SATHYABAMA
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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF PHYSICS

Mechanics – SPH1112

I. Dynamics

Rigid body:

In physics, a rigid body (also known as a rigid object) is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.
When force is applied on a rigid body, there will be no change in the shape or size of the rigid body. In case of a non-rigid body the force will distort shape and/or size of the body.

Moment of inertia:

What is Inertia? It is the property of a body by virtue of which it resists change in its state of rest or motion.
What causes inertia in a body?
Inertia in a body is due to it mass. More the mass of a body more is the inertia. For instance, it is easier to throw a small stone farther than a heavier one. Because the heavier one has more mass, it resists change more, that is, it has more inertia.

Moment of Inertia Definition

So we have studied that inertia is basically mass. In rotational motion, a body rotates about a fixed axis. Each particle in the body moves in a circle with linear velocity, that is, each particle moves with an angular acceleration. Moment of inertia is the property of the body due to which it resists angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.

Formula for Moment of Inertia can be expressed as:
∴ Moment of inertia $I = \Sigma\, m_i r_i^2$

The moment of inertia depends on:
a) mass of the body
b) shape and size of the body
c) distribution of mass about the axis of rotation
All the factors together determine the moment of inertia of a body.

we derived expressions of moments of inertia (MI) for different object forms as :
1. For a particle: $I = m r^2$
2. For a system of particles: $I = \sum m_i r_i^2$
3. For a rigid body: $I = \int r^2 \, đm$

Radius of Gyration

As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation, we define a new parameter known as the radius of gyration**.** It is related to the moment of inertia and the total mass of the body.
we can write $I = Mk^2$
where *k* has the dimension of length.
Therefore, the radius of gyration is the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis. Therefore, the moment of inertia depends not only on the mass, shape, and size of the body but also the distribution of mass in the body about the axis of rotation.

In this module, we shall evaluate MI of different regularly shaped rigid bodies:

1. Moment of inertia of a solid cylinder

Moment of inertia of a solid cylinder about its centre can be found using the following equation or formula;

$I = 1/2MR^2$
Here, M = total mass and R = radius of the cylinder.

Derivation of Moment Of Inertia Of Solid Cylinder

We will take a solid cylinder with mass M, radius R and length L. We will calculate its moment of inertia about the central axis.

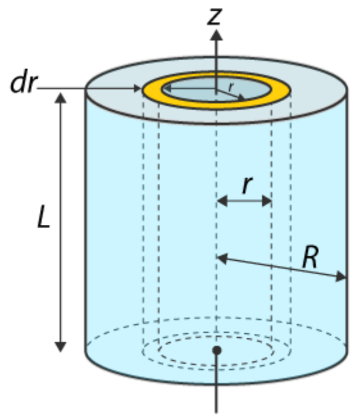


Fig .1.1

Here we have to consider a few things:

- The solid cylinder has to be cut or split into infinitesimally thin rings.
- Each ring consists of the thickness of dr with length L.
- We have to sum up the moments of infinitesimally these thin cylindrical shells.

We will follow the given steps.

1. We will use the general equation of moment of inertia:

$dI = r^2\ dm$

Now we move on to finding the dm. It is normally given as;

$dm = \rho\ dV$

In this case, the mass element can be expressed in terms of an infinitesimal radial thickness dr by;

$dm = 2r\ L\ dr$

In order to obtain dm we have to calculate dv first. It is given as;

$dV = dA\ L$

Meanwhile, dA is the area of the big ring (radius: r + dr) minus the smaller ring (radius: r). Hence;

$dA = \pi\ (r + dr)^2 - \pi\ r^2$ $\quad dA = \pi\ (r + 2rdr + (dr)^2) - \pi r^2$

Notably, here the (dr)2 = 0.

$dA = 2\pi rdr$

2. Substitution of dA into dV we get;

$dV = 2\pi Ldr$

Now, we substitute dV into dm and we will have;

dm = $2\pi Ldr$

The dm expression is further substituted into the dI equation and we get;

dl = $2\pi r^3 Ldr$

3. Alternatively, we have to find the expression for density as well. We use the equation;

p = M/V

Now,

p=M/(πR^2L)

4. The final step involves using integration to find the moment of inertia of the solid cylinder. The integration basically takes the form of a polynomial integral form.

I = $2P\pi rL \int_{R2}^{R1} r^3 dr$ I = $2P\pi L\frac{r^4}{4}$ I = 2π[M/(πR^2L)]L$\frac{R^4}{4}$

Therefore, I = ½ MR^2

<u>2. Calculation of moment of inertia of an uniform solid sphere</u>

Derivation of moment of inertia of an uniform solid sphere

An uniform solid sphere has a radius R and mass M. calculate its moment of inertia about any axis through its centre.

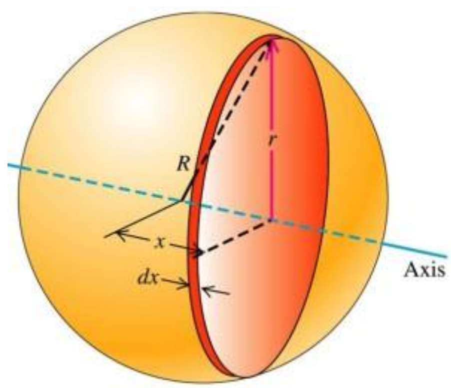


Fig. 1.2

First, we set up the problem.

1. Slice up the solid sphere into infinitesimally thin solid cylinders
2. Sum from the left to the right

the moment of inertia for a solid cylinder:

$I=\frac{1}{2}MR^2$

Hence, for this problem,

$dI=\frac{1}{2}r^2dm$

Now, we have to find dm,

$dm = \rho\, dV$

Finding dV,

$dV = \pi r^2 dx$

Substitute dV into dm,

$dm = \rho\pi r^2 dx$

Substitute dm into dI,

$dI = (1/2)\rho\pi r^4 dx$

Now, we have to force x into the equation. Notice that x, r and R makes a triangle above. Hence, using Pythagoras' theorem,

$r^2=R^2-x^2$

Substituting,

$dI=(1/2)\rho\pi(R^2-x^2)^2dx$

Hence,

$I=(1/2)\rho\pi \int_{-R}^{R} (R^2 - x^2)^2 dx$

After expanding out and integrating, you'll get

$I=(1/2)\rho\pi(16/15)R^5$

Now, we have to find what is the density of the sphere:

$\rho = M/V$

$\rho= \frac{M}{\frac{4}{3}\pi r^3}$

Substituting, we will have:

$I = (2/5) MR^2$

3.Calculation of moment of inertia of a thin spherical shell

Derivation of moment of inertia of a thin spherical shell

A thin uniform spherical shell has a radius of R and mass M. Calculate its moment of inertia about any axis through its centre.

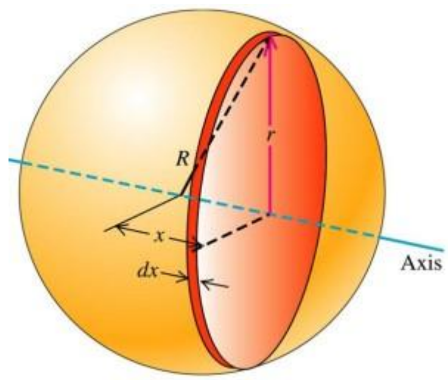


Fig. 1.3

Notice that the thin spherical shell is made up of nothing more than lots of thin circular hoops.

Recall that from <u>Calculation of moment of inertia of cylinder</u>:

Moment of inertia for a thin circular hoop: $I=Mr^2$
Hence,

$dI=r^2dm$ (1)
In order to continue, we will need to find an expression for dm in Equation 1.
$dm=(M/A)\ dA$ (2)
where A is the total surface area of the shell – $4\pi R^2$
Finding dA

If A is the total surface area of the shell, dA is the area of one of the many thin circular hoops. With reference to the picture, each thin circular hoops can be thought to be a thin rectangular strip. The area for each hoop, dA, is the product of the "length" (circumference of the hoop) and the "breadth" (dx in the picture or known as the arc length). [The equation for normal arc length is $R\theta$.]
dA can be expressed with:
dA =length × breadth
=circumference × arc length
$=2\pi r\times Rd\theta$ (3)

Now, in Equation 3, notice that you will have different r for different hoops. Hence, we have to find a way to relate r with θ.
Relating r with θ

Consider the above picture, notice that there is a right-angle triangle with angle θ at the centre of the circle. Hence,

$\sin\theta = r/R$

$r = R\sin\theta$ (4)

Substitutions

Hence, using Equation 4 in Equation 3, dA can be expressed by:

$dA = 2\pi R^2 \sin\theta d\theta$ (5)

Substituting the Equation 5 into the Equation 2, we have:

$dm = ((M\sin\theta)/2)\, d\theta$ (6)

Substituting Equation 6 and the Equation 4 into Equation 1, we have:

$dI = (MR^2/2) \sin^3\theta d\theta$

Integrating with the proper limits, (from one end to the other)

$I = (MR^2/2) \int_0^{\pi} sin^3\theta d\theta$

Now, we split the $\sin^3\theta$ into two,

$I = (MR^2/2) \int_0^{\pi} sin^2\theta \sin\theta d\theta$

$I = (MR^2/2) \int_0^{\pi} (1 - cos^2\theta) \sin\theta d\theta$

Now, at this point, we will use the substitution: u= cosθ. Hence,

$I = (MR^2/2) \int_1^{-1} u^2 - 1\, du$ (7)

$I = (2/3)\, MR^2$ (8)

Compound Pendulum: Measurement of acceleration due to gravity (g) by a compound pendulum

OBJECTIVE:

Use the compound pendulum to find:

1) The acceleration due to gravity g.
2) The moment of inertia of the rod.

THEORY:

Any object mounted on a horizontal axis so as to oscillate under the force of gravity is a compound pendulum. The one used in this experiment is a uniform rod suspended at different locations along its length. The period T of a compound pendulum is given by

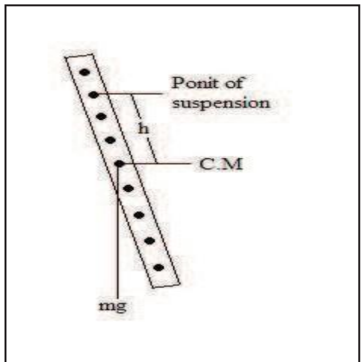


Fig. 1.4

$$\mathbf{T} = 2\upsilon\sqrt{\frac{I}{Mgh}} \tag{1}$$

Where:
I is the rotational inertia of the pendulum about the axis of suspension
M is the pendulum mass
And h is the distance between the suspension point and the center of mass.

Using the parallel axis theorem

$$I = I_G + Mh^2 \tag{2}$$

I_G is the rotational inertia of the body about its center of mass and it is given by

$$I_G = MK^2 \tag{3}$$

Substituting equation 3 in equation 2

$$I = M(h^2 + K^2) \tag{4}$$

Where K is the radius of gyration. substituting equation 4 in equation 1

$$\mathbf{T} = 2\upsilon\sqrt{\frac{h^2 + K^2}{gh}} \tag{5}$$

The period of the simple pendulum is given by

$$\mathbf{T} = 2\upsilon\sqrt{\frac{L}{g}} \tag{6}$$

The period of a compound pendulum equals the period of a simple pendulum of a length

$$\mathbf{L} = \frac{h^2 + K^2}{h} \tag{7}$$

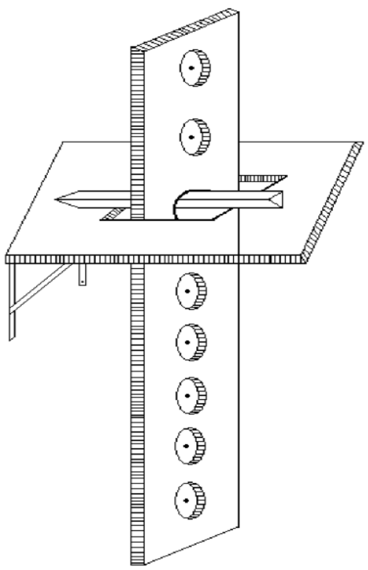
This equation can be solved to find L and K:

$$L = h_1 + h_2 \tag{8}$$

$$K = \sqrt{h_1 h_2} \tag{9}$$

Description:

The bar pendulum consists of a metallic bar of about one meter long. A series of circular holes each of approximately 5 mm in diameter are made along the length of the bar. The bar is suspended from a horizontal knife-edge passing through any of the holes (Fig. 2). The knife-edge, in turn, is fixed in a platform provided with the screws. By adjusting the rear screw the platform can be made horizontal.



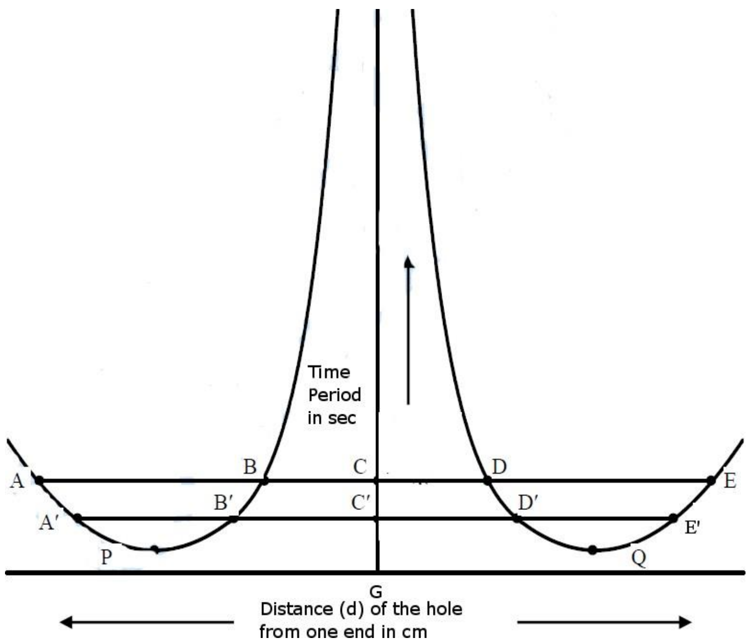


Fig. 1.5

Procedure:

(i) Suspend the bar using the knife edge of the hook through a hole nearest to one end of the bar. With the bar at rest, focus a telescope so that the vertical cross-wire of the telescope is coincident with the vertical mark on the bar.

(ii) Allow the bar to oscillate in a vertical plane with small amplitude (within 4^{o} of arc).

(iii) Note the time for 20 oscillations by a precision stop-watch by observing the transits of the vertical line on the bar through the telescope. Make this observation three times and find the mean time t for 20 oscillations. Determine the time period T.

(iv) Measure the distance d of the axis of the suspension, i.e. the hole from one of the edges of the bar by a meter scale.

(v) Repeat operation (i) to (iv) for the other holes till C.G of the bar is approached where the time period becomes very large.

(vi) Invert the bar and repeat operations (i) to (v) for each hole starting from the extreme top.

(vii) Draw a graph with the distance d of the holes as abscissa and the time period T as ordinate. The nature of graph will be as shown in Fig. 3.

Draw the horizontal line ABCDE parallel to the X-axis. Here A, B, D and E represent the point of intersections of the line with the curves. Note that the curves are symmetrical about a vertical line which meets the X-axis at the point G, which gives the position of the C.G of the bar. This vertical line intersects with the line ABCDE at C. Determine the length AD and BE and find the
length L of the equivalent simple pendulum from

L = AD + DE / 2.

Find also the time period T corresponding to the line ABCDE and then compute the value of g. Draw several horizontal lines parallel to X-axis and adopting the above procedure find the value of g for each horizontal line. Calculate the mean value of g. Alternatively, for each horizontal line obtain the values of L and T and draw a graph with T^2 as abscissa and L as ordinate. The graph would be a straight line. By taking a convenient point on the graph, g may be calculated.

Similarly, to calculate the value of K, determine the length AC, BC or CD, CE of the line ABCDE and compute $\sqrt{AC \text{X} BC}$ or $\sqrt{CD \text{X} CE}$. Repeat the procedure for each horizontal line. Find the mean of all K.

Observations: Table 1-Data for the T versus d graph

Serial no of holes from one end		Distance d of the hole from one end (cm)	Time for 20 oscillations (sec)	Mean time t for 20 oscillations (sec)	Time period T = t/20 (sec)
One side of C.G	1	…..	…. …. ….	…..	…..
	2	…..	….. …… ….	….	……
	3 … …	……	…. ….. ……	…..	……..
Other side of C.G	1	…..	…. …. ….	…..	…..
	2	…..	….. …… ……	….	……
	… …	……	…… ……. ……	…..	……..

TABLE 2- The value of g and K from T vs. d graph

No. of obs.	L (cm)	T (sec)	$g = 4\pi^2 \frac{L}{T^2}$ (cm/sec^2)	Mean 'g' (cm/sec^2)	K (cm)	Mean 'K' (cm)
1. ABCDE	(AD+BE)/2	..	..		$\sqrt{AC \text{X} BC}$ or $\sqrt{CD \text{X} CE}$	
2.	..	..	..		..	
3.	..	..	..		..	

Reversible (Kater's) Pendulum

A physical pendulum with two adjustable knife edges for an accurate determination of "g".

What It Shows:

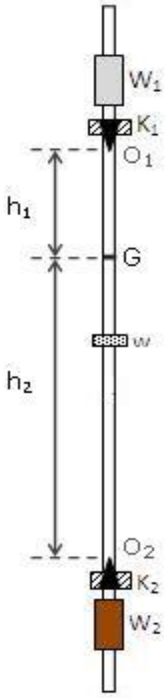
An important application of the pendulum is the determination of the value of the acceleration due to gravity. By adding a second knife-edge pivot and two adjustable masses to the physical pendulum described in the Physical Pendulum demo, the value of g can be determined to 0.2% precision.

How It Works:

An improvement in the precision of the measurement of g was developed in 1817 by Kater. He realised that by using a compound pendulum and suspending it from each end in turn the requirement to measure the distance from the centre of mass to the pivot could be removed. He made a very accurate measurement of g in London, a value that was used to define the metre for many years. The version of Kater's reversible pendulum used in this experiment has a knife-edge for suspension at either end: thus there are two distances, l1 and l2, and two periods T1 and T2.:

Theory

Kater's pendulum, shown in Fig. 1, is a physical pendulum composed of a metal rod 1.20 m in length, upon which are mounted a sliding metal weight W_1, a sliding wooden weight W_2, a small sliding metal cylinder w, and two sliding knife edges K_1 and K_2 that face each other. Each of the sliding objects can be clamped in place on the rod. The pendulum can suspended and set swinging by resting either knife edge on a flat, level surface. The wooden weight W_2 is the same size and shape as the metal weight W_1. Its function is to provide as near equal air resistance to swinging as possible in either suspension, which happens if W_1 and W_2, and separately K_1 and K_2, are constrained to be equidistant from the ends of the metal rod. The centre of mass G can be located by balancing the pendulum on an external knife edge. Due to the difference in mass between the metal and wooden weights W_1 and W_2, G is not at the centre of the rod, and the distances h_1 and h_2 from G to the suspension points O_1 and O_2 at the knife edges K_1 and K_2 are not equal. Fine adjustments in the position of G, and thus in h_1 and h_2, can be made by moving the small metal cylinder w.



In Fig. 1, we consider the force of gravity to be acting at G. If h_i is the distance to G from the suspension point O_i at the knife edge K_i, the equation of motion of the pendulum is

$$I_i\ddot{\theta} = -Mgh_i \sin\theta$$

where I_i is the moment of inertia of the pendulum about the suspension point O_i, and i can be 1 or 2. Comparing to the equation of motion for a simple pendulum

$$Ml_i^{\,2}\ddot{\theta} = -Mgl_i \sin\theta$$

we see that the two equations of motion are the same if we take

$$Mgh_i / I_i = g / l_i \qquad (1)$$

It is convenient to define the radius of gyration of a compound pendulum such that if all its mass M were at a distance from O_i, the moment of inertia about O_i would be I_i , which we do by writing

$$I_i = Mk_i^{\,2}$$

Inserting this definition into equation (1) shows that

$$k_i^{\,2} = h_i l_i \qquad (2)$$

If I_G is the moment of inertia of the pendulum about its centre of mass G, we can also define the radius of gyration about the centre of mass by
writing

$$I_G = Mk_G^{\,2}$$

The parallel axis theorem gives us

$$k_i^{\,2} = h_i^{\,2} + k_G^{\,2}$$

so that, using (2), we have

$$l_i = \frac{h_i^{\,2} + k_G^{\,2}}{h_i}$$

The period of the pendulum from either suspension point is then

$$T_i = 2\pi\sqrt{\frac{l_i}{g}} = 2\pi\sqrt{\frac{h_i^2 + k_G^2}{g h_i}} \qquad (3)$$

Squaring (3), one can show that

$$h_1 T_1^2 - h_2 T_2^2 = \frac{4\pi^2}{g}\left(h_1^2 - h_2^2\right)$$

and in turn,

$$\frac{4\pi^2}{g} = \frac{h_1 T_1^2 - h_2 T_2^2}{(h_1 + h_2)(h_1 - h_2)} = \frac{T_1^2 + T_2^2}{2(h_1 + h_2)} + \frac{T_1^2 - T_2^2}{2(h_1 - h_2)}$$

which allows us to calculate g,

$$g = 8\pi^2\left[\frac{T_1^2 + T_2^2}{h_1 + h_2} + \frac{T_1^2 - T_2^2}{h_1 - h_2}\right]^{-1} \qquad (4)$$

Applications

Pendulums are used to regulate pendulum clocks, and are used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length. The problem with using pendulums proved to be in measuring their length.
A fine wire suspending a metal sphere approximates a simple pendulum, but the wire changes length, due to the varying tension needed to support the swinging pendulum. In addition, small amounts of angular momentum tend to creep in, and the centre of mass of the sphere can be hard to locate unless the sphere has highly uniform density. With a compound pendulum, there is a point called the centre of oscillation, a distance l from the suspension point along a line through the centre of mass, where l is the length of a simple pendulum with the same period. When suspended from the centre of oscillation, the compound pendulum will have the same period as when suspended from the original suspension point. The centre of oscillation can be located by suspending from various points and measuring the periods, but it is difficult to get an exact match in the period, so again there is uncertainty in the value of l.

With equation (4), derived by Friedrich Bessel in 1826, the situation is improved. $h_1 + h_2$, being the distance between the knife edges, can be measured accurately. $h_1 – h_2$ is more difficult to measure accurately, because accurate location of the centre of mass G is difficult. However, if T_1 and T_2 are very nearly equal, the second term in (4) is quite small compared to the first, and $h_1 – h_2$ does not have to be known as accurately as $h_1 + h_2$.
Kater's pendulum was used as a gravimeter to measure the local acceleration of gravity with greater accuracy than an ordinary pendulum, because it avoids having to measure l. It was popular from its invention in 1817 until the 1950's, when began to be possible to directly measure the acceleration of gravity during free fall using a Michelson interferometer. Such an absolute gravimeter is not particularly portable, but it can be used to accurately calibrate a relative gravimeter consisting of a mass hanging from a spring adjacent to an accurate length scale. The relative gravimeter can then be carried to any location where it is desired to measure the acceleration of gravity.

Procedure

Real Lab

- Shift the weight W_1 to one end of katers pendulum and fix it. Fix the knife edge K_1 just below it.
- Keep the knife edge K_2 at the other end and fix the wooden weight W_2 symmetrical to other end. Keep the small weight 'w' near to centre.
- Suspend the pendulum about the knife edge 1 and take the time for about 10 oscillations. Note down the time t_1 using a stopwatch and calculate its time period using equation $T_1=t_1/10$.
- Now suspend about knife edge K_2 by inverting the pendulum and note the time t_2 for 10 oscillations. Calculate T_2 also.
- If $T2 \neq T1$, adjust the position of knife edge K_2 so that $T2 \approx T1$.
- Balance the pendulum on a sharp wedge and mark the position of its centre of gravity. Measure the distance of the knife-edge K_1 as h_1 and that of K_2 as h_2 from the centre of gravity.

Observations and Calculations

Table:3 TodetermineT$_1$ andT$_2$:

Knife edge	Time for 10 oscillations			Time period (s)
	1(s)	2(s)	Mean t(s)	
K_1				
K_2				

Distance of K_1 from C.G,h_1 = ...m.
Distance of K_2 from C.G,h_2 =..............m.

$$g = \frac{8\pi^2}{\frac{T_1^2+T_2^2}{h_1+h_2}+\frac{T_1^2-T_2^2}{h_1-h_2}}$$

Acceleration due to gravity, g =..............ms^{-2}.

Question Bank:

Part – A

1. Define moment of inertia.
2. What is the role of moment of inertia in rotational motion?
3. What is meant by radius of gyration? What is its physical significance?
4. State the theorem of principle axis and prove it.
5. State the theorem of parallel axis and prove it.
6. Write the analogous parameters in translational and rotational motion.
7. Calculate moment of inertia for a plane laminar body.
8. Evaluate the moment of inertia of a uniform circular disc about a diameter.
9. Evaluate the moment of inertia of a solid sphere about a tangent.
10. A solid cylinder of mss 2kg and radius 0.17m is rotating about its axis. Calculate moment of inertia of the cylinder.
    Solution: $I = MR^2 /2$; $I = (2 \times (0.17)2)/2 = 0.0289$ kg-m^2 .
11. Define 'centre of suspension' and 'centre of oscillation'.
12. What is a compound pendulum? Write an expression for its time period.
13. Distinguish between a simple and compound pendulum.
14. Give advantages of compound pendulum over simple pendulum in determining the value of g.
15. What is Kater's reversible pendulum?

Part – B

1. Define moment of inertia. What is its physical significance? Show that, in rotator motion, moment of inertia plays the same role as mass does in linear motion.
2. Calculate the moment of inertia of a circular disc (i) about an axis through its centre and perpendicular to its plane (ii) about a diameter.
3. Find the moment of inertia of a solid cylinder (i) about its own axis of cylindrical symmetry (ii) about the axis through its centre and perpendicular to its axis of cylindrical symmetry.
4. Calculate the moment of inertia of a solid sphere (i) about a diameter (ii) about a tangent.
5. Find the moment of inertia of a hollow sphere (i) about a diameter (ii) about a tangent.
6. Derive an expression for the time period of a compound pendulum and establish the inter-changeability of the centre of oscillation and suspension.
7. Describe Kater's reversible pendulum and determining acceleration due to gravity.

II. Statics and Hydrostatics

Statics:

Statics is the study of internal and external forces in a structure. **Statics** is the branch of mechanics that deals with bodies at rest. The study of systems in which momentum does not change is called **statics**, whereas dynamics involves the study of changes in momentum.

It is also defined as the branch of mechanics that is concerned with the analysis of loads (force and torque, or "moment") acting on physical systems that do not experience an acceleration (a=0), but rather, are in static equilibrium with their environment. The application of Newton's second law to a system gives:

$$F = ma$$

Where bold font indicates a vector that has magnitude and direction. F is the total of the forces acting on the system, m is the mass of the system and a is the acceleration of the system.
The summation of forces will give the direction and the magnitude of the acceleration and will be inversely proportional to the mass. The assumption of static equilibrium of a = 0 leads to:

$$F = 0$$

The summation of forces, one of which might be unknown, allows that unknown to be found. So when in static equilibrium, the acceleration of the system is zero and the system is either at rest, or its center of mass moves at constant velocity. Likewise the application of the assumption of zero acceleration to the summation of moments acting on the system leads to:

$$M = I\alpha = 0$$

Here, M is the summation of all moments acting on the system, I is the moment of inertia of the mass and $\alpha = 0$ the angular acceleration of the system, which when assumed to be zero leads to:

$$M = 0$$

The summation of moments, one of which might be unknown, allows that unknown to be found. These two equations together, can be applied to solve for as many as two loads (forces and moments) acting on the system.
From Newton's first law, this implies that the net force and net torque on every part of the system is zero. The net forces equaling zero is known as the first condition for equilibrium, and the net torque equaling zero is known as the second condition for equilibrium.

Centre of gravity:

Centre of gravity, in physics, an imaginary point in a body of matter where, for convenience in certain calculations, the total weight of the body may be thought to be concentrated. The concept is sometimes useful in designing static structures (e.g., buildings and bridges) or in predicting the behaviour of a moving body when it is acted on by gravity.

The center of gravity (not to be confused with center of mass) of a body is a point where the weight of the body acts and total gravitational torque on the body is zero.
To determine the center of gravity (CG) of an irregularly shaped body (say a cardboard), we take a narrow tipped object (say a sharp pencil). Now by trial and error, we can locate a point **G** on the cardboard, where it is balanced on the tip of the pencil. This point of balance is

the center of gravity of the cardboard. The tip of the pencil at **G** provides the normal reaction **R** to the total weight **mg** of the cardboard. The cardboard is in translational equilibrium, as **R** = **mg**.

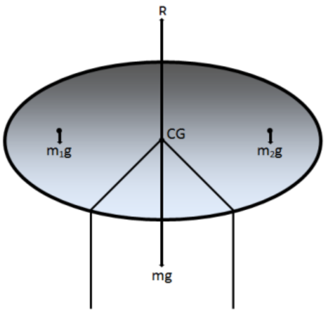


Fig. 2.1

Also, the cardboard is in rotational equilibrium because if it were not so, the cardboard would tilt and fall due to an unbalanced torque. Force of gravity like m_1g, m_2g, m_3g... etc. act on individual particles of the cardboard. They make up the torque on the cardboard. For its particle of mass $\mathbf{m_i}$, the force of gravity is $\mathbf{m_i\, g}$. If $\mathbf{r_i}$ is the position vector of this particle from CG of the cardboard, the torque about the CG is

$$\vec{\tau}_i = \vec{r}_i \times m_i\ \vec{g}$$

CG of the cardboard is so located that the total torque on it to forces of gravity on all the particles is zero. Thus, total torque is:

$$\vec{\tau}\ = \sum_{i=1}^{n} \vec{\tau}_i = \sum_{i=1}^{n} \vec{r}_i \times\ m_i\, \vec{g} = 0$$

As g is a non-zero value and same for all particles of the body, so the above equation can be written as

$$\sum_{i=1}^{n} m_i \vec{r}_i = 0$$

This is the condition, when center of mass of the body lies at the origin. As position vectors are taken with respect to the CG, the center of gravity of the body coincides with the center of mass of the body.

However, if the body is so extended that **g** varies from part to part of the body, then the center of gravity will not coincide with center of mass of the body. For a body of small size, having uniform density throughout, the CG of the body coincides with the center of mass. In case of solid sphere, both CG and center of mass lie at center of the sphere. For a body of very large dimensions and having non-uniform density, the center of gravity does not coincide with the center of mass.

Uniform solid tetrahedron, pyramid and cone

Definition. A median of a tetrahedron is a line from a vertex to the centroid of the opposite face.
Theorem I. The four medians of a tetrahedron are concurrent at a point ¾ of the way from a vertex to the centroid of the opposite face.
Theorem II. The centre of mass of a uniform solid tetrahedron is at the meet of the medians.
Theorem I can be derived by a similar vector geometric argument used for the plane triangle. It is slightly more challenging than for the plane triangle, and it is left as an exercise for the reader. I draw two diagrams (figure 1). One shows the point C1 that is ¾ of the way from the vertex A to the centroid of the opposite face. The other shows the point C2 that is ¾ of the way from the vertex B to the centroid of its opposite face. You should be able to show that
C1 = (A + B + D)/4.

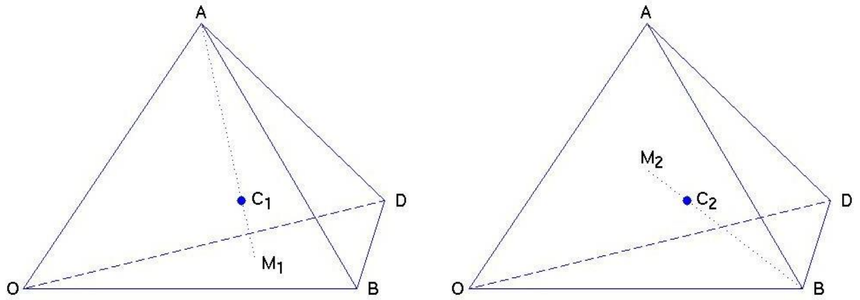


Fig. 2.2

In fact this suffices to prove Theorem I, because, from the symmetry between **A**, **B** and **D**, one is bound to arrive at the same expression for the three-quarter way mark on any of the four medians. But for reassurance you should try to show, from the second figure, that

$$\mathbf{C}_2 = (\mathbf{A} + \mathbf{B} + \mathbf{D})/4$$

The argument for Theorem II is easy, and is similar to the corresponding argument for plane triangles.

Pyramid

A right pyramid whose base is a regular polygon (for example, a square) can be considered to be made up of several tetrahedra stuck together. Therefore, the centre of mass is 3/4 of the way from the vertex to the mid-point of the base.

Cone

A right circular cone is just a special case of a regular pyramid in which the base is a polygon with an infinite number of infinitesimal sides. Therefore, the centre of mass of a uniform right circular cone is 3/4 of the way from the vertex to the centre of the base. We can also find the position of the centre of mass of a solid right circular cone by calculus. We can find its volume by calculus, too, but we'll suppose that we already know, from the theorem of Pappus, that the volume is $\frac{1}{3} \times$ base $\times$ height.

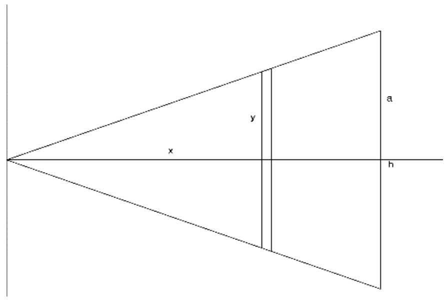


Fig. 2.3

Consider the cone in figure 2.3, generated by rotating the line $y = ax/h$ (between $x = 0$ and $x = h$) through 360° about the x axis. The radius of the elemental slice of thickness δx at x is ax/h. Its volume is $\pi a^2 x^2 \delta x/ h$. Since the volume of the entire cone is $\pi a^2 h/3$, the mass of the slice is

$$M \text{ x } \pi a^2 x^2 \delta x/ h^2$$

$$\mathrm{M} \times \frac{\pi \mathrm{a}^2 \mathrm{x}^2 \delta \mathrm{x}}{\mathrm{h}^2} \div \frac{\pi \mathrm{a}^2 \mathrm{h}}{3} = \frac{3\mathrm{Mx}^2 \delta \mathrm{x}}{\mathrm{h}^3}$$

where M is the total mass of the cone. The first moment of mass of the elemental slice with respect to the y axis is $3Mx^3\delta x/h^3$.

The position of the centre of mass is therefore

$$\bar{x} = \frac{3}{h^3}\int_0^h x^3 dx = \frac{3}{4}h$$

Hollow cone:

The surface of a hollow cone can be considered to be made up of an infinite number of infinitesimally slender isosceles triangles, and therefore the centre of mass of a hollow cone (without base) is 2/3 of the way from the vertex to the midpoint of the base.

Uniform solid hemisphere:

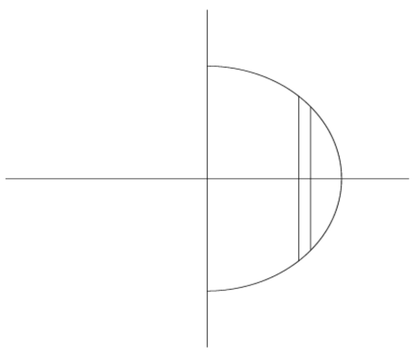


Fig. 2.4

Above Figure will serve. The argument is exactly the same as for the cone. The volume of the elemental slice is $\pi y^2\delta x = \pi\left(a^2 - x^2\right)\delta x$, and the volume of the hemisphere is $2\pi a^3/3$, so the mass of the slice is

$$M \times \pi(a^2 - x^2)\delta x \div \left(\frac{2\pi a}{3}\right) = \frac{3M(a^2 - x^2)\delta x}{2a^3}$$

where M is the mass of the hemisphere. The first moment of mass of the elemental slice is x times this, so the position of the centre of mass is

$$\bar{x} = \frac{3}{2a^3}\int_0^a x\,(a^2 - x^2)dx = \frac{3a}{8}$$

We may note to begin with that we would expect the centre of mass to be further from the base than for a uniform solid hemisphere. Again, figure I.4 will serve. The area of the elemental annulus is $2\pi a\delta x$ (NOT $2\pi y\delta x$!) and the area of the hemisphere is $2\pi a^2$. Therefore the mass of the elemental annulus is

$$M \times 2\pi a\delta x \div (2\pi a^2) = M\delta x / a$$

The first moment of mass of the annulus is x times this, so the position of the centre of mass is

$$\bar{x} = \int_0^a \frac{x}{a}dx = \frac{a}{2}$$

Triangular lamina: 2/3 of way from vertex to midpoint of opposite side Solid Tetrahedron, Pyramid, Cone: 3/4 of way from vertex to centroid of opposite face. Hollow cone: 2/3 of way from vertex to midpoint of base. Solid hemisphere: $3a/8$ Hollow hemisphere: $a/2$

Centre of Pressure

When the fluid is in static condition, there will not be any relative motion between adjacent fluid layers. The velocity gradient as well as shear stress will be zero. The forces acting on fluid particles will be due to pressure of fluid normal to the surface and due to gravity.

Consider a plane surface of arbitrary shape immersed vertically in a static mass of fluid as shown in Fig. 3.1.

Let,

C = Centre of gravity
P= Centre of pressure
$\bar{x}$ = depth of centre of gravity from free liquid surface 1-1
$\bar{h}$ = depth of centre of pressure from free surface of liquid 1-1

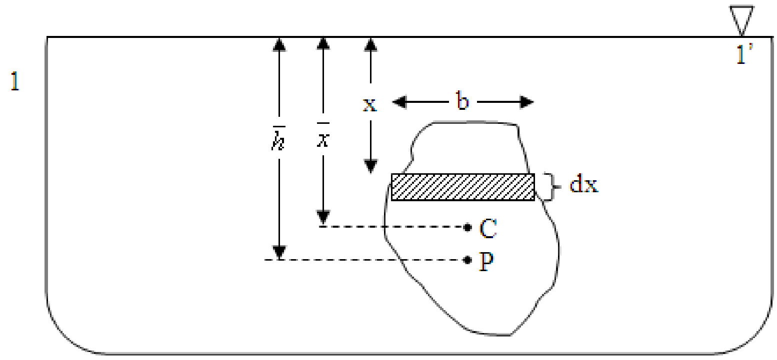


Fig. 2.5 A plane surface of arbitrary shape immersed vertically in a static fluid

The distance of centre of gravity from free surface is $\bar{x}$. Let P be the centre of pressure at which the resultant pressure on the rectangular plate acts.
Consider an elementary strip of dx thickness and width b at the distance x from the free surface of liquid. Let pressure acting on the strip is p. If density of the liquid is ρ, then
Then total pressure force (F) acting on the elementary strip = p b dx

$$= \rho\, g\, x\, b\, dx$$

Total pressure force acting on whole area $= \rho g \int x\, b\, dx$

Since $\int x\, b\, dx$ = moment of surface area about the free liquid surface, we can take its value surface as $\bar{x}$.A
Then total pressure force, F = $\rho g \bar{x}$ .A. (where A is area of plate)
At point C pressure = $\rho g \bar{x}$
Total pressure force is equal to total area multiplied by the pressure at the centre of gravity of the plate surface immersed in the liquid.

Location of Centre of Pressure

The pressure on the immersed surface increases with depth. As shown in the following figure the pressure will be minimum at the top and maximum at the bottom of immersed plane.

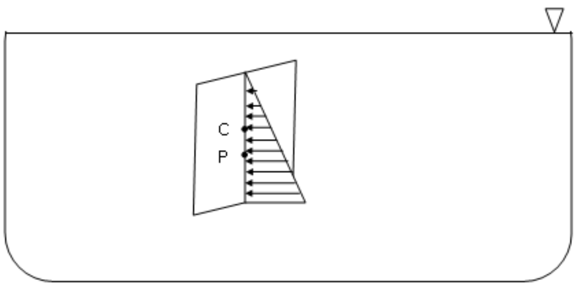


Fig. 2.6 Location of centre of pressure

Suppose centre of pressure (P) is $\bar{h}$ from the free surface of the liquid;
Thus the resultant pressure will act at a point P much below the centre of gravity. Point P is known as the centre of pressure at which the resultant pressure acts on the immersed surface.

Total pressure force acting on the elementary strip = p b dx
= ρ g x b dx

Moment of pressure force about free liquid surface = ρgx^2 b dx
Total moment of pressure force for entire area = ρg$\int x^2 b\, dx$
Since $\int x^2\ b\ dx$= moment of inertia of entire surface about free surface 1-1 = I_o
Total moment of pressure force for entire area = ρgI_o
The sum of moment of pressure force = F$\bar{h}$
Equating equation (i) & (ii):

$F\bar{h} = \rho g I_o$

$\rho g A\, \bar{x}\, \bar{h} = \rho g I_o$

$$\bar{h} = \frac{I_o}{A\bar{x}}$$

From theorem of parallel axis for moment of inertia we have,

$$I_o = I_c + A\bar{x}^2$$

Here,

I_c = moment of inertia of area about an axis passing through centre of gravity.

$\bar{x}$ = distance of centre of gravity from free liquid surface

Placing value of I_o into (iii),

$$\bar{h} = \frac{I_c}{A\bar{x}} + \bar{x}$$

Where, I_c is moment of inertia of the immersed figure;

For rectangular surface $I_c = \frac{bd^3}{12}$ (b= base of rectangle, d= height or depth).

Laws of Floatation:

A body floating freely in a fluid, must obey the following laws known as laws of floatation:

1. A body floats only if its weight is equal to the weight of fluid displaced by its immersed part.
2. For the body to float in upright position, the centre of gravity of the floating body and the centre of buoyancy of the fluid displaced by the immersed part of the body must lie on the same straight line.
   1st law stated above is necessary for the body to float whereas the 2nd law stated above is necessary for the body to float in upright position.

What are laws of floatation?

Floatation depends upon the density. If an object has density less than the density of water, it floats. Principle of floatation is stated by the Archimedes. This article deals with what is floatation, laws of floatation, its applications and examples.

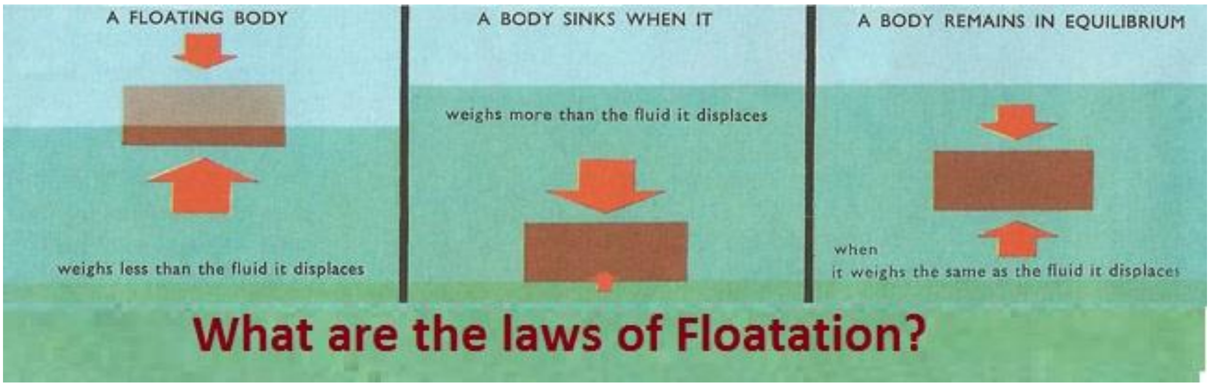


Fig. 2.6

Archimedes, the Ancient Greek scientist first stated the principle of floatation. According to him, all the objects placed in a liquid experience an upward force which allows the body to float if it displaces water with the weight equal to the weight of the body. This upward force is known as **buoyant force** and the law is known as **the law of buoyancy.** Mainly, floatation depends upon the density. If an object has a density less than the density of water, it floats. Like, leaf of a plant floats on the water because the density of leaf is less than the density of water. A stone thrown in water sinks because the density of stone is more than the density of water.

Have you ever thought that a ship weighing several tons floats while a needle sinks? This can be explained as follows: A ship is made up of iron and steel, but it has a lot of space filled with air. This causes the ship to displace water with a weight equal to the weight of the ship. On the other hand, the needle displaces more water than its weight and hence it sinks.

What is Archimedes Principle?

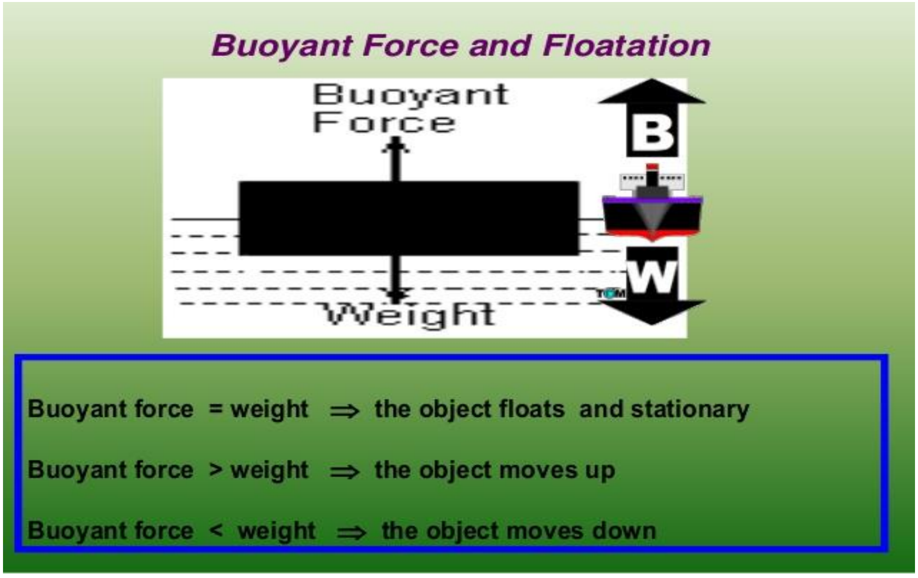


Fig. 2.7

When a body is immersed partly or wholly in a liquid, there is an apparent loss in the weight of the body which is equal to the weight of liquid displaced by the body.

Laws of Floatation

A body floats in a liquid if:
- Density of the material of the body is less than or equal to the density of the liquid.
- If density of material of body is equal to density of liquid, the body floats fully submerged in liquid in neutral equilibrium.
- When body floats in neutral equilibrium, the weight of the body is equal to the weight of displaced liquid.
- The centre of gravity of the body and centre of gravity of the displaced liquid should be in one vertical line.

Centre of Buoyancy: The centre of gravity of the liquid displaced by a body is known as centre of buoyancy.

Meta centre: When a floating body is slightly tilted from equilibrium position, the centre of buoyancy shifts. The point at which the vertical line passing through the new position of centre of buoyancy meets with the initial line is called meta centre.

What are the conditions for stable equilibrium for floating body?
a. The meta centre must always be higher than the centre gravity of the body.
b. The line joining the centre of gravity of the body and centre of floatation should be vertical.
We know density is mass per unit volume. Its S.I unit is kg/m3 and relative density is density of material per density of water at four degree Celsius. Relative density is measured by **Hydrometer.**
- The density of sea water is more than that of normal water. That is why it is easier to swim in seawater.
- When ice floats in water, its 1/10th the part remain outside the water.
- If ice floating in water in a vessel melts, the level of water in the vessel does not change.
- Purity of milk is measured by **lactometer.**

From the article we have learnt that what is floatation, how can we swim, why some objects floats instead of sinking, what are the laws of floatation etc.

Definition of *metacenter*
: the point of intersection of the vertical through the center of buoyancy of a floating body with the vertical through the new center of buoyancy when the body is displaced

Illustration of *metacenter*

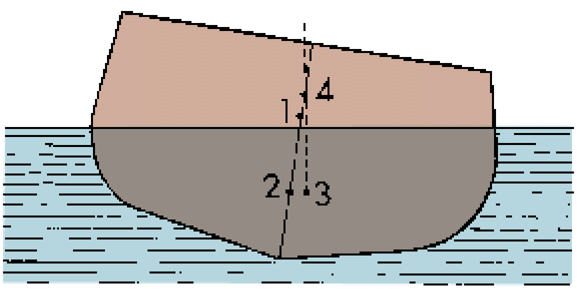


Fig. 2.8

metacenter: *1* center of gravity, *2* center of buoyancy, *3* new center of buoyancy when floating body is displaced, *4* point of intersection

The **metacentric height** (**GM**) is a measurement of the initial static stability of a floating body. It is calculated as the distance between the centre of gravity of a ship and its metacentre. A larger metacentric height implies greater initial stability against overturning. The metacentric height also influences the natural period of rolling of a hull, with very large metacentric heights being associated with shorter periods of roll which are uncomfortable for passengers. Hence, a sufficiently, but not excessively, high metacentric height is considered ideal for passenger ships.

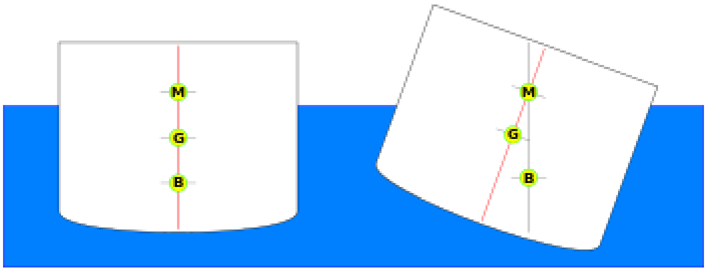


Fig. 2.9

Determination of metacentric height of a ship model:

Apparatus used for determine metacentric height of a ship are Water bulb, Metacentric height apparatus and Scale or measuring tube.

Concepts:
Metacenter:

When a floating body is given a small displacement it will rotate about a point, so the point at which the body rotates is called as the Metacenter.
“OR”
The intersection of the lines passing through the original center of buoyancy and center of gravity of the body and the vertical line through the new center of buoyancy.

Metacentric height:

The distance between center of gravity of a floating body and Metacenter is called as Metacentric height.

Why to find Metacentric height?

It is necessary for the stability of a floating body, If metacenter is above center of gravity body will be stable because the restoring couple produced will shift the body to its original position.

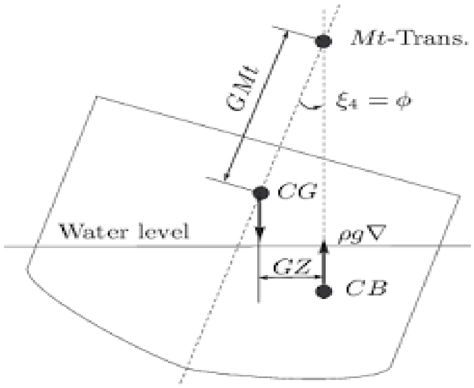


Fig. 2.10

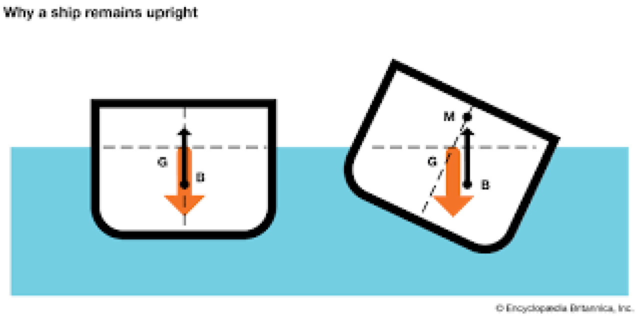


Fig. 2.11

Center of buoyancy:

The point though which the force of buoyancy is supposed to pass is called as the center of buoyancy.
"OR"
The center of area of the immersed portion of a body is called its center of buoyancy.

Procedure:

1. First of all I adjust the movable weight along the vertical rod at a certain position and measured the distance of center of gravity by measuring tape.
2. Then I brought the body in the water tube and changed the horizontal moving load distance first towards right.
3. The piston tilted and suspended rod gave the angle of head, I noted the angle for respective displacements.
4. I did the same procedure for movable mass by changing its position towards left.
5. Then I took the body from water tube and find another center of gravity by changing the position of vertically moving load.
6. I again brought the body in the water tube and find the angle of head by first keeping the movable load towards right and then towards left.
7. I repeated the above procedure for another center of gravity.
8. I calculated the metacentric height by the following formula:

MH = w * d / W * tanØ

Where
MH = Metacentric height
w = Horizontally movable mass
d = Distance of movable mass at right or left of center
W = Mass of assemble position
Ø = Respective angle of heel

Table 2.1: observation

S.No.	W kg	W1 Kg	W2 kg	X1 cm	X2 cm	W1X1 kgcm	W2X2 kgcm	W1X1-W2X2 Kgcm	θ	Tan θ	W tan θ	GM cm
1.												
2.												
3.												

Question Bank:

Part A

1. Define centre of gravity and centroid.
2. Write the centroid of solid tetrahedron, solid and hollow hemisphere.
3. Define centre of pressure.
4. State the Principle of Archimedes
5. State the law of floatation.
6. Define centre of buoyancy.
7. What is metacentre and metacentric height?
8. Find the condition for stability of equilibrium of a floating body.
9. Define atmospheric pressure.
10. What is barometer?

Part B

1. Find the position of centre of gravity of a solid tetrahedron.
2. Find the position of centre of gravity of solid and hollow hemisphere.
3. Find the location of the centroid of the volume of the cone.
4. Determine the position of centre of pressure of a rectangular lamina immersed vertically (a) with one edge coinciding with the free surface (b) with one edge parallel to the free surface and at a depth c below the free surface.
5. Derive an expression for the metacentric height of a ship and give the experimental method for its determination.
6. Discuss how the atmospheric pressure changes with altitude above the surface of the earth.
7. State the Principle of Archimedes and define clearly the terms (i) Centre of buoyancy (ii) Metacentre and (iii) Metacentric Height. Discuss in detail the conditions for the stable equilibrium of a floating body, with particular reference to a floating ship

III. Frame of Reference

Laws of Mechanics:

The following are the fundamental laws of mechanics:
1.Newton’s first law
2.Newton’s second law
3.newton’s third law
4.Newton’s law of gravitation
5.Law of transmissibility of forces, and
6.Parallelogram law of forces

1.Newton’s first law

It states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it.
This leads to the definition of force as the external agency which changes or tends to change the state of rest or uniform linear motion of the body.

2.Newton’s second law

It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it.
Thus according to this law,
Force ∞ mass x acceleration

$$F = ma$$

3.Newton’s third law

It states that for every action ther is an equal and opposite reaction.
Consider the two bodies in contact with each other. Let one body applies a force F on another. According to this law the second body develops a reactive force R which is equal in magnitude to force F and acts in the line same as F but in the opposite direction.

4.Newton’s law of gravitation

Everybody attracts the other body. The force of attraction between any two bodies is dirctly proportional to their masses and inversely proportional to the square of the distance between them. According to this law the force of attraction between the bodies of mass m1 and mass m2 at a distance d as shown in fig. is
$F=Gm_1m_2/d^2$

5.Law of Transmissibility of force

According to this law the state of rest or motion of the rigid body is unaltered if a force acting on the body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.

6.Parallelogram law of forces

The parallelogram law of forces enables us to determine the single force called resultant force.
This law states that if two forces acting simultaneously on a body at a point are presented in magnitude and direction by the two adjacent sides of a parallelogram, their resultant is

represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces.

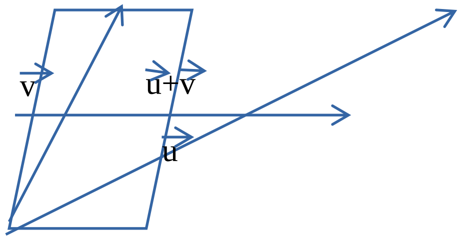


Fig. 3.1

What is a frame of reference?
We have learned about velocity, acceleration, and displacement. But all these quantities need a frame of reference from which they are measured.

In physics, a frame of reference consists of an abstract coordinate system and the set of physical reference points that uniquely fix the coordinate system and standardize measurements within that frame.

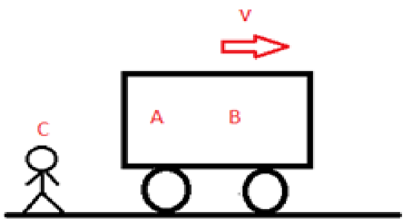


Fig. 3.2

If we ask A what velocity of B is, he will say it is at rest. But if we ask the same question to C, he will say B is moving with a velocity V in the positive X direction. So we can see before specifying the velocity we have to specify in which frame we are or in simple terms, we need to define a frame of reference.

Types of frame of reference
Once we have chosen our reference, they can be of two types:

- Inertial Frame of Reference
- Non inertial Frame of Reference

Inertial Frame of Reference

An inertial frame of reference is a frame where Newton's law holds true. That means if no external force is acting on a body it will stay at rest or remain in uniform motion. Suppose a body is kept on the surface of the earth, for a person on earth it is at rest while for a person on the moon it is in motion so which is my inertial frame here?

Actually, the term inertial frame is relative i.e. first we assume a reference frame to be the inertial frame of reference. So a more general definition of an inertial frame would be: Inertial frame is at rest or moves with constant velocity with respect to my assumed inertial reference frame.

Inertial frame of reference- the systems in which the Newton's First law of motion holds good are called inertial frame of reference.

Example: moving in a car on cruise control
sitting at your desk

Non-inertial Frame of Reference

Now we can define non-inertial frame as a frame which is accelerated with respect to the assumed inertial frame of reference. Newton's law will not hold true in these frames. So in the above example if I assume earth to be an inertial reference frame the moon becomes a non-inertial reference frame as it is in accelerated motion with respect to earth. But if we want to make Newton's law hold here we need to take some mysterious forces also known as pseudo forces.

Frame of reference:

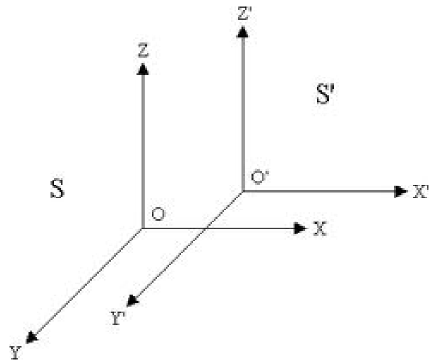


Fig. 3.3

A system of co-ordinate axes which defines the position of a particle in two- or three-dimensional space is called a *frame of reference*. The simplest frame of reference is the familiar Cartesian system of co-ordinates, in which the position of the particle is specified by its three co-ordinates x,y,z, along the three perpendicular axes. In fg 1.2 we have indicated two observers O and O' and a particle P. These observers use frames of reference XYZ and X'Y'Z', respectively. If O and O' are at rest, they will observe the same motion of P. But if O and O' are in relative motion, their observation of the motion of P will be different.
Unaccelerated reference frames in uniform motion of translation relative to one another are called Galilean frames or inertial frames.
Accelerated frames are called non-inertial frames.

Simulation for Frame of Reference

(i) <u>Reference frame is at rest</u>: -

The acceleration of the mass will be, say, $\vec{a}_{rest}$.

Therefore, the force on it will be $\vec{F}_{rest}$.

We will reason that

$$\vec{F}_{rest} = m\vec{a}_{rest}$$

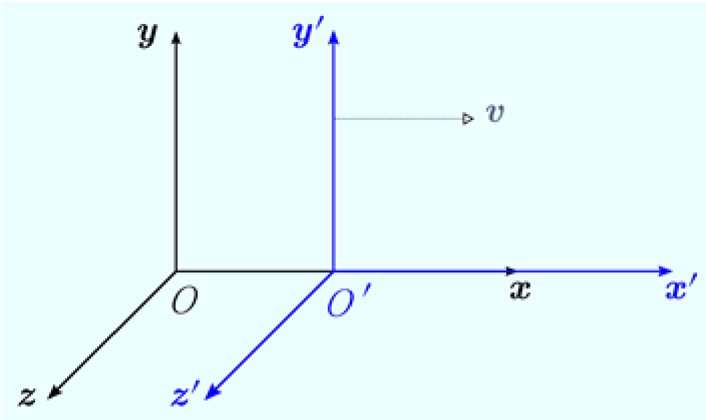


Fig. 3.4

(ii) <u>Reference frame starts moving with constant velocity vector-v</u>: -
The acceleration of frame = $\vec{a} = 0$
Thus, acceleration of mass m relative to frame is given by
$\vec{a}_{inertial} = \vec{a}_{rest} - \vec{a} = \vec{a}_{rest}$
Force on it will be $\vec{F}$inertial and we will reason that
$\vec{F}_{inertial} = m\vec{a}_{inertial} = m\vec{a}_{rest} = \vec{F}_{rest}$

(iii) <u>Reference frame moves with constant acceleration</u>:-
Let the acceleration of frame be $\vec{a}_{frame}$.
Thus, acceleration of mass relative to frame will be $\vec{a}_{rel}$.
$\vec{a}_{rel} = \vec{a}_{inertial} - \vec{a}_{frame} = \vec{a}_{rest} - m\vec{a}_{frame}$
Let there be force $\vec{F}$ frame on mass we will reason, that
$\vec{F}_{frame} = m\vec{a}_{rel} = m\vec{a}_{rest} - m\vec{a}_{frame}$
$= \vec{F}_{rest} + m(-\vec{a}_{frame}) = \vec{f}_{rest} + \vec{F}_{pseudo}$
We see that the force is not the same as that in the inertial frames.
Therefore, we postulate that under observation from an accelerated reference frame we substitute the inertial forces on the body with those same initial forces plus an additional force which numerically equal to the mass of the body under observation times the acceleration of the frame taken in the opposite direction. This force we call as pseudo force.

Newtonian Relativity:

The Newtonian principle of relativity may be stated as “Absolute motion, which is the translation of a body from one absolute place to another absolute place, can never be detected. Translatory motion can be perceived only in the form of motion relative to other material bodies”. This implies that if we are drifting along at a uniform speed in a closed spaceship, all the phenomena observed and all the experiments performed inside the ship will appear to be the same as if the ship were not in motion. This means that the fundamental physical laws and principles are identical in all inertial frames of reference. This is the concept of Newtonian relativity.

Galilean transformation equations:

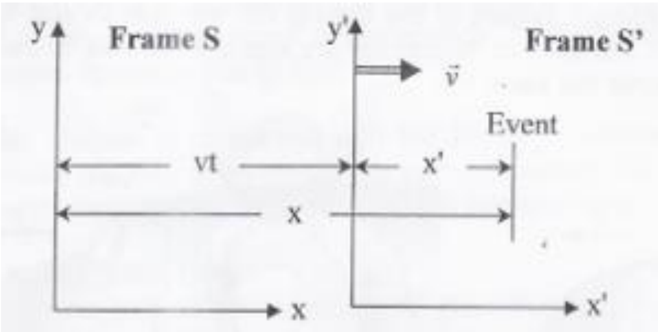


Fig. 3.5

Let S and S’ be two inertial frames fig3.5.

Let S be at rest and S’ move with uniform velocity v along the positive X direction. We assume that v<<c. Let the origins of the two frames coincide at t-=0. Suppose some event occurs at the point P. The observer O in frame S determines the position of the event by the coordinates x, y, z. The observer O’ in frame S’ determines the position of the even by the coordinates x’, y’, z’. there is no relative motion between S and S’ along the axes of Y and Z. Hence, we have y=y’ and z=z’. let the time proceed at the same rate in both frames.
The distance moved by S’ in the positive X-direction in time t=vt. So, the X coordinates of the two frames differ by vt. Hence x’=x-vt.

Then the transformation equations from S to S’ are given by

$$x' = x - vt \quad ....(1)$$
$$y' = y \quad ....(2)$$
$$z' = z \quad ....(3)$$
$$t' = t \quad ......(4)$$

Conservation of momentum

The law of conservation of momentum states that, in the absence of outside forces, the total momentum of objects that interact does not change.
The amount of momentum is the same before and after they interact.
The total momentum of any group of objects remains the same, or is conserved, unless outside forces act on the objects.
Friction is an example of an outside force.

The **law of conservation of linear momentum** states that if no external forces act on the system of two colliding objects, then the vector sum of the linear momentum of each body remains constant and is not affected by their mutual interaction.

Alternatively, it states that if net external force acting on a system is zero, the total momentum of the system remains constant.
Proof:
Let us consider a particle of mass ‘m’ and acceleration ‘a’. Then, from 2^{nd} law of motion,

$$F = \frac{dP}{dt}$$

If no external force acts on the body then, F=0,

$$F = \frac{dP}{dt} = 0$$

Therefore, ‘P’ is constant or conserved.

Deduction of Law of Conservation of linear momentum for two colliding bodies:

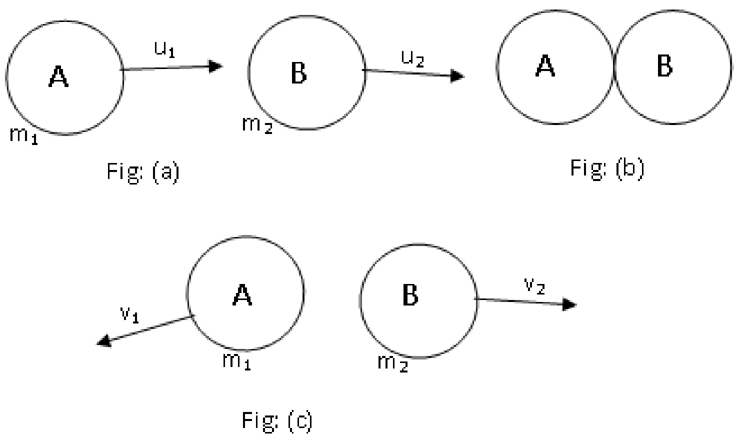


Fig. 3.6

Let us consider two bodies of masses m_1 and m_2 moving in straight line in the same direction with initial velocities u_1 and u_2. They collide for a short time Δt. After collision, they move with velocities v_1 and v_2.
From 2^{nd} law of motion,
Force applied by A on B = Rate of change of momentum of B

$$F_{AB} = (m_2v_2-m_2u_2)/\Delta t$$

Similarly,
Force applied by B on A = Rate of change of momentum of A

$$F_{BA} = (m_1v_1-m_1u_1)/\Delta t$$

From Newton's 3^{rd} law of motion,

$$F_{AB} = -F_{BA}$$

Or, $(m_2v_2-m_2u_2)/\Delta t = -(m_1v_1-m_1u_1)/\Delta t$
Or, $m_2v_2-m_2u_2 = -m1v1+m1u1$
Or, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$
This means the total momentum before collision is equal to total momentum after collision. This proves the principle of co conservation of linear momentum.

Non-inertial frame of reference:

The systems in which the Newton's First law of motion does not holds good are called non-inertial frame of reference.
A frame of reference which is in accelerated motion with respect to an inertial frame is called non-inertial frame of reference.

Fictitious force:

A **fictitious force** (also called a pseudo **force**, d'Alembert **force**, or inertial **force**) is a **force** that appears to act on a mass whose motion is described using a non-inertial frame of reference, such as an accelerating or rotating reference frame.
The fictitious force is an apparent force that acts on an object on a moving frame of reference. This frame of reference could be a car or a rotating frame of reference, like the moon orbiting the Earth.
The reason why this fictitious force occurs in objects is because of inertia, the resistance of an object to acceleration. For example, when a person in a car stops at an intersection, then presses the accelerator, inertia wants the person to be still since the person stopped first before accelerating. This also applies to rotating objects because the moon experiences centripetal acceleration, acceleration that keeps an object rotating around a point. However, inertia wants the object to go in the direction tangent to the circular motion and results in an apparent outward force. If this property did not exist, then the moon would not even form.

Centrifugal force:

In Newtonian mechanics, the centrifugal force is an inertial force that appears to act on all objects when viewed in a rotating frame of reference. It is directed away from an axis passing through the coordinate system's origin and parallel to the axis of rotation.

Coriolis force

The Coriolis force is a pseudo force that operates in all rotating frames. One way to envision it is to imagine a rotating platform (such as a merry-go-round or a phonograph turntable) with a perfectly smooth surface and a smooth block sliding inertially across it. The block, having no (real) forces acting on it, moves in a straight line at constant speed in inertial space. However, the platform rotates under it, so that to an observer on the platform, the block appears to follow a curved trajectory, bending in the opposite direction to the motion of the platform. Since the motion is curved, and hence accelerated, there appears, to the observer, to be a force operating. That pseudoforce is called the Coriolis force.

Derivation of the centrifugal and Coriolis force

Let us start with the principal equation defining angular velocity in three dimensions,

$$\dot{r} = \omega \times r.$$

(This can be derived roughly by considering a centripetal force acting on a particle. Note that this equation applies symmetrically in inertial and rotating reference frames.)

Notice that we can in fact generalise this statement in terms of r for an arbitrary vector aa that is known to be fixed in the rotating body.

Transformation between inertial and rotating frames

Now consider a vector aa, which we can write in Cartesian coordinates (fixed within the body) as

$$a = a_x \hat{\imath} + a_y \hat{\jmath} + a_z \hat{k}.$$

In Newtonian mechanics, scalar quantities must be invariant for any given choice of frame, so we can say

$$\left.\frac{da_x}{dt}\right|_I = \left.\frac{da_x}{dt}\right|_R$$

where I indicates the value is for the inertial frame, and RR that the value is for the rotating frame. Equivalent statements apply for a_y and a_z, of course. Hence, any transformation of aa between frames must be due to changes in the unit vectors of the basis.

Now by the product rule,

$$\left.\frac{da_x}{dt}\right|_I = \frac{d}{dt}(\text{ax}\,\hat{\imath} + \text{ay}\,\hat{\jmath} + \text{az}\,\hat{k})$$

$$= \left(\frac{\text{dax}}{dt}\,\hat{\imath} + \frac{\text{day}}{dt}\,\hat{\jmath} + \frac{\text{daz}}{dt}\,\hat{k}\right) + \left(\text{ax}\,\frac{d\hat{\imath}}{dt} + \text{ay}\,\frac{d\hat{\jmath}}{dt} + \text{az}\,\frac{d\hat{k}}{dt}\right)$$

Using the previous equation for angular velocity, we then have

$$\left.\frac{da_x}{dt}\right|_I = \left(\frac{\text{dax}}{dt}\,\hat{\imath} + \frac{\text{day}}{dt}\,\hat{\jmath} + \frac{\text{daz}}{dt}\,\hat{k}\right) + (\text{ax}\,\omega \times \hat{\imath} + \text{ay}\,\omega \times \hat{\jmath} + \text{az}\,\omega \times \hat{k})$$

$$= \left.\frac{da}{dt}\right|_R + \omega \times R$$

Now consider a position vector on the surface of a rotating body. We can write

$$V\, I = \frac{dr}{dt}\Big|_I = \frac{dr}{dt}\Big|_R + \omega \times R$$

and similarly, for a=v I,

$$\frac{d^2r}{dt^2}\Big|_I = \left(\frac{d}{dt}\Big|_R + \omega \times\right)^2 r$$

$$= \frac{d^2r}{dt^2}\Big|_R + 2\,\omega \times \frac{dr}{dt}\Big|_R + \omega \times (\omega \times r).$$

Forces on body in rotating frame

Now consider a force acting on an object at position r (for example, gravity). Newton's third law states

$$F = m\, \frac{d^2r}{dt^2}\Big|_I$$

And so substituting this into the previous equation for $\frac{d^2r}{dt^2}\Big|_I$ and rearranging we get

$$F_{net} = m\, \frac{d^2r}{dt^2}\Big|_R$$

$$= F - 2m\omega \times v_R - m\omega \times (\omega \times r)$$

$$= F - 2m\omega \times v_R + m\omega^2 r.$$

And here we have it. The second term on the right is the **Coriolis force**, and the third term is the **centrifugal force** (clearly pointing away from the centre of rotation). Any interpretation of the Coriolis and centrifugal forces then follow naturally from this single important equation.

Coriolis effect and its causes:

Coriolis effect is used to describe the Coriolis force experienced by the moving objects such that the force is acting perpendicular to the direction of motion and to the axis of rotation. The earth's rotation is the main cause for Coriolis effect as the earth rotates faster at the equator and near the poles the rotation is sluggish. This is because the rate of change in the diameter of the earth's latitude increases near the poles.

The air currents in the Northern hemisphere bend to the right making the objects deflect to the right whereas in the Southern hemisphere the air currents bend to the left making the objects deflect to the left. The Coriolis effect is noticed only for the motions occurring at large-scale such as movement of air and water in the ocean. Example of Coriolis effect is change in weather patterns.

Question Bank:

Part A

1. What is a frame of reference? Name the types of frame of reference.
2. Discuss the limitations of Newton's laws f motion.
3. Explain what is meant by an inertial frame of reference.
4. What are the other names of inertial frame of reference?
5. Show that all other frames of reference, with constant velocity relative to it, are also inertial frames.
6. What are the characteristics properties and its importance?
7. What do you understand by Galilean transformation and Galilean invariance?
8. What is meant by non- inertial frame of reference?
9. How do you differ non-inertial frame from inertial one?
10. What is fictitious force? Why is this force so called? Write the other names of fictitious force.
11. Can a non-inertial frame of reference serve as an inertial frame of reference? If so, under what condition?
12. What is Coriolis force? Is the centrifugal force fictitious one?
13. What are transformation equations?

Part – B

1. Enunciate Newton's laws of motion and discuss their limitations.
2. Explain in detail how the two laws of motion hold good in inertial frame of reference.
3. Explain what do you understand by Galilean transformation and Galilean invariance? Show that length and acceleration are both invariant to Galilean transformation.
4. Show that position and velocity are not invariant to Galilean transformation.
5. Enunciate the laws of conservation of momentum and energy and show that they are both invariant to Galilean transformation.
6. What is fictitious force? Why is this force so called? Under what conditions does it come into play?
7. What is Coriolis force? Under what conditions does it come into play?
8. Explain in detail if no force acing on a particle in an inertial frame, a force seems to be acting it, as observed in a non-inertial frame, either in linear or circular motion with respect to the inertial frame

IV. Special Theory of Relativity

Michelson -Morley Experiment

Sound waves need a medium through which to travel. In 1864 James Clerk Maxwell showed that light is an electromagnetic wave. Therefore, it was assumed that there is an *ether* which propagates light waves. This ether was assumed to be everywhere and unaffected by matter. This ether could be used to determine an absolute reference frame (with the help of observing how light propagates through the ether).

The Michelson-Morley experiment (circa 1885) was performed to detect the Earth's motion through the ether as follows:

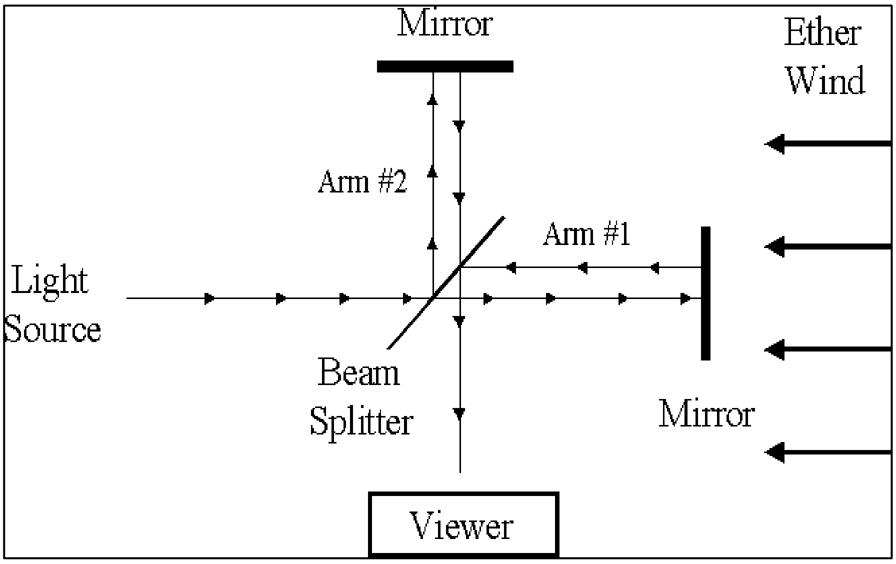


Fig. 4.1

Light beam from the source 'S' is incident at a beam splitter, which is a semi silvered glass plate. The plate splits the beam into two coherent beams and out of them one is transmitted and other one is reflected. The transmitted rays strike the mirror M1 and from there it is reflected back to plate. The reflected beam strikes the mirror M2 and it is also reflected back to plate. The returned beams from mirror M1 and M2 reach the telescope T. The superposition of these two rays produces interference pattern, which are seen through the telescope T.

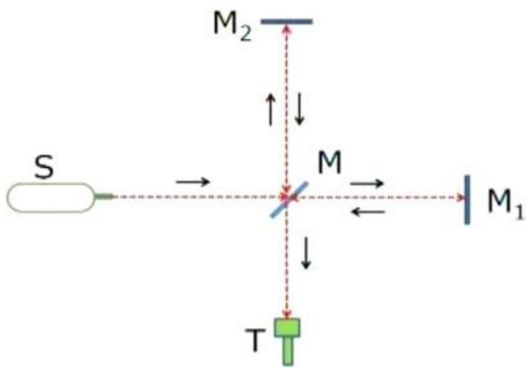
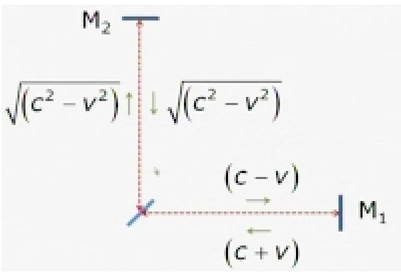


Fig. 4.2

The separation between the P and M_1 and P and M_2 is same and that is equal to 'l' and this separation is called length of the arm. The light will be reflected back from mirrors M_1 and M_2 respectively and will interfere at P. This interference pattern is noticed by Telescope T.



Time taken by the light to travel to mirror M1 and came back to plate P

As the apparatus and the light both are moving in same direction that is when light is going towards M'2. Thus the relative velocity will be c – v. After reflection, the apparatus and the light both are moving in the opposite direction that is when light is going towards P. Thus the relative velocity will be c + v.

$$t_1 = \frac{l}{c-v} + \frac{l}{c+v}$$

$$t_1 = \frac{l(c+v+c-v)}{c^2-v^2}$$

$$t_1 = l\left(\frac{2c}{c^2-v^2}\right)$$

$$t_1 = \frac{2lc}{c^2}\left(\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$t_1 = \frac{2l}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$t_1 = \frac{2l}{c}\left[1 - \left(\frac{v^2}{c^2}\right)\right]^{-1}$$

Apply Binomial theorem and neglect higher terms

$$t_1 \approx \frac{2l}{c}\left[1 + \left(\frac{v^2}{c^2}\right)\right] \quad ....(1)$$

time taken for light to travel to M2 and back to plate

$$t_2 = \frac{l}{\sqrt{c^2 - v^2}} + \frac{l}{\sqrt{c^2 - v^2}}$$

$$t_2 = \frac{2l}{\sqrt{c^2 - v^2}}$$

$$t_2 = \frac{2l}{c}\left[\frac{1}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}\right]$$

$$t_2 = \frac{2l}{c}\left[1 - \left(\frac{v^2}{c^2}\right)\right]^{-\frac{1}{2}}$$

Apply Binomial theorem and neglect higher terms

$$t_2 \approx \frac{2l}{c}\left[1 + \left(\frac{v^2}{2c^2}\right)\right] \quad ......(2)$$

Therefore the time difference between the transmitted and reflected rays will be

$$\Delta t = t_2 - t_1$$

Using equations (1) and (2)

$$\Delta t = \frac{2l}{c}\left(1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right)$$

$$\Delta t = \frac{l}{c}\left(\frac{v^2}{c^2}\right) \quad ....(3)$$

After this the apparatus is rotated through 90^o so that mirrors will exchange their positions.

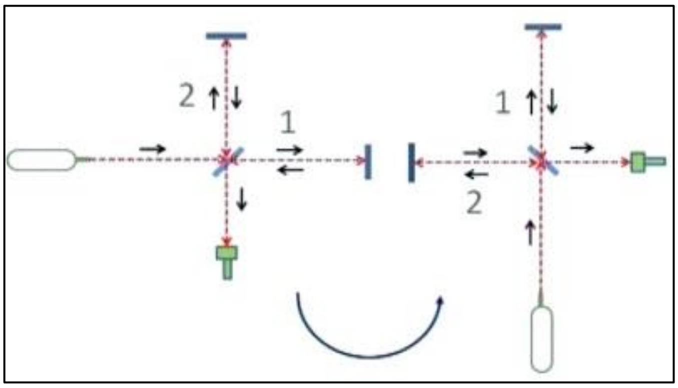


Fig. 4.3

In a rotated position the time difference the same two arms would be given by

$$\Delta t' = -\frac{l}{c}\left(\frac{v^2}{c^2}\right) \quad \text{...(4)}$$

The time delay varies as the apparatus is rotated. The total delay in 90^o rotation is given by

$$\Delta t - \Delta t' = \frac{2l}{c}\left(\frac{v^2}{c^2}\right) \quad \text{......(5)}$$

this would cause fringe pattern to move during rotation, which can be observed experimentally.

therefore, path difference in 90^o rotation is given by

$$\Delta = (\Delta t = \Delta t').c = 2l\left(\frac{v^2}{c^2}\right)$$

and phase difference

$$\Delta\delta = \frac{2\pi}{\lambda}\Delta = \frac{2\pi}{\lambda}2l\left(\frac{v^2}{c^2}\right)$$

Fringe Shift

The total amount of fringe shift N on rotation can be calculated from the time difference

$$N = \frac{\Delta\delta}{2\pi}$$

$$N = \frac{2l}{\lambda}\left(\frac{v^2}{c^2}\right) \quad \text{.......(6)}$$

Then N is calculated by putting l = 11m, v = 10^{-4} c and λ = 5500 angstroms

Thus N = 0.4 fringes. The experiment was performed day and night and at different times of year. Even though the sensitivity of the set up is to detect a shift of 0.01 fringe, no such shift was observed. Similar experiments were repeated by several groups but the result was same. The above experiment shows that the speed of light is constant in space irrespective of the direction and speed of inertial frame.

Explanation for negative results

The following explanations were given for the negative result of Michelson–Morley experiment.

(i) Ether drag theory: The moving bodies drag the surrounding ether with them. So, we can say that there is no relative motion between ether and earth

(iii) Light velocity hypothesis: This hypothesis shows that the velocity of light from a moving source is the vector sum of velocity of light and velocity of source light. Based on some astronomical evidences, this hypothesis was also rejected. In 1905, Einstein proposed that the motion through ether is a meaningless concept. He does not completely rejected the idea of ether but expressed that it can never be detected. The motion of an object relative to a frame of reference has a physical concept.

The postulates of special theory of relativity are as follows:

i) The laws of physics are the same in all inertial frames of reference.
ii) The velocity of light in free space is a constant in all the frames of reference.

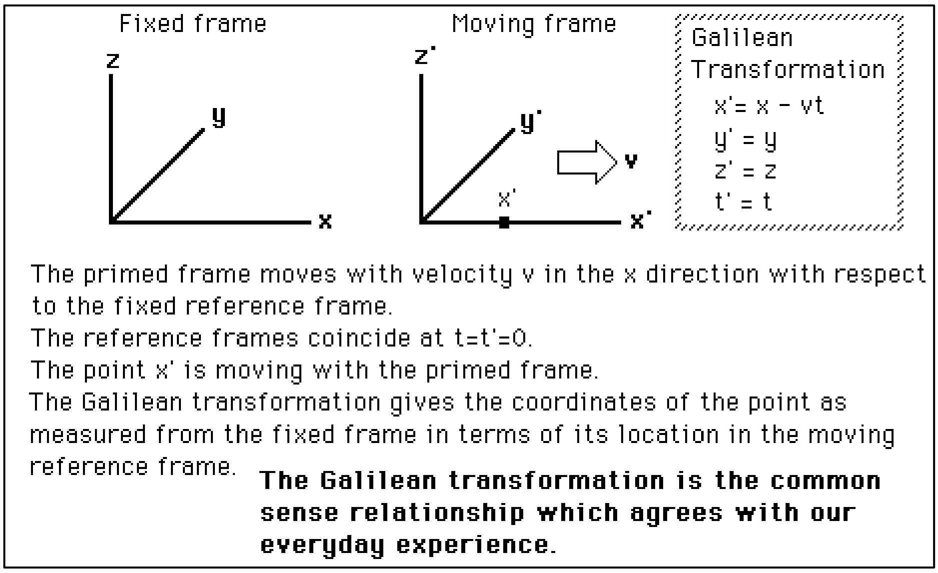
Special theory of relativity or special relativity is a physical theory which states the relationship between space and time. This is often termed as STR theory. This is theory is based on two postulates –

1. Laws of Physics are invariant
2. Irrespective of the light source, the speed of light in a vacuum is the same in any other space.

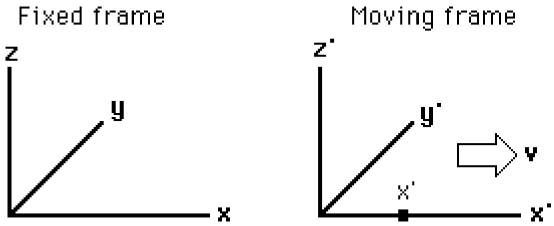
Albert Einstein originally proposed this theory in one of his paper "On the Electrodynamics of Moving Bodies". Special relativity implies consequences like mass-energy equivalence, relativity of simultaneity, length contraction and a universal speed limit. The conventional notion of absolute universal time is replaced by the notion of a time that is dependent on the reference frame and spatial position.

In relative theory, Reference frames play a vital role. It is used to measure a time of events by using a clock. An event is nothing but an occurrence that refers to a location in space corresponding to reference frame. For instance, the explosion of a fire flower can be considered as an event.

Galilean Transformation:



Lorentz Transformation



$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The primed frame moves with velocity v in the x direction with respect to the fixed reference frame. The reference frames coincide at t=t'=0. The point x' is moving with the primed frame.

The reverse transformation is:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

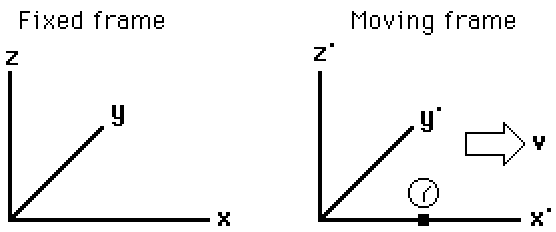
$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Much of the literature of relativity uses the symbols β and γ as defined here to simplify the writing of relativistic relationships.

Time Dilation

A clock in a moving frame will be seen to be running slow, or "dilated" according to the Lorentz transformation. The time will always be shortest as measured in its rest frame. The time measured in the frame in which the clock is at rest is called the "proper time".



If the time interval $T_0 = t'_2 - t'_1$ is measured in the moving reference frame, then $T = t_2 - t_1$ can be calculated using the Lorentz transformation.

$$T = t_2 - t_1 = \frac{t'_2 + \frac{vx'_2}{c^2} - t'_1 + \frac{vx'_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The time measurements made in the moving frame are made at the same location, so the expression reduces to:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} = T_0\gamma$$

Eor small velocities at which the relativity factor is very close to 1, then the time dilation can be expanded in a binomial expansion to get the approximate expression:

$$T \approx T_0\left[1 + \frac{v^2}{2c^2}\right]$$

Length Contraction

The length of any object in a moving frame will appear foreshortened in the direction of motion, or contracted. The amount of contraction can be calculated from the Lorentz transformation. The length is maximum in the frame in which the object is at rest.

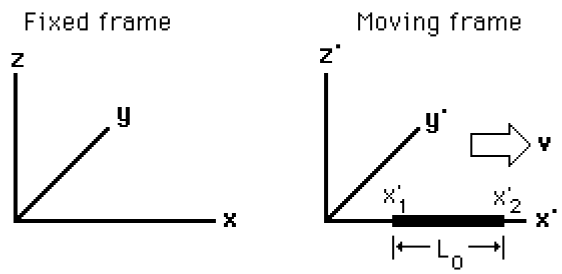


Fig. 4.4

If the length $L_0 = x'_2 - x'_1$ is measured in the moving reference frame, then $L = x_2 - x_1$ can be calculated using the Lorentz transformation.

$$L_0 = x'_2 - x'_1 = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

But since the two measurements made in the fixed frame are made simultaneously in that frame, $t_2 = t_1$, and

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

Length contraction

Relativistic Doppler Shift

The normal Doppler shift for waves such as sound which move with velocities v much less than c is given by the expression.

Doppler effect for sound

$$f_{observed} = \left[\frac{\mathrm{v}}{\mathrm{v} \pm \mathrm{v}_{source}}\right] f_{source}$$

which when changed to the normal notation for light waves becomes

$$\nu_{observed} = \nu_{source}\left[\frac{1}{1 \pm \frac{\mathrm{v}_{source}}{\mathrm{v}_{wave}}}\right]$$

where the plus sign is taken for waves traveling away from the observer. For light and other electromagnetic waves, the relationship must be modified to be consistent with the Lorentz transformation and the expression becomes

Doppler effect for light

$$\nu_{observed} = \left[\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} \right] \nu_{source}$$

which can be rearranged to the form

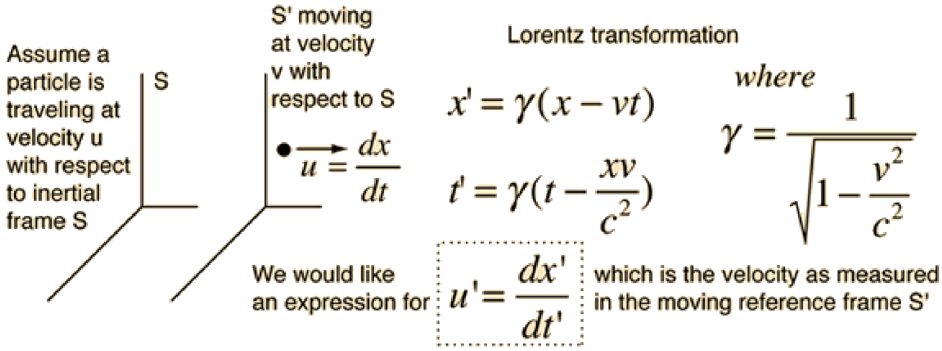
$$\nu_{observed} = \nu_{source}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

or in common relativity notation: $\nu_{observed} = \nu_{source}\sqrt{\frac{1+\beta}{1-\beta}}$

Here v is the relative velocity of source and observer and v is considered positive when the source is approaching.

Relativistic Velocity Transformation

No two objects can have a relative velocity greater than c! But what if I observe a spacecraft traveling at 0.8c and it fires a projectile which it observes to be moving at 0.7c with respect to it!? Velocities must transform according to the Lorentz transformation, and that leads to a very non-intuitive result called Einstein velocity addition.



Just taking the differentials of these quantities leads to the velocity transformation. Taking the differentials of the Lorentz transformation expressions for x' and t' above gives

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{vdx}{c^2})} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\frac{dx}{dt}}{c^2}}$$

Putting this in the notation introduced in the illustration above:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

The reverse transformation is obtained by just solving for u in the above expression. Doing that gives

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Applying this transformation to the spacecraft traveling at 0.8c which fires a projectile which it observes to be moving at 0.7c with respect to it, we obtain a velocity of 1.5c/1.56 = 0.96c rather than the 1.5c which seems to be the common-sense answer.

Question Bank:

Part A

1. What is theory of relativity?
2. What are the two parts of theory of relativity? Explain it.
3. Explain the physical significance of the negative result.
4. State the basic postulates of Einstein's special theory of relativity.
5. How Lorentz transformation follows directly from the postulates of the special theory of relativity?
6. What are the two important kinematical effects which derive from the special theory of relativity?
7. What is meant by 'length contraction' due to relativistic effect?
8. What do you understand by 'time dilation'?
9. Write the Lorentz velocity transformation equations.
10. Write the Doppler's relativistic formula for light waves in vacuum.
11. A rod has length 100 cm. When the rod is in a satellite moving with a velocity that is one half of the velocity of light relative to laboratory, what is the length of the rod as determined by an observer
    (a) in the satellite, and
    (b) in the laboratory.
    Solution:
    (a) The observer in the satellite is at rest relative the rod hence the length of the rod as measured by him is 100 cm.
    (b) The length 'l' of the rod in the laboratory is given by
    $l = l' (1 - v^2 /c^2)^{1/2} = 100 (1- (0.5c/c)^2)^{1/2} = 86.6cm$
12. Calculate the momentum of a photon whose energy is 1.00×10^{-29} joules.
    $p = E/c$
    $p = 1.00 \times 10^{-19} / 3 \times 10^8 = 3.33 \times 10^{-28}$ kg.m/sec.
13. Calculate the fringe shift in Michelson-Morley experiment. Given effective length of each path is 10m, velocity of earth 3×10^4 m/s and wavelength of light used is 6000A°.
    $2\Delta = 2lv^2 /c^2\lambda = 0.33$ fringe.

14. In Michelson-Morley experiment the length of the paths of the two beans is 11 meters each. The wavelength of the light used is 6000 A°. If the expected fringe shift is 0.4 fringe, calculate the velocity of earth relative to ether. $2\Delta = 2lv^2 / c^2\lambda$ from this we have $v^2 = 2\Delta c^2 \lambda / 2l$ $v = 3.1 \times 10^5$ m/s.
15. What will be the apparent length of meter stick measured by an observer at rest, when the stick is moving along its velocity equal to $(\sqrt{3}/2)c$.
   $l = l_o \sqrt{1 - v^2/c^2}$
   $l_o = 1$m ; $v = (\sqrt{3}/2)c$
   $l = 0.5$ m.
16. How, fast would a rocket have to go relative to an observer for its length to be contracted to 99% of its length at rest.
   $l = lo \sqrt{1 - v^2/c^2}$
   $l = (99/100)lo$; $l/l_o = 99/100$
   $v = 0.1416$ c.
17. A young lady decides on her twenty fifth birth day that it is time to slenderize. She weights 100kg. She has heard that if she moves fast enough, she will appear thinner to her stationary friends.
   (i) How fast must she move to appear slenderized by a factor of 50%.
   (ii) If she maintains her speed until the day she calls her twenty ninth birthday, how old will her stationary friends claim shows according to their measurements?

   Solution:
   (i) $l = l_o \sqrt{1 - v^2/c^2}$
   $l/l_o = 99/100$; $v = 0.866c$.
   (iii) $\Delta t' = \Delta t \sqrt{1 - v 2 c 2}$; $\Delta t = 4 years$; $\sqrt{1 - v^2/c^2} = 1 2$
   $\Delta t' = 4/(1/2) = 8$ years
   The lady will appear to be (25+ 8) = 33 years old.
18. An electron is moving with a speed of 0.85c in a direction opposite to that of a moving photon. Calculate the relative velocity of the electron with respect to photon.

   Solution:
   The speed of the photon = c
   The speed of the electron = 0.85c
   $v = 0.85c$, $u' = c$;
   $u = (u' + v) / (1 + vu/c^2) = (c + 0.85c)/(1 + 0.85) = c$
19. A space-ship moving away from earth with velocity of 0.4c fires a rocket whose velocity relative to space-ship is 0.6c, away from the earth. What will be the velocity of the rocket as observed from the earth.

   Solution:
   Velocity of rocket relative to earth = u
   Velocity of space-ship relative to earth = v
   Velocity of rocket relative to space-ship = u'x
   $v = 0.4c$, $u'x = 0.6c$;
   $u_x = (u_x' + v) / (1 + vu_x'/c^2) = c / (1 + 0.24) = 0.8$ c.

Part – B

1. Describe the Michelson – Morley experiment and show how the negative results obtained are interpreted.
2. State the fundamental postulates of special theory of relativity and deduce the Lorentz transformations.
3. What is length contraction in special theory of relativity? Explain it by Lorentz transformations.
4. What is time dilation in special theory of relativity? Explain it by Lorentz transformations.
5. State and deduce the law of addition of relativistic velocities and show that in no case can the resultant velocity of a material particle be greater than c.
6. Deduce the Doppler's relativistic formula for light from Lorentz transformation equation.
7. Derive Lorentz transformation for moment and energy.

V. Simple Hormonic Oscillations

Introduction: We see different kinds of motion every day. The motion of the hands of a clock, motion of the wheels of a car, etc. Did you ever notice that these types of motion keep repeating themselves? Such motions are periodic in nature. One such type of periodic motion is simple harmonic motion (S.H.M.). But what is S.H.M.? Let's find out.

Periodic Motion and Oscillations

A motion that repeats itself in equal intervals of time is *periodic***.** We need to know what periodic motion is to understand simple harmonic motion. Periodic motion is the motion in which an object repeats its path in equal intervals of time. We see many examples of periodic motion in our day-to-day life. The motion of the hands of a clock is periodic motion. The rocking of a cradle, swinging on a swing, leaves of a tree moving to and fro due to wind breeze, these all are examples of periodic motion. The particle performs the same set of movement again and again in a periodic motion. One such set of movement is called an Oscillation. A great example of oscillatory motion is Simple Harmonic Motion.

Simple Harmonic Motion (S.H.M.)

When an object moves to and fro along a line, the motion is called simple harmonic motion**.** Have you seen a pendulum? When we swing it, it moves to and fro along the same line. These are called oscillations. Oscillations of a pendulum are an example of simple harmonic motion.

Now, consider there is a spring that is fixed at one end. When there is no force applied to it, it is at its equilibrium position. Now,

- If we pull it outwards, there is a force exerted by the string that is directed towards the equilibrium position.
- If we push the spring inwards, there is a force exerted by the string towards the equilibrium position.

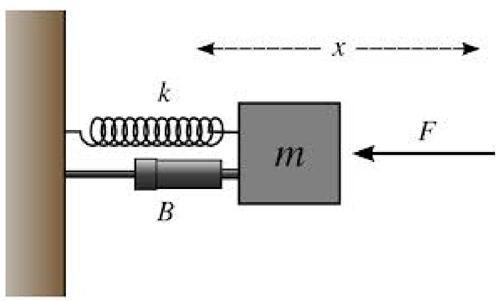


Fig. 5.1

In each case, we can see that the force exerted by the spring is towards the equilibrium position. This force is called the restoring force. Let the force be F and the displacement of the string from the equilibrium position be x. Therefore, the restoring force is given by, F= – kx (the negative sign indicates that the force is in opposite direction). Here, k is the constant called the force constant. Its unit is N/m in S.I. system and dynes/cm in C.G.S. system.

Linear simple harmonic motion

Linear simple harmonic motion is defined as the linear periodic motion of a body in which the restoring force is always directed towards the equilibrium position or mean position and its magnitude is directly proportional to the displacement from the equilibrium position. All simple harmonic motions are periodic in nature but all periodic motions are not simple harmonic motions. Now, take the previous example of the string. Let its mass be m. The acceleration of the body is given by,

$a = F/m = – kx/m = – \omega^2 x$

Here, $k/m = \omega^2$ (ω is the angular frequency of the body)

Concepts of Simple Harmonic Motion (S.H.M)

- **Amplitude**: The maximum displacement of a particle from its equilibrium position or mean position is its amplitude. Its S.I. unit is the metre. The dimensions are $[L^1M^0 T^0]$. Its direction is always away from the mean or equilibrium position.
- **Period**: The time taken by a particle to complete one oscillation is its period. Therefore, period if S.H.M. is the least time after which the motion will repeat itself. Thus, the motion will repeat itself after nT. where n is an integer.
- **Frequency**: Frequency of S.H.M. is the number of oscillations that a particle performs per unit time. S.I. unit of frequency is hertz or r.p.s(rotations per second). Its dimensions are $[L^0M^0T^{-1}]$.
- **Phase**: Phase of S.H.M. is its state of oscillation. Magnitude and direction of displacement of particle represent the phase. The phase at the beginning of the motion is known as Epoch(α)

Periodic and Oscillatory Motion:

We come across various kinds of motions in our daily life. You have already studied some of them like linear and projectile motion. However, these motions are non-repetitive. Here, we are going to learn about periodic and oscillatory motion.

Energy in simple harmonic motion

Each and every object possesses energy, either while moving or at rest. In the simple harmonic motion, the object moves to and fro along the same path. Do you think an object possesses energy while travelling the same path again and again? Yes, it is energy in simple harmonic motion. Let’s learn how to calculate this energy and understand its properties.

The total energy that a particle possesses while performing simple harmonic motion is energy in simple harmonic motion. Take a pendulum for example. When it is at its mean position, it is at rest. When it moves towards its extreme position, it is in motion and as soon as it reaches its extreme position, it comes to rest again. Therefore, in order to calculate the energy in simple harmonic motion, we need to calculate the kinetic and potential energy that the particle possesses.

Kinetic Energy (K.E.) in S.H.M

Kinetic energy is the energy possessed by an object when it is in motion. Let's learn how to calculate the kinetic energy of an object. Consider a particle with mass *m* performing simple harmonic motion along a path AB. Let O be its mean position. Therefore, OA = OB = a.

The instantaneous velocity of the particle performing S.H.M. at a distance x from the mean position is given by

$v = \pm \omega \sqrt{a^2 - x^2}$

$\therefore\ v^2 = \omega^2 (a^2 - x^2)$

$\therefore$ Kinetic energy= $1/2\ mv^2 = 1/2\ m\ \omega^2 (a^2 - x^2)$

As, $k/m = \omega^2$ (1)

$\therefore k = m\ \omega^2$ (2)

Kinetic energy= $1/2\ k\ (a^2 - x^2)$. The equations 1 and 2 can both be used for calculating the kinetic energy of the particle.
Learn how to calculate Velocity and Acceleration in Simple Harmonic Motion.
Potential Energy(P.E.) of Particle Performing S.H.M.

Potential energy is the energy possessed by the particle when it is at rest. Let's learn how to calculate the potential energy of a particle performing S.H.M. Consider a particle of mass *m* performing simple harmonic motion at a distance x from its mean position. You know the restoring force acting on the particle is F= -kx where k is the force constant.

Now, the particle is given further infinitesimal displacement dx against the restoring force F. Let the work done to displace the particle be dw. Therefore, The work done dw during the displacement is

$dw = -fdx = -(-kx)\ dx = kxdx$ (3)

Therefore, the total work done to displace the particle now from 0 to x is

$\int dw = \int kxdx = k \int x\ dx$ (4)

Hence Total work done = 1/2 K x^2 = 1/2 m $\omega^2 x^2$

The total work done here is stored in the form of potential energy.

Therefore Potential energy = 1/2 kx^2 = 1/2 m $\omega^2 x^2$

Equations 3 and 4 are equations of potential energy of the particle. Thus, potential energy is directly proportional to the square of the displacement, that is P.E. α x^2.

Learn the Difference between Periodic and Oscillatory Motion.

Total Energy in Simple Harmonic Motion (T.E.)

The total energy in simple harmonic motion is the sum of its potential energy and kinetic energy.

Thus, T.E. = K.E. + P.E. = 1/2 k ($a^2 - x^2$) + 1/2 K x^2 = 1/2 k a^2

Hence, T.E.= E = 1/2 m $\omega^2 a^2$

Equation III is the equation of total energy in a simple harmonic motion of a particle performing the simple harmonic motion. As ω^2, a^2 are constants, the total energy in the simple harmonic motion of a particle performing simple harmonic motion remains constant. Therefore, it is independent of displacement x.

As $\omega = 2\pi f$, E= 1/2 m $(2\pi f)^2 a^2$

$\therefore$ E= $2m\pi^2 f^2 a^2$

As 2 and π^2 constants, we have T.E. $\sim$ m, T.E. $\sim f^2$, and T.E. $\sim a^2$

Thus, the total energy in the simple harmonic motion of a particle is:

- Directly proportional to its mass
- Directly proportional to the square of the frequency of oscillations and
- Directly proportional to the square of the amplitude of oscillation.

The law of conservation of energy states that energy can neither be created nor destroyed. Therefore, the total energy in simple harmonic motion will always be constant. However, kinetic energy and potential energy are interchangeable. Given below is the graph of kinetic and potential energy vs instantaneous displacement.

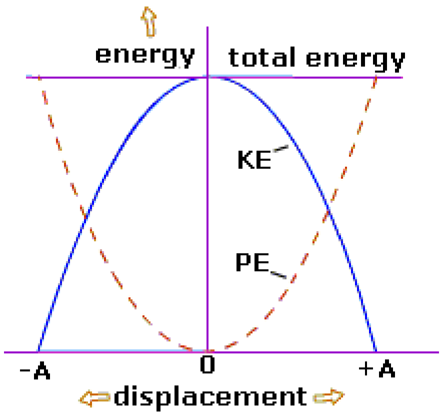


Fig. 5.2

In the graph, we can see that,

- At the mean position, the total energy in simple harmonic motion is purely kinetic and at the extreme position, the total energy in simple harmonic motion is purely potential energy.
- At other positions, kinetic and potential energies are interconvertible and their sum is equal to $1/2\ k\ a^2$.
- The nature of the graph is parabolic.

Periodic Motion

What is common in the motion of the hands of a clock, wheels of a car and planets around the sun? They all are repetitive in nature, that is, they repeat their motion after equal intervals of time. A motion which repeats itself in equal intervals of time is periodic.

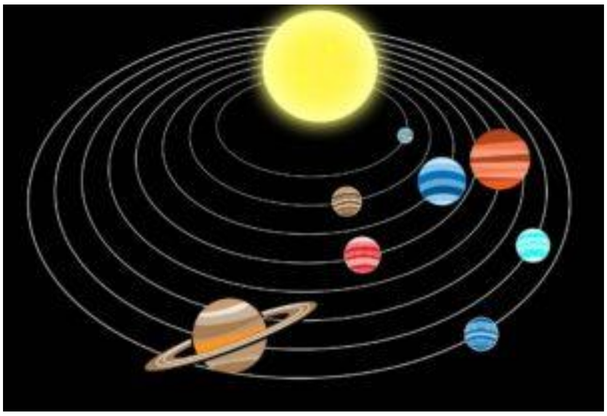


Fig. 5.3

A body starts from its equilibrium position(at rest) and completes a set of movements after which it will return to its equilibrium position. This set of movements repeats itself in equal intervals of time to perform the periodic motion.
Circular motion is an example of periodic motion. Very often the equilibrium position of the object is in the path itself. When the object is at this point, no external force is acting on it. Therefore, if it is left at rest, it remains at rest.

Period and Frequency of Periodic Motion

We know that motion which repeats itself after equal intervals of time is periodic motion. The time period(T) of periodic motion is the time interval after which the motion repeats itself. Its S.I.unit is second.

The reciprocal of T gives the number of repetitions per unit time. This quantity is the frequency of periodic motion. The symbol υ represents frequency. Therefore, the relation between υ and T is

$\upsilon = 1/T$

Thus, the unit of υ is s^{-1} or hertz(after the scientist Heinrich Rudolf Hertz). Its abbreviation is Hz. Thus, 1 hertz = 1 Hz = 1 oscillation per second = 1 s^{-1} The frequency of periodic motion may not be an integer. But it can be a fraction.

Oscillatory Motion

Fig. 5.4

Oscillatory motion is the repeated to and fro movement of a system from its equilibrium position. Every system at rest is in its equilibrium position. At this point, no external force is acting on it. Therefore, the net force acting on the system is zero. Now, if this system is displaced a little from its fixed point, a force acts on the system which tries to bring back the system to its fixed point. This force is the restoring force and it gives rise to oscillations or vibrations.

For example, consider a ball that is placed in a bowl. It will be in its equilibrium position. If displaced a little from this point, it will oscillate in the bowl. Therefore, every oscillatory motion is periodic but all periodic motions are not oscillatory. For instance, the circular motion is a periodic motion but not oscillatory. Moreover, there is no significant difference between oscillations and vibrations. In general, when the frequency is low, we call it oscillatory motion and when the frequency is high, we call it vibrations. Furthermore, simple harmonic motion is the simplest type of oscillatory motion. This motion takes place when the restoring force acting on the system is directly proportional to its displacement from its equilibrium position. In practice, oscillatory motion eventually comes to rest due to damping or frictional forces. However, we can force them by means of some external forces. Also, a number of oscillatory motions together form waves like electromagnetic waves.

Displacement in Oscillatory Motion

Displacement of a particle is a change in its position vector. In an oscillatory motion, displacement simply means a change in any physical property with time.

Consider a block attached to a spring, which in turn is fixed to a rigid wall. We measure the displacement of the block from its equilibrium position. In an oscillatory motion, we can represent the displacement by a mathematical function of time. One of the simplest periodic functions is given by,

$$f(t) = A \cos \omega t$$

If the argument, ωt, is increased by an integral multiple of 2π radians, the value of the function remains the same. Therefore, it is periodic in nature and its period T is given by,

$$T = 2\pi/\omega$$

Thus, the function f(t) is periodic with period T. $\therefore f(t) = f(t + T)$. Now, if we consider a sine function, the result will be the same. Further, taking a linear combination of sine and cosine functions is also a periodic function with period T.

$$f(t) = A \sin \omega t + B \cos \omega t$$

Taking $A = D\cos\emptyset$ and $B = D\sin\emptyset$ equation V becomes, $f(t) = D \sin (\omega t + \emptyset)$. In this equation D and $\emptyset$ are constant and they are given by,

$$D = \sqrt{}A^2 + B^2 \text{ and } \emptyset = \tan^{-2}(B/A)$$

Therefore, we can express any periodic function as a superposition of sine and cosine functions of different time periods with suitable coefficients. The period of the function is $2\pi/\omega$.

Table 5.1: Difference between Periodic and Simple Harmonic Motion

Periodic Motion	Simple Harmonic Motion
In the periodic motion, the displacement of the object may or may not be in the direction of the restoring force.	In the simple harmonic motion, the displacement of the object is always in the opposite direction of the restoring force.
The periodic motion may or may not be oscillatory.	Simple harmonic motion is always oscillatory.
Examples are the motion of the hands of a clock, the motion of the wheels of a car, etc.	Examples are the motion of a pendulum, motion of a spring, etc.

Damped Harmonic Motion

We know that when we swing a pendulum, it will eventually come to rest due to air pressure and friction at the support. This motion is damped simple harmonic motion. Let's understand what it is and how it is different from linear simple harmonic motion.
When the motion of an oscillator reduces due to an external force, the oscillator and its motion are **damped**. These periodic motions of gradually decreasing amplitude are damped simple harmonic motion. An example of a damped simple harmonic motion is a simple pendulum. In the damped simple harmonic motion, the energy of the oscillator dissipates continuously. But for a small damping, the oscillations remain approximately periodic. The forces which dissipate the energy are generally **frictional forces**.

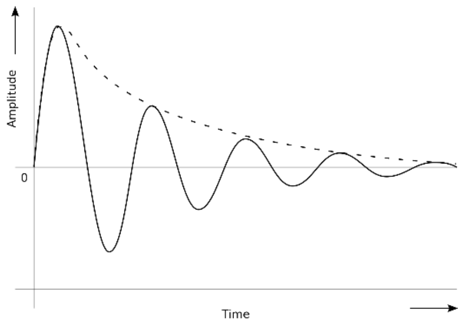


Fig. 5.5

Expression of damped simple harmonic motion

Let's take an example to understand what a damped simple harmonic motion is. Consider a block of mass *m* connected to an elastic string of spring constant k. In an ideal situation, if we push the block down a little and then release it, its angular frequency of oscillation is $\boldsymbol{\omega = \sqrt{k}/ m}$.

However, in practice, an external force (air in this case) will exert a damping force on the motion of the block and the mechanical energy of the block-string system will decrease. This energy that is lost will appear as the heat of the surrounding medium.

The damping force depends on the nature of the surrounding medium. When we immerse the block in a liquid, the magnitude of damping will be much greater and the dissipation energy is much faster. Thus, the damping force is proportional to the velocity of the bob and acts opposite to the direction of the velocity. If the damping force is F_d, we have,

$$F_d = -b\upsilon \qquad \text{(I)}$$

where the constant b depends on the properties of the medium(viscosity, for example) and size and shape of the block. Let's say O is the equilibrium position where the block settles after releasing it. Now, if we pull down or push the block a little, the restoring force on the block due to spring is $F_s = -kx$, where x is the displacement of the mass from its equilibrium position.

Therefore, the total force acting on the mass at any time t is, $F = -kx - b\upsilon$.

Now, if a(t) is the acceleration of mass *m* at time t, then by Newton's Law of Motion along the direction of motion, we have

$$ma(t) = -kx(t) - b\upsilon(t) \qquad \text{(II)}$$

Here, we are not considering vector notation because we are only considering the one-dimensional motion. Therefore, using first and second derivatives of s(t), v(t) and a(t), we have,

$$m(d^2x/dt^2) + b(dx/dt) + kx = 0 \qquad \text{(III)}$$

This equation describes the motion of the block under the influence of a damping force which is proportional to velocity. Therefore, this is the expression of damped simple harmonic motion. The solution of this expression is of the form

$$x(t) = Ae^{-bt/2m}\cos(\omega' t + ø) \qquad \text{(IV)}$$

where A is the amplitude and ω' is the angular frequency of damped simple harmonic motion given by,

$$\omega' = \sqrt{(k/m - b^2/4m^2)} \qquad \text{(V)}$$

The function x(t) is not strictly periodic because of the factor $e^{-bt/2m}$ which decreases continuously with time. However, if the decrease is small in one-time period T, the motion is then approximately periodic. In a damped oscillator, the amplitude is not constant but depends on time. But for small damping, we may use the same expression but take amplitude as $Ae^{-bt/2m}$

$$\therefore E(t) = 1/2\ kAe^{-bt/2m} \qquad \text{(VI)}$$

This expression shows that the damping decreases exponentially with time. For a small damping, the dimensionless ratio $(b/\sqrt{km})$ is much less than 1. Obviously, if we put b = 0, all equations of damped simple harmonic motion will turn into the corresponding equations of undamped motion.

Forced Simple Harmonic Motion

When we displace a pendulum from its equilibrium position, it oscillates to and fro about its mean position. Eventually, its motion dies out due to the opposing forces in the medium. But can we force the pendulum to oscillate continuously? Yes. This type of motion is known as forced simple harmonic motion. Let's find out what forced simple harmonic motion is.

Definition of Forced Simple Harmonic Motion

When we displace a system, say a simple pendulum, from its equilibrium position and then release it, it oscillates with a natural frequency ω and these oscillations are free oscillations. But all free oscillations eventually die out due to the ever-present damping forces in the surrounding.
However, an external agency can maintain these oscillations. These oscillations are known as **forced** or **driven** oscillations. The motion that the system performs under this external agency is known as **Forced Simple Harmonic Motion**. The external force is itself periodic with a frequency ω_d which is known as the **drive frequency**.

A **very important point** to note is that the system oscillates with the driven frequency and not its natural frequency in Forced Simple Harmonic Motion. If it oscillates with its natural frequency, the motion will die out. A good example of forced oscillations is when a child uses his feet to move the swing or when someone else pushes the swing to maintain the oscillations.

Expression of Forced Simple Harmonic Motion

Consider an external force F(t) of amplitude F_0 that varies periodically with time. This force is applied to a damped oscillator. Therefore, we can represent it as,

$F(t) = F_0 \cos \omega_d t$ (I)

Thus, at this time, the forces acting on the oscillator are its restoring force, the external force and a time-dependent driving force. Therefore,

$ma(t) = -kx(t) - b\upsilon(t) + F_0 \cos \omega_d t$ (II)

We know that acceleration = d^2x/dt^2. Substituting this value of acceleration in equation II , we get,

$m(d^2x/dt^2) + b(dx/dt) + kx = F_0 \cos \omega_d t$ (III)

Equation III is the equation of an oscillator of mass m on which a periodic force of frequency ω_d is applied. Obviously, the oscillator first oscillates with its natural frequency. When we apply the external periodic force, the oscillations with natural frequency die out and the body then oscillates with the driven frequency. Therefore, its displacement after the natural oscillations die out is given by:

$x(t) = A\cos(\omega_d + \o)$ (IV)

where t is the time from the moment, we apply external periodic force.

Resonance

The phenomenon of increase in amplitude when the driving force is close to the natural frequency of the oscillator is known as resonance. To understand the phenomenon of resonance, let us consider two pendulums of nearly equal (but not equal) lengths (therefore, different amplitudes) suspended from the same rigid support.

When we swing the first pendulum which is greater in length, it oscillates with its natural frequency. The energy of this pendulum transfers through the rigid support to the second pendulum which is slightly smaller in length. Therefore, the second pendulum starts oscillating with its natural frequency first.

At one point, the frequency with the second pendulum vibrates becomes nearly equal to the first one. Therefore, the second pendulum now starts with the frequency of the first one, which is the driven frequency. When this happens, the amplitude of the oscillations is **maximum**. Thus, resonance takes place.

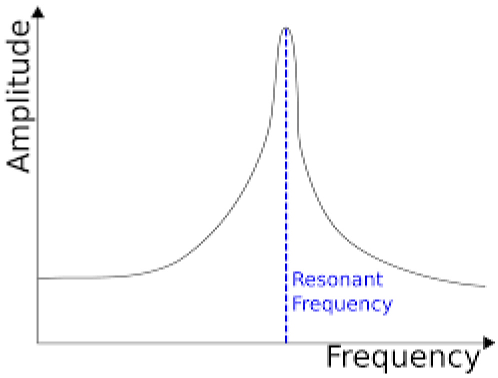


Fig. 5.6

Resonance:

The sharpness of resonance can be understood better by understanding resonance. Resonance is defined as the tendency of a system to vibrate with an increase in amplitude at the excitation of frequencies. Resonance frequency or resonant frequency is the maximum frequency at which the amplitude is relatively maximum. The Q factor is used to define the sharpness of the resonance.

Sharpness of resonance:

The sharpness of resonance is defined using the Q factor which explains how fast energy decay in an oscillating system. The sharpness of resonance depends upon:

- Damping: Effect due to which there is a reduction in amplitude of vibrations
- Amplitude: Maximum displacement of a point on a vibrating body which is measured from its equilibrium position.

The sharpness of resonance increases or decreases with an increase or decrease in damping and as the amplitude increases, the sharpness of resonance decreases.

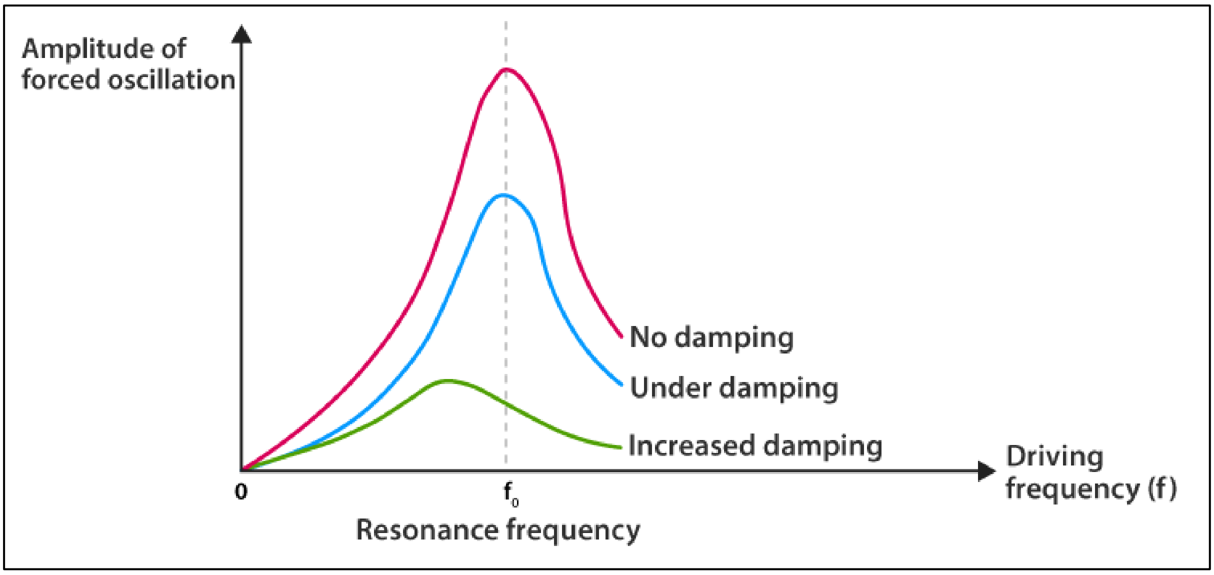


Fig. 5.7

Q Factor:

Q factor or quality factor is a dimensionless parameter that is used to describe the underdamped resonator and characterizes the bandwidth and center frequency of the resonator.

The mathematical representation is:
$Q = E_{stored}/E_{lost\ per\ cycle}$

Power dissipation:

The decrement of rate of change of average energy with respect to time is known as power dissipation.

<u>Question Bank:</u>

<u>Part A</u>

1. Define simple harmonic motion.
2. What are the conditions needed for linear simple harmonic motion?
3. What are the importances of S.H.M?
4. Define damping coefficient.
5. Define damped harmonic oscillator. 6. State the condition of resonance.
6. Define forced vibration. Distinguish between free and forced vibrations.
7. Write power dissipation equation in damped harmonic oscillator.
8. Define quality factor in damped harmonic oscillator.
9. What is meant by sharpness of resonance?

Part – B

1. Derive the equation of SHM and its solution.
2. Show that the total energy of particle executing simple harmonic motion is proportional to (a) square of its amplitude (b) the square of its frequency.
3. Derive expressions for the period and amplitude of damped harmonic motion.
4. Explain in detail power dissipation in damped harmonic oscillator.
5. Explain in detail Quality factor in damped harmonic oscillator.
6. Derive an expression for forced harmonic oscillator with steady state solution.
7. What is resonance? Give some important examples of resonance. Explain sharpness of the resonance.

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