SMTA5401	FLUID DYNAMICS	L	T	P	CREDIT
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Objective of the Course: The ability to identify, reflect upon, evaluate and apply different types of information and knowledge to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

UNIT 1: KINEMATICS 10 Hrs

Kinematics of fluids in motion: Real fluids and ideal fluids, velocity of a fluid at a point, streamlines and path lines, Steady and unsteady flows. The velocity potential, the vorticity vector, Local and particle rates of change, the equation of continuity, worked examples, acceleration of a point of a fluid.

UNIT 2: MOTION OF A FLUID 10 Hrs

Equations of motion of a fluid: pressure at a point in a fluid at rest, Pressure at a point in a moving fluid, Conditions at a boundary of two inviscid Immiscible fluids, Euler's equations of motion, Bernoulli's equation, worked examples, some flows involving axial symmetry, Some special two-dimensional flows, Impulsive motion.

UNIT 3: TWO DIMENSIONAL FLOWS-I 10 Hrs

Some two-dimensional flows: Meaning of two- dimensional flow, use of cylindrical polar coordinates, The stream function. The complex potential for two-dimensional irrotational, incompressible flow, complex velocity potential for standard two-dimensional flows, uniform stream, line sources and line sinks, line doublets, line vortices, worked examples.

UNIT 4: TWO DIMENSIONAL FLOWS-II 10 Hrs

Some two- dimensional flows (Continued): Two- dimensional image systems, The Milne Thomson circle theorem, some application of the circle theorem, extension of the circle theorem, the theorem of blasius, the use of conformal transformation – some hydro dynamical aspects of conformal transformation worked example, vortex rows – single infinite rows of line vortices, The karman vortex street.

UNIT 5: THREE DIMENSIONAL FLOWS 10 Hrs

Some three- dimensional flows: Introduction, sources, sinks and doublets, Images in a rigid infinite plane, Axi-symmetric flows, stokes stream function, some special form of the stream function for axi-symmetric irrotational motions.

Max Hours: 50 Hrs

REFERENCE BOOKS:

1. F.Chorlton, Textbook of Fluid Dynamics, CBS Publication and Distribution, 2004.
2. M.D. Raisinghania, Fluid Dynamics, S. Chand,2008.
3. G.K.Batchelor, An Introduction to Fluid Mechanics, Foundation Books,1984.

Course Outcomes: At the end of the course, learners would acquire competency in the following skills.

CO1	Define Real Fluids, Ideal Fluids, Streamlines, Path lines, Vortex Lines, Source, Sinks, Doublets, Potential Flow, Irrotational Flow, 2D Flow, 3D Flow, Impulsive Motion
CO2	Derive Equation of Continuity, Euler's equation of motion, Bernoulli's equation, Milne Thomson Circle Theorem, Theorem of Blasius. Explain Axisymmetric flows, Karman Vortex Street.
CO3	Prepare Conditions at boundary of two inviscid Immiscible Fluid, Two-Dimensional Image system. Application of Circle Theorem.
CO4	Analyze fluid motion in General. Discuss Steady & Unsteady Flow, Compressible & Incompressible Fluid, Viscous & Inviscid Fluid, Sources, sinks & doublets.
CO5	Evaluate the velocity potential, streamlines, path lines, equi-potential surface, steam function, complex potential for two dimensional, irrotational, incompressible flow.
CO6	Determine the relation between local and particle rate of change, Develop Stokes stream function, Determine the flow characteristics, hydrodynamical aspects of Conformal transformation, pressure at a point in a fluid.

SCHOOL OF SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I - KINEMATICS – SMTA5401

UNIT-1

<u>Kinematics of Fluid motion</u>

Fluid dynamics is the study of fluid in motion.

Fluid means the substance that flow we have two kinds of fluid.

1) Liquid-incompressible fluid
   i.e., their volumes do not change as the pressure changes.
2) Gas-compressible fluid,
   i.e., change in volume whenever the pressure changes.

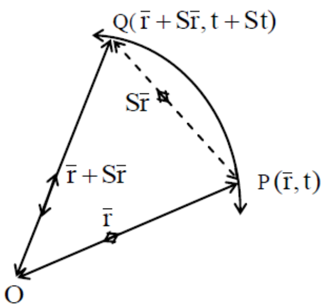
Stresses:

- Two types of force acts on a fluid element one of them is body force and the other is surface force.
- The body force is proportional to the mass of the body on which it acts while the surface area and acts on the boundary of the force.
- The normal force per unit area is called normal stress.
- Tangential force per unit area is called shearing stress.

Viscosity:

- It is the internal friction between the particles of the fluid, that resists to the deformation of the fluid.
- Fluid with viscosity is known as viscous or real fluid.
- Fluid without viscosity is known as inviscous or ideal fluid.

Velocity of fluid at a point:



Let at time t the particle be at the point P. $\overrightarrow{OP} = r$ and at the time t. $t + \delta t$ the particle P' such that $OP' = r + \delta_r$ .In the interval δt ,moment of the particle is $\overrightarrow{PP'} = \delta_r$ Then, the particle $\vec{q} = \lim_{\delta_r \to 0}\left(\frac{\delta t}{\delta_r}\right) = \frac{dr}{dt}$,q is dependent on both r and t

Streamlines:

It is the curve drawn in the fluid such that direction of the tangent to it at any point coincides with the direction of fluid velocity at that point.

Derivation of streamlines:

At any point P, let $\vec{q} = [u, v, w]$ be the velocity at the point P(x,y,z) of the fluid. The direction ratios of the tangent to the curve at P(x,y,z) are

$d\vec{r} = [dx, dy, dz]$

Since, the tangent and the velocity at P have the same direction, we have

$$\vec{q} \times d\vec{r} = 0$$

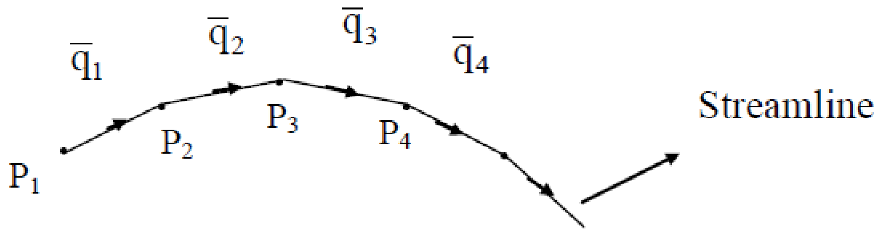
i.e., $(vdw - wdy)\vec{i} - (udz - wdx)\vec{j} + (udy - vdx)\vec{k} = 0$

$$vdw - wdy = 0 \qquad udz - wdx = 0 \qquad udy - vdx = 0$$

$$\frac{dz}{w} = \frac{dy}{v} \qquad \frac{dz}{w} = \frac{dx}{u} \qquad \frac{dy}{v} = \frac{dx}{u}$$

$$\text{i.e., } \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

These are the differential equation for the streamlines. i.e., their solution gives the streamlines



$\vec{q}_1, \vec{q}_2, \vec{q}_3, ...$ denote the velocities at neighbouring points $P_1, P_2, ...$ then, the small straight line segments $P_1P_2, P_2P_3, P_3P_4 .....$ collectively give the approximate form of streamline.

Pathlines:

When the fluid motion is steady, so that pattern of flow does not very with time. The paths of the fluid particle coincide with the streamlines, though the streamline through any point P does touch the pathline through P.

In case of unsteady motion, the flow pattern varies with time and the paths of the particle do not coincide with the streamline.

In case of unsteady motion, the flow pattern varies with time and the paths of the particle do not coincide with the streamline.

Pathlines are the curve described by the fluid particles during their motion.

i.e., these are the paths of the particle the differential equation of pathlines are,

$$\frac{d\vec{r}}{dt} = \vec{q}$$

$$\frac{dx}{dt} = u\,, \qquad \frac{dy}{dt} = v\,, \qquad \frac{dz}{dt} = w$$

where x,y,z are the Cartesian co-ordinates of the fluid and not a fixed point of space.

Notes:

Streamlines give motion of a particle at a given instant, whereas the pathlines gives the motion of a given particle at each instant.

Streamtube:

If we draw the streamline through every point of a closed curve in the fluid we obtain streamtube.

Velocity potential:

Let the fluid velocity at time t is q=[u,v,w] in Cartesian form.

The equation of streamlines at one instant is $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

The curve cut the surface $udx + vdy + wdz = 0$

Suppose, the expression $udx + vdy + wdz = 0$ is an exact differential say(-dφ)

i.e., $\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial t} dz + \frac{\partial \phi}{\partial t} dt = udx + vdy + wdz$

where φ is φ(x,y,z,t) is some scalar function, uniform throughout the entire field.

Therefore, $u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y}, w = \frac{-\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} = 0$

$$\Rightarrow \phi(x, y, z) = \phi$$

$$q = u\vec{i} + v\vec{j} + w\vec{k}$$

$$= \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$= -\nabla \phi$$

Note:

- $q = -\nabla \phi$, the negative sign is a convention and it ensures a flow takes place from higher or lower potential.
- The level surface φ(x,y,xz,t)=constant are called equi-potentials or equi-potential surfaces.
- When curl q=0, the flow is said to be irrotational of potential kind. For such flow, the field of q is conservated and q is lamellar vector.

Vorticity:

q=[u v w] be the velocity vector of a fluid particle then the vector $\vec{\varepsilon} = \nabla \times \vec{q}$ is called vortex vector or vorticity.

The components are $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\text{i.e., } \varepsilon_1 = \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

$$\varepsilon_2 = \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

$$\varepsilon_3 = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Note:

- The fluid motion is said to be rotational if $\varepsilon = \nabla \times \vec{q} \neq 0$
- If $\varepsilon = \nabla \times \vec{q} = 0$ then the fluid motion is irrotational or of potential kind.

Vortex line:

It is the curve in the fluid there exist the tangent at any point on the curve has the direction of vorticity vector.

Vortex tube:

It is the surface formed by drawing vortex lines through each point of the closed curve in the fluid.

A vortex tube with small cross-section called a vortex filament.

Problem:

At the point in an incompressible fluid having spherical polar co-ordinates(r,φ,ψ), the velocity component are $\left[2\mu r^{-3}\cos\theta, \mu r^{-3}\sin\theta, 0\right]$ **where M is a constant. Show that velocity is of potential kind. Find the velocity potential and the equations of streamlines.**

Solution:

Taking $ds = dr\hat{r} + rd\theta\hat{\theta} + r\sin d\psi\hat{\psi}$

$$q = 2\mu r^{-3}\cos\theta\hat{r} + \mu r^{-3}\sin\theta\hat{\theta} + 0$$

We obtain curl q $\Rightarrow \nabla \times q$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\psi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ 2\mu r^{-3}\cos\theta & \mu r^{-3}\sin\theta & 0 \end{vmatrix}$$

$\nabla \times q = 0.$

Thus, the flow is of potential kind. Let $\phi(r,\theta,\psi)$ be the approximate velocity potential.

Then, $\frac{-\partial\phi}{\partial r} = 2\mu r^{-3}\cos\theta, \frac{-\partial\phi}{r\partial\theta} = \mu r^{-3}\sin\theta, \frac{-\partial\phi}{r\sin\theta\psi} = 0$

$$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\psi}d\psi$$

$$= -(2Mr^{-3}\cos\theta)dr - (M\,r^{-2}\sin\theta)\,d\,\theta + 0d\psi$$

$$\phi = 2Mr^{-3}\cos\theta$$

The streamlines are given by $\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r\,d\,\theta}{M\,r^{-2}\sin\theta} = \frac{r\sin\theta d\psi}{0}$

$$d\psi = 0$$

$$2\cot\theta d\theta = \left(\frac{1}{r}\right)dr$$

Int, $\psi = \text{cost}$

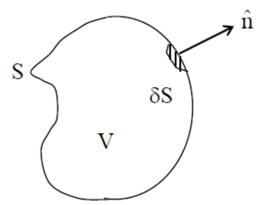
$$r = A\sin^2\theta$$

Equation of continuity:

When a region of a fluid contains neither source nor sinks i.e.,when there is no inlets and outlets through which the fluid can enter or have the region the mass contained inside a given volume of fluid remains constant throughout the motion.

Let Δ be a closed surface drawn in the fluid and taken fixed in space.

Let it contains a volume Δv of the fluid and let ρ=ρ(x,y,z,t) be the fluid density at any point (x,y,z) of the fluid in Δv at any time t.



Let n be unit outward drawn normal at any surface element δs ofΔs. δs ≤Δs.

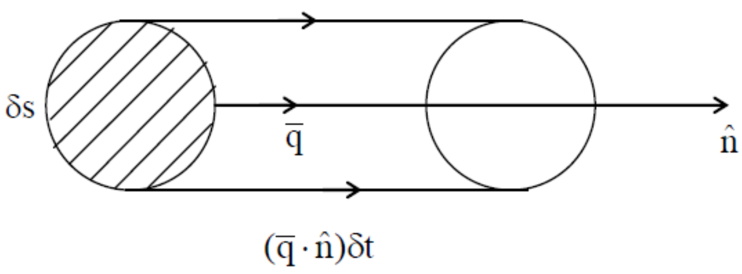
Then if q is the fluid velocity at the element δs , the normal component of q measured outward from Δv is $n.\vec{q}$.

We consider the mass of fluid which leaves Δv by flowing across an element δs ofΔs in time δt.

This quantity is exactly that which is contained in a small cylinder of cross section δs of length $(q.\hat{n})\delta t$

Mass of the fluid=density*volume

$$=\rho(\vec{q}.\hat{n})\delta t.\delta s$$



Hence, the rate of at which fluid leaves ∇v by flowing across the element δs is $\rho(\vec{q}.\hat{n})\delta s$

Summing over all such elements δs, we obtain the rate of flow of fluid coming out of ∇v across the entire surface Δs.

The rate at which mass flows out of the region Δv is

$$\Delta s = \int_s \rho(\vec{q}.\hat{n})\, ds$$

$$= \int_s \rho\vec{q}.\hat{n}\delta s$$

$$= \int_v \nabla.\rho\vec{q} dv \rightarrow (1)$$

The mass M of the fluid possessed by the volume ∇v of the fluid is

$$M = \int_v \rho\, dv$$

where, ρ=ρ(x,y,z,t) with (x,y,z) Cartesian co-ordinate of a general point of ∇v.

Since, the space co-ordinated are independent of time t.

Therefore, the rate of increase of mass within ∇v is,

$$\frac{dM}{dt} = \frac{d}{dt}\int_v \rho\, dv = \int_v \frac{\partial\rho}{\partial t} dv \rightarrow (2)$$

But ∇v does not change with respect to time.

But, the considered region is free from source or sink.

[i.e., mass is neither created nor destroyed]

Therefore, the total rate of change of mass is zero.

$$\int_v \frac{\partial\rho}{\partial t} dv + \int_v \nabla.(\rho\vec{q})\, dv = 0$$

$$\int_v \left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{q})\right] dv = 0$$

Since,∇v is arbitrary

$$\left[\frac{\partial\rho}{\partial t}+\nabla.(\rho\overrightarrow{q})\right]=0\to(3)$$

This is known as equation of continuity which must always hold at any points of a fluid free from sources and sinks.

Other form of equation of continuity

1) $\frac{\partial\rho}{\partial t}+\rho\nabla q+q.\nabla\rho=0\to(4)$

2) $\frac{\partial\rho}{\partial t}+\rho\nabla q=0$

We consider the differential following fluid motion

From equation(4)

The equation of continuity

$$\frac{\partial\rho}{\partial t}+\rho(\nabla.q)+(q.\nabla)\rho=0$$

$$\left[\frac{\partial}{\partial t}+q.\nabla\right]\rho+(q.\nabla)\rho=0$$

$$\frac{D\rho}{Dt}+(q.\nabla)\rho=0..............(5)$$

3) Equation (5) can be written as

$$\frac{1}{\rho}.\frac{D\rho}{Dt}+\nabla.\vec{q}=0$$

$$\frac{D\rho}{Dt}(\log\rho)+\nabla.\vec{q}=0$$

4) When the motion of fluid steady

$$\frac{\partial\rho}{\partial t}=0$$

From (5) the equation of continuity

$$\nabla.\rho\vec{q}=0$$

Here ρ is a function of time

i.e. ρ=ρ(x,y,z)

5) When the fluid is incompressible then ρ=constant and thus $\frac{D\rho}{Dt} = 0$

From (5) equation of continuity

$$\nabla.\rho\vec{q} = 0$$

$$\nabla\vec{q} = 0$$

The same is for homogenous and incompressible fluid.

6) If the fluid is homogenous, incompressible and the flow is of potential kind.

i.e., $\vec{q} = -\nabla\phi$

then, the equation of continuity becomes

$$\nabla\vec{q} = 0$$

$$\nabla(-\nabla\phi) = 0$$
$$\nabla^2\phi = 0$$

This is Laplace equation.

Example:

Text whether the motion specified by $q = \frac{K^2(x_j - y_i)}{x^2 + y^2}$, k is constant is a possible motion for incompressible fluid. If so determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Solution:

To prove the given velocity is possible motion of Incompressible fluid

i.e to prove ∇.q=0

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{q} = \frac{K^2(x\vec{j} - y\vec{i})}{x^2 + y^2} \text{ or } \frac{-K^2 y\vec{i}}{x^2 + y^2} + \frac{K^2 x\vec{j}}{x^2 + y^2}$$

$$\nabla.q = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{K^2(x\vec{j} - y\vec{i})}{x^2 + y^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-K^2 y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{K^2 x}{x^2 + y^2}\right)$$

$$= \frac{K^2 y(2x)}{\left(x^2 + y^2\right)^2} + \frac{(-2x)K^2 y}{\left(x^2 + y^2\right)^2}$$

$$= 0$$

$$\therefore \nabla.q = 0$$

Therefore the given velocity is possibly the motion of incompressible fluid.

Streamlines

The equation of streamlines are given by

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{\frac{-K^2 y}{x^2 + y^2}} = \frac{dy}{\frac{K^2 x}{x^2 + y^2}} = \frac{dz}{0}$$

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0}$$

$$xdx = -ydy \quad \ldots\ldots\ldots\ldots 1$$

$$dz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

Integrating we get

From 1

$$x^2 = -y^2 + c \ldots\ldots\ldots\ldots\ldots 3$$

From2

Z = constant

$Z = c$…………………4

3 and 4 are required equation of streamline.

Potential kind

Suppose that 'q' is the velocity at any time at a point in a fluid.

$$\bar{q} = u\hat{i} + v\hat{j} + w\hat{k} = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = -\nabla\phi$$

where ϕ is termed as the **velocity potential** and the flow of such type is called flow of potential kind.

To prove the motion is of potential kind.

$\nabla \times q = 0$

$$\nabla \times q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-K^2 y}{x^2+y^2} & \frac{K^2 x}{x^2+y^2} & 0 \end{vmatrix} = 0$$

Now, to find velocity potential $\phi(x, y, z)$

$$\frac{-\partial\phi}{\partial x} = \frac{-K^2 y}{x^2+y^2},\ \frac{\partial\phi}{\partial y} = \frac{-K^2 x}{x^2+y^2} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..5$$

Integrating w.r.t x

$$\phi = \int \frac{K^2 y}{x^2+y^2}dx$$

$$= yK^2\left\{\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right\} + f(y) \because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= K^2\tan^{-1}\left(\frac{x}{y}\right) + f(y)\text{..........}6$$

Consider

$$\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial y}\left[K^2\tan^{-1}\left(x/y\right)+f(y)\right]$$

$$=K^2\frac{1}{1+\left(x/y\right)^2}\left(-x/y^2\right)+f'(y)$$

$$=K^2\frac{y^2}{x^2+y^2}\left(-x/y^2\right)+f'(y)$$

$$=\frac{-K^2x}{x^2+y^2}+f'(y)..........................7$$

$$\text{From}(7)\text{and}(5)$$

$$f'(y)=0$$

$$f(y)=\text{constant}$$

$$\phi=K^2\tan^{-1}\left(\frac{x}{y}\right)+c$$

The equipotential is thus given by the plane x=cy through z.

Example

For an incompressible fluid q=[-wy,wx,0],w=constant. Discuss the nature of flow.

The flow is said to be of potential kind if $\nabla\times q=0$

$$\vec{q}=-wy\vec{i}+wx\vec{j}+o\vec{k}$$

$$\nabla\times q=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -wy & wx & 0\end{vmatrix}$$

$$=\left(-wx\frac{\partial}{\partial z}\right)\vec{i}-\left(wy\frac{\partial}{\partial z}\right)\vec{j}+\left(\frac{\partial}{\partial x}wx+\frac{\partial}{\partial y}wy\right)\vec{k}$$

$$=\vec{k}(w+w)=2w\vec{k}\neq 0$$

which it is rotational.

The flow is not of potential kind. It is a rigid body rotating about the z-axis with constant vector angular velocity $w\vec{k}$.

i.e., for the velocity at (x,y,z) in the body is $= -wy\vec{i} + wx\vec{j}$

Equation of streamlines are

$$\frac{dx}{-wy} = \frac{dy}{wx} = \frac{dz}{0}$$

Therefore, the streamlines are the circle

$x^2+y^2=c$ and $z=c$

Example

For a fluid moving in a fine tube of variable section prove from 1st principle that the equation of continuity is $A\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial s}(A\rho v) = 0$ where v is the speed at the point P of the fluid and s is the length of the tube upto P. What does this become for the steady incompressible fluid.

Solution:

Let OPP’ be the central streamline of the tube.

Let S and S+δS be the arc length of OP and OP’.

Let v be the velocity of the fluid at P.

Let A be the area of the section at P. We assume the conditions are constant over the section A.

So that rate of mass flux over A in the sense of S increasing is ρvA.

At the neighbouring section, A’ through P’ the mass flux per unit time in the direction of S increasing is $\rho vA + \delta s.\frac{\partial}{\partial s}(\rho vA)$

At the same instant of time t . Thus, the net rate of flow of mass into the element between the section A and A+δA.

$$\Rightarrow -\rho vA * \delta s \frac{\partial}{\partial s}(\rho vA) + \rho vA$$
$$\Rightarrow -\delta s \frac{\partial}{\partial s}(\rho vA)$$

But at time t, the mass between the sections is $\rho A \delta s$

Rate of increases, $\frac{\partial}{\partial t}(\rho A \delta s)$

$$\frac{\partial \rho}{\partial t} A \delta s$$

In the absence of sources and sinks

$$-\delta s \frac{\partial}{\partial s}(\rho vA) = \frac{\partial \rho}{\partial t} A \delta s$$
$$\text{i.e., } A\frac{\partial \rho}{\partial t} \neq \frac{\partial \rho}{\partial s} vA = 0$$

For steady incompressible flow, the ρ is constant

$$A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(\rho vA) = 0$$
$$\rho = c$$
$$\frac{\partial}{\partial s}(vA) = 0$$
$$\text{i.e., } \frac{d}{ds}(vA) = 0$$

vA=constant

i.e., volume of fluid crossing every section per unit time is constant.

Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + Kx^2/a$ about x-axis (-a≤x≤a). If the liquid enters at the end x=-a of the pipe with velocity v. Show that the time taken by liquid particle to traverse the entire length of the pipe from x=-a to x=a is

$$\frac{2a}{v(1+k)^2}\left(1+\tfrac{2}{3}k+\tfrac{1}{5}k^2\right)$$

Solution:

Let v_0 be the velocity at the section at the section x=0

The area of the section x=-a $\pi a^2(1+k)^2$

The area of the section x=0 πa^2

The area of the section distant x from 0 $\pi\left(a+\frac{kx^2}{a}\right)^2$

By equation of continuity $\pi a^2(1+k)^2 = \pi a^2 v_0 = \pi\left\{\left(a+\frac{kx^2}{a}\right)\right\}^2 .x^0$

Since, x^0 is the velocity across the plane distant x from o

$$dt = \left(a+\frac{kx^2}{a}\right)^2 . \frac{dx}{(1+k)^2 v}$$

$$t = \frac{2}{v(1+k)^2}\int_0^a \left(a+\frac{kx^2}{a}\right)^2 dx$$

Which gives the stated result.

Local and particle rate of change.

Suppose a particle of fluid moves form p(x,y,z) at time t to $P'(x+\delta x, y+\delta y, z+\delta z)$ at time t+δt.Let f(x,y,z,t) be a scalar function associated with some properties of fluid.Then, the motion of the particle from p to p' the total change of is

$$\delta f = \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y + \frac{\delta f}{\delta z}\delta z + \frac{\delta f}{\delta t}\delta t$$

Thus, the total rate of change of f at a point P at a time t.

In the motion of the particle,

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{\delta f}{\delta x}\right)$$
$$= \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} + \frac{\delta f}{\delta z}\frac{dz}{dt} + \frac{\delta f}{\delta t}$$
$$= u\frac{\delta f}{\delta x} + v\frac{\delta f}{\delta y} + w\frac{\delta f}{\delta z} + \frac{\delta f}{\delta t}$$

If q=[u,v,w] is the velocity of the fluid particle at P

$$\frac{df}{dt} = q.\nabla f + \frac{\partial f}{\partial t} \to (1)$$

Similiarlly, for a velocity function F(x,y,z,t) associtated with some property of a fluid.

$$\frac{dF}{dt} = q.\nabla F + \frac{\partial F}{\partial t} \to (2)$$

Hence, both the scalar and vector function of position and time,

By operation equality $\frac{d}{dt} = q.\nabla + \frac{\partial}{\partial t} \to (3)$,provided that those functions are associated with the properties of the moving fluid.

In the obtaining equation (1) and (2), we considered total change. When the fluid particle moves from p(x,y,z) to $P'(x+\delta x, y+\delta y, z+\delta z)$ in time δt.

Thus, $\frac{df}{dt},\frac{dF}{dt}$ are a total differentiation following the fluid particles are called particle rates of change.

On the other hand, partical time derivative $\frac{\partial f}{\partial t},\frac{\partial F}{\partial t}$ are only the time rates of change at the point p(x,y,z)

Consider fixed in space at a point p(x,y,z) they are the local rates of change. It follows that $q.\nabla f$ or $q.\nabla F$

We presents the rate of change due to the motion of particle along it path.

Then, nothing the arc length of the a path by S and PP' by δS.

Simillarly, for a function F Here we use, $\delta\hat{s}.\nabla = \delta/\partial s$

Conditions at a rigid boundary

P is the point on the boundary where the fluid velocity is q, boundary has the velocity U.

If n specifies a unit normal direction at P then q.n=U.n .Since, there is no relative normal velocity at P between boundary and fluid.

Note:

- for invicous fluid the above condition exist.
- For viscous fluid, there is no slip, tangential components must be equal.
- If the boundary is at rest, q.n=0. Every point of boundary.

UNIT -2

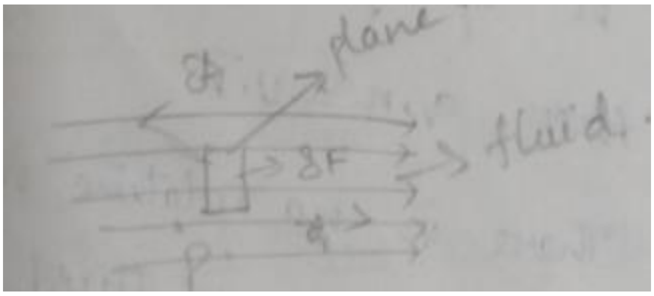
SCHOOL OF SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – MOTION OF A FLUID – SMTA5401

Let P be a point in a fluid moving with velocity $\vec{q}$. We insert an elementary rigid plane area δA into this fluid at point P. This Plane area also moves with the velocity $\vec{q}$ of the local fluid at P

If $\delta\vec{F}$ denotes the force exited on one side of δA by the fluid particles on the other side, then this is will act normal to δA



We assume that $\underset{\delta A \to 0}{lt} \dfrac{\delta\vec{F}}{\delta A}$ exists uniquely.

Then, this limit is called the (hydrodynamic) fluid pressure at point P and is denoted by P.

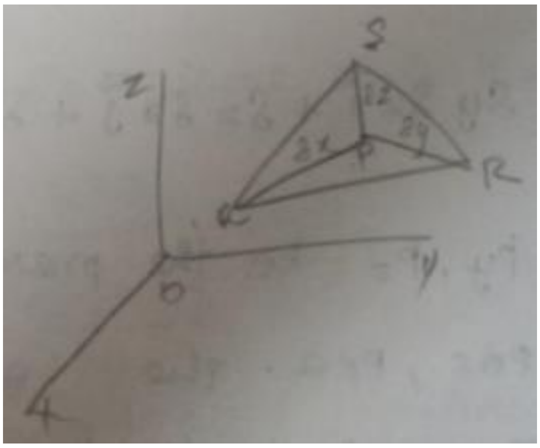
1. Pressure P at a point P in a moving fluid is some in all direction.

Or

Pressure p at P in a moving fluid is independent of the orientation of δA

Let $\vec{q}$ be the velocity of the fluid. We consider an elementary tetrahedron. PQRS of the fluid at a point P of the moving fluid.

Let the edges of the tetrahedron be PQ=δx, PR=δy, PS=δz at time t, where $\delta x, \delta y, \delta z$ are taken along the co-ordinate axes ox,oy,oz resp. This tetrahedron is also moving with the velocity $\vec{q}$ of the local fluid at P.



Let p be the pressure on the face QRS where are is δs. Suppose (l,m,n) are the directional cosines of the normal to δs drawn outwards from the tetrahedron.

$l\delta s$ = projection of the arc δs on yz plane

=are of the face PRS

$= \frac{1}{2}\,\delta x\,\delta z$

lly, $m\delta s = \frac{1}{2}\,\delta x\,\delta z$ $\qquad$ $n\delta s = \frac{1}{2}\,\delta x\,\delta y$

$\therefore$ The total force exerted by the fluid outside the tetrahedron, on the force QRS is

$=-p\delta s\,(l\vec{i} + m\vec{j} + n\vec{k})$

$=-p(l\delta s\,\vec{i} + m\delta s\,\vec{j} + n\delta s\,\vec{k})$

$=-p/2(\delta y\,\delta z\,\vec{i} + \delta z\,\delta x\,\vec{j} + \delta x\,\delta y\,\vec{k})$

Let px,py,pz be the pressure on the faces PRS,PQS,PRQ. The exerted on these faces by the exterial fluid are $\frac{1}{2}px\,\delta y\,\delta z\,\vec{i}$, $\frac{1}{2}py\,\delta z\,\delta x\,\vec{j}$, $\frac{1}{2}pz\,\delta x\,\delta y\,\vec{k}$ resp.

Thus, the total surface on the tetrahedron.

$=-p/2\,(\delta y\delta z\vec{i} + \delta z\delta x\vec{j} + \delta x\delta y\vec{k}) + \frac{1}{2}px\delta x\delta y\vec{i} + \frac{1}{2}py\delta z\delta x\vec{j} + \frac{1}{2}pz\delta x\delta y\vec{k})$

$= \frac{1}{2}\left[(px-p)\delta y\delta z\vec{i} + (py-p)\delta z\delta x\vec{j} + (pz-p)\delta x\delta y\vec{k}\right]$

In addition to surface force (fluid), the fluid may be subjected to body force which are due to external causes such as gravity. Let $\vec{F}$ be the mean body is per unit mass within the tetrahedron.

Volume of the tetrahedron PQRS is 1/2 hδs ie, 1/6 $\delta x\delta y\delta z$ where h is the perpendicular r from p on the face QRS. Thus, the total is acting on the tetrahedron PQRS=1/6 $\ell\vec{F}$ $\delta x\delta y\delta z$ ………… (2)

ℓ - mean density of the fluid.

The net force acting on the tetrahedron is

$$\delta \tfrac{1}{2}\left[(px-p)\delta y\delta z\vec{i}+(py-p)\delta z\delta x\vec{j}+(pz-p)\delta x\delta y\vec{k}\right]+\tfrac{1}{6}\ell\vec{\vec{F}}\delta x\delta y\delta z$$

Let Q be the velocity of P then $\frac{dQ}{dt}$ is the acceleration of P.

If the mass $\tfrac{1}{6}\ell\delta x\delta y\delta z$ stays constant the equation of motion is

$$\tfrac{1}{2}\left[(px-p)\delta y\delta z\vec{i}+(py-p)\delta z\delta x\vec{j}+(pz-p)\delta x\delta y\vec{k}\right]+\tfrac{1}{6}\ell\vec{F}\delta x\delta y\delta z=\tfrac{1}{6}\ell$$
$$\vec{F}\delta x\delta y\delta z\frac{dQ}{dt}$$

$$(px-p)l\delta s\vec{i}+(py-p)m\delta s\vec{j}+(pz-p)n\delta s\vec{k}+\tfrac{1}{6}\ell\vec{F}h\delta s=\tfrac{1}{6}\ell\vec{F}\delta s\frac{dQ}{dt}$$

$\div\,\delta s$ and letting the tetrahedron to zero about P in which h$\rightarrow$0

[This equation contains quantities of the second d third orders of small quantities when we make the edge of the tetrahedron vanishingly small we have 2nd order

$$\tfrac{1}{2}\{(px-p)\delta y\delta z\vec{i}+(py-p)\delta z\delta x\vec{i}+(pz-p)\delta x\delta y\overline{k}\}=0$$

Equating co-eff of the unit vectors

Px=py=pz=p

Since the choice of axes is quite arbitrary this establishes that at any point p of the moving fluid the pressure p is the same in all direction.

- **Condition at a boundary of two in viscid Immiscible fluids.**

Two fluids rerated by a plane boundary, their velocity at p on the boundary being q_1,q_2 respectively.

Consider a small cylindrical hat-box shaped element of normal section δs containing P, and projection into both fluids. Its generators being normal to the surface. Since there is no fluid transfer across the boundary.

$$P_1 \delta s = P_2 \delta s$$

$$P_1 = P_2$$

Ie, In case of liquid in contact with the atmosphere, the pressure at the free surface is the same as that of the atmosphere.

- **Equation of motion Fp+Fg=mxa**
  To obtain Euler's dynamical equation, we use newton's second law of motion.

Consider a region τ of fluid bounded by a closed surface S which consists of the same fluid particles at all time.

Let Q be the velocity and ℓ be the density of the fluid. Then, $\ell d\tau$ is an element of man within S and it remain constant.

The linear momentum of volume τ is M= $\int_{\tau} \vec{q} \ell d\tau$

Rate of change of momentum

$$\frac{dM}{dt} = \frac{d}{dt}\int_{\tau} \vec{q}\ell d\tau = \int_{\tau} \frac{dq}{dt}\ell dt$$

The fluid within τ is acted upon by two types of forces.

The first types of forces are the surface forces which are due to the fluid exterior to τ. Since the fluid is ideal, the surface is simply the pressure p directed along the inward normal at all point of s.

The total surafe force on S is

$$\int_{s} p(-n)ds = -\int_{s} p\vec{n}ds = -\int_{\tau} \nabla p dt \qquad \text{……………….(1)}$$

By gauss div theorem,

The second type of force are the body force which are due to some external agent.

Let $\vec{F}$ be the body force per unit mass acting on the fluid. Then $\vec{F}\ \ell\ d\tau$ is the body force on the element of mass $\ell\ d\tau$

The total body force on the mass within τ is $\int_\tau \vec{F}\rho d\tau$

…………….(2)

By Newton's second law of motion, we have Rate of change of momentum = total force

$$-\nabla\Omega - \frac{1}{\ell}\nabla p$$

$$\int_\tau \frac{dq}{dt}\rho d\tau = \int_\tau \vec{F}\rho d\tau - \int_\tau \nabla p d\tau$$

$$\int_\tau \left(\frac{dq}{dt}\rho - \vec{F}\rho + \nabla p\right) d\tau = 0$$

Since $d\tau$ is arbitrary, we get

$$\frac{dq}{dt}\rho - \vec{F}\rho + \nabla p = 0 \qquad \frac{dq}{dt}\rho - \vec{F}\rho + \nabla p = 0$$

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

……………(*)

Which holds at every point of the fluid and is known as Euler's dynamical equation for an ideal fluid.

- **Other forms of Euler's Equation of motion**

$$\frac{d}{dt} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q}.\nabla$$

$$\therefore \frac{d}{dt}\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p \ \therefore \frac{d}{dt}\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

$$\left[\frac{\partial}{\partial t} + \vec{q}.\nabla\right]\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\wedge(\nabla\wedge q) = F - \frac{1}{\rho}\nabla p$$

$$\left(q.\nabla\right) = \nabla\left(\frac{1}{2}q^2\right)-(\nabla \text{X} q)\text{X}\vec{q}$$

- **Bernoulli's Equation**

The Euler's dynamical equation is $\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$

……………….(1)

Where $\vec{q}$ is velocity, $\vec{F}$ is the body force, p and ℓ are pressure and density

$\vec{F}$ be conservative so that it can be expressed in terms of a body force potential function Ω as $\vec{F} = -\nabla\Omega$

………………..(2)

When the flow is steady then $\frac{\partial\vec{q}}{\partial t} = 0$

………………..(3)

$\therefore$ In case of steady motion with a conservative body force equation (1) on using (2) and (3) gives

$$\nabla\left(\frac{1}{2}q^2\right) - \vec{q}\,\text{X}(\nabla \text{X}\vec{q}) = -\nabla\Omega - \frac{1}{\rho}\nabla p$$

$$\because \frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q}$$

$$\frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^{\,2}\right) - \vec{q}\,\text{X}\nabla\text{X}\vec{q}$$

$$\frac{\partial q}{\partial t} = 0$$

$$\nabla\left(\frac{1}{2}\vec{q}^{\,2} + \Omega\right) + \frac{1}{\rho}\nabla p = \vec{q}\,\text{X}\nabla\text{X}\vec{q}$$

………………….(4)

The density is a function of pressure p only.

$$\frac{1}{\rho}$$

$$\frac{1}{\rho}\nabla p = \nabla\int\frac{dp}{\rho}$$

Using in (4), we get

$$\nabla[\tfrac{1}{2}\vec{q}^{\,2}+\Omega+\int\frac{dp}{\rho} = \vec{q}\,X(\nabla X\vec{q})$$

……………………..(5)

Multiplying (5) scalarly by $\vec{q}$

$\vec{q}.(\vec{q}\,X\ \text{curl}\,\vec{q})=(\vec{q}\,X\vec{q}).\text{curl}\ \vec{q}=0$

$$\vec{q}.\nabla[\frac{1}{2}\vec{q}^{\,2}+\Omega+\int\frac{dp}{\rho}=0$$

…..………………(6)

If $\vec{s}$ is a unit vector along the streamline through general point of the fluid and S measure distance along this streamline, then since S is parallel to $\vec{q}$

$\therefore$ eqn (6) gives $\frac{\partial}{\partial s}[\frac{1}{2}\vec{q}^{\,2}+\Omega+\int\frac{dp}{\rho}]=0$ $\qquad\because \vec{s}\,\square\,\vec{q}$

$\vec{q}=ks$

$\vec{s}\nabla=\frac{\partial}{\partial s}$

Hence along any particular streamline, we have

$$\frac{1}{2}\vec{q}^{\,2}+\Omega+\int\frac{dp}{\rho}=c$$

………………..(7)

Where c is constant which takes different values for different streamlines. (7) is called Bernoulli's equation

- **Some potential theorem.**

  An irrotational motion is called acyclic if the velocity potential φ is a single valued function.

ie, when at every field point, a unique velocity potential exists, otherwise the irrotational motion is said to be cyclic.

For a possible fluid motion, ever if φ is multivalued at a particular point, the velocity at that point must be single valued.

We prove number of theorems for steady irrotational incompressible flows for which the velocity potential φ satisfies $\nabla^2\phi = 0$.

Mean value of velocity potential over spherical surface.

THEOREM:

The mean value of a ϕ over any spherical surface S drawn in the fluid throughout whose interior

$\nabla^2\phi = 0$, is equal to the value of ϕ at the center of the sphere.

If S is the boundary of a spherical surface lying wholly within the fluid, then the mean value of the velocity potential is equal to its value at the center of the sphere.

Let $\phi(p)$ be the value of ϕ at the center P of a spherical surface S of radius r, wholly lying in the liquid. Let $\bar{\phi}$ denotes the mean value of ϕ over S.

Let us draw another concentric sphere ω of unit radius.. Then, a cone with vertex P which intercepts are ds form the sphere S intercepts an area $d\omega$ from the sphere ω.

$$\frac{ds}{dw} = \frac{r^2}{1^2} \qquad\qquad ds=r^2 d\omega$$

Now by definition

$$\bar{\phi} = \frac{\int_s \phi ds}{\int_s ds} = \frac{1}{4\pi r^2}\int_s \phi ds$$

$$\frac{\partial\phi}{\partial r} = \frac{1}{4\pi r^2}\int_c \frac{\partial\phi}{\partial r} ds$$

Since the normal $\hat{n}$ to the surface is along the radius r

$\therefore$ on s, we have $\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial n} = \nabla\varphi.\vec{n}$

$$\therefore \frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi r^2} \int_s \nabla \phi . \hat{n}\, ds$$

$$= \frac{1}{4\pi r^2} \int_s \nabla \phi . \hat{n}\, d\tau \qquad \text{(Gauss theorem)}$$

$$\Rightarrow \frac{1}{4\pi r^2} \int_\tau \nabla^2 \phi d\tau = 0 \qquad \because \nabla \phi = 0 \because \nabla \phi = 0$$

Where τ is the volume enclosed by surface S Thus $\frac{\partial \bar{\varphi}}{\partial r} = 0$

$\Rightarrow \bar{\varphi} = \text{constant}$

Hence, $\bar{\phi}$ is independent of r, so that the mean value of ϕ is the same over all sphere having the same center and therefore is equal to its value at the center.

THEOREM:

If $\sum$ is the solid boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surface, then the mean value of ϕ on $\sum$ is of the form

$$\Phi = \left(M/_R \right) + C$$

Where M and C are constant , provided that the fluid extends to infinitely and is at rest there

(or)

If the liquid of infinite extent is in irrotational motion and is bounded internally by of one or more closed surfaces s, the mean value of ϕ over a large sphere $\sum$, of radius R, which enclose S is of the form $\Phi = \frac{M}{R} + C$ where M and C are constants provided that the liquid is at rest at infinity.

PROOF:

Suppose that the volume of fluid acrossing each of internal surface contained within $\sum$, per unit time is a finite quantity say $-4\pi m$ [$-4\pi m$ represent the flux of fluid across $\sum$ or s]

Since the fluid velocity at any point of $\sum$ is $\frac{\partial \phi}{\partial R}$ radially outwards, the equation f continuity gives

$$\int_{\Sigma} \frac{\partial \varphi}{\partial R} d\Sigma = -4\pi M$$

$$d\Sigma = R^2 dw$$

$$\therefore \frac{1}{4\pi} \int_{\Sigma} \frac{\partial \phi}{\partial R} R^2 dw = -m$$

$$\frac{1}{4\pi} \int_{\Sigma} \frac{\partial \phi}{\partial R} dw = \frac{-m}{R^2}$$

$$\frac{1}{4\pi} \frac{\partial}{\partial R} \int_{\Sigma} dw = \frac{-m}{R^2}$$

Integrating with resp. to R we get,

$$\frac{1}{4\pi} \int_{\Sigma} \phi dw = \frac{M}{R} + C$$

Where c is independent of R , $\frac{1}{4\pi} \int_{\Sigma} \phi \left(\frac{d\Sigma}{R^2} \right) = \frac{M}{R} + C$

$$\frac{\int_{\Sigma} \phi d\Sigma}{4\pi R^2} = \frac{M}{R} + C$$

$$\bar{\phi} = \frac{M}{R} + C$$

………………(2)

To show that c is an absolutes constant, we have to prove that it is independent of co-ordinates of center of sphere Σ.

Let the centre of the sphere Σ be displaced by distance δx in an arbitrary direction while keeping R constant.

$$\frac{\partial \bar{\phi}}{\partial x} = \frac{\partial c}{\partial x}$$

………………….(3)

$\because R$ is constant

Also,

$$\frac{\partial \bar{\phi}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{4\pi}\int_{\Sigma} \phi dw\right] = \frac{1}{4\pi}\int_{\Sigma} \frac{\partial \phi}{\partial x} dw$$

$$= 0$$

From, (3) we get,

$\frac{\partial c}{\partial x} = 0 \Rightarrow c$ is an absolute constant.

Hence,

$\bar{\phi} = \frac{M}{R} + C$ where M and C are constant.

***THEOREM:**

With the radiation of theorem 3, if the fluid is at rest at infinity and if each surface δm is rigid, then the kinetic energy of the moving fluid is

$$T = \tfrac{1}{2}\rho\int_{v} q^2 dv = \tfrac{1}{2}\rho.\sum_{m=1}^{k}\int_{\delta m} \phi\frac{\partial \phi}{\partial n}ds$$

(or)

If Σ is the solid boundary of a large spherical surface of Radius R, containing fluid in motion and also enclosing one or more closed surface, is ϕ_p denotes the potential at any point P of the fluid, if the fluid is at rest ∞ and if each surface X_m, S_m is rigid then the kinetic energy of the moving fluid is

$$T = \tfrac{1}{2}\rho\int_{v} q^2 dv = \tfrac{1}{2}\rho.\sum_{m=1}^{k}\int_{\delta m} \phi\frac{\partial \phi}{\partial n}ds$$

PROOF:

Kinetic energy, from of energy that an object or a particle has by reason T=1/2 of its motion is

$$\text{K.E} = \frac{1}{2}mv^2$$

$\therefore$ Kinetic energy of a fluid particle of mass $\rho\delta v$ is $\frac{1}{2}\rho s v q^2$

$$= \frac{1}{2}\rho\left(-\nabla\phi\right)^2 dv \qquad \left(\because \vec{q} = -\nabla q\right)$$

Hence for entire fluid, $T = \frac{1}{2}\rho\int_v (\nabla\phi)^2 dv$

$$(\nabla\phi)^2 = \nabla\phi.\nabla\phi + \phi\nabla^2\phi$$

$$= \nabla\phi.\nabla\phi$$

$$\nabla.(\phi\nabla\phi) = \nabla\phi.\nabla\phi + (\nabla.\nabla\phi)$$

$$\therefore T = \frac{1}{2}\rho\int_v \nabla.(\phi\nabla\phi)dv$$

$$T = \frac{1}{2}\rho\int_s \hat{n}.(\phi\nabla\phi)\,ds$$

$$= \frac{\rho}{2}\int_s \phi\left(\nabla\phi.\hat{n}\right)ds \quad = \frac{\rho}{2}\int_s \phi\left(\nabla\phi.\hat{n}\right)ds$$

$$= \frac{\rho}{2}\int_\Sigma \phi\frac{\partial\phi}{\partial n}ds + \frac{\rho}{2}\sum_{m=1}^{k}\int_{s_m}\phi\frac{\partial\phi}{\partial n}ds$$

………………..(1)

Denotes the sum of the outer boundary surface S_0, and the inner boundaries S_1, S_2, S_3,……..S_m, The equation of continuity for the entire region V is

$$\int_\Sigma \frac{\partial\phi}{\partial n}ds + \sum_{m=1}^{k}\int_{s_m}\frac{\partial\phi}{\partial n}ds = 0$$

……………..(2)

From (1)&(2).

$$T = \frac{1}{2}\rho\int_\Sigma (\phi - c)\frac{\partial\phi}{\partial n}ds + \frac{1}{2}\rho\sum_{m=1}^{k}\int_{s_m}(\phi - c)\frac{\partial\phi}{\partial n}ds$$

$T = \frac{1}{2}\rho\int_\Sigma (\phi - c)\frac{\partial\phi}{\partial n}ds + \frac{1}{2}\rho\sum_{m=1}^{k}\int_{s_m}(\phi - c)\frac{\partial\phi}{\partial n}ds$ where c is a constant Taking C to be the value ϕ when

$p \to \infty$ as $R \to \infty$, $T = \frac{1}{2}\rho\sum_{m=1}^{k}\int_{s_m}(\phi - c)\frac{\partial\phi}{\partial n}ds$ on each rigid surface S_m, there is no flow so that

$$\int_{s_m} \frac{\partial \phi}{\partial n} ds = 0,$$

Then, $T = \frac{1}{2}\rho \sum_{m=1}^{k} \int_{s_m} \phi \frac{\partial \phi}{\partial n} ds$

Hence Proved

UNIQUENESS THEOREM:1

If $\sum$ is the solid boundary of large spherical surface of Radius R, cont. fluid in motion and enclosing one or more closed surface then $\overline{\varphi} = \frac{M}{R} + c$ provided the fluid extends to infinity and is at rest, if each surface S_m is rigid and K.E of the moving fluid.

$$T = \frac{1}{2}\rho \int_v q^2 dv = \frac{\rho}{2} \sum_{m=1}^{k} \int_{s_m} \phi \frac{\partial \phi}{\partial n} ds \quad T = \frac{1}{2}\rho \int_v q^2 dv = \frac{\rho}{2} \sum_{m=1}^{k} \int_{s_m} \phi \frac{\partial \phi}{\partial n} ds$$

Then, if either ϕ or $\frac{\partial \phi}{\partial n}$ is prescribed on each surface S_m, then, ϕ is determine uniquely through an arbitrary constant.

PROOF:

From previous theorem, $\int_v (\nabla \phi)^2 dv = \sum_{m=1}^{k} \int_{s_m} \phi \frac{\partial \phi}{\partial n} ds$

………………(1)

Within v $\int_v (\nabla \psi)^2 dv = \sum_{m=1}^{k} \int_{s_m} \psi \frac{\partial \psi}{\partial n} ds \quad \nabla^2 \phi = 0$

………………..(2)

Suppose $\phi = \phi_1, \phi = \phi_2$ are the two solutions of (2) subject to (1).

Write, $\phi = \phi_1 - \phi_2$ then. $\nabla^2 \psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0$ in V

Thus, ψ is a harmonic function satisfying similar condition to ϕ_1, ϕ_2

Hence, $\int_v (\nabla \psi)^2 dv = \sum_{m=1}^{k} \int_{s_m} \psi \frac{\partial \psi}{\partial n} ds$

……………(3)

On each S_m, either $\phi_1 = \phi_2$ or $\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}$ ie, either $\psi = 0$ or $\frac{\partial \psi}{\partial n} = 0$ thus in all cases $\psi \frac{\partial \psi}{\partial n} = 0$

On each S_m, so that the RHS of (3) is zero. $\int_v (\nabla \psi)^2 dv = 0$

………………(4)

Since, $(\nabla \psi)^2 \geq 0$, the condition (4) holds only if, $\nabla \psi = 0$ in V

………………(5)

(5) gives ψ is constant $\phi_1 - \phi_2 = \text{constant}$

*** UNIQUENESS THEOREM:2**

If the fluid of theorem V is in uniform motion at infinity and if $\frac{\partial \phi}{\partial v}$ is prescribed on each surface S_m then ϕ is uniquely determined throughout V.

Let V be the velocity of the fluid at infinity suppose a velocity –v on the entire system so as to reduce the velocity of the fluid at infinity to rest. Then the condition of then V prevails in which the velocity at all points of each S_m are known in a fluid which is at rest at infinity. This leads to a unique value of ϕ at each point of v

If the region occupied b the fluid is infinite and fluid is at rest at infinity, prove that only one irrotational motion is possible when internal boundaries have prescribed velocities.

Let there be two irrotational motions gives by two different velocity potential ϕ_1, ϕ_2

The condition are boundaries are $\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}$

…………………(1)

$$\overrightarrow{q_1} = \overrightarrow{q_2} = \vec{0} \qquad \text{at infinity}$$

………………(2)

$$\phi = \phi_1 - \phi_2$$

…………………(3)

$$\nabla^2 \phi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0 - 0 = 0$$

Motion given by ϕ is also irrotational

Further from (3) we get

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0$$

$\vec{q}.\vec{n} = 0 \qquad \Rightarrow \quad \vec{q} = \vec{n}$ on the surface .

Also,

$$\vec{q} = -\nabla\phi = -\nabla\phi_1 + \nabla\phi_2$$

$q_1 - q_2 = 0$ at ∞ using (2)

$\therefore \ \vec{q} = \vec{0}$ everyone on the surface and also at infinity .

Hence we get $\phi =$ constant

$$\phi_1 - \phi_2 = \text{constant}$$
…………………(4)

We can take the constant on RHS (4) to be zero (it gives no motion) and thus we get, $\phi_1 = \phi_2$

Some flows involving axial symmetry

If the region occupied by the fluid is infinite then only one irrotational motion of the fluid exists when the boundaries have prescribed velocities .

(or)

Show that there cannot be 2 different forms of a cyclic irrotational motion of a given liquid whose boundaries have prescribed velocities . Let ϕ_1 and ϕ_2 be two different velocity potentials representing two motions then

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2$$

Since , the kinetic conditions at the boundaries are satisfied by both flows therefore at each point of s $\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n}$

Let $\phi = \phi_1 - \phi_2$

$\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ at each point of fluid $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ at each point of S.

Kinetic energy is given by

$$\frac{\rho}{2}\int_{\Sigma} q^2 d\tau = \frac{\rho}{2}\int_{s}\phi\frac{\partial\phi}{\partial n}ds = 0 \qquad \left(\frac{\partial\phi}{\partial n} = 0\right)$$

$\vec{q} = 0$ at each point of fluid.

$$\begin{aligned}\nabla\phi &= 0\\ \nabla\phi_1 - \nabla\phi_2 &= 0\\ \nabla\phi_1 &= \nabla\phi_2\end{aligned} \qquad \begin{aligned}\nabla\phi &= 0\\ \nabla\phi_1 - \nabla\phi_2 &= 0\\ \nabla\phi_1 &= \nabla\phi_2\end{aligned}$$

Which shows that the motion are the same. Moreover ϕ is unique apart from an additive constant which gives vise be no velocity and thus can be taken as zero.

- **Some flows involving axial symmetry.**

Let $\phi(r,\theta,\psi)$ be the velocity potential at any point having spherical polar co-ordinates (r,θ,ψ) in a fluid of steady irrotational incompressible flow laplace equation $\nabla^2\phi$=0

Since,

$$\sin\theta\left(\frac{\partial}{\partial r}\right)\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin\theta}\left(\frac{\partial^2}{\partial\psi^2}\right) = 0$$

when there is symmetric about the line $\theta = 0$ then $\phi = \phi(r,\theta)$ and the becomes $\sin\theta\left(\frac{\partial}{\partial r}\right)\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$ special solution of (1) are

$\phi = r\cos\theta, \phi = \frac{1}{r^2}\cos\theta$

Thus the move general solution,

$$\phi(r,\theta) = \left(Ar + Br^{-2}\right)\cos\theta$$

- **Stationary sphere in a uniform stream sphere at rest in a uniform stream.**

Consider an impermeable solid of radius a at rest with its center and at the pole of system of spherical polar co-ordinates (r,θ,ψ)

The sphere is immersed in an infinite homogenous liquid with constant density ℓ in the absence of the sphere, would be flowing as a uniform stream with speed U along the direction $\theta = 0$

The presence of the sphere will produce a local perturbation of the uniform streaming motion such that the disturbance diminishes with increasing distance r from the centre of sphere.

We say that the perturbation of the uniform stream tends to zero as $r \to \infty$ In this problem Z-axis is the and symmetry.

Undisturbed velocity of incompressible fluid $U\vec{k}$ ie, $\vec{q} = U\vec{k}$

The velocity potential ϕ_0 due to such a uniform flow would be $\phi_0 = -Uz = -Ur\cos\theta$ when the sphere is inserted, the undisturbed potential -Urcosθ of uniform stream has to be modified by perturbation potential due to the presence of the sphere.

This must have the using properties.

i) It must satisfy laplace equation for the case of axial symmetry.

ii) It must tend to zero at large distance from the sphere.

So, we write $\phi(r,\theta)$=-$Ur\cos\theta + \phi_1(r,\theta)$ r>=a

Where ϕ_1 satisfies the laplace equation together with boundary condition.

$\frac{\partial\theta}{\partial r} = -U\cos\theta + \frac{\partial\phi_1}{\partial r}$ $\frac{\partial\phi}{\partial r} = 0$ (velocity normal at sphere is zero at r=a)

$\frac{\partial\phi_1}{\partial r} = U\cos\theta$ $r = a, a \le \theta \le \pi$

$|\nabla\phi_1| \to 0$ as $r \to \infty$

Hence a suitable form of function ϕ_1 is $\phi_1 = Br^{-2}\cos\theta$

We assume that

$\phi(r,\theta) = -Ur\cos - \frac{B}{r^2}\cos\theta$
……………….(1)

Constant B is to be determined from the fact that there is no flow normal to the surface

r=a ie, $\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = 0$

from (1) $\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{2B}{\alpha^3}\cos\theta$

r=a

$$0 = -U\cos\theta + \frac{2B}{\alpha^3}\cos\theta$$

$$B = \tfrac{1}{2}Ua^3$$

$\therefore$ (1) becomes

$$\phi(r,\theta) = -Ur\cos\theta - \frac{Ua^3}{2r^2}\cos\theta$$

$$= -U\left[r + \frac{a^3}{2r^2}\right]\cos\theta$$

………………(2)

Now, the uniqueness theorem in for that the velocity potential in (2) is unique.

The velocity components at $p(r,\theta,\psi)(r \geq a)$ are

$$q_r = \frac{-\partial\phi}{\partial r} = U\left[1 - \frac{a^3}{r^3}\right]\cos\theta$$

$$q_\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = -U\left[1 + \frac{a^3}{2r^3}\right]\sin\theta$$

$$q_\psi = \frac{-1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

- **Different terms related to motion stagnation points**

Stagnation points are those points in the flow where the velocities vanishes ie, $\vec{q} = \vec{0}$

Thus, these points are obtained by solving the equations

$$U\left[1-\frac{a^3}{r^3}\right]\cos\theta = 0$$

$$U\left[1+\frac{a^3}{2r^3}\right]\sin\theta = 0$$

………………….(4)

Which are satisfied only by r=a, $\theta = 0, \pi$. Thus the stagnation points are $(r = a, \theta = 0)\, and\, (r = a, \theta = \pi)$ on the sphere There are referred to respectively as the near and forward stagnation points.

- **Stream lines**

The equation of streamlines $\dfrac{dr}{q_r} = \dfrac{rd\theta}{q_\theta} = \dfrac{r\sin\theta d\psi}{q_\psi}$

$$\frac{dr}{U\left[1-\frac{a^3}{r^3}\right]\cos\theta} = \frac{rd\theta}{-U\left[1+\frac{a^3}{2r^3}\right]\sin\theta} = \frac{r\sin\theta d\psi}{0} \qquad r \ge a$$

$$d\psi = 0$$

ψ =constant

$$r\left[1-\frac{a^3}{r^3}\right]\cos\theta d\theta = -\left[1+\frac{a^3}{2r^3}\right]\sin\theta dr$$

$$r\left[\frac{r^3-a^3}{r^3}\right]\cos\theta d\theta = -\left[\frac{2r^3+a^3}{2r^3}\right]\sin\theta dr$$

$$\frac{1}{r}\left[\frac{2r^2+\frac{a^3}{r}}{r^2-\frac{a^3}{r}}\right]dr = -2\cot\theta d\theta$$

$$\frac{1}{r}r\left[\frac{2r^2+\frac{a^3}{r^2}}{r^2-\frac{a^3}{r}}\right]dr = -2\cot\theta d\theta$$

Integrating we get,

$$\log\left(r^2 - a^3 r^{-1}\right) = -2\log\sin\theta + \log c$$

$$\log\left(\frac{r^3 - a^3}{r}\right) = -\log\sin^2\theta + \log c$$

$$\sin^2\theta = \frac{cr}{r^3 - a^3}\left(c \geq 0\right)$$

For each value of C_1, above equation given a streamline in the plane ψ =constant.

The choice of C=0 corresponds to the sphere and axis of symmetric.

- **Pressure at any point (stagnation pressure)**

We find stagnation pressure at any point by applying Bernoulli's equation along the streamline.

The pressure at any point of the fluid is obtained by applying Bernoulli's equation along the streamline through the point, taking the pressure at ∞ to be of constant value p_∞ Thus in the absence of body force, the Bernoulli's equation for homogeneous steady flow is

$$p = p_\infty + \frac{1}{2}\rho U^2\left[\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin\theta - 1\right]$$

$$p = p_\infty + \frac{1}{2}\rho U^2\left[\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin\theta - 1\right]$$

$$\frac{p}{\rho} + \frac{1}{2}\left(-\nabla\phi\right)^2 = C$$

…..………….(a)

At infinity $P = p_\infty$ and $-\nabla\phi = U\vec{k}$ we get $C = \frac{p_\infty}{\rho} + \frac{1}{2}U^2$

……………..(b)

From (a) and (b)

$$\frac{p}{\rho} + \frac{1}{2}\left(\nabla\phi\right)^2 = \frac{p_\infty}{\rho} + \frac{1}{2}U^2$$

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(\nabla\phi\right)^2$$

$$= p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left[U^2\left(1-\frac{a^3}{r^3}\right)\cos^2\theta + \left(1+\frac{a^3}{2r^3}\right)^2\sin\theta\right]$$

$$\left(\nabla\phi = -\vec{q}\right)$$

$$p = p_\infty + \frac{1}{2}\ell U^2\left[\left(1-\frac{a^3}{r^3}\right)\cos^2\theta + \left(1+\frac{a^3}{2r^3}\right)^2\sin\theta - 1\right]$$

…..……..(5)

Which gives the pressure at any point of the fluid of particular interest is the distribution of pressure on the boundary of the sphere. It is obtained by putting r=a in (5)

$$p = p_\infty - \frac{1}{2}\rho U^2\left[\left(1+\frac{a^3}{2a^3}\right)^2\sin^2\theta - 1\right]$$

$$= p_\infty - \frac{1}{2}\rho U^2\left[\left(\frac{3a^3}{2a^5}\right)^2\sin^2\theta - 1\right]$$

$$= p_\infty + \frac{1}{8}\rho U^2\left[\frac{9}{4}\sin^2\theta - 1\right]$$

$$= p_\infty + \frac{1}{8}\rho U^2\left[4 - 9\sin^2\theta\right]$$

$$= p_\infty + \frac{1}{8}\rho U^2\left[9\cos^2\theta - 5\right]$$

The maximum pressure occurs at the stagnation point where $\theta = 0$ or π

$$p_{\max} = p_\infty + \frac{1}{2}\rho U^2 \;\; p_{\max}$$

$p_{\max}$ is also called stagnation pressure.

The minimum pressure occurs along the equation circle of sphere where $\theta = \frac{\pi}{2}$

$$p_{\min} = p_\infty - \frac{5}{8} pu^2$$

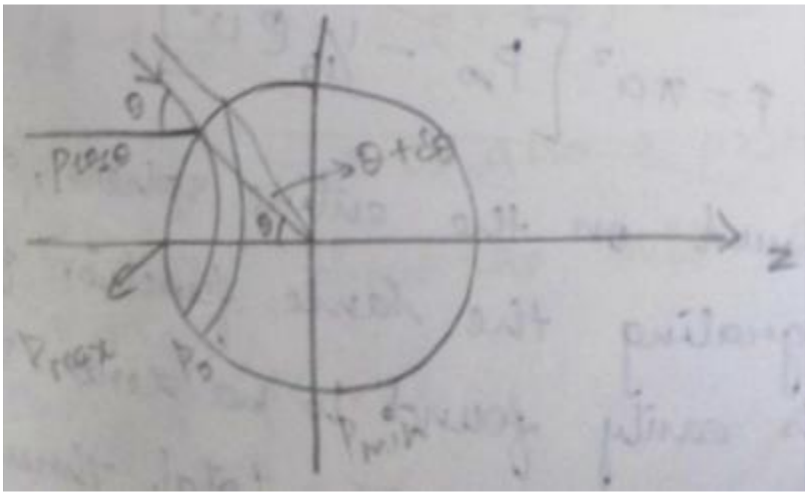
A fluid is pressured to be incapable of sustaining a negative pressure then $p_{\min} = 0 \Rightarrow U = \sqrt{\frac{8p\infty}{5\rho}}$. At this stage the fluid will tend to break away from the surface of the sphere and cavitation is said to occur.

Ie, vaccum is formed.

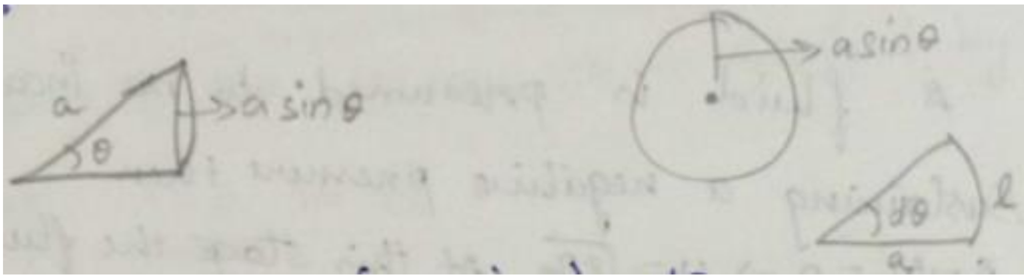
- **Thrust on the hemisphere**

We find thrust (force) on the hemisphere on which the liquid impinges r=a, $0 \le \theta \le \frac{\pi}{2}$

Let δ_s be a small element at $p_0\left(a,\theta,\psi\right)$ of the hemisphere bounded by circles at r=a and at angular distance θ and $\theta + \delta\theta$ from the axis of symmetry (z-axis)



The component of the thrust on δ_s is $p\cos\delta_s$. Hence total thrust on the hemisphere is along Z'o and is given by



$$ds = (2\pi a \sin\theta)\, a d\theta$$

$$F = \int p \cos ds$$

$$= \int_0^{\frac{\pi}{2}} p \cos(2\pi a \sin\theta) a d\theta$$

$$= \int_o^{\frac{\pi}{2}} 2\pi a^2 \sin\theta \cos\theta \left[p_\infty + \frac{1}{8}\rho U^2 \left[9\cos^2\theta - 5 \right] \right] d\theta$$

$$= \int_o^{\frac{\pi}{2}} 2\pi a^2 \sin\theta \cos\theta \left[p_\infty + \frac{1}{2}\rho U^2 \left[\frac{9}{4}\sin^2\theta - 1 \right] \right] d\theta$$

Using value of p at boundary

$$f = \pi a^2 \left[p_\infty - \tfrac{1}{16}\rho U^2 \right]$$

The thrust on the entire sphere, obtained by integrating the same function form $\theta = 0$ to π is easily found to be zero. This result can be generalized. The total thrust on a rigid body of any sphere in a uniform stream is zero. This is called Alembert’s paradox.

- **Sphere in motion in fluid at rest of infinity**

Let a solid sphere of radius ‘a’ centered at O be moving with uniform velocity $-U\vec{k}$ in incompressible fluid of infinity extent, which is at rest at infinity, z-axis is the axis of symmetry and $\vec{k}$ is unit vector in this direction. As the sphere is moving with velocity $-U\vec{k} \Rightarrow$ the relative velocity of fluid if the sphere be considered to be at rest is $U\vec{k}$.

The boundary value problem for ϕ is now to solve, $\nabla^2\phi = 0$ ……………(1)

Such that $\frac{-\partial\phi}{\partial r} = -U\cos$ (r=a)
……………(2)

And $(\nabla\phi) = 0$ $(r \to \infty)$
…………….(3)

The present case is also a problem with axial symmetry about the axis $\theta = 0, \pi$ so, $\phi = \phi(r, \theta)$

Also, Since, $p_1(\cos\theta) = \cos\theta$

Legendre function and the boundary condition (2) implies that the dependence of ϕ an θ must be like $\cos\theta$,

$\therefore \phi$ has the form

$$\phi = -(Ar + \frac{B}{r^2}) p_1(\cos\theta)$$

$$= -(Ar + \frac{B}{r^2}) \cos\theta$$

To satisfy (3), it is necessary that A=0 and then from (2) we get $B = \frac{1}{2} U a^3$

Thus, the solution for ϕ is

$$\phi = \frac{-Ua^3}{2r^2} \cos\theta$$

From here, the velocity component are obtained to be

$$q_r = \frac{-\partial\phi}{\partial r} = \frac{-Ua^3}{r^3} \cos\theta$$

$$q_\theta = \frac{-1}{r} \frac{\partial\phi}{\partial r} = \frac{-Ua^3}{2r^3} \sin\theta$$

$$q_\psi = 0 \; q_\psi = 0$$

Where (r, θ, ψ) are spherical polar co-ordinates the various terms of particular importance related to this motion.

- **Streamline**

The differential equations for streamlines are $\frac{dr}{q_r}=\frac{rd\theta}{q_\theta}=\frac{r\sin\theta d\psi}{q_\psi}$

$$\frac{dr}{\left[\frac{-Ua^3}{r^3}\right]\cos\theta}=\frac{rd\theta}{\left[\frac{-Ua^3}{2r^3}\right]\sin\theta}=\frac{r\sin\theta d\psi}{0}$$

$$d\psi=0\Rightarrow \qquad \psi=\text{constant}$$

$\frac{dr}{r}=2\cot\theta d\theta$, $\quad \log r=2\log\sin\theta+\log C$

$r=c\sin^2\theta$

Streamline, line are $r=c\sin^2\theta$, ψ =constant.

- **Kinetic energy of the liquid.**

Let S be the surface of sphere and ℓ be the density of liquid, then kinetic energy is given by

$$T_1=\frac{\rho}{2}\int_s \phi\frac{\partial\phi}{\partial n}ds$$

Where $\vec{n}$ is the outward unit normal. But for the sphere $\vec{n}$ is along radius vector

$$\therefore\left(\phi\frac{\partial\phi}{\partial n}\right)_s=\left(-\phi\frac{\partial\phi}{\partial r}\right) \qquad \text{(r=a)}$$

$$=\left(\frac{1}{2}Ua\cos\theta\right)U\cos\theta$$

$$=\frac{1}{2}U^2a\cos^2\theta$$

$$\therefore T_1=\frac{\rho}{2}\int_s\frac{1}{2}v^2a\cos^2\theta ds$$

$$=\frac{\rho av^2}{4}\int_0^\pi\cos^2\theta 2\pi a\sin\theta ad\theta$$

$$=\frac{\pi\rho a^3v^2}{2}\int_0^\pi\cos^2\theta\sin^2\theta d\theta$$

$$= \frac{\pi\rho a^3 v^2}{2}\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{1}{3}\pi\rho a^3 v^2 = \left(4\!\!\diagup_3 \pi\rho a^3\right)\left(\frac{v^2}{4}\right)$$

$$= \frac{1}{4} m^1 v^2$$

……………….(6)

Where $m^1 = \frac{4}{3}\pi\rho a^3$ is the mass of the liquid displaced by the sphere.

Also, F.E of the sphere moving with speed v is given by, $T_2 = \frac{1}{2} m v^2$

…………………(7)

$M = \frac{4}{3}\pi a^3$ is the mass of the sphere σ being the density of the material of the sphere.

From (6) and (7), Total Kinetic Energy T is $T = T_1 + T_2$

$$m + \frac{m^1}{2}$$

$T = \frac{1}{2}[m + \frac{1}{2}\mathrm{m}^1]\mathrm{U}^2(t)$ $T = \frac{1}{2}[m + \frac{m^1}{2}]\mathrm{U}^2$ he quantity $m + \frac{m^1}{2}$ is called the virtual mass of the sphere.

- **Accelerating sphere moving in a fluid at rest at infinity**

The solution derived above for ϕ is applicable when the sphere translates unsteadily along a stream line

$U = U(\rho)$ and the velocity potential has $\phi = \phi(r, \theta, t) = \frac{-U(t)a^3}{2r^2}\cos\theta$

…………….(1)

The instantaneous values of velocity comp and F.E at time t are given by

$q_r = \frac{-U(t)a^3}{r^3}\cos\theta$, $q_\theta = \frac{-U(t)a^3}{2r^3}\sin\theta$, $q_\psi = 0$ (similar to steady case)

$$T=\frac{1}{2}[m+\frac{1}{2}\mathrm{m}^{1}]\mathrm{U}^{2}\left(t\right)$$

……………..(2)

The pressure at any point of the fluid is obtained by using Bernoulli’s equation for unsteady flow of a homogenous liquid, in the absence of body force an,

$$\frac{p}{\rho}+\frac{1}{2}\vec{U}^{2}-\frac{\partial\phi}{\partial t}=f\left(t\right)$$

…………………(3) $\frac{p}{\rho}=\frac{p_{\infty}}{\rho}-\frac{1}{2}\vec{U}^{2}+\frac{\partial\phi}{\partial t}$

Where f(t) is a function of time t only let to be the presence at infinity where the fluid is at rest

From (3) we get,

$$f(t)=\frac{p_{\infty}}{\partial t}$$

$$\frac{\partial r}{\partial t}=U\cos\theta$$

$$\frac{p}{\rho}=\frac{p_{\infty}}{\rho}-\frac{1}{2}\vec{U}^{2}+\frac{\partial\phi}{\partial t} \qquad \text{……………..(4)}$$

To find $\frac{\partial\phi}{\partial t}$

$\vec{U}=-U\vec{k}=-U\left(t\right)\vec{k}$ is the velocity of sphere the velocity potential given in (1) can be expressed in the from $\phi=\frac{1}{2}\frac{a^{3}\left(\vec{u}.\vec{r}\right)}{r^{3}}$ ……………….(5)

Since $\vec{r}$ is the position vector of a fixed point P of the fluid relative to the moving centre O of the sphere, it is the $\vec{U}=\frac{\partial}{\partial t}\left(\overrightarrow{-r}\right)$

………………(6)

$r^{2}=\vec{r}.\vec{r}=r\frac{\partial\vec{r}}{\partial t}=\vec{r}\frac{\partial\vec{r}}{\partial t}=-\vec{r}.\vec{u}$ using (6)

$$=-\vec{r}.\left(-u\vec{k}\right)$$

$$=ru\left(\vec{r}.\vec{k}\right)$$

$$= ru\cos\theta$$

$$\frac{\partial r}{\partial t} = U\cos\theta$$

……………...(7)

Diff (5) with respect to 't'

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}a^3[\frac{-U^2}{r^3} - \frac{\cos\theta}{r^2}\frac{\partial U}{\partial t} + \frac{3U^2}{r^3}\cos^2\theta]$$

$$= \frac{-a^3}{2}[\frac{\mathrm{U}\cos\theta}{r^2} + \frac{U^2}{r^3} - \frac{3U^2}{r^3}\cos^2\theta]$$

$$U = \frac{\partial u}{\partial t}$$

Also,

$$\vec{U}^2 = q_r^2 + q_\theta^2 = \frac{U^2a^6}{r^6}\cos^2\theta + \frac{U^2a^6}{4r^6}\sin^2\theta \; \frac{U^2a^6}{r^6}[\cos^2\theta + \frac{1}{4}\sin^2\theta]$$

$$= \frac{U^2a^6}{r^6}[\cos^2\theta + \frac{1}{4}\sin^2\theta]$$

The pressure at any point of the fluid can be obtained from equation (4) $U = \frac{\partial u}{\partial t}$

In particular at a point on the sphere r=a

$$\frac{\partial\phi}{\partial t} = \frac{-1}{2}[Ua\cos\theta + U^2 - 3U^2\cos^2\theta]$$

$$\overline{U} = \frac{U^2}{4}[4\cos^2\theta + \sin^2\theta]$$

And the corresponding pressure is given by

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \tfrac{1}{2}Ua\cos\theta + \tfrac{1}{8}U^2(9\cos^2\theta - 5)$$

………………(5)

The forces (thrust) acting on the sphere is given by

$$\vec{F} = \int_0^\pi p\cos\theta(2\pi a\sin\theta)ad\theta\vec{k}$$

$$\frac{p}{\rho}=\frac{p_{\infty}}{\rho}-\tfrac{1}{2}Ua\cos\theta+\tfrac{1}{8}U^{2}(9\cos^{2}\theta-5)$$

$$\overline{U}=\frac{U^{2}}{4}[4\cos^{2}\theta+\sin^{2}\theta]$$

$$=2\pi a^{2}\vec{k}\int_{0}^{\pi}[p_{\infty}-\tfrac{1}{2}\rho\dot{U}\,a\cos+\tfrac{1}{8}pU^{2}(9\cos^{2}\theta-5)]\cos\theta\sin\theta d\theta$$

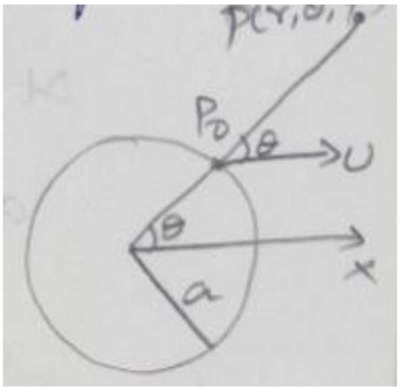
$$=\frac{2}{3}\pi\rho a^{3}\dot{U}\,\vec{k}=\tfrac{1}{2}(\frac{4}{3}\pi\rho a^{3})\dot{U}\,\vec{k}$$

$$=\tfrac{1}{2}m^{1}\dot{U}\,\vec{k}$$

$m=\dfrac{4}{3}\pi a^{3}\rho$ is mass of the liquid displaced.

This shows that the force acts in the direction appointing the sphere's motion.

- **Sphere moving with constant velocity which is otherwise at rest.**



We consider a solid sphere with centre O moving with uniform velocity U_i in incompressible fluid of infinite extent which is at rest at infinity OX is the axis of symmetry and the direction of unit vector i. We take ϕ to be finite at infinity then, the velocity potential at $P(r,\theta,\psi)$ where r>=a will be in the form

$$\phi(r,\theta)=\mathrm{Ar}^{-2}\cos\theta$$

……………….(1)

This satisfies the axiom symmetric form of laplace equation in spherical polar co-ordinates are

$$\frac{-\partial\phi}{\partial r}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}$$

From (1),

$$q_r = \frac{-\partial\phi}{\partial r} = 2Ar^{-3}\cos\theta$$
$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = Ar^{-3}\sin\theta$$
$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$
$$\therefore q_r = 2Ar^{-3}\cos\theta$$
$$q_\theta = Ar^{-3}\sin\theta$$
$$q_\psi = 0$$

At $p_0(a,\theta,\psi)$

$$q_r = \frac{2A}{a^3}\cos\theta$$
$$q_\theta = \frac{A}{a^3}\sin\theta$$
$$q_\psi = 0$$

Now, the velocity at P_0 is U_i and hence

$$q_r = U\cos\theta$$
…………………(B)

From (A) and (B),

$$\frac{2A}{a^3}\cos\theta = U\cos\theta$$

$$U = \frac{2A}{a^3} \quad A = \frac{U}{2}a^3$$

$$A = \frac{U}{2}a^3$$

Now (C) in (1),

$$\phi(r,\theta) = \frac{U}{2}a^3r^{-2}\cos\theta$$

$$\phi(r,\theta) = \frac{U}{2r^2}a^3\cos\theta$$
…………….(*)

To find kinetic energy of a fluid

We consider,

$$T = \frac{1}{2}\rho\int_s \phi\frac{\partial\phi}{\partial r}ds$$

$$= \frac{1}{2}\rho\int_s \phi(\frac{-\partial\phi}{\partial r})ds$$

From (*)

$$\frac{\partial\phi}{\partial r} = \frac{-Ua^3}{r^3}\cos\theta$$

$$\therefore\ T = \frac{1}{2}\rho\int_s \phi\left(\frac{-\partial\phi}{\partial r}\right)ds$$

$$\phi\left(\frac{-\partial\phi}{\partial r}\right) = \frac{Ua^3}{-2r^2}\cos\theta\left(\frac{Ua^3}{r^3}\cos\theta\right)$$

$$= \left[\frac{U^2a^6}{2r^5}\cos^2\theta\right] \qquad \text{r=a}$$

$$= \frac{U^2a}{2}\cos^2\theta$$

$$\therefore \qquad T = \frac{1}{2}\rho\int_s \frac{U^2a}{2}\cos^2\theta ds$$

$$= \frac{U^2a\rho}{4}\int_s \cos^2\theta ds$$

$$= \frac{\rho aU^2}{4}\int_s \cos^2\theta(2\pi a^2\sin\theta d\theta)$$

$$= \frac{\rho a^3\pi U^2}{2}\int_s \cos^2\theta\sin\theta d\theta$$

$$= \frac{\rho a^3 \pi U^2}{2} \int_0^{\pi} \cos^2\theta d(\cos\theta)$$

$$= \frac{\rho a^3 U^2 \pi}{2} \left[\frac{\cos^3\theta}{3} \right]_0^{\pi}$$

$$= \frac{\rho a^3 U^2 \pi}{2} \left[\frac{-1}{3} - \frac{-1}{3} \right]$$

$$= \frac{\rho a^3 U^2 \pi}{2} \left[\frac{-2}{3} \right]$$

$$= \frac{-\rho a^3 U^2 \pi}{3}$$

- **Accelerating sphere moving in fluid at ret at infinity**

(or)

A sphere of centre I and radius a moves through an infinite liquid of constant density ρ at rest at infinity O describes a straight lines with velocity V(t) if there are no body force show that the pressure P at points on the surface of sphere in a plane perpendicular to the straight lines at a distance x from O measure positively in a direction of v given by.

$$P = p_o - \frac{5}{8}\rho v^2 + \frac{9}{8}\rho v^2 (\frac{x^2}{a^2}) + \frac{1}{2}\rho x \frac{dv}{dt}$$

p_o **is pressure at infinity reduce that the thrust on sphere is** $\frac{1}{2} M^1 \frac{dv}{dt}$ **where M^1 is the man of liquid having the volume of sphere.**

Sol:

Let $p(r, \theta, \psi)$ be a point such that, ϕ at P is given by

$\phi(r, \theta, \psi) = A(t) r^{-2} \cos\theta$ which satisfies the spherical polar form of laplace equation.

$$q_r = -\left(\frac{\partial \phi}{\partial r} \right) = -[A(t)(-2r^{-3}\cos\theta)]$$

$$= 2A(t) r^{-3} \cos\theta$$

But, by boundary condition,

$$q_r = v(t)\cos\theta$$

$$\therefore 2A(t)r^{-3}\cos\theta = v(t)\cos\theta$$

$$A(t) = \left[\frac{1}{2}v(t)r^3\right] \qquad\qquad r=a$$

$$A(t) = \frac{1}{2}v(t)a^3$$

$$\therefore \phi(r,\theta,\psi) = \frac{1}{2}v(t)a^3r^{-2}\cos\theta$$

At $\phi(r,\theta,\psi)$ for r>=a

$$q_o = \frac{-\partial\phi}{\partial r} = -[v(t)a^3(-2r^{-3})\cos\theta]$$

$$= v(t)a^3r^{-3}\cos\theta$$

$$\frac{1}{2r^3}v(t)a^3\sin\theta$$

$$q_\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = -[\frac{-1}{r}.\frac{1}{2}v(t)a^3r^{-2}\sin\theta]$$

$$= \frac{1}{2r^3}v(t)a^3\sin\theta$$

$$q_\psi = \frac{-1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

$$q^2 = v(t)[a^6r^{-6}\cos^2\theta + \frac{1}{4r^6}a^6\sin^2\theta]$$

$$= \frac{1}{4}v(t)[a^6r^{-6}(4\cos^2\theta + \sin^2\theta)]$$

Consider,

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}a^3\frac{\partial}{\partial t}\left(\frac{v(t)}{r^2}\cos\theta\right)$$

$$=\frac{1}{2}a^3[\frac{\cos\theta}{r^2}\frac{dv}{dt}+v\frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right)=\frac{\partial}{\partial t}\left(\frac{\vec{r}.i}{r^3}\right)$$

$$=\frac{\partial}{\partial t}\left(\vec{r}.i\right)r^{-3}$$

$$=i\frac{\partial\bar{r}}{\partial t}r^{-3}+\left(\vec{r}.t\right)(-3r^{-4})\frac{\partial r}{\partial t}$$

$$\overline{po}=-\bar{r}$$

$$\frac{\partial}{\partial t}(-\bar{r})=\text{vel of O =V}_{\text{i}}$$

$$\frac{\partial\bar{r}}{\partial t}=-v_i$$

$$r^2=\overline{r^2}$$

$$2r\frac{\partial r}{\partial t}=2\bar{r}\frac{\partial\bar{r}}{\partial t}$$

$$\frac{\partial r}{\partial t}=\frac{1}{r}\left(-v_i,\bar{r}\right)$$

$$\therefore\ \frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right)=\frac{v}{r^3}(-1+3\cos^2\theta)$$

$\therefore$ The Bernoulli's equation,

$$\frac{\rho}{2}+\frac{1}{2}a^2-\frac{\partial\phi}{\partial t}=f(t)$$

$$\frac{p}{\rho}+\frac{1}{8}v(t)\mathrm{a}^6r^{-6}(4cos^2\theta+\sin^2\theta)-\frac{a^3}{2r^3}[r\cos\theta\frac{dv}{dt}-v^2+3v^2\cos^3\theta]d\theta$$

Putting , $r=a$, $x=a\cos\theta$

$$p=p_o-\frac{5}{8}\rho v^2+\frac{9}{8}\rho v^2\frac{x^2}{a^2}+\frac{1}{2}\rho x\frac{dv}{dt}$$

SCHOOL OF SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – TWO DIMENSIONAL FLOWS-I – SMTA5401

UNIT-III

Some two dimensional flow use of cyclindrical polar co-ordinates.

For an incompressible irrotational flow of uniform density, the equation of continuity $\Delta^2\phi = 0$ for the velocity potential $\phi(r,\theta,z)$ in cylindrical polar co-ordinates (r,θ,z) is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \rightarrow (1)$

If the flow is two dimensional and the co-ordinate axes are to so chosen that all physical quantities associated with the fluid are independent of z then $\phi = \phi(r,\theta)$

$\therefore$ (1) becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \rightarrow (2)$$

Let $\phi(r,\theta) = -f(r)g(\theta)$ be the solution of equ (2) for separation of variables.

Thus, we get $g(\theta)\frac{1}{r}\frac{d}{dr}[rf'(r)]+\frac{1}{r^2}f(r)g''(\theta)=0$

$$\frac{r\frac{d}{dr}[rf'(r)]}{f(r)} = -\frac{g''(\theta)}{g(\theta)} \rightarrow (4)$$

Thus, L.H.S of (4) is a function of r only and RHS is a function of θ only.

As r,θ are independent variables. So, each side of equ(4) is a constant say λ.

$$\frac{r^2f''(r)+rf'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda$$

i.e. , $r^2f''(r)+rf'(r)-\lambda f(r)=0 \rightarrow (5)$

$$g''(\theta)+\lambda g(\theta)=0 \rightarrow (6)$$

Equation (6) has periodic solution when λ>0 normally the physical problem requires that $g(\theta+2\pi)=g(\theta)$ and this is satisfied when $\lambda=n^2$ for n=1,2,3,⋯

The basic solution of equ (6) are

$$g(\theta)=C_1\cos n\theta+C_2\sin n\theta\rightarrow(7)$$

Equ (5) is of Euler homogeneous type and it is reduced to a linear different equation of constant co-efficient by putting

$$\begin{aligned}&r=e^t,\\&t=\log r,\\&\frac{dt}{dr}=\frac{1}{r}\end{aligned}$$

$$f'(r)=\frac{df}{dr}=\frac{df}{dt}.\frac{dt}{dr}=\frac{1}{r}\frac{df}{dt}$$

$$f''(r)=\frac{d^2f}{dr^2}=\frac{1}{r}\frac{d}{dr}\left(\frac{df}{dt}\right)+\left(\frac{-1}{r^2}\right)\frac{df}{dt}$$

$$=\frac{1}{r}\left[\frac{d}{dt}\left(\frac{df}{dt}\right)\frac{dt}{dr}\right]-\frac{1}{r^2}\frac{df}{dt}$$

$$=\frac{1}{r^2}\frac{d^2f}{dt^2}-\frac{1}{r^2}\frac{df}{dt}$$

$$r^2f''(r)=\frac{d^2f}{dt^2}-\frac{df}{dt}$$

Equation (5) reduces to

$$\frac{d^2f}{dt^2}-\frac{df}{dt}+\frac{df}{dt}-n^2f=0$$

$$\frac{d^2f}{dt^2}-n^2f=0$$

Solution is $f=e^{\pm nt}=\left(e^t\right)^{\pm n}=r^{\pm n}$

$$=c_3r^n+c_4r^{-n}\rightarrow(8)$$

A special solution of equ(2) is obtained by equ(7) and (8) as

$$\phi(r,\theta) = -f\left(r\right)g\left(\theta\right)$$
$$\phi(r,\theta) = -\left(c_3 r^n + c_4 r^{-n}\right)\left(c_1 \cos n\theta + c_2 \sin n\theta\right) \to (9)$$

The most general solution is

$$\phi(r,\theta) = -\sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\theta + D_n \sin n\theta\right) \to (10)$$

Particular case,

For n=0 we have,

$$f = k_1 + k_2 t = k_1 + k_2 \log r$$
$$g = k_3 + k_4\theta$$

So the another solution of equ(2) is

$$\phi(r,\theta) = -\left(k_1 + k_2 \log r\right)\left(k_3 + k_4\theta\right)$$

For n=1

$$\phi = -r\cos\theta \qquad \phi = -r\sin\theta$$
$$\phi = -r^{-1}\cos\theta \qquad \phi = -r^{-1}\sin\theta$$

Discuss the uniform flow part as infinitely long circular cylinder.

Let P be a point with cylindrical polar co-ordinates $\left(r,\theta,z\right)$ in the flow region of an unbounded.

Incompressible fluid of uniform density moving irrotationally with uniform velocity $-U\vec{i}$ at infinity past the fixed solid cyclinder r≤a.

When the cyclinder r=a is introduced, it will produce a perturbation which is such as to satisfy laplace equation and to become vanishingly small for large r.

This suggests taking the velocity potential for r≤a,0≤θ≤2π in the form

$$\phi\left(r,\theta\right) = Ur\cos\theta - Ar^{-1}\cos\theta \to (1)$$

Where the velocity potential of the uniform stream is $Ux = Ur\cos\theta$

and due to perturbation it is $-Ar^{-1}\cos\theta$ which tends to zero as r→∞ and gives rise to a velocity pattern which is symmetrical about θ=0,π(the term $r^{-1}\sin\theta$ is not there since it does not give symmetric flow)

As there is no flow across r=a_1 so the boundary condition on the surface is

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{When } r=a \rightarrow (2)$$

$$\phi = Ur\cos\theta - Ar^{-1}\cos\theta$$

$$\frac{\partial\phi}{\partial r} = U\cos\theta + Ar^{-2}\cos\theta$$

When r=a, $\frac{\partial\phi}{\partial r} = 0$,0≤θ≤2π

$$0 = U + Aa^{-2}$$
$$0 = Ua^2 + A$$
$$A = Ua^2$$

Thus, velocity potential for an uniform flow part a fixed infinite cylinder is

$$\phi(r,\theta) = Ur\cos\theta + U\frac{a^2}{r}\cos\theta$$

$$= U\cos\theta\left(r + \frac{a^2}{r}\right) \rightarrow (3), r > a, 0 \le \theta \le 2\pi$$

From here, the cylindrical components of velocity are $\left(\vec{q} = \nabla\phi\right)$

$$q_r = \frac{-\partial\phi}{\partial r} = -U\cos\theta\left[1 - \frac{a^2}{r^2}\right]$$
$$q_\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-1}{r}U\sin\theta\left[r + \frac{a^2}{r}\right]$$

$$= U\sin\theta\left(1+\frac{a^2}{r^2}\right)$$

$$q_z = \frac{-\partial\phi}{\partial z} = 0$$

We note that as r→∞, $q_r = -U\cos\theta, q_\theta = U\sin\theta$ which are consistent with the velocity at infinity $-U\vec{i}$ of the uniform stream.

Stream function:

When motion is the same in all planes parallel to xy plane and there is no velocity parallel to the x-axis i.e., when u,v are function of x,y,t and w=0. The motion is regarded as two-dimensional.

Now to consider the flow across a curve in this plane, we mean the flow across unit length of a cylinder where trace on the xy plane is the curve in question, the generations of the cylinder being parallel to the z-axis.

For a two-dimensional motion in x-y plane, $\vec{q}$ is a function of x,y,t only and the differential equation of the streamline are

$$\frac{dx}{u} = \frac{dy}{v} \text{i.e.,} \, vdx - udy = 0 \rightarrow (1)$$

and corresponding equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow (2)$$

Equation (2) is condition of exactness of (1)

It is that (1) must be exact differential $vdx - udy = d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$

$$u = \frac{-\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}$$

This function ψ is called the stream function or the current function or lagranges stream function. Then streamlines are given by the solution of (1) is dψ=0, i.e, ψ=constant.

Thus, the stream function is constant along a streamlines.

Note:

- It is clear that the existence of stream function is merely a consequence of the continuity and incompressibility of fluid.
- The stream function always exists is all types of two dimensional motion whether rotational or irroational.

Physical interpretation of stream function.

Let P be a point on a curve C in x-y plane. Let an element ds of the curve makes an angle θ with x-axis. The direction cosines of the normal at P are

$$\left(\cos\left(90+\theta\right),\cos\theta,0\right)$$
$$\text{i.e.,}\left(-\sin\theta,\cos\theta,0\right)$$

The flow across the curve C from right to left is

$$\int_C \vec{q}.\hat{n}ds \quad \text{where} \quad \begin{array}{l}\hat{n}=-\sin\theta\vec{i}+\cos\theta\vec{j}\\ \vec{q}=u\vec{i}+v\vec{j}\end{array}$$

$$=\int_C\left(-u\sin\theta+v\cos\theta\right)ds$$
$$=\int_C\left(\frac{\partial\psi}{\partial y}\sin\theta+\frac{\partial\psi}{\partial x}\cos\theta\right)ds\left(\because u=\frac{\partial\psi}{\partial y},v=\frac{\partial\psi}{\partial x}\right)$$
$$=\int_C\left(\frac{\partial\psi}{\partial y}\frac{dx}{ds}+\frac{\partial\psi}{\partial x}\frac{dy}{ds}\right)ds\left(\because\cos\theta=\frac{dx}{ds},\sin\theta=\frac{dy}{ds}\right)$$
$$=\int_C\frac{\partial\psi}{\partial y}dx+\frac{\partial\psi}{\partial x}dy$$
$$=\int_C d\psi=\psi_B-\psi_A$$

Where $\psi_A \text{and} \psi_B$ are the values of ψ at the initial and final points of the curve.

Thus, the difference of the values of a stream function at any two points represents the flow across the curve, joining the two points.

Corolloary:

Suppose that the curve C be the streamline, then no fluid crosses its boundary, then $\psi_B-\psi_A=0\Rightarrow\psi_B=\psi_A$

Ψ is constant along C.

Relation between φ and ψ:

The velocity potential φ is given by

$$\vec{q} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

$$u = \frac{-\partial\phi}{\partial x} \qquad v = \frac{-\partial\phi}{\partial y} \quad \rightarrow(1)$$

The stream function Ψ is given by

$$u = \frac{-\partial\psi}{\partial y} \qquad v = \frac{-\partial\psi}{\partial x} \quad \rightarrow(2)$$

Form equ(1) and (2)

$$\frac{-\partial\phi}{\partial x} = \frac{-\partial\psi}{\partial y}, \; \frac{-\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x} \quad \rightarrow(3)$$

i.e., $\nabla^2\phi = 0 \text{and} \nabla^2\psi = 0$

i.e., φ and ψ are harmonic functions

∇ φ=grad φ= $\vec{q} = -\left(u\vec{i} + v\vec{j}\right)$

$$= -\left(\frac{-\partial\psi}{\partial y}\vec{i} + \frac{\partial\psi}{\partial x}\vec{j}\right)$$

$$= \frac{\partial\psi}{\partial y}\vec{i} - \frac{\partial\psi}{\partial x}\vec{j}$$

$$= \frac{\partial\psi}{\partial y}\overrightarrow{(j\times k)} - \frac{\partial\psi}{\partial x}\overrightarrow{(i\times k)}$$

$$= \left(\frac{\partial\psi}{\partial x}\vec{i} + \frac{\partial\psi}{\partial y}\vec{j}\right)\times\vec{k}$$

$$= \nabla\psi\times\vec{k}$$

$$= \text{grad}\psi\times\vec{k}$$

i.e., grad $\varphi = \text{grad}\psi\times\vec{k}$

$$= -\vec{k}\times\text{grad}\psi$$

$$\nabla\phi = \nabla\psi \times \vec{k} \rightarrow (4)$$

Again from (3), we have

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y}\left(\frac{-\partial\phi}{\partial y}\right)$$
$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\cdot\frac{\partial\phi}{\partial y} = 0$$
$$\nabla\phi.\nabla\psi = 0$$

Thus for irrotational incompressible two-dimensional flow (steady or unsteady). $\phi(x,y), \psi(x,y)$ are harmonic functions and family of curves

φ=constant(equipotential)

ψ=constant(streamlines) intersect orthogonally.

Complex potential

We consider irrotational plane flows of incompressible fluid of uniform density for which the velocity potential $\phi(x,y)$ and the stream function $\psi(x,y)$ exist.

(x.y) specify two dimensional Cartesian co-ordinates in a plane of flow

$$w = \phi(x,y) + i\psi(x,y) \rightarrow (1)$$

All four first order partial derivatives of φ and ψ with respect to x,y exist and are continuous throughout the plane of flow.

Velocity $\vec{q}$ =(u,v) has components satisfying $\vec{q} = -\nabla\phi$

$$u = \frac{-\partial\phi}{\partial x} = \frac{-\partial\psi}{\partial y} \qquad v = \frac{-\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \quad \rightarrow (2)$$

Thus, φ and ψ satisfy the C-R equations and so w must be an analytic function of z=x+iy.

Therefore, we can write (1) as

$$w = f(z) = \phi + i\psi \rightarrow (3)$$

The function w=f(z) is called complex potential of the plane flow. Complex velocity differential partially with respect to x

$$w = \phi + i\psi$$
$$z = x + iy$$

$$\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \qquad z = x + iy$$

$$\frac{\partial z}{\partial x} = 1$$

$$= u + iv$$

$$\text{but} \cos\theta - iq\sin\theta$$

therefore, $\frac{dw}{dz} = -u + iv, \frac{-dw}{dz} = u - iv$

$$= q\cos\theta - iq\sin\theta$$
$$= q(\cos\theta - i\sin\theta)$$
$$= qe^{-i\theta}$$

The combination u-iv is known as complex velocity.

Speed $q = \left|\frac{-dw}{dz}\right| = \sqrt{u^2 + v^2}$ and for stagnation points $\frac{dw}{dz} = 0$.

Discuss the flow for which complex potential is $w=z^2$

We have $w=z^2$

$=(x+iy)^2$

$=x^2-y^2+2ixy$

$$\phi(x, y) = x^2 - y^2$$
$$\psi(x, y) = 2xy$$

The equtipotentials φ=constant are the rectangular hyperbola x^2-y^2=constant having asymptotes y=±x.

The streamline ψ=constant are the rectangular hyperbola xy=constant having the axes x=0 and y=0 as asymptotes.

Consider $\frac{dw}{dz} = 2z$, therefore the only stagnation point is the origin.

The two tamilips of curves cut each other orthogonally. Both φ and ψ are harmonic and the flow is irroational.

Complex velocity potential for standard two-dimensional flows.

We consider flow patterns due to a uniform stream, a line source and sink and a line labels.

Complex potential for a uniform tream we first consider the uniform stream having velocity $-U\vec{i}$

This gives rise to a velocity potential $\phi = U_x$

$$w = Uz = U(x + iy)$$

The stream function ψ=imaginary(w). So that the lines y=constant are the streamlines.

Secondly, Let the uniform stream advance with a velocity having magnitude U and being inclined at angle α, to the positive direction of x-axis.

$$u = U\cos\alpha \qquad v = U\sin\alpha$$

$$\frac{-dw}{dz} = u - iv$$

$$= U\cos\alpha - iU\sin\alpha$$
$$= Ue^{-i\alpha}$$

The simplest form of w, ignoring the constant of integrating is $w = -Uze^{-i\alpha}$

$$\phi + i\psi = -U(x+iy)(\cos\alpha - i\sin\alpha)$$

$$= -U(x\cos\alpha + y\sin\alpha) - Ui(y\cos\alpha - x\sin\alpha)$$

Equating real and imaginary part, we get

$$\phi = -U(x\cos\alpha + y\sin\alpha)$$
$$\psi = -U(y\cos\alpha - x\sin\alpha)$$

Equations of equipotentials are $-U(x\cos\alpha + y\sin\alpha)$=constant $\rightarrow(1)$

These equations represent a family of parallel streamlines.

The equation of streamlines are $y\cos\alpha - x\sin\alpha$=constant $\rightarrow(2)$

These equation represent another family of parallel streamlines inclined at angle α to the positive x-direction

The two family of streamlines intersect orthogonally.

Line source and line sink:

Line source and line sink are the two dimensional analogues of the three dimensional simple source and simple sinks.

Let A be any point of the considered plane of flow and C be any closed curve surrounding A.

We construct a cylinder having its generators through the points of C and normal to the plane of flow.

Suppose that in each plane of flow, fluid is emitted radically and symmetrically from all points on the infinite line through a normal to the plane of flow and such that the rate of emission from all such points as A is the same.

Then the line through A is called a line source.

We take the closed curve C to be a circle having centre A and radius r.

Suppose the line source exits fluid at of the source per unit time, in all length of the source per unit time, in all directions in the plane of flow(x-y plane). We define the strength of the line source to be m.

A line source of strength –m is called a line sink.

Complex potential for a line source:

Let there be a line source of strength m per unit length at z=0. Since the flow is radial, the velocity has the radial component q_r only. Then the flow across a circle of radius r is (by law of conversation of mass)

$$\left(2\pi r q_r\right)\rho = 2\pi m\rho$$

$$q_r = \frac{2\pi m\rho}{2\pi r\rho} = \frac{m}{r}$$

The complex potential is obtained from the relation

$$\frac{-dw}{dz} = u - iv = q_r\cos\theta - iq_r\sin\theta$$

$$\begin{aligned}&= q_r\left(\cos\theta - i\sin\theta\right)\\&= q_r e^{-i\theta}\\&= \frac{m}{r}e^{-i\theta}\end{aligned}$$

$$\frac{-dw}{dz} = \frac{-m}{r}e^{-i\theta} = \frac{-m}{re^{i\theta}} = \frac{-m}{z}$$

Integrating

$w = -m\log z$ ignoring constant of intergration

$$\phi + i\psi = -m\log(re^{i\theta})$$

$$\begin{aligned}&= -m[\log r + \log e^{i\theta}]\\&= -m\left[\log r + i\theta\right]\\&= -m\log r - im\theta\end{aligned}$$

$$\begin{aligned}\phi &= -m\log r\\ \psi &= -m\theta\end{aligned}$$

Then the equipotentials and streamlines have the respective form

$$\text{i.e., } x^2+y^2 = \text{constant} \qquad \text{i.e., } \tan^{-1}\left(y/x\right) = \text{constant}$$

$$x^2+y^2 = C_1 \qquad y = C_1 x$$

Thus, the equipotential are circles and streamlines are straight lines passing through origin.

If the line source is at $z=z_0$ instead of z=0 then the complex potential is

$$w = m\log\left(z-z_0\right)$$

If there are a number of line source at $z = z_1, z_2, \cdots, z_n$ of respective strengths $m_1, m_2, \cdots, m_n$ per unit length then the complex potential is

$$z = m_1 \log\left(z-z_1\right) - m_2 \log\left(z-z_2\right) - \cdots - m_n \log\left(z-z_n\right)$$

Complex potential for a line doublet:

The combination of a line source and a line sink of equal strength when placed close to each other gives a line doublet.

Let us take a line source of strength m per unit length at $z = ae^{i\alpha}$ and a line sink of strength m per unit length at z=0.

$\therefore$ The complex potential due to the combination is

$$w = -m\log\left(z-ae^{i\alpha}\right) + m\log(z-0)$$

$$= -m\log\left(\frac{z-ae^{i\alpha}}{z}\right)$$

$$= -m\log\left(1-\frac{ae^{i\alpha}}{z}\right)$$

$$= m\left[\log\left(1-\frac{ae^{i\alpha}}{z}\right)^{-1}\right]$$

$$= m\left[\frac{ae^{i\alpha}}{z} + \frac{a^2e^{2i\alpha}}{z^2} + \cdots\right]$$

Suppose m become vary large and the distance between source and sink 'a' becomes small.

Then $ma \to \mu$

i.e., $m \to \infty$ and $a \to 0$, $ma \to \mu$

$$w = \frac{\mu e^{i\alpha}}{z}$$

If the line sink is situated at $z=z_0$ then the complex potential is

$$w = \frac{\mu e^{i\alpha}}{z - z_0}$$

Notes:

- If $\alpha=0$ then the line source is an x-axis and thus $w = \frac{\mu}{z - z_0}$
- If there are number of line doublets of strength $\mu_1, \mu_2, \cdots, \mu_n$ per unit length with line sinks at points $z_1, z_2, \cdots, z_n$ and their axis being inclined at angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ with the positive direction of x-axis then the complex potential is given by

$$w = \mu_1 \frac{e^{i\alpha_1}}{z - z_1} + \mu_2 \frac{e^{i\alpha_2}}{z - z_2} + \cdots + \mu_n \frac{e^{i\alpha_n}}{z - z_n}$$

Example:

Discuss the flow due to a uniform line doublet at origin of strength μ per unit length and its axis being along x-axis.

Solution:

We know that the complex potential for a doublet is $w = \frac{\mu e^{i\alpha}}{z - z_0}$.

when the doublet is at origin having its axis along z-axis then $\alpha=0, z=0$

$$w = \frac{\mu}{z} = \frac{\mu}{x + iy} = \frac{\mu(x - iy)}{x^2 + y^2}$$

$$= \frac{\mu x}{x^2+y^2} - \frac{i\mu y}{x^2+y^2}$$

$$\phi = \frac{\mu x}{x^2+y^2} \qquad \psi = \frac{-\mu y}{x^2+y^2}$$

Thus, the equipotentials φ=constant are co-axial circle

$$x^2+y^2 = 2k_1 x \rightarrow (1)$$

And the stream lines ψ=constant are co-axial circle

$$x^2+y^2 = 2k_2 x \rightarrow (2)$$

Family (1) have centre $(k_1,0)$ and radii k_1 and family (2) have centres $(0,k_2)$ and radii k_2.

The two families are orthogonal.

Line vortices:

Two dimensional flow $\vec{q} = u\vec{i} + v\vec{j}$ where, $\begin{matrix} u = u(x,y) \\ v = v(x,y) \end{matrix}$ then the vorticity vector and is given by $\varepsilon = \nabla \times q = \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\vec{k}$

This shows that the two dimensional flow the vorticity vector is perpendicular to the plane of flow.

Discuss the two dimensional flows for which $w = \left(ik/2\pi\right)\log z$ **,k is real constant.**

Let $z = re^{i\theta}$

$$w = \frac{ik}{2\pi}\log(re^{i\theta})$$

$$w = \frac{ik}{2\pi}\log r - \frac{k}{2\pi}\theta$$

$$\phi = \frac{-k}{2\pi}\theta \qquad \psi = \frac{k}{2\pi}\log r$$

Thus the streamlines in the plane of flow are the concentric circles r=constant.

Equipotentials are the radii vectors θ=constant through the origin.

The two families are mutually orthogonal and φ,ψ are both harmonic function.

The radial and transverse velocity components are

$$q_r = \frac{-\partial\phi}{\partial r} = 0 \quad q_r = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = \frac{k}{2\pi r}$$

The circulation Γ round any closed curve and surrounding the origin and in the plane of flow is given by $\Gamma = \int_{\varepsilon} q.ds$

$$q = \frac{k}{2\pi r}\vec{\theta}$$
$$ds = dr.\vec{r} + rd\theta.\vec{\theta}$$

$$q.ds = \frac{k}{2\pi}d\theta$$

$$\Gamma = \frac{k}{2\pi}\int_{\varepsilon} d\theta = \frac{k}{2\pi}\times 2\pi = k$$

If ε does not surround, then Γ is easily shown to be zero.

Hence, a two dimensional distribution having a complex velocity potential $w = \frac{ik}{2\pi}\log z$ gives a circulation round any closed curve ε in the plane and enclosing the origin O of amount K.

Also,round any other curve in the plane of flow which does not enclose O the circulation is zero.

The streamlines are the concentric circles r=constant and the equipotentials the lines θ=constant.

Uniform line vortex.

The uniform disturbance along an infinite line such that the circulation round any curve ε in any plane perpendicular to that line is a constant K when ε enclosed the intersection of the vortex and plane and is zero, when ε does not contain the intersection.

The strength of such a uniform line vortex is defined to be k and its complex velocity potential is $\frac{ik}{2\pi}\log z$. when the origin is taken at the intersection of the plane with the line.

Two-dimensional image system.

In a two dimensional fluid motion, if the flow across a curve C is zero then the system of line sources, sinks, doublets etc., on one side of the curve C is said to form image of line source, sinks, doublets etc., on the other side of C.

Image of a line source in a plane:

Without loss of generality, we take the rigid impermeable plane to be x=0 and perpendicular to the plane of flow(xy plane).To determine the image of the line source of strength m per unit length at A(a,0) with respect to the streamline OY. Place a line source per unit length A'(-a,0)The complex potential of strength at a point P due to the system of line sources is given by

$$w = -m\log(z-a) - m\log(z+a)$$

$$\begin{aligned} &= -m\log[(z-a)(z+a)] \\ &= -m\log[r_1 e^{i\theta_1}.r_2 e^{i\theta_2}] \\ &= -m\log[r_1 r_2 e^{i\theta_1}.e^{i\theta_2}] \end{aligned}$$

$$\begin{aligned} &\phi + i\psi = -m\log r_1 r_2 = -im(\theta_1+\theta_2) \\ &\psi = -m(\theta_1+\theta_2) \end{aligned}$$

If P lies on y-axis then

$$\begin{aligned} &PA = PA' \\ &\angle PA'A = \angle PAA' \\ &\theta_2 = \pi - \theta_1 \\ &\text{i.e.,}\ \pi = \theta_1 + \theta_2 \end{aligned}$$

Therefore, ψ=-mπ=constant

Which shows that y-axis is a streamline.

Hence, the image of a line source of strength m per unit length of a line at A(a,0) is a source of strength m per unit length at A'(-a,0).

In other words, image of line source with respect to a plane (a streamline) is a line source of equal strength situated on opposite side of the plane (streamline) at an equal distance.

Image of a line doublets in a plane:

Consider the rigid plane x=0 and perpendicular to the plane of flow (xy-plane).

Thus, to determine the image of a line doublet with respect to the streamlines OY.

Let there be line sources at the points A and B, taken close together, of strength –m and m per unit length.

Their respective image in OY are –m at A', m at B' where A' and B' are the reflections of A,B in OY.

The line $\overline{AB}$ makes angle α with $\overline{OX}$. Thus $\overline{A'B'}$ makes angle with OX.

In the limiting case, as m→∞,AB→0, we have equal line doublets at A and A' with their axes inclined at α,(π-α) to $\overline{OX}$.

Image of vortex in a plane:

Let us consider two line vortices of strength K and –K per unit length at $A(z=z_1)$ and $B(z=z_2)$ respectively.

The complex potential due to there line vortices.

$$w = ik\log(z-z_1) - ik\log(z-z_2)$$

$$= ik\log\left(\frac{z-z_1}{z-z_2}\right)$$

$$\text{i.e.,}\ \phi + i\psi = ik\log\left(\frac{z-z_1}{z-z_2}\right)$$

$$\psi = k\log\left(\frac{z-z_1}{z-z_2}\right)$$

$$\psi = k\log\frac{r_1}{r_2}$$

If $r_1 = r_2$ then ψ=k log(1)

=0

Thus, the plane boundary OP is a streamline so that there is no flow across OP.

Hence, the line vortex at B with strength –k per unit length is the image of the line per unit length is the image of the line vortex at A is strength K per unit length so that A and B are at equal distance from OP.

Milne Thomson Cricle theorem:

Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities within the circle $|z| = a$. Then on introducing the solid circular cylinder $|z| = a$,with impermeable boundary into the flow, the new complex potential for the fluid outside the cylinder is given by

$$W = f(z) + \bar{f}\left(a^2/z\right), |z| \geq a$$

The complex potential due to a line source, a line doublet and a line vertex each have the respective forms,

$$-m\log(z-z_1), \frac{\mu e^{i\alpha}}{z-z_1}, \frac{ik}{2\pi}\log(z-z_1)$$

Each of these function has a singularity at $z = z_1$ elsewhere each is analytic.

Proof:

Let C be the cross section of the cylinder with equation $|z| = 1$.

Therefore, on the circle C, $|z| = a$

$$z\bar{z} = a^2 \Rightarrow \bar{z} = a^2/z$$

Where $\bar{z}$ is the image of the point z with respect to the circle.

If z is outside the circle, then $\bar{z} = a^2/z$ is inside the circle.

All singularities of f(z) lie outside C and singularities of $f\left(a^2/z\right)$.

Therefore, $\bar{f}\left(a^2/z\right)$ lies inside C

$\therefore \bar{f}\left(a^2/z\right)$ introduced no singularity outside the cylinder.

Thus, the function f(z) and f(z)+ $\bar{f}\left(a^2/z\right)$ both have the same singularities outside C.

The conditions satisfied by f(z) in the absence of the cylinder and satisfied by f(z)+ $\bar{f}\left(a^2/z\right)$ in the presence of the cylinder.

The complex potentials, after insertion of the cylinder, $|z| = a$ is

$$w = f(z) + \bar{f}\left(a^2/z\right)$$

$$= f\left(ae^{i\theta}\right) + \bar{f}\left(ae^{-i\theta}\right)$$

$$= f\left(ae^{i\theta}\right) + \overline{f\left(ae^{-i\theta}\right)}$$

i.e., $w = f(z) + f(\bar{z})$

= a purely real quantity

But, we know that w=φ+iψ

It is that ψ=0

This shows that the circular boundary in streamline across which no fluid flows. Hence, z=a is a possible boundary for a new flow $w = f(z) + \bar{f}\left(a^2/z\right)$ is a approximate complex velocity for a new flow.

Uniform flow part a stationary cylinder

Uniform stream having velocity =Ui gives rise to a complex potential Uz

$$f(z)=Uz$$

$$\text{then} \quad \bar{f}(z)=Uz$$

$$\text{and so} \quad \bar{f}\left(\frac{a^2}{z}\right)=\frac{Ua^2}{z}$$

Then, on introducing the cylinder of circular section $|z|=0$ into the stream, the complex potential for the region $|z|\ge a$ becomes

$$w=f(z)+\bar{f}\left(a^2/z\right)$$

$$=Uz+\frac{Ua^2}{z}$$

$$z=re^{i\theta}$$

$$=Ure^{i\theta}+Ua^2r^{-1}e^{-i\theta}$$

$$=U\left[r(\cos\theta+i\sin\theta)+a^2r^{-1}(\cos\theta-i\sin\theta)\right]$$

$$=U\cos(r+a^2r^{-1})iU\sin(r-a^2r^{-1})$$

Uniform stream at incidence α to $\overline{OX}$

Complex potential for a uniform stream of velocity O at incidence to **α** to $\overline{OX}$, i.e., $Uze^{-i\alpha}$

$$f(z)=Uze^{-i\alpha}$$

$$f(z)=Uze^{i\alpha}$$

$$f(a^2/z)=U\frac{a^2}{z}e^{i\alpha}$$

Hence, when the cylinder of section $|z| = a$ is introduced the complex potential is $|z| \geq a$

$$w = Uze^{-i\alpha} + U\frac{a^2}{z}e^{-i\alpha}$$

$$w = U\left\{ze^{-i\alpha} + \frac{a^2}{z}e^{-i\alpha}\right\}$$

Image of a line source in a circular cylinder show that image of line source in a right circular cylinder is an equal line source through the inverse point in the circular section in the inverse point in the circular section in the plane of flow together with an equal line sink through the centre of the section.

Solution:

Suppose there is a uniform line source of strength m per unit length through the point z=d,d>a.

Then, the complex potential at a point

$$f(z) = -m\log(z-d)$$

$$\bar{f}(z) = -m\log(z-d)$$

$$\bar{f}\left(a^2/z\right) = -m\log\left(\frac{a^2}{z} - d\right)$$

On introducing the circular cylinder of section $|z| = a$, the complex velocity potential in the region $|z| \geq a$

$$w = -m\log(z-d) - m\log\left(\frac{a^2}{z} - d\right)$$

Alternative form:

$$w = -m\log(z-d) - m\log\left(\frac{a^2}{z} - d\right) + m\log z + \text{constant}$$

The term $-m\log\left(\frac{a^2}{z}-d\right)$ is the complex velocity potential due to a uniform line source of strength m per unit length through the point $z = a^2/d$.

This point $z = a^2/d$ is the inverse of the point z=d in the circle $|z| = a$.

The term m logz is the complex potential due to a line sink of strength m per unit length through the centre.

The image of a line source in a right circular cylinder is an equal line source through the inverse point in the circular section in the plane of flow together with an equal line sink through the centre of the section.

Line doublet parallel to the axis of a right circular cylinder.

Suppose there is a uniform line doublet of strength μ per unit length through the point z=d>a. Suppose further that the axis of the line doublet is set at angle $\boldsymbol{\alpha}$ to $\overline{OX}$.

The complex velocity potential due to this distribution is

$$f(z) = \frac{\mu e^{-i\alpha}}{(z-d)}$$

$$\bar{f}(z) = \frac{\mu e^{i\alpha}}{(z-d)}$$

$$\bar{f}\left(a^2/z\right) = \frac{\mu e^{-i\alpha}}{\frac{a^2}{z}-d}$$

Thus, the total complex velocity potential obtained when the circular cylinder, of section $|z| = a$ is introduced by

$$w = \frac{\mu e^{i\alpha}}{(z-d)} + \frac{\mu e^{-i\alpha}}{\frac{a^2}{z}-d}$$

Theorem of Blasius:

An incompressible fluid moves steadily and irroationaly under no external force to the z-plane, past a fixed cylinder whose section in that plane is bounded by a closed curve ε the complex potential for the flow is W then, the action of the fluid pressure on the cylinder is equivalent to'a' force per unit length having component[x,y] and a couple per unit length of a momentum M, where

$$y+ix=\frac{-\rho}{2}\int_{\varepsilon}\left(\frac{dw}{dz}\right)^2 dz$$

$$M=\operatorname{Re}\left\{\frac{-\rho}{2}\int_{\varepsilon} z\left(\frac{dw}{dz}\right)^2 dz\right\}$$

Proof:

Let ds be a element of arc at a point P(x,y) and tangent at P makes an angle θ with x-axis.

The pressure at P(x,y),P→pressure/unit length

p ds acts along the inward normal to the cylindrical surface and its component along its co-ordinate axes

$$pds\cos(90+\theta), pds\cos\theta$$
$$-pds\sin\theta, pds\cos\theta$$

Therefore, the pressure at a element ds

dF=dx+idy

$$=-p\sin\theta+ip\cos\theta ds$$
$$=ip[\cos\theta+i\sin\theta]ds$$
$$=ip\left[{}^{dx}\!/_{ds}+i\,{}^{dy}\!/_{ds}\right]ds$$

$$dF=ip(dx+idy)=ipdz\rightarrow(1)$$

The pressure equation in the absence of external force is $\frac{p}{\rho}+\frac{1}{2}q^2$ =constant.

$$p=-\frac{1}{2}\rho q^2+k\rightarrow(2)$$

$$\frac{dw}{dz}=-u+iv$$

$$= -q\cos\theta + iq\sin\theta$$
$$= -q(\cos\theta - i\sin\theta)$$
$$= qe^{-i\theta} \rightarrow (3)$$

$$dz = dx + idy$$

$$= \left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds$$
$$= (\cos\theta + i\sin\theta)\,ds$$
$$= e^{i\theta}\,ds \rightarrow (4)$$

The pressure of the cylinder is obtained by integrating (1)

$$(1) \Rightarrow dF = dx + idy = ipdz$$

$$F = x + iy = \int_{\varepsilon} ipdz$$

$$= \int_{\varepsilon} i\left[\frac{-1}{2}\rho q^2 + k\right]dz$$
$$= \frac{-i\rho}{2}\int_{\varepsilon} q^2 dz$$
$$= \frac{-i\rho}{2}\int_{\varepsilon} q^2 e^{i\theta} ds$$
$$= \frac{i\rho}{2}\int_{\varepsilon} q^2 e^{-i\theta} ds$$
$$= \frac{i\rho}{2}\int_{\varepsilon} (q^2 e^{-2i\theta}) e^{i\theta} d\theta$$
$$= \frac{i\rho}{2}\int_{\varepsilon}\left(\frac{dw}{dz}\right)^2 dz$$

Using (3) and (4)

The moment M is given by

$$M = \int_{\varepsilon} \left|\vec{r} \times d\vec{F}\right|$$

$$= \int_{\varepsilon} \left(xpds\cos\theta + ypds\sin\theta \right)$$

$$= \int_{\varepsilon} xp\frac{dx}{ds}ds + yp\frac{dy}{ds}ds$$

$$= \int_{\varepsilon} p(xdx + ydy)$$

$$= \int_{\varepsilon} \left(k - \frac{1}{2}\rho q^2 \right)\left(xdx + ydy \right)$$

$$= k\int_{\varepsilon} xdx + ydy - \frac{1}{2}p\int q^2\left(xdx + ydy \right)$$

$$= \frac{-p}{2}\int q^2\left(xdx + ydy \right)$$

$$= R.P\left[\frac{-p}{2}\int_{\varepsilon} q^2(x + iy)(\cos\theta - i\sin\theta)ds \right]$$

$$= R.P\left[\frac{-p}{2}\int_{\varepsilon} q^2 ze^{-i\theta}ds \right]$$

$$= R.P\left[\frac{-p}{2}\int_{\varepsilon} z(q^2 e^{-2i\theta})e^{i\theta}ds \right]$$

$$= R.P\left[\frac{-p}{2}\int_{\varepsilon} z\left(\frac{dw}{dz} \right)^2 dz \right]$$

Hence proved.

Circulation about a circular cylinder in a uniform stream.

Let a liquid be in motion with a velocity –U along the x-axis. The complex potential due to the stream is Uz.

If the circular cylinder of radius a is introduced inside the liquid, then the complex potential , by circle theorem becomes

$$Uz + \frac{Ua^2}{z}$$

Let there be a circulation and about the cylinder. The complex potential due to circulation is ik(logz).

Thus, the complex potential of the whole system is

$$w = Uz + \frac{Ua^2}{z} + ik \log z$$

$$-q^2 = \frac{dw}{dz} = U - \frac{Ua^2}{z^2} + \frac{ik}{z}$$

At the stagnation point $\vec{q} = 0$

$$\frac{dw}{dz} = 0$$

$$U - \frac{Ua^2}{z^2} + \frac{ik}{z} = 0$$

$$z = \frac{-ik \pm \sqrt{(-k)^2 + 4U.Ua^2}}{2U}$$

$$= \frac{-ik \pm \sqrt{(-k)^2 + 4U.Ua^2}}{2U}$$

$$= \frac{-ik}{2U} \pm \frac{\sqrt{(-k)^2 + 4U^2a^2}}{2U}$$

$$= \frac{-ik}{2U} \pm \frac{2Ua\sqrt{\frac{k^2}{4U^2a^2} - 1}}{2U}$$

$$= \frac{-ik}{2U} \pm a\sqrt{\frac{k^2}{4U^2a^2} - 1}$$

Since a and U are constants.

Therefore, the flow potential term depends very much on the magnitude of K.

Case(i)

When $K \subset 2aU$

i.e., $\frac{k^2}{4a^2U^2} < 1$

$$\frac{k^2}{4a^2U^2} = \sin^2\beta$$

$$z = \frac{-ik}{2U} \pm a\sqrt{1 - \frac{k^2}{4a^2U^2}}$$

$$= \frac{-i2aU\sin\beta}{2U} \pm a\sqrt{1-\sin^2\beta}$$

$$= -ia\sin\beta \pm a\cos\beta$$

$$z = -ia\sin\beta + a\cos\beta \qquad\qquad z = -ia\sin\beta - a\cos\beta$$

$$= a[\cos\beta - i\sin\beta] \qquad\qquad = -a[\cos\beta + i\sin\beta]$$

$$= ae^{-i\beta} \qquad\qquad = -ae^{i\beta}$$

If the above inequality holds, there are two distinct stagnation points at $z = z_1, z = z_2$ on the surface of cylinder.

The diagram shows the pattern of streamlines formed in such a case A_1, A_2 be the stagnation points.

At A_1, A_2 the pressure is a maximum and so the effect produced on the direction $\overline{OY}$.

Then, the force is $M = \frac{-1}{2}\rho \times \mathrm{Re}\left\{2\pi i\left(-k^2 - 2U^2a^2\right)\right\} = 0$

When k=0 the streamlines flow is symmetrical about the plane y=0 and there is no such tendency.

Circulation may be produced in practice by rotating the cylinder about its axis.

The viscosity of the real fluid would then produce such circulation. This lifting effect produced by the circulation is called the magnus effect.

We know that at the stagnation points (critical points) there are two branches of the streamline which are at right angles to each other.

Thus, the liquid inside the loop formed at the stagnation points will not be carried by the stream but will circulate round the cylinder.

The pressure (forces) on the circular cylinder.

$$W = Uz + \frac{Ua^2}{z} + ik\log z$$
$$\frac{dw}{dz} = U - \frac{Ua^2}{z^2} + \frac{ik}{z}$$

Therefore, by Blasius theorem,

$$x - iy = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

$= -\pi\rho$ [sum of residues of $\left(\frac{dw}{dz}\right)^2$ with in the circle $|z| = a$]

[by cauchy's residue theorem as $\left(\frac{dw}{dz}\right)^2$ is a meromorphic function]

Where x,y are components of the pressure of the liquid and ρ is the density of the liquid

$$\left(\frac{dw}{dz}\right)^2 = U^2\left[1 - \frac{a^2}{z^2}\right] + \frac{2ikU}{z}\left[1 - \frac{a^2}{z^2}\right] - \frac{k^2}{z^2}$$

The only pole inside the cylinder $|z| = a$ is z=0 is a simple pole. The residue at z=0 is 2ikU

$$x - iy = -\pi\rho(2ikU)$$
$$x = 0, y = 2\pi k\rho U$$

This represent an upward thrust on the cylinder due to circulation. The lifting tendency is called the magnus effect.

Problems on theorem of Blasius:

1.) Infinite circular cylinder in uniform stream with circulation.

Forms previous problem write the lines

$$w = U(z + a^2 z^{-1}) + ik \log z$$

$$\frac{dw}{dz} = U\left[1 - \frac{a^2}{z^2}\right] + \frac{ik}{z}$$

$$y + ix = \frac{-\rho}{2}\int_{\varepsilon}\left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{-1}{2}\rho \times 2\pi i \times \text{residue of integrand at z=0}$$

$$= \frac{-1}{2}\rho \times 2\pi i \times 2iUk = 2\pi\rho kU$$

The moment about O,

$$M = \frac{\rho}{2}\operatorname{Re}\int_{\varepsilon} z\left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{\rho}{2}\operatorname{Re}\int_{\varepsilon} z\left(U + \frac{ik}{z} - \frac{Ua^2}{z^2}\right)^2 dz$$

Co-efficient of (1/2) in integrand=co-efficient of ($1\backslash z^2$) in

$$\left(U + \frac{ik}{z} - \frac{Ua^2}{z^2}\right)^2$$

$$= k^2 - 2U^2 a^2$$

$$M = \frac{-1}{2}\rho \times \operatorname{Re}[2\pi i(-k^2 - 2U^2 a^2)] = 0$$

$$x = 0, y = 2\pi\rho kU, M = 0$$

This shows that the cylinder experiences an uplifting forces.

2. **verify that** $w = ik\log\left(\frac{z - ia}{z + ia}\right)$ **k and a both real is the complex potential of a steady flow of liquid about a circular cylinder, the plane y=0 being a rigid boundary. Find the force exerted by the liquid on unit length of the cylinder.**

W=φ+iψ

$$\phi + i\psi = ik\log\left(\frac{z-ia}{z+ia}\right)$$

$$= ik\left[\log\left(\frac{z-ia}{z+ia}\right) + i\tan^{-1}\frac{y-a}{x} - i\tan^{-1}\frac{y-a}{x}\right]$$

$$\psi = k\log\left(\frac{z-ia}{z+ia}\right)$$

The streamlines ψ=constant are given by $\left|\frac{z-ia}{z+ia}\right|$=constant=λ

For λ≠1, there are non-intersecting coaxial of circle having z=±ia as the limiting points i.e.,circles of zero radius.

In particular, for λ=1, we get a streamlines which is the perpendicular bisector of the line segment joining the points ±ia and it is the radical axis of the co-axial system.

Coaxial circle:

A system of circles, every pair of which have the same radical axis.

Radical axis:

The radical axis of two circles is the locus of points at which tangent drawn to both circles have the same length.

No fluid crosses a streamlines and so a rigid boundary may be introduced along any circle λ=constant of the coaxial system,including the perpendicular bisector λ=1.

We note that for λ=1

$$|z-ia| = |z+ia|$$
$$x^2 + (y-a)^2 = x^2 + (y+a)^2$$
$$y = 0$$

Hence, we can introduce rigid boundaries along

i. A particular circle λ=constant
ii. Along the plane y=0(λ=1)

This establishes the result of the first part of the question.

SCHOOL OF SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – TWO DIMENSIONAL FLOWS-II – SMTA5401

The circular section C of the cylinder and the rigid plane y=0 are shown.

Circle C is any member of the above mentioned, λ-system of coaxial circles and it enclosed the point A(0,a) whereas the point B(0,-a) is external to it.

Use of conformal transformation:

Suppose z and t are two complex variables defined as z = x+iy, t = $\xi + i\eta$ where x, y, ξ, η are real numbers. We can form diagram for the loci of z and t, x and ξ horizontally and y and η vertically.

Let z describes a curve ξ in the (x-y) plane and suppose that t is related to z by means of the transformation t = g(z)

If g(z) is a single values function of z, then to each point in the z - plane we can obtain a corresponding point in the t – plane.

Therefore, the curve ξ in the z – plane is mapped into a curve ξ' in the t – plane.

Suppose g(z) is analytic. Let P, P_1, P_2 be the neighboring points in the z – plane and t

OP = z, $OP_1 = z+\delta z_1$, $OP_2 = z+\delta z_2$

Under the given transformation

t=f(z), suppose that P, P_1, P_2 map into the points Q, Q_1, Q_2 in the t – plane.

OQ = t, $OQ_1 = t+\delta t_1$, $OQ_2 = t+\delta t_2$

It is assumed that $|\delta z_1|, |\delta z_2|, |\delta t_1|, |\delta t_2|$ are small.

Since g(z) is analytic.

$\frac{dt}{ds}$ is unique at P.

Thus, to the first order of smallness

$$\frac{\delta t_1}{\delta x_1} = \frac{\delta t_2}{\delta x_2}$$

$$\frac{\delta t_1}{\delta t_2} = \frac{\delta x_1}{\delta x_2}$$

$$\left|\frac{\delta t_1}{\delta t_2}\right| = \left|\frac{\delta x_1}{\delta x_2}\right| \qquad (1)$$

$\arg \delta t_1 - \arg \delta t_2 = \arg \delta z_1 - \arg \delta z_2$ (2)

from (1)

$$\frac{QQ_1}{QQ_2} = \frac{PP_1}{PP_2} \qquad (3)$$

From (2) , on reversing the signs of each other

$$\underline{|Q_2QQ_1} = \underline{|P_2PP_1} \qquad (4)$$

The equations (3) and (4) shows that the triangles Q_2QQ_1, $P_2\,PP_1$ are similar.

i.e., within the neighborhood of any point p in the z – plane and its corresponding mapping in the t – plane, angle remain unaltered.

From (4) the ratios of corresponding linear dimension when the mapping takes place under a transformation of the form t = f(z), where f(z) is analytic at p and within its neighborhood.

Such a transformation is said to be conformal.

1) Theorems concerning the conformal transformation of the line distribution under conformal transformation a uniform line source maps into another uniform line source of the same length.

Let there be a uniform line source of strength m per unit length through the point $z = z_0$. Suppose the conformal t = g(z) is made from the z – plane to the t – plane so that the point $z = z_0$ maps the point $t = t_0$.

Let ξ be a closed curve in the z – plane containing the point $z = z_0$ and suppose ξ maps ξ' in the t – plane.

Then $t = t_0$ lies within ξ'

The complex velocity potential ω is the same for both systems and has the forms.

$\omega = \phi + i\varphi$ for the z - plane

$\omega = \phi' + i\varphi'$ for the t - plane

Thus $\phi = \phi', \varphi = \varphi'$. Since φ is the same at corresponding points of ξ, ξ'.

$$\int_{\xi} d\varphi = \int_{\xi'} d\varphi' \qquad (1)$$

Now in the z – plane ω = - m log(z-z_0)

$$d\omega = -m \frac{dz}{z - z_0}$$

$$\int_{\xi} d\omega = -m \int \frac{dz}{z - z_0} = -m 2\pi i$$

Since the integrand has a residue of 1 at z = z_0

$$d\omega = d\phi + i d\varphi$$

$$\int_{\xi} d\varphi = -2\pi m \qquad (2)$$

The numerical value of this is the volume of fluid crossing unit thickness of ξ per unit t time.

Equation (1) and (2) shows that the same volume crosses unit thickness of ξ' per unit time which implies an equal line source of strength m per unit length through t = t_0.

2) Under conformal transformation a uniform line vortex maps into another uniform line vortex of the same strength.

Let there be a uniform line vortex of strength k per unit length through z = z_0. Suppose the conformal transformation t = g(z) is made from the z – plane to t – plane so that the point z = z_0 maps into t=t_0.

Let ξ be its map in the t – plane then ξ' contain t = t_0.

The complex velocity potential ω is the same for both systems and has the forms.

$\omega = \phi + i\varphi$ for the z - plane

$\omega = \phi' + i\varphi'$ for the t - plane

Thus $\phi = \phi', \varphi = \varphi'$. Since φ is the same at corresponding points of ξ, ξ'.

$$\int_{\xi} d\varphi = \int_{\xi'} d\varphi' \qquad (1)$$

In the z – plane the complex potential is

$$\omega = \frac{ik}{2\pi}\log(z - z_0)$$

$$\int_{\xi} d\omega = \frac{ik}{2\pi}\int_{\xi}\frac{dz}{z - z_0} = \frac{ik}{2\pi}2\pi i = -k$$

Equating real parts gives,

$$\int_{\xi} d\phi = -k \qquad ; \qquad -\int_{\xi} d\phi = k \qquad (2)$$

The integral on the LHS is the circulation round ξ.
Equation (1) and (2) shows that the circulation round ξ' is also k.

Thus, the line source through $z = z_0$ of strength k per unit length maps into an equal line source through $t = t_0$.

3) Under conformal transformation a uniform line doublet maps into another uniform line doublet of the different length.

Let there be a uniform line doublet of strength μ per unit length through P where $z = z_0$.

Suppose that under the conformal transformation t = g(z), P maps into Q where

$t = t_0$.

Let the line doublet be replaced by equivalent line sources of strength -m, m per unit length through P, P' where $PP' = \delta z, \mu = m|\delta z|$

PP' is in the direction of the axis of the line doublet.

Suppose P' maps into Q'

Then the line source of strength -m, +m per unit length through PP' maps into ones of strength -m, +m per unit length through QQ'.

If $QQ' = \delta t, \delta t = g'(z)\delta z$

So that,

$$|\delta t| = |g'(z)||\delta z|$$

$$\arg \delta t = \arg g'(z) + \arg \delta z$$

Hence the two line sources through QQ' gives a line doublet at Q of strength μ'.

$$\xi' = m|\delta t| = \mu|g'(z)|$$

The inclination of the axis of the line doublet to the real axis is increased by $\arg g'(z)$.

4) Single infinite row of vertices:

To find complex potential of an infinite row of parallel vertices of strength k and distance 'a' apart.

Proof:

Let there be 2n+1 vertex with their centers on x – axis and the middle vortex having its center at the origin.

The vertices are placed at points z = Ina where n=0,1,2… symmetrical about y – axis.

The complex potential due to these vertices is

$$\omega = ik\log z + ik\log(z-a) + ik\log(z-2a) + \ldots + ik\log(\mathrm{z-na}) + \mathrm{iklog(z+a)} + ik\log(z+2a) + \ldots + ik\log(z+na)$$

$$= ik\log z + ik\log(z^2 - a^2) + ik\log(z^2 - 2^2a^2) + \ldots + ik\log(z^2 - n^2a^2)$$

$$= \frac{\pi}{a}\frac{a}{\pi}\left(ik\log z + ik\log(z^2 - a^2) + ik\log(z^2 - 2^2a^2) + \ldots + ik\log(z^2 - n^2a^2)\right)$$

$$= \left[ik\,\mathrm{logz} + ik\log(-a^2)\left(1 - \frac{z^2}{a^2}\right) + ik\log(-2^2a^2)\left(1 - \frac{z^2}{2^2a^2}\right) + \ldots + ik\log(-n^2a^2)\left(1 - \frac{z^2}{n^2a^2}\right)\right]$$

$$= ik\log\frac{\pi z}{a}\left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right)\ldots\left(1 - \frac{z^2}{n^2a^2}\right) + \left[ik\log\frac{a}{\pi}(-1)^n(a^2)(2^2a^2)\ldots(n^2a^2)\right]$$

Ignoring constant terms and having $\frac{\pi z}{a} = \theta$

$$\omega = ik\log\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\left(1-\frac{\theta^2}{n^2\pi^2}\right) \qquad as\ n\to\infty$$

$$\omega = ik\log\sin\theta$$

$$= ik\log\sin\frac{\pi z}{a} \qquad (1)$$

The velocity of vertices at origin is given by

$$q_\theta = -\frac{d}{dz}\left[\omega - ik\log\frac{\pi z}{a}\right]$$

$$= -\frac{d}{dz}\left[ik\log\sin\frac{\pi z}{a} - ik\log\frac{\pi z}{a}\right]_{z=0}$$

$$= -ik\left[\frac{\pi}{a}\frac{\cos\frac{\pi z}{a}}{\sin\frac{\pi z}{a}} - \frac{1}{z}\right]_{z=0}$$

Indeterminate form $\to\infty$ as $z\to 0$

Hence, velocity at $z\to 0$ is zero.

Thus, the infinite row of vertices does not induce any velocity by itself.

The velocity at any point of the fluid other than the vertices is given by,

$$q = u - iv = -\frac{d\omega}{dz} = -\frac{ik\pi}{a}\cot\frac{\pi z}{a}$$

$$= \frac{ik\pi}{a}\cot\left(\frac{\pi}{a}(x+iy)\right)$$

$$= \frac{ik\pi}{a}\frac{\cos\left(\frac{\pi}{a}(x+iy)\right)}{\sin\left(\frac{\pi}{a}(x+iy)\right)}$$

$$= \frac{ik\pi}{a}\frac{2\cos\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}{2\sin\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}$$

$$= -ik\frac{\pi}{a}\left[\frac{\sin\frac{2\pi x}{a} - \sin\frac{2\pi yi}{a}}{\cos\frac{2\pi yi}{a} - \cos\frac{2\pi x}{a}}\right]$$

$$= -ik\frac{\pi}{a}\left[\frac{\sin\frac{2\pi x}{a} - i\sinh\frac{2\pi y}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}}\right]$$

$$u = \frac{-k\dfrac{\pi}{a}\sinh\dfrac{2\pi y}{a}}{\cosh\dfrac{2\pi y}{a} - \cos\dfrac{2\pi x}{a}}$$

$$v = \frac{-k\dfrac{\pi}{a}\sin\dfrac{2\pi x}{a}}{\cosh\dfrac{2\pi y}{a} - \cos\dfrac{2\pi x}{a}}$$

We have,

$$\omega = \phi + i\varphi = ik\log\sin\frac{\pi z}{a}$$

$$\phi - i\varphi = -ik\log\sin\frac{\pi\bar{z}}{a}$$

$$(\phi + i\varphi) - (\phi - i\varphi) = ik\log\sin\frac{\pi z}{a} - \left(ik\log\sin\frac{\pi\bar{z}}{a}\right)$$

$$2i\varphi = 2ik\log\sin\frac{\pi z}{a}\sin\frac{\pi\bar{z}}{a}$$

Streamline φ= constant, are found to be $\cosh\dfrac{2\pi y}{a} - \cos\dfrac{2\pi x}{a}$ is constant.

5) Karman vortex street:

This consist of two parallel infinite rows of line vortices arranged as follows

The first row consists of line vortices of strength k at the points having cartesian coordinates.

$$(na, \frac{1}{2}b) \qquad where\ n = 0 \pm 1, \pm 2, \ldots$$

The second row consists of line vortex of strength -k at the points

$$\left(\frac{1}{2}(2n+1)a, -\frac{1}{2}b\right) \quad where\ \ n = 0, \pm 1, \pm 2, \ldots$$

Such arrangement is called as Karman vortex street.

Proof:

This consist of two parallel infinite row AA' and BB' of vortices of equal spacing ‘a’ so arranged that each vertex of strength k of AA' is exactly above the midpoint of the join of two vertices of BB' each of strength -k.

Therefore, the complex potential

$$\omega = -\frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z - ib) - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{a}{2} + ib\right)$$

By considering the point z_1 = ib, z_2 = - ib

The velocity of the vertex at z = ib is

$$u_1 - iv_1 = \frac{\pi}{a}\frac{ik}{2\pi}\cot\frac{\pi}{a}(ib - \frac{a}{2} + ib)$$

$$= \frac{\pi}{a}\frac{ik}{2\pi}\cot\frac{\pi}{a}a\left[\frac{2ib}{a} - \frac{1}{2}\right]$$

$$= \frac{\pi}{a}\frac{ik}{2\pi}\cot\pi\left[\frac{2ib}{a} - \frac{1}{2}\right]$$

$$= \frac{\pi}{a}\frac{ik}{2\pi}\cot\left[\frac{2ib\pi}{a} - \frac{\pi}{2}\right]$$

$$= -\frac{\pi}{a}\frac{ik}{2\pi}\cot\left[\frac{\pi}{2} - \frac{2ib\pi}{a}\right]$$

$$= -\frac{\pi}{a}\frac{ik}{2\pi}\tan\left[\frac{2ib\pi}{a}\right]$$

$$= \frac{\pi k}{2\pi a}\tanh\left[\frac{2b\pi}{a}\right]$$

$$u_1 = \frac{\pi k}{2\pi a}\tanh\left[\frac{2b\pi}{a}\right] \qquad where \;\; v_1 = 0$$

The same can be shown to each of vortices at the row AA' and BB' move with the same velocity. This means that the vertex configuration remains unaltered at all times. Since, both AA' and BB' have the same velocity $\frac{\pi k}{2\pi a}\tanh\left(\frac{2\pi b}{a}\right)$ in x – direction.

Hence the street moves through the liquid with two velocity.

This shows that Karman vortex street is realized when a flat plate moves broadside through a liquid.

6) Schwartz – Christoffel transformation:

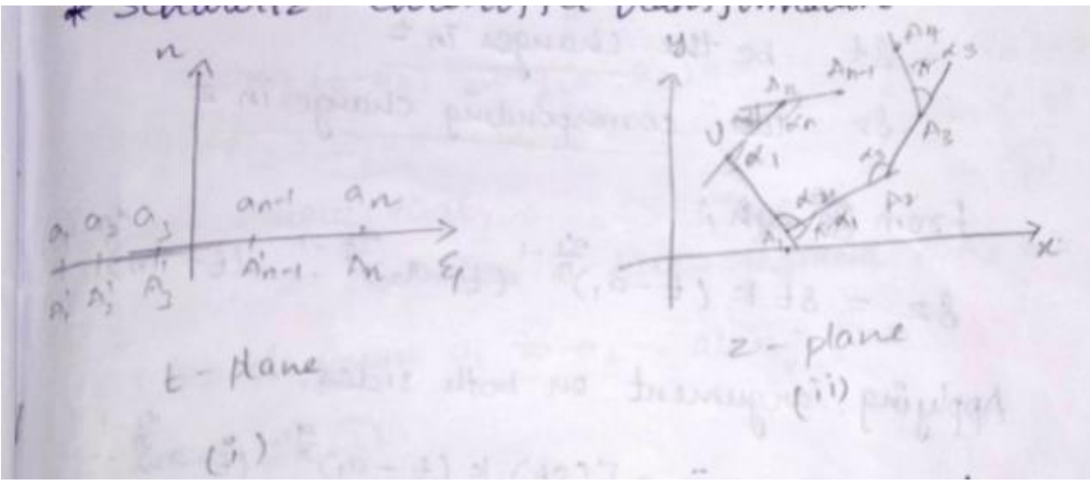


Fig (i) shows n points $A_1', A_2', A_3', \ldots A_n'$ on the real axis t = ξ+iη, ξ and η are real.

The points A_r' are specified by t = a_r (r = 1,2,…n)

The ordering is such that,

$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$

Now suppose that, t mapped onto the z – plane

$$\frac{dz}{dt} = k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}\ldots(t-a_n)^{\alpha_n/\pi - 1} \qquad (1)$$

Which is conformal everywhere, save at t = a_r (r = 1,2,…n)

The number $\alpha_1, \alpha_2, \ldots \alpha_n$ *are real constants*

$$let\ \frac{\delta z}{\delta t} = k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}\ldots(t-a_n)^{\alpha_n/\pi - 1} \qquad (*)$$

Where k be the complex constant

δt be the changes in t

δz the corresponding change in z.

From (*),

$\delta z = \delta t k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}\ldots(t-a_n)^{\alpha_n/\pi - 1}$

Applying argument on both sides,

$$\arg(\delta z) = \arg[\delta t k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}\ldots(t-a_n)^{\alpha_n/\pi - 1}]$$

$$= \arg(\delta t)[\arg(k) + (\alpha_1/\pi - 1)\arg(t-a_1) + (\alpha_2/\pi - 1)\arg(t-a_2) + \ldots + (\alpha_n/\pi - 1)\arg(t-a_n)]$$

First suppose we say, t travels from $-\infty$ to a_1, along η=0

Then, $\arg(\delta t) = 0, \quad \arg(t-a_1) = \arg(t-a_2) = \ldots = \arg(t-a_n) = \pi$

Hence,

$$(\arg \delta z)_1 = \arg k + (\alpha_1 - \pi) + (\alpha_2 - \pi) + \ldots + (\alpha_n - \pi) = constant \qquad (2)$$

Equation (2), z describes the straight line segment.

UA_1 as t travels from $-\infty$ to a_1, along η=0

Next t travels from a_1 to a_2, along η=0

Then, $\arg(\delta t) = 0,$

$$\arg(t-a_1) = 0 \text{ and } \arg(t-a_2) = \ldots = \arg(t-a_n) = \pi = constant \qquad (3)$$

This shows that, z describes a line segment A_1, A_2, as t travels from a_1 to a_2, along η=0

From (2) and (3),

$(\arg \delta z)_2 = (\arg \delta z)_1 = -\alpha_1 + \pi$

The angle between the direction UA_1, A_1, A_2 is $\pi - \alpha_1$. This means t travels along the axis η=0 from $-\infty$ part A_1, similarly at t moves from a_2 to a_3, along η=0

Then, $\arg(\delta t) = 0$,

$$\arg(t-a_1) = 0, \ \arg(t-a_2) = 0 \text{ and } \arg(t-a_3)... = \arg(t-a_n) = \pi = constant \qquad (4)$$

Equation (4) shows that z describes a line segment A_2, A_3, as t travels from a_2 to a_3, along η=0

From (3) and (4)

$(\arg \delta z)_3 = (\arg \delta z)_2 = \pi - \alpha_2$

This shows that direction z changes by $A - \alpha_2$ in the positive sense as z passes through A_2.

Continuing in this way, z describes the sides $\cup A_1,\ A_1,\ A_2,\ ....A_{n-1},\ A_n,\ A_n \cup$ of a closed polygon on z – plane , as t moves along the real axis $-\infty$ to a_1, a_1 to a_2, a_2 to a_3, …….. a_{n-1} to a_n, a_n to $+\infty$ respectively.

Therefore, the angles turned through in the positive sense at A_1, A_2, …. A_{n-1}, A_n are $\pi - \alpha_1, \pi - \alpha_2, ...\pi - \alpha_n$

So, the internal angles of the polygon at these points are $\alpha_1, \alpha_2, ...\alpha_n$

Suppose that, $\alpha_1 + \alpha_2 + ... + \alpha_n = (n-2)\pi$

Then , A_1, A_2, …. A_{n-1}, A_n is a closed polygon , the portion, $A_n \cup A_1$ beast line .

Further, since the half plane, img (t)>0 is along the positive direction of the axis η=0. The corresponding area in the z plane is the interior of polygon.

If $a_1 < a_2 < a_3 < ... < a_{n-1} < a_n$ are the n distinct points on the img (t) =0 in the t – plane.

If $\alpha_1, \alpha_2, ...\alpha_n$ are n real constants $\alpha_1 + \alpha_2 + ... + \alpha_n = (n-2)\pi$, where n is the positive integer.

Then the transformation,

$$dz = \left[k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}...(t-a_n)^{\alpha_n/\pi - 1} \right] dt$$

On integrating both sides, we get

$$z = \left[k\int (t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}...(t-a_n)^{\alpha_n/\pi - 1} \right] dt$$

Here, k are the constants that maps the entire line img (t) = 0 onto the boundary

A_1, A_2, …. A_{n-1}, A_nA_1 of a closed polygon on n – sides.

Such a way, that points t = a_k transform into A_k at which the internal angles are k = 1,2,3,..n and the half plane img (t)>0 transforms into the interior.

This transformation is called the Schwartz - Christoffel transformation.

Note;

Suppose the real line t is infinity then, if $a_n = \infty$

$$\frac{dz}{dt} = k(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}...(t-a_n)^{\alpha_n/\pi - 1}$$

Put k = $\mathrm{A}(-a_n)^{1-\alpha_n/\pi}$

Then , $k(t-a_1)^{\alpha_n/\pi - 1} = \mathrm{A}\left\{1-\frac{t}{a_n}\right\}^{\frac{\alpha_n}{\pi}-1}$

$$= \mathrm{A}\left\{1-\frac{t}{a_n}\right\}^{\frac{\alpha_n}{\pi}-1} \qquad \to \mathrm{A}, \text{ as } \mathrm{a}_n \to \infty$$

The Schwartz – Christoffel transformation becomes

$$\frac{dz}{dt} = A(t-a_1)^{\alpha_1/\pi - 1}(t-a_2)^{\alpha_2/\pi - 1}...(t-a_n)^{\alpha_n/\pi - 1}$$

This shows $\mathrm{a}_n = \infty$, the factor $(t-a_n)^{\alpha_n/\pi - 1}$ is suppressed. A similar result for $\mathrm{a}_n = -\infty$

SCHOOL OF SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – THREE DIMENSIONAL FLOWS – SMTA5401

1) Stress components in a real fluid :

Let δs be a small rigid plane area inserted at a point P in a viscous fluid.

Cartesian coordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ.

Suppose that $\delta\vec{F}_n$ is the forces exerted by the moving fluid on one side of δs, the unit vector $\hat{n}$ being taken to specify the normal at P to δs on this side.

In case, of inviscid fluid, $\delta\vec{F}_n$ is aligned with $\hat{n}$ for a viscous fluid, the frictional forces are called into play between fluid and the surface so that $\delta\vec{F}_n$ will have a component tangential to δs.

Suppose the cartesian components of $\delta\vec{F}_n$ be $(\delta F_{nx}, \delta F_{ny}, \delta F_{nz})$ so that

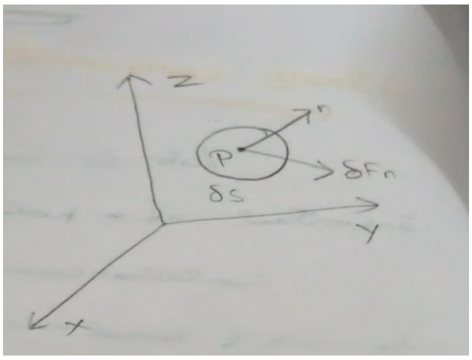
$$\delta\vec{F}_n = \delta F_{nx}\,\vec{\mathrm{i}}, \delta F_{ny}\,\vec{\mathrm{j}}, \delta F_{nz}\,\vec{\mathrm{k}}$$

Components of stress parallel to the axes are defined to be $\sigma_{nx}, \sigma_{ny}, \sigma_{nz}$.

$$\sigma_{nx} = \underset{\delta s \to 0}{lt} \frac{\delta F_{nx}}{\delta s} = \frac{dF_{nx}}{ds}$$

$$\sigma_{ny} = \underset{\delta s \to 0}{lt} \frac{\delta F_{ny}}{\delta s} = \frac{dF_{ny}}{ds}$$

$$\sigma_{nz} = \underset{\delta s \to 0}{lt} \frac{\delta F_{nz}}{\delta s} = \frac{dF_{nz}}{ds}$$



In the components $\sigma_{nx}, \sigma_{ny}, \sigma_{nz}$, the first suffix n denote the direction of the normal to the elements of plane δs, second suffix x or y or z denotes the direction in which components measured.

$\hat{n}$ in turn with the unit vector $\vec{i}, \vec{j}, \vec{k}$ in $\vec{OX}, \vec{OY}, \vec{OZ}$ which is achieved by suitably re-orientating δs.

Three set of stress components,

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{matrix}$$

The diagonal element $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ of this array are called normal or direct stresses.

The remaining six elements are called shearing stresses.

For an inviscid fluid,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yx} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} 0$$

We consider the normal stresses as positive when they are tensile and negative when they are compressive, so that p is the hydrostatic pressure.

The matrix,

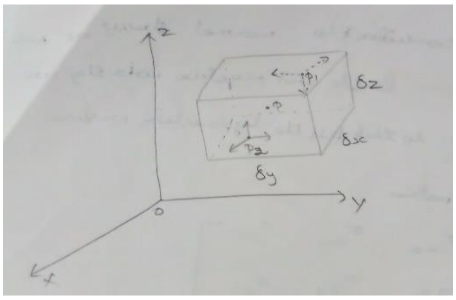
$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

are called the stress matrix.

The quantities σ_{ij} (i, j= x, y, z) are called the components of the stress tensor where matrix is of above form.

2) Derive the relations between cartesian components of stress:
   We consider the motion of a small rectangular parallelepiped of viscous fluid, its center being P(x, y, z) and its edges of lengths $\delta x, \delta y, \delta z$ parallel to fixed cartesian axes. The mass $p\delta x\delta y\delta z$ of the fluid element remains constant and the element is presumed to move

along with the fluid. In the diagram the points P_1, P_2 have coordinates $(x-\frac{1}{2}\delta x, y, z), (x+\frac{1}{2}\delta x, y, z)$



At P(x, y, z) the force components parallel to $\vec{OX}, \vec{OY}, \vec{OZ}$ on the surface of area $\delta y\delta z$ through P and having i as unit normal are $\left[p_{\partial xx}\delta y\delta z, p_{\partial xy}\delta y\delta z, p_{\partial xz}\delta y\delta z\right]$

At $P_2(x+\frac{1}{2}\delta x, y, z)$, since i is the unit normal measured outwards from the fluid, the corresponding force components across the parallel plane of area $\delta y\delta z$ are

$$\left[\left\{p_{xx}+\frac{1}{2}\delta x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\delta z, \left\{p_{xy}+\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\delta z, \left\{p_{xz}+\frac{1}{2}\delta x\left(\frac{\partial p_{xz}}{\partial x}\right)\right\}\delta y\delta z\right]$$

For the parallel plane through $P_1(x-\frac{1}{2}\delta x, y, z)$, since -i is the unit normal drawn outwards from the fluid element, the corresponding components are,

$$\left[-\left\{p_{xx}-\frac{1}{2}\delta x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\delta z, -\left\{p_{xy}-\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\delta z, -\left\{p_{xz}-\frac{1}{2}\delta x\left(\frac{\partial p_{xz}}{\partial x}\right)\right\}\delta y\delta z\right]$$

The forces on the parallel planes through P_1, P_2 are equivalent to a single force at P with components,

$$\left[\frac{\partial p_{xx}}{\partial x}, \frac{\partial p_{xy}}{\partial x}, \frac{\partial p_{xz}}{\partial x}\right]\partial x\partial y\partial z$$

Together with couples whose moments (to the third order) are

$$\begin{cases} -p_{xz}\delta x\delta y\delta z & \text{about oy;} \\ +p_{xy}\delta x\delta y\delta z & \text{about oz} \end{cases}$$

Similarly, the pair of faces perpendicular to the y axis gives a force at P having components

$$\left[\frac{\partial p_{yx}}{\partial y},\frac{\partial p_{yy}}{\partial y},\frac{\partial p_{yz}}{\partial y}\right]\delta x\delta y\delta z$$

Together with couple couples of moments,

$$\begin{cases} -p_{yx}\delta x\delta y\delta z & \text{about oz;} \\ +p_{yz}\delta x\delta y\delta z & \text{about ox} \end{cases}$$

The pair of faces perpendicular to the z axis give a face at P having components,

$$\left[\frac{\partial p_{zx}}{\partial z},\frac{\partial p_{zy}}{\partial z},\frac{\partial p_{zz}}{\partial z}\right]\delta x\delta y\delta z$$

Together with couple couples of moments,

$$\begin{cases} -p_{zy}\delta x\delta y\delta z & \text{about ox;} \\ +p_{zx}\delta x\delta y\delta z & \text{about oy} \end{cases}$$

Taking into account the surface forces on all six faces of the cuboid, we see that they reduce to a single force at P having components,

$$\left[\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right),\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right),\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\right]\delta x\delta y\delta z$$

Together with a vector couple having cartesian components,

$$\left[\left(p_{yz}-p_{zy}\right),\left(p_{zx}-p_{xz}\right),\left(p_{xy}-p_{yx}\right)\right]\delta x\delta y\delta z$$

Now, suppose the external body forces acting at P are [X, Y, Z] per unit mass, so that the total body force on the element has components $\left[X,Y,\mathrm{Z}\right]\rho\delta x\delta y\delta z$. Take moments about the i direction through P, then

Total moment of forces =(moment of inertia about axis) *(angular acceleration)

$$\text{i.e., } \left(p_{yz}-p_{zy}\right)\delta x\delta y\delta z+0_4=0_5$$

here 0_n satisfies a quantity of n^{th} order of smallness in $\delta x,\delta y,\delta z$. Thus to the third order of smallness,

$$\left(p_{yz}-p_{zy}\right)\delta x\delta y\delta z=0$$

So that as the element becomes vanishingly small, we obtain,

$$p_{yz}=p_{zy}$$

Similarly,

$$p_{zx}=p_{xz}, p_{xy}=p_{yx}$$

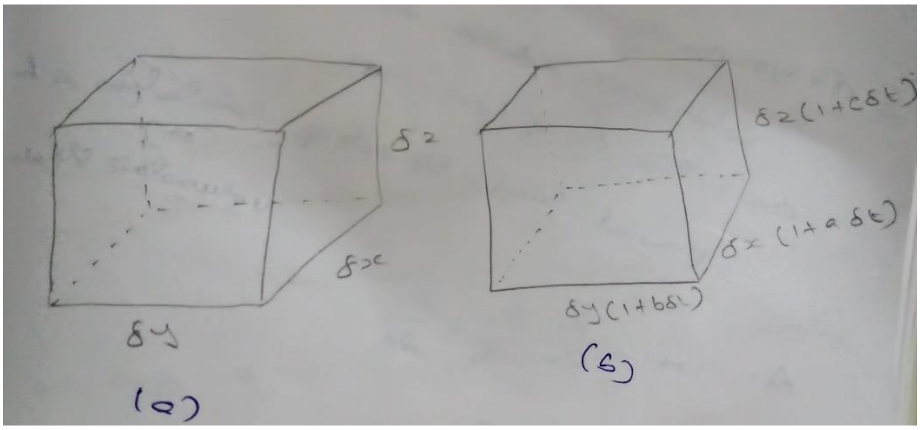
Thus the stress matrix is diagonally symmetric and contains only six unknowns.

3) Relation between stress and rate of strain :

Using the analysis of a foregoing section, we now link together the stress and rates of strain in a viscous fluid in motion.

Suppose fig (a) represents a particle of fluid at time t in the shape of a rectangular parallelepiped of edges $\delta x\delta y\delta z$, parallel to fixed cartesian axes.

At the time t the velocity components in the x direction at the corner (x, y, z) of the bios is u and so that at the corner $(x+\delta x, y, z)$ is $u+\left(\frac{\partial u}{\partial x}\right)\delta x \Rightarrow u+a\delta x$



Thus at time (t+δt) , therefore the edge δx has grown to length $\delta x + a\delta x\delta t$, since aδx is the relation velocity increases between its two ends.

Similarly, the edges δy, δz have grown to lengths $\delta y(1+b\delta t), \delta z(1+c\delta t)$ respectively.

Thus, volumetric increment in the interval δt is

$$\delta x\delta y\delta z(1+a\delta t)(1+b\delta t)(1+c\delta t) - \delta x\delta y\delta z \approx (a+b+c)\delta x\delta y\delta z\delta t$$

Which gives a dilatation or volumetric strain in time δt

Of (a+b+c) δt.

Hence at time t, the rate of dilatation is Δ (or) volumetric stress where

$$\Delta = a+b+c = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

= dig q

This quantity have been seen to be invariant at each point of the fluid its volume is also A+B+C in terms of principle rates of strain.

Further, w.k.t, the equation of continuity for an incompressible fluid is Δ=0, but for a compressible one Δ≠0, we discuss the two cases separately,

In the case of incompressible fluid, we suppose the principle stresses p_1,p_2,p_3 differ from this mean value -p by quantities proportional to the rates of distortion A, B, C in the principal direction.

Thus we write,

$$\left.\begin{aligned} p_1 &= -p + 2\mu A \\ p_2 &= -p + 2\mu B \\ p_3 &= -p + 2\mu C \end{aligned}\right\} \qquad (1)$$

Where μ is a constant.

In the case of compressible fluid we have the additional affect of the rate of dilatation Δ manifesting itself equally in the directions.

This effect we represent by adding to the R.H.S. of each of the equation (1) the quantity λΔ, where λ is a constant.

So that in the case

$$\left.\begin{aligned} p_1 &= -p + 2\mu A + \lambda\Delta \\ p_2 &= -p + 2\mu B + \lambda\Delta \\ p_3 &= -p + 2\mu C + \lambda\Delta \end{aligned}\right\} \qquad (2)$$

Adding together equation (2) and using Δ=A+B+C, we find, since $p_1+p_2+p_3=-3p$ and Δ≠0 and $\lambda = -\frac{2}{3}\mu$ (3)

The equation (1), (2) link principal stresses with principal rate of strain.

We next evaluate non principal stress $p_{xx},\ldots p_{yz},\ldots$ in terms of non principal rates of strain.

The equation of motion in stress analysis is

$$p_{xx} = l_1^2 p_1 + l_2^2 p_2 + l_3^2 p_3$$

Using the equation (2), we obtain,

$$p_{xx} = p(l_1^2 + l_2^2 + l_3^2) + 2\mu(l_1^2 A + l_2^2 B + l_3^2 \mathrm{C}) + \lambda\,\mathrm{A}(l_1^2 + l_2^2 + l_3^2)$$

$$= -p + 2\mu A + \lambda\Delta$$

$$= -p + 2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda\Delta$$

Permuting the symbols gives the equation,

$$\left.\begin{aligned} p_1 &= -p + 2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda\Delta \\ p_2 &= -p + 2\mu\left(\frac{\partial v}{\partial y}\right) + \lambda\Delta \\ p_3 &= -p + 2\mu\left(\frac{\partial w}{\partial z}\right) + \lambda\Delta \end{aligned}\right\} \qquad (3)$$

Where Δ= div q, Δ=0 for incompressible flow and $\lambda = -\frac{2}{3}\mu$ for the compressible flow. From analysis stress in fluid motion, we get

$$p_{yz} = m_1 n_1 p_1 + m_2 n_2 p_2 + m_3 n_3 p_3$$

$$= (-p + \lambda\Delta)(m_1 n_1 + m_2 n_2 + m_3 n_3) + 2\mu(m_1 n_1 A + m_2 n_2 B + m_3 n_3 \mathrm{C})$$

$$= 2\mu f$$

$$= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Thus we obtain the three equation

$$\left.\begin{aligned} p_{yz} &= p_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ p_{zx} &= p_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ p_{xy} &= p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \qquad (4)$$

Which are true for compressible and incompressible fluid.

4) The Navier- stokes equation of motion of a viscous fluid:

The translational equation of motion in the form,

$$\frac{du}{dt} = X + \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right)$$

On substituting,

$$p_{xx} = -p + 2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda\Delta$$

$$p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$p_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

We obtain,

$$\frac{du}{dt} = X - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) + V\nabla^2 u + \left(V + \frac{\lambda}{\rho}\right)\frac{\partial v}{\partial x}$$

Since $\lambda = -\frac{2}{3}\mu$ for the compressible fluid and since Δ=0 for incompressible fluid this equation may be written unambiguously for the two cases in the form,

$$\frac{du}{dt} = X - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) + V\nabla^2 u + \frac{1}{3}V\frac{\partial v}{\partial x}$$

Thus the equation of motion in the three directions may be written X_{FD}

$$\left.\begin{aligned}\frac{du}{dt} &= X - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) + V\nabla^2 u + \frac{1}{3}V\frac{\partial \Delta}{\partial x}\\ \frac{dv}{dt} &= Y - \frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right) + V\nabla^2 u + \frac{1}{3}V\frac{\partial \Delta}{\partial y}\\ \frac{dw}{dt} &= Z - \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) + V\nabla^2 u + \frac{1}{3}V\frac{\partial \Delta}{\partial z}\end{aligned}\right\} \qquad (1)$$

The tensor form of these equation is

$$\frac{du_i}{dt} = X_i - \frac{1}{\rho}p_{,i} + Vu_{i,jj} + \frac{1}{3}V\Delta_{,i}$$ (i) writing,

$F = [X, Y, \mathrm{Z}], \mathrm{q} = [\mathrm{x}, \mathrm{y}, \mathrm{z}]$ the vectoral form of equation (1) is clearly

$$\frac{dq}{dt} = F - \nabla\int\frac{dp}{\rho} + V\nabla^2 q + \frac{1}{3}V\nabla(\nabla.q) \qquad (2)$$

Now, $$\frac{dq}{dt} = \left(\frac{\partial q}{\partial t}\right) + (q.\nabla)q = \left(\frac{\partial q}{\partial t}\right) + \nabla\left(\frac{1}{2}q^2\right) - q\wedge(\nabla\wedge q),$$
$$\nabla\wedge(\nabla\wedge q) = \nabla(\nabla.q) - \nabla^2 q,$$

So that (2) may also written as

$$\frac{dq}{dt} + \nabla\left(\frac{1}{2}q^2\right) - q\wedge(\nabla\wedge q) = F - \nabla\int\frac{dp}{\rho} + \frac{4}{3}(V\Delta(\nabla.q) - V\nabla\wedge(\nabla\wedge q)) \qquad (3)$$

any of the forms (1), (2) and (3) are called the Navier- stokes equation of motion.

For incompressible flow, the forms (2), (3) gives,

$$\frac{dq}{dt} = F - (1-\rho)\nabla p + V\Delta^2 q$$

$$= F - (1-\rho)\nabla p - V\Delta \wedge (\nabla \wedge q) \qquad (4)$$

Where, as before $\frac{dq}{dt}$ may be developed in the form on the L.H.S. of (3)

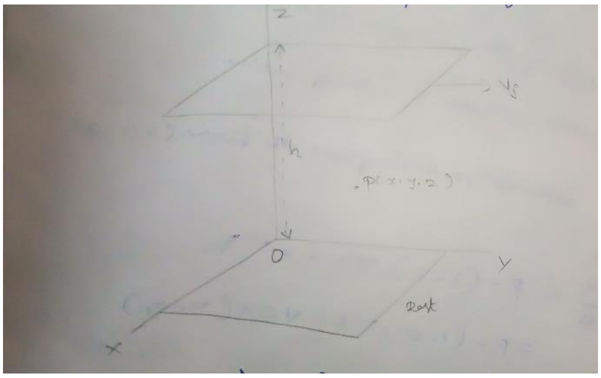
The equation (4) shows that for incompressible flow of the motion differs from Euler's equation in inviscid flow by the terms $-V\Delta \wedge (\nabla \wedge q)$. This term, due to viscosity, increases the complexity by boundary condition is required.

This furnished by the condition that there must be no slip between a viscous fluid and its boundary. For this reason, we cannot obtain the solution to the corresponding flow problem by solving (4) and then letting V→0

5) Some solvable problems in viscous flow:

There is no general solution to the Navier- stokes equation, nevertheless there are some special problems which can be solved. We consider a few here. The problem treated all relate to incompressible fluids.

Steady motion between parallel planes;



The region $0 \le z \le h$ between the planes z=0, z=h is filled with incompressible viscous fluid as shown above in diagram. The plane z=0 is held at rest and the plane z=h moves with constant velocity v_j. it is required to determine the nature of the flow the conditions are steady, assuming there is no slip between the fluid and either boundary, neglecting body forces.

Let P(x, y, z) be any point within the fluid . then the velocity q at P will be of the form.

q=v(y, z)j (1)

the equation n of the continuity $\nabla.q$ =0, gives

$$\frac{\partial v}{\partial y} = 0 \qquad (2)$$

and from (1), (2), we infer that,

q=v(z)j (3)

with no body forces, the Navier – stokes vector equation of the motion may be taken in the form

$$\frac{\partial q}{\partial t} + (q.\nabla)q = -\frac{1}{\rho}\nabla p + V\nabla^2 q \qquad (4)$$

the form (4) is the suitable here since we are dealing with axes fixed in space. Since the flow is steady, $\frac{\partial q}{\partial t}$ =0 also,

$$(q.\nabla)\,\mathrm{q} = (\mathrm{v}\frac{\partial}{\partial y})v(z)\,j = 0$$

and

$$\nabla^2 q = v^*(z)\,j$$

hence (4) gives,

$$0 = -\left(\frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j + \frac{\partial p}{\partial z}k\right) + \mu v^*(z)\,j$$

Equating coefficients of the unit vector,

$$\frac{\partial p}{\partial x} = 0 \qquad (5)$$

$$\frac{\partial p}{\partial y} = \mu\frac{d^2 v}{dz^2} \qquad (6)$$

$$\frac{\partial p}{\partial z} = 0 \qquad (7)$$

Equation (5), (7) shows that p=p(y). hence (6) becomes,

$$\frac{dp(y)}{dy} = \mu\frac{d^2 v(z)}{dz^2} \qquad (8)$$

The L.H.S. of equation (8) is a function of y only; the R.H.S. is a function of z only. Hence each is a constant. as the fluid is moving in the positive y direction the pressure p(y) should decrease as y increases. Hence $\frac{dp(y)}{dy} < 0$ and so we take

$$\frac{dp(y)}{dy} = \mu\frac{d^2 v(z)}{dz^2} = -P$$

Where P>0. Solving for v gives,

$$v(z) = A + B(z) - \left(\frac{p}{2\mu} z^2 \right) \qquad (9)$$

When z=0, v=0 and when z=h, v=V. hence we find

$$v(z) = \left(\frac{V}{h} + \frac{Ph}{2\mu} \right) - \left(\frac{p}{2\mu} z^2 \right) \qquad (10)$$

Equation (10) shows that the velocity profile between the plates is parabolic.

The total flow per unit breadth across a plane perpendicular to Oy is

$$\int_0^h v(z)dz = \frac{1}{2}Vh + \frac{1}{12}\frac{Ph^3}{\mu}$$

And the mean velocity across such a section is

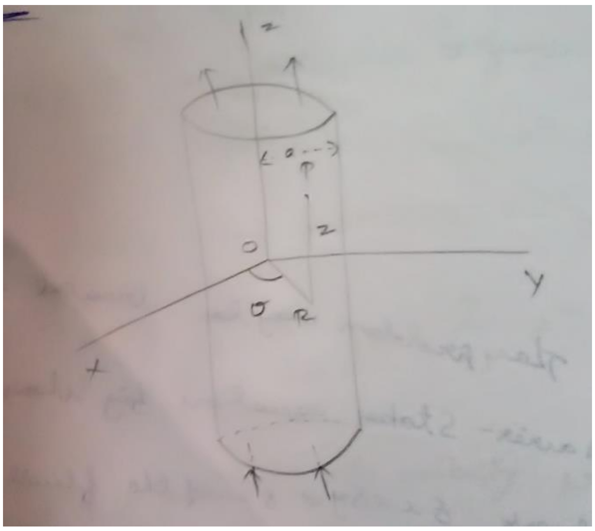
$$\frac{1}{h}\int_0^h v(z)dz = \frac{1}{2}V + \frac{1}{12}\frac{Ph^2}{\mu}$$

The tangential stress at any point P(x, y, z) is

$$\mu\frac{dv}{dz} = \frac{V}{h} + \frac{Ph}{2\mu} - \frac{Pz}{\mu}$$

Thus the drag per unit area in the lower plane is $\frac{V}{h} + \frac{Ph}{2\mu}$ and that on the upper plane is $\frac{V}{h} - \frac{Ph}{2\mu}$

6) Steady flow through tube of uniform circular cross section (Poiseuille flow)



The figure illustrates the steady flow of an inviscid incompressible fluid through a circular tube of radius a . P is a point in the fluid having cylindrical polar coordinates (R, θ, z) refereed to the origin O on the axis of the tube which is taken as the z- axis . we assume there are no body forces. Then continuity considerations applied to an annular shaped element of radii R, R+δR of the fluid indicate that the fluid velocity is of the form

$$q=w(R)k \qquad (1)$$

let us take the Navier – stokes vector equation in the form

$$\frac{\partial q}{\partial t}+(q.\nabla)q=-\frac{1}{\rho}\nabla p-V\nabla\wedge(\nabla\wedge q) \qquad (2)$$

here this form is adopted with the last term $-V\nabla\wedge(\nabla\wedge q)$ and not $+V\nabla^2 q$ as in the previous case, because the axes are not fixed in space. For the same reason the form give on the L.H.S. is more useful than $\frac{dq}{dt}$ then we have,

$$\frac{\partial q}{\partial t}=0 \quad (steady\ conditions)$$

$$(q.\nabla)q=(w\frac{\partial}{\partial z})[\mathrm{w(R)\,k}]=0$$

$$\nabla p=\frac{\partial p}{\partial R}\hat{R}+\frac{1}{R}\frac{\partial \phi}{\partial \theta}\hat{\theta}+\frac{\partial p}{\partial z}k$$

in the above $(q.\nabla)q$ is easy to evaluate since the operand involves the constant unit vector k. also,

$$\nabla\wedge q=\frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\theta} & k\\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ 0 & 0 & w(R)\end{vmatrix}=-w'(\mathrm{R})\hat{\theta}$$

hence,

$$\nabla\wedge(\nabla\wedge q)=\frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\theta} & k\\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ 0 & -Rw'(\mathrm{R}) & 0\end{vmatrix}=-\frac{1}{R}\frac{d}{dR}(\mathrm{Rw'})\,\mathrm{k}$$

thus the equation (2), becomes,

$$0=-\frac{1}{\rho}\left[\frac{\partial p}{\partial R}\hat{R}+\frac{\partial p}{R\partial\theta}+\frac{\partial p}{\partial z}k\right]+\frac{v}{R}\frac{d(Rw')}{dR}k$$

equating coefficients of the unit vectors gives,

$$\frac{\partial p}{\partial R}=0 \qquad (3)$$

$$\frac{\partial p}{\partial \theta}=0 \qquad (4)$$

$$\frac{\partial p}{\partial z} = \frac{\mu}{R}\frac{d(Rw')}{dR} \qquad (5)$$

equation (3) and (4) shows that p=p(z) so that (5) becomes

$$\frac{dp(z)}{dz} = \frac{\mu}{R}\frac{d}{dR}[Rw'(R)] \qquad (6)$$

the L.H.S. of (6) is a function of z only the R.H.S. is a function of R only. Hence each is constant. As flow is supposes to occur in the positive direction, we suppose $\frac{\partial p}{\partial z}$ <0. Take each side of (6) to be -P where P is a positive constant. Then

$$\frac{d}{dR}(Rw') = -\frac{PR}{\mu}$$

or

$$\frac{Rdw}{dR} = A - \frac{PR^2}{2\mu}$$

hence

$$\frac{dw}{dR} = \frac{A}{R} - \frac{1}{2}\frac{PR}{\mu}$$

and so, $w(R) = B + A\log R - \frac{1}{4}\left(\frac{PR^2}{\mu}\right)$ (7)

now w is finite on R=0. Thus we require A=0. Also on R=a, w=0 since there is no slip.

Then , Hence

$$w(R) = \frac{1}{4}\left(\frac{P}{\mu}\right)\left(a^2 - R^2\right) \qquad (8)$$

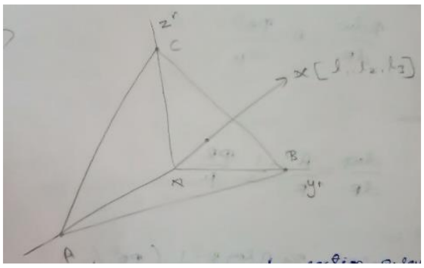
Then for, (8) shows that the velocity [profile is parabolic. i.e., the plot of w against R from R=0 to a is parabolic shape.

The volume of fluid discharged over any section per unit time is

$$Q = \int_0^a w(R)2\pi R dR = \frac{\pi P a^4}{8\mu}$$

If p denotes the pressure difference at two points on the axis of the tube distant l apart, then $P = \frac{p}{l}$

7) Analysis stress in fluid motion:



With a notation of section a plane perpendicular to P_x cuts the principal axes Px^n, Py^n, Pz^n of the rate of strain quadric in A,B,C to form a small tetrahedron of fluid PABC in fig. If δA denotes this area of the face ABC, then $l_1\delta A, l_2\delta A, l_3\delta A$ are the areas of the faces PBC, PCA, PAB. Since these last three are principal planes, using the notation it follows that the only forces on them are the normal forces $p_1 l_1 \delta A, \mathrm{p}_2\, l_2\delta A, \mathrm{p}_3\, l_3 \delta A$. The forces on the face ABC are
$p_{xx}\delta A, \mathrm{p}_{xy}\, \delta A, \mathrm{p}_{xz}\, \delta A$
in the x-y-z direction.

The equation of motion in the x direction (assuming the element to be a particle of fixed mass moving with the fluid) is

$$p_{xx}\delta A - l_1\left(p_1 l_1 \delta A\right) - l_2\left(p_2 l_2 \delta A\right) - l_3\left(p_3 l_3 \delta A\right) = 0_3$$

Where 0_3 is a term of the third order. Thus in the limit as the volume of the element tends to zero, we obtain

$$\left.\begin{aligned} p_{xx} &= \left(p_1 l^2{}_1\right) + \left(p_2 l^2{}_2\right) + \left(p_3 l^2{}_3\right) \\ &\textit{and similarly} \\ p_{yy} &= \left(p_1 m^2{}_1\right) + \left(p_2 m^2{}_2\right) + \left(p_3 m^2{}_3\right) \\ p_{zz} &= \left(p_1 n^2{}_1\right) + \left(p_2 n^2{}_2\right) + \left(p_3 n^2{}_3\right) \end{aligned}\right\} \qquad (1)$$

Adding the equations (1) together we find

$$p_{xx} + p_{yy} + p_{zz} = p_1 + p_2 + p_3 (= -3p, say)$$

Thus the sum of the normal stresses across any three perpendicular planes at a point is an invariant. We denote this sum by -3p, so that p denotes the mean pressure at the point.

Resolving in the direction and neglecting third order terms,

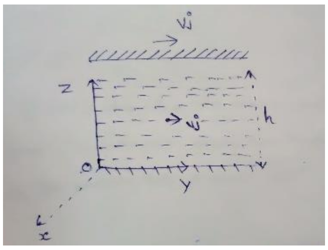
$$\left.\begin{aligned} p_{xy} &= \left(l_1 m_1 p_1\right) + \left(l_2 m_2 p_2\right) + \left(l_3 m_3 p_3\right) \\ &\textit{and similarly} \\ p_{yz} &= \left(m_1 n_1 p_1\right) + \left(m_2 n_2 p_2\right) + \left(m_3 n_3 p_3\right) \\ p_{zx} &= \left(n_1 l_1 p_1\right) + \left(n_2 l_2 p_2\right) + \left(n_3 l_3 p_3\right) \end{aligned}\right\} \qquad (2)$$

The equation (1) and (2) express the six distinct components of stress matrix in terms of the principal stresses.

8) Principal plane and principle stress:

The plane where the maximum normal stress exist and shear stress is zero is called principal plane and these maximum negative values of normal stress is called principal stress.

9) The co-efficient of viscosity and laminar flow:



Two parallel planes z=0, z=h , a small distance h apart, the space between containing a thin film of viscous fluid. The plane z=0 is held fixed whilst the upper plane is given a constant velocity right wards of amount v_j. then provided V is not excessively large, the layers of liquid in contact with x=0 are at rest whilst those in contact with z=h are moving with the velocity v_j.

i.e., there is no slip between fluid and either surface. A velocity gradient is set up in the fluid between the planes. At some point P(x, y, z) in between the planes the fluid velocity will be v_j where$0<v<V$ an v is independent of x, y. thus when z is fixed, v is fixed.

i.e., the fluid moves in layers parallel to the two planes. Such flow is termed laminar. Due to the viscosity of the fluid there is friction between these parallel layers.

Experimental works shows that the shearing stress on the moving plane is proportional to $\frac{V}{h}$ when h is sufficiently small. Thus we write this stress in the form $\mu'\frac{V}{h}$, where μ' is a constant called the coefficient of viscosity. Now suppose $h\to 0$. Then the stress on the fixed plane is

$$P_{zy} = \mu' \lim_{h\to 0}\left(\frac{V}{h}\right) = \mu'\frac{dv}{dx} \qquad (1)$$

w.k.t, we put u=0, w=0, v=v(x) in the equation of shearing stress

we obtain, $p_{zy} = \mu\frac{dv}{dx}$, p_{xz}=0, p_{yx}=0 (2)

from (1) and (2), $\mu' = \mu$ (3)

i.e., the constant μ is the coefficient of viscosity from (1) we find the dimensions of μ. Thus,

$$[\mu] = \frac{[p_{zy}]}{\left[\frac{dv}{dz}\right]} = \frac{\left(MLT^{-2}\right)/L^2}{\left(LT^{-1}\right)L^{-1}} = ML^{-1}T^{-1} \qquad (4)$$

where M, L, T signifies mass, length and time.

In aerodynamics a rather more important quantity is the kinematic coefficient of viscosity v defined by

$$v = \frac{\mu}{\rho} \qquad (5)$$

Thus, $[v] = L^2T^{-1}$

For most fluids μ depends on the pressure and temperature. For gases, according to kinetic theory, μ is independent of the pressure but decreases with the temperature.