SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMTA5403- ADVANCED OPERATIONS RESEARCH

Objective of the Course: The ability to identify, reflect upon, evaluate and apply different types of information and knowledge to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

UNIT 1: Sensitivity Analysis: Introduction – Sensitivity Analysis – Change in Objective Function Coefficient – Change in the Availability of Resources – Changes in the Input Output Coefficients – Addition of New Variable – Addition of New Constraint

UNIT 2: Integer Linear Programming: Introduction – Types of Integer Programming Problems – Enumeration and Cutting Plane Solution Concept – Gomory's All Integer Cutting Plane Method - Gomory's Mixed Integer Cutting Plane Method.

UNIT 3: Goal Programming : Introduction – Difference between LP and GP approach – Concept of Goal Programming - Goal Programming model formulation – Single Goal with Multiple sub Goals – Equally ranked Multiple Goals – Ranking and Weighting of Unequal Multiple Goals - General GP Model – Graphical Solution method of GP – Modified Simplex Method of GP.

UNIT 4: Decision and Game Theory: Decision Theory – Introduction – Steps of Decision making process – Types of Decision Making Environments – Decision Making Under Uncertainty - Decision Making Under Risk - Expected Monetary Value. Theory of Game – Introduction – Two Person Zero Sum Games – Games with Saddle

Point – Rules to determine Saddle point - Games without Saddle Point - related problems – Principles of Dominance – Solution method for Games without Saddle point- Graphical Method.

UNIT 5: Dynamic Programming: Introduction – Dynamic Programming Terminology– Developing Optimal Decision Policy – The General Algorithm - Dynamic Programming Under Certainty – Model-I Shortest Route Problem – Model-II, Multiple Separable Return Function and Single Additive Constraint Dynamic Programming Approach for Solving Linear Programming Problems.

Course Outcomes: At the end of the course, the student will be able to:

1. Define sensitivity analysis, Integer linear programming, Goal programming, Two person zero sum games, Dynamic programming.

2. Explain change in objective function, input output coefficients, addition of new constraints, principal of dominance, algorithm of dynamic programming under certainty.

3. Choose an appropriate method and solve the problems in sensitivity analysis and prepare an integer programming table and goal programming to solve the problem.

4. Distinguish between ILP and GP and analyze the methods and also estimate problems of game theory using graphical method.

5. Evaluate problems on Sensitivity analysis, Integer programming problem, Goal programming and Dynamic Programming.

6. Design solutions using iteration method and graphical method and also Developing ILP, GP and DP model.

UNIT – I – Sensitivity Analysis

I. Introduction

In an LP model, the coefficients (also known as parameters) such as:

1. profit (cost) contribution (c_j) per unit of a decision variable, x_j .
2. availability of a resources (b_j), and
3. consumption of resource per unit of decision variables (a_{ij}), are assumed to be known constant.

However, in real-world situations, these input parameters value may change due to dynamic nature of the business environment. Such changes in any of these parameters may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision-maker, in such situations, would like to know how changes in these parameters may affect the optimal solution and the range within which the optimal solution will remain unchanged.

II . Sensitivity Analysis

Sensitivity analysis helps in evaluating the effect on optimal solution of any LP problem due to changes in its parameters, one at a time.

Aim of sensitivity analysis is to determine the range (or limit) within which the LP model parameters can change without affecting the current optimal solution. For this, instead of solving an LP problem again with new values of parameters, the current optimal solution is considered as an initial solution to determine the ranges, both lower and upper, within which a parameter may assume a value.

The sensitivity analysis is also referred to as post-optimality analysis because it does not begin until the optimal solution to the given LP model has been obtained.

Different parametric changes in an LP problem discussed in this chapter are:

1. Profit (or cost) per unit (c_j) associated with both basic and non-basic decision variables (i.e., coefficients in the objective function).
2. Availability of resources (i.e., right-hand side constants, b_i in constraints).
3. Consumption of resources per unit of decision variables x_i (i.e., coefficients of decision variables in the constraints, (a_{ij}).

4. Addition of a new variable to the existing list of variables in LP problem. item Addition of a new constraint to the existing list of constraints in the LP problem.

III Change in Objective Function Coefficient (c_j)

The coefficient, c_j in the objective function of an LP model represents either the profit or the cost per unit of an activity (variable) x_j . The question that may now arise is: What happens to the optimal solution and the objective function value when this coefficient is changed? Given an optimal basic feasible solution, suppose that the coefficient c_k of a variable x_k in the objective function is changed from c_k to $c_k+ \Delta c_k$, where Δc_k represents the positive (or negative) amount of change in the value of c_k . In optimal simplex table, the feasibility of the solution remains unaffected due to changes in the coefficients, c_j of basic variables in the objective function. However, any change in these coefficients (c_j ‘ s) only affect the optimality of the solution. Thus, such a change requires recomputing z_j values in $z_j – c_j$ row of the optimal simplex table. The cost coefficients (c_j) associated with basic variables x_1 , x_2 , and s_2 in the objective function are $c_B = (c_1, c_2, c_3) = (3, 5, 0)$. The changes in c_j can be classified as under:

Case I: Change in the coefficient of a non-basic variable .

Case II: Change in the coefficient of a basic variable

Case III: Change in the coefficient of non-basic variables

Example 1 :

Use simplex method to solve the following LPP :

Max $Z = 3x_1+5x_2$

Subject to the constraints

(i) $3x_1+2x_2\leq18$
(ii) $x_1\leq4$
(iii) $x_2\leq6$ and
$x_1, x_2 \geq0$

Discuss the change in c_j on the optimality of the optimal basic feasible solution.

Solution :

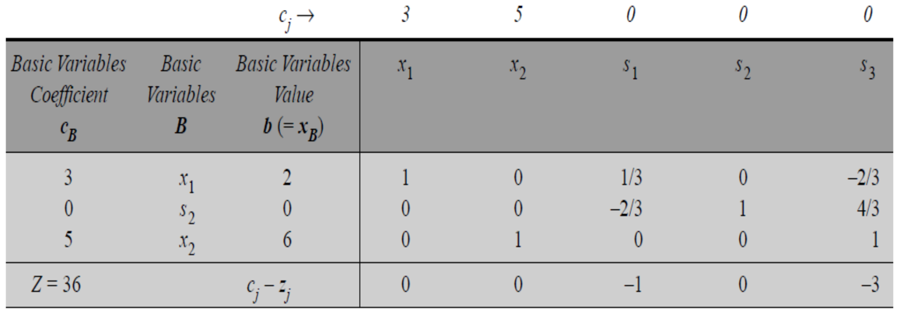
The standard form of the given LP problem is stated as follows:

$$\text{Maximize } Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

(i) $3x_1 + 2x_2 + s_1 = 18$, (ii) $x_1 + s_2 = 4$ (iii) $x_2 + s_3 = 6$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$,



		$c_j \rightarrow$	3	5	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b (= x_B)$	x_1	x_2	s_1	s_2	s_3
3	x_1	2	1	0	1/3	0	–2/3
0	s_2	0	0	0	–2/3	1	4/3
5	x_2	6	0	1	0	0	1
$Z = 36$		$c_j - z_j$	0	0	–1	0	–3

The optimal solution: $x_1 = 2$, $x_2 = 6$ and Max Z = 36 .
The cost coefficients (c_j) associated with basic variables x_1, x_2, and s_2 in the objective function are
$\mathbf{C_B} = (c_1, c_2, c_4) = (3, 5, 0)$. The changes in c_j can be classified as under:

***Changes in the coefficients c_j* (i.e. *c_3 and c_5*) *of non-basic variables s_1 and s_3*:**

Thus, the new values of $c3 – z3$ and $c5 – z5$ will become $\Delta c3 – 1$ and $\Delta c5 – 3$, respectively. In order to maintain optimality, we must have $\Delta c3 – 1 \leq 0$ and $\Delta c5 – 3 \leq 0$ or $\Delta c3 \leq 1$ and $\Delta c5 \leq 3$

***Change in the coefficients c_j* (i.e. *c_1, c_2 and c_4*) *of basic variables x_1, x_2* and *s_2*:**

$$\text{Min}\left\{\frac{c_j - z_j}{y_{kj} < 0}\right\} \geq \Delta\, c_{Bk} \geq \text{Max}\left\{\frac{c_j - z_j}{y_{kj} > 0}\right\}$$

For $k = 1$ (i.e. basic variable x_1 in row 1), we have

$$\text{Min}\left\{\frac{-3}{-2/3}\right\} \geq \Delta c_1 \geq \text{Max}\left\{\frac{-1}{1/3}\right\}; \quad j = 3, 5 \quad \text{or} \quad 9/2 \geq \Delta c_1 \geq -3$$

The current optimal solution will not change as long as

$$\left(3 + \frac{9}{2}\right) \geq c_1 \geq (3 - 3) \;\; \text{or} \;\; \frac{15}{2} \geq c_1 \geq 0$$

Hence, the current optimal solution will not change as long as: $0 \leq c_1 \leq 15/2$.

For $k = 3$ (i.e. basic variable x_2 in row 3), we have

$$\text{Min}\left\{\frac{-1}{0}\right\} \geq \Delta c_2 \geq \left\{\frac{-3}{1}\right\}; \quad \text{or} \quad \infty \geq \Delta c_2 \geq -3, \quad \text{for} \quad j = 3, 5$$

Hence, the current optimal solution will not change as long as:

$$(5 + \infty) \geq c_2 \geq (5 - 3) \text{ or } \infty \geq c_2 \geq 2.$$

IV Change in the Availability of Resources (*bi*)

Case I : When slack variable is not a basic variable

The procedure for finding the range for 'resource values' within which the current optimal solution remains unchanged is summarized below.

(a) Treat the slack variable corresponding to *resource value* as an entering variable in the solution. For this, calculate exchange ratio (minimum ratio) for every row.

$$\text{Minimum ratio} = \frac{\text{Basic variable value, } \mathbf{x_B}}{\text{Coefficients in a slack variable column}}$$

(b) Find both the lower and upper sensitivity limits.

Lower limit = Original value – Least (smallest) positive ratio or – ∞

Upper limit = Original value + Smallest absolute negative ratio or ∞

Case II: When a slack variable is a basic variable.

The range of variation for the corresponding *resource value* (RHS in a constraint)
is as follows:

Lower limit = Original value – Solution value of slack variable

Upper limit = Infinity (∞)

Case III: Changes in right-hand side when constraints are of the mixed type

- *When surplus variable is not in the basis* (*Basic variable column*, **B**)

Lower limits = Original value – Smallest absolute value of negative minimum ratios or –∞
Upper limit = Original value + Smallest positive minimum ratio or ∞

- *When surplus variable is in the basis* (*Basic variable column*, **B**)

Lower limit = Minus infinity (– ∞)
Upper limit = Original value + Solution value of surplus variable.

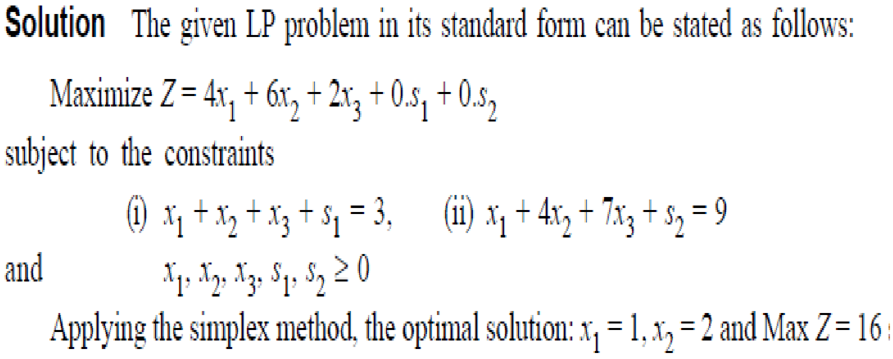
Example 2

Solve the following LP problem : Maximize $Z = 4x_1 + 6x_2 + 2x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \leq 3$, (ii) $x_1 + 4x_2 + 7x_3 \leq 9$ and $x_1, x_2, x_3 \geq 0$.

Discuss the effect of discrete change in the availability of resources from $[3, 9]^T$ to $[9, 6]^T$.



Solution The given LP problem in its standard form can be stated as follows:

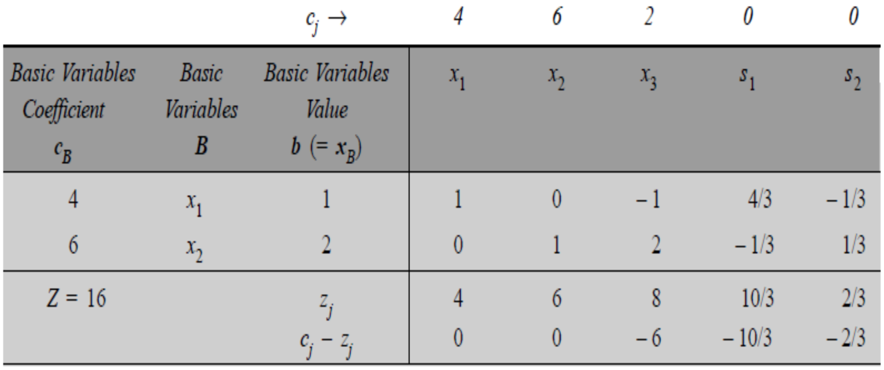
Maximize $Z = 4x_1 + 6x_2 + 2x_3 + 0.s_1 + 0.s_2$

subject to the constraints

(i) $x_1 + x_2 + x_3 + s_1 = 3$, (ii) $x_1 + 4x_2 + 7x_3 + s_2 = 9$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

Applying the simplex method, the optimal solution: $x_1 = 1, x_2 = 2$ and Max $Z = 16$

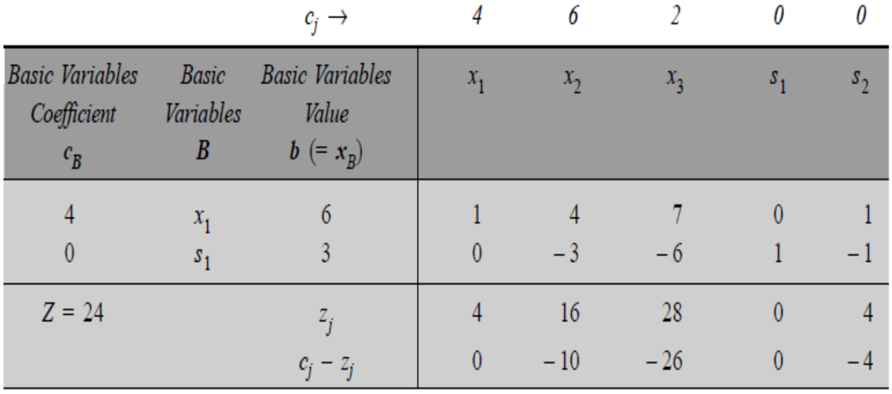


		$c_j \to$	4	6	2	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	– 1	4/3	– 1/3
6	x_2	2	0	1	2	– 1/3	1/3
$Z = 16$		z_j	4	6	8	10/3	2/3
		$c_j - z_j$	0	0	– 6	– 10/3	– 2/3

If new values of the right-hand side constants in the constraints are [9, 6]^{T,} then the new values of the basic variables ($\mathbf{x_B}$ = $\mathbf{B}^{-1}$ $\mathbf{b}$), then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \quad \text{or} \quad x_1 = 10, \text{ and } x_2 = -1$$

Since the value of x_2 is negative, the optimal solution is not feasible. Apply dual simplex to remove this infeasibility.



		$c_j \rightarrow$	4	6	2	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2
4	x_1	6	1	4	7	0	1
0	s_1	3	0	−3	−6	1	−1
$Z = 24$		z_j	4	16	28	0	4
		$c_j - z_j$	0	−10	−26	0	−4

The solution: $x_1 = 6$, $x_2 = 0$, $x_3 = 0$ and Max $Z = 24$.

V Changes in the Input-Out Coefficients (a_{ij}'s)

Suppose that the elements of coefficient matrix **A** are changed. Then two cases arise

(i) Change in a coefficient, when variable is a basic variable, and
(ii) Change in a coefficient, when variable is a non-basic variable.

The range for the discrete change Δa_{ij} in the coefficients of non-basic variable, x_j in
constraint, *i* can be determined by solving following linear inequalities:

$$\text{Max}\left\{\frac{c_j - z_j}{\mathbf{c}_B\,\boldsymbol{\beta}_i > 0}\right\} \le \Delta a_{ij} \le \text{Min}\left\{\frac{c_j - z_j}{\mathbf{c}_B\,\boldsymbol{\beta}_i < 0}\right\}$$

Here $\boldsymbol{\beta}_i$ is the *i*th column unit matrix **B**. If $\mathbf{c}_B\,\boldsymbol{\beta}_i = 0$, then Δa_{ij} is unrestricted in sign.

Suppose a basic variable column $\mathbf{a}_k \in \mathbf{B}$ is changed to $\mathbf{a}_{k^*}$. Then conditions to maintain both feasibility and optimality of the current optimal solution are:

$$\text{(a)}\quad \underset{k \neq p}{\text{Max}}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} > 0}\right\} \le \Delta a_{ij} \le \underset{k \neq p}{\text{Min}}\left\{\frac{-x_{Bk}}{x_{Bk}\,\beta_{pi} - x_{Bp}\,\beta_{ki} < 0}\right\}$$

$$\text{(b)}\quad \text{Max}\left\{\frac{c_j - z_j}{(c_j - z_j)\beta_{pi} - y_{pj}\,\mathbf{c}_B\,\boldsymbol{\beta}_i > 0}\right\} \le \Delta a_{ij} \le \text{Min}\left\{\frac{c_j - z_j}{(c_j - z_j)\beta_{pi} - y_{pj}\,\mathbf{c}_B\,\boldsymbol{\beta}_i < 0}\right\}$$

Example 3. Solve the following LP problem

Maximize $Z = -x_1 + 3x_2 - 2x_3$
subject to the constraints
(i) $3x_1 - x_2 + 2x_3 \leq 7$, (ii) $-2x_1 + 4x_2 \leq 12$, (iii) $-4x_1 + 3x_2 + 8x_3 \leq 10$
and $x_1, x_2, x_3 \geq 0$.

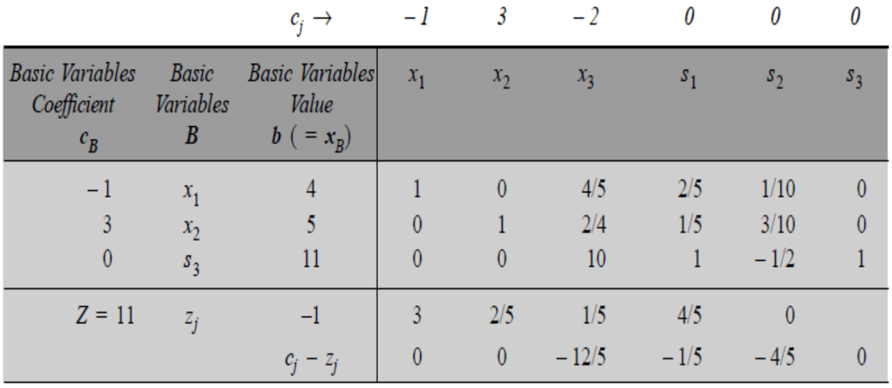
Discuss the effect of the following changes in the optimal solution.

a. Determine the range for discrete changes in the coefficients a_{13} and a_{23} consistent with the optimal solution of the given LP problem.
b. '$x3$'-column in the LP problem is changed from $[\ 2, 0, 8\ ]^T$ to $[\ 3, 1, 6\ ]^T$.

Solution :

Write the given LPP in standard form and apply Simplex (Big M) method. The optimal basic feasible solution shown in the following table is:

$$x_1 = 4,\ x_2 = 5,\ x_3 = 0 \text{ and Max } Z = 11.$$



		$c_j \rightarrow$	-1	3	-2	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	4	1	0	4/5	2/5	1/10	0
3	x_2	5	0	1	2/4	1/5	3/10	0
0	s_3	11	0	0	10	1	$-1/2$	1
$Z = 11$	z_j	-1	3	2/5	1/5	4/5	0	
		$c_j - z_j$	0	0	$-12/5$	$-1/5$	$-4/5$	0

$$\mathbf{c_B}\,\boldsymbol{\beta}_1 = -1(2/5) + 3(1/5) + 0(1) = 1/5$$
$$\mathbf{c_B}\,\boldsymbol{\beta}_2 = -1(1/10) + 3(3/10) + 0(-1/2) = 8/10$$
$$\mathbf{c_B}\,\boldsymbol{\beta}_3 = -1(0) + 3(0) + 0(1) = 0$$

(a) Ranges for discrete change in coefficients a_{13} and a_{23} in the x_3-column vector of computed as:

$$\text{Max}\left\{\frac{c_3 - z_3}{\mathbf{c}_B\, \boldsymbol{\beta}_1}\right\} = \text{Max}\left\{\frac{-12/5}{1/5}\right\} \le \Delta a_{13} \quad \text{or} \quad \Delta a_{13} \ge -12$$

and $$\text{Max}\left\{\frac{c_3 - z_3}{\mathbf{c}_B\, \boldsymbol{\beta}_2}\right\} = \text{Max}\left\{\frac{-12/5}{8/10}\right\} \le \Delta a_{23} \quad \text{or} \quad \Delta a_{23} \ge -3$$

(b) Suppose column vector **a**3 (*x*3-column in Table 6.18) of original LP model is changed from $[2, 0, 8]^T$ to $[3, 1, 6]^T$.

Then new value of $c_3 - z_3^*$ for this column is:

$$\mathbf{B}^{-1}\,\mathbf{a}_3^* = \begin{bmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 13/10 \\ 9/10 \\ 17/2 \end{bmatrix}$$

$$c_3 - z_3^* = c_3 - c_B\,\mathbf{B}^{-1}\,\mathbf{a}_3^* = -2 - [-1, 3, 0]\begin{bmatrix} 13/10 \\ 9/10 \\ 17/2 \end{bmatrix} = -\frac{34}{10}$$

Hence the optimum Solution is obtained.

VI **Addition of a New Variable (Column):**

Example 4:

Discuss the effect on optimality by adding a new variable to the following LP problem with
column coefficients $(3, 3, 3)^T$ and coefficient 5 in the objective function

Minimize $Z = 3x_1 + 8x_2$ subject to the constraints

(i) $x_1 + x_2 = 200$, (ii) $x_1 \le 80$, (iii) $x_2 \ge 60$ and $x_1, x_2 \ge 0$.

Solution The given LP problem in its standard form can be expressed as:

Minimize $Z = 3x_1 + 8x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

(i) $x_1 + x_2 + A_1 = 200$, (ii) $x_1 + s_1 = 80$, (iii) $x_2 - s_2 + A_2 = 60$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Applying the simplex method to obtain optimal solution: $x_1 = 80, x_2 = 120$ and Min $Z = 1{,}200$

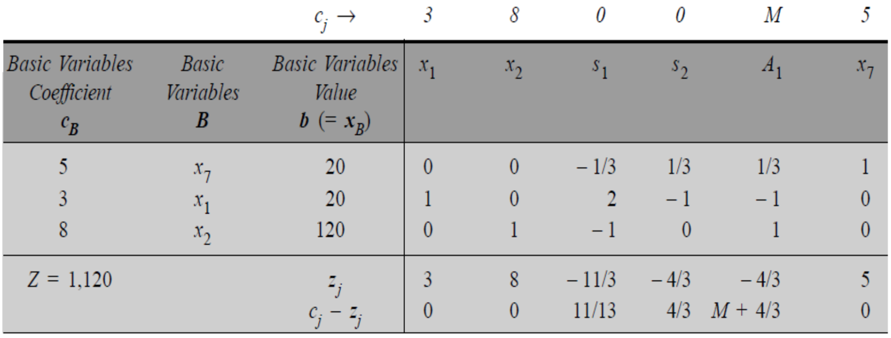
		$c_j \rightarrow$	3	8	0	0	M
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1
0	s_2	60	0	0	-1	1	1
3	x_1	80	1	0	1	0	0
8	x_2	120	0	1	-1	0	1
$Z = 1{,}200$		z_j	3	8	-5	0	8
		$c_j - z_j$	0	0	5	0	$M - 8$

Given that $c_7 = 5$ and the column, $\mathbf{a}_7 = (3, 3, 3)^T$, the changes in the optimal solution can be evaluated as follows:

$$c_7 - z_7 = c_7 - \mathbf{c_B}\, \mathbf{B}^{-1}\, \mathbf{a}_7 = 5 - (0,\ 3,\ 8) \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -4$$

$$\mathbf{a}_7^{*} = \mathbf{B}^{-1}\, \mathbf{a}_7 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

		$c_j \rightarrow$	3	8	0	0	M	5	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1	x_7	Min. Ratio x_B/x_7
0	s_2	60	0	0	-1	1	1	③	60/3 →
3	x_1	80	1	0	1	0	0	3	80/3
8	x_2	120	0	1	-1	0	1	0	—
$Z = 1{,}200$		z_j	3	8	-5	0	8	9	
		$c_j - z_j$	0	0	5	0	$M - 8$	-4	
								↑	



		$c_j \rightarrow$	3	8	0	0	M	5
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	A_1	x_7
5	x_7	20	0	0	– 1/3	1/3	1/3	1
3	x_1	20	1	0	2	– 1	– 1	0
8	x_2	120	0	1	– 1	0	1	0
Z = 1,120		z_j	3	8	– 11/3	– 4/3	– 4/3	5
		$c_j - z_j$	0	0	11/13	4/3	M + 4/3	0

Since all $c_j - z_j \geq 0$ in the above table , the solution is optimal, with $x_1 = 20$, $x_2 = 120$, $x_7 = 20$ and Min $Z = 1{,}120$.

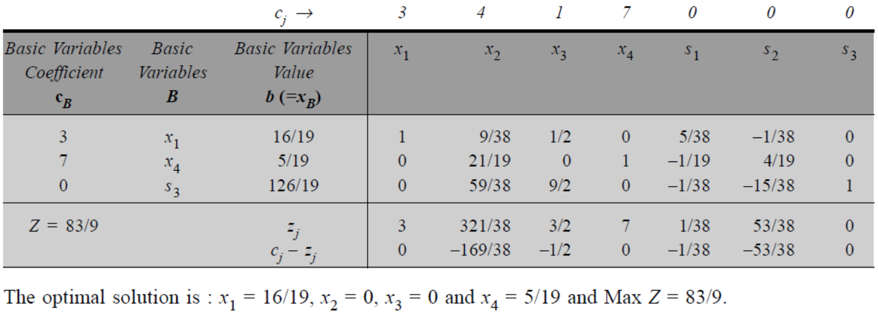
VII Addition of a New Constraint (Row):

Example 5. Solve the following LP problem

Maximize $Z = 3x_1 + 4x_2 + x_3 + 7x_4$

subject to the constraints
(i) $8x_1 + 3x_2 + 4x_3 + x_4 \leq 7$, (ii) $2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$, (iii) $x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8$ and $x_1, x_2, x_3, x_4 \geq 0$.
Discuss the effect on the optimal solution of the LP problem of adding an additional constraint: $2x_1 + 3x_2 + x_3 + 5x_4 \leq 4$.

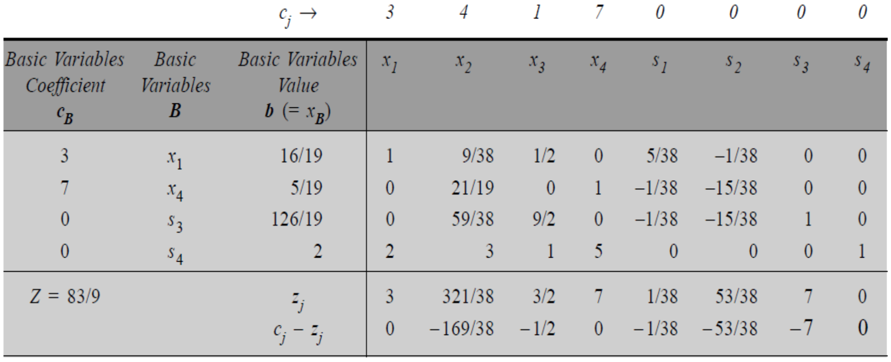
Solution : The optimal solution of the LP problem is



		$c_j \rightarrow$	3	4	1	7	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (=x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3
3	x_1	16/19	1	9/38	1/2	0	5/38	–1/38	0
7	x_4	5/19	0	21/19	0	1	–1/19	4/19	0
0	s_3	126/19	0	59/38	9/2	0	–1/38	–15/38	1
Z = 83/9		z_j	3	321/38	3/2	7	1/38	53/38	0
		$c_j - z_j$	0	–169/38	–1/2	0	–1/38	–53/38	0

The optimal solution is : $x_1 = 16/19$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 5/19$ and Max $Z = 83/9$.

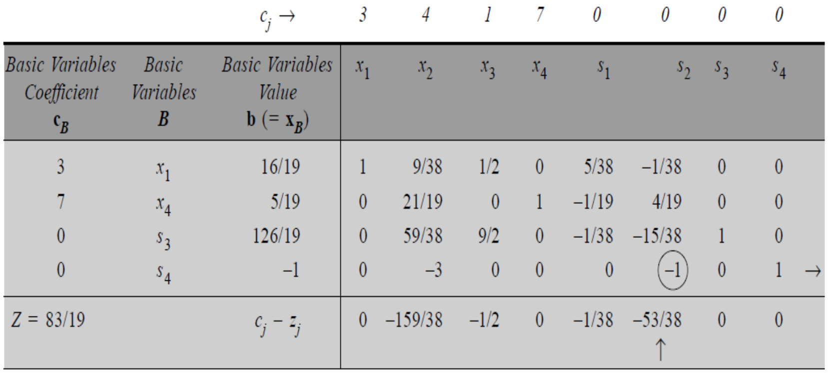
Addition of New Constraint



		$c_j \rightarrow$	3	4	1	7	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	16/19	1	9/38	1/2	0	5/38	–1/38	0	0
7	x_4	5/19	0	21/19	0	1	–1/38	–15/38	0	0
0	s_3	126/19	0	59/38	9/2	0	–1/38	–15/38	1	0
0	s_4	2	2	3	1	5	0	0	0	1
Z = 83/9		z_j	3	321/38	3/2	7	1/38	53/38	7	0
		$c_j - z_j$	0	–169/38	–1/2	0	–1/38	–53/38	–7	0

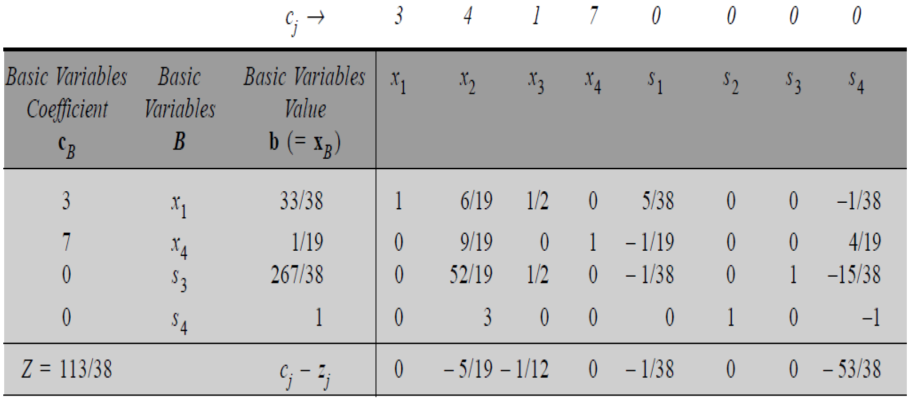
The basis matrix B has been disturbed due to Row 4. Thus the coefficients in Row 4 under column x_1 and x_4 should become zero. This can be done by applying following row operations.

$$R_4(\text{new}) = R_4\ (\text{old}) - 2R_1 - 5R_2$$



		$c_j \rightarrow$	3	4	1	7	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	16/19	1	9/38	1/2	0	5/38	–1/38	0	0
7	x_4	5/19	0	21/19	0	1	–1/19	4/19	0	0
0	s_3	126/19	0	59/38	9/2	0	–1/38	–15/38	1	0
0	s_4	–1	0	–3	0	0	0	⓵ –1	0	1 →
Z = 83/19		$c_j - z_j$	0	–159/38	–1/2	0	–1/38	–53/38	0	0
								↑		

The solution shown is not feasible because $s_4 = -1$. Thus the dual simplex method is applied to the obtained optimal basic feasible solution. The new solution is shown in the following table



		$c_j \rightarrow$	3	4	1	7	0	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	33/38	1	6/19	1/2	0	5/38	0	0	–1/38
7	x_4	1/19	0	9/19	0	1	– 1/19	0	0	4/19
0	s_3	267/38	0	52/19	1/2	0	– 1/38	0	1	–15/38
0	s_4	1	0	3	0	0	0	1	0	–1
Z = 113/38		$c_j - z_j$	0	– 5/19	– 1/12	0	– 1/38	0	0	– 53/38

Since all $c_j - z_j \leq 0$ and all $x_{Bi} \geq 0$, therefore the current solution is optimal. The new

solution value are:

$x_1 = 33/38$, $x_2 = 0$, $x3 = 0$, $x_4 = 1/19$ and Max $Z = 113/38$.

UNIT – II – Integer Programming Problem

I Indroduction

In linear programming, each decision variable, slack and/or surplus variable is allowed to take any discrete or fractional value. However, there are certain real-life problems in which the fractional value of the decision variables has no significance. For example, it does not make sense to say that 1.5 men will be working on a project or 1.6 machines will be used in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum value of the variables to the nearest integer value. This approach can be easy in terms of economy of effort, time, and the cost that might be required to derive an integer solution. This solution, however, may not satisfy all the given constraints. Secondly, the value of the objective function so obtained may not be the optimal value. All such difficulties can be avoided if the given problem, where an integer solution is required, is solved by integer programming techniques.

II TYPES OF INTEGER PROGRAMMING PROBLEMS

Linear integer programming problems can be classified into three categories:
(i) ***Pure (all) integer programming problems*** in which all decision variables are restricted to integer
values.
(ii) ***Mixed integer programming problems*** in which some, but not all, of the decision variables are
restricted to integer values.
(iii) ***Zero-one integer programming problems*** in which all decision variables are restricted to integer values
of either 0 or 1.

The pure integer linear programming problem in its standard form can be stated as follows:
Maximize $Z = c_1 x_1 + c_2 x_2 + . . . + c_n x_n$
subject to the constraints
$a_{11} x_1 + a_{12} x_2 + . . . + a_{1n} x_n = b_1$
$a_{21} x_1 + a_{22} x_2 + . . . + a_{2n} x_n = b_2$
.
.
.
$a_{m1} x_1 + a_{m2} x_2 + . . . + a_{mn} x_n = b_m$
and $x_1, x_2, . . ., x_n \geq 0$ and are integers.

III ENUMERATION AND CUTTING PLANE SOLUTION CONCEPT

This method is based on creating a sequence of linear inequalities called ***cuts.*** Such a *cut* reduces a part of the feasible region of the given

LP problem, leaving out a feasible region of the integer LP problem. The hyperplane boundary of a cut is called the ***cutting plane.***

Example 1:

Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

subject to the constraints

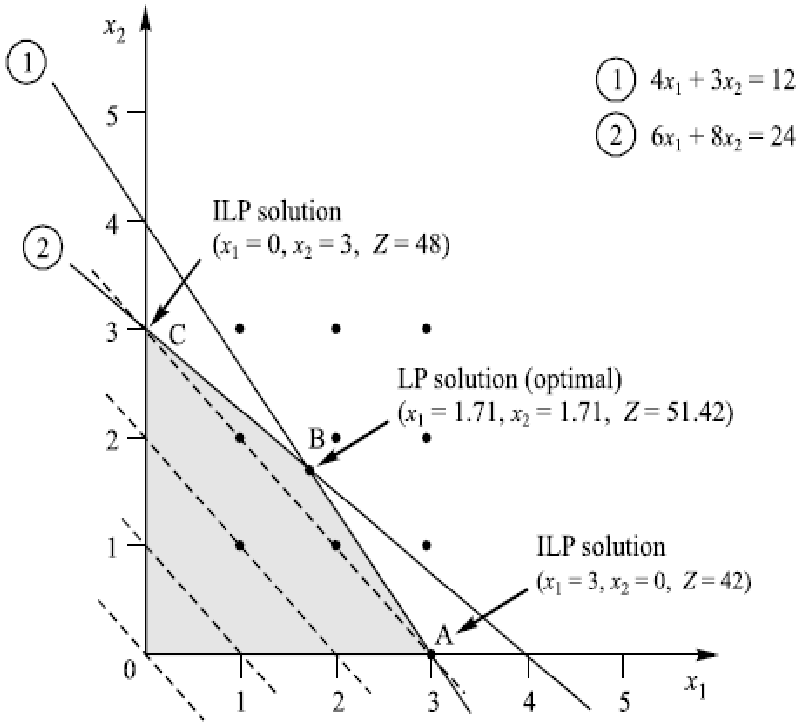
(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

and x_1, $x_2 \geq 0$ and are integers.

Relaxing the integer requirement, the problem is solved graphically. The optimal solution to this LP problem is:

$x_1 = 1.71$, $x_2 = 1.71$ and Max $Z = 51.42$.

This solution does not satisfy the integer requirement of variables x_1 and x_2. Rounding off this solution to $x_1 = 2$, $x_2 = 2$ does not satisfy both the constraints and therefore, the solution is infeasible. The dots in Fig. also referred to as *lattice points,* represent all of the integer solutions that lie within the feasible solution space of the LP problem. However, it is difficult to evaluate every such point in order to determine the value of the objective function.



The optimal integer solution is: $x_1 = 0$, $x_2 = 3$ and Max $Z = 48$. The lattice point, *C* is not even adjacent to the most desirable LP problem solution corner, B.

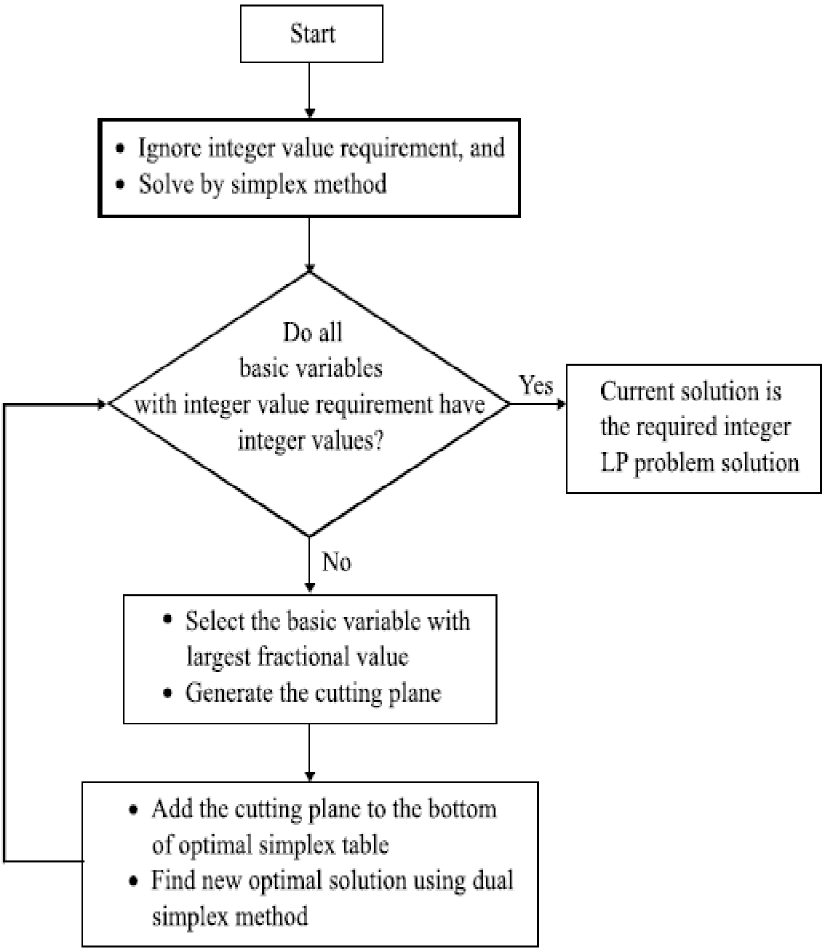
IV GOMORY'S ALL INTEGER CUTTING PLANE METHOD

Gomory's algorithm has the following properties.

(i) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.
(ii) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem/

Steps of Gomory's All Integer Programming Algorithm

1. **Step 1: Initialization** Formulate the standard integer LP problem.
2. **Step 2: Test the optimality.**
3. **Step 3: Generate cutting plane**
4. **Step 4: Obtain the new solution**



Example 2 : Solve the following Integer LP problem using Gomory's cutting plane method.

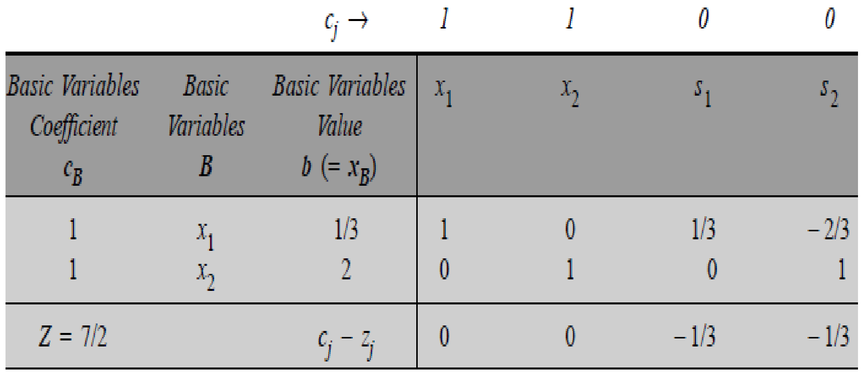
Maximize $Z = x_1 + x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \leq 5$, (ii) $x_2 \leq 2$

and $x1$, $x2 \geq 0$ and are integers.

Solution : Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.



		$c_j \rightarrow$	1	1	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	– 2/3
1	x_2	2	0	1	0	1
$Z = 7/2$		$c_j - z_j$	0	0	– 1/3	– 1/3

The optimal solution of LP problem is: $x_1 = 1/3$, $x_2 = 2$ and Max $Z = 7/2$.

To obtain an optimal solution satisfying integer value requirement Gomory cut as follows:

$$\frac{1}{3} = x_1 + 0.x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2 \qquad (x_1\text{-source row})$$

The factoring of numbers (integer plus fractional) in the x_1-source row gives

$$\left(0+\frac{1}{3}\right) = (1+0)\,x_1 + \left(0+\frac{1}{3}\right) s_1 + \left(-1+\frac{1}{3}\right) s_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

Rearranging all of the integer coefficients on the left-hand side, we get

$$\frac{1}{3} + (s_2 - x_1) = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Since value of variables x_1 and s_2 is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3}s_1 + \frac{1}{3}s_2 \qquad \text{(Ref. Eq. 4)}$$

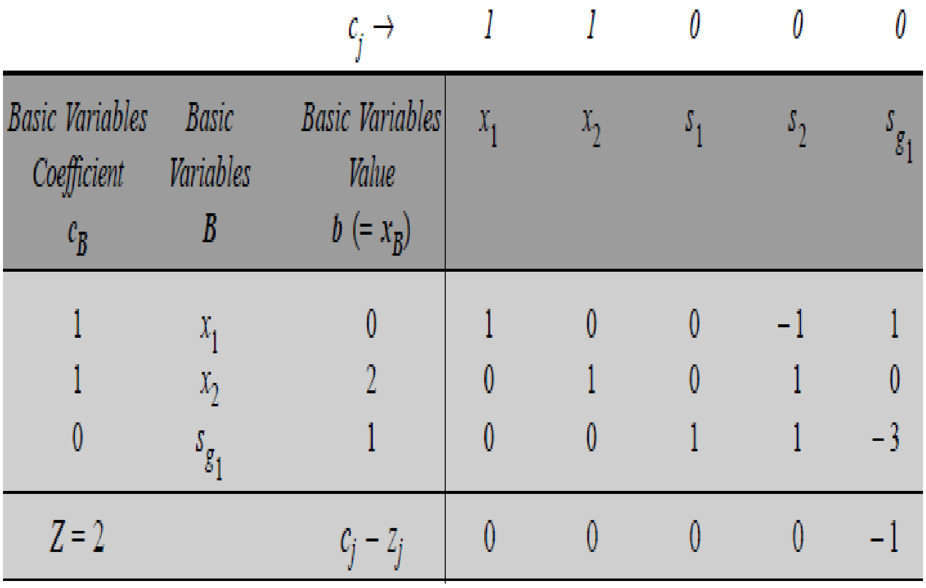
$$\frac{1}{3} + s_{g1} = \frac{1}{3}s_1 + \frac{1}{3}s_2 \quad \text{or} \quad s_{g1} - \frac{1}{3}s_1 - \frac{1}{3}s_2 = -\frac{1}{3} \qquad \text{(Cut I)}$$

where s_{g1} is the new non-negative (integer) slack variable.

Adding this equation (also called Gomory cut) at the bottom of Table 7.1, the new values so obtained

		$c_j \rightarrow$	1	1	0	0	0
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* b (= x_B)	x_1	x_2	s_1	s_2	s_{g1}
1	x_1	1/3	1	0	1/3	– 2/3	0
1	x_2	2	0	1	0	10	0
0	s_{g1}	– 1/3	0	0	– 1/3	– 1/3	1 →
Z = 7/2		$c_j - z_j$	0	0	– 1/3	– 1/3	0
		Ratio: Min $(c_j - z_j)/y_{3j}$ (< 0)	–	–	1 ↑	1	–

The new solution is obtained by applying the following row operations. $R3$(new) → R_3(old) × –3; R_1(new) → R_1(old) – (1/3) R_3(new)



		$c_j \rightarrow$	1	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	s_{g_1}	1	0	0	1	1	-3
$Z = 2$		$c_j - z_j$	0	0	0	0	-1

The solution: $x_1 = 0$, $x_2 = 2$, $s_{g1} = 1$ and Max $Z = 2$, is an optimal basic feasible solution of the given ILP problem.

V GOMORY'S MIXED-INTEGER CUTTING PLANE METHOD

Solve the following mixed-integer programming problem
Maximize $Z = x_1 + x_2$
subject to the constraints
(i) $3x_1 + 2x_2 \leq 5$, (ii) $x_2 \leq 2$
and $x_1, x_2 \geq 0$, x_1 non-negative integer.

Solution Converting given LP problem into its standard form as follows:
Maximize $Z = x_1 + x_2 + 0s_1 + 0s_2$
subject to the constraints
(i) $3x_1 + 2x_2 + s_1 = 5$, (ii) $x_2 + s_2 = 2$
and $x_1, x_2 \geq 0$; x_1 is non-negative integer

Apply simplex method to obtain an optimal solution ignoring the integer restriction on x_1. The optimal noninteger solution is: $x_1 = 1/3$, $x_2 = 2$ and Max $Z = 7/2$.

		$c_j \rightarrow$	1	1	0	0
Basic Variables Coefficient c_B	Basic Variables B	Solution Value $b\ (= x_B)$	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	−2/3
1	x_2	2	0	1	0	1
$Z = 7/3$		$c_j - z_j$	0	0	−1/3	−1/3

Since in the current optimal solution the variable x_1, that is restricted to take integer value, is not an integer, therefore generating Gomory cut considering x_1-row as follows:

$$\frac{1}{3} = x_1 + \frac{1}{3}s_1 - \frac{2}{3}s_2 \quad (x_1\text{-source row})$$

Since the coefficient of non-basic variable, s_1 is positive, therefore after applying rule (15) we

$$f_{13}^* = \frac{1}{3}\left[f_{rj}^* = a_{rj}\ ;\ a_{rj} \geq 0, \text{ where } r = 1 \text{ and } j = 3\right]$$

The coefficient of non-basic variable, s_2 is negative, so by applying rule (15), we get:

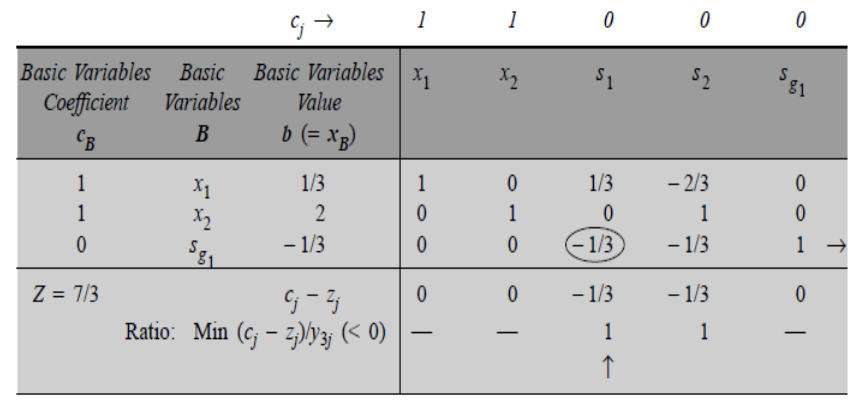
$$f_{14}^* = \left(\frac{f_r}{f_r - 1}\right)a_{rj} = \left\{\frac{1/3}{(1/3) - 1}\right\}\left(-\frac{2}{3}\right) = \frac{1}{3}$$

Thus, Gomory's mixed integer cut (Ref. Eq. 14) becomes

$$s_{g_1} = -\frac{1}{3} + \left(\frac{1}{3}s_1 + \frac{1}{3}s_2\right)$$

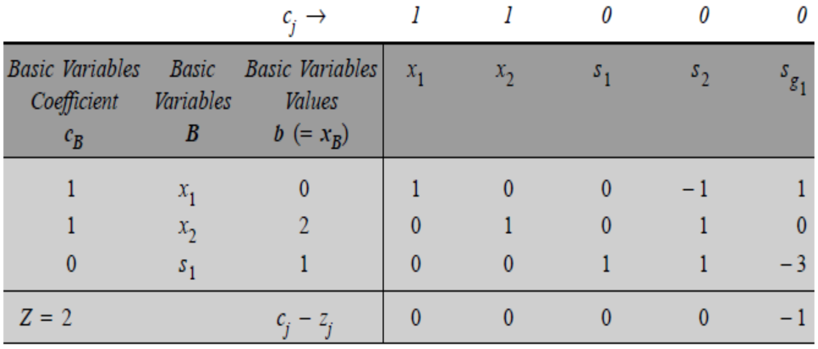
or
$$-\frac{1}{3}s_1 - \frac{1}{3}s_2 + s_{g_1} = -\frac{1}{3} \quad \text{(Mixed integer cut I)}$$

where s_{g_1} is Gomory's slack variable.



		$c_j \rightarrow$	1	1	0	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}	
1	x_1	1/3	1	0	1/3	− 2/3	0	
1	x_2	2	0	1	0	1	0	
0	s_{g_1}	− 1/3	0	0	$\boxed{-1/3}$	− 1/3	1	→
$Z = 7/3$		$c_j - z_j$	0	0	− 1/3	− 1/3	0	
	Ratio:	Min $(c_j - z_j)/y_{3j}\ (< 0)$	—	—	1	1	—	
					↑			

Applying the dual simplex method, we obtain the revised solution



		$c_j \rightarrow$	1	1	0	0	0
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Values $b\ (= x_B)$	x_1	x_2	s_1	s_2	s_{g_1}
1	x_1	0	1	0	0	− 1	1
1	x_2	2	0	1	0	1	0
0	s_1	1	0	0	1	1	− 3
$Z = 2$		$c_j - z_j$	0	0	0	0	− 1

Therefore the required mixed integer optimal solution: $x_1 = 0$, $x_2 = 2$, $s_1 = 1$ and Max $Z = 2$.

UNIT – III – Goal Programming

I Introduction

Goal programming (GP) technique (or approach) is used for solving a multi-objective optimization problem that balances trade-off in conflicting objectives, i.e., GP technique helps in attaining the ‘satisfactory’ level of all objectives. The method of formulating a mathematical model of GP is same as that of LP problem. However, while formulating multiple, and often conflicting, incommensurable (dimension of goals and unit of measurement may not be same) goals, in a particular priority order (hierarchy) are taken into consideration. A particular priority level (or order) is decided in accordance with the importance of each goal and sub-goals given in a problem. The priority structure helps to deal with all goals that cannot be completely and/or simultaneously achieved, in such a manner that more important goals are achieved first, at the expense of the less important ones.

Linear programming has two major limitations from its application point of view:

(i) single objective function, and
(ii) same unit of measurement of various resources

II Goal Programming

In GP, instead of trying to minimize or maximize the objective function directly, as in the case of an LP, the deviations from established goals within the given set of constraints are minimized. In the simplex algorithm of linear programming such deviational variables are called *slack variables* and they are used only as dummy variables. In GP, these slack variables take on a new significance. The deviational variables are represented in two dimensions – both positive and negative deviations from each goal and subgoal. These deviational variables represent the extent to which the target goals are not achieved. The objective function then becomes the minimization of a sum of these deviations, based on the relative importance within the pre-emptive priority structure assigned to each deviation.

The deviational variables in goal programming model are equivalent to slack and surplus variables (the amount by which the objective is below or above the target) in linear programming model.

Model Formulation :

Example 1: A manufacturing firm produces two types of products: A and B. The unit profit from product A is Rs 100 and that of product B is Rs 50. The goal of the firm is to earn a total profit of exactly Rs 700 in the next week.
Formulation :

To interpret the profit goal in terms of subgoals, which are sales volume of products, let

x_1 and x_2 = number of units of products A and B to be produced, respectively.The single goal of profit maximization is stated as:

Maximize (profit) $Z = 100x_1 + 50x_2$

Since the goal of the firm is to earn a target profit of Rs 700 per week, the profit goal can be restated to allow for underachievement or overachievement as:

$$100x_1 + 50x_2 + d_1^{-} d_1^{+} = 700$$

Now the goal programming model can be formulated as follows:

$$\text{Minimize } Z = d_1^{-} + d_1^{+}$$

subject to the constraints

$$100x_1 + 50x_2 + d_1^{-} - d_1^{+} = 700$$

and $x_1, x_2, d_1^{-}, d_1^{+} \geq 0$

where d_1^{-} = underachievement of the profit goal of Rs 700

d_1^{+} = overachievement of the profit goal of Rs 700

If the profit goal is not completely achieved, then the slack in the profit goal will be expressed by a negative deviational (underachievement) variable, d_1^{-}, from the goal. But if the solution shows a profit in excess of Rs 700, the surplus in the profit will be expressed by positive deviational (overachievement) variable d_1^{+} from the goal. If the profit goal of exactly Rs 700 is achieved, both d_1^{+} and d_1^{-} will be zero.

In the given example, there are an infinite number of combinations of x_1 and x_2 that will achieve the profit goal. The required solution will be any linear combination of x_1 and x_2 between the two points:

$$x_1 = 7, x_2 = 0 \text{ and } x_1 = 0, x_2 = 14.$$

This straight line is exactly the iso-profit function line when the total profit is Rs 700.

III Steps to Formulate GP Model

The procedure (algorithm) to formulate a GP model is summarized below:

1. Identify the goals and constraints based on the availability of resources (or constraints) that may restrict achievement of the goals (targets).
2. Determine the priority to be associated with each goal in such a way that goals with priority level P_1 are most important, those with priority level P_2 are next most important, and so on.
3. Define the decision variables.
4. Formulate the constraints in the same manner as in LP model.
5. For each constraint, develop a equation by adding deviational variables d_i^{+} and d_i^{-}. These variables indicate the possible deviations below or above the target value (right-hand side of each constraint).
6. Write the objective function in terms of minimizing a prioritized function of the deviational variables.

IV GRAPHICAL SOLUTION METHOD FOR GOAL PROGRAMMING

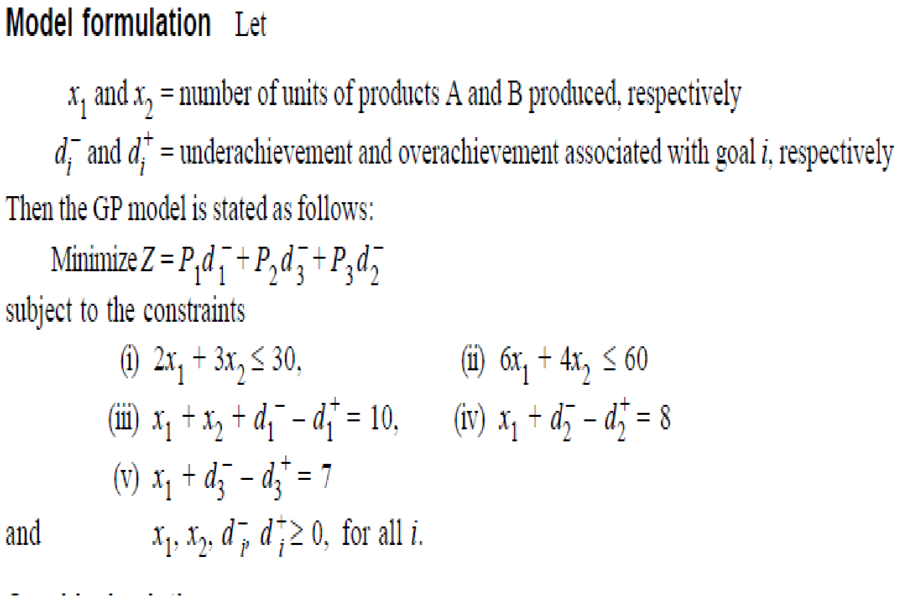
The graphical solution method for goal programming model is similar to the graphical solution method for linear programming model. In this case the feasible solution space (region) is indicated by goal priorities in such a way that the deviation from the goal with the highest priority is minimized to the fullest extent possible, before the deviation from the next priority goal is minimized. If goal constraints are stated only in terms of deviational variables, then such constraints must be restated in terms of the real variables, before proceeding with the graphical solution.

Example 2

A firm produces two products A and B. Each product must be processed through two departments namely 1 and 2. Department 1 has 30 hours of production capacity per day, and department 2 has 60 hours. Each unit of product A requires 2 hours in department 1 and 6 hours in department 2. Each unit of product B requires 3 hours in department 1 and 4 hours in department 2. Management has rank ordered the following goals it would like to achieve in determining the daily product mix:

*P*1 : Minimize the underachievement of joint total production of 10 units.
*P*2 : Minimize the underachievement of producing 7 units of product B.
*P*3 : Minimize the underachievement of producing 8 units of product A.

Formulate this problem as a GP model and then solve it by using the graphical method.



Model formulation Let

x_1 and x_2 = number of units of products A and B produced, respectively

d_i^- and d_i^+ = underachievement and overachievement associated with goal i, respectively

Then the GP model is stated as follows:

$$\text{Minimize } Z = P_1 d_1^- + P_2 d_3^- + P_3 d_2^-$$

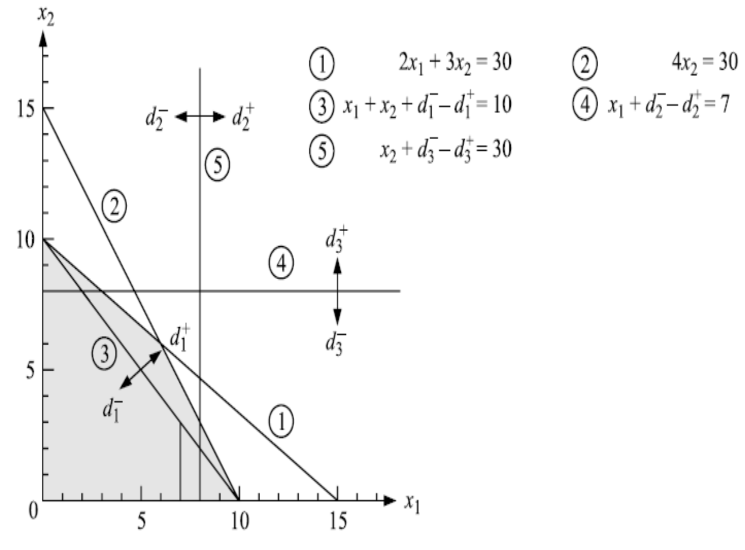
subject to the constraints

(i) $2x_1 + 3x_2 \leq 30,$ (ii) $6x_1 + 4x_2 \leq 60$

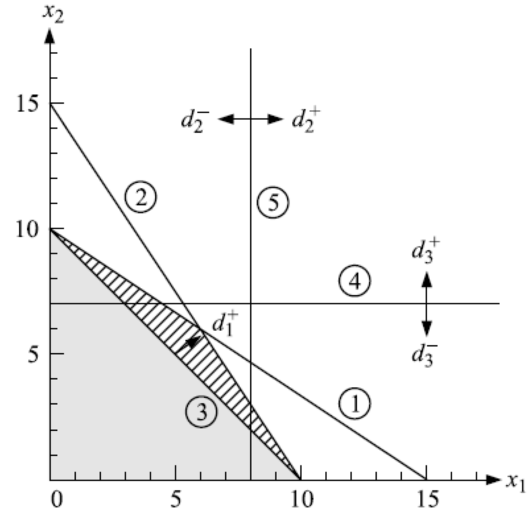
(iii) $x_1 + x_2 + d_1^- - d_1^+ = 10,$ (iv) $x_1 + d_2^- - d_2^+ = 8$

(v) $x_1 + d_3^- - d_3^+ = 7$

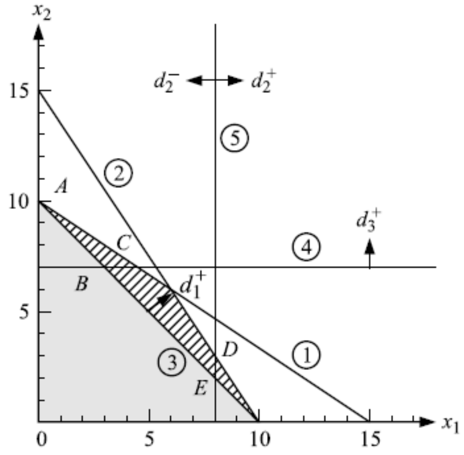
and $x_1, x_2, d_i^-, d_i^+ \geq 0,$ for all i.



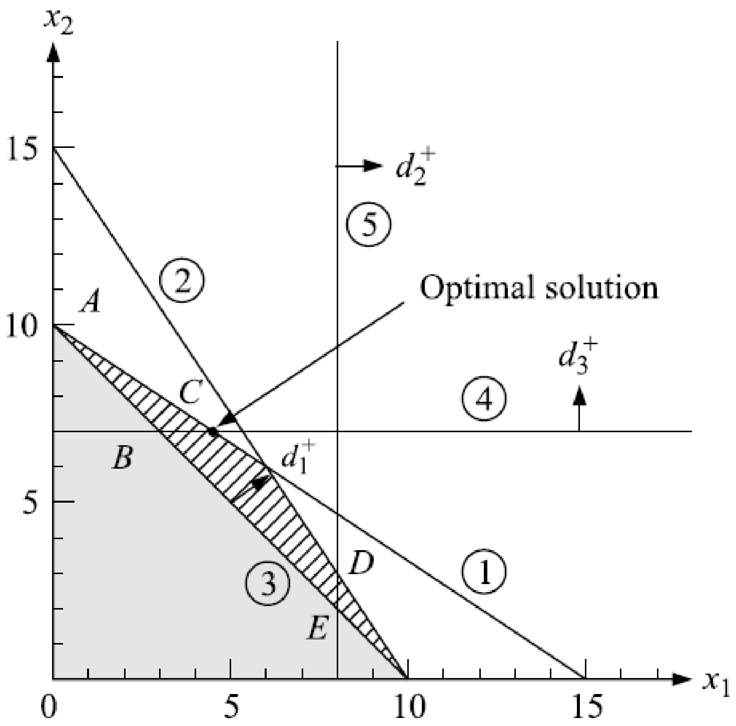
(a)



(b) First Goal Achieved by Eliminating d_1^- Area



(c) Second Goal Achieved by Eliminating d_2^- Area



(d) Third Goal Achieved by Eliminating d_3^- Area

V MODIFIED SIMPLEX METHOD OF GOAL PROGRAMMING

The simplex method for solving a GP problem is similar to that of an LP problem. The features of the simplex method for the GP problem are:

1. The z_j and $c_j - z_j$ values are computed separately for each of the ranked goals, P_1, P_2, ... This is because different goals are measured in different units. These are shown from bottom to top, i.e. first priority goal (P_1) is shown at the bottom and least priority goal at the top. The optimality criterion z_j or $c_j - z_j$ becomes a matrix of $k \times n$ size, where k represents the number of pre-emptive priority levels and n is the number of variables including both decision and deviational variables.
2. First examine $c_j - z_j$ values in the $P1$-row. If all $c_j - z_j \leq 0$ at the highest priority levels in the same column, then the optimal solution been obtained.

If $c_j - z_j > 0$, at a certain priority level, and there is no negative entry at higher unachieved priority levels, in the same column, the current solution is not optimal.

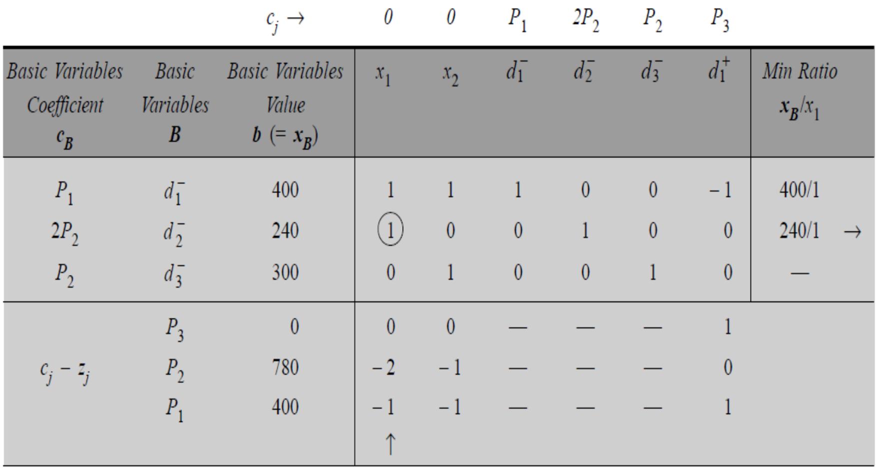
3. If the target value of each goal in **x_B**-column is zero, the solution is optimal.
4. To determine the variable to be entered into the new solution mix, start examining ($c_j - z_j$) row of highest priority (P_1) and select the largest negative value. Otherwise, move to the next higher priority (P_2) and select the largest negative value.

5. Apply the usual procedure for calculating the ‘minimum ratio’ to choose a variable that needs to leave the current solution mix (basis).
6. Any negative value in the $(c_j - z_j)$ row that has positive $(c_j - z_j)$ value under any lower priority rows are ignored. This is because that deviations from the highest priority goal would be increased with the entry of this variable in the solution mix.

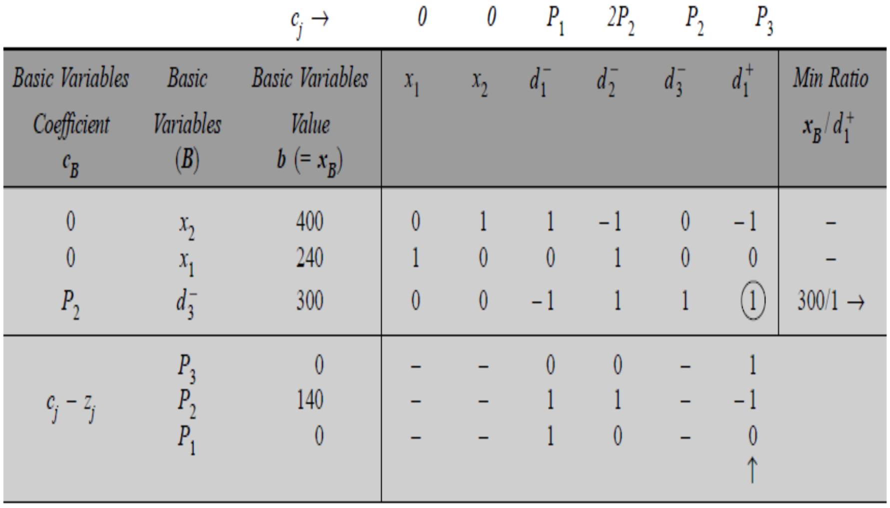
Example 3

Use modified simplex method to solve the following GP problem.
Minimize $Z = P_1\, d_1^{\,-} + P_2(2\, d_2^{\,-} + d_{\,3}^{\,-}) + P_3\, d_{\,1}^{\,+}$
subject to the constraints
(i) $x_1 + x_2 + d_1^- - d_1^+ = 400$, (ii) $x_1 + d_{\,2}^{\,-} = 240$,
(iii) $x_1 + d_{\,3}^{\,-} = 300$
and x_1 , x_2 , d_1^- , d_1^+ , d_2^- , $d_3^- \geq 0$

Solution : The initial simplex table for this problem is presented .The basic assumption in formulating the initial table of the GP problem is the same as that of the LP problem. In goal programming, the pre-emptive priority factors and differential weights correspond to the c_j values in linear programming.

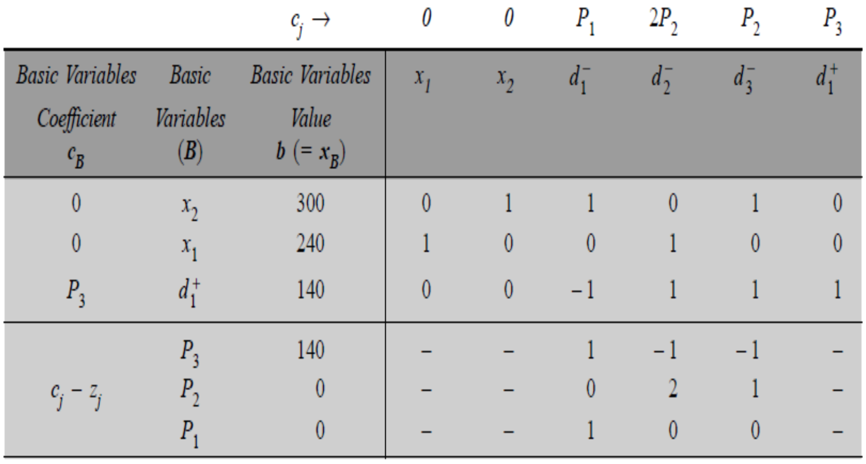


		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b\ (= x_B)$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Min Ratio x_B/x_1
P_1	d_1^-	400	1	1	1	0	0	–1	400/1
$2P_2$	d_2^-	240	①	0	0	1	0	0	240/1 →
P_2	d_3^-	300	0	1	0	0	1	0	—
$c_j - z_j$	P_3	0	0	0	—	—	—	1	
	P_2	780	–2	–1	—	—	—	0	
	P_1	400	–1	–1	—	—	—	1	
			↑						



		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
Basic Variables Coefficient c_B	Basic Variables (B)	Basic Variables Value $b\ (= x_B)$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Min Ratio x_B / d_1^+
0	x_2	400	0	1	1	–1	0	–1	–
0	x_1	240	1	0	0	1	0	0	–
P_2	d_3^-	300	0	0	–1	1	1	①	300/1 →
$c_j - z_j$	P_3	0	–	–	0	0	–	1	
	P_2	140	–	–	1	1	–	–1	
	P_1	0	–	–	1	0	–	0	
								↑	

The largest negative value in P_2-row is selected in order to determine the key column. All $c_j - z_j$ values in the P_2-row are either positive or zero. Thus, the second goal (P_2) is fully achieved.



		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3
Basic Variables Coefficient c_B	Basic Variables (B)	Basic Variables Value $b\ (= x_B)$	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+
0	x_2	300	0	1	1	0	1	0
0	x_1	240	1	0	0	1	0	0
P_3	d_1^+	140	0	0	−1	1	1	1
$c_j - z_j$	P_3	140	–	–	1	−1	−1	–
	P_2	0	–	–	0	2	1	–
	P_1	0	–	–	1	0	0	–

It may be noted in Table that there are two negative values in the P_3-row. However, we could not choose d_2^- or d_3^- as the key column because there is already a positive value at a higher priority level (P_2). Hence, the optimal Solution

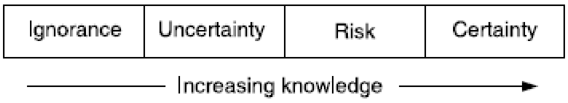
$$x_1 = 240,\ \ x_2 = 300,\ d_1^- = d_2^- = d_3^- = 0,\ d_1^+ = 140\ .$$

UNIT – IV – Decision and Game Theory

DecisionTheory

I INTRODUCTION :

The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making acceptable decisions on time. To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action. Decision theory is both descriptive and prescriptive business modelling approach to classify the degree of knowledge and compare expected outcomes due to several courses of action. The degree of knowledge is divided into four categories: *complete knowledge (i.e. certainty), ignorance, risk and uncertainty*



II STEPS OF DECISION-MAKING PROCESS

The decision-making process involves the following steps:

1. Identify and define the problem.

2. List all possible future events (not under the control of decision-maker) that are likely to occur.

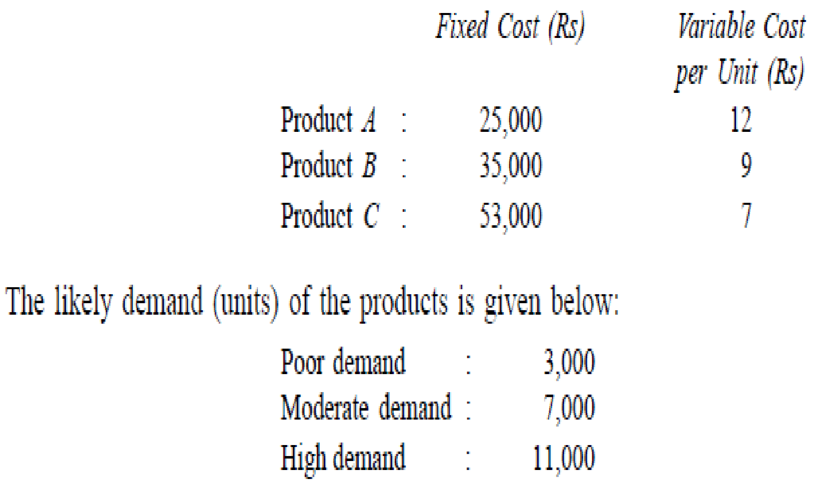
3. Identify all the *courses of action* available to the decision-maker.

4. Express the payoffs (p_{ij}) resulting from each combination of course of action and state of nature.

5. Apply an appropriate decision theory model to select the best course of action from the given list on the basis of a criterion (measure of effectiveness) to get optimal (desired) payoff.

Example :

A firm manufactures three types of products. The fixed and variable costs are given below:



	Fixed Cost (Rs)	Variable Cost per Unit (Rs)
Product *A* :	25,000	12
Product *B* :	35,000	9
Product *C* :	53,000	7

The likely demand (units) of the products is given below:

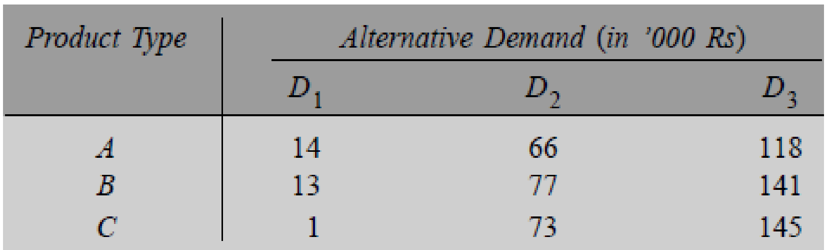
Poor demand :	3,000
Moderate demand :	7,000
High demand :	11,000

If the sale price of each type of product is Rs 25, then prepare the payoff matrix.

Solution :

Let D_1, D_2 and D_3 be the poor, moderate and high demand, respectively. The payoff is determined as:
Payoff = Sales revenue – Cost
The calculations for payoff (in ’000 Rs) for each pair of alternative demand (course of action) and the
types of product (state of nature) are shown below:
$D_1\ A = 3 \times 25 - 25 - 3 \times 12 = 14$
$D_2\ A = 7 \times 25 - 25 - 7 \times 12 = 66$
$D_1\ B = 3 \times 25 - 35 - 3 \times 9 = 13$
$D_2\ B = 7 \times 25 - 35 - 7 \times 9 = 77$
$D_1\ C = 3 \times 25 - 53 - 3 \times 7 = 1$
$D_2\ C = 7 \times 25 - 53 - 7 \times 7 = 73$
$D_3\ A = 11 \times 25 - 25 - 11 \times 12 = 118$
$D_3\ B = 11 \times 25 - 35 - 11 \times 9 = 141$
$D_3\ C = 11 \times 25 - 53 - 11 \times 7 = 145$

The payoff values are



Product Type	*Alternative Demand (in ’000 Rs)*		
	D_1	D_2	D_3
A	14	66	118
B	13	77	141
C	1	73	145

III TYPES OF DECISION-MAKING ENVIRONMENTS

1. Decision-Making under Certainty
2. Decision-Making under Risk
3. Decision-Making under Uncertainty

The following criteria of decision-making under uncertainty have been discussed in this section.

(i) Optimism (Maximax or Minimin) criterion
(ii) Pessimism (Maximin or Minimax) criterion
(iii) Equal probabilities (Laplace) criterion
(iv) Coefficient of optimism (Hurwiez) criterion
(v) Regret (salvage) criterion

Example A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S_1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S_2), or may mall a small change in the composition of the existing product, backing it with the word 'New' and a negligible increase in price (S_3). The three possible states of nature or events are: (i) high increase in sales (N_1), (ii) no change in sales (N_2) and (iii) decrease in sales (N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table:

	States of Nature		
Strategies	N_1	N_2	N_3
S_1	7,00,000	3,00,000	1,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

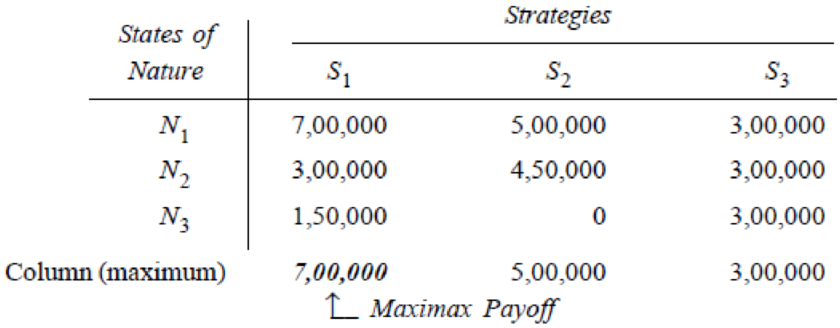
Which strategy should the concerned executive choose on the basis of
(a) Maximin criterion (b) Maximax criterion
(c) Minimax regret criterion (d) Laplace criterion?

Solution The payoff matrix is rewritten as follows:
Maxmin Criterion

States of Nature	*Strategies*		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column (minimum)	1,50,000	0	**3,00,000**← *Maximin Payoff*

The maximum of column minima is 3,00,000. Hence, the company should adopt strategy S_3.

Maxmax Criterion

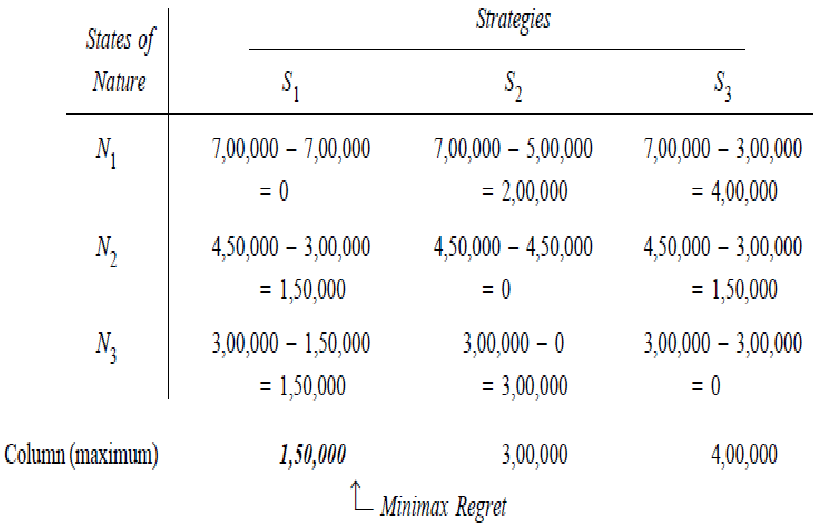


States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column (maximum)	*7,00,000*	5,00,000	3,00,000

↑ *Maximax Payoff*

The maximum of column maxima is 7,00,000. Hence, the company should adopt strategy S_1.

Minimax Regret Criterion Opportunity loss table is shown below:

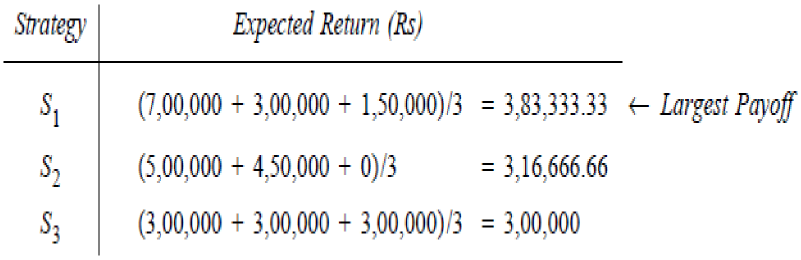


States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000 – 7,00,000 = 0	7,00,000 – 5,00,000 = 2,00,000	7,00,000 – 3,00,000 = 4,00,000
N_2	4,50,000 – 3,00,000 = 1,50,000	4,50,000 – 4,50,000 = 0	4,50,000 – 3,00,000 = 1,50,000
N_3	3,00,000 – 1,50,000 = 1,50,000	3,00,000 – 0 = 3,00,000	3,00,000 – 3,00,000 = 0
Column (maximum)	*1,50,000*	3,00,000	4,00,000

↑ *Minimax Regret*

Hence the company should adopt minimum opportunity loss strategy, S_1.

Laplace Criterion Assuming that each state of nature has a probability 1/3 of occurrence. Thus,



Strategy	Expected Return (Rs)	
S_1	(7,00,000 + 3,00,000 + 1,50,000)/3 = 3,83,333.33	← *Largest Payoff*
S_2	(5,00,000 + 4,50,000 + 0)/3 = 3,16,666.66	
S_3	(3,00,000 + 3,00,000 + 3,00,000)/3 = 3,00,000	

Since the largest expected return is from strategy *S*1, the executive must select strategy S_1.

The widely used criterion for evaluating decision alternatives (courses of action) under risk is the *Expected Monetary Value* (EMV) or *Expected Utility*.

Expected Monetary Value (EMV)

The expected monetary value (EMV) for a given course of action is obtained by adding payoff values multiplied by the probabilities associated with each state of nature. Mathematically, EMV is stated as follows:

$$\text{EMV } (\textit{Course of action, } S_j) = \sum_{i=1}^{m} p_{ij}\, p_i$$

where m = number of possible states of nature

p_i = probability of occurrence of state of nature, N_i

p_{ij} = payoff associated with state of nature N_i and course of action, S_j

The Procedure

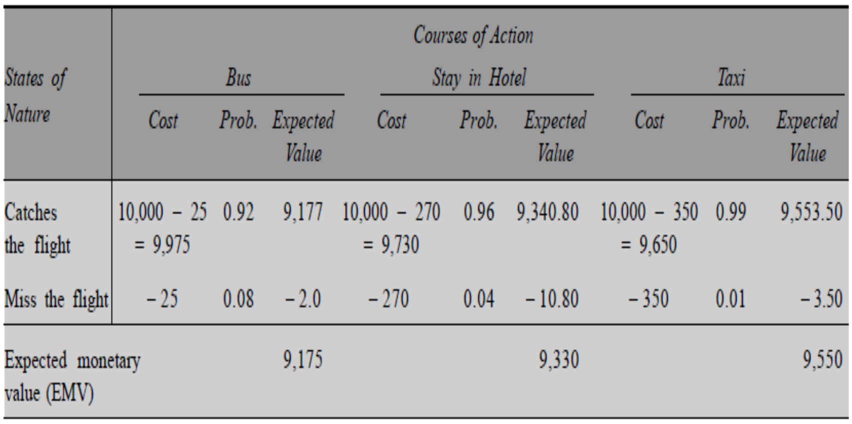
1. Construct a payoff matrix listing all possible courses of action and states of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature.

2. Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and adding these weighted values for each course of action.

3. Select the course of action that yields the optimal EMV.

Example 11.5 Mr X flies quite often from town A to town B. He can use the airport bus which costs Rs 25 but if he takes it, there is a 0.08 chance that he will miss the flight. The stay in a hotel costs Rs 270 with a 0.96 chance of being on time for the flight. For Rs 350 he can use a taxi which will make 99 per cent chance of being on time for the flight. If Mr X catches the plane on time, he will conclude a business transaction that will produce a profit of Rs 10,000, otherwise he will lose it. Which mode of transport should Mr X use? Answer on the basis of the EMV criterion.

Solution Computation of EMV associated with various courses of action is shown in Table



States of Nature	Courses of Action								
	Bus			Stay in Hotel			Taxi		
	Cost	Prob.	Expected Value	Cost	Prob.	Expected Value	Cost	Prob.	Expected Value
Catches the flight	10,000 − 25 = 9,975	0.92	9,177	10,000 − 270 = 9,730	0.96	9,340.80	10,000 − 350 = 9,650	0.99	9,553.50
Miss the flight	− 25	0.08	− 2.0	− 270	0.04	− 10.80	− 350	0.01	− 3.50
Expected monetary value (EMV)			9,175			9,330			9,550

Since EMV associated with course of action ‘Taxi’ is largest (= Rs 9,550), it is the logical alternative.

Game Theory

Introduction:

Game theory came into existence in 20th Century. However, in 1944 John Von Neumann and Oscar Morgenstern published a book named *Theory of Games and Economic Behavior*, in which they discussed how businesses of all types may use this technique to determine the best strategies given a competitive business environment. The author’s approach was based on the principle of ‘*best out of the worst*’. The models in the *theory of games* can be classified based on the following factors:

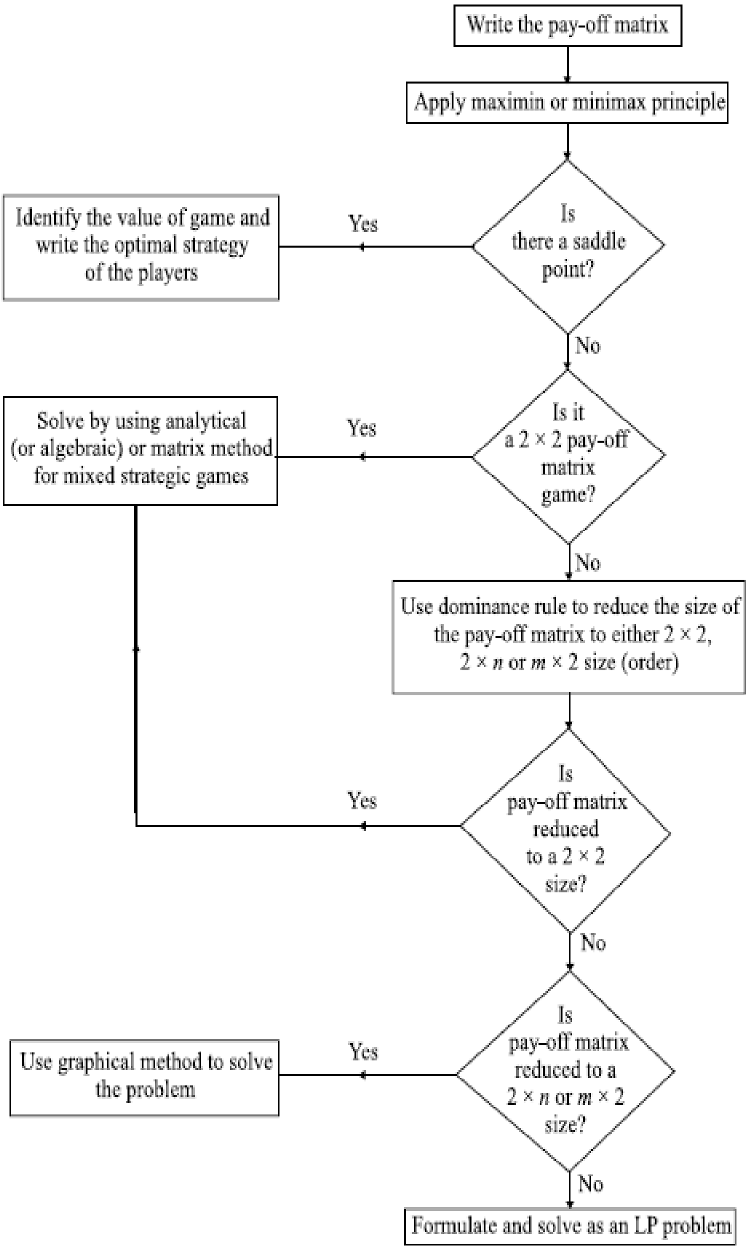
Number of players If a game involves only two players (competitors), then it is called a *two-person game*. However, if the number of players are more, the game is referred to as *n-person game*.

Sum of gains and losses If, in a game, the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a *zero-sum game*. Otherwise it is said to be *non-zero sum game*.

Strategy The strategy for a player is the list of all possible actions (moves, decision alternatives or courses of action) that are likely to be adopted by him for every payoff (outcome). It is assumed that the players are aware of the rules of the game governing their decision alternatives (or strategies). The outcome resulting from a particular strategy is also known to the players in advance and is expressed in terms of numerical values (e.g. money, per cent of market share or utility). The particular strategy that optimizes a player’s gains or losses, without knowing the competitor’s strategies, is called *optimal strategy*.

The expected outcome, when players use their optimal strategy, is called *value of the game.*

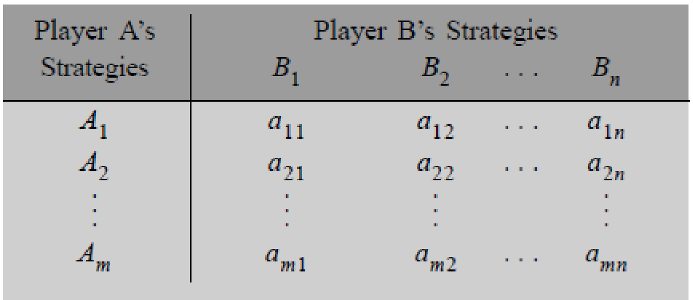
Generally, the following two types of strategies are followed by players in a game:
(a) *Pure Strategy* A particular strategy that a player chooses to play again and again regardless of other player’s strategy, is referred as *pure strategy*. The objective of the players is to maximize their gains or minimize their losses.
(b) *Mixed Strategy* A set of strategies that a player chooses on a particular move of the game with some fixed probability are called *mixed strategies*. Thus, there is a probabilistic situation andobjective of the each player is to maximize expected gain or to minimize expected loss by making the choice among pure strategies with fixed probabilities.



TWO-PERSON ZERO-SUM GAMES

A game with only two players, say *A* and *B*, *is* called a *two-person zero-sum game*, only if one player’s gain is equal to the loss of other player, so that total sum is zero.

Payoff matrix The payoffs (a quantitative measure of satisfaction that a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the payoff matrix.



Player A’s Strategies	Player B’s Strategies			
	B_1	B_2	$\dots$	B_n
A_1	a_{11}	a_{12}	$\dots$	a_{1n}
A_2	a_{21}	a_{22}	$\dots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	a_{m1}	a_{m2}	$\dots$	a_{mn}

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

Maximin principle For player *A* the minimum value in each row represents the least gain (payoff) to him, if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player *A* is called the *maximin principle,* and the corresponding gain is called the *maximin value of the game.*

Minimax principle For player *B*, who is assumed to be the looser, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategy. These are written in the payoff matrix by column maxima. He will then select the strategy that gives the minimum loss among the column maximum values. This choice of player *B* is called the *minimax principle*, and the corresponding loss is the *minimax value of the game.*

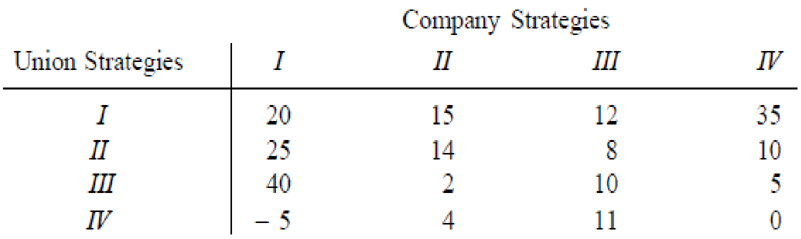
Optimal strategy A course of action that puts any player in the most preferred position, irrespective of the course of action his competitor(s) adopt, is called as optimal strategy. In other words, if the maximin value equals the minimax value, then the game is said to have a *saddle* (*equilibrium*) *point* and the corresponding strategies are called *optimal strategies.*

Value of the game This is the expected payoff at the end of the game, when each player uses his optimal strategy, i.e. the amount of payoff, *V*, at an equilibrium point. A game may have more than one saddle points. A game with no saddle point is solved by choosing strategies with fixed probabilities.

Example 12.2 A company management and the labour union are negotiating a new three year settlement. Each of these has 4 strategies:

I : Hard and aggressive bargaining
II : Reasoning and logical approach
III : Legalistic strategy
IV : Conciliatory approach

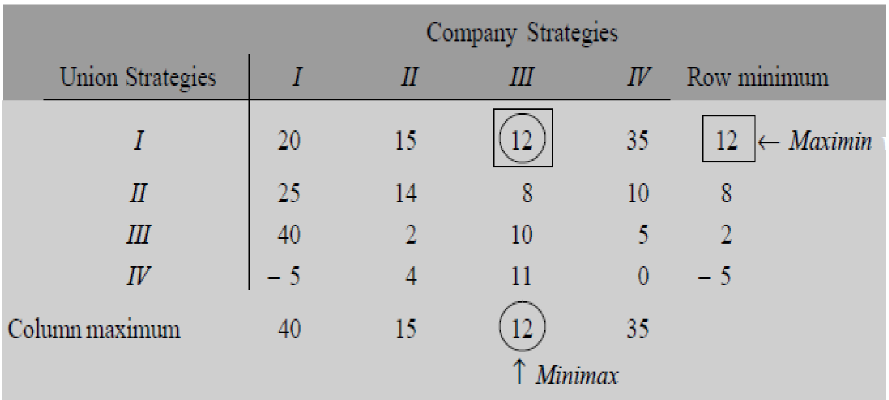
The costs to the company are given for every pair of strategy choice.



	Company Strategies			
Union Strategies	*I*	*II*	*III*	*IV*
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	– 5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution Applying the rule of finding out the saddle point, we obtain the saddle point that is enclosed both in a circle and a rectangle, as shown below



	Company Strategies				
Union Strategies	*I*	*II*	*III*	*IV*	Row minimum
I	20	15	$\boxed{⑫}$	35	$\boxed{12}$ ← *Maximin*
II	25	14	8	10	8
III	40	2	10	5	2
IV	– 5	4	11	0	– 5
Column maximum	40	15	⑫	35	
			↑ *Minimax*		

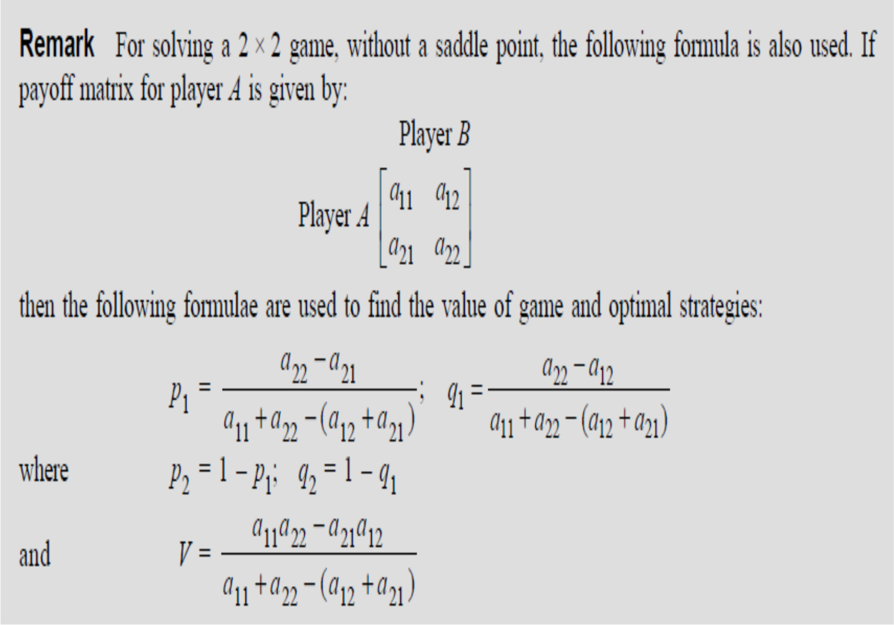
since Maximin = Minimax = Value of game = 12, therefore the company will always adopt strategy III – Legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

In certain cases, no saddle point exists, i.e. maximin value ≠ minimax value. In all such cases, players must choose the mixture of strategies to find the value of game and an optimal strategy. The value of game obtained by the use of mixed strategies represents the least payoff, which player *A* can expect to win and the least which player *B* can expect to lose. The expected payoff to a player in a game with payoff matrix [*aij*] of order *m* × *n* is defined as:

$$E(p,\ q)\ =\ \sum_{i=1}^{m}\ \sum_{j=1}^{n}\ p_i\ a_{ij}\ q_j\ = \mathbf{P}^T\,\mathbf{A}\ \mathbf{Q}\ \text{(in matrix notation)}$$

where **P** = (p_1, p_2, . . ., p_m) and **Q** = (q_1, q_2, . . ., q_n) denote probabilities (or relative frequency with which a strategy is chosen from the list of strategies) associated with *m* strategies of player *A* and *n* strategies of player, *B* respectively, where $p_1 + p_2 + . . . + p_m = 1$ and $q_1 + q_2 + . . . + q_n = 1$.



Remark For solving a 2 × 2 game, without a saddle point, the following formula is also used. If payoff matrix for player *A* is given by:

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

then the following formulae are used to find the value of game and optimal strategies:

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};\quad q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

where $\quad p_2 = 1 - p_1;\ \ q_2 = 1 - q_1$

and $\quad V = \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$

THE RULES (PRINCIPLES) OF DOMINANCE

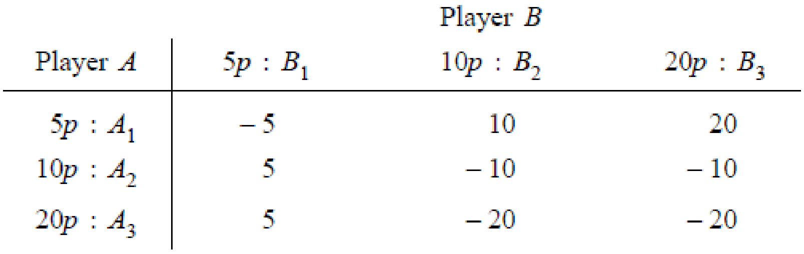
The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix that are inferior (less attractive) to at least one of the remaining rows and/or columns (strategies), in terms of payoffs to both the players. Rows and/or columns once deleted can never be used for determining the optimum strategy for both the players. The rules of dominance are especially used for the evaluation of two-person zero-sum games without a saddle (equilibrium) point. Certain dominance principles are stated as follows:

1. For player *B*, who is assumed to be the loser, if each element in a column, say *Cr* is greater than or equal to the corresponding element in another column, say *Cs* in the payoff matrix, then the column *Cr* is said to be dominated by column *Cs* and therefore, column *Cr* can be deleted from the payoff matrix. In other words, *player B will never use the strategy that corresponds to column Cr because he will loose more by choosing such strategy*.
2. For player *A*, who is assumed to be the gainer, if each element in a row, say *Rr*, is less than or equal to the corresponding element in another row, say *Rs*, in the payoff matrix, then the row *Rr* is said to be dominated by row *Rs* and therefore, row *Rr* can be deleted from the payoff matrix. In other words, *player A will never use the strategy corresponding to row Rr, because he will gain less by choosing such a strategy*.

3. A strategy say, *k* can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy *k* can be deleted. If strategy *k* dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination would be decided as per rules 1 and 2 above.

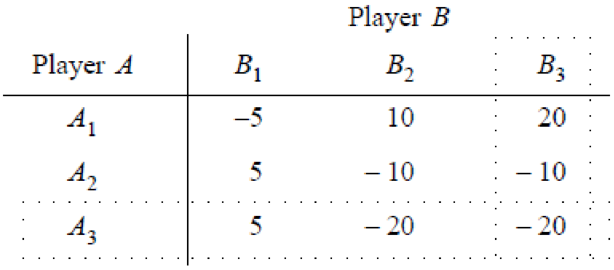
Example Players *A* and *B* play a game in which each has three coins, a 5*p*, 10*p* and a 20*p*. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, then *A* wins *B*'s coin. But, if the sum is even, then *B* wins *A*'s coin. Find the best strategy for each player and the values of the game.

Solution The payoff matrix for player *A* is



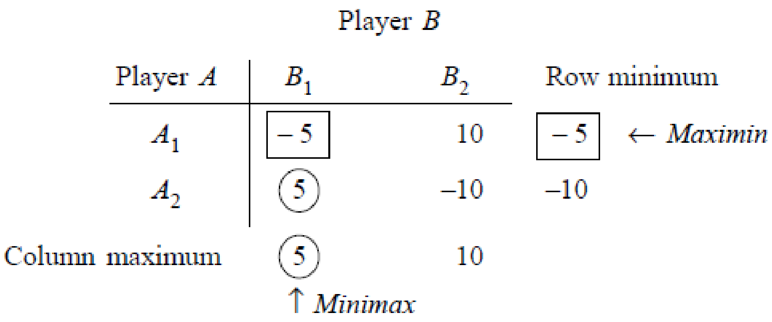
		Player *B*	
Player *A*	$5p : B_1$	$10p : B_2$	$20p : B_3$
$5p : A_1$	-5	10	20
$10p : A_2$	5	-10	-10
$20p : A_3$	5	-20	-20

It is clear that this game has no saddle point. Therefore, further we must try to reduce the size of the given payoff matrix as further as possible. Note that every element of column *B*3 (strategy *B*3 for player *B*) is more than or equal to every corresponding element of row *B*2 (strategy *B*2 for player *B*). Evidently, the choice of strategy *B*3, by the player *B*, will always result in more losses as compared to that of selecting the strategy *B*2. Thus, strategy *B*3 is inferior to *B*2. Hence, delete the *B*3 strategy from the payoff matrix. The reduced payoff matrix is shown below:



		Player *B*	
Player *A*	B_1	B_2	B_3
A_1	-5	10	20
A_2	5	-10	-10
A_3	5	-20	-20

After column *B*3 is deleted, it may be noted that strategy *A*2 of player *A* is dominated by his *A*3 strategy, since the profit due to strategy *A*2 is greater than or equal to the profit due to strategy *A*3, regardless of which strategy player *B* selects. Hence, strategy *A*3 (row 3) can be deleted from further consideration. Thus, the reduced payoff matrix becomes:



	Player B			
Player A	B_1	B_2	Row minimum	
A_1	$\boxed{-5}$	10	$\boxed{-5}$	← *Maximin*
A_2	⑤	–10	–10	
Column maximum	⑤	10		
	↑ *Minimax*			

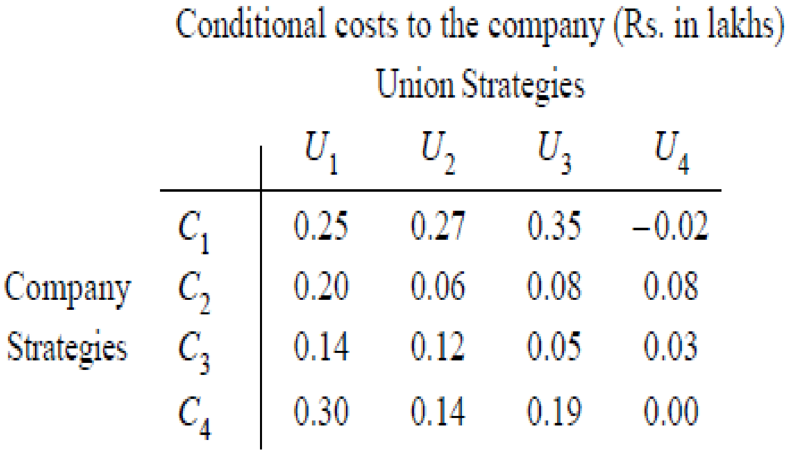
As shown in the reduced 2 × 2 matrix, the maximin value is not equal to the minimax value. Hence, there is no saddle point and one cannot determine the point of equilibrium. For this type of game situation, it is possible to obtain a solution by applying the concept of mixed strategies.

SOLUTION METHODS FOR GAMES WITHOUT SADDLE POINT

Algebraic Method

This method is used to determine the probability of using different strategies by players *A* and *B*. This method becomes quite lengthy when a number of strategies for both the players are more than two.

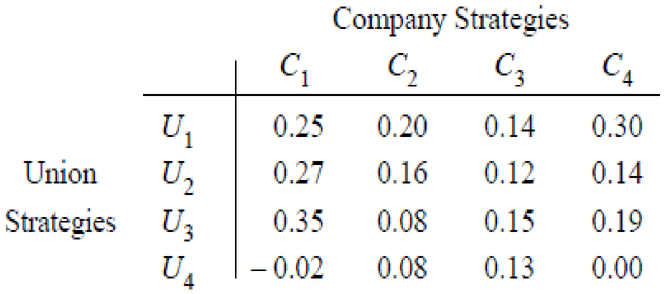
Example A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in table represent wage increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union? What is the game value?



Conditional costs to the company (Rs. in lakhs)

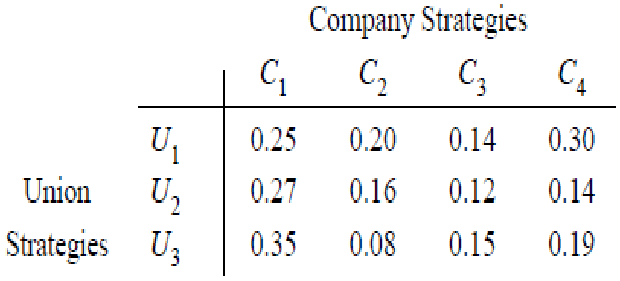
		Union Strategies			
		U_1	U_2	U_3	U_4
	C_1	0.25	0.27	0.35	−0.02
Company	C_2	0.20	0.06	0.08	0.08
Strategies	C_3	0.14	0.12	0.05	0.03
	C_4	0.30	0.14	0.19	0.00

Solution Suppose, Company is the gainer player and Union is the looser player. Transposing payoff matrix because company's interest is to minimize the wage increase while union's interest is to get the maximum wage increase.



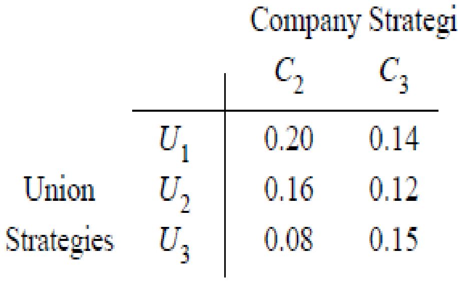
		Company Strategies			
		C_1	C_2	C_3	C_4
	U_1	0.25	0.20	0.14	0.30
Union	U_2	0.27	0.16	0.12	0.14
Strategies	U_3	0.35	0.08	0.15	0.19
	U_4	– 0.02	0.08	0.13	0.00

In this payoff matrix strategy U_4 is dominated by strategy U_1 as well as U_3. After deleting this strategy, we get



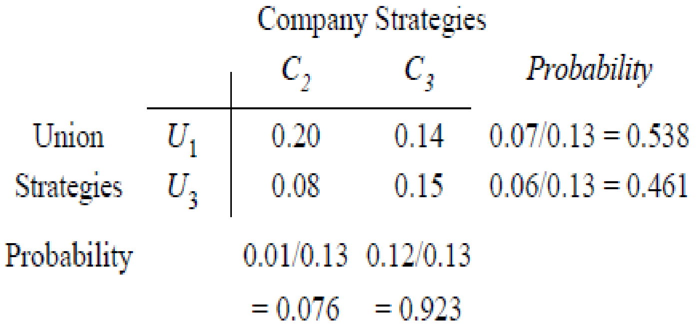
		Company Strategies			
		C_1	C_2	C_3	C_4
	U_1	0.25	0.20	0.14	0.30
Union	U_2	0.27	0.16	0.12	0.14
Strategies	U_3	0.35	0.08	0.15	0.19

Company's point of view, strategy C_1 is dominated by C_2 as well as C_3, while C_4 is dominated C_3. Deleting strategies C_1 and C_4 we get



		Company Strategies	
		C_2	C_3
	U_1	0.20	0.14
Union	U_2	0.16	0.12
Strategies	U_3	0.08	0.15

Again strategy U_2 is dominated by U_1 and is, therefore, deleted to give



		Company Strategies		
		C_2	C_3	*Probability*
Union	U_1	0.20	0.14	0.07/0.13 = 0.538
Strategies	U_3	0.08	0.15	0.06/0.13 = 0.461
Probability		0.01/0.13 = 0.076	0.12/0.13 = 0.923	

Optimal strategy for the company : (0, 0.076, 0.923, 0)
Optimal strategy for the union : (0.538, 0, 0.461, 0)
Value of the game, V : 0.538 × 0.20 + 0.461 × 0.08 = Rs. 14360

Arithmetic Method

The arithmetic method (also known as *short-cut method*) provides an easy method for finding optimal strategies for each player in a payoff matrix of size 2 × 2, without saddle point.

Example Two competitors are competing for the market share of the similar product. The payoff matrix in terms of their advertising plan is shown below:

	Competitor *B*		
Competitor *A*	*No Advertising*	*Medium Advertising*	*Heavy Advertising*
No Advertising	10	5	–2
Medium Advertising	13	12	13
Heavy Advertising	16	14	10

Suggest optimal strategies for the two firms and the net outcome thereof.

Solution : Applying rules of dominance to delete first column (dominated by second column) and then first row (dominated by second as well as third rows) from the payoff matrix, we obtain the following reduced payoff matrix:

	Firm *B*	
Firm *A*	*Medium Advt.* B_2	*Heavy Advt.* B_3
Medium Advt. A_2	12	15
Heavy Advt. A_3	14	10

$$\begin{array}{c|cc} & \text{Firm } B & \\ \text{Firm } A & B_2 & B_3 \\ \hline A_2 & 12 & 15 \\ A_3 & 14 & 10 \end{array}$$

$$14 - 10 = 4,\ p(A_2) = \frac{4}{4+3} = \frac{4}{7}$$

$$15 - 12 = 3,\ p(A_3) = \frac{3}{4+3} = \frac{3}{7}$$

$$15 - 10 = 5 \qquad 14 - 12 = 2$$

$$p(B_2) = \frac{5}{5+2} = \frac{5}{7} \qquad p(B_3) = \frac{2}{5+2} = \frac{2}{7}$$

Expected Gain to Firm A
(i) 12 × (4/7) + 14 × (3/7) = 90/7, Firm *B* adopt B_2
(ii) 15 × (4/7) + 10 × (3/7) = 90/7, Firm *B* adopt B_3
Expected Loss to Firm B
(i) 12 × (5/7) + 15 × (2/7) = 90/7, Firm *A* adopt A_2
(ii) 14 × (5/7) + 10 × (2/7) = 90/7, Firm *A* adopt A_3

Matrix Method

If the game matrix is in the form of a square matrix, then the optimal strategy mix as well as value of the game may be obtained by the matrix method. The solution of a two-person zero-sum game with mixed strategies with a square payoff matrix may be obtained by using the following formulae:

$$\text{Player } A\text{'s optimal strategy} = \frac{[1 \quad 1]\, P_{\text{adj}}}{[1 \quad 1]\, P_{\text{adj}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\text{Player } B\text{'s optimal strategy} = \frac{[1 \quad 1]\, P_{\text{cof}}}{[1 \quad 1]\, P_{\text{adj}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Value of the game = (Player *A*'s optimal strategies) × (Payoff matrix p_{ij}) × (Player *B*'s optimal strategies) where P_{adj} = adjoint matrix, P_{cof} = cofactor matrix. Player *A*'s optimal strategies are in the form of a row vector and *B*'s optimal strategies are in the form of a column vector. This method can be used to find a solution of a game with size of more than 2 × 2. However, in rare cases, the solution violates the non-negative condition of probabilities, i.e. $p_i \geq 0$, $q_j \geq 0$, although the requirement $p_1 + p_2 + \ldots + p_m = 1$ or $q_1 + q_2 + \ldots + q_n = 1$ is met.

Example Solve the following game after reducing it to a 2 × 2 game

Player *B*

Player *A*	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

In the given game matrix, the third row is dominated by the second row and in the reduced matrix third column is dominated by the first column. So, after elimination of the third row and the third column the game matrix becomes:

Player *B*

Player *A*	B_1	B_2
A_1	1	7
A_2	6	2

For this reduced matrix, let us calculate P_{adj} and P_{cof} as given below:

$$P_{\text{adj}} = \begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix} \quad \text{and} \quad P_{\text{cof}} = \begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}$$

$$\text{Player } A\text{'s optimal strategies} = \frac{[1 \;\; 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}}{[1 \;\; 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{[-4 \;\; -6]}{-10} = \frac{[4 \;\; 6]}{10}$$

This solution can be broken down into the optimal strategy mix for player A as $p_1 = 4/10 = 2/5$ and $p_2 = 6/10 = 3/5$, where p_1 and p_2 represent the probabilities of player A's using his strategies A_1 and A_2, respectively. Similarly, the optimal strategy mixture for player B is obtained as:

$$\text{Player } B\text{'s optimal strategies} = \frac{[1 \;\; 1]\begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}}{[1 \;\; 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{[-5 \;\; -5]}{-10} = \frac{[5 \;\; 5]}{10}$$

This solution can also be broken down into the optimal strategy mixture for player B as $q_1 = 5/10 = 1/2$ and $q_2 = 5/10 = 1/2$, where q_1 and q_2 represent the probabilities of player B's using his strategies B_1 and B_2, respectively. Hence:

$$\text{Value of the game, } V = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 4$$

Graphical Method

The graphical method is useful for the game where the payoff matrix is of the size 2 × n or m × 2, i.e. the game with mixed strategies that has only two undominated pure strategies for one of the players in the two-person zero-sum game. Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be used.

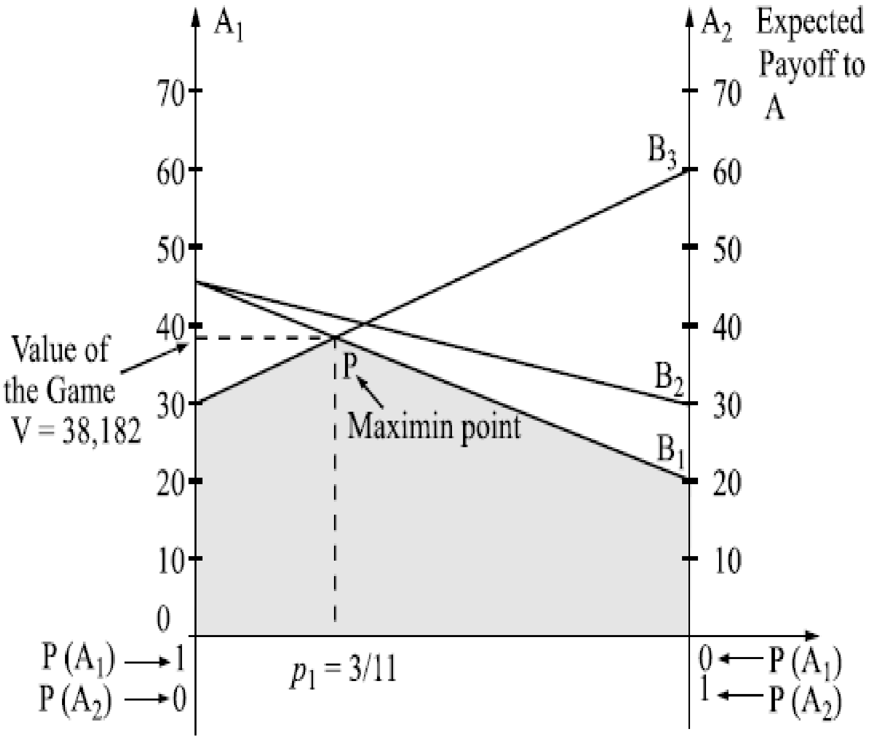
Example 12.15 Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs 400 per colour set, or 150 colour and 150 black & white sets, or 300 black &

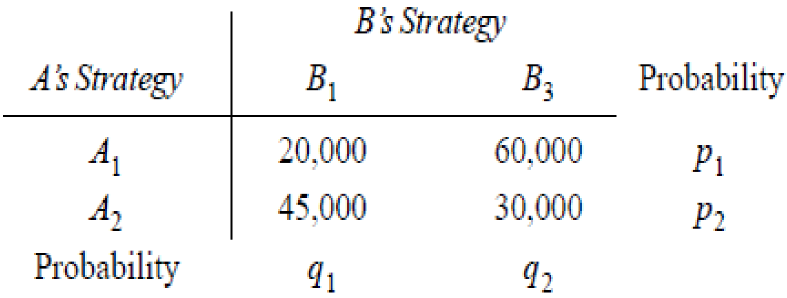
white sets per week. It also has the same profit margin on the two sets as *A*. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set. Write the pay-off matrix of *A* per week. Obtain graphically *A's* and *B's* optimum strategies and value of the game.

Solution For firm *A*, the strategies are:
A_1 : make 150 colour sets, *A*2 : make 150 black & white sets.
For firm *B*, the strategies are:
B_1 : make 300 colour sets, *B*2 : make 150 colour and 150 black & white sets.
B_3 : make 300 black and white sets.
For the combination *A*1*B*1, the profit to firm *A* would be: {150/(150 + 300)} × 150 × 400 = Rs 20,000 wherein
150/(150 + 300) represents share of market for *A*, 150 is the total market for colour television sets and 400 is the profit per set. In a similar manner, other profit figures may be obtained as shown in the following pay-off matrix:

	B's Strategy		
A's Strategy	B_1	B_2	B_3
A_1	20,000	30,000	60,000
A_2	45,000	45,000	30,000

This pay-off table has no saddle point. Thus to determine optimum mixed strategy, the data are plotted on graph





	B's Strategy		
A's Strategy	B_1	B_3	Probability
A_1	20,000	60,000	p_1
A_2	45,000	30,000	p_2
Probability	q_1	q_2	

The optimal mixed strategies of player *A* are: A_1 = 3/11, A_2 = 8/11. Similarly, the optimal mixed strategies for *B* are: B_1 = 6/11, B_2 = 0, B_3 = 5/11. The value of the game is *V* = 38,182.

UNIT – V – DYNAMIC PROGRAMMING

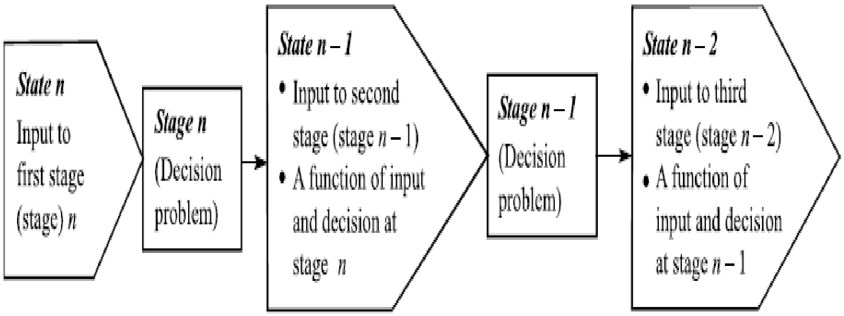
I INTRODUCTION :

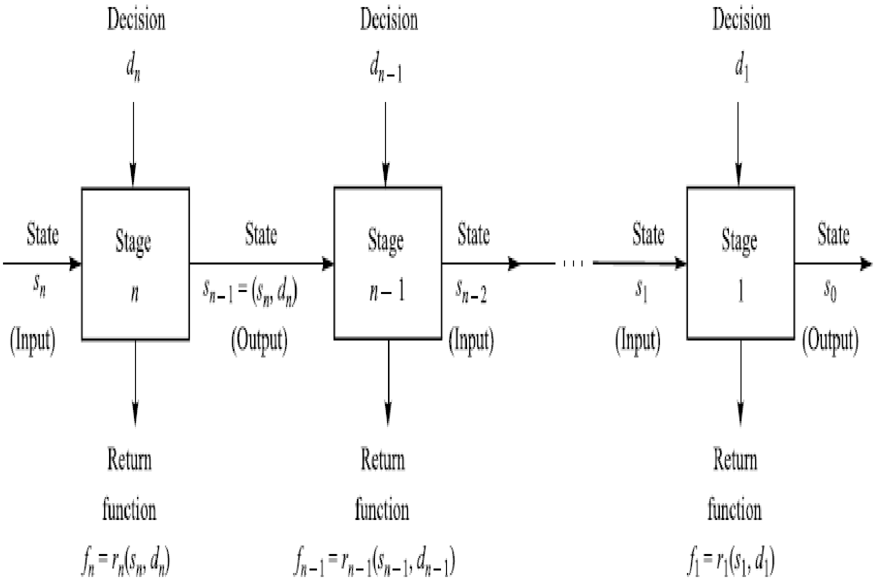
The mathematical technique of optimizing a sequence of interrelated decisions over a period of time is called *dynamic programming.* The dynamic programming approach uses the idea of recursion to solve a complex problem, broken into a series of interrelated (sequential) decision stages (also called *subproblems*) where the outcome of a decision at one stage affects the decision at each of the following stages. The word *dynamic* has been used because time is explicitly taken into consideration.

Dynamic programming (DP) differs from linear programming in two ways:
(i) In DP, there is no set procedure (algorithm) as in LP to solve any decision-problem. The DP technique allows to break the given problem into a sequence of smaller subproblems, which are then solved in a sequential order (stage).
(ii) LP approach provides one-time period (single stage) solution to a problem whereas DP approach is useful for decision-making over time and solves each subproblem optimally.

DYNAMIC PROGRAMMING TERMINOLOGY

1. Stage
2. State
3. Return Function





where n = stage number

s_n = state input to stage n from stage $n + 1$. Its value is the status of the system resulting from the previous $(n + 1)$ stage decision.

d_n = decision variable at stage n (independent of previous stages). This represents the range of alternatives available when making a decision at stage n.

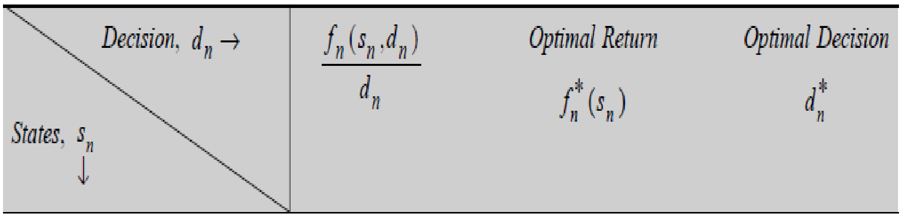
$f_n = r_n(s_n, d_n)$ = return (or objective) function for stage n.

III DEVELOPING OPTIMAL DECISION POLICY

The General Procedure

The procedure for solving a problem by using the dynamic programming approach can be summarized in the following steps:

Step 1: Identify the problem decision variables and specify the objective function to be optimized under certain limitations, if any.
Step 2: Decompose (or divide) the given problem into a number of smaller sub-problems (or stages). Identify the state variables at each stage and write down the transformation function as a function of the state variable and decision variable at the next stage.
Step 3: Write down a general recursive relationship for computing the optimal policy. Decide whether to follow the forward or the backward method for solving the problem.
Step 4: Construct appropriate tables to show the required values of the return function at each stage as shown
Step 5: Determine the overall optimal policy or decisions and its value at each stage. There may be more than one such optimal policy.



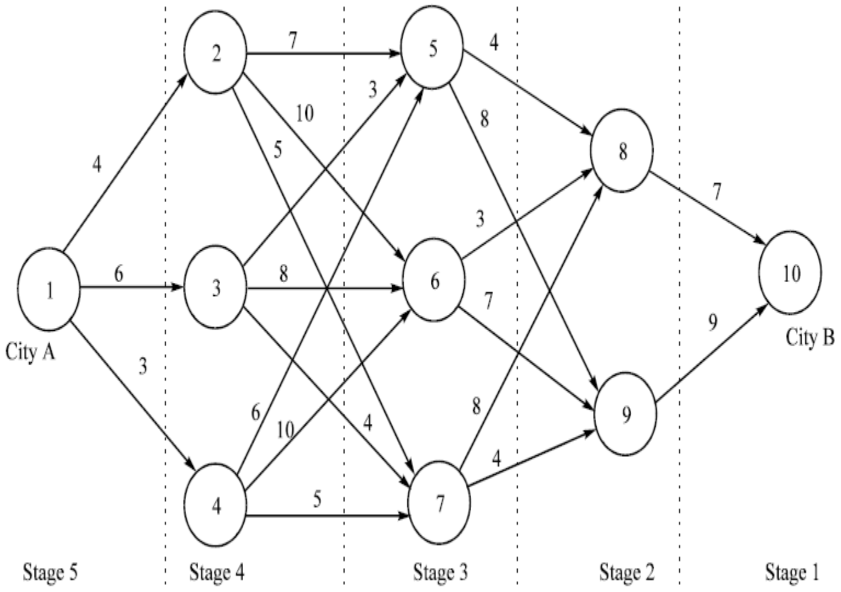
Decision, $d_n \rightarrow$ States, $s_n \downarrow$	$\frac{f_n(s_n, d_n)}{d_n}$	Optimal Return $f_n^*(s_n)$	Optimal Decision d_n^*

DYNAMIC PROGRAMMING UNDER CERTAINTY

The decision problems where conditions (constraints) at each stage, (i.e. state variables) are known with certainty, can by solved by dynamic programming.

Model I : Shortest Route Problem

Example A salesman located in a city A decided to travel to city B. He knew the distances of alternative routes from city A to city B. He then drew a highway network map as shown .The city of origin A, is city 1. The destination city B, is city 10. Other cities through which the salesman will have to pass through are numbered 2 to 9. The arrow representing routes between cities and distances in kilometers are indicated on each route. The salesman's problem is to find the shortest route that covers all the selected cities from A to B.



Solution To solve the problem, we need to define problem stages, decision variables, state variables, return function and transition function. For this particular problem, the following definitions will be used to denote various the state and decision variables.

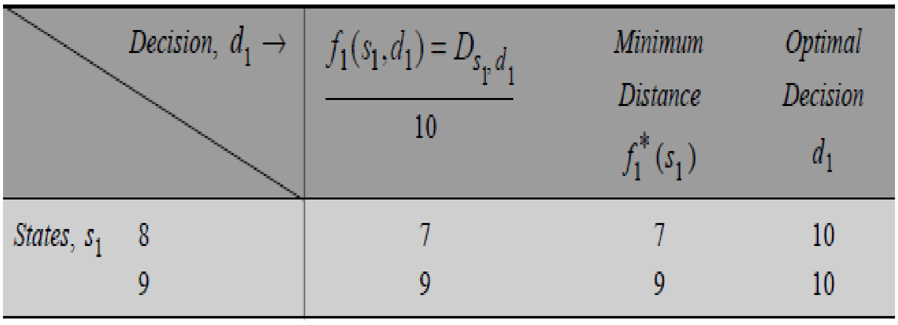
d_n = decision variables that define the immediate destinations when there are $n(n = 1, 2, 3, 4,)$ stages to go.

s_n = state variables describe a specific city at any stage.

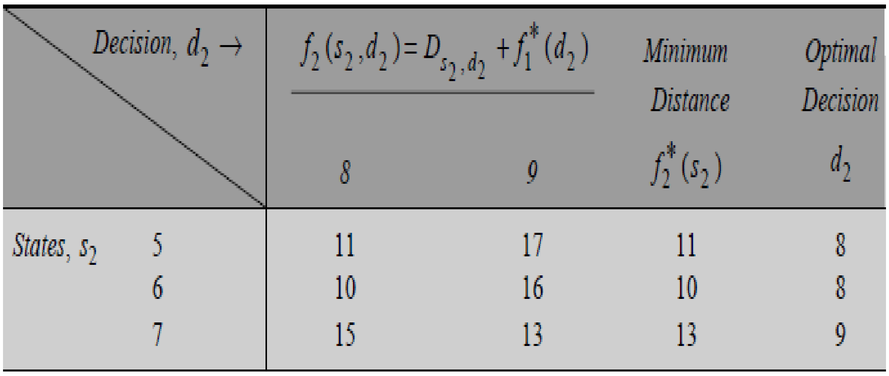
D_{sn}, d_n = distance associated with the state variable, s_n, and the decision variable, d_n for the current nth stage.

f_n (s_n , d_n) = minimum total distance for the last n stages, given that salesman is in state s_n and selects d_n as immediate destination.
f_n*(s_n) = optimal path (minimum distance) when the salesman is in state s_n with n more stages to go for reaching the final stage (destination). We start calculating distances between a pair of cities from destination city 10 (= x_1) and work backwards $x_5 \longrightarrow x_4 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1$ to find the optimal path. The recursion relationship for this problem can be stated as follows:

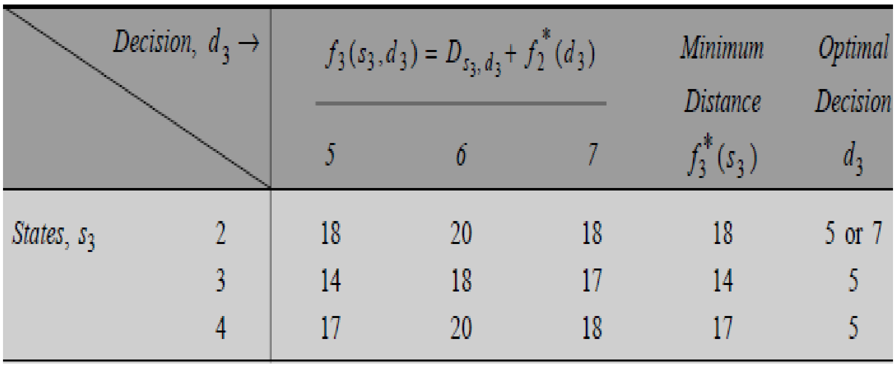
$$f_n^*(s_n) = \underset{d_n}{\text{Min}} \left\{ D_{s_n, d_n} + f_{n-1}^*(d_n) \right\}; \quad n = 1, 2, 3, 4$$



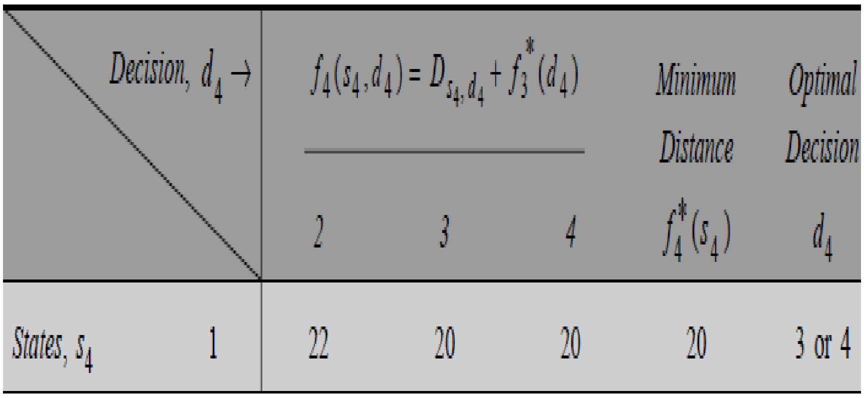
Decision, $d_1 \rightarrow$		$f_1(s_1,d_1) = D_{s_1,d_1}$	Minimum Distance $f_1^*(s_1)$	Optimal Decision d_1
		10		
States, s_1	8	7	7	10
	9	9	9	10



Decision, $d_2 \rightarrow$		$f_2(s_2,d_2) = D_{s_2,d_2} + f_1^*(d_2)$		Minimum Distance $f_2^*(s_2)$	Optimal Decision d_2
		8	9		
States, s_2	5	11	17	11	8
	6	10	16	10	8
	7	15	13	13	9



Decision, $d_3 \rightarrow$		$f_3(s_3,d_3) = D_{s_3,d_3} + f_2^*(d_3)$			Minimum Distance $f_3^*(s_3)$	Optimal Decision d_3
		5	6	7		
States, s_3	2	18	20	18	18	5 or 7
	3	14	18	17	14	5
	4	17	20	18	17	5



Decision, $d_4 \rightarrow$	$f_4(s_4,d_4)=D_{s_4,d_4}+f_3^*(d_4)$			Minimum Distance	Optimal Decision
	2	3	4	$f_4^*(s_4)$	d_4
States, s_4 1	22	20	20	20	3 or 4

The above optimal results at various stages can be summarized as below:

$$\begin{array}{ll} & \text{Entering states (nodes)} \\ \text{Sequence} & \begin{cases} 10 \quad 8 \quad 5 \quad 3 \quad 1 \\ 10 \quad 8 \quad 5 \quad 4 \quad 1 \end{cases} \\ \text{Distances} & \begin{cases} 7 \quad 4 \quad 3 \quad 6 \;=\; 20 \\ 7 \quad 4 \quad 6 \quad 3 \;=\; 20 \end{cases} \end{array}$$

From the above, it is clear that there are two alternative shortest routes for this problem, both having a minimum distance of 20 kilometres.

Model II Multiplicative Separable Return Function and Single Additive Constraint

Example : Consider the problem of designing electronic devices to carry five power cells, each of which must be located within three electronic systems. If one system's power fails, then it will be powered on an auxiliary basis by the cells of the remaining systems. The probability that any particular system will experience a power failure depends on the number of cells originally assigned to it. The estimated power failure probabilities for a particular system are given below:

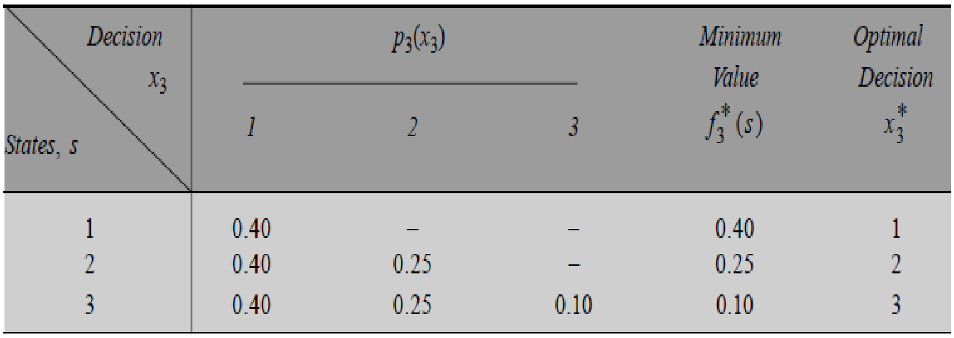
Power Cells	*Probability of System Power Failure*		
	System 1	*System 2*	*System 3*
1	0.50	0.60	0.40
2	0.15	0.20	0.25
3	0.04	0.10	0.10
4	0.02	0.05	0.05
5	0.01	0.02	0.01

Determine how many power cells should be assigned to each system in order to maximize the overall system reliability.

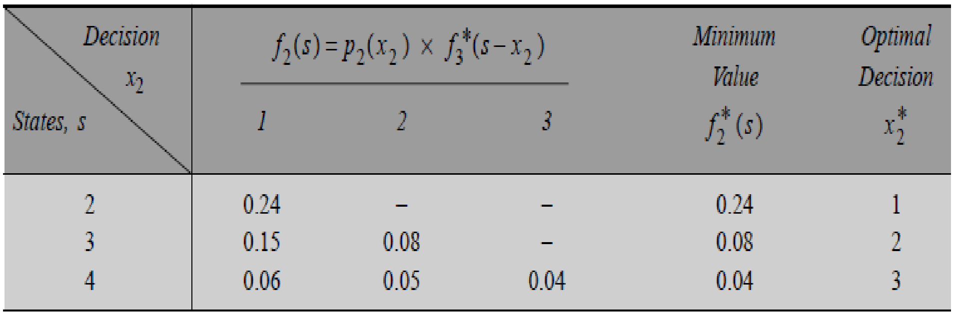
Solution Let us adopt the following notations:
x_n = number of power cells assigned to stage
$p_n(x_n)$ = probability of power failure for the system n, when it is assigned *xn* power cells
$f_n(s)$ = probability that nth and all higher systems will fail, while entering state s
Here the stages correspond to systems and state s is the number of power cells available for allocation at different stages. We shall start from state (power cell) 1. The recursive equation for this problem may be given by:

$$f_n(s) = \underset{x_n \leq 5}{\text{Min}} \left\{ p_n(x_n) \times f_{n+1}(s - x_n) \right\}, \quad n = 1, 2$$

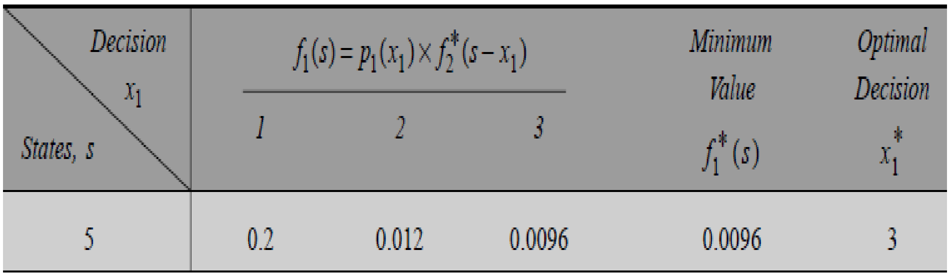
subject to the constraint
$x_1 + x_2 + . . . + x_n = 5$
The dynamic programming calculations are as follows:



Decision x_3 States, s	$p_3(x_3)$			Minimum Value	Optimal Decision
	1	2	3	$f_3^*(s)$	x_3^*
1	0.40	–	–	0.40	1
2	0.40	0.25	–	0.25	2
3	0.40	0.25	0.10	0.10	3



Decision x_2 States, s	$f_2(s) = p_2(x_2) \times f_3^*(s - x_2)$			Minimum Value	Optimal Decision
	1	2	3	$f_2^*(s)$	x_2^*
2	0.24	–	–	0.24	1
3	0.15	0.08	–	0.08	2
4	0.06	0.05	0.04	0.04	3



Decision x_1 States, s	$f_1(s) = p_1(x_1) \times f_2^*(s - x_1)$			Minimum Value	Optimal Decision
	1	2	3	$f_1^*(s)$	x_1^*
5	0.2	0.012	0.0096	0.0096	3

At stage 1, value of $f_1(s) = 0.0096$ is minimum when $x_1 = 3$ corresponds to system 1. Then for $x_1 = 3$, $x_2 + x_3 = 2$. But at stages 2 and 1, optimal values of x_2 and x_3 are 1 and 1, respectively. Thus, the optimal solution is: $x_1 = 3$, $x_2 = 1$, $x_3 = 1$, with the smallest probability of total power failure, $f_1(5) = 0.0096$.

DYNAMIC PROGRAMMING APPROACH FOR SOLVING LINEAR PROGRAMMING PROBLEM

Example Use dynamic programming to solve the following linear programming problem.
Maximize $Z = 3x_1 + 5x_2$
subject to the constraints
(i) $x_1 \le 4$, (ii) $x_2 \le 6$, (iii) $3x_1 + 2x_2 \le 18$
and $x_1, x_2 \ge 0$

Solution This linear programming problem can be considered as a two-stage, three-state problem because there are two decision variables and three constraints with available resources. The optimal value of $f_1(b_1, b_2, b_3)$ at the first stage is given by:

$$f_1(b_1, b_2, b_3) = \underset{0 \le x_1 \le b}{\text{Max}} \{3x_1\}, \quad \text{where} \quad b_1 = 4,\ b_2 = 6 \text{ and } b_3 = 18.$$

The feasible value of x_1 is a non-negative value that satisfies all the given constraints $x_1 \le b_1$ (= 4), $3x1 \le b_3$ (= 18). Thus, the maximum value of b that x_1 can assume is, $b = \text{Min}(4, 18/3) = 4$. Therefore

$$f_1(4, 6, 18) = \underset{0 \le x_1 \le 4}{\text{Max}} \{3x_1\} = 3\text{Min}\left\{4, 6 - \frac{2}{3} \cdot x_2\right\}$$

$$x_1^* = \text{Min}\left\{4, 6 - \frac{2}{3} \cdot x_2\right\}$$

The recursive relation for optimization of this two-stage problem is:

$$f_2(b_1, b_2, b_3) = \underset{0 \le x_2 \le b}{\text{Max}} \left\{5x_2 + f_1^*(b_1, b_2 - x_2, b_3 - 2x_2\right\}$$

where the maximization of x_2, satisfying the conditions of $x_2 \le b_2$ (= 6) and $2x_2 \le b_3$ (= 18) , is the minimum of $b = \min(6, 9) = 6$. Therefore, the recurrence relationship can be expressed as:

$$f_2(4, 6, 18) = \underset{0 \le x_2 \le 6}{\text{Max}} \left\{5x_2 + f_1^*(4, 6 - x_2, 18 - 2x_2)\right\}$$

$$= \underset{0 \le x_2 \le 6}{\text{Max}} \left\{5x_2 + 3\text{Min}\left(4, 6 - \frac{2}{3}x_2\right)\right\}$$

Since, $\text{Min}\left(4,\ 6-\frac{2}{3}\cdot x_2\right) = \begin{cases} 4; & 0 \le x_2 \le 3 \\ 6-\frac{2}{3}x_2; & 3 \le x_2 \le 6 \end{cases}$

we get $\text{Max}\left\{5x_2 + 3\text{Min}\left(4,\ 6-\frac{2}{3}x_2\right)\right\} = \begin{Bmatrix} 5x_2+12; & 0 \le x_2 \le 3 \\ 18+3x_2; & 3 \le x_2 \le 6 \end{Bmatrix}$

Now the maximum value of $5x_2 + 12 = 27$ at $x_2 = 3$ and maximum value of $18 + 3x_2 = 36$ at $x_2 = 6$. Therefore, the optimal value of $f_2* = (4, 6, 18) = 36$, is obtained at $x_2 = 6$. Since,

$$x_1^* = \text{Min}\left\{4, 6-\frac{2}{3}x_2\right\} = \text{Min}\left\{4, 6-\frac{2}{3}\times 6\right\} = 2$$

The optimum solution to the given LP problem is: $x_1 = 2$, $x_2 = 6$ and Max $Z = 36$.

Bloom’s Taxonomy

01 KNOWLEDGE:	02 UNDERSTAND:	03 APPLY:	04 ANALYZE:	05 EVALUATE:	06 CREATE:
Define,	Summarize,	Solve,	Contrast,	Criticize,	Design,
Identify,	Interpret,	Change,	Connect,	Reframe,	Modify,
Describe,	Classify,	Relate,	Relate,	Judge,	Role-Play,
Recognize,	Compare,	Complete,	Devise,	Defend,	Develop,
Tell,	Contrast,	Use,	Correlate,	Appraise,	Rewrite,
Explain,	Infer,	Sketch,	Illustrate,	Value,	Pivot,
Recite,	Relate,	Teach,	Distill,	Prioritize,	Modify,
Memorize,	Extract,	Articulate,	Conclude,	Plan,	Collaborate,
Illustrate,	Paraphrase,	Discover,	Categorize,	Grade,	Invent,
Quote	Cite	Transfer	Take Apart	Reframe	Write