\Rightarrow \int \vec{\nabla}\cdot\vec{F}\, dv = \int \vec{F}\cdot ds$$

Gauss Divergence theorem

Curl

Curl is a vector quantity

Curl operator on vector only.

Curl of function $\vec{F}$ can be calculated by

$$\text{Curl}(\vec{F}) = \vec{\nabla}\times\vec{F}$$

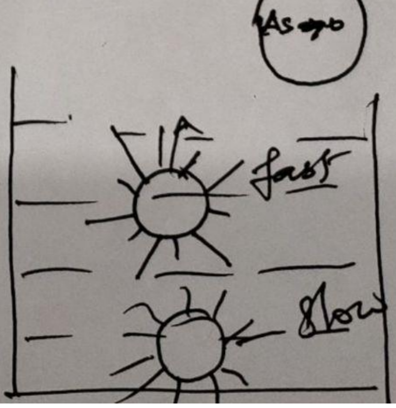
$$\text{if } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\text{Curl}(\vec{F}) = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\Rightarrow \vec{\nabla}\times\vec{F} = \lim_{\Delta s \to 0} \frac{\int \vec{F}\cdot dl}{\Delta S}$$

$$\int \vec{\nabla}\times\vec{F}\, ds = \int \vec{F}\cdot dl$$

Stokes theorem



1.3 COORDINATE SYSTEMS

Coordinate system is defined as a system which is used to represent a point in space. Basically coordinate systems are of three types.

1. Cartesian (or) Rectangular coordinate system.
2. Circular cylindrical coordinate system.
3. Spherical coordinate system.

The simplest among these coordinate systems is the cartesian coordinate system.

1.3.1 Cartesian (or) Rectangular Coordinate System

The cartesian coordinates are represented in figure 1.3. The three axes x, y, z are mutually perpendicular to each other. These are said to be orthogonal to each other. The unit vectors along the coordinate axes are represented by $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$.

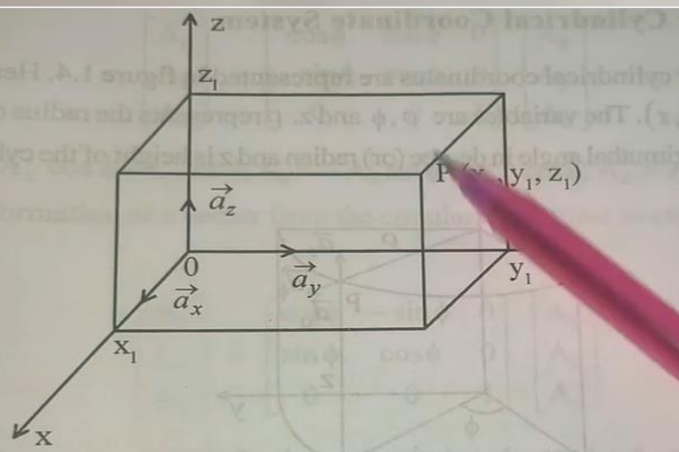
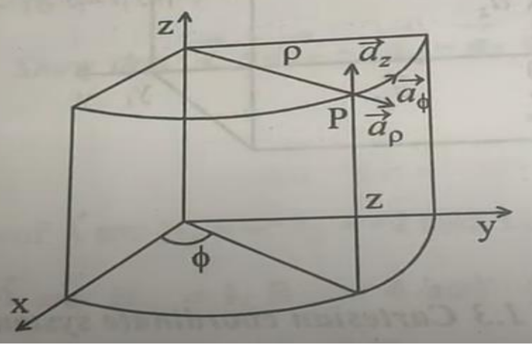


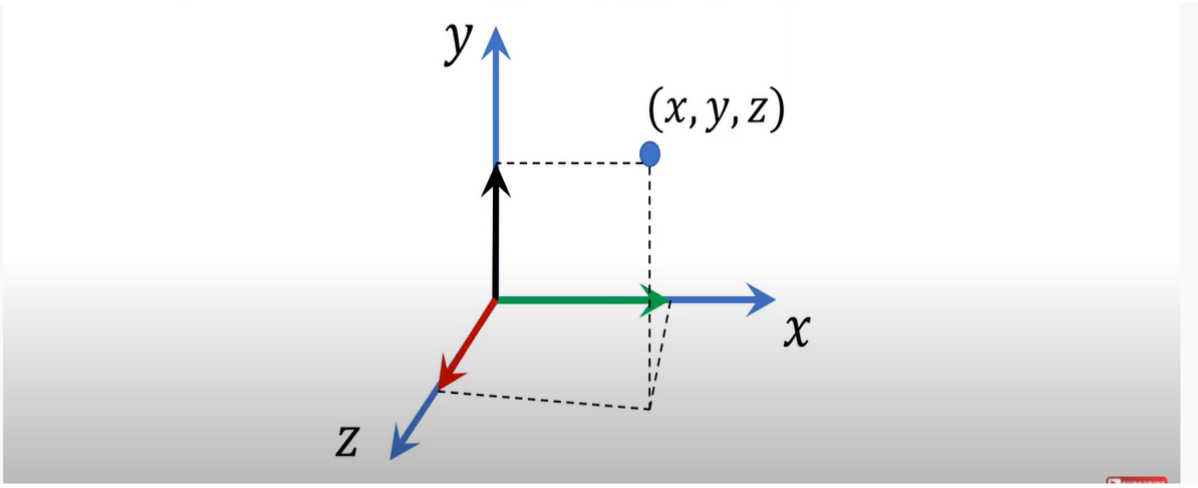
Figure 1.3 Cartesian coordinate system

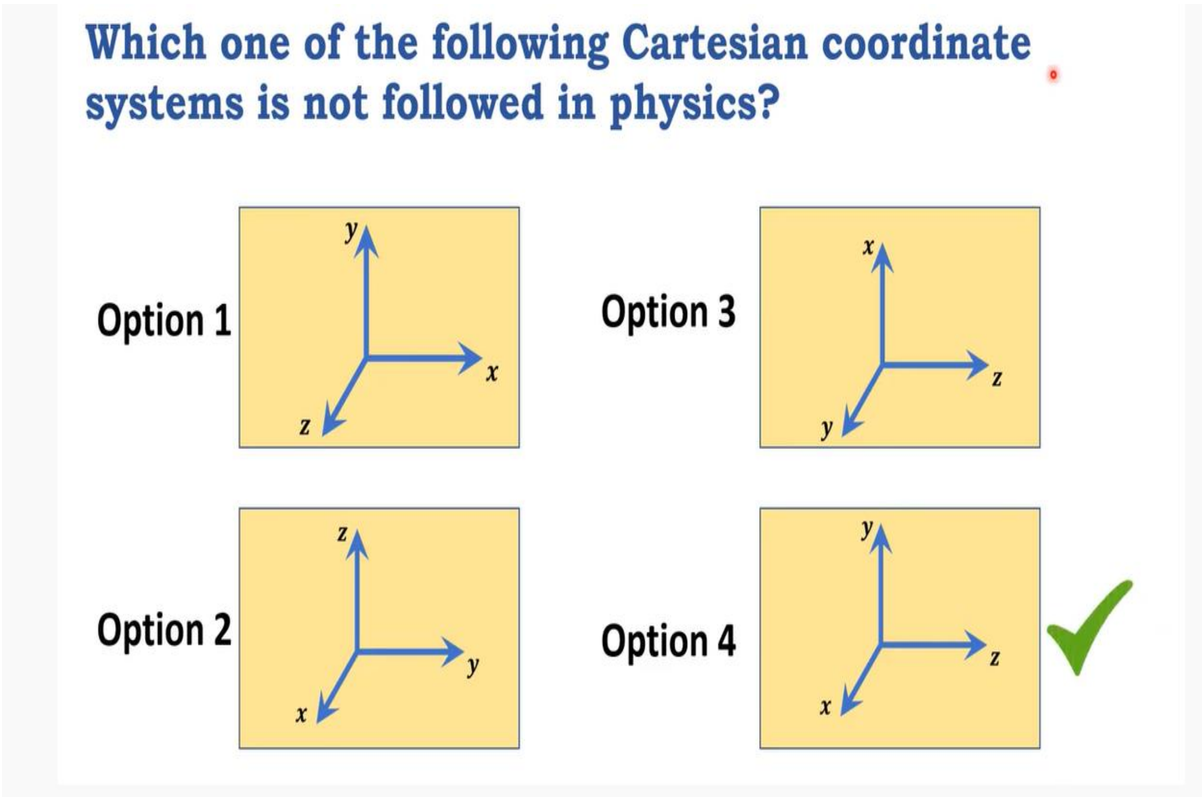
Here a point, P is represented by P (x, y, z). The variables are x, y and x represents the width of the rectangular in metre, y is length of the rectangular in metre and z is height of the rectangular in metre.

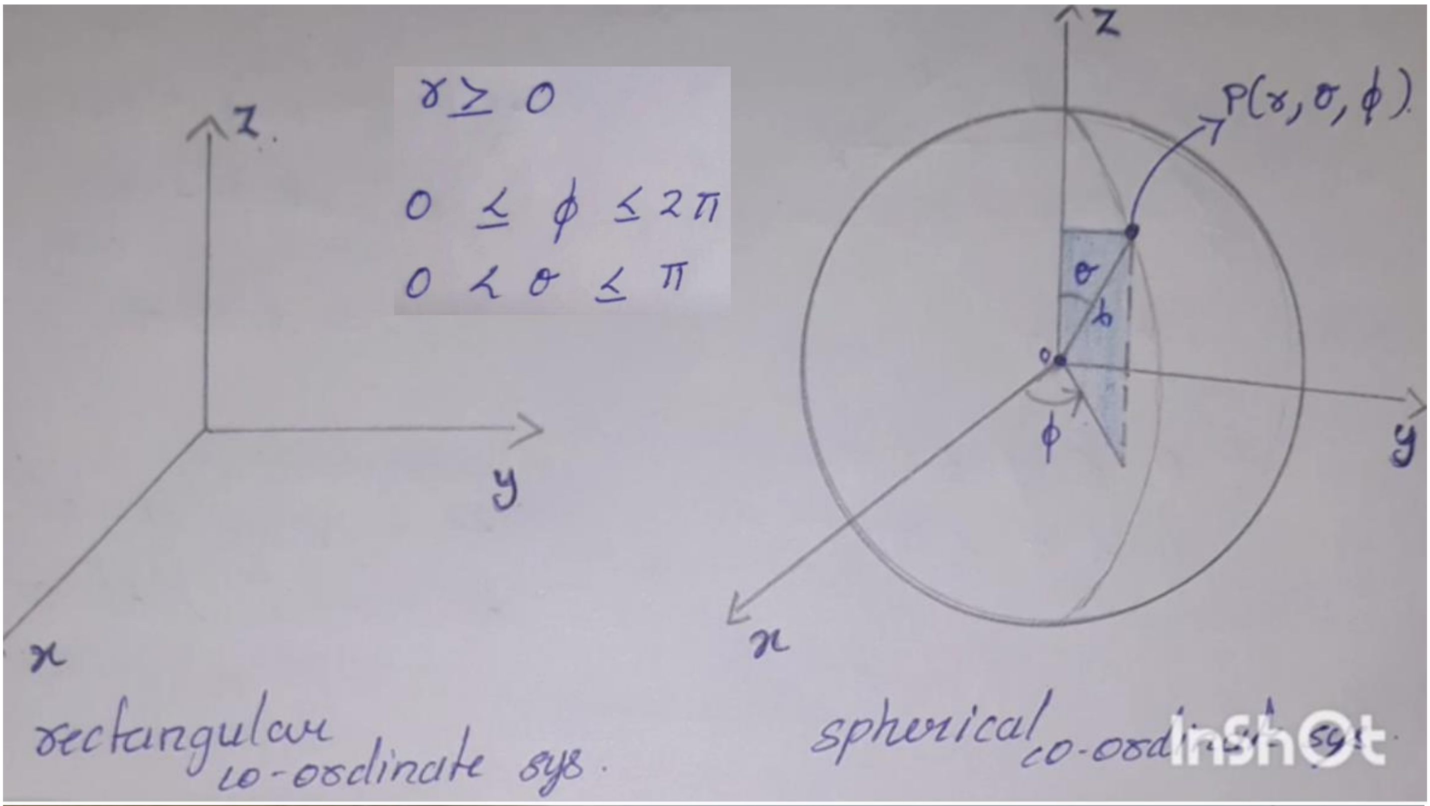
1.3.2 Circular Cylindrical Coordinate System

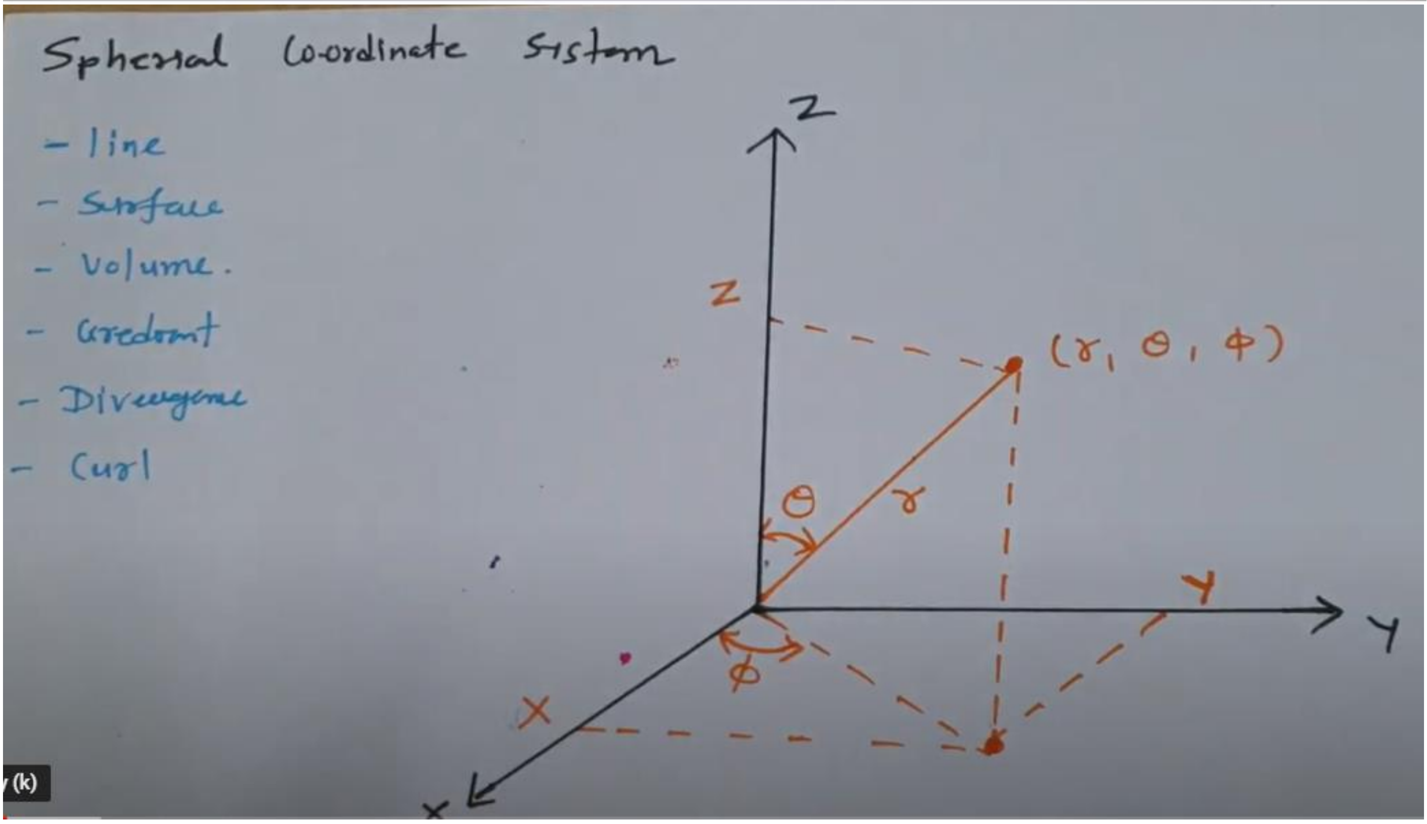
The circular cylindrical coordinates are represented in figure 1.4. Here a point is represented P (ρ, ϕ, z). The variables are ρ, ϕ and z. ρ represents the radius of a cylinder in metre, ϕ is called azimuthal angle in degree (or) radian and z is height of the cylinder in metre.

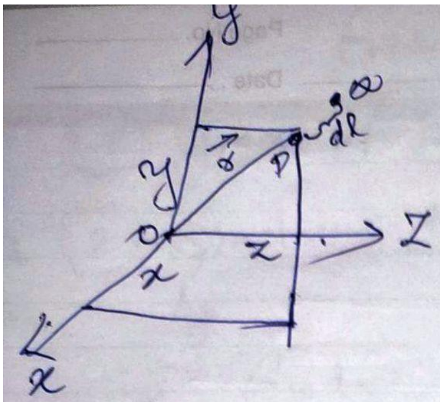



y
(x, y, z)
x
z


Which one of the following Cartesian coordinate systems is not followed in physics?
Option 1
y
x
z
Option 3
x
z
y
Option 2
z
y
x
Option 4
y
z
x


r ≥ 0
0 ≤ ϕ ≤ 2π
0 ≤ θ ≤ π
z
y
x
P(r, θ, ϕ)
rectangular co-ordinate sys.
spherical co-ordinate sys.


Spherical Coordinate System
- line
- Surface
- volume.
- Gradient
- Divergence
- Curl
Z
(r, θ, ϕ)
θ
r
ϕ
X
Y



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Volume element

$$d\tau = dx\,dy\,dz$$

Cartesian

Gradient

$t(x, y, z)$

$$\vec{\nabla} t = \frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

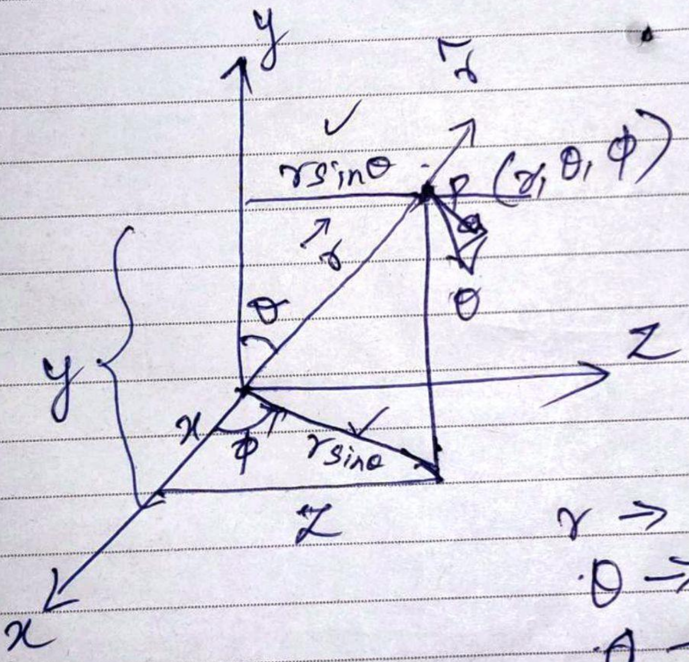
Curl

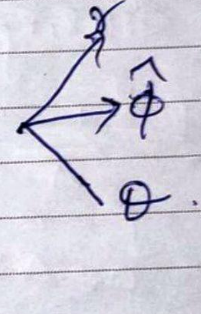
$\vec{\nabla} \times \vec{V}$

Laplacian $\Rightarrow$ div gradient of a function

$= \nabla^2 (\vec{\nabla} \cdot \vec{\nabla})$

Spherical Co-ordinate System



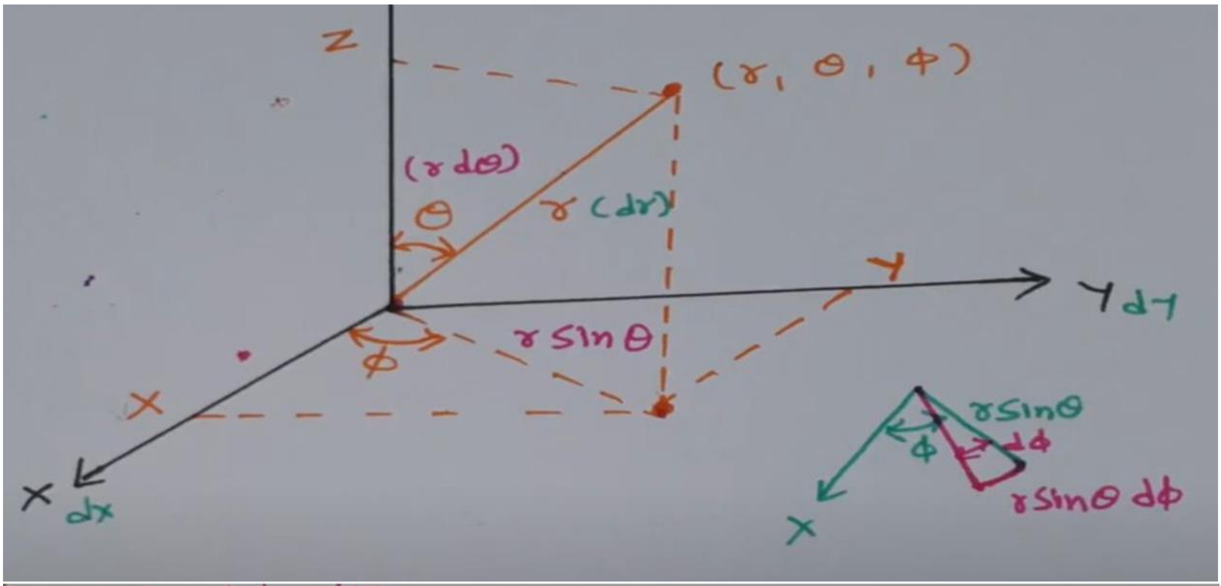


$r \rightarrow 0$ to ∞

$\theta \rightarrow 0$ to π

$\phi \rightarrow 0$ to 2π

$|\vec{r}|$



- For line Integration

$$d\vec{l} = dr\,\hat{i}_r + r\,d\theta\,\hat{i}_\theta + r\sin\theta\,d\phi\,\hat{i}_\phi$$

- For Surface Integration

$$d\vec{S} = r\,dr\,d\theta\,\hat{i}_\phi$$
$$= r\sin\theta\,dr\,d\phi\,\hat{i}_\theta$$
$$= r^2\sin\theta\,d\theta\,d\phi\,\hat{i}_r.$$

- For vol.^{n} Integration

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

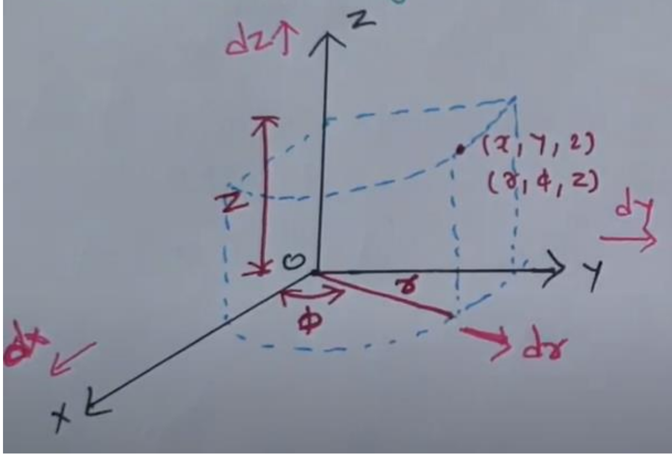
- For del operator.

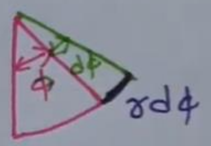
$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{i}_r + \frac{\partial}{r\,\partial\theta}\hat{i}_\theta + \frac{\partial}{r\sin\theta\,\partial\phi}\hat{i}_\phi$$

- For Curl $(\vec{F} = a_r\hat{i}_r + a_\theta\hat{i}_\theta + a_\phi\hat{i}_\phi)$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i}_r & \hat{i}_\theta & \hat{i}_\phi \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ a_r & a_\theta & a_\phi \end{vmatrix}$$

- Cylindrical Co-ordinate system for line, Surface, Volume, Gradient, Divergence and Curl Calculation





→ Cartesian Co-ordinate System | → Cylindrical Co-ordinate System

→ line

→ $d\vec{l} = dx\,\hat{i}_x + dy\,\hat{i}_y + dz\,\hat{i}_z$

→ $d\vec{l} = dr\,\hat{i}_r + r\,d\phi\,\hat{i}_\phi + dz\,\hat{i}_z$

→ Surface

→ $d\vec{S} = dx\,dy\,\hat{i}_z$
$= dx\,dz\,\hat{i}_y$
$= dy\,dz\,\hat{i}_x$

→ $d\vec{S} = r\,dr\,d\phi\,\hat{i}_z$
$= dr\,dz\,\hat{i}_\phi$
$= r\,d\phi\,dz\,\hat{i}_r$

→ Vol.^{m}

→ $dV = dx\,dy\,dz$

→ $dV = r\,d\phi\,dr\,dz$

→ Gradient

→ Curl $(\vec{\nabla}\times\vec{F})$

$\vec{F} = F_x\hat{i}_x + F_y\hat{i}_y + F_z\hat{i}_z$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

→ $\vec{F} = F_r\hat{i}_r + F_\phi\hat{i}_\phi + F_z\hat{i}_z$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i}_r & \hat{i}_\phi & \hat{i}_z \\ \partial/\partial r & \partial/r\partial\phi & \partial/\partial z \\ F_r & F_\phi & F_z \end{vmatrix}$$

→ Divergence

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial(rF_r)}{r\,\partial r} + \frac{\partial F_\phi}{r\,\partial\phi} + \frac{\partial F_z}{\partial z}$$

Velocity & acceleration in spherical Co-ordinates

(r, θ, ϕ) $\quad \hat{r}, \hat{\theta}, \hat{\phi}$

$\cancel{\vec{r} = r\hat{r}}$ $\quad \vec{l} = l\hat{l}$

Try to find $\vec{dl}$ using $l\hat{l}$

$d\vec{l} = dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta$

(In Cartesian Co-ordinates
$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$d\vec{l} = d(l\hat{l}) = dl\,\hat{l} + l\,d\hat{l} \quad -(1)$$

$(\vec{l} = l\hat{l})$

$$\hat{l} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k} \quad -(2)$$

$$d\hat{l} = d(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\vec{k})$$

$$= \hat{i}[\cos\theta(d\theta)\cos\phi] + \hat{i}[\sin\theta(-\sin\phi\,\boxed{d\phi})]$$

$$\hat{j}[(\cos\theta\,d\theta)\sin\phi] + \hat{j}[\sin\theta\cos\phi\,\boxed{d\phi}]$$

$$+[-\sin\theta\,d\theta\,\hat{k}]$$

$$= d\theta\overbrace{[\cos\theta\cos\phi\,i + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}]}^{\hat{\theta}} +$$

$$+ d\phi[-\sin\theta\sin\phi\,d\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j}]$$

$$= \hat{\theta}\,d\theta + d\phi\sin\theta\underbrace{[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}]}_{\hat{\phi}}$$

$$= \hat{\theta}\,d\theta + d\phi\sin\theta\,\hat{\phi}$$

Put in eqn (1)

Put (2) in (1)

$\cancel{\vec{dr} = dr\hat{r}}$.

$$d\vec{r} = d(r\hat{r}) = dr\,\hat{r} + r\,d\hat{r}$$

$$= dr\,\hat{r} + r(d\theta\,\hat{\theta} + d\phi\sin\theta\,\hat{\phi})$$

$$= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\left(\vec{l} = l\hat{l}\right)$$

$$d\vec{l} = dl\,\hat{l} + l\,d\hat{l}$$

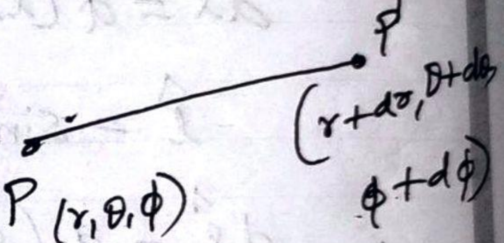
$$= dl\,\hat{l} + l\,(d\theta\,\hat{\theta} + d\phi \sin\theta\,\hat{\phi})$$

$$= dl\,\hat{l} + l\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

Displacement

$\vec{v}$ in spherical polar co-ordinates.

$$\vec{v} = \frac{d\vec{r}}{dt}$$



$$\cancel{d\vec{l} = dl} \quad d\vec{r} = dr\,\hat{r} + r\,d\hat{r}$$

$$d\vec{r} = dr\,\hat{l} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\right)\hat{r} + r\left(\frac{d\theta}{dt}\right)\hat{\theta} + r\sin\theta\left(\frac{d\phi}{dt}\right)\hat{\phi}$$

$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

Magnitude of $\vec{v} = |v| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\phi})^2}$

$(r, \theta, \phi) \quad \hat{r}, \hat{\theta}, \hat{\phi}$

$d\vec{l} = dx\,\vec{i} + dy\,\vec{j} + \ldots$

$\frac{\partial l}{\partial t} =$

Try to find $d\vec{r}$ using r and $\hat{r}$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,\hat{\phi}$$

$$d(\vec{r}) = d(r\hat{r}) = dr\,\hat{r} + r\,d\hat{r} \quad -(1)$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$d\hat{r} = d\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$$

$$= \left[\cos\theta\,\boxed{d\theta}\cos\phi\,\hat{i} + \hat{i}\sin\theta\,(-\cos\phi\,d\phi)\right] +$$

$$= \left[\cos\theta\,\boxed{d\theta}\sin\phi\,\hat{j} + \hat{j}\sin\theta\,(\cos\phi\,d\phi)\right]$$

$$+ \left[-\sin\theta\,\boxed{d\theta}\,\hat{k}\right]$$

$$\hat{\theta}$$

$$d\hat{r} = d\theta\left[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right] +$$

$$d\phi\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j}\right] =$$

$$= \cancel{\hat{\theta}\,d\theta} + d\phi\sin\theta\left[\cancel{\cos\phi\,\hat{i} + \cos\phi\,\hat{j}}\right]$$

$$= \hat{\theta}\,d\theta + d\phi\sin\theta\,\underbrace{\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]}_{\hat{\phi}}$$

$$d\hat{r} = \hat{\theta}\,d\theta + d\phi\sin\theta\,\hat{\phi} \quad -(2)$$

Acceleration of the particle

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}\right)$$

$$\frac{d}{dt}(\dot{r}\hat{r}) = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}}$$ (Both $\dot{r}$ $\hat{r}$ can vary with time)

$$\frac{d}{dt}(r\dot{\theta}\hat{\theta}) = (\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}})$$

$$\frac{d}{dt}(r\sin\theta\,\dot{\phi}\,\hat{\phi}) = \dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi} + r(\cos\theta\,\dot{\theta})\,\dot{\phi}\,\hat{\phi} + r\sin\theta\,\ddot{\phi}\,\hat{\phi} + r\sin\theta\,\dot{\phi}\,\dot{\hat{\phi}}$$

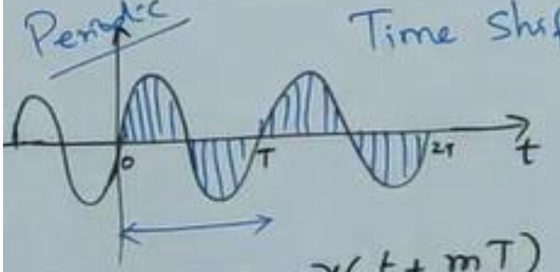
SCHOOL OF SCIENCE AND HUMANITIES

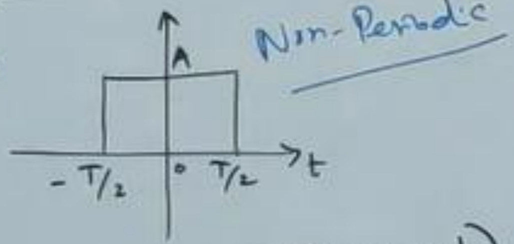
DEPARTMENT OF PHYSICS

UNIT – IV - Fourier Series – SPH1215

Periodic Signal: A Signal is said to be Periodic, if it repeats after a fixed time period.

$$\boxed{x(t) = x(t+T) \quad | \quad x(n) = x(n+N)}$$





$x(t) = x(t + mT)$

$m = 1, 2, 3 \ldots$

Periodic with Period $T, 2T, 3T \ldots$

$T = \frac{1}{f}$

$T \rightarrow$ can be any Value

$x(n) = x(n + mN)$

$m: 1, 2, 3 \ldots$

Periodic with Period $N, 2N, 3N \ldots$

N - Not defined

N - is only integer

1) $x(t) = \cos(t + \pi/4)$

$x(t) = \cos(2\pi f t + \phi)$

$2\pi f = 1$

$\frac{2\pi}{T} = 1$ $\quad \boxed{T = 2\pi}$

Periodic with $T = 2\pi$

2) $x(t) = \sin\left(\frac{2\pi}{3} t\right)$

$x(t) = \sin(2\pi f t + \phi)$

$2\pi f = \frac{2\pi}{3}$

$\frac{2\pi}{T} = \frac{2\pi}{3}$ $\quad$ Periodic with Period $T = 3$

$\boxed{T = 3}$

$$x(t) = x_1(t) + x_2(t)$$

$$x_1(t) = x_1(t + mT_1)$$

$$x_2(t) = x_2(t + kT_2)$$

$$x(t) = x(t + T)$$

$$T = mT_1 = kT_2$$

$$\boxed{\frac{T_1}{T_2} = \frac{k}{m} : \text{rational}}$$

---

3) $x(t) = \underbrace{\cos\left(\frac{\pi t}{3}\right)}_{x_1(t)} + \underbrace{\sin\left(\frac{\pi t}{4}\right)}_{x_2(t)}$

$$2\pi f_1 = \frac{\pi}{3} \qquad 2\pi f_2 = \frac{\pi}{4}$$

$$\frac{2\pi}{T_1} = \frac{\pi}{3} \qquad \frac{2\pi}{T_2} = \frac{\pi}{4}$$

$$\boxed{T_1 = 6} \qquad \boxed{T_2 = 8}$$

$$\frac{T_1}{T_2} = \frac{6}{8} \ (\text{rational})$$

4) $x(t) = \cos t + \sin\sqrt{2}t$

$x_1(t)$ $\quad$ $x_2(t)$

$2\pi f_1 = 1$ $\quad$ $2\pi f_2 = \sqrt{2}$

$\frac{2\pi}{T_1} = 1$ $\quad$ $\frac{2\pi}{T_2} = \sqrt{2}$

$\boxed{T_1 = 2\pi}$ $\quad$ $\boxed{T_2 = \frac{2\pi}{\sqrt{2}}}$

$\frac{T_1}{T_2} = \frac{2\pi}{2\pi}\sqrt{2}) = \sqrt{2}$ not a rational number

Non-Periodic

$x(n) = \cos(2\pi f n)$

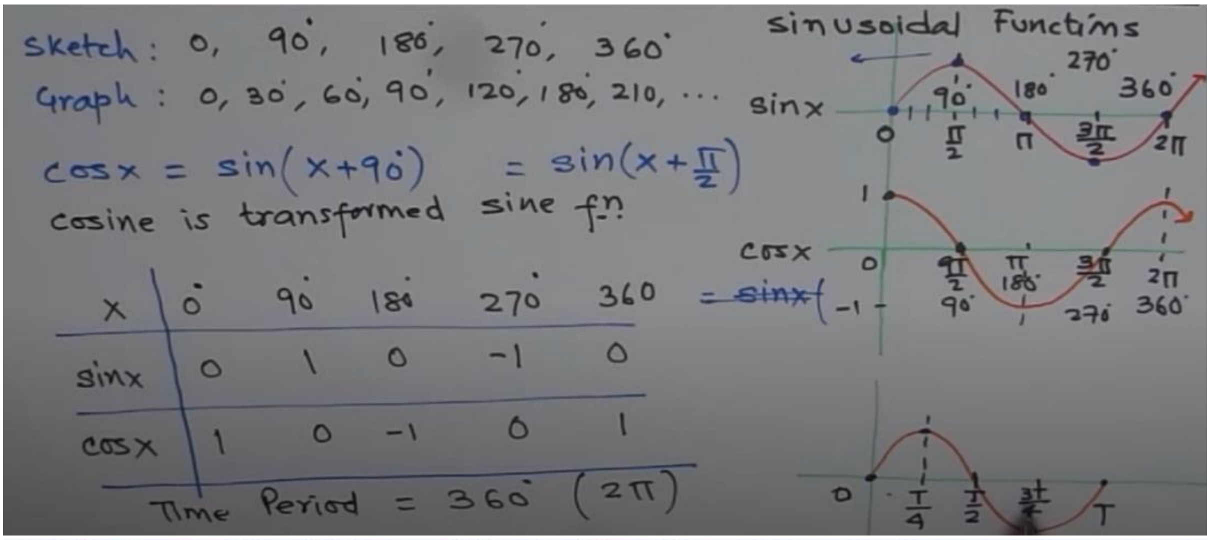
$x(n+N) = \cos[2\pi f(n+N)]$

$= \cos[\underbrace{2\pi f n}_{A} + \underbrace{2\pi f N}_{B}]$

$\cos 2\pi f n \cos 2\pi f N - \sin 2\pi f n \sin 2\pi f N$

$2\pi f N = 2\pi K$

$\boxed{f = \frac{K}{N}}$ = rational



Orthogonality of Sine and Cosine Functions

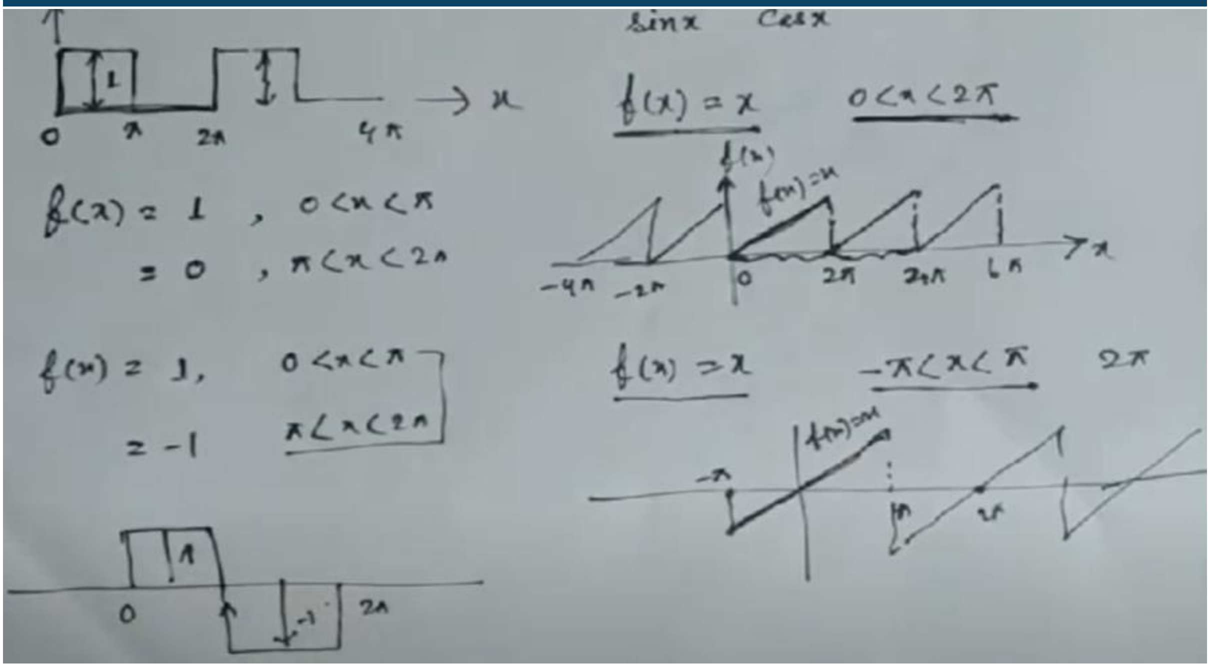
Mutually orthogonal functions

- The condition for two functions f (x) and g (x) to be mutually orthogonal in interval [a , b] is $\int_a^b f(x)\, g(x)\, dx = 0$

The set of functions

$$\left\{ 1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, ..., \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, ...\right\}$$

is linearly independent and mutually orthogonal in an interval $[\alpha, \alpha + 2L]$



$\underline{\text{Taylor}}$: $f(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$, $c_n = \frac{f^{(n)}(a)}{n!}$

$\underline{\text{Fourier}}$: $f(x) = a_0 + a_1\cos(1x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$ $\quad a_0 = \frac{1}{2\pi} \cdots (x)\,d$

$+ b_1\sin(1x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

on $[-\pi,\pi]$

$$\int_{-\pi}^{\pi} f(x)\,dx = \underbrace{\int_{-\pi}^{\pi} a_0\,dx}_{2\pi a_0} + \underbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nx)\,dx}_{0} + \underbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\sin(nx)\,d}_{0}$$

$\underline{\text{note}}$: $\int_{-\pi}^{\pi}\cos(nx)\,dx = 0$

$= \frac{1}{n}\sin(nx)\Big]_{-\pi}^{\pi}$

$= \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(n\cdot -\pi)$

$= \frac{1}{n}$

$\int_{-\pi}^{\pi}\sin(nx)\,dx = 0$

$\underline{\text{Fourier}}$: $f(x) = a_0 + a_1\cos(1x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$ $\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$+ b_1\sin(1x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

on $[-\pi,\pi]$

$m \in \mathbb{Z}^+$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \underbrace{\int_{-\pi}^{\pi} a_0\cos(mx)\,dx}_{0} + \underbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nx)\cos(mx)\,dx}_{a_m\pi} + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\sin(nx)\cos(mx)\,dx$$

$$\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$\underline{\text{Taylor}}$: $f(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$, $c_n = \frac{f^{(n)}(a)}{n!}$

$\underline{\text{Fourier}}$: $f(x) = a_0 + a_1\cos(1x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$

$+ b_1\sin(1x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,d \end{aligned}}$$

on $[-\pi,\pi]$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \int_{-\pi}^{\pi} a_0\sin(mx)\,dx + \underbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nx)\sin(mx)\,dx}_{0} + \underbrace{\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\sin(nx)\sin(mx)\,dx}_{b_m\pi}$$

$$\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

Problems to be solved:

1. $f(x) = x^2, \quad [-\pi, \pi]$

2. $f(x) = e^{-x}, \quad [0, 2\pi]$

3. $f(x) = x - x^2, \quad [-\pi, \pi]$

4. $f(x) = \sqrt{(1-\cos x)}, \quad [0, 2\pi]$

5. $f(x) = x \sin x, \quad [0, 2\pi]$

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Solⁿ Here, $f(x) = x - x^2$ and $b - a = 2\pi$

The fourier series for $f(x) = x - x^2$ is

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2} - \frac{\pi^3}{3} + \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \frac{\pi^3}{3}\right] = -\frac{2\pi^3}{3\pi} = \boxed{-\frac{2\pi^2}{3} = a_0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} + (1-2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi-\pi^2)\frac{\sin n\pi}{n} - (-\pi-\pi^2)\frac{\sin n(-\pi)}{n} + (1-2\pi)\frac{\cos n\pi}{n^2} - (1+2\pi)\frac{\cos n(-\pi)}{n^2} + 2\frac{\sin n\pi}{n^3} - 2\frac{\sin n(-\pi)}{n^3}.\right]$$

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$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} + (1-2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]$$

$$= \frac{1}{\pi}\left[(\pi-\pi^2)\cancelto{0}{\frac{\sin n\pi}{n}} - (-\pi-\pi^2)\cancelto{0}{\frac{\sin n(-\pi)}{n}} + (1-2\pi)\frac{\cos n\pi}{n^2} - (1+2\pi)\frac{\cos n(-\pi)}{n^2} + 2\cancelto{0}{\frac{\sin n\pi}{n^3}} - 2\cancelto{0}{\frac{\sin n(-\pi)}{n^3}}.\right]$$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - (1+2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{n^2\pi}\left[\cancel{(-1)^n} - 2\pi(-1)^n - \cancel{(-1)^n} - 2\pi(-1)^n\right] = \boxed{\frac{-4(-1)^n}{n^2} = a_n}$$

$$-\frac{4\pi(-1)^n}{n^2\cancel{\pi}}$$

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$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x^2-x)\frac{\cos nx}{n} + (1-2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{\cos n\pi}{n} - (\pi^2+\pi)\frac{\cos n(-\pi)}{n} + (1-2\pi)\frac{\cancel{}}{n^2}\right.$$

$$\left. - (1+2\pi)\frac{\cancel{\sin n(-\pi)}}{n^2} - \cancel{2\frac{\cos n\pi}{n^3}} + \cancel{2\frac{\cos n(-\pi)}{n^3}}\right]$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{(-1)^n}{n} - (\pi^2+\pi)\frac{(-1)^n}{n}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[\cancel{\pi^2} - \pi - \cancel{\pi^2} - \pi\right] = \frac{(-1)^n}{n\cancel{\pi}}(-2\cancel{\pi})$$

$$\therefore \boxed{b_n = -\frac{2(-1)^n}{n}}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

UNIT – V - Frobenius Method and Some Special Integrals – SPH1215

Unit V

Frobenius Method and Some Special Integrals

The second order differential equation is $f(x)y'' + g(x)y' + r(x)y = 0 \dots\dots \to (1)$

(i) $f(a) \neq 0$

$x = a$ is an ordinary points

(ii) $f(a) = 0$

$x = a$ is a singular point

(i) Regular singular point

(ii) Irregular singular point.

Regular singular point:

$(1) \Rightarrow$

$$f(x)y'' + g(x)y' + r(x)y = 0$$

$$y'' + \frac{g(x)}{f(x)}y' + \frac{r(x)}{f(x)}y = 0$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + \frac{g(x)}{f(x)}y' + \frac{r(x)}{f(x)}y = 0$$

$$y'' + P(x)y' + Q(x)y = 0$$

(i) $\underset{x \to a}{Lt}\ (x-a)P(x)$ exists and finite value.

(ii) $\underset{x \to a}{Lt}\ (x-a)^2 Q(x)$ exists and finite value.

$\therefore x = a$ is a regular singular point.

Irregular singular point:

Either $\underset{x\to a}{Lt}(x-a)P(x)$ (or) $\underset{x\to a}{Lt}(x-a)^2 Q(x)$ does not exists then the point $x=a$ is an irregular singular point.

$$xy'' + (1-x)y' + \frac{1}{x}y = 0$$

... regular singular points of

$$xy'' + (1-x)y' + \frac{1}{x}y = 0$$

$\Rightarrow$

$$xy'' + (1-x)y' + \frac{1}{x}y = 0 \quad \ldots\ldots \to (1)$$

Here $f(x) = x$

$f(0) = 0$

$\therefore x = 0$ is a singular point

$$xy'' + (1-x)y' + \frac{1}{x}y = 0$$

$$y'' + \frac{1-x}{x}y' + \frac{1}{x^2}y = 0$$

$$P(x) = \frac{1-x}{x}, \quad Q(x) = \frac{1}{x^2}$$

(i) $\underset{x\to a}{Lt}(x-a)P(x) = \underset{x\to 0}{Lt}\, \cancel{x} \cdot \frac{1-x}{\cancel{x}} = \underset{x\to 0}{Lt}(1-x) = 1$

(ii) $\underset{x\to a}{Lt}(x-a)^2 Q(x) = \underset{x\to 0}{Lt}\, \cancel{x^2} \cdot \frac{1}{\cancel{x^2}} = \underset{x\to 0}{Lt}\, 1 = 1$

(i) & (ii) both are exists.

$\therefore x = 0$ is a regular singular point.

at $x = 1$

(i) $\underset{x \to a}{Lt} (x-a)\, p(x) = \underset{x \to 1}{Lt} \cancel{(x-1)} \dfrac{1}{(x+1)(x-1)^{\cancel{2}}}$

$$= \underset{x \to 1}{Lt} \frac{1}{(x+1)(x-1)}$$

$$= \frac{1}{0}$$

$$= \infty$$

$\therefore x = 1$ is an irregular singular point.

Frobenius Method

$$3x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$y'' + P y' + Q y = 0$$

$\div 3x$

$$y'' + \frac{2}{3x} y' + \frac{y}{3x} = 0$$

$$(x - x_0) P(x)\Big|_{x = x_0} = \cancel{x} \times \frac{2}{3\cancel{x}} = \frac{2}{3} \neq \infty$$

$$(x - x_0)^2 Q(x)\Big|_{x = x_0} = x^{\cancel{2}} \times \frac{2}{3\cancel{x}}\Big|_{x = x_0} = 0 \times \frac{2}{3} = 0 \neq \infty$$

$$3x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y'=\sum_{n=0}^{\infty}(n+k)\,a_n x^{n+k-1}$$

$$y''=\sum_{n=0}^{\infty}(n+k)(n+k-1)\,a_n x^{n+k-2}$$

$$\sum_{n=0}^{\infty}\Big[3a_n(n+k)(n+k-1)\,x^{n+k-1}+2a_n x^{n+k-1}(n+k)$$

$$a_n x^{n+k}\Big]$$

The coeff of lowest degree term and put $n=0$

The coeff of next Lowest term x^{k}

$n=1$

$$[3a_1(k+1)k + 2a_1(k+1)] + a_0 = 0$$

$$a_1 = \frac{-a_0}{(3k+2)(k+2)}$$

Equate $x^{n+k} = 0$

$$3a_{n+1}(n+k+1)(n+k) + 2a_{n+1}(n+k+1) + a_n = 0$$

$$a_{n+1} = \frac{-a_n}{(n+k+1)(3n+3k+2)}$$

Put $k=0$

$a_1 = -\frac{a_0}{2}$ $a_2 = \frac{a_0}{20}$ $a_3 = \frac{-a_0}{480}$

$y_1 = \sum a_n x^{n+k}$

$= a_0\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \dots\right)$

Put $k = 1/3$

$a_1 = \frac{-a_0}{4}$ $a_2 = \frac{a_0}{56}$ $a_3 = \frac{-a_0}{1680}$

$a_0 x^{1/3}\left[1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \dots\right]$

Gamma Function

factors Gamma function is used to give us

$\sqrt{3} = 2! = 2\times 1 = 2$

$\sqrt{7} = 6! = 6\times5\times4\times3\times2\times1$

$\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx.$

(i) $\sqrt{1} = 1$ ii $= \sqrt{n+1}$

$\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx$ $n>0$

~~Put n=n+1~~

$\Gamma_1 = \int_0^\infty e^{-x} e^{1-1} dx$

$= \int_0^\infty e^{-x} e^0 dx = \int_0^\infty e^{-x} dx$

$$= \left[\cancel{x} \frac{e^{-x}}{-1} \right]_0^{\infty}$$

$$= \left[\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1} \right]$$

$e^{\infty} = \infty$

$e^{-\infty} = 0$

$$= \left[\frac{0}{-1} - \frac{1}{-1} \right]$$

$$= 1$$

$$\Gamma n+1 = \int_0^{\infty} e^{-x} x^{n+1-1} dx$$

$$\Gamma n+1 = \int_0^{\infty} e^{-x} x^n dx$$

$$\int uv\,dx = u \int v\,dx - \int \left[\left[\frac{du}{dx} \right] \cdot \int v\,dx \right] dx$$

$$\Gamma n+1 = \int_0^{\infty} x^n \cdot e^{-x} dx$$

$$= x^n \int_0^{\infty} e^{-x} dx - \int_0^{\infty} \left[\frac{d}{dx} (x^n) \int e^{-x} dx \right] dx$$

$$= \left[x^n \frac{e^{-x}}{-1} \right]_0^{\infty} - \int_0^{\infty} n x^{n-1} \frac{e^{-x}}{-1} dx$$

Expt. Name:

$$\lim_{x \to \infty} \left[x^n e^{-x} \right]$$

$$\lim_{x \to \infty} \frac{x^n}{e^x}$$

$$= 0$$

$$= 0 + n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

$$\Gamma(1) = 1$$

$$\boxed{\cancel{\Gamma(n+1) = n!}}$$

$$\Gamma(5/2) = \Gamma\left(\tfrac{3}{2}+1\right) \quad (\Gamma(n+1))$$

$$= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{3}{2}\Gamma\left(\tfrac{1}{2}+1\right)$$

$$= \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\tfrac{1}{2}\right)$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n!$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

=