SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMTA5304 ADVANCED NUMERICAL METHODS

Course Outcomes

At the end of the course, the student will be able to:

CO1 Recall transcendental and polynomial equations and solve it by different methods such as Chebyshev Method, Multipoint Iteration Methods, Beirge Vieta Method, Baristow Method, Graeffe's root Squaring Method to the Second degree equation

CO2 Summarize the different method for solving eigen value, eigen vector and inverse of A^{-1} for system of algebraic equations

CO3 Choose appropriate interpolation formula such as Lagrange, Newton's Bivariate interpolation etc and solve problems based on it.

CO4 Analyze trapezoidal and Simpsons rule for double integration

CO5 Evaluate the ordinary differential equations by numerical methods such as Euler method, Runge Kutta method.

CO6 Formulate Numerical integration based on GaussLegendreandGauss ChebyshevIntegration Methods

Syllabus

UNIT 1 Transcendental and Polynomial Equations

Transcendental and Polynomial Equations: Iteration method based on Second degree equations: The Chelyshev Method – Multipoint Iteration Methods – The Bridge Vieta Method – The Baristow Method – Graeffe'sroot Square Method.

UNIT 2 The System of Algebraic Equations and Eigen Value Problems

The System of Algebraic Equations and Eigen Value Problems: Iteration Methods-Jacobi Method, Guass Seidel Method, Successive Over Relaxation Method – Iterative Method for A-1 – Eigen Values and Eigen Vectors – Jacobi Method for symmetric Matrices , Power Method.

UNIT 3 Interpolation and Approximation

Interpolation and Approximation – Hermite Interpolation – Piecewise cubic Interpolation and cubic Spline interpolation – Bivariate interpolation – Lagrange and Newton's Bivariate interpolation – Least Square approximation – Gram-Schmidt Orthogonalizing Process.

UNIT 4 Differentiation and Integration

Differentiation and Integration; Numerical Differentiation – Methods Based on Interpolation – Partial Differentiation – Numerical Integration – Methods Based on Interpolation – Methods Based on Undetermined Coefficients – Gauss Quadrature methods - Gauss Legendre and Gauss Chebyshev Integration Methods – Double Integration – Trapezoidal and Simpson’s Rule – Simple Problems.

UNIT 5 Ordinary Differential Equations

Ordinary Differential Equations: Numerical Methods – Euler Method – Backward Euler Method – Mid-Point Method – Rungekutta Methods – Implicit RungeKutta Methods – Predictor – Corrector Methods.

UNIT – I - ADVANCED NUMERICAL METHODS – SMTA5304

UNIT I

Transcendental and Polynomial Equations

Polynomial

An equation formed with variables, exponents, and coefficients together with operations and an equal sign is called a polynomial equation.

Examples

$p(x) = ax + b \quad p(x) = a_0 x^n + a_1 x^{n-1} + --- + a_n \quad p(x) = ax + b$

Equation:

An equation is a mathematical statement with an 'equal to' symbol between two algebraic expressions that have equal values.

$2x + 5 = 1$

monomial /linear equation

Binomial /quadratic equation

Trinomial /Cubic equation

transcendental equations

An equation containing transcendental functions (for example, exponential, logarithmic, trigonometric, or inverse trigonometric functions) of the unknowns. Examples of transcendental equations are $\sin x + \log x = x$ and $2x - \log x = \arccos x$ *Note*

Elementary transcendental functions are the exponential, logarithmic, trigonometric, reverse trigonometric, and hyperbolic functions

iterative method

An iterative method is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the n-th approximation is derived from the previous ones.

Chebyshev polynomial Approximation

Chebyshev method:

Problem 1: Find the roots of the equation $x = e^x$ nearer to $x = 0.5$ correct up to four decimal places using Chebyshev's method.

Solution: given $f(x) = x - e^{x}$ ---------(1)

$f'(x) = 1 - e^{x}$ ----(2)

$f''(x) = -e^{x}$ ----(3)

Chebyshev method. $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}\left[\frac{f(x_k)}{f'(x_k)}\right]^2 \left[\frac{f''(x_k)}{f'(x_k)}\right]$ $x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\left[\frac{f_k}{f'_k}\right]^2 \frac{f''_k}{f'_k}$

Here $\left.\begin{array}{l} f(0) = 0 - e^{0} = -1(-ve) \\ f(1) = 1 + e^{-1} = 1.36788(+ve) \end{array}\right\} \Rightarrow the\,root\, x_k \in (0,1)$

Let $x_0 = 0.5(given)$

k	$f_k \equiv f(x_k)$	$f'_k \equiv f'(x_k)$	$f''_k \equiv f''(x_k)$	$\frac{f_k}{f'_k} = \frac{f(x_k)}{f'(x_k)}$	$\frac{f''_k}{f'_k} = \frac{f''(x_k)}{f'(x_k)}$	x_{k+1}
K=0	$0.5 - e^{-0.5} = -0.10653$	$1 + e^{-0.5} = 1.60653$	$-e^{-0.5} = -0.60653$	-0.06631	-0.37754	0.56673
K=1	$x_1 - e^{-x_1} = -0.00065$	$1 + e^{-x_1} = 1.56738$	$-e^{-x_1} = -0.56738$	-0.00041	-0.36199	0.56714
K=2	$x_2 - e^{-x_2} = -0.00005$	$1 + e^{-x_2} = 1.56715$	$-e^{-x_2} = -0.56715$	-0.000003	-0.36189	0.56714

$f''(x) = -e^{x}$ ----(3)

Problem 2: Find the positive root of the equation x^3-4x+1=0

Correct to four places of decimal using Chebyshev method.

Solution:

$$f(x) = x^3 - 4x + 1, f'(x) = 3x^2 - 4, f''(x) = 6x$$

$$\left.\begin{array}{l} f(0) = 1(+ve) \\ f(1) = -2(-ve) \end{array}\right\} \Rightarrow the\,root\, x_k \in (0,1)$$

the root lies between 0 and 1

let $x_0 = 0$

Chebyshev's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}\left[\frac{f(x_k)}{f'(x_k)}\right]^2 \left[\frac{f''(x_k)}{f'(x_k)}\right] \qquad x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\left[\frac{f_k}{f'_k}\right]^2 \frac{f''_k}{f'_k}$$

I st iteration: When k=0,

$$x_1 = x_0 - \frac{f_0}{f_0'} - \frac{1}{2}\left[\frac{f_0}{f_0'}\right]^2 \frac{f_0''}{f_0'}$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left[\frac{f(x_0)}{f'(x_0)}\right]^2 \left[\frac{f''(x_0)}{f'(x_0)}\right]$$

$f(x_0) = 1, f'(x_0) = -4, f''(x_0) = 0,$

$\boldsymbol{x_1 = 0.25}$

2 nd iteration: when k=1

$$x_2 = x_1 - \frac{f_1}{f_1'} - \frac{1}{2}\left[\frac{f_1}{f_1'}\right]^2 \frac{f_1''}{f_1'} \quad \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\left[\frac{f(x_1)}{f'(x_1)}\right]^2 \left[\frac{f''(x_1)}{f'(x_1)}\right]$$

x_1=0.25,f_1=0.01563,f_1'=-3.8125,f_1''=1.5,

$f_1^{(2)} = 0.000244, [f_1']^3 = -55.41528$

$\boldsymbol{x_2 = 0.2541}$

3 rd iteration:

when k=2,

$$x_3 = x_2 - \frac{f_2}{f_2'} - \frac{1}{2}\left[\frac{f_2}{f_2'}\right]^2 \frac{f_2''}{f_2'} \quad \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} - \frac{1}{2}\left[\frac{f(x_2)}{f'(x_2)}\right]^2 \left[\frac{f''(x_2)}{f'(x_2)}\right]$$

$x_2 = 0.2541, f_2 = 0.00001, f_2' = -3.80630, f_2'' = 1.52460,$

$f_2^{(2)} = 0, [f_2']^3 = -55.14536,$

$\boldsymbol{x_3 = 0.2541}$

Problem 3: Using Chebyshev's method find the root of the equation f(x)=cos x - xe^x=0, correct to 6 decimal places. [Ans=0.517757]

Problem 4: Determine 2 iteration of Chebyshev's method to find an approximate value of

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Multipoint Iteration:

Method 1:

Step 1: $x_{k+1}^* = x_k - \frac{1}{2}\frac{f_k}{f_k'}$, or $x_{k+1}^* = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}, f_{k+1}^* = f(x_{k+1}^*)$ step 2: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^*)}$

$x_{k+1}^* = x_k - \frac{f_k}{2f_k'}$; $x_{k+1} = \frac{f_k}{f_{k+1^*}'}$; $f_{k+1^*}' = f'(x_{k+1}^*)$

Method 2:

Step1: $x_{k+1}^* = x_k - \frac{f_k}{f_k'}$, or $x_{k+1}^* = x_k - \frac{f(x_k)}{f'(x_k)}, f_{k+1}^* = f(x_{k+1}^*)$ step 2: $x_{k+1} = x_{k+1}^* - \frac{f(x_{k+1}^*)}{f'(x_k)}$

$x_{k+1}^* = x_k - \frac{f_k}{f_k'}$; $x_{k+1} = x_{k+1}^* - \frac{f_{k+1}'}{f_k'}$; $f_{k+1}^* = f(x_{k+1}^*)$

Problem:

1. Perform 3 iteration of multipoint iteration method to find the smallest positive root of the equation f(x)=x^3 – 5x +1,

Solution: $f(x) = x^3 - 5x + 1, f'(x) = 3x^2 - 5$

$$\left.\begin{aligned} f(0) &= 1(+ve) \\ f(1) &= -3(-ve) \end{aligned}\right\} \Rightarrow the\, root\, x_k \in (0,1)$$

the root lies between 0 and 1

let $x_0 = 0$

Method 1: $x_{k+1}^* = x_k - \frac{f_k}{2f_k'}$; $x_{k+1} = x_k - \frac{f_k}{f_{k+1^*}'}$; $f_{k+1^*}' = f'(x_{k+1}^*)$

1st Iteration: Step1: take k = 0;

$$x_1^* = x_0 - \frac{1}{2}\frac{f(x_0)}{f'(x_0)}, f_{k+1}^* = f(x_1^*)$$

$$\Rightarrow x_1^* = 0.5 - \frac{1}{2}\frac{0.5^3 - 5(0.5) + 1}{3(0.5^2) - 5} = 0.33824, f_{k+1}^{*(1)} = f'(x_1^*) = 3(x_1^*)^2 - 5 = -4.65678$$

step 2: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^*)} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_1^*)} \Rightarrow x_1 = 0.5 - \frac{(x_0)^3 - 5(x_0) + 1}{3(x_1^*)^2 - 5} = 0.205445$

$\boldsymbol{x_1 = 0.20474}$

2nd Iteration: Step1: take k = 1;

$$x_2^* = x_1 - \frac{1}{2}\frac{f(x_1)}{f'(x_1)}, f_{k+1}^* = f(x_2^*)$$

$$\Rightarrow x_2^* = 0.20474 - \frac{1}{2}\frac{(x_1)^3 - 5(x_1) + 1}{3(x_1)^2 - 5} = 0.203185, f_{k+1}^{*(1)} = f'(x_2^*) = 3(x_2^*)^2 - 5 = -4.87615$$

step 2: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^*)} \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_2^*)} \Rightarrow x_2 = 0.5 - \frac{(x_1)^3 - 5(x_1) + 1}{3(x_2^*)^2 - 5} = 0.201640$

$\mathbf{x_2 = 0.201640}$

3rd Iteration: Step1: take k = 2

$$x_3^* = x_2 - \frac{1}{2}\frac{f(x_2)}{f'(x_2)}, f_{k+1}^* = f(x_3^*)$$

$$\Rightarrow x_3^* = 0.201640 - \frac{1}{2}\frac{(x_2)^3 - 5(x_2) + 1}{3(x_2)^2 - 5} = 0.201640, f_{k+1}^{*(1)} = f'(x_3^*) = 3(x_3^*)^2 - 5 = -4.87802$$

step 2: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^*)} \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_3^*)} \Rightarrow x_3 = x_2 - \frac{(x_2)^3 - 5(x_2) + 1}{3(x_3^*)^2 - 5} = 0.201640$

$$x_3 = 0.201640$$

Method 2: $x^*_{k+1} = x_k - \frac{f_k}{f_k'}$; $x_{k+1} = x^*_{k+1} - \frac{f^*_{k+1}}{f_k'}$; $f^*_{k+1} = f(x^*_{k+1})$

$f(x) = x^3 - 5x + 1; f'(x) = 3x^2 - 5$

1st Iteration: Step1: take k = 0;

$$x^*_{k+1} = x_k - \frac{f_k}{f_k'}, or\ x^*_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, f^*_{k+1} = f(x^*_{k+1})$$

$$x_1^* = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5^3 - 5(0.5) + 1}{3(0.5^2) - 5} = 0.17647, f^*_{k+1} = f(x_1^*) = (x_1^*)^3 - 5(x_1^*) + 1 = 0.12314$$

step 2: $x_{k+1} = x^*_{k+1} - \frac{f(x^*_{k+1})}{f'(x_k)} \Rightarrow x_1 = x_1^* - \frac{f(x_1^*)}{f'(x_0)} = 0.17647 - \frac{0.12314}{(-4.25)} = 0.205444$

$$x_1 = 0.205444$$

2nd Iteration: Step1: take k = 1;

$$x_2^* = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - 5(x_1) + 1}{3(x_1)^2 - 5} = 0.201568, f^*_{k+1} = f(x_2^*) = (x_2^*)^3 - 5(x_2^*) + 1 = 3.4964 \times 10^{-4}$$

step 2: $x_{k+1} = x^*_{k+1} - \frac{f(x^*_{k+1})}{f'(x_k)} \Rightarrow x_2 = x_2^* - \frac{f(x_2^*)}{f'(x_2)} = 0.201568 - \frac{3.4964 \times 10^{-4}}{(-4.9999)} = 0.20164$

$$x_2 = 0.201640$$

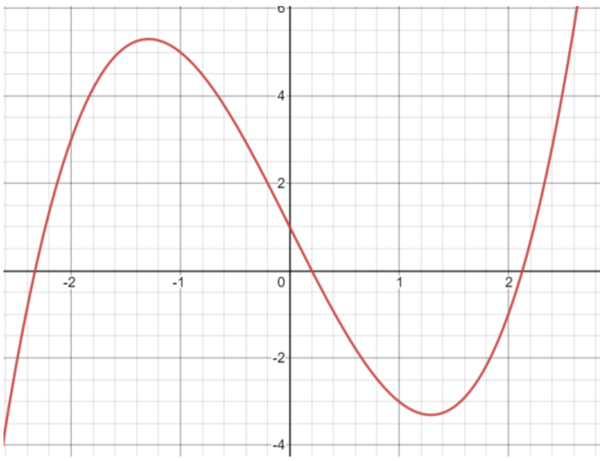
3rd Iteration: Step1: take k = 2;

$$x_3^* = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - 5(x_2) + 1}{3(x_2)^2 - 5} = 0.2016397, f^*_{k+1} = f(x_3^*) = (x_3^*)^3 - 5(x_3^*) + 1 = -1.5818 \times 10^{-6}$$

step 2: $x_{k+1} = x^*_{k+1} - \frac{f(x^*_{k+1})}{f'(x_k)} \Rightarrow x_3 = x_3^* - \frac{f(x_3^*)}{f'(x_2)} = 0.2016397 - \frac{-1.5818 \times 10^{-6}}{(-4.8780)} = 0.2016393$

$$x_3 = 0.201640$$

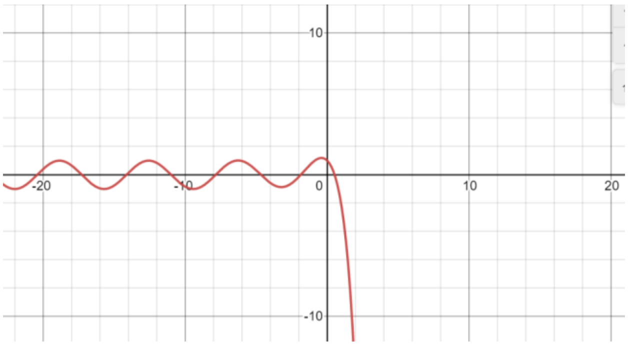
	$=x_0$	x^3-5x+1 $=f(x_k)$	$3x^2-5$ $=f'(x_k)$	$x_{k+1}^*=x_k-\frac{1}{2}\frac{f(x_k)}{f'(x_k)}$	$f'(x_{k+1}^*)$	$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_{k+1}^*)}$
	1.0	--1.0	3.0	1.166667	4.083336	1.244798
	1.053333	--0.071158	4.648566	1.252452	4.705908	1.259919
	1.259919	−0.00001	4.762188	1.259920	4.762195	1.259921
				$x_{k+1}^*=x_k-\frac{f(x_k)}{f'(x_k)}$	$f(x_{k+1}^*)$	$x_{k+1}=x_{k+1}^*-\frac{f(x_{k+1}^*)}{f'(x_k)}$
	1.0	--1.0	3.0	1.333333	0.370369	1.209877
	1.209877	−0.228979	4.391407	1.262020	0.010012	1.259740
	1.259740	−0.000862	4.760835	1.259921	0	1.259921



1. Perform 3 iteration of multipoint iteration method to find the smallest positive root of the equation $\cos x = xe^x$

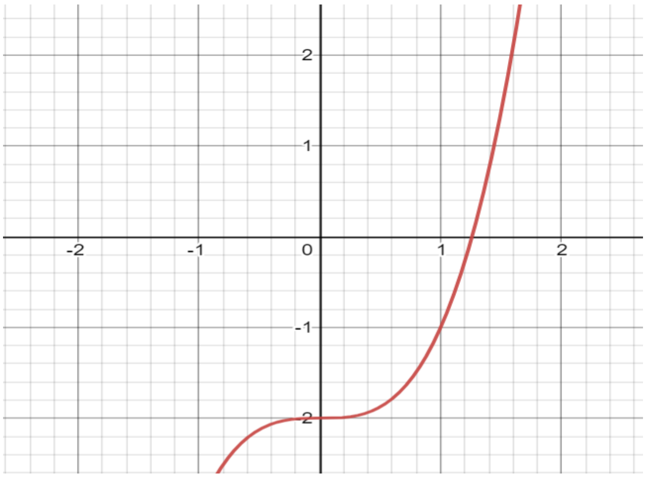
	$=x_0$	$\cos x - xe^x$ $=f(x_k)$	$-\left[\sin x+(x+1)e^x\right]$ $=f'(x_k)$	$x_{k+1}^*=x_k-\frac{1}{2}\frac{f(x_k)}{f'(x_k)}$	$f'(x_{k+1}^*)$	$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_{k+1}^*)}$
	0.5	0.053222	--2.952507	0.5785692	--3.362181	0.515830
	0.515830	0.005854	--3.032315	0.527176	--3.090347	0.517724
		0.000101	--3.041954	0.5178778	--3.042737	0.517757
		0.000001	--3.042122	0.517759	--3.042132	0.517757
				$x_{k+1}^*=x_k-\frac{f(x_k)}{f'(x_k)}$	$f(x_{k+1}^*)$	$x_{k+1}=x_{k+1}^*-\frac{f(x_{k+1}^*)}{f'(x_k)}$
	0.5	0.053222	--2.952507	0.518026 0.509013	--0.000817	0.517749

		0.000025	--3.042081	0.517757 0.	--0.000001	0.517757
		0.000001	--3.042122	0.517757	--0.000001	0.517757



2. Carry out two iterations of the multipoint method for finding the root of equation f(x)=$x^3 - 2$=0 with x_0=1, [Ans=1.259]

	$= x_0$	$x^3 - 2$ $= f(x_k)$	$3x^2$ $= f'(x_k)$	$x_{k+1}^* = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}$	$f'(x_{k+1}^*)$	$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^*)}$
	1.0	--1.0	3.0	1.166667	4.083336	1.244798
	1.053333	--0.071158	4.648566	1.252452	4.705908	1.259919
	1.259919	−0.00001	4.762188	1.259920	4.762195	1.259921
				$x_{k+1}^* = x_k - \frac{f(x_k)}{f'(x_k)}$	$f(x_{k+1}^*)$	$x_{k+1} = x_{k+1}^* - \frac{f(x_{k+1}^*)}{f'(x_k)}$
	1.0	--1.0	3.0	1.333333	0.370369	1.209877
	1.209877	−0.228979	4.391407	1.262020	0.010012	1.259740
	1.259740	−0.000862	4.760835	1.259921	0	1.259921



Birge vieta method:

$$\boldsymbol{p_{k+1} = p_k - \frac{b_n}{c_{n-1}}};$$

The deflated polynomial is arrived from the above equation in the following manner

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = (x-p)(b_0x^{n-1} + b_2x^{n-2} + \cdots + b_n) + R$ where R is the residual depends on p , $R = p_n(p) = b_n = a_n + pb_{n-1}$

Therefore, the deflated polynomial is

$$Q_{n-1}(x) = (b_0x^{n-1} + b_2x^{n-2} + \cdots + b_n)$$

Problems:

Problem 1:Use synthetic division to perform 2 iteration of the Birge-Vieta method to find the smallest positive root of the polynomial $p_3(x) = 2x^3 - 5x + 1 = 0$,and hence find the deflated polynomial.

Solution: Given

$p_3(x) = 2x^3 - 5x + 1 = 0$, n = 3, $\boldsymbol{p_{k+1} = p_k - \frac{b_n}{c_{n-1}}}$

$$p_{k+1} = p_k - \frac{b_n}{c_{n-1}}$$

Taking initial value as $p_0 = 0.5$

0.5	2	0	-5	1	
	0	1	0.5	-2.25	
0.5	2	1	-4.5	-1.25	$\Rightarrow b_3 = -1.25(b_n)$
	0	1	1		
	2	2	-3.5	$\Rightarrow c_2 = -3.5(c_{n-1})$	

1 st iteration: Take k=0,

$$p_1 = p_0 - \frac{b_3}{c_2}; here\ b_3 = -1.25;\ c_2 = -3.5$$

$$\boldsymbol{p_1 = 0.1429}$$

Taking initial value as $p_1 = 0.1429$

0.1429	2	0	-5	1	
	0	0.2858	0.0408	-0.7087	
0.1429	2	0.2858	-4.9592	0.2913	$\Rightarrow b_3 = 0.2913(b_n)$
	0	0.2858	0.0817		
	2	0.5716	-4.8775	$\Rightarrow c_2 = -4.87752(c_{n-1})$	

2 nd iteration: Take k=1,

$$p_2 = p_1 - \frac{b_3}{c_2}; here\ b_3 = 0.2913;\ c_2 = -4.8775$$

$$\boldsymbol{p_2 = 0.2026}$$

To find deflated polynomial

0.2026	2	0	-5	1	

	0	0.4052	0.0821	-0.9964	
0.2026	2	0.4052	-4.9179	0.00363	$\Rightarrow b_3 = 0.2913(b_n)$

Deflated polynomial: $p_3(\mathrm{x}) = (\mathrm{x}-\mathrm{p})Q_{n-1}(\mathrm{x})$

$Q_{n-1}(x) = 2x^2 + 0.4052\,x - 4.9179$

Problem2: Find the positive root of $2x^3 - 5x + 1 = 0$ using Birge-Vieta Method

Solution: the iteration formula of Birge-Vieta Method $p_{k+1} = p_k - \dfrac{b_n}{c_{n-1}}$

Given $p_n(x) = 2x^3 - 5x + 1$ with $n = 3$

$$p_n(x) = 2x^3 - 5x + 1 \begin{matrix} p_n(0) = 1(+ve) \\ p_n(1) = -2(-ve) \end{matrix} \Big\} the\,root\; p_k \in (0,1)$$

Hence the root lies between 0 and 1

let $p_0 = 0.5$ (initial approximation)

	First iteration				
Step 1	$a_0 = 2$	$a_1 = 0$	$a_2 = -5$	$a_3 = 1$	
	0	$b_0 p_0 = 1$	$b_1 p_0 = 0.5$	$b_2 p_0 = -2.25$	
Step2	$b_0 = a_0 = 2$	$b_1 = a_1 + b_0 p_0 = 1$	$b_2 = a_2 + b_1 p_0 = -4.5$	$b_3 = a_3 + b_2 p_0 = -1.25$	$\Rightarrow b_3 = -1.25(b_n)$
	0	$p_0 c_0 = 1$	$p_0 c_1 = 1$		
Step3	$c_0 = b_0 = 2$	$c_1 = b_1 + p_0 c_0 = 2$	$c_2 = b_2 + p_0 c_1 = -3.5$	$\Rightarrow c_2 = -3.5(c_{n-1})$	
Step4	$p_{k+1} = p_k - \dfrac{b_n}{c_{n-1}} \Rightarrow p_1 = p_0 - \dfrac{b_3}{c_2} = 0.5 - \dfrac{1.25}{3.5} = 0.14285714 \sim 0.1429 = p_1$				

	2^{nd} iteration			
	$a_0 = 2$	$a_1 = 0$	$a_2 = -5$	$a_3 = 1$
$1429 = p_1$	0	$p_1 b_0 = 0.2858$	$p_1 b_1 = 0.0408$	$p_1 b_2 = -0.7087$
	$b_0 = a_0 = 2$	$b_1 = a_1 + p_1 b_0 = 0.2858$	$b_2 = a_2 + p_1 b_1 = -4.9592$	$b_3 = a_3 + p_1 b_2 = 0.2913$
	0	$p_1 c_0 = 0.2858$	$p_1 c_1 = 0.0817$	
	$c_0 = b_0 = 2$	$c_1 = b_1 + p_1 c_0 = 0.5716$	$c_2 = b_2 + p_1 c_1 = -4.8775$	$\Rightarrow c_2 = -4.8775(c_{n-1})$
	$p_{k+1} = p_k - \dfrac{b_n}{c_{n-1}} \Rightarrow p_2 = p_1 - \dfrac{b_3}{c_2} = 0.1429 - \dfrac{0.2913}{(-4.8775)} = 0.2026232 \sim 0.2026$			

	3rd iteration			
Step 1	$a_0 = 2$	$a_1 = 0$	$a_2 = -5$	$a_3 = 1$
$0.2026 = p_2$	0	$p_1 b_0 = 0.4052$	$p_1 b_1 = 0.0821$	$p_1 b_2 = -0.9964$
Step2	$b_0 = a_0 = 2$	$b_1 = a_1 + p_2 b_0 = 0.4052$	$b_2 = a_2 + p_1 b_1 = -4.9179$	$b_3 = a_3 + p_1 b_2 = 0.00363$
	0	$0.8104 = p_2 c_0$	$0.2463 = p_2 c_1$	
Step3	$c_0 = b_0 = 2$	$c_1 = b_1 + p_2 c_0 = 1.2156$	$c_2 = b_2 + p_1 c_1 = -4.6716$	$\Rightarrow c_2 = -4.6716 (c_{n-1})$
Step4	$p_{k+1} = p_k - \frac{b_n}{c_{n-1}} \Rightarrow p_3 = p - \frac{b_3}{c_2} = 0.2026 - \frac{0.00363}{(-4.6716)} = 0.2033770 \sim 0.2034$			

	4th iteration			
Step 1	$a_0 = 2$	$a_1 = 0$	$a_2 = -5$	$a_3 = 1$
$0.2034 = p_3$	0	$p_3 b_0 = 0.4068$	$p_3 b_1 = 0.0828$	$p_3 b_2 = -1.0002$
Step2	$b_0 = a_0 = 2$	$b_1 = a_1 + p_3 b_0 = 0.4068$	$b_2 = a_2 + p_3 b_1 = -4.9172$	$b_3 = a_3 + p_1 b_2 = -0.00016$
	0	$0.4068 = p_3 c_0$	$0.1655 = p_3 c_1$	
Step3	$c_0 = b_0 = 2$	$c_1 = b_1 + p_3 c_0 = 0.8136$	$c_2 = b_2 + p_3 c_1 = -4.7517$	$\Rightarrow c_2 = -4.7517 (c_{n-1})$
Step4	$p_{k+1} = p_k - \frac{b_n}{c_{n-1}} \Rightarrow p_3 = p - \frac{b_3}{c_2} = 0.2034 - \frac{-0.00016}{(-4.7517)} = 0.2033663 \sim 0.2034$			

The smallest positive root is $\Rightarrow x = 0.2034 (4 decimals)$

The deflated polynomial $Q_{n-1}(x) = b_0 x^2 + b_1 x + b_2 = 0$

Problem3: Find the **positive root of** $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$ using Birge-Vieta Method

Solution: the iteration formula of Birge-Vieta Method $p_{k+1} = p_k - \frac{b_n}{c_{n-1}}$

Given $p_n(x) = x^4 - 3x^3 + 3x^2 - 3x + 2$ with $n = 4$

$$\left.\begin{array}{l} p_n(0) = 2(+ve) \\ p_n(1) = 0(-ve) \end{array}\right\} the\,root\ p_k \in (0,1)$$

Hence the root lies between 0 and 1

let $p_0 = 0.5$ (initial approximation)

	First iteration				
Step 1	$1 = a_0$	$--3 = a_1$	$3 = a_2$	$-3 = a_3$	$2 = a_4$
	0	$0.5 = b_0 p_0$	$--1.25 = b_1 p_0$	$0.375 = b_2 p_0$	$--1..0625 = b_2 p_0$

Step2	$1=b_0=a_0$	$-2.5=b_1=a_1+b_0p_0$	1.75 $=b_2=a_2+b_1p_0$	$-2.125=b_3=a_3+b_2p_0$	$\Rightarrow b_4=0.9375(R)$
	0	$0.5=p_0c_0$	$--1.0=p_0c_1$	$0.375=p_0c_1$	
Step3	$=c_0=b_0$	$--2.0=c_1=b_1+p_0c_0$	0.75 $=c_2=b_2+p_0c_1$	$--1.750=c_3=\dfrac{dR}{dp}(c_{n-1})$	
Step4	$p_{k+1}=p_k-\dfrac{b_n}{c_{n-1}}\Rightarrow p_1=p_0-\dfrac{b_3}{c_2}=0.5-\dfrac{0.9375}{-1.750}=1.035714\sim 1.0357=p_1$				

	2nd iteration				
Step 1	$1=a_0$	$--3=a_1$	$3=a_2$	$-3=a_3$	$2=a_4$
$1.0357=p_1$	0	$1.0357=p_1b_0$	$-2.0344=p_1b_1$	$1.00005=p_1b_2$	$-2.0714=p_1b$
Step2	1 $=b_0=a_0$	$-1.9643=b_1=a_1+p_1b_0$	$0.9656=b_2=a_2+p_1b_1$	$-1.9999=b_3=a_3+p_1b_2$	$-0.0714=b_4=$
	0	$1.0357=p_1c_0$	$--0.9618=p_1c_1$	$0.003986=p_1c_2$	
Step3	$1=c_0$	$-0.9286=c_1=b_1+p_1c_0$	$0.00834=c_2=b_2+p_1c_1$	$-1.9959=c_3=(c_{n-1})$	
Step4	$p_{k+1}=p_k-\dfrac{b_n}{c_{n-1}}\Rightarrow p_2=p_1-\dfrac{b_3}{c_2}=1.0357-\dfrac{-0.0714}{(-1.9959)}=0.9999267\sim 1.0$				

	3rd iteration					
Step 1	$1=a_0$	$--3=a_1$	$3=a_2$	$-3=a_3$	$2=a_4$	
$1.0=p_2$	0	$1.0=p_1b_0$	$-2=p_1b_1$	$1=p_1b_2$ 0.9964	$2.01=p_1b_3$	
Step2	$1=b_0=a_0$	$-2=b_1=a_1+p_2b_0$	$1=b_2=a_2+p_1b_1$	$-2=b_3=a_3+p_1b_2$	$0=b_4=R$	

The deflated polynomial $Q_{n-1}(x)=b_0x^3+b_1x^2+b_2x+b_3$ (using step 2 of third iteration)

The smallest positive root is $\Rightarrow x=1.0(4decimals)$

The deflated polynomial $Q_{n-1}(x)=b_0x^2+b_1x\ +b_2=0$

Problem 2: compute the smallest positive root of $x^4-3x^3+3x^2-3x+2=0$ using Birge-Vieta Method with initial iteration $p_0=0.5$. Hence find deflated polynomial. Ans $p_1=1.0356, p_k=1.0$

Problem 3: compute the smallest positive root of $x^5-x+1=0$ using Birge-Vieta Method with initial iteration $p_0=-1.5$. Hence find deflated polynomial. Ans $p_1=-1.2909, p_2=-1.1903,$ $p_3=-1.1683 p_4=-1.1671, x^4-1.1671x^3+1.3919x^2-1.5893x+0.8547=0$

Problem 3: compute the negative root of $3x^4+5x^3+6x+10=0$ using Birge-Vieta Method with initial iteration $p_0=-1.9$. Hence find deflated polynomial. Ans $p_1=-1.7465, p_2=-1.6810,$ $p_3=p_4=, x^4-1.1671x^3+1.3919x^2-1.5893x+0.8547=0$

BAIRSTOW METHOD:

We extract a quadratic factor $x^2 + px + q$ from the given polynomial

$p_n(x) = a_0 x^n + a_1 x^{n-1} + --- + a_n$ we get a complex pair or a real pair of roots.

Let $x^2 + px + q$ be not a exact factor then $p_n(x) = (x^2 + px + q)Q_{n-2}(x) + Rx + S$ where $Q_{n-2}(x) = b_o x^{n-2} + b_1 x^{n-3} + --- + b_{n-3}x + b_{n-2}$.

We find a value of p,q as follows: let $p_1 = p_0 + h_1$ $q_1 = q_0 + h_2$ where p_0, q_0 are the initial approximation and

$$h_1 = \frac{b_{n-1}c_{n-2} - b_n c_{n-3}}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}$$

$$h_2 = \frac{b_n c_{n-2} - b_{n-1}(c_{n-1} - b_{n-1}}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}$$

We get the value of $b_o, b_1, b_2, \dots\dots$ and $c_o, c_1, c_2, \dots\dots$ Using the following synthetic division.

$-p_0$	a_0	a_1	$a_2 \dots\dots\dots$	a_{n-1}	a_n
$-q_0$	0	$-p_0 b_0$	$-p_0 b_1 \dots\dots$	$-p_0 b_{n-2}$	$-p_0 b_{n-1}$
	0	0	$-q_0 b_0 \dots\dots$	$-q_0 b_{n-3}$	$-q_0 b_{n-2}$
	b_0	b_1	$b_2 \dots\dots\dots$	b_{n-1}	b_n
	0	$-p_0 c_0$	$-p_0 c_1$	$-p_0 c_{n-1}$	
	0	0	$-q_0 c_0$	$-p_0 c_{n-3}$	
	c_0	c_1	c_2	c_{n-1}	

PROBLEMS

1. Perform 2 iteration of Bairstow method to extract a following quadratic factor $x^2 + px + q$ from the polynomial $3p_3(x) = x^3 + x^2 - x + 2 = 0$ use the initial approximation $p_0 = -0.9$, $q_0 = 0.9$ and hence find the deflated polynomial.

Solution:

$$-p_0 = 0.9 \qquad -q_0 = -0.9$$

	1	1	-1	2
0.9	0	0.9	1.71	-0.171
-0.9	0	0	-0.9	-1.71
	1	1.9	-0.19	0.119
	0	0.9	2.52	
	0	0	-0.9	
	1	2.8	1.43	

$$h_1 = \frac{b_{n-1}c_{n-2} - b_n c_{n-3}}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}$$

$$h_2 = \frac{b_n c_{n-2} - b_{n-1}(c_{n-1} - b_{n-1})}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}$$

$h_1 = -0.10466$ $\qquad$ $h_2 = 0.103055$

$p_1 = -1.0047$ $\qquad$ $q_1 = 1.0031$

2nd iteration

$p_2 = p_1 + h_1$ $\qquad\qquad$ $q_2 = q_1 + h_2$

1.0047	1	1	-1	2
-1.0031	0	1.0047	2.0141	0.0111
	0	0	-1.0031	-2.01090
	1	2.0047	0.0110	0.0002
	0	1.0047	3.02355	
	0	0	-1.0031	
	1	3.0094	2.0314	

$h_1 = 0.0047$ $\qquad\qquad$ $h_2 = -0.0031$

$p_2 = -1$ $\qquad\qquad$ $q_2 = 1$

The extracted the polynomial factor is

$$x^2 + p_2 x + q_2 = x^2 - x + 1 \ = \ 0$$

The exact factor is $x^2 - x + 1$

Deflated polynomial

1	1	1	-1	2
-1	0	0	2	0
	0	0	-1	-2
	1	2	0	0

Deflated polynomial $x + 2 \ = \ 0$

$$x \ = \ -2$$

Solving $x^2 - x + 1 = 0, \quad a = 1, b = 1, c = 1$

$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} \qquad x = -2, \frac{1 \pm i\sqrt{3}}{2}$

Problem 2: Execute 3 iteration of Bairstow method to extract a following quadratic factor $x^2 + px + q$ from the polynomial $p_4(x) = x^4 - x - 10 = 0$ use the initial approximation $p_0 = 0.1, q_0 = 0.1$ and hence find the deflated polynomial. solu

$p_k = -0.15811256, q_k = -3.149803 (k = 5)$

$\alpha_1 = -1.697, \alpha_2 = 1.856, Q_{n-2}(x) = x^2 + 0.157x + 3.17, \alpha_3 = -0.079 + 1.78i, \alpha_4 = -0.079 - 1.78i$

Graeffe's Root Squaring Method

This is a direct method and it is used to find the roots of a polynomial equation with real coefficient that is equation of the form.

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} \dots \dots \dots . + a_n = 0$

where a_i's are real.

The roots may be real, distinct or equal or complex we separate the roots of the equation by forming another polynomial equation, by the method of roots squaring process, whose roots are high process of the roots of the equation

The equation is separate so that even power of are on one side and odd powers of x are on the other side; then squaring both sides, we get

$(a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \cdots)^2 = (a_1 x^{n-1} + a_3 x^{n-3} + \cdots)^2$

Simplifying we get

$(a_0^2 x^{2n} + (a_1^2 - 2a_0 a_2) x^{2n-2} + (a_2^2 - 2a_j a_3 + a_0 a_4) x^{2n-4} + \cdots + (-1)^n a_n^2 = 0$

setting $-x^2 = z$ then reduces to

$$b_0z^n + b_1z^{n-1} + b_2z^{n-2} + \cdots + b_{n-1}z + b_n = 0$$

where $b_0 = a_0^2$

$b_1 = a_1^2 - 2a_0a_2$

$b_2 = a_2^2 - 2a_1a_3 + 2a_0a_4$

....

....

$b_n = a_n$

now all b_i's are got in terms of a_i's

the root of the equation are $-\alpha_1^2, -\alpha_2^2 \ldots\ldots -\alpha_n^2$ if $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ are the roots of the equation.

The coefficient b_i's of equation can be easily got from the following tables

a_0	a_1	a_2	a_3	a_4	a_5	....	a_n
a_0^2	a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	....	a_n^2
	$-2a_0a_2$	$-2a_1a_3$	$-2a_2a_4$	$-2a_3a_5$	$-2a_4a_6$		
		$2a_0a_4$	$2a_1a_5$	$2a_2a_6$	$2a_3a_7$		
			$-2a_0a_6$				
B_o	B_1	B_2	B_3	B_4	B_5		B_n

The $(\gamma+1)^{th}$ column in the above tables is got as follow. The terms occurring in the $(r+1)^{th}$ columns alternate in sign starting with the positive sign for a_r^2. The second term is twice the product of the immediate product of the next neighbouring coefficient a_{r-2} and a_{r+2} and this procedures is continues until there are no available coefficients to get the product terms . The sum of all such terms will be b_{r+1}.

If this procedure is repeated m times, we get the equation

$B_0x^n+B_1x^{n-1}+B_2x^{n-2}+\ldots\ldots\ldots+B_n = 0$

Whose roots are $R_1, R_2, R_3, \ldots\ldots R_n$ power of the roots of the equation with sign changed.

That is $R_i = -\alpha_i^{2m}$, $i=1,2,3,\ldots\ldots n$

Case:1 Suppose we assume $|\alpha_1| > |\alpha_2| \ldots\ldots > ||\alpha_n|$ then $|R_1| >> |R_2| >> |R_3| \ldots\ldots >> |R_n|$

If the roots of (1) differ in magnitude, then the 2^{mth} power of the roots are separate widely for higher values of m

$$\Sigma R_i = -\frac{B_1}{B_0}$$

$$\sum R_i R_j \approx R_1 R_2 = \frac{B_2}{B_0}$$

$$\sum R_i R_j R_k \approx R_1 R_2 R_3 = \frac{B_3}{B_0}$$

$$R_1, R_2, R_3, \dots \dots, R_k = (-1)^n \frac{B_n}{B_0}$$

$$|R_i| = |\alpha_i| 2^m = \frac{B_i}{B_{i+1}}$$

$$Log\ |\alpha_i| = 2^{m-1}(\log|B_i| - \log|B_{i+1}|) \qquad i = 1,2,3, \dots \dots m$$

From this we can find the values of α_i substituting α_i or $-\alpha_i$ in (1) we can determine the sign of the roots of the eqn. The process of squaring is stopped when another process of squaring produces new coefficients which are approximately the squares of the corresponding coefficient Bi's

Case:2 After a few squaring processes, if the magnitude of the coefficient B_1 is half the square of the magnitude of corresponding coefficient in the previous eqn, then this indicates that α_i is a double root.

$$|R_i| \approx |\frac{B_i}{B_{i-1}}| R_{i+1} \approx |\frac{B_{i+1}}{B_i}|$$

$$R_i R_{i+1}| \approx R_i^2 \approx |\frac{B_{i+1}}{B_{i-1}}|$$

$$R_i^2 = \alpha_i^{\ 2^{m+1}} \approx |\frac{B_{i+1}}{B_{i-1}}|$$

From this we can get double roots. The sign of its can be got as before by substituting in equation.

Case 3 : If α_k, α_{k+1} are two complex conjugate roots, then this would make the coefficient of x^{n-k}

in the successive squaring to fluctuate both in magnitude and sign.

If $\alpha_k, \alpha_{k+1} = \beta_k(\cos\emptyset_k + i \sin\emptyset_k)$ is the complex pair of roots, then coefficient will fluctuate in magnitude and sign by the quantity $2\beta_k^m \cos\emptyset_k$.

A complex pair is located only by the such oscillation. If m is large β_k can be got from

$$\beta_k^{2(2)^m} \approx \left|\frac{B_{k+1}}{B_{k-1}}\right| \text{ and } \emptyset \text{ is got from } \quad 2\beta_k^m \cos m\,\emptyset_n \approx |\frac{B_{k+1}}{B_{k-1}}$$

if the eqn. Possesses only two complex roots p+iq we have $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{k-1} + 2p + \alpha_{k+2} + \ldots + \alpha_n = -a_1$

this gives the values of p.

Since$|\beta_k|^2 = p^2 + q^2$ and $|\beta_k$ is known already q is known from the relation.

Note : $p^{(m+1)}(z) = (-1)^m p^{(m)}(x) p^{(m)}(-x) : z = x^2$ so that the roots of $p^{(m)}(z)$ are those of $p(x)$ raised to the power 2^m.

Problem 1: Find all the roots of the equation $x^3 - 9x^2 + 18x - 6 = 0$ by Graeffe's root square method (roots squaring 3 times)

Solution:

m = 0,1,2,3

m	2^m	Coefficient			
0	1	1 1	-9 81 -36	18 324 -108	-6 36
1	2	1 1	45 2025 -432	216 46656 -3240	36 1296
2	4	1 1	1593 2537649 -886832	43416 188494905 -4129056	1296 1679616
3	8	1	2450817	1880820000	1679616

The magnitude of the coefficient of b_0, b_1, b_2, b_3 is not half of the square of the magnitude of the corresponding coefficient in the previous eqn.

The roots are real and unequal.

We know that

$$|R_i| = |\alpha_i|^{2^m} = \frac{B_i}{B_{i-1}} \Rightarrow |\alpha_i|^8 = \frac{2450817}{1} \Rightarrow |\alpha_i| = [2450817]^{1/8} = 6.2901914$$

When $i = 1$

$$|R_1| = |\alpha_i|^{2^3} = \frac{B_1}{B_0}$$

When $i = 2$

$$|R_2| = |\alpha_i|^{2^3} = \frac{B_2}{B_1}$$

When $i = 3$

$$|R_3| = |\alpha_i| 2^3 = \frac{B_3}{B_2}$$

$\alpha_1 = \pm 6.2901914$

$\alpha_2 = \pm 2.29419085$

$\alpha_3 = \pm 0.415774496$

$f(\alpha_1) = 0.00578$ $\qquad f(-\alpha_1) = -724.2029$

$f(\alpha_2) = 0.0000671$ $\qquad f(-\alpha_2) = -106.74028$

$f(\alpha_3) = 0.0000007$ $\qquad f(-\alpha_3) = -15.11163$

the roots are 6.2901914, 2.29419085,0.415774496.

Problem 2: Determine the roots of the equation $x^3 + 3x^2 = 4$ by Graeffe's root square method

Solution :

m	2^m	coefficients			
0	2^0	$1 = a_0$	$3 = a_1$	$0 = a_2$	$--4 = a_3$
		$1 = a_0^2$	$9 = a_1^2$	$0 = a_2^2$	$16 = a_3^2$
			$0 = -2a_0a_2$	$24 = -2a_1a_3$	
1	2^1	1	9	24	16
		1	81	576	256
			--48	--288	
2	2^2	1	33	288	256
		1	1089	82944	65536
			--576	--16896	

3	2^3	1	513	66048	65536
		1	263169	4362328304	4294967296
			-132096	--67239936	
4	2^4	1	131073	4295098368	4294967296
		1	1.71801313×10^{10}	1.844786999×10^{19}	1.844674407×10^{19}
			--8590196736	$--1.125908497\times10^{15}$	
5	2^5	1	8589934564	1.844674408×10^{19}	1.844674407×10^{19}
		B_0	B_1	B_2	B_3

From the above table its observed that the magnitude of the coefficient B_1 is half the square of the magnitude of corresponding coefficient in the previous equation this indicates that α_i is a double root.

To find the double root α_i

$$|R_i|^2 = |\alpha_i|^{2^{m+1}} = \left|\frac{B_{i+1}}{B_{i-1}}\right| = \left|\frac{B_2}{B_0}\right| = \left|\frac{1.844674408\times10^{19}}{1}\right|$$

$$|\alpha_i|^{2^6} = \left|\frac{1.844674408\times10^{19}}{1}\right| \Rightarrow |\alpha_i| = \left|\frac{1.844674408\times10^{19}}{1}\right|^{1/64} = 2.0$$

$$\alpha_i = \pm 2.0$$

Since $f(2) = 2^3 + 3(2^2) - 4 = 16$ and $f(-2) = (-2\ )^3 + 3(-2\ )^2 - 4 = 0$

The double root is $\alpha_i = -2.0 (i = 1, 2)$

To find the third root

$$|R_i| = |\alpha_i|^{2^m} = \left|\frac{B_i}{B_{i-1}}\right| \Rightarrow |\alpha_3|^{2^m} = \left|\frac{B_3}{B_2}\right| \Rightarrow |\alpha_3|^{32} = \left|\frac{1.844674407\times10^{19}}{1.844674408\times10^{19}}\right|$$

since $f(2) = 0$ the real root is since

$$|\alpha_3| = \left|\frac{1.844674407\times10^{19}}{1.844674408\times10^{19}}\right|^{1/32} = 0.999999 \Rightarrow \alpha_3 = \pm 1$$

$$f(1) = 1^3 + 3(1^2) - 4 = 0 \quad \alpha_3 = 1$$

All the roots are -2,-2,1

Problem 3: Determine the roots of the equation $x^3 - 4x^2 + 5x - 2 = 0$ by Graeffe's root square method

Solution :

m	2^m	coefficients			
0	2^0	$1 = a_0$	$--4 = a_1$	$5 = a_2$	$--2 = a_3$
		$1 = a_0^2$	$16 = a_1^2$	$25 = a_2^2$	$4 = a_3^2$
			$--10 = -2a_0a_2$	$--16 = -2a_1a_3$	
1	2^1	1	6	9	4
		1	36	81	16
			--18	--48	
2	2^2	1	18	33	16
		1	324	1089	256
			--66	--576	
3	2^3	1	258	513	256
		1	66564	263169	65536
			-1026	--131073	
4	2^4	1	65538	131073	65536
		1	4295229444	17180131329	4294967276
			--262146	--8590196736	
5	2^5	1	4294967298	8589934593	4294967276
		B_0	B_1	B_2	B_3

From the above table its observed that the magnitude of the coefficient $B_2 (k=2)$ is half the square of the magnitude of corresponding coefficient in the previous equation this indicates that $\alpha_i \ (i=2)$ is a double root.

To find the double root α_i

$$|R_i|^2 = |\alpha_i|^{2^{m+1}} = \left|\frac{B_{i+1}}{B_{i-1}}\right| = \left|\frac{B_3}{B_1}\right| = \left|\frac{4294967276}{4294967298}\right|$$

$$|\alpha_i|^{2^6} = \left|\frac{4294967276}{4294967298}\right| \Rightarrow |\alpha_i| = \left|\frac{4294967276}{4294967298}\right|^{1/64} = 1.0$$

$$\alpha_i = \pm 1.0$$

Since $f(1) = (1)^3 - 4(1)^2 + 5(1) - 2 = 0$

The double root is $\alpha_i = 1.0 (i = 2,3)$

To find the third root

$$|R_i| = |\alpha_i|^{2^m} = \left|\frac{B_i}{B_{i-1}}\right| \Rightarrow |\alpha_1|^{2^m} = \left|\frac{B_1}{B_0}\right| \Rightarrow |\alpha_1|^{32} = \left|\frac{4294967298}{1}\right|$$

$$|\alpha_1| = \left|\frac{4294967298}{1}\right|^{1/32} = 2.0 \Rightarrow \alpha_1 = \pm 2$$

$f(2) = (2)^3 - 4(2)^2 + 5(2) - 2 = 0$ since $f(2) = 0$ the real root is $\alpha_1 = 2$

Determine the roots of the equation $x^3 - x^2 - x = 2$ by Graeffe's root square method

Solution :

m	2^m	coefficients			
0	2^0	$1 = a_0$	$--1 = a_1$	$--1 = a_2$	$--2 = a_3$
		$1 = a_0^2$	$1 = a_1^2$	$1 = a_2^2$	$4 = a_3^2$
			$2 = -2a_0a_2$	$--4 = -2a_1a_3$	
1	2^1	1	3	--3.0	4
		1	9	9	16
			6	--24	
2	2^2	1	15	--15	16
		1	225	225	256
			30	--480	
3	2^3	1	255	--255	256
		1	65025	65025	65536
			510	--130560	
4	2^4	1	65535	--65535	65536
		1	4.2948362×10^9	4.2948362×10^9	4.2949673×10^9
			131070	$--8.5898035 \times 10^9$	
5	2^5	1	4.2949673×10^9	$--4.2949673 \times 10^9$	4.2949673×10^9
		B_0	B_1	B_2	B_3

From the above table its observed that the magnitude of the coefficient B_0 becomes constant, while the magnitude of the coefficient $B_2 (k = 2)$ oscillates, hence α_1 is a real root and α_2, α_3, are complex.

To find the real root

$$|\alpha_1|^{2^5} = \frac{B_1}{B_0} \Rightarrow |\alpha_1| = \left[\frac{4.2949673 \times 10^9}{1}\right]^{1/32} = 2.0$$ since $f(2) = 0$ the real root is α_1 =2.0

To find the complex root

$$\beta_k^{2(2)^m} = \left|\frac{B_{k+1}}{B_{k-1}}\right| \Rightarrow \beta_2^{2(2)^5} = \left|\frac{B_3}{B_1}\right|, \Rightarrow \beta_2^{64} = \left[\frac{4.2949673\times10^9}{4.2949673\times10^9}\right] = \beta_2 = 1.0 --(1) \text{ where } \phi \text{ is given by}$$

$$2\beta_k^m \cos m\phi_n = \frac{B_{k+1}}{B_{k-1}} \text{ and } |\beta_k|^2 = p^2+q^2 ---(2)$$

Let the complex roots be $\alpha_2 = p+iq, \alpha_3 = p-iq$,then $\alpha_1 + \alpha_2 + \alpha_3 = -\frac{a_1}{a_0} \Rightarrow p = -0.5 --(3)$

From $(1),(2)$ and $(3)\, p^2+q^2 = \beta_k^2 = 1 \Rightarrow q = \frac{\sqrt{3}}{2}$

The roots are 2, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Problem 5: Determine the roots of the equation
$x^4 - 3x + 1 = 01.1892, 0.3377, -0.734365 + i1.3811173$ by Graeffe's root square method

Solution :

m	2^m	coefficients				
0	2^0	$1=a_0$	$0=a_1$	$0=a_2$	$--3=a_3$	$1=a_4$
		$1=a_0^2$	$0=a_1^2$	$0=a_2^2$	$9=a_3^2$	$1=a_4^2$
			$0=-2a_0a_2$	$0=-2a_1a_3$	$0=-2a_2a_4$	
				$2=+2a_0a_4$	$=+2a_1a_5$	
1	2^1	1	0	2	9	1
		1	0	4	81	1
			--4	0	--4	
				2		
2	2^2	1	16	6	77	1
		1	256	36	5929	
			--12	--2464	--12	
				2		
3	2^3	1	244	--2426	5917	1
		1	59536	5885476	35010889	
			4852	--2887496	2426	
				2		
4	2^4	1	64388	2997982	35013315	1
		1	4145814544	8.987896072×10^{12}	1.225932227×10^{15}	1
			--5995964	$--4.508874652\times10^{12}$	--5995964	
				2		
5	2^5	1	4139818580	4.47902142×10^{12}	1.225932221×10^{15}	1
		B_0	B_1	B_2	B_3	

UNIT – II - ADVANCED NUMERICAL METHODS – SMTA5304

UNIT II

Gauss Seidel Iteration Method:

Consider the system of iteration

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = 0$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = 0$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots = 0$

To apply Gauss Seidel method, we have to rewrite the equation in such a way that a set of equations satisfies diagonally dominant.

i.e) $|a_{11}| > |a_{12}| + |a_{13}| + \cdots$

$|a_{22}| > |a_{21}| + |a_{23}| + \cdots$

$|a_{nn}| > |a_{a1}| + |a_{n2}| + \cdots$

Problems:

Solve the system of equation by Gauss Seidel method to approximate the solution to 4 significant digits 28x+4y-z=32, 4x+3y+10z=24, 2x+17y+4z=35.

Solution:

The given system of equation is not diagonally dominant.

Interchanging the last two rows, we get the equation:

28x+4y-z=32

2x+17y+4z=35

4x+3y+10z=24

$x = (1/28)\ [32-4y+z]$

$y = (1/17)\ [35-2x-4z]$

$z = (1/10)\ [24-4x-3y]$

Initial values are taken it as y=0,z=0

$x = (1/28)\ [32-0+0]= 1.14286$

$y = (1/17)\ [35-2(1.14286)-0]= 1.92437$

$z = (1/10)\ [24-4(114286)-3(1.92437)]= 1.36555$

ITERATION	X	Y	Z
0	1.14286	1.92437	1.36555
1	0.91672	1.62967	154441
2	096521	1.58188	1.53935
3	0.97185	1.58229	1.53657
4	0.97169	1.58296	1.53645
5	1.997159	1.58300	1.53646
6	0.97159	1.58300	1.53646

$x = 0.9716$

$y = 1.5830$

$z = 1.5365$

Gauss Jacobi:

1. Using Gauss Jacobi method find the solution for the following:
   $10x_1+2x_2+x_3=9$

   $x_1+10x_2-x_3=-22$

   $-2x_1+3x_2+10x_3=22$

Solution

The system of equation is diagonally dominant.

$x_1 = (1/10)\ [9-2x_2-x_3]$

$x_2 = (1/10)\ [-22-x_1+x_3]$

$x_3 = (1/10)\ [22+2x_1-3x_2]$

Initially we take $x_1=0, x_2=0, x_3=0$

$x_1 = (1/10)\ (9)= 0.9$

$x_2 = (1/10)\ (-22)= -2.2$

$x_3 = (1/10)\ (22)=2.2$

ITERATION	x_1	x_2	x_3
0	0.9	-2.2	2.2
1	1.12	-2.07	3.04
2	1.01	-2.008	3.045
3	0.9971	-1.9965	3.0044
4	0.9989	-1.9992	2.9984
5	1	-2.0000	2.99954
6	100001	-2.0000	3

$x_1=1,\quad x_2=-2,\quad x_3=3$

by using Gauss Jacobi method:

Initially we take $x_1=0, x_2=0, x_3=0$

$x_1 = (1/4)\ (2)= 0.5$

$x_2 = (1/5)\ (-6)$

$x_3 = (1/3)\ (-4)=-1.3$

ITERATION	x_1	x_2	x_3
0	0.5	-1.2	-1.3
1	1.125	--0.78	-0.7
2	0.87	-1.145	-1.1883
3	1.0833	-0.8987	-0.86
4	0.9397	-1.0727	-1.0953
5	1.042	-0.9498	-0.9314
6	0.9703	-1.0358	-1.0475
7	1.0201	-0.9751	-0.9662
8	0.9853	-1.018	-1.023
9	1	-1	-1

$x_1 = 1, \quad x_2 = -1, \quad x_3 = -1$

Iteration Methods:

Gauss Jacobi and seidel method:

AX=b

$x^{(k+1)}=Hx^{(k)}+c$ [H is a Iteration matrix]

Jacobi iterative method:

$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots a_{2n}x_n=b_2$

$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots a_{nn}x_n=b_n$

$x_1^{(k+1)} = (1/a_{11})\,[b_1-(a_{12}x_2^{(k)}+a_{13}x_3^{(k)}+\ldots\ldots+a_{1n}x_n^{(k)})]$

$x_2^{(k+1)} = (1/a_{22})\,[b_2-(a_{21}x_1^{(k)}+a_{23}x_3^{(k)}+\ldots\ldots+a_{2n}x_n^{(k)})]$

$x_n^{(k+1)} = (1/a_{nn})\,[b_n(a_{n1}x_1^{(k)}+a_{n2}x_2^{(k)}+\ldots\ldots+a_{nn-1}x_{n-1}^{(k)})]$

$A=D+L+U$

$AX=b$

$(D+L+U)X=b$

$DX=-(L+U)X+b$

$DX^{(k+1)}=-(L+U)X^{(K)}+b$

$X^{(k+1)}=-D^{-1}(L+U)X^{(K)}+D^{-1}b$

$X^{(K+1)}=HX^{(K)}+C$

Error :

Where $H =-D^{-1}(L+U)$
$C = D^{-1}b$

Computing Procedure (If Error is Given)

$$x^{(k+1)} = - D^{-1} (L+U) x^{(k)} + D^{-1} b$$

$$= x^{(k)} – x^{(k)} – D^{-1} (L+U) x^{(k)} + D^{-1} b$$

$$= x^{(k)} – [I + D^{-1} (L+U)] x^{(k)} + D^{-1} b$$

$$= x^{(k)} – [DD^{-1} + D^{-1} (L+U)] x^{(k)} + D^{-1} b$$

$$= x^{(k)} – D^{-1} [D+L+U] x^{(k)} + D^{-1} b$$

$$x^{(k+1)} - x^{(k)} = - D^{-1} A x^{(k)} + D^{-1} b$$

$$x^{(k+1)} - x^{(k)} = D^{-1} [b – A x^{(k)}]$$

$$v^{(k)} = D^{-1} r^{(k)}$$

Where $v^{(k)} = x^{(k+1)} - x^{(k)}$

$r^{(k)} = b – A x^{(k)}$

Formulas to be Known :

1. For finding Iterative matrix $H = - D^{-1} (L+U)$ and $C = D^{-1} b$.
   Hence $x^{(k+1)} = Hx^{(k)} + c$.
2. If Error is given , $v^{(k)} = x^{(k+1)} - x^{(k)}$ and $r^{(k)} = b – A x^{(k)}$. Hence $v^{(k)} = D^{-1} r^{(k)}$

GAUSS SEIDAL ITERATIVE METHOD:

$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots..a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots..a_{2n}x_n=b_2$

$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots...a_{nn}x_n=b_n$

$x_1^{(k+1)} = (1/a_{11})\ [b_1-(a_{12}x_2^{(k)}+a_{13}x_3^{(k)}+\ldots\ldots.+a_{1n}x_n^{(k)})]$

$x_2^{(k+1)} = (1/a_{22})\ [b_2-(a_{21}x_1^{(k+1)}+a_{23}x_3^{(k)}+\ldots\ldots..+a_{2n}x_n^{(k)})]$

$x_n^{(k+1)} = (1/a_{nn})\ [b_n(a_{n1}x_1^{(k+1)}+a_{n2}x_2^{(k+1)}+\ldots\ldots\ldots+a_{nn-1}x_{n-1}^{(k+1)})]$

$x_1^{(k+1)} = (1/a_{11})\ [b_1-(a_{12}x_2^{(k)}+a_{13}x_3^{(k)}+\ldots\ldots.+a_{1n}x_n^{(k)})]$

$x_2^{(k+1)}+a_{21}/a_{22}x_1^{(k+1)} = (1/a_{22})\ [b_2-(a_{21}x_1^{(k+1)}+a_{23}x_3^{(k)}+\ldots\ldots..+a_{2n}x_n^{(k)})]$

$x_n^{(k+1)}+a_{n1}/a_{nn}x_1^{(k+1)} = (1/a_{nn})\ [b_n(a_{n1}x_1^{(k+1)}+a_{n2}x_2^{(k+1)}+\ldots\ldots\ldots+a_{nn-1}x_{n-1}^{(k+1)})]$

$a_{11}x_1^{(k+1)} = [b_1-(a_{12}x_2^{(k)}+a_{13}x_3^{(k)}+\ldots\ldots.+a_{1n}x_n^{(k)})]$

$a_{22}x_2^{(k+1)} = [b_2-(a_{23}x_3^{(k)}+\ldots\ldots..+a_{2n}x_n^{(k)})]$

$a_{nn}x_n^{(k+1)} + a_{n1}x_1^{(k+1)}+\ldots\ldots\ldots..+a_{nn\text{-}}1x_{n-1}^{(k+1)}=b_n$

$(D+L)X^{(k+1)} = b+UX^{(K)}$

$X^{(k+1)} = (D+L)^{-1}b+(D+L)^{-1}UX^{(K)}$

$X^{(k+1)} = (D+L)^{-1}UX^{(K)}+(D+L)^{-1}b$

$X^{(K+1)} = HX^{(K)}+C$

Where $H =-(D+L)^{-1}U$

$C = (D+L)^{-1}b$

Computing (error is given)

$x^{(k+1)} \quad = x^{(k)}-x^{(k)}-(D+L)^{-1}Ux^{(k)}+(D+L)^{-1}b$

$x^{(k+1)}-x^{(k)} = [-I-(D+L)^{-1}U]x^{(k)}+(D+L)^{-1}b$

$= [-(D+L)(D+L)^{-1}-(D+L)^{-1}U]x^{(k)}+(D+L)^{-1}b$

$= -(D+L)^{-1}[-(D+L)+U]x^{(k)}+(D+L)^{-1}b$

$x^{(k+1)}-x^{(k)} = -(D+L)^{-1}Ax^{(K)}+(D+L)^{-1}b$

$v^{(k)} = (D+L)^{-1}[b-Ax^{(k)}]$

$= (D+L)^{-1}r^{(k)}$

$v^{(k)} = x^{(k+1)}-x^{(k)}$

$r^{(k)} = b-Ax^{(k)}$

$v^{(k)} = (D+L)^{-1}r^{(k)}$

Formulas to be Known:

1. For finding Iterative matrix $H = -(D+L)^{-1} U$ and $C = (D+L)^{-1} b$. Hence $x^{(k+1)} = Hx^{(k)} + c$.
2. If Error is given, $v^{(k)} = x^{(k+1)} - x^{(k)}$ and $r^{(k)} = b - A x^{(k)}$. Hence $v^{(k)} = (D+L)^{-1} r^{(k)}$

   Where $v^{(k)}$ is the error approximation.

   $r^{(k)}$ is the residual vector

1. Solve the system of equation $4x_1+x_2+x_3= 2$, $x_1+5x_2+2x_3= -6$, $x_1+2x_2+3x_3= -4$, using Gauss Jacobi & Gauss Seidel methods.

Solution:

Assume the initial approximation as $x^{(k)}= [0.5, -0.5, -0.5]^T$. Perform 3 iterations and hence obtain the Iteration matrix and the error approximation for the given system of equation.

The system of equation is diagonally dominant.

$$x_1^{(k+1)}= \frac{1}{4}[2 - x_2^{(k)} - x_3^{(k)}]$$

$x_2^{(k+1)} = \frac{1}{5}[-6 - x_1^{(k)} - 2x_3^{(k)}]$

$x_3^{(k+1)} = \frac{1}{3}[-4 - x_1^{(k)} - 2x_2^{(k)}]$

Jacobi:

Given, $x_1^{(0)} = 0.5$ $x_2^{(0)} = -0.5$ $x_3^{(0)} = -0.5$

1st Iteration: k=0

$x_1^{(1)} = \frac{1}{4}[2 - x_2^{(0)} - x_3^{(0)}] = 0.75$

$x_2^{(1)} = \frac{1}{5}[-6 - x_1^{(0)} - 2x_3^{(0)}] = -1.1$

$x_3^{(1)} = \frac{1}{3}[-4 - x_1^{(0)} - 2x_2^{(0)}] = -1.1667$

2nd Iteration: k=1

$x_1^{(2)} = \frac{1}{4}[2 - x_2^{(1)} - x_3^{(1)}] = 1.0667$

$x_2^{(2)} = \frac{1}{5}[-6 - x_1^{(1)} - 2x_3^{(1)}] = -0.8833$

$x_3^{(2)} = \frac{1}{3}[-4 - x_1^{(1)} - 2x_2^{(1)}] = -0.85$

3rd Iteration: k=2

$x_1^{(3)} = \frac{1}{4}[2 - x_2^{(2)} - x_3^{(2)}] = 0.9333$

$$x_2^{(3)} = \frac{1}{5}[-6 - x_1^{(2)} - 2x_3^{(2)}] = -1.0733$$

$$x_3^{(3)} = \frac{1}{3}[-4 - x_1^{(2)} - 2x_2^{(2)}] = -1.1000$$

Method 2:

Iterative Method:

$$x^{(k+1)} = Hx^{(k)} + c$$

$$H = -D^{-1}(L + U)$$

$$C = D^{-1}b$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L+U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$H = -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$H = -\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{5} & 0 & \frac{2}{5} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$x^{(k+1)} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{2}{3} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{-2}{3} & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix} \qquad = \begin{bmatrix} 0.25 \\ 0.1 \\ 0.1667 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1666 \end{bmatrix}$$

III iteration:

$$x^{(2)} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{-2}{3} & 0 \end{bmatrix} x^{(1)} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{-2}{3} & 0 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1666 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5667 \\ 0.31664 \\ 0.4833 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix} = \begin{bmatrix} 1.0667 \\ -0.8834 \\ -0.8500 \end{bmatrix}$$

III Iteration:

$$x^{(3)} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{-2}{3} & 0 \end{bmatrix} x^{(2)} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{5} & 0 & \frac{-2}{5} \\ \frac{-1}{3} & \frac{-2}{3} & 0 \end{bmatrix} \begin{bmatrix} 1.0667 \\ -0.8834 \\ -0.8500 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0.4334 \\ 0.1266 \\ 0.2334 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-6}{5} \\ \frac{-4}{3} \end{bmatrix} = \begin{bmatrix} 0.9334 \\ -1.0733 \\ -1.0993 \end{bmatrix}$$

Method 3:

$$v^{(k)} = x^{(k+1)} - x^{(k)}$$

$$x^{(k+1)} = x^{(k)} + v^{(k)}$$

$$v^{(k)} = D^{-1} r^{(k)}$$

$$r^{(k)} = b - Ax^{(k)}$$

When k=0

$$r^{(0)} = b - Ax^{(0)}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$v^{(0)} = D^{-1} r^{(0)}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{-3}{5} \\ \frac{-2}{3} \end{bmatrix}$$

$$x^1 = x^0 + v^0$$

$$= \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{-2}{5} \\ \frac{-2}{3} \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}$$

When k=1,

$$n^{(1)} = b - Ax^{(1)}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 0.7333 \\ -7.0834 \\ -4.9501 \end{bmatrix} = \begin{bmatrix} 1.2667 \\ 1.0834 \\ 0.9501 \end{bmatrix}$$

$$v^{(1)} = D^{-1}r^{(1)}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1.2667 \\ 1.0834 \\ 0.9501 \end{bmatrix} = \begin{bmatrix} 0.3167 \\ 0.2167 \\ 0.3167 \end{bmatrix}$$

$$x^2 = x^1 + v^1$$

$$= \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix} + \begin{bmatrix} 0.3167 \\ 0.2167 \\ 0.3167 \end{bmatrix} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.8500 \end{bmatrix}$$

When k=2,

$$n^{(2)} = b - Ax^{(2)}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.8500 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 2.5335 \\ -5.0498 \\ -3.2499 \end{bmatrix} = \begin{bmatrix} -0.5335 \\ -0.9502 \\ -0.7501 \end{bmatrix}$$

$$v^{(2)} = D^{-1}r^{(2)}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -0.5335 \\ -0.9502 \\ -0.7501 \end{bmatrix} = \begin{bmatrix} -0.1334 \\ -0.1900 \\ -0.2500 \end{bmatrix}$$

$$x^3 = x^2 + v^2$$

$$= \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.8500 \end{bmatrix} + \begin{bmatrix} -0.1334 \\ -0.1900 \\ -0.2500 \end{bmatrix} = \begin{bmatrix} 0.9333 \\ -1.0733 \\ -1.1000 \end{bmatrix}$$

Seidel:

Given, $x_1^{(0)} = 0.5$ $x_2^{(0)} = -0.5$ $x_3^{(0)} = -0.5$

1st Iteration: k=0

$$x_1^{(1)} = \frac{1}{4}[2 - x_2^{(0)} - x_3^{(0)}] = 0.75$$

$$x_2^{(1)} = \frac{1}{5}[-6 - x_1^{(1)} - 2x_3^{(0)}] = -1.15$$

$$x_3^{(1)} = \frac{1}{3}[-4 - x_1^{(1)} - 2x_2^{(1)}] = -0.8167$$

2nd Iteration: k=1

$$x_1^{(2)} = \frac{1}{4}[2 - x_2^{(1)} - x_3^{(1)}] = 0.9917$$

$$x_2^{(2)} = \frac{1}{5}[-6 - x_1^{(1)} - 2x_3^{(1)}] = -1.0717$$

$$x_3^{(2)} = \frac{1}{3}[-4 - x_1^{(1)} - 2x_2^{(1)}] = -0.9494$$

3rd Iteration: k=2

$$x_1^{(3)} = \frac{1}{4}[2 - x_2^{(2)} - x_3^{(2)}] = 1.0053$$

$$x_2^{(3)} = \frac{1}{5}[-6 - x_1^{(2)} - 2x_3^{(2)}] = -1.0218$$

$$x_3^{(3)} = \frac{1}{3}[-4 - x_1^{(2)} - 2x_2^{(2)}] = -0.9876$$

Method 2:

$$x^{(k+1)} = Px^{(k)} + q$$

$$P = -(D + L)^{-1}U$$

$$q = (D + L)^{-1}b$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D+L = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} & 0 \\ \frac{a_{21}a_{32} - a_{31}a_{22}}{a_{11}a_{22}a_{33}} & -\frac{a_{32}}{a_{22}a_{33}} & \frac{1}{a_{33}} \end{bmatrix}$$

$$(D+L)^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{-1}{20} & \frac{1}{5} & 0 \\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3} \end{bmatrix}$$

$$P = -(D+L)^{-1}U$$

$$= -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{-1}{20} & \frac{1}{5} & 0 \\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{20} & \frac{-7}{20} \\ 0 & \frac{1}{20} & \frac{19}{60} \end{bmatrix}$$

$$q = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{-1}{20} & \frac{1}{5} & 0 \\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.3 \\ -0.6333 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -13/10 \\ -19/30 \end{bmatrix}$$

1st Iteration: $k = 0 \quad X^{(1)} = PX^{(0)} + q \qquad\qquad X^{(k+1)} = PX^{(k)} + q$

$$X^{(1)} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{20} & \frac{-7}{20} \\ 0 & \frac{1}{20} & \frac{19}{60} \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -1.3 \\ -0.6333 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/20 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -13/10 \\ -19/30 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -23/20 \\ -17/15 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \\ -1.15 \\ -0.8166 \end{bmatrix}$$

2nd Iteration: $k = 1 \quad X^{(2)} = PX^{(1)} + q \qquad\qquad X^{(k+1)} = PX^{(k)} + q$

$$X^{1} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{20} & \frac{-7}{20} \\ 0 & \frac{1}{20} & \frac{19}{60} \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.15 \\ -0.8166 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -1.3 \\ -0.6333 \end{bmatrix} = \begin{bmatrix} 0.9917 \\ -1.0717 \\ -0.9494 \end{bmatrix}$$

3rd Iteration:

$$X^{2} = \begin{bmatrix} 0 & \frac{-1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{20} & \frac{-7}{20} \\ 0 & \frac{1}{20} & \frac{19}{60} \end{bmatrix} \begin{bmatrix} 0.9917 \\ -1.0717 \\ -0.9494 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -1.3 \\ -0.6333 \end{bmatrix} = \begin{bmatrix} 1.0053 \\ -1.0213 \\ -0.9876 \end{bmatrix}$$

Method 3:

$$\mathrm{v}^{(\mathrm{k})} = x^{(\mathrm{k}+1)} - ɲ^{(\mathrm{k})} x^{(k+1)} = x^{(\mathrm{k})} + \mathrm{v}^{(\mathrm{k})}$$

$$\mathrm{v}^{(\mathrm{k})} = (\mathrm{D} + \mathrm{L})^{-1} ɲ^{(\mathrm{k})} ɲ^{(\mathrm{k})} = \mathrm{b} - \mathrm{A}x^{(\mathrm{k})}$$

When k=0

$$\jmath^{(0)} = \mathrm{b} - \mathrm{A}x^{(0)}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$\mathrm{v}^{(0)} = (\mathrm{D} + \mathrm{L})^{-1} \jmath^{(0)}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{-1}{20} & \frac{1}{5} & 0 \\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -0.65 \\ -0.3167 \end{bmatrix}$$

$$x^1 = x^0 + \mathrm{v}^0$$

$$= \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ -0.65 \\ -0.3167 \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.15 \\ -0.8167 \end{bmatrix}$$

When k=1

$$\jmath^{(1)} = \mathrm{b} - \mathrm{A}x^{(1)}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.15 \\ -0.8167 \end{bmatrix} = \begin{bmatrix} 0.9667 \\ 0.6334 \\ 0.0001 \end{bmatrix}$$

$$\mathrm{v}^{(1)} = (\mathrm{D} + \mathrm{L})^{-1} \jmath^{(1)}$$

$$=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ \frac{-1}{20} & \frac{1}{5} & 0\\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0.9667\\0.6334\\0.0001\end{bmatrix}=\begin{bmatrix}0.2417\\0.0783\\-0.1528\end{bmatrix}$$

$$x^{(2)}=x^{(1)}+\mathrm{v}^{(1)}$$

$$=\begin{bmatrix}0.75\\-1.15\\-0.8167\end{bmatrix}+\begin{bmatrix}0.2417\\0.0783\\-0.1528\end{bmatrix}=\begin{bmatrix}0.9917\\-1.0717\\-0.9495\end{bmatrix}$$

When k=2

$$\jmath\!\jmath^{(2)}=\mathrm{b}-\mathrm{A}x^{(2)}$$

$$=\begin{bmatrix}2\\-6\\-4\end{bmatrix}-\begin{bmatrix}4 & 1 & 1\\1 & 5 & 2\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}0.9917\\-1.0717\\-0.9495\end{bmatrix}=\begin{bmatrix}0.0544\\0.2658\\0.0002\end{bmatrix}$$

$$\mathrm{v}^{(2)}=(\mathrm{D}+\mathrm{L})^{-1}\mathrm{r}^{(2)}$$

$$=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ \frac{-1}{20} & \frac{1}{5} & 0\\ \frac{-1}{20} & \frac{-2}{15} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0.0544\\0.2658\\0.0002\end{bmatrix}=\begin{bmatrix}0.0136\\0.0504\\-0.0381\end{bmatrix}$$

$$x^{(3)}=\jmath\!\jmath^{(2)}+\mathrm{v}^{(2)}$$

$$=\begin{bmatrix}0.9917\\-1.0717\\-0.9495\end{bmatrix}+\begin{bmatrix}0.0136\\0.0504\\-0.0381\end{bmatrix}=\begin{bmatrix}1.0053\\-1.0213\\-0.9876\end{bmatrix}$$

Successive Over Relaxation Method [SOR]

SOR is a method of solving a linear system of Equation AX=b. Derived by extrapolating the Gauss Seidel Method.

This extrapolation takes the form of the weighted average between the previous iterate and the computed Gauss Seidel iterate successively for each component

$x_i^k = w_{\bar{x_i}}^{(k)} + (1-w)x_i^{(k-1)}$ where $\bar{x}$ denotes the gauss seidal iterate and *w* is the extrapolation factor.

The idea is to choose the value for *w* that will accelerate the rate of convergent of the iterates to the solution

In matrix terms, SOR algorithm can be written as $x^k = (D-wL)^{-1}[wU + (1-w)D$ $x^{(k+1)} + w(D-wL)^{-1}b$ where the matrix D, L, U represent diagonal matrix, strictly lower triangle matrix, strictly upper triangle matrix A, respectively.

Note – 1:-

If w=1, the SOR simplifies to Gauss Seidel method.

Note – 2:-

At theorem due to Kahan shows that SOR fails to converge if w is outside the interval (0,2)

Note – 3:-

If w =0, there is no iteration

Note – 4:-

If 0 < w < 1, then it is under relaxation

Note – 5:-

If 1 < w < 2, or 0 < w < 2, then it is over relaxation.

Note – 6:-

If ≤ 2, then it is divergent.

Note – 7:-

Over relaxation methods are called succussive over relaxation [SOR]

Theorem:-

If a is positive definite and relaxation parameter w satisfying 0 < w < 2, then the SOR iteration converges for any initial vector x^o

1) Consider the linear system AX = b where

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} b = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

   a) Check that the SOR method with value w = 1.25 of the relaxation parameters can be used to solve the system.
   b) Compute the its iteration by the SOR method starting at the point $x^{(0)} = (0,0,0)^T$

Solution:-

   a) All leading principal minor are positive and so the Matrix a is positive Definite. W.K.T the SOR method converges for 0 < w < 2.
      Therefore w = 1.25 can be used to solve the system.
   b) Write the system of an Equation

$$3x_1 - x_2 + x_3 = -1$$
$$-x_1 + 3x_2 + x_3 = 7$$
$$x_1 - x_2 + 3x_3 = -7$$

by Gauss Seidel Method,

$$x_1^{(k+1)} = \frac{1}{3}[-1 + x_2^{(k)} - x_3^{(k)}]$$
$$x_2^{(k+1)} = \frac{1}{3}[7 + x_1^{(k+1)} + x_3^{(k)}]$$
$$x_3^{(k+1)} = \frac{1}{3}[-7 - x_1^{(k+1)} + x_2^{(k+1)}]$$
$$x_3^{(k+1)} = wx_i^{(k+1)} + (1 - w)x_i^{(k)}]$$

$$x_1^{(k+1)} = \frac{w}{3}\left[-1 + x_2^{(k)} - x_3^{(k)}\right] + (1 - w)x_1^{(k)}$$
$$x_2^{(k+1)} = \frac{w}{3}\left[7 + x_1^{(k+1)} + x_3^{(k)}\right] + (1 - w)x_2^{(k)}$$
$$x_3^{(k+1)} = \frac{w}{3}\left[-7 - x_1^{(k+1)} + x_2^{(k+1)}\right] + (1 - w)x_3^{(k)}$$

Put k = 0

$x_1^{(1)} = -0.4167$
$x_2^{(1)} = 2.7430$
$x_3^{(1)} = -1.6001$

Put k = 1
$x_1^{(2)} = 1.4972$
$x_2^{(2)} = 3.5218$
$x_3^{(2)} = -1.6732$

Put k = 2
$x_1^{(3)} = 1.3738$
$x_2^{(3)} = 1.9117$
$x_3^{(3)} = -2.2745$

$k = 3, x_1^{(4)} = 0.9842$
$x_2^{(4)} = 1.9013$
$x_3^{(4)} = -1.9661$

$k = 4, x_1^{(5)} = 0.9488$
$x_2^{(5)} = 2.0176$
$x_3^{(5)} = -1.9299$

EXAMPLE – 2 Solve the system

$$10x + 2y - z = 7$$

$$1x + 8y + 3z = -4$$

$$-2x - y + 10z = 9$$

Using Jacobi, Gauss-Seidel and Successive Over-Relaxation methods.

SOLUTION

Given

$$10x + 2y - z = 7$$

$$1x + 8y + 3z = -4$$

$$-2x - y + 10z = 9$$

The above equations can be written as the matrix form

$$AX = B$$

Let

$$\begin{pmatrix} 10 & 2 & -1 \\ 1 & 8 & 3 \\ -2 & -1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 9 \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & 2 & -1 \\ 1 & 8 & 3 \\ -2 & -1 & 10 \end{pmatrix}$$

the given matrix A is diagonally dominant (i.e $|a_{ii}| \geq \sum |a_{ij}|$), (ie$10 \geq 3$, $8 \geq 4$, $10 \geq 3$) hence, we apply the above said these iterative methods

To solve these equations by iterative methods, we are rewrite them as follows,

$$x = \frac{1}{10}(7 - 2y + z)$$

$$y = \frac{1}{8}(-4 - x - 3z)$$

$$z = \frac{1}{10}(9 + 2x + y)$$

The results are given in the table 2(a)

Table 2(a) – Number of iterations of the iterative methods

JACOBI METHOD				GAUSS-SEIDEL METHOD			SOR METHOD		
Iterations	X	Y	Z	x	Y	Z	X	Y	Z
0	0	0	0	0	0	0	0	0	0
1	0.7	-0.5	0.9	0.7	-0.5875	0.98125	0.77	-0.55	0.99
2	0.89	-0.925	0.99	0.9156	-0.9824	0.98488	0.9229	-1.0303	0.9807
3	0.984	-0.9825	0.9855	0.995	-0.9937	0.9996			
4	0.9951	-0.9926	0.9986	0.9987	-0.9997	1.1997			
5	0.9984	-0.9989	0.9998						
6	0.99976	-0.9997	0.9998						
7	0.9999	-0.9999	0.9999						
8	0.9999	-0.9999	0.9999						
9	0.9999	-0.9999	0.9999						
10	0.9999	-0.9999	0.9999						

Table -2(b) Number of iterations for the SOR, GAUSS-SEIDEL AND JACOBI ITERATIVE METHODS

METHODS	NUBMER OF ITERATIONS
SOR METHOD	2
GAUSS SEIDEL METHOD	4
JACOBI METHOD	10

Example1

$$A=\begin{bmatrix}16 & 3\\ 7 & -11\end{bmatrix}, b=\begin{bmatrix}11\\ 3\end{bmatrix} X^{(0)}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$the matrix is strictly\ diagonally do\min ant but not\ positive definite$$

$$the\,solution\ \ X=\begin{bmatrix}0.8122\\ -0.6650\end{bmatrix}$$

Seidel Iteration	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$		
x_1	**0.5000**	**0.8494**	**0.8077**	**0.8127**	**0.8121**	**0.8122**		
x_2	**-0.8636**	**-0.6413**	**-0.6678**	**-0.6646**	**-0.6650**	**-0.6650**		

$$A=\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}, b=\begin{bmatrix}11\\ 3\end{bmatrix} X^{(0)}=\begin{bmatrix}1.1\\ 2.3\end{bmatrix}$$

$$the matrix is\ neither\ \ diagonally do\min ant\ nor\ \ positive definite$$

$$the\,convergence is not guaranted\ in\,this\ case$$

Seidel Iteration	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$
x_1	**0.6**	**2.32727**	**-0.98727**	**0.878864**
x_2	**1.03018**	**2.03694**	**-1.01446**	**0.984341**
x_3	**1.00659**	**2.00356**	**-1.00253**	**0.998351**

x_4	**1.00086**	**2.00030**	**-1.00031**	**0.99985**

Exact solution (1,2,-1,1)

$$A=\begin{bmatrix}10 & -1 & 2 & 0\\ -1 & 11 & -1 & 3\\ 2 & -1 & 10 & -1\\ 0 & 3 & -1 & 8\end{bmatrix}, b=\begin{bmatrix}6\\ 25\\ -11\\ 15\end{bmatrix} X^{(0)}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

the matrix is diagonally dominant

the convergence is guaranted in this case

Seidel Iteration	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$
x_1	**2.050**	**4.911**	**0.8077**	
x_2	**0.393**	**-0.6413**	**-1.651**	

Power Method:

Power method is normally used to determine the largest Eigen value [in magnitude] and its Eigen vector of the system $AX = \lambda X$

Let $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots \ldots . \lambda_n$ be the distinct Eigen values such that $|\lambda_1| > |\lambda_2| \ldots .. |\lambda_n|$ and $v_1, v_2, \ldots \ldots . v_n$ be the corresponding Eigen vectors, then we have the algorithm has

$$Y_{k+1} = AV_k$$

$$V_{k+1} = \frac{Y_{k+1}}{m_{k+1}}$$

Where $m_{k+1} = \max\limits_{r} |(y_{k+1})_r|$

Then $\lambda_1 = \lim\limits_{k\to\infty} \frac{(y_{k+1})_r}{(v_k)_r}; r = 1,2,3 \ldots \ldots \ldots n$

$V_{k+1} = Eigen\ vector$

Remark:

In fact, Max {|λj|: j =1, 2, …, n} is called the spectral radius of the matrix A which is of great importance in Mathematics. The Power method works under the assumptions: 1. The matrix A has n eigenvalues λ1, λ2, …, λn such that |λ1| > |λ2| > |λ3| > …. > |λn|. 2. The matrix A has n linearly independent eigenvectors {x1, x2, …, xn}. This will be possible if A has all n distinct eigenvalues or if A is symmetric. However, these restrictions on A are not essential.

Inverse Power method works under the following assumptions: 1. A is an n x n real non-singular (invertible) matrix. 2. A has n eigenvalues λ1, λ2, …, λn such that |λ1| < |λ2| < |λ3| < …. < |λn|. 3. The matrix A has n linearly independent eigenvectors {x1, x2, …, xn}.

(Inverse Power method): Iterative steps: 1. Use Gauss- Jorden method to compute A-1 from augmented matrix [A|I]. 2. Use power method on A-1 to find its dominant eigenvalue λ and corresponding eigenvector. 3. Then the eigenvalue of A with minimum absolute value is 1/ λ and corresponding eigenvector remaining same.

If the initial column vector $X_0$is an eigen vector of A other than that corresponding to the dominant eigen eigenvalue then the method fails.

The speed of convergence of the power method depends on the ratio

$$\left|\frac{\lambda_{\max}}{\lambda_{next\ \max}}\right| = \begin{cases} small \rightarrow converge\, slowly \\ nearer\, to\, 1 \rightarrow converge\ fastly \end{cases}$$

The power method gives only dominant eigen value

Problem 1: Find the dominant Eigen value and corresponding Eigen vector of the matrix

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

Solution:

Assume $V_0 = [1 \quad 1 \quad 1]^T$

$Y_{k+1} = AV_k$

$If \quad k = 0, y_1 = AV_0$

$$y_1 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 42 \\ 5 \end{bmatrix}$$

$$m_{k+1} = \max |(y_{k+1})_r|$$

$$k = 0, \qquad m_1 = \max \ |(y_1)_r|$$

$$m_1 = 42$$

$$V_{k+1} = \frac{y_{k+1}}{m_{k+1}}$$

$$V_1 = \frac{y_1}{m_1} = \begin{bmatrix} 0.1190 \\ 1 \\ 0.1190 \end{bmatrix} = \frac{\begin{bmatrix} 5 \\ 42 \\ 5 \end{bmatrix}}{42}$$

$$if \ \ k = 1,$$

$$y_2 = AV_1 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.1190 \\ 1 \\ 0.1190 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 1.4760 \\ 40.2380 \\ 1.4760 \end{bmatrix}$$

$$k = 1 \ in \ the \ equation$$

$$m_2 = \max \ |(y_2)_r| = 40.2380$$

$$k = 1 \ in \ the \ equation$$

$$V_2 = \frac{Y_2}{m_2} = \begin{bmatrix} 0.0367 \\ 1 \\ 0.0367 \end{bmatrix}$$

$$if \ \ k = 2,$$

$$m_3 = \max \ |(y_2)_r| = 40.0734$$

$$V_3 = \frac{Y_3}{m_3} = \begin{bmatrix} 0.0286 \\ 1 \\ 0.0286 \end{bmatrix}$$

$$if \ \ k = 3,$$

$$y_4 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.0286 \\ 1 \\ 0.0286 \end{bmatrix} = \begin{bmatrix} 1.1144 \\ 40.0572 \\ 1.1144 \end{bmatrix}$$

$$m_4 = 40.0572$$

$$V_4 = \begin{bmatrix} 0.0278 \\ 1 \\ 0.0278 \end{bmatrix}$$

$if \;\; k = 4,$

$$y_5 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.0278 \\ 1 \\ 0.0278 \end{bmatrix} = \begin{bmatrix} 1.1112 \\ 40.0556 \\ 1.1112 \end{bmatrix}$$

$m_5 = 40.0556$

$$V_5 = \begin{bmatrix} 0.0277 \\ 1 \\ 0.0277 \end{bmatrix}$$

$$\lambda_i: ratios = \frac{(y_5)_r}{(v_4)_r}, \frac{y_5}{(v_4)_r}, \frac{(y_5)_r}{(v_4)_r}$$

$$= \frac{1.11120}{0.0278}, \frac{40.0556}{1}, \frac{1.11120}{0.0278}$$

$$= (39.9712, 40.556, 39.9712)$$

Error = 39.9712 – 40.0556

= 0.0844

Largest Eigen value = 40.0556

Corresponding Eigen vector = $V_5 = (0.0277 \quad 1 \quad 0.0277)$

Example 1: Analysis the convergence of power method

$$A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

The eigen values are $\lambda_1 = 1, \lambda_2 = -1$

Here $|\lambda_1| = |\lambda_2|$ but not $|\lambda_1| > |\lambda_2|$ hence no dominant eigen value

Example 2: Analysis the convergence of power method

$$A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The eigen values are $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 1$

Here $|\lambda_1| = |\lambda_2| > |\lambda_3|$ but not $|\lambda_1| > |\lambda_2| > |\lambda_3|$ hence A has no dominant eigen value

Applying power method to find dominant eigen value and corresponding eigen vector of the matrix

$$A=\begin{bmatrix}1 & 2 & 0\\ -2 & 1 & 2\\ 1 & 3 & 1\end{bmatrix} \quad X^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$Y_{(1)} = AX^{(0)} = \begin{bmatrix}1 & 2 & 0\\ -2 & 1 & 2\\ 1 & 3 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\5\end{bmatrix} = 5\begin{bmatrix}0.6\\0.20\\1\end{bmatrix} = \lambda\ X$$

$$X^{(1)} = \begin{bmatrix}0.6\\0.20\\1\end{bmatrix}$$

$$Y_{(2)} = AX^{(1)} = \begin{bmatrix}1 & 2 & 0\\ -2 & 1 & 2\\ 1 & 3 & 1\end{bmatrix}\begin{bmatrix}0.6\\0.20\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2.2\end{bmatrix} = 2.2\begin{bmatrix}0.45\\0.45\\1\end{bmatrix} = \lambda\ X$$

$$X^{(2)} = \begin{bmatrix}0.45\\0.45\\1\end{bmatrix}$$

$$Y_{(3)} = AX^{(1)} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.45 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.36 \\ 1.55 \\ 2.82 \end{bmatrix} = 2.82 \begin{bmatrix} 0.48 \\ 0.55 \\ 1 \end{bmatrix} = \lambda \ X$$

$$X^{(3)} = \begin{bmatrix} 0.48 \\ 0.55 \\ 1 \end{bmatrix}$$

$$Y_{(4)} = AX^{(3)} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.55 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.58 \\ 1.58 \\ 3.13 \end{bmatrix} = 3.13 \begin{bmatrix} 0.51 \\ 0.51 \\ 1 \end{bmatrix} = \lambda \ X$$

$$X^{(4)} = \begin{bmatrix} 0.51 \\ 0.51 \\ 1 \end{bmatrix}$$

$$Y_{(6)} = AX^{(5)} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.49 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.49 \\ 1.49 \\ 2.99 \end{bmatrix} = 2.99 \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} = \lambda \ X$$

$$X^{(6)} = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

$$Y_{(7)} = AX^{(6)} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} = \lambda \ X$$

$$X^{(7)} = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

	Approximation						
Iteration	$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$
x_1	**1**	**0.6**	**0.45**	**0.48**	**0.51**	**0.50**	**0.50**
x_2	**1**	**0.20**	**0.45**	**0.55**	**0.51**	**0.49**	**0.50**
x_3	**1**	**1.0**	**1.0**	**1.0**	**1.0**	**1.0**	**1.0**
λ		**5.0**	**2.2**	**2.82**	**3.13**	**3.02**	**2.99**

The dominant eigenvalue $\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r} \right] = \left(\frac{1.5}{0.5}, \frac{1.5}{0.5}, \frac{3}{1} \right) = 3$

The dominant eigen vector $X = \begin{bmatrix} 0.50 \\ 0.50 \\ 1 \end{bmatrix}$ dominant eigen value $\lambda = 3$

To find numerically smallest eigen value and its eigen vector of A

The dominant eigen vector and dominant eigen value of A^{-1}

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} -5 & -2 & 4 \\ 4 & 1 & -2 \\ -7 & -1 & 5 \end{bmatrix} X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Y_{(1)} = A^{-1}X^{(0)} = \begin{bmatrix} -5/3 & -2/3 & 4/3 \\ 4/3 & 1/3 & -2/3 \\ -7/3 & -1/3 & 5/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \lambda\ X$$

$$X^{(1)} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

The dominant eigen vector $X \cong \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ dominant eigen value $\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r}\right] = (\quad) \Rightarrow \lambda = 1$

Note: $\lambda^3 - 3\lambda^2 + \lambda + 3 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = i, \lambda_3 = -i$

Example 3: Analyse the convergence of power method

$$A = \begin{bmatrix} 4 & 5 \\ 6 & 5 \end{bmatrix}$$

The eigen values are $\lambda_1 = 10, \lambda_2 = -1$

Here $|\lambda_1| > |\lambda_2| \Rightarrow A$ has a dominant eigen value

$\frac{|\lambda_1|}{|\lambda_2|} = 0.1\, which\, is\, small \Rightarrow$ power method converges quickly (within 6 iterations).

Example 3: Analyse the convergence of power method

$$A = \begin{bmatrix} -4 & 10 \\ 7 & 5 \end{bmatrix}$$

The eigen values are $\lambda_1 = 10, \lambda_2 = -9$

Here $|\lambda_1| > |\lambda_2| \Rightarrow A$ has a dominant eigen value

$\dfrac{|\lambda_1|}{|\lambda_2|} = 0.9 \, which \; is \, nearer \, to \, 1 \Rightarrow$ power method converges slowly (after 68 iterations)

Problem 4: Applying power method to find dominant eigen value and corresponding eigen

solution: vector of the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}$ $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y_{(1)} = AX^{(0)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda\ X_1 \ where X^{(1)} = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

$$Y_{(2)} = AX^{(1)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -4 \\ 0.5 \end{bmatrix} = 4\begin{bmatrix} 0.875 \\ -1 \\ 0.125 \end{bmatrix} = \lambda\ X_2 \ where\ X^{(2)} = \begin{bmatrix} 0.875 \\ -1 \\ 0.125 \end{bmatrix}$$

$$Y_{(3)} = AX^{(2)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.875 \\ -1 \\ 0.125 \end{bmatrix} = \begin{bmatrix} 3.625 \\ -6.125 \\ 1.125 \end{bmatrix} = 6.125\begin{bmatrix} 0.5918 \\ -1 \\ 0.1837 \end{bmatrix} = \lambda\ X_3 \ where\ X^{(3)} = \begin{bmatrix} 0.5918 \\ -1 \\ 0.1837 \end{bmatrix}$$

$$Y_{(4)} = AX^{(3)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.5918 \\ -1 \\ 0.1837 \end{bmatrix} = \begin{bmatrix} 2.7755 \\ -5.7347 \\ 0.1837 \end{bmatrix} = 5.7347\begin{bmatrix} 0.4840 \\ -1 \\ 0.2064 \end{bmatrix} = \lambda\ X_4 \ where\ X^{(4)} = \begin{bmatrix} 0.4840 \\ -1 \\ 0.2064 \end{bmatrix}$$

$$Y_{(5)} = AX^{(4)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.4840 \\ -1 \\ 0.2064 \end{bmatrix} = \begin{bmatrix} 2.452 \\ -5.5872 \\ 1.2064 \end{bmatrix} = 5.5872\begin{bmatrix} 0.4389 \\ -1 \\ 0.2159 \end{bmatrix} = \lambda\ X$$

$$Y_{(6)} = AX^{(5)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.4389 \\ -1 \\ 0.2159 \end{bmatrix} = \begin{bmatrix} 2.3166 \\ -5.5255 \\ 1.2159 \end{bmatrix} = 5.5255\begin{bmatrix} 0.4193 \\ -1 \\ 0.2201 \end{bmatrix} = \lambda\ X$$

$$Y_{(11)} = AX^{(10)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.4043 \\ -1 \\ 0.2232 \end{bmatrix} = \begin{bmatrix} 2.2128 \\ -5.4782 \\ 1.2232 \end{bmatrix} = 5.4782\begin{bmatrix} 0.4039 \\ -1 \\ 0.2233 \end{bmatrix} = \lambda\ X$$

$$Y_{(12)} = AX^{(11)} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.4039 \\ -1 \\ 0.2233 \end{bmatrix} = \begin{bmatrix} 2.2118 \\ -5.4777 \\ 1.2233 \end{bmatrix} = 5.4777\begin{bmatrix} 0.4038 \\ -1 \\ 0.2233 \end{bmatrix} = \lambda\ X$$

	Approximation								
Iteration	$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$	$X^{(12)}$	$X^{(13)}$
x_1	**1**	**1**	**0.875**	**0.5918**	**0.4840**	**0.4389**	**0.4193**	**0.4038**	**0.4037**
x_2	**1**	**-0.5**	**-1**	**-1**	**-1**	**-1**	**-1**	**-1**	**-1**
x_3	**1**	**0**	**0.125**	**0.1837**	**0.2064**	**0.2159**	**0.2201**	**0.2233**	**0.2233**
λ		**2**	**4**	**6.125**	**5.7347**	**5.5872**	**5.255**	**5.4777**	**5.4775**

The dominant eigen vector $X = \begin{bmatrix} 0.4037 \\ -1 \\ 0.2233 \end{bmatrix} \cong \begin{bmatrix} 0.40 \\ -1 \\ 0.22 \end{bmatrix}$ dominant eigen value

$$\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r} \right] = \left(\frac{2.2118}{0.4039}, \frac{-5.4777}{-1}, \frac{1.2233}{0.2233} \right) \Rightarrow \lambda = 5.48$$

To find numerically smallest eigen value and its eigen vector of A

The dominant eigen vector and dominant eigen value of A^{-1}

$$A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \quad X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Y_{(1)} = A^{-1}X^{(0)} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 0.33 \\ 0.93 \\ 1 \end{bmatrix} = \lambda\ X\ where X^{(1)} = \begin{bmatrix} 0.33 \\ 0.93 \\ 1 \end{bmatrix}$$

$$Y_{(2)} = A^{-1}X^{(1)} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.33 \\ 0.93 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.27 \\ 12.47 \\ 13.47 \end{bmatrix} = 13.47 \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix} = \lambda\ X\ where\ X^{(2)} = \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix}$$

$$Y_{(3)} = A^{-1}X^{(2)} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.24 \\ 12.41 \\ 13.41 \end{bmatrix} = 13.41 \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix} = \lambda\ X\ where\ X^{(3)} = \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix}$$

The dominant eigen vector $X \cong \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix}$ dominant eigen value of A^{-1}

$$\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r} \right] = \left(\frac{4.24}{0.32}, \frac{12.41}{0.93}, \frac{13.41}{1} \right) \Rightarrow \lambda = 13.41$$

Hence numerically smallest eigen value and its eigen vector of A $= \frac{1}{\lambda} = \frac{1}{13.41} = 0.0746$

Note: $\lambda^3 - 8\lambda^2 + 14\lambda - 1 = 0 \Rightarrow \lambda_1 = 0.0746, \lambda_2 = 2.4481, \lambda_3 = 5.4774$

Applying power method to find dominant eigen value and corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} X^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

solution: $Y_{(k+1)} = A\ X_{(k)}$

$$Y_{(1)} = A\ X_{(0)} = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda\ X_1\ where\ X^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Y_{(2)} = A\ X_{(0)} = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix} = 8\begin{bmatrix} -1 \\ 0.75 \end{bmatrix} = \lambda\ X_2\ where\ X^{(2)} = \begin{bmatrix} -1 \\ 0.75 \end{bmatrix}$$

$$Y_{(3)} = A\ X_{(2)} = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} -6.75 \\ 5 \end{bmatrix} = 6.75\begin{bmatrix} -1 \\ 0.74 \end{bmatrix} = \lambda\ X_3\ where\ X^{(3)} = \begin{bmatrix} -1 \\ 0.74 \end{bmatrix}$$

$$Y_{(4)} = A\ X_{(3)} = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0.74 \end{bmatrix} = \begin{bmatrix} -6.7 \\ 4.96 \end{bmatrix} = 6.7\begin{bmatrix} -1 \\ 0.74 \end{bmatrix} = \lambda\ X_4\ where\ X^{(4)} = \begin{bmatrix} -1 \\ 0.74 \end{bmatrix}$$

The dominant eigen vector $X = \begin{bmatrix} -1 \\ 0.74 \end{bmatrix}$ dominant eigen value

$$\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r}\right] = \left(\frac{-6.7}{-1}, \frac{4.96}{0.74}\right) \Rightarrow \lambda \cong 6.7$$

To find numerically smallest eigen value and its eigen vector of A

The dominant eigen vector and dominant eigen value of A^{-1}

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} X^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y_{(1)} = A^{-1}X^{(0)} = \begin{bmatrix} 2 & 5/2 \\ 1 & 3/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 2.5 \end{bmatrix} = 4.5\begin{bmatrix} 1 \\ 0.56 \end{bmatrix} = \lambda\ X_1\ where\ X_1 = \begin{bmatrix} 1 \\ 0.56 \end{bmatrix}$$

$$Y_{(2)} = A^{-1}X^{(1)} = \begin{bmatrix} 2 & 5/2 \\ 1 & 3/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.56 \end{bmatrix} = \begin{bmatrix} 3.39 \\ 1.83 \end{bmatrix} = 3.39\begin{bmatrix} 1 \\ 0.54 \end{bmatrix} = \lambda\ X_2\ where\ X_2 = \begin{bmatrix} 1 \\ 0.54 \end{bmatrix}$$

$$Y_{(3)} = A^{-1}X^{(2)} = \begin{bmatrix} 2 & 5/2 \\ 1 & 3/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 3.35 \\ 1.81 \end{bmatrix} = 3.35\begin{bmatrix} 1 \\ 0.54 \end{bmatrix} = \lambda\ X_3 \ where\ X_3 = \begin{bmatrix} 1 \\ 0.54 \end{bmatrix}$$

The dominant eigen vector $X \cong \begin{bmatrix} 1 \\ 0.54 \end{bmatrix}$ dominant eigen value of A^{-1}

$$\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r}\right] = \left(\frac{3.35}{1}, \frac{1.81}{0.54}\right) \Rightarrow \lambda = 3.35$$

Hence numerically smallest eigen value and its eigen vector of A $= \frac{1}{\lambda} = \frac{1}{3.35} = 0.2983$

Note: $\lambda^2 - 7\lambda + 2 = 0 \Rightarrow \lambda_1 = 0.2984, \lambda_2 = 6.7016$

Problem 5: Find the numerically smallest eigen value of the matrix $A = \begin{pmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{pmatrix}$

i) by finding A^{-1}

ii) Without finding A^{-1} given that the numerically largest eigen value of A is -20

$$|A| = \begin{vmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{vmatrix}$$

$$= (-15)(-24 + 24) - 4(20 - 120) + 3(-40 + 240)$$
$$= 0 + 400 + 600 = 1000$$

$$A^{-1} = \frac{1}{1000}\begin{pmatrix} 0 & -20 & 60 \\ 100 & -90 & 120 \\ 200 & 20 & 140 \end{pmatrix} = \begin{pmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{pmatrix}$$

$$let\ X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Then\ A^{-1}X_0 = \begin{pmatrix} 0.04 \\ 0.13 \\ 0.36 \end{pmatrix} = 0.361\begin{pmatrix} 0.111 \\ 0.361 \\ 1 \end{pmatrix} = 0.361\ X_1 \text{where } X_1 = \begin{pmatrix} 0.111 \\ 0.361 \\ 1 \end{pmatrix}$$

$$A^{-1}X_1 = \begin{pmatrix} 0.053 \\ 0.099 \\ 0.169 \end{pmatrix} = 0.169\begin{pmatrix} 0.314 \\ 0.586 \\ 1 \end{pmatrix} = 0.169\ X_2$$

$$A^{-1}X_2 = 0.215\ X_3$$

$$A^{-1}X_3 = 0.194\ X_4$$

$$A^{-1}X_4 = 0.203\ X_5$$

$$A^{-1}X_5 = 0.199\,X_6$$

$$A^{-1}X_6 = 0.200\,X_7$$

$$A^{-1}X_7 = 0.200\,X_8$$

Since $X_7 = 0.2 = X_8$ convergence has occurred the dominant eigen value of $A^{-1} = 0.2 = \frac{1}{5}$

Therefore, the Numerically smallest eigen value of A=5.

To find the Numerically smallest eigen value of A [without finding A^{-1}]

we find the dominant eigen value of B = A – λI

Given largest eigen value λ of A = -20 (ie) $\lambda = -20$

Therefore B = A+20I

$$B = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 \\ 10 & 8 & 6 \\ 20 & -4 & 22 \end{bmatrix}$$

$$let\ X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$BX_0 = \begin{pmatrix} 12 \\ 24 \\ 38 \end{pmatrix} = 38\begin{pmatrix} 0.3158 \\ 0.6316 \\ 1 \end{pmatrix} = 38X_1 \text{ where } X_1 = \begin{pmatrix} 0.3158 \\ 0.6316 \\ 1 \end{pmatrix}$$

$$BX_1 = \begin{pmatrix} 7.1054 \\ 14.2108 \\ 25.8616 \end{pmatrix} = 25.8618\begin{pmatrix} 0.2747 \\ 0.5495 \\ 1 \end{pmatrix} = 25.8616\,X_2$$

$$BX_2 = 25.2960\,X_3$$

$$BX_3 = 25.1176\,X_4$$

$$BX_4 = 25.0468\,X_5$$

$$BX_5 = 25.0196\,X_6$$

$$BX_6 = 25.0072\,X_7$$

$$BX_7 = 25.0020\,X_8$$

$$BX_8 = 25.0012\,X_9$$

$BX_9 = 24.9996\, X_{10}$

$BX_{10} = 25\, X_{11}$

$BX_{11} = 25\, X_{12}$

Since, $25X_{11} = 25X_{12}$, the dominant eigen value of B =25.

Therefore, The numerically smallest eigen value of $A = 25 + (-20) = 5$

solution: vector of the matrix $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y_{(1)} = AX^{(0)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 18 \end{bmatrix} = 18 \begin{bmatrix} -0.44 \\ 0.22 \\ 1 \end{bmatrix} = \lambda\ X$$

$$X^{(1)} = \begin{bmatrix} -0.44 \\ 0.22 \\ 1 \end{bmatrix}$$

$$Y_{(2)} = AX^{(1)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.44 \\ 0.22 \\ 1 \end{bmatrix} = \begin{bmatrix} 10.56 \\ -1.11 \\ -7.78 \end{bmatrix} = 10.56 \begin{bmatrix} 1 \\ -0.11 \\ -0.74 \end{bmatrix} = \lambda\ X$$

$$Y_{(3)} = AX^{(2)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.11 \\ -0.74 \end{bmatrix} = \begin{bmatrix} -17.63 \\ 6.84 \\ 18.95 \end{bmatrix} = 18.95 \begin{bmatrix} -0.93 \\ 0.36 \\ 1 \end{bmatrix} = \lambda\ X$$

$$Y_{(4)} = AX^{(3)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0.93 \\ 0.36 \\ 1 \end{bmatrix} = \begin{bmatrix} 18.4 \\ -7.64 \\ -18.06 \end{bmatrix} = 18.4 \begin{bmatrix} 1 \\ -0.42 \\ -0.98 \end{bmatrix} = \lambda\ X$$

$$Y_{(5)} = AX^{(4)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.42 \\ -0.98 \end{bmatrix} = \begin{bmatrix} -19.6 \\ 9.09 \\ 19.7 \end{bmatrix} = 19.7 \begin{bmatrix} -1 \\ 0.46 \\ 1 \end{bmatrix} = \lambda\ X$$

$$Y_{(6)} = AX^{(5)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 19.77 \\ -9.49 \\ -19.75 \end{bmatrix} = 19.77 \begin{bmatrix} 1 \\ -0.48 \\ -1 \end{bmatrix} = \lambda\ X$$

$$Y_{(11)} = AX^{(10)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} -20 \\ 9.99 \\ 20 \end{bmatrix} = 20 \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = \lambda\ X$$

$$Y_{(12)} = AX^{(11)} = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ -9.99 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \lambda\ X$$

The dominant eigen vector $X \cong \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$ dominant eigen value of A

$$\lambda = \left[\frac{\left(AX^{(k)}\right)_r}{\left(X^{(k)}\right)_r} \right] = \left(\frac{20}{-1}, \frac{-9.99}{0.5}, \frac{-20}{1} \right) \Rightarrow \lambda = -20$$

Note: $\lambda^3 + 25\lambda^2 + 50\lambda\ - 1000 = (\lambda - 5)(\lambda + 20)(\lambda + 10)$

Determine all the eigen values of the matrix A using power method and method of deflation

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

$$Y_{(1)} = AX_{(0)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(1)}$$

$$Y_{(2)} = AX_{(1)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 0.07 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(2)}$$

$$Y_{(3)} = AX_{(2)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.64 \\ 7.86 \end{bmatrix} = 7.86 \begin{bmatrix} -0.21 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(3)}$$

$$Y_{(4)} = AX_{(3)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} -0.21 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.05 \\ 8.42 \end{bmatrix} = 8.42 \begin{bmatrix} -0.36 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(4)}$$

$$Y_{(4)} = AX_{(3)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} -0.36 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.81 \\ 8.42 \end{bmatrix} = 8.72 \begin{bmatrix} -0.44 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(5)}$$

$$Y_{(6)} = AX_{(5)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} -0.44 \\ 1 \end{bmatrix} = \begin{bmatrix} -4.18 \\ 8.87 \end{bmatrix} = 8.87 \begin{bmatrix} -0.47 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(6)}$$

$$Y_{(7)} = AX_{(6)} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} -0.47 \\ 1 \end{bmatrix} = \begin{bmatrix} -4.36 \\ 8.94 \end{bmatrix} = 8.94 \begin{bmatrix} -0.49 \\ 1 \end{bmatrix} = \lambda_{(0)} X_{(7)}$$

	Approximation							
Iteration	$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$	$X^{(12)}$	$X^{(13)}$
x_1	**1**	**0.5**	**0.07**	**-0.21**	**-0.36**	**-0.44**	**-0.47**	**-0.5**	**-0.5**
x_2	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
λ		**6**	**7**	**7.86**	**8.42**	**8.72**	**8.87**	**9**	**9**

By Method of deflation

$$A_{(1)} = A - \lambda uu^T \ where\, u = \frac{X}{\|X\|} = \frac{}{\sqrt{\ }} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow A_{(1)} = \begin{bmatrix} 5 & -2 \\ 2 & 8 \end{bmatrix} - \frac{9}{\sqrt{5}\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}^T =$$

$$\Rightarrow A_{(1)} = \begin{bmatrix} 3.2 & 1.6 \\ 1.6 & 0.8 \end{bmatrix}$$

$$Y_{(1)} = A_{(1)} X_{(0)} = \begin{bmatrix} 3.2 & 1.6 \\ 1.6 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 2.4 \end{bmatrix} = 4.8 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \lambda_{(0)} X_{(1)}$$

$$Y_{(1)} = A_{(1)} X_{(0)} = \begin{bmatrix} 3.2 & 1.6 \\ 1.6 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \lambda_{(0)} X_{(1)}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda^3 - 3\lambda^2 + 3\lambda \ = 0 \Rightarrow \lambda_1 \ = 0, \lambda_2 =, \lambda_3 =$$

$$A = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix}$$

Note: $\lambda^3 - 8\lambda^2 + 14\lambda \ - 1 = 0 \Rightarrow \lambda_1 \ = 0.0746, \lambda_2 \ = 2.4481, \lambda_3 \ = 5.4774$

UNIT – III - ADVANCED NUMERICAL METHODS – SMTA5304

UNIT III

Interpolation And Approximation

Hermite Interpolation:

Given the value of $f(x)$ and $f^{'}(x)$ at the distinct points x_i , $i = 1,2,3, \dots \dots ., n$. $x_0 < x_1 < \cdots \dots \dots < x_n$

We determine a unique polynomial of degree which satisfies the conditions.

$$P(x_i) = f_i$$

$$P^{'}(x_i) = f_i^{'}, i = 0,1,2, \dots \dots . . n$$

The required polynomial is given by

$$P(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f^{'}(x_i)$$

Where $A_i(x)$, $B_i(x)$ are polynomials of degree $2n + 1$ are given by

$$A_i(x) = \left(1 - 2(x - x_i) l_i^{'}(x)\right) l_i^2(x)$$

$$B_i(x) = (x - x_i) l_i^2(x)$$

and $l_i(x)$ is the lagrange fundamental polynomial.

$$l_i(x) = \frac{(x - x_0)(x - x_1) \dots \dots (x - x_{i-1})(x - x_{i+1}) \dots . . (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots . . (x_i - x_n)}$$

(or)

$$l_i(x) = \frac{w(x)}{(x - x_i) w^{'}(x)}$$

1. Construct an interpolation polynomial that fits the data

x	1	2
$f(x)$	2	17
$f^{'}(x)$	4	32

Solution:

$$l_0(x) = 2 - x$$
$$l'_0(x) = -1$$
$$l_1(x) = x - 1$$
$$l'_1(x) = 1$$
$$A_i(x) = \left(1 - 2(x - x_i)l_i'(x)\right) l_i^2(x)$$

Put i=0, we get

$$A_0(x) = \left(1 + 2(x - 1)\right)(2 - x)^2$$

Put i=1, we get

$$A_1(x) = (5 - 2x)(x - 1)^2$$

Set i=0, in $B_i(x) = (x - x_i)l_i^2(x)$

$$B_0(x) = (x - 1)(2 - x)^2$$

$$B_1(x) = (x - 2)(x - 1)^2$$

Hermite interpolating polynomial is

$$P(x) = (2 - x)^2(8x - 6) + (x - 1)^2(21 - 2x)$$

2. Express y as a polynomial in x from the following data using Hermits interpolation formula

x	0	1	2
y	1	3	21
y'	0	6	36

Answer: $y = x^4 + x^2 + 1$

Piecewise Interpolation:

In order to keep the degree of interpolating polynomial small and also to achieve accurate results we use piecewise Interpolation.

We subdivide the given interval, $[a, b]$ $a = x_o < x_1 < \dots < x_n$= b in to a number of non-overlapping sub intervals each containing 2 or 3 or 4 nodal points. Then we construct the corresponding linear or quadratic or cubic interpolation polynomial fitting the given data the piecewise linear or quadratic or cubic interpolating polynomials respectively.

Piecewise Cubic Interpolation:

Let the number of distinct nodal points be $3n+1$ with a = $x_o < x_1 < \ldots <$

x_{3n+1} = b we consider groups of 4 nodal points as $[x_o, x_1, x_2, x_3]$, $[x_3, x_4, x_5, x_6]$….. $[x_{3n-2}$, x_{3n-1}, x_{3n}, $x_{3n+1}]$ on each of the subintervals we write the cubic interpolating polynomial.

We use Newton's divided difference interpolation to obtain the interpolating polynomial.

1. Obtain the piecewise cubic Interpolation for the function $f(x)$ define by the data

x	-3	-2	-1	1	3	6	7
$f(x)$	369	222	171	165	207	990	1779

Obtain the approximate the values of $f(-2.5)$ and $f(6.5)$

Solution:

We consider the group of nodal points as { -3, -2, -1, 1} and { 1,3,6,7 }

We shall use the Newtons divided difference interpolation.

x	$f(x)$	1st d.d	2nd d.d	3rd d.d
-3	369			
-2	222	-147	48	
-1	171	-51	16	-8
1	165	-3		

NDDF on [-3,1]

$$P(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$= -8x^3 + 5x + 168$$

$f(-2.5) = 280.5$

x	$f(x)$	1st d.d	2nd d.d	3rd d.d
1	165			
3	207	21	48	
6	990	261	132	14
7	1779	789		

NDDF on [1,7]

$$P(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$P(x) = 14x^3 - 92x^2 + 207x + 36$$

$f(6.5) = 1339.25$

2. Obtain the piecewise cubic interpolation polynomials for the function f(x) defined by the given data at the indicated points.

x	-5	-4	-2	0	1	3	4
$f(x)$	275	-94	-334	-350	-349	-269	-94

Cubic Spline Interpolation

The name spline is derived from the Draughtsman's Spline. A thin flexible rubber chord or wooden or plastic piece which is used to draw smooth curves through a set of points.

Let $f(x)$ be a cubic spline (a polynomial of degree 3). In the interval (x_{i-1}, x_i) where i ranges from 1 to n with respect to the points $x_0, x_1, x_2, \dots \dots .. x_n$. Since $f(x)$ is a cubic polynomial and $f'(x)$ is a second degree polynomial and $f''(x)$ is a linear function.

$$f(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)\left[f_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}(x - x_{i-1})[f_i - \frac{h^2}{6}M_i]$$

Where $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \quad i = 1,2, \dots . n-1$

$$M_0 = 0 \; and \; M_n = 0 \; and \; M_i = f''(x_i)$$

1. Find the cubic spline for the data. Hence evaluate y(1.5) given that $y_0'' = y_2'' = 0$

Solution:

$M_0 = 0 \; and \; M_n = 0$

To find M_1

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \quad i = 1,2, \dots . n-1$$

Put i=1 we get

$$M_0 + 4M_1 + M_2 = \frac{6}{1}[y_0 - 2y_1 + y_2]$$

$M_1 = 18$

To find f(x)

$$f(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)\left[f_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}(x - x_{i-1})[f_i - \frac{h^2}{6}M_i]$$

$f(x) = -3x^3 + 27x^2 - 61x + 37$

To find y(1.5)

Put x = 1.5 we get y(1.5) = -4.6250

2. Obtain a cubic spline approximation for function define by the data

c	0	1	2	3
f(x)	1	2	33	244

Hence find an estimate of f(2.5).

Answer: f(2.5)=121.25

Bivariate Interpolation:

Lagrange Bivariate Interpolation

If the value of the functions f(x,y) at (m+1)(n+1) distinct point (x_i,y_i) i=0,1,…,n are given then the polynomial P(x,y) of degree atmost m in x and n in y.
It satisfies the condition
$P(x_i, y_j) = f(x_i, y_j) = f_{i,j}$

$where\ i = 0,1,2, \dots..m\ and\ j = 0,1,2, \dots \dots n$

$$P_{mn}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m} x_{m,i}(x) y_{n,j}(y) f_{i,j}$$

$where\ x_{m,i}(x) = \frac{w\ (x)}{(x-x_i)w'(w_i)}$ where $w(x) = (x - x_o)(x - x_1) \dots..(x - x_m)$

$y_n{,}j\ (y) = \frac{w*(y)}{(y-y_j)w'*(y_j)}$

Problems :-

1. The following data represents a function f (x, y)

x \ y	0	1	4
0	1	4	49
1	1	5	53
3	1	13	85

Obtain bivariate interpolation polynomial which fits the data.

Solution:

$$P_{22}(x,y) = \sum_{j=0}^{2}\sum_{i=0}^{2} x_{2,i}(x) y_{2,j}(y) f_{i,j}$$

$m = 2, i = 0$

$$x_{2,0} = \frac{w\ (x)}{(x-x_i)\ \ (w'(x_i)\ )}$$

$$= \frac{[(x-x_0)\ (\ x-\ x_1).......(x-x_n)]}{(\ x-x_i)\ \ [(x_i - x_0)(x_i-x_1).......(x_i-\ x_m)\ ]}$$

$m = 2, i = 1$

$$x_{2,1} = \frac{(\ x-\ x_0)\ (\ x-x_1)\ (\ x-\ x_2)}{(x-x_1)\ \ [\ (x_1-\ x_0)\ (x_1-\ x_2)]}$$

$$x_{2,1} = \frac{x^2-\ 4x}{-3}$$

$m = 2, i = 2$

$$x_{2,2} = \frac{(\ x-\ x_0)\ (\ x-x_1)\ (\ x-\ x_2)}{(x-x_2)\ \ [\ (x_2-\ x_0)\ (x_2-\ x_1)]}$$

$$= \frac{(x-0)(x-1)(x-\ 4)}{(x-4)\ (4-0)\ (4-1)}$$

$$y_{2,2} = \frac{x^2-\ x}{12}$$

$$y_{n,j} = \frac{w^{*}y}{(y-y_j)\ (w'^{*}(y_j))}$$

$n = 2, j = 0$

$$y_{2,0} = \frac{(y - y_0)(y - y_1)(y - y_2)}{(y - y_0)\ [(y_0 - y_1)(y_0 - y_2)]}$$

$$y_{2,0} = \frac{y^2 - 4y + 3}{3}$$

$$y_{2,1} = \frac{(y - y_0)(y - y_1)(y - y_2)}{(y - y_1)\ [(y_1 - y_0)(y_1 - y_2)]}$$

$$y_{2,1} = \frac{y^2 - 3y}{-2}$$

$$y_{2,2} = \frac{(y - y_0)(y - y_1)(y - y_2)}{(y - y_2)\ [(y_2 - y_0)(y_2 - y_1)]}$$

$$y_{2,2} = \frac{y^2 - y}{6}$$

$$P_{22}\ (x, y) = y_{2,0}\ y_{2,0}\ f_{0,0} + y_{2,1}\ f_{0,1} + 4_{2,2}\ f_{0,2}]$$

$$+ y_{2,1}\ [y_{2,0}\ f_{1,0} + y_{2,1}\ f_{1,1} + 4_{2,2}\ \ f_{1,2}]$$

$$+ y_{2,2}\ [y_{2,0}\ f_{2,0} + y_{2,1}\ f_{2,1} + 4_{2,2}\ \ f_{2,2}]$$

$$= \frac{36\,x^2 + 12\,xy^2 + 12}{12}$$

$$P_{22}\ (x, y) = 3x^2 + xy^2 + 1$$

Newton's Bivariate interpolation for Equi spaced Points

With equispased points with spacing h in x and k in y we defined

$$\Delta_x f(x,y) = f(x+h,y) - f(x,y)$$

$$\Delta_y f(x,y) = f(x,y+k) - f(x,y)$$

$$\Delta_{xx} f(x,y) = \Delta_x . \Delta_x f(x,y)$$

$$= f(x+2h,y) - 2f(x+h,y) + f(x,y)$$

$$\Delta_{yy} f(x,y) = \Delta_y . \Delta_y f(x,y)$$

$$= f(x,y+2k) - 2f(x,y+k) + f(x,y)$$

$$\Delta_{xy} f(x,y) = \Delta_x . \Delta_y f(x,y)$$

$$= f(x+h,y+k) - 2f(x,y+k) - f(x+h,y) + f(x,y)$$

Let $x = x_0 + nh \; and \; y = y_0 + nk$

Then,

$$P(x,y) = f(x_0,y_0) + \left[\frac{1}{h}(x-x_0)\Delta_x + \frac{1}{k}(y-y_0)\Delta_y\right] f(x_0,y_0) + \frac{1}{2!}\left[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta_{xx} + \frac{1}{k^2}(y-y_0)(y-y_1)\Delta_{yy}\right] + (x_0,y_0) + \ldots\ldots.$$

1. Obtain the Newton's Bivariate interpolating polynomial that fix the following data

x \ y	1	2	3
1	4	18	56
2	11	25	63
3	30	44	82

Let $x = x_0 + nh \quad and \quad y = y_0 + nk$

$$p\,(x,y) = f\,(x_0,y_0) + \left[\frac{1}{h}\,(x-x_0)\Delta_x + \frac{1}{k}(y-y_0)\Delta_y\right] f\,(x_0,y_0$$
$$+\frac{1}{2}\left[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta_{xx} + \frac{2}{hk}(x-x_0)(y-y_0)\Delta_{xy}\right.$$
$$\left.+\frac{1}{k^2}(y-y_0)(y-y_1)\Delta_{yy}\right] f(x_0,y_0) + \;\ldots\ldots$$

$$\Delta_x\, f(x_0,y_0 = f(x_0+h,y_0) - \,f\,(x_0,y_0) = \;18-4 = 14$$

$$\Delta_y\, f(x_0,y_0 = f(x_0,y_0+k) - \,f\,(x_0,y_0) \; = \;11-4 = 7$$

$$\Delta_{xx}\, f(x_0,y_0) = f(x_0+\;2h,y_0) - 2f\,(x_0+h,y_0) + f(x_0,y_0) = \;f(x_2,y_0) - 2f(x_0,y_1) + f\,(x_0,y_0) = 56-36+4 = 24$$

$$\Delta_{yy}\, f(x_0,y_0) = f(x_0,y_0+2k) - 2f\,(x_0,y_0+k) + f(x_0,y_0) = \;f(x_0,y_2) - 2f(x_0,y_1) + f\,(x_0,y_0) = 30-22+4 = 12$$

$$\Delta_{xy}\, f(x_0,y_0) = f(x_0+h,y_0+k) - f\,(x_0,y_0+k) - f(x_0,+h,y_0) + f(x_0,y_0)$$

$$\Delta_{xy}\, f(x_0,y_0) = 0$$

$$P(x,y) = 12x^2 + 6y^2 - 22x - 11y + 19$$

3. Using the following data, construct the (*i*) Lagrange and (*ii*) Newton's bivariate interpolating polynomials.

x \ y	0	1	2
0	1	3	7
1	3	6	11
2	7	11	17

Least squaring Approximations:

Least squaring Approximation are used for approximating a function f(x) which may be given in tabular form over a given interval.
For functions which are continuous on [a,b] are given as
$I(c_0, c_1, \dots \dots .., c_n) = \int_a^b w(x)[f(x) - \sum_{i=0}^{n} c_i\varphi_i(x)]^2 dx$ = minimum
The function $\varphi_i(x)$ are usually choosen as $\varphi_i(x) = x^i, i = 0,1, \dots \dots ., n$ and w(x)=1
The necessary condition to have a minimum value is $\frac{\partial I}{\partial c_i} = 0, i = 0,1, \dots \dots .., n$
This gives a system of n+1 linear equations in n+1 unknowns $c_0, c_1, \dots \dots .., c_n$. These equations are called normal equation.

1. Obtain the least square approximation of degree one and two for $x^{1/2}$ on (0,1)

   Solution:
   For degree one

   $$P(x) = c_0 + c_1 x$$

   $$I(c_0, c_1, \dots \dots .., c_n) = \int_a^b w(x)[f(x) - \sum_{i=0}^{n} c_i\varphi_i(x)]^2 dx$$

   $$I(c_0, c_1) = \int_0^1 (1)[x^{1/2} - \sum_{i=0}^{1} c_i x^i]^2 dx$$

   Differentiating with respect to c_0

   $$\frac{\partial I}{\partial c_0} = 0$$

   ie.

   $$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1 x\right](-1)dx = 0$$

   Thus $2c_0 + c_1 = \frac{4}{3}$

   Differentiating with respect to c_1

   $$\frac{\partial I}{\partial c_1} = 0$$

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1 x\right](-x)dx = 0$$

Thus $c_0 + 2\frac{c_1}{3} = \frac{4}{5}$

Solving the above equations, we get

$$P(x) = \frac{4}{15} + \frac{4}{5}x$$

For degree two we have

$$P(x) = c_0 + c_1 x + c_2 x^2$$

$$I(c_0, c_1, \dots \dots .., c_n) = \int_a^b w(x)[f(x) - \sum_{i=0}^{n} c_i \varphi_i(x)]^2 dx$$

$$I(c_0, c_1, c_2) = \int_0^1 (1)[x^{1/2} - \sum_{i=0}^{2} c_i x^i]^2 dx$$

Differentiating with respect to c_0

$$\frac{\partial I}{\partial c_0} = 0$$

ie.

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1 x - c_2 x^2\right](-1)dx = 0$$

Thus $2c_0 + c_1 + \frac{2}{3}c_2 = \frac{4}{3}$

Differentiating with respect to c_1

$$\frac{\partial I}{\partial c_1} = 0$$

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1 x - c_2 x^2\right](-x)dx = 0$$

Thus $c_0 + 2\frac{c_1}{3} + \frac{1}{2}c_2 = \frac{4}{5}$

Differentiating with respect to c_2

$$\frac{\partial I}{\partial c_2} = 0$$

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1 x - c_2 x^2\right](-x^2)dx = 0$$

Thus $\frac{2}{3}c_0 + \frac{c_1}{2} + \frac{2}{5}c_2 = \frac{4}{7}$

Solving the above equations, we get

$$P(x) = \frac{6}{35} + \frac{48}{35}x - \frac{20}{35}x^2$$

2. Evaluate the least squares straight line fit to the following data

x	0.2	0.4	0.6	0.8	1
f(x)	0.447	0.632	0.775	0.894	1

Solution:

x	$f(x)$	$xf(x)$	x^2
0.2	0.447	0.0894	0.4
0.4	0.632	0.2528	0.16
0.6	0.775	0.4650	0.36
0.8	0.894	0.7152	0.64
1	1	1	1
$\sum x_i = 3$	$\sum f(x_i) = 3.748$	$\sum x_i f(x_i) = 2.5224$	$\sum {x_i}^2 = 2.2$

$$\sum f(x_i) = c_0 \sum x_i + nc_1$$

(ie) $3.748 = c_0(3) + 5c_1$

$$\sum x_i f(x_i) = c_0 \sum x_i^{\,2} + c_1 \sum x_i$$

(ie) $2.5224 = c_0(2.2) + 3c_1$

Solving the above equations, we get

$$c_1 = 0.3392 \; and \; c_0 = 0.6840$$

$$P_1(x) = c_0 x + c_1$$

$$P_1(x) = o.684x + 0.3392$$

Least square Error

$$\sum_{i=0}^{4} [f(x_i) - (c_0 x + c_1)]^2$$

(ie) $2.3395x^2 - 2.8073x + 1.0317$

$$x = \frac{2.8073 \pm 1.3318i}{4.6790}$$

Gram- Schmidt Orthogonalizing Process:

Given the polynomial $\varphi_i(x)$ of degree I, the polynomials $\varphi_i^*(x)$ of degree I which are orthogonal over [a,b] with respect to the weight function W(x) can be generated recursively from the relation

$$\varphi_i^*(x) = x^i - \sum_{r=0}^{i-1} a_{ir} \varphi_r^*(x) \quad i = 1,2, \dots \dots, n$$

Where $a_{ir} = \frac{\int_a^b W(x) x^i \varphi_r^*(x) dx}{\int_a^b W(x) \varphi_r^*(x)^2 dx}$ $and \; \varphi_0^*(x) = 1$

1. Utilize the Gram-Schmidt orthogonalization process, compute the first three orthogonal polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$ which are orthogonal on [0, 1] with respect to the weight function w(x)=1. Using these polynomials obtain the least square approximation of second degree for $f(x) = x^{\frac{1}{2}}$ on [0, 1].

Solution:

$\varphi_i^*(x) = x^i - \sum_{r=0}^{i-1} a_{ir}\varphi_r^*(x) \quad i = 1,2, \dots \dots, n$

Put i=0 and $\varphi^* = P$

$P_0(x) = x - a_{10}P_0(x)$

We know that

$$a_{ir} = \frac{\int_a^b W(x)x^i\varphi_r^*(x)dx}{\int_a^b W(x)\varphi_r^*(x)^2dx} \; and \; \varphi_0^*(x) = 1$$

Here, i=1, r=0

$$a_{10} = \frac{\int_0^1 (1)x^{(1)}(x)dx}{\int_0^1 (1)[1]^2dx} = \frac{1}{2}$$

Therefore $P_1(x) = x - \frac{1}{2}$

Put i=2 and $\varphi^* = P$

$P_2(x) = x^2 - a_{20}P_0(x) - a_{21}P_1(x)$

Put i=2 and r=0

$$a_{20} = \frac{\int_0^1 (1)x^{(2)}(1)dx}{\int_0^1 (1)[1]^2 dx} = \frac{1}{3}$$

Put i=2 and r=1

$$a_{21} = \frac{\int_0^1 (1)x^{(2)}(x-1/2)dx}{\int_0^1 (1)[x-1/2]^2 dx} = 1$$

Therefore $P_2(x) = x^2 - x + \frac{1}{6}$

Using Least square approximation

$$I(c_0, c_1, c_2) = \int_0^1 (1)[x^{1/2} - \sum_{i=0}^{2} c_i x^i]^2 dx$$

Differentiating with respect to c_0

$$\frac{\partial I}{\partial c_0} = 0$$

ie.

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1(x - \tfrac{1}{2}) - c_2(x^2 - x + \tfrac{1}{6})\right](-1)dx = 0$$

Thus $c_0 = \frac{2}{3}$

Differentiating with respect to c_1

$$\frac{\partial I}{\partial c_1} = 0$$

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1\left(x - \frac{1}{2}\right) - c_2\left(x^2 - x + \frac{1}{6}\right)\right](x - \frac{1}{2})dx = 0$$

Thus

$$c_1 = \frac{4}{5}$$

Differentiating with respect to c_2

$$\frac{\partial I}{\partial c_2} = 0$$

$$\int_0^1 (2)\left[x^{\frac{1}{2}} - c_0 - c_1(x - \frac{1}{2}) - c_2(x^2 - x + \frac{1}{6})\right]\left(x^2 - \frac{1}{2}\right) dx = 0$$

Thus

$$c_2 = \frac{-4}{7}$$

Solving the above equations, we get

$$P(x) = \frac{6}{5} + \frac{48}{5}x - \frac{20}{5}x^2$$

Using the Jacobi Method, find all the eigen values and the corresponding eigen vectors of

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Solution:-

Iteration – I

Largest of diagonal element $= 2 = a_{13} = a_{31} \;\; a_{11} = a_{33} = 1$

$$\theta = \frac{1}{2} tan^{-1} \left[\frac{2a_{13}}{a_{11} - a_{33}}\right] = \frac{1}{2} tan^{-1} [\frac{2*2}{1-1}]$$

$$\theta = \frac{1}{2} tan^{-1}(\infty)$$

$$\theta = \frac{\pi}{4}$$

Rotation Matrix:-

$S_1 = I$, except the elements corresponding to these 4 elements

$$S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix}$$

$$S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A_1 = S_1^T A S_1$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Iteration - II

Largest of diagonal element = $2 = a_{13} = a_{31}\;\; a_{11} = a_{33} = 3$

$$\theta = \frac{1}{2}tan^{-1}\left[\frac{2a_{13}}{a_{11}-a_{33}}\right] = \frac{1}{2}tan^{-1}[\frac{2*2}{3-3}]$$

$$\theta = \frac{1}{2}tan^{-1}(\infty)$$

$$\theta = \frac{\pi}{4}$$

$$S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = S_2^T A_1 S_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = Diagonal\ Matrix.$$

Eigen Value = 5,1,-1

The corresponding Eigen vectors are given by the Columns of

$$S = S_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The eigen vectors corresponding to 5,1,-1 are

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}^T, \begin{bmatrix}-\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}^T, \begin{bmatrix}-\frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^T$$

UNIT – IV - ADVANCED NUMERICAL METHODS – SMTA5304

UNIT IV

Numerical Differentiation and Integration

Partial Differentiation:

Consider here a function f(x,y) of two variables. Let the values of the function f(x,y) be given at the set of points (x_i, y_j) in the (x, y) plane with spacing h and k in x and y direction respectively. We have $x_i = x_0 + ih \; and \; y_j = y_0 + jk \quad i, j = 1,2,3, \ldots.$

Then $(\frac{\partial f}{\partial x})_{(x_i,y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$

Where $f_{i,j} = f(x_i, y_j)$

$(\frac{\partial f}{\partial y})_{(x_i,y_j)} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$ with h=k=1

1. Find the Jacobian matrix of the system of equation $f_1(x,y) = x^2 + y^2 - x$ and $f_{20}(x,y) = x^2 - y^2 - y$ at the point (1,1). Using the methods $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

   Solution:

   The Jacobian matrix is given as

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$(\frac{\partial f}{\partial x})_{(x_i,y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$
$$(\frac{\partial f}{\partial x})_{(1,1)} = \frac{f_{2,1} - f_{0,1}}{2h} \quad h = 1$$

$$f_1(x,y) = x^2 + y^2 - x$$
$$f_1(2,1) = 3$$
$$f_1(0,1) = 1$$
$$(\frac{\partial f}{\partial x})_{(x_i,y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$
$$(\frac{\partial f_1}{\partial x})_{(1,1)} = 1$$
$$(\frac{\partial f_2}{\partial x})_{(1,1)} = 2$$
$$(\frac{\partial f}{\partial y})_{(x_i,y_j)} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

$(\frac{\partial f_1}{\partial y})_{(1,1)} = 2$
$(\frac{\partial f_2}{\partial y})_{(1,1)} = -3$
$Hence\ J = -7$

Methods based on undetermined coefficients –Gauss quadrature methods:

$$\int_a^b w(x)f(x)dx = \sum_{k=0}^{n} \lambda_k f_k$$

The nodes $x_k's$ and the weights $\lambda_k's$ k=0 to n, can also be obtained by making the formula exact for polynomials of degree up to m

When the nodes are known ie m=n the corresponding methods are called Newton Codes method.

When the nodes are also to be determined we have m=2n+1 and the methods are called Gaussian Integration methods.

Any finite interval [a,b] can always be transform to [-1,1] using the transformation $x = (\frac{b-a}{2})t + (\frac{b+a}{2})$

Guass-Lagendre Integration Methods:

Consider the integral in the form

$\int_{-1}^{1} w(x)f(x)dx = \sum_{k=0}^{n} \lambda_k f_k$ for w(x)=1

The above equation reduces to

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} \lambda_k f_k$$

In this case all nodes and weights are unknown

One point formula [n=0]

$$\int_{-1}^{1} f(x)dx = 2f(0)$$

Two point formula [n=1]

$$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Three point formula [n=2]

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}[5f\left(\frac{-\sqrt{3}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{3}}{5}\right)]$$

1. Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using the Gauss –Legendre three point formula.

   Solution:
   First we transform the interval [0,1] to the interval[-1,1]
   Using the transformation $x = (\frac{b-a}{2})t + (\frac{b+a}{2})$
   We get $2x = t + 1$
   Differentiating both sides

$$\frac{dx}{dt} = 2$$

$$I = \int_0^1 \frac{dx}{1+x} = \int_{t=-1}^{1} \frac{dt/2}{1 + (\frac{t+1}{2})} = \int_{-1}^{1} \frac{dt}{t+3}$$

Three point formula [n=2]

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}[5f\left(\frac{-\sqrt{3}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{3}}{5}\right)]$$

$$\int_{-1}^{1} \frac{dt}{t+3} = o.6931$$

2. Evaluate the integral I = $\int_1^2 \frac{2xdx}{1+x^4}$ using the Gauss-legendre 1-point, 2-point and 3-point quadrature rules. Compare with the exact solution.

Gauss Chebyshev Integration Method:

Let the weight function be $w(x) = \frac{1}{\sqrt{1-x^2}}$

Then the method

$$\int_{-1}^{1} w(x)f(x)dx = \sum_{k=0}^{n} \lambda_k f_k$$

Where w(x)>0 $-1 \leq x \leq 1$ reduces to

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \sum_{k=0}^{n} \lambda_k f_k$$

$$One\ Point\ formula\ n = 0$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \pi f(0)$$

$$Two\ point\ formula\ n = 1$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2}[f(\frac{-1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}})]$$

$$Three\ point\ formula\ n = 2$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{3}[f(\frac{-\sqrt{3}}{2}) + f(0) + f(\frac{\sqrt{3}}{2})]$$

1. Evaluate the integral I = $\int_{-1}^{1}(1-x^2)^{\frac{3}{2}} \cos x dx$ using the Gauss-Chebyshev 1-point, 2-point and 3-point quadrature rules. Evaluate it also using the Gauss-Legendre 3-point formula.

Solution:

$$I = \int_{-1}^{1}(1-x^2)^{\frac{3}{2}} \cos x dx$$

Reduce this to

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \sum_{k=0}^{n} \lambda_k f_k$$

Thus $f(x) = (1-x^2)^2 \cos x$

$$One\ Point\ formula\ n = 0$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \pi f(0)$$

$$(ie)\ I = \int_{-1}^{1} \left(1-x^2\right)^{\frac{3}{2}} \cos x dx = 3.14159$$

$$Two\ point\ formula\ n = 1$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}[f(\frac{-1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}})]$$

$$(ie)\ I = \int_{-1}^{1} \left(1-x^2\right)^{\frac{3}{2}} \cos x dx = 0.59709$$

$$Three\ point\ formula\ n = 2$$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{3}[f(\frac{-\sqrt{3}}{2}) + f(0) + f(\frac{\sqrt{3}}{2})]$$

$$(ie)\ I = \int_{-1}^{1} \left(1-x^2\right)^{\frac{3}{2}} \cos x dx = 1.13200$$

Double Integration:

The problem of double integration is to evaluate an integration of the form

$I = \int_c^d (\int_a^b f(x,y)dx)dy$

Over the rectangle x=a, x=b, x=c, x=d

y \ x	y_0	$y_1 = y_0 + k$	$y_2 = y_0 + 2k$	……	$y_m = y_0 + mk$
x_0	f_{00}	f_{01}	f_{02}	…….	f_{0m}
$x_1 = x_0 + h$	f_{10}	f_{11}	f_{12}	…….	f_{1m}
$x_2 = x_0 + 2h$	f_{20}	f_{21}	f_{22}	…….	f_{2m}
⋮	⋮	⋮	⋮	⋮	⋮
$x_n = x_0 + nh$	f_{n0}	f_{n1}	f_{n2}	…….	f_{nm}

$$f_{00} = f(x_0, y_0)$$

$f_{01} = f(x_0, y_1)$

$$I = \int_c^d (\int_a^b f(x,y)dx)dy$$

$$I = \frac{hk}{4}[f_{00} + f_{01} + f_{10} + f_{11}]$$

$$= \frac{h}{2}\int_{y_0}^{y_0+k} [f(x_0 + h, y) + f(x_0, y)]dy$$

$$= \frac{hk}{4}\,[\,f\,(x_0 + h, y_0 + \,k\,) + \,f\,(\,x_0, y_0 + \,k\,) + \,f\,(x_0 + \,h, y_0) + \,f\,(x_0, y_0)]$$

Problem :

1) Using Trapezoidal Rule evaluate $\int_{1.4}^{2.0} \int_{1.0}^{1.5} \log(\,x + 2y\,)dydx\ using\ h = 0.15\ and\ k = 0.25.$

x \ y	$y_0 = 1.0$	$y_1 = 1.25$	$y_2 = 1.5$
$x_0 = 1.4$	$F_{00} = 1.2238$	$f_{01} = 1.3610$	$f_{02} = 1.4816$
$x_1 = 1.55$	$F_{10} = 1.2669$	$f_{11} = 1.3987$	$f_{12} = 1.5151$
$x_2 = 1.70$	$F_{20} = 1.3083$	$f_{21} = 1.4350$	$f_{22} = 1.547$
$x_3 = 1.85$	$F_{30} = 1.3480$	$f_{31} = 1.4702$	$f_{32} = 1.5790$
$x_4 = 2$	$F_{40} = 1.3863$	$f_{41} = 1.5041$	$f_{42} = 1.6094$

$$I = \frac{hk}{4}\,[\,f_{00} + \,f_{02} + f_{40} + f_{42} + \,2\,[\,f_{01} + \,f_{10} + \,f_{20} + \,f_{30} + \,f_{41} + \,f_{12} + \,f_{22} + \,f_{32}\,] + 4\,[\,f_{11} + \,f_{21} + \,f_{31}\,]\,]$$

$$I \;= (\,0.0094\,)[\,507011 + 22.8588 + 17.2156\,]$$

$$I = 0.4303$$

Differentiation and integration:

Numerical differentiation: -

3 techniques

a) Methods based on interpolation
b) Methods based on finite difference operators
c) Methods based on undetermined co-efficient.

1) Methods based on interpolation:-

Non-uniform nodal points :-

(i) Linear interpolation: -

$$p_1'(x) = \frac{f_1 - f_0}{x_1 - x_0} \forall x \in [x_0, x_1)$$

Error is

$$E_1'(x_0) = \frac{x_0 - x_1}{2} f''(\varepsilon)$$

$$E_1'(x_1) = \frac{x_1 - x_0}{2} f''(\varepsilon), x_0 < \varepsilon < x_1$$

(ii) Quadratic interpolation: [-

$$p_2'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} fo + \frac{x_0 - x_1}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{x_0 - x_1}{(x_2 - x_0)(x_1 - x_1)} f2$$

Error is:-

$$E_2'(x_0) = \frac{1}{6}(x_0 - x_1)(x_0 - x_2) + f'''(\varepsilon), x_0 < \varepsilon < x_2$$

Similarly $p_2''(x) = 2[\frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}]$

Error is:-

$$E''(x_0) = \frac{1}{3}(2x_0 - x_1 - x_2)f''(\varepsilon) + \frac{1}{24}(x_0 - x_1)(x_0 - x_2)[f^{iv}(\eta_1) + f^{iv}\eta_2$$

Where $\eta_1, \eta_2, x_1 \in (x_0, x_2)$

Uniform Nodal Points :-

i) Linear interpolation: -

$$f^1(x_0) = p_1'(x_0) = \frac{f_1 - f_0}{h}$$

$$f^1(x_1) = p_1'(x_1) = \frac{f_1 - f_0}{h}$$

Error is:-

$$E_1'(x_0) = -\frac{h}{2} f''(\varepsilon), x_0 < \varepsilon < x_1$$
$$E_1'(x_1) = -\frac{h}{2} f''(\eta), x_0 < \eta < x_1$$

ii) Quadratic Interpolation: -

$$f'(x_0) = \frac{1}{2h}[-3f_0 + 4f_1 - f_2$$

$$f'(x_1) = \frac{1}{2h}[f_2 - f_0$$

$$f'(x_2) = \frac{1}{2h}[f_0 + 4f_1 - 3 f_2$$

$$E_2''(x_0) = -h f'''(\varepsilon_1) ;\ x_0 < \varepsilon_1 < x_2$$

$$E_2''(x_1) = -\frac{h^2}{12} f^{iv}(\varepsilon_2) ;\ x_0 < \varepsilon_2 < x_2$$

$$E_2''(x_2) = h f'''(\varepsilon_3),\ x_0 < \varepsilon_3 < x_2$$

Error is :-

$$E_2'(x_0) = -\frac{h^2}{3} f'''(\varepsilon),\ x_0 < \varepsilon < x_2$$

$$E_2'(x_1) = -\frac{h^2}{6} f'''(\eta_1),\ x_0 < x_1 < x_2$$

$$E_2'(x_2) = -\frac{h^2}{3} f'''(\eta_2),\ x_0 < \eta_2 < x_2$$

Non – Uniform nodal P^5:-

Given the following values of f (x) = ln (x_1) find the approximate value of $f'(2.0)$& $f''(2.0)$ using the methods based on linear & quadratic interpolation. Also obtain an upper bound on the error

i	0	1	2
x_i	2.0	2.2	2.6
f_i	0.69315	0.78846	0.95551

Solution

(i) Using the method [Linear interpolation]

$$f'(x_0) = \frac{f_1 - f_0}{x_1 - x_0}$$

$$f'(2.0) = 0.47655$$

(ii) If we use the method [quadratic interpolation]

$$f'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}(f_0) + \frac{x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)}(f_1)$$

$$+\frac{x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}(f_2)$$

$$f'(2.0) = 0.47655$$

Exact value of $f'(x) = \frac{1}{x}$

$$f'(2.0) = \frac{1}{x2.0} = 0.5$$

(iii) Similarly using the method [Quadratic interpolation]

$$f''(x_0) = 2[\frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$f''(2.0) = 2\ \frac{0.69315}{(2 - 2.2)(2 - 2.6)} + \frac{0.78846}{(2.2 - 2)(2.2 - 2.6)} + \frac{0.95551}{(2.6 - 2)(2.6 - 2.2)}$$

$$f''(2.0) = -0.19642$$

Exact Value :- $f(x) = lnx$

$$f'(x) = \frac{1}{2}\ ,\ f''(x) = \frac{-1}{x^2}$$

$$f''(2.0) = \frac{-1}{(2)^2} = -0.25$$

Error associated with linear interpolation are given by

$$E_1'(x_0) = \frac{x_0 - x_1}{2} f''(\varepsilon), x_0 < \varepsilon < x_1$$

Error associated with quadratic interpolation is

$$E_2'(x_0) = \frac{1}{6}(x_0 - x_1)(x_0 - x_2)f''(\varepsilon), x_0 < \varepsilon < x_2$$

$$E_2''(x_0) = \frac{1}{3}(2x_0 - x_1 - x_2)f'''(\varepsilon) + \frac{1}{24}(x_0 - x_1)(x_0 - x_2)$$

$$\left[f^{iv}(\eta_1) + f^{iv}(\eta_1)\right] x_0 < \varepsilon, \eta_1, \eta_2, < x_2$$

UNIT – V - ADVANCED NUMERICAL METHODS – SMTA5304

UNIT – V

Ordinary Differential Equations

Euler Method:

$$U_{j+1} = U_j + hf_i\ ; j = 0,1,2, \dots\dots, N-1$$

Where $f_j = f\left(t_j, u_j\right)$

Backward Euler method:-

$$U_{j+1} = U_j + hf_j + 1$$

$$f_{j+1} = f(\, t_{j+1}, U_{j+1})$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$U_{j+1}^{(s+1)} = u_{j+1}^{(s)} - \frac{f(u_{j+1}^{(s)})}{f'(u_{j+1}^{(s)})}$$

$$x = u_{j+1}$$

$$n = s$$

$$f_{j+1} = f(t_{j+1}, u_{j+1}$$

$$f\left(t_{j+1}, u_{j+1}\right) = -\,2t_{j+1}\, u_{j+1}^2$$

$$u_{j+1} = u_j + h\,[-2t_{j+1}\, u_{j+1}^2]$$

Mid point Method:-

$$u_{j+1} = u_{j-1} + 2hf_j$$

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ldots\ldots..$$

$$u(t+h) = u(t) + \frac{h}{1!}u'(t) + \frac{h^2}{2!}u''(t) + \ldots\ldots..$$

$$= u\,(t_0) + \frac{h}{1!}u'(t_0)$$

$\boldsymbol{R-K\ \ Method}$

$$u_{j+1} = u_j + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf\ [t_j, u_j]$$

$$k_2 = hf\ [t_{j+\frac{h}{2}} + u_{j+\frac{k_1}{2}}]$$

$$k_3 = hf\ [t_{j+\frac{h}{2}} + u_{j+\frac{k_2}{2}}]$$

$$k_4 = hf\ [t_{j+h} + u_{j+k_2}]$$

Problems

1. Use the Euler method to solve numerically the initial value problem $u' = -2tu^2, u\,(0) = 1, with\ h = 0.2, 0.1, 0.05\ and\ the\ interval\ [0,1]$.

   Solution:

$$u_{j+1} = u_j + hf_j\ \ j = 0,1,2, \ldots.., N-1$$

Where $f_j = f(t_j, u_j)$

$$f_j = -2t_j{u_j}^2$$

$$u_{j+1} = u_j + h(-2t_j{u_j}^2)\ \ j = 0,1,2, \ldots.., N-1$$

With h=0.2 and u(0)=1 ie t_0=0 and u_0=1

(i) $u_1 = u_0 + h(-2t_0{u_0}^2)$
$u(0.2) = 1$

(ii) $u_2 = u_1 + h(-2t_1{u_1}^2)$
$u(0.4) = 0.92$

(iii) $u_3 = u_2 + h(-2t_2{u_2}^2)$
$u(0.6) = 0.7846$

(iv) $u_4 = u_3 + h(-2t_3{u_3}^2)$
$u(0.8) = 0.6369$

(v) $u_5 = u_4 + h(-2t_4{u_4}^2)$
$u(1) = 0.6369$

With h=0.1 and u(0)=1 ie t_0=0 and u_0=1

(i) $u_1 = u_0 + h(-2t_0{u_0}^2)$
$u(0.1) = 1$

(ii) $u_2 = u_1 + h(-2t_1{u_1}^2)$
$u(0.2) = 0.98$

(iii) $u_3 = u_2 + h(-2t_2{u_2}^2)$
$u(0.3) = 0.94158$

(iv) $u_4 = u_3 + h(-2t_3{u_3}^2)$
$u(0.4) = 0.88939$

(v) $u_5 = u_4 + h(-2t_4{u_4}^2)$
$u(0.5) = 0.82525$
Similarly
$u(1.0) = 0.50364$

With h=0.05 and u(0)=1 ie t_0=0 and u_0=1

(vi) $u_1 = u_0 + h(-2t_0{u_0}^2)$
$u(0.05) = 1$

(vii) $u_2 = u_1 + h(-2t_1{u_1}^2)$
$u(0.1) = 0.995$

(viii) $u_3 = u_2 + h(-2t_2{u_2}^2)$
$u(0.15) = 0.9851$

(ix) $u_4 = u_3 + h(-2t_3{u_3}^2)$
$u(0.2) = 0.97054$

(x) $u_5 = u_4 + h(-2t_4{u_4}^2)$
$u(0.25) = 0.9517$
Similarly
$u(1) = 0.50179$

2. Solve the initial value problem $u' = -2tu^2$ u(0)=1 with h=0.2 on the interval[0,0.4] using the backward Euler method.

Solution:

$$U_{j+1} = U_j + hf_j + 1$$

$$f_{j+1} = f(t_{j+1}, U_{j+1})$$

$$U' = 2tu^2, u(o) = 1, h = 0.2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$U_{j+1}^{(s+1)} = u_{j+1}^{(s)} - \frac{f(u_{j+1}^{(s)})}{f'(u_{j+1}^{(s)})}$$

$$x = u_{j+1}$$

$$n = s$$

$$f_{j+1} = f(t_{j+1}, u_{j+1})$$

$$f\left(t_{j+1}, u_{j+1}\right) = -2t_{j+1} u_{j+1}^2$$

$$u_{j+1} = u_j + h\,[-2t_{j+1} u_{j+1}^2]$$

$$f\left(u_{j+1}\right) = u_{j+1} - u_j - 2ht_{j+1}, u_{j+1}^2 = 0$$

$$f'\left(u_{j+1}\right) = 1 - 0 - 2ht_{j+1}(2u_{j+1})$$

Thus we have

$$u_1^{(1)} = 0.9310$$

$$u_1^{(2)} = 0.9307$$

$$u_1^{(3)} = 0.9307$$

Take t_2=0.4

$$u_2^{(0)} = 0.9307$$

$$u_2^{(1)} = 0.8239$$

$$u_2^{(2)} = 0.8224$$

$$u_2^{(3)} = 0.8224$$

$$u_2^{(4)} = 0.8224$$

Thus u(0.4)= 0.8224

3. Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ using the mid point method with h=0.02 over the interval [a,b]. Use the Taylor series method of second order to compute u(0.2). Determine the percentage relative error at t=1.

Solution:

$$u_{j+1} = u_{j-1} + 2hf_j \qquad u' = 2tu^2, u(0) = 1, h = 0.2 \ [0.1]$$

$$u' = 2tu^2, \qquad u(0) = 1, \qquad h = 0.02$$

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \ldots\ldots..$$

$$u(t+h) = u(t) + \frac{h}{1!}u'(t) + \frac{h^2}{2!}u''(t) + \ldots\ldots..$$

$$= u\,(t_0) + \frac{h}{1!}u'(t_0)$$

$$u_1 = u(0.2) = 0.96$$

$$u_2 = u(0.4) = 0.85254$$

$$u_3 = u(0.6) = 0.72741$$

$u_4 = u(0.8) = 0.5985$

$u_5 = u(1.0) = 0.49811$

Exact solution is

$$\frac{1}{u} = t^2 + c$$

Where c=1

$$u(t) = \frac{1}{1+t^2}$$

$u(1) = 0.5$

The percentage error is defined as

$$P.E = \frac{|u - u^*|}{|u|}100$$

P.E = .38

Where u is the exact solution

And u^* is the approximate solution

4. Solve the initial value problem $u' = -2tu^2\,, u(0) = 1\; with\; h = 0.2$ on the interval [0,0.4] use the second order implicit Runge-kutta method.

   Solution:

$$u_{j+1} = u_j + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf\,[t_j, u_j]$$

$$k_2 = hf\,[t_{j+\frac{h}{2}} + u_{j+\frac{k_1}{2}}]$$

$$k_3 = hf\,[t_{j+\frac{h}{2}} + u_{j+\frac{k_2}{2}}]$$

$$k_4 = hf\,[t_{j+h} + u_{j+k_2}]$$

$u_1 = u(0.2) = 0.9615$

$u_2 = u(0.4) = 0.8617$

Implicit Runge – Kutta Method:

The Second order implicit Runge-Kutta method becomes $u_{j+1} + \; u_{j+k}$

$k = hf\ [t_j + \frac{h}{2}, u_j + \frac{k_1}{2}]$

For obtaining the value of k_1, we need to solve a non-linear algebraic equation in one variable k_1

1) Solve the initial value problem $u' = 2tu^2, u(o) = 1$ with h=0.2 on the interval [0,0.4] use the fourth order classical Runge – Kutta Method compare with the exact solution.

Solution : -

$t_0 = 0, \qquad u_0 = 1$

$k_1 = hf(t_0, u_0) = h(-2t_0u_0^2)$

= 0

$$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right)$$

= - 0.04

$$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right)$$

= -0.038416

$k_4 = hf(t_0 + h, u_0 + k_3)$

= - 0.0739715

$$u(0.2) = u_1 = u_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$u_1 = 0.9615328$$

For j = 1

$$t_1 = 0.2, u_1 = 0.9615328$$

$$k_1 = hf(t_1, u_1)$$

= - 0.0739636

$$k_2 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{k_1}{2}\right)$$

$$k_2 = -0.1025753$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{k_2}{2}\right)$$

$$k_3 = -0.0994255$$

$$k_4 = hf(t_1 h, u_1 + k_3)$$

$$k_4 = -0.1189166$$

$$u(0.4) = u_2 = u_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051506 - 0.1988510 - 0.1189166$$

$$u(0.4) = 0.8620525$$

Absolute errors in the numerical solutions are

$$\epsilon(0.2) = |0.961539 - 0.961533|$$

$$= 0.000006$$

$$\epsilon(0.4) = |0.862069 - 0.862053|$$

=0.000016

Predictor – corrector Methods

$P(EC)^m E$ Method :

The Predictor – corrector method may be written as

$P\ :\ Predict\ some\ value\ u_{j+1}^{(0)}$

$E\ :\ Evaluate\ f(t_{j+1}, u_{j+1}^{(0)})$

$C\ :\ Correct\ u_{j+1}^{(1)} \sum_{i=1}^{k} \left(u_{j-i+1} + hb_i f_{j-i+1}\right) + hb_0\ f(t_{j+1}, u_{j+1}^{(0)}$

$E\ :\ Evaluate\ \ u_{j+1}^{(2)} = \sum_{i=1}^{k} \left(a_i u_{j-i+1} + hb_i f_{j-i+1}\right) + hb_0\ f(t_{j+1}, u_{j+1}^{(0)}$

The sequence of operations

PECECE………..

Is denoted by $P(EC)^m\ E\ and\ is\ called\ a\ predictor - corrector\ Method$

$PM_p\ \ CM_c\ Method$:

We can use the estimate of the truncation error to modify the predicted and corrected values. Thus we may write this procedure as $PM_p\ CM_c$. This is called the modified predictor corrector method.

The modified P-C method becomes

Predicted Value $\quad P_{j+1} = \sum_{i=1}^{K} [a_i^{(0)} u_{j-i+1} + hb_i^{(0)}\ f_{j-i+1}]$

Modified Value $\quad M_{j+1} = p_{j+1} + c_{j+1}^{*} - (c_{j+1} - c_{j+1}^{*})^{-1}(p_j - c_j)$

Corrected Value $\quad C_{j+1} = \sum_{i=1}^{K} \left[a_i u_{j-1+1} + hb_i u'_{j-1+1} + 1\right) + hb_0 m'_{j+1}$

Final Value $\quad u_{j+1} = c_{j+1} + c_{j+1}(c_{j-1} - c_{j+1}^{*})^{-1}(p_{j+1} - c_{j+1})$

The quantity $p_j\ - c_j = 0$

Example:

Consider the P – C Method

$$P\ :\ u_{j+1} =\ u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$

$$C \ : \ u_{j+1} = u_j + \frac{h}{2}(u'_{j+1} + u'_j)$$

Modified predictor – corrector method is

$$P_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$
$$M_{j+1} = p_{j+1} - \frac{5}{6}(p_j - c_j)$$
$$C_{j+1} = u_j + \frac{h}{2}(m'_{j+1} + u'_j)$$
$$U_{j+1} = c_{j+1} + \frac{1}{6}\left(p_{j+1} - c_{j+1}\right) \qquad j = 1,2, \dots \dots$$

1. Solve the initial value problem $u' = 2tu^2,\ u(o) = 1$ $with\ h = 0.2\ on\ the\ interval$ $[0,0.4]$ $using\ the\ P - C\ method.$

Solution:

$$P \ : \ u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$
$$C \ : \ u_{j+1} = u_j + \frac{h}{2}(u'_j - u'_j)$$
$as\ (i)\ \ P(EC)^m$ E, m=2

$(ii)\ PM_p\ \ CM_c$

Solution

To use the predictor, we need the values of $u'(t)at = 0.2$

The values obtained from the enact solution is $u(t) = \frac{1}{1+t^2}$ are $u(0.2) = u_1 = \frac{1}{1+(0.2)^2}$

= 0.9615385

We have $u'(t) = -2tu^2$

$$u'(0.2) = -2(0.2)\ (0.9615385)^2$$

$$u'(0.2) = u'_1 = -0.3698225$$

$for\ j = 1 \qquad t_0 = 0, \qquad t_1 = 0.2, \qquad t_2 = 0.4$

Equation became s

$P\ :\ u_2^{(0)} = u_1 + \frac{h}{2}(3u_1' - u_0')$

Therefore: $u_0' = -2t_0u_0^2$

$= \ -2(0) = 0$

$u_2^{(0)} = \ 0.8505918$

$E\ :f(t_2, u_2^{(0)}$

$u_2'^{(0)} = \ -2t_2(u_2^{(0)})^2$

$u_2^{(0)} = 0.5788051$

Equation becomes

$C\ :\ u_2^{(1)} = u_1 + \frac{h}{2}[u_2'^{(0)} + u_1'$

$u_2^{(1)} = 0.8666756$

$E\ :f(t_2, u_2^{(1)}$

$u_2^{(1)} = \ -2t_2(u_2^{(1)})^2$

$= 0.6009015$

Equation becomes

$C\ :\ u_2^{(2)} = u_1 + \frac{h}{2}[u_2^{\prime(1)} + u_1'$

$u_2^{(2)} = 0.8644661$

$u(0.4) = \ 0.8644661$

(iii) $for\ PM_p\ \ CM_c\ method,$

We have

$$P_{j+1} = u_j + \frac{h}{2}\left(3u_j' - u_{j-1}'\right)$$
$$M_{j+1} = p_{j+1} - \frac{5}{6}\left(p_j - c_j\right)$$
$$C_{j+1} = u_j + \frac{h}{2}\left(m_{j+1}' + u_j'\right)$$
$$U_{j+1} = c_{j+1} + \frac{1}{6}\left(p_{j+1} - c_{j+1}\right),\ \ j = 1,2, \dots ..$$

To start the method, we need the values of $u(t) and\ u'(t)'\ at\ t = 0.2.$

The exact value are
$t_1 = 0.2, \qquad u_1 = 0.9615385,\ \ u_1' = 0.3698225$

For J = 1

Equation becomes $p_2 = u_1 + \frac{h}{2}(3u_1' - u_0')$

$$p_2 = 0.9615385 + \frac{0.2}{2}\left(3(-0.36982 - 0)\right) = 0.8505918$$

Equation becomes $m_2 = p_2 - \frac{5}{6}(p_1 - c_1)$

Taking $p_1 - c_1 = 0$

We obtain $m_2 = p_2$

$$m_2 = 0.8505918$$
$$m_2' = -2t_2 m_2^2$$
$$= 0.5788051$$

Equation becomes $c_2 = u_1 + \frac{h}{2}(m_2' + u_1')$

$$= 0.9615385 + \frac{0.2}{2}(-0.578805) + (-0.3698225)$$

$c_2 = 0.8666757$

Equation becomes $u_2 = c_2 + \frac{1}{6}(p_2 - c_2)$

$u_2 = 0.8639951$

$u(0.4) = u_2 = 0.8639951$

Exact solution:

$$u(t) = \frac{1}{1 + (0.4)^2} = 0.86207$$