SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT I– Functional Analysis -SMTA5302

Unit-I

Banach spaces

A Banach space is a linear space which is also, in a special way, a complete metric space. This combination of algebraic and metric structures opens up the possibility of studying linear transformations of one Banach space into another which have the additional property of being continuous,

Most of our work in this chapter centers around three fundamental theorems relating to continuous linear transformations. The HahnBanach theorem guarantees that a Banach space is richly supplied with continuous linear functionals, and makes possible an adequate theory of conjugate spaces. The open mapping theorem enables us to give a satisfactory description of the projections on a Banach space, and has the important closed graph theorem as one of its consequences. We use the uniform boundedness theorem in our discussion of the conjugate of an operator on a Banach space, and this in turn provides the setting for our treatment in the next chapter of the adjoint of an operator on a Hilbert space.

Virtually all this theory had its origins in analysis. Our present interest, however, lies in the study of form and structure, not in exploring the many applications of these ideas to specific problems. This chapter is therefore strongly oriented toward the algebraic and topological aspects of the matters at hand,

21 i

1. THE DEFINITION AND SOME EXAMPLES

We begin by restating the definition of a Banach space.

A normed linear space is a linear space N in which to each vector c there corresponds a real number, denoted by llcll and called the norm of c, in such a manner that (i) norm of any vector is greater than or equal to zero and zero if and only if that vector itself is zero (ii) norm of scalar multiplication of a vector is equal to absolute value of the scalar in to norm the vector and (iii) norm of sum of vectors is less than or equal to the sum of norm of the vectors

The non-negative real number is to be thought of as the length of the vector c. If we regard as a real function defined on N, this function is called the norm on N. It is easy to verify that the normed linear space N is a metric space with respect to the metric d defined by d(c,y) c — y". A Banach space is a complete normed linear space. Our main interest in this chapter is in Banach spaces, but there are several points in the body of the theory at which it is convenient to have the basic definitions and some of the simpler facts formulated in terms of normed linear spaces. For this reason, and also to emphasize the role of completeness in theorems which require this assumption, we work in the more general context whenever possible. The reader will find that the deeper theorems, in which completeness hypotheses are necessary, often make essential use of Baire's theorem.

Several simple but important facts about a normed linear space are based on the following inequality:

$$| \|x\| - \|y\| | \leq \|x - y\|. \qquad (1)$$

To prove this, it suffces to prove that

$$\|x\| - \|y\| \le \|x - y\|; \qquad (2)$$

for it follows from (2) that we also have

$$-(\|x\| - \|y\|) = \|y\| - \|x\| \le \|y - x\| = \|-(x - y)\| = \|x - y\|,$$

which together with (2) yields (1). We now prove (2) by observing that $\|x\|$= (c ‾ Y) + — -l- The main conclusion we draw from (1) is that the norm is a continuous function:

$$x_n \to x \Rightarrow \|x_n\| \to \|x\|.$$

This is clear from the fact that I llcnll $- \|x\| \mid \le \|x_n - x\|$, since $x_n \to x$ means that — —+ O. In the same vein, we can prove that addition and scalar multiplication are jointly continuous (see Problem 22-5), for

cn -4 C and y ⇒ x_n+ C Y and an a and T n → x ⇒anCn ax.

These assertions follow from

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \le \|x_n - x\| + \|y_n - y\|$$

and

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \le |\alpha_n| \, \|x_n - x\| + |\alpha_n - \alpha| \, \|x\|.$$

Our first theorem exhibits one of the most useful ways of forming new normed linear spaces out of old ones.

Theorem A. Let M be a closed linear subspace of a normed linear space N. If the norm of a coset c + M in the quotient space N/M is defined by

$\|x$+ Mll = inf { 'lc + mll : m e M}, (3)

then N/M is a normed linear space. Further, if N is a Banach space, then so is N/M.

PROOF. We first verify that (3) defines a norm in the required sense. It is obvious that lc + M Il 2 0; and since M is closed, it is easy to see that + Mll = O there exists a sequence {me} in M such that h[i] c + mkll O c is in M —c + M = M = the zero element of N/M. Next, we have Il(c + M) 4- (y -l- M)" = Il(z 4- y) 4- Mll = inf { llc + y + ml : m e M}= inf + y + m + m'll :m and m' E M} = inf {Il(c +m) + (y + : m and $m' \varepsilon M\}$ ≤ inf + + + : m and E M} = inf {lic -k ml : m e M) + inf -k m'll:m' e M} = "c + Mll -k "y + M ll. The proof of la(c 4- M)" = [al lc 4- Mll is similar.

Finally, we assume that N is complete, and we show that N/ M is also complete. If we start with a Cauchy sequence in N/ M, then by Problem 12-2 it sumces to show that this sequence has a convergent subsequence. It is clearly possible to find a subsequence {cn + M} of the original Cauchy sequence such that (Cl + M) — ($2 + M) Il < h, $\|$($2 -k M) — ($3 + M)ll < h, and, in general, Il (cn + M) — (cn+l + M) Il < $1/2^n$. We prove that this sequence is convergent in N/ M. We begin by choosing any vector in -k M, and we select in -l- M such that — < h. We next select a vector in 4- M such that — < k. Continuing in this way, we obtain a sequence {yn} in N such that (lyn — < $1/2^n$. If •m < n, then

— ynll = (ym — ym+l) + (ym+l — $y_{m+2}) + \cdots$

Ym — — + $\|y_{m+1} - y$ y,n+211 + . . .

+ — ynll < 1 pm + 1/2+1 + . . . + $1/2^{n-i} < 1/2^{m-1}$

,

so {yn} is a Cauchy sequence in N. Since N is complete, there exists a vector y in N such that yn y. It now follows from Il (cn -I- M) —

$(y + M)\| \leq \|y_n$— yl that + M — y + M, so N/M is complete.

In the following sections and chapters, we shall often have occasion to consider the quotient space of a normed linear space with respect to a closed linear subspace. In accordance with our theorem, a quotient space of this kind can always be regarded as a normed linear space in its own right.

We now describe some of the main examples of Banach spaces. In each of these, the linear operations are understood to be defined either coordinatewise or pointwise, whichever is appropriate in the circumstances.

Example 1. The spaces R and C—the real numbers and the complex numbers—are the simplest of all normed linear spaces. The norm of a number c is of course defined by = lcl, and each space is a Banach space.

Example 2. The linear spaces R" and C^n of all n-tuples c = $2,. , .cn) of real and complex numbers can be made into normed linear spaces in an infinite variety of ways, as we shall see below. If the norm is defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \qquad (4)$$

then we get the n-dimensional Euclidean and unitary spaces familiar to us from our earlier work. We denoted these spaces by R" and C^n in Part 1 of this book, and we know by the theorems of Sec. 15 that both are Banach spaces.

Each of the following examples consists of n-tuples of scalars, sequences of scalars, or scalar-valued functions defined on some nonempty set, where the scalars are the real numbers or the complex numbers. We do not normally specify which system of scalars is to be used, and it should be emphasized that both possibilities are allowed unless the contrary is clearly stated. Also, we make no distinction in notation between the real case and the complex case. When it turns out to be necessary to distinguish these two cases, we do so verbally, by referring, for instance, to "the complex space — These conventions are in accord with the standard usage preferred by most mathematicians, and they enable us to avoid a good deal of cumbersome notation and many unnecessary case distinctions.

Example 3. Let p be a real number such that 1 p < m. We denote by the space of all n-tuples c = ($1, c2, , cn) of scalars, with the norm defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}. \qquad (5)$$

Formula (4) is obviously the special case of (5) which corresponds to p — 2, so the real and complex spaces are the n-dimensional Euclidean and unitary spaces R" and C". It is easy to see that (5) satisfies conditions (1) and (3) required by the definition of a norm. In Problem 4 we outline a proof of the fact that (5) also satisfies condition (2), that is, that -l- yllp + llyllp. The completeness of follows from substantially the same reasoning as that used in the proof of Theorem 15-A, so is a Banach space.

Example 4. We again consider a real number p with the property that 1 p < 00, and we denote by the space of all sequences

$$x = \{x_1, x_2, \ldots, x_n, \ldots\}$$

of scalars such that $\Sigma_{n=1}^{\infty} |x_n|^p < \infty$, with th < 00 , with the norm defined by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right) \quad \text{llp} \qquad (6)$$

The reader will observe that the real and complex spaces are precisely the infinite-dimensional Euclidean and unitary spaces R^{∞} and C" defined in Problem 154. The proof of the fact that actually is a Banach space requires arguments similar to those used in Problems 15-3 and 15-4.

The Banach spaces discussed in these examples are all special cases of the important LP spaces studied in the theory of measure and integration. A detailed treatment of these spaces is outside the scope of this book, but we can describe them loosely as follows. An LP space essentially consists of all measurable functions f defined on a measure space X with measure m which are such that is integrable, with

(7)

taken as the norm. In order to include the spaces and within the theory of LP spaces, we have only to consider the sets {1, 2,· · · ,and {1, 2, . n, . . . } as measure spaces in which each point has measure 1, and to regard n-tuples and sequences of scalars as functions defined on these sets. Since integration is a generalized type of summation, formulas (5) and (6) are special cases of formula (7). [1]

Example 5, Just as in Example 3, we start with the linear space of all n-tuples c = ($1, $2, . . . , cn) of scalars, but this time we define the

[1] Several remarks and examples relating to LP spaces are scattered about in this and the next chapter. This fragmentary material is not essential for an understanding of these chapters, and may be disregarded by any reader without the necessary background. Brief sketches of the relevant ideas can be found in Taylor [41, chap. 7) and Loomis [27, chap. 3]. For more extended treatments, see Halmos [18], Zaanen [45], or Kolmogorov and Fomin [26, vol. 2].

norm by

$\|x\|$= max {ICII, • •, (8) This Banach space is commonly denoted by it, and the symbol is occasionally used for the norm given by (8), The reason for this practice lies in the interesting fact that

$$\|x\|_\infty = \lim \|x\|_p \quad \text{as } p \to \infty,$$

that is, that max lim (2 as $p \to \infty$. (9)

We briefly inspect the case n = 2 to see why this is true. Letc = (Cl, $2) be an ordered pair of real numbers with and 2 0. It is clear that = max {$1, $2} (CIP + $2p)1/p If = $2, then lim llcll = lim ('2C2P)11P = lim 2 11PC2 = = llcll„. And if < $2, then

lim = lim (CIP + lip = 11m • ([(C1/C2) P + 1]C2P) 1/p

= lim [(C1/C2)P + = — $\|x\|_\infty$.Example 6. Consider the linear space of all bounded sequences c = {x_1,

, x_n, ·. . } of scalars. By analogy with Example 5, we define the norm by $\|x\|$ =sup 'cnl,

and we denote the resulting Banach space by 190. The set c of all convergent sequences is easily seen to be a closed linear subspace of and is therefore itself a Banach space. Another Banach space in this family is the subset co of c which consists of all convergent sequences with limit O.

Example 7. The Banach space of primary interest to us is the space e(X) of all bounded continuous scalar-valued functions defined on a topological space X, with the norm given by

=sup$x$ II.

This norm is sometimes called the uniform norm, because the statement that fn converges to f with respect to this norm means that fn converges to f uniformly on X. The fact that this space is complete amounts to the fact that if f is the uniform limit of a sequence of bounded continuous functions, then f itself is bounded and continuous. If, as above, we consider n-tuples and sequences as functions defined on {1, 2, · · · and {1, 2, . · · , n, ·,· then the spaces and are the special cases of e(X) which correspond to choosing X to be the sets just mentioned, each with the discrete topology.

ı The real space e(X) and the complex space e(X) are, of course, the spaces previously denoted by e(X,R) and e(X,C).

Many important properties of a Banach space are closely linked to the shape of its closed unit sphere, that is, the set S = {c: 1}. One basic property of S is that it is always convex, in the sense (see Problem 32-5) that if c and y are any two vectors in S, then the vector z = ac -k ßy is also in S, where a and ß are non-negative real numbers such that a -kß = 1; for llzll —$\|\alpha x + \beta y\| \leq \alpha\|x\| + \beta\|y\| \leq \alpha + \beta = 1.$

In this connection, it is illuminating to consider the shape of S for certain simple examples. Let our underlying linear space be the real linear

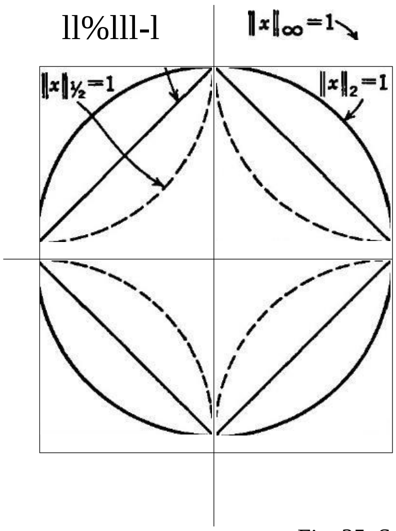


Fig. 35. Some closed unit spheres.

space R^2 of all ordered pairs c = (cu,C2) of real numbers. As we have seen, there are many different norms which can be defined on R^2, among which are the following: —— + IC21 ; IIc11 2 = + and $\|x\|_\infty$— max {Ictl, Figure 35 illustrates the closed unit sphere which corresponds to each of these norms. In the first case, S is the square with vertices (1,0), (0,1), (—1,0), (O, —1); in the second, it is the circular disc of radius 1; and in the third, it is the square with vertices (1,1), (−1,1), (−1,−1), (1,-1). If we consider the norm defined by $\|x$Il (ICIlp + IX21P) 1/P,

where 1 p < 00 , and if we allow p to increase from 1 to 00 , then the corresponding S's swell continuously from the first square mentioned to the second. We note that S is truly "spherical" p = 2. These considerations also show quite clearly why we always assume that p 2 1; for if we were to define Itcllp by formula (12) with p < 1, then S = {c: llcllp

1} would not be convex (see the star-shaped inner portion of Fig. 35). For p < 1, therefore, formula (12) does not yield a norm.

In the above examples, we have exhibited several different types of Banach spaces, and there are yet others which we have not mentioned. Amid this diversity of possibilities, it is well to realize that any Banach space can be regarded—from the point of view of its linear and norm structures alone—as a closed linear subspace of e(X) for a suitable compact Hausdorff space X. We prove this below, in our discussion of the natural imbedding of a Banach space in its second conjugate space.

Problems

1. Let N be a non-zero normed linear space, and prove that N is a Banach space $\Leftrightarrow \{x : \|x\| = 1\}$ is complete,

2, Let a Banach space B be the direct sum of the linear subspaces M and . N, so that B = M O N. If z = c + y is the unique expression of a vector z in B as the sum of vectors c and y in M and N, then a new norm can be defined on the linear space B by llzll'$= \|x\|$

Prove that this actually is a norm. If B' symbolizes the linear space B equipped with this new norm, prove that B' is a Banach space if M and N are closed in B.

3. Prove Eq. (9) for the case of an arbitrary positive integer n.

4. In this problem we sketch the proofs—and we ask the reader to fill in the details—of some important inequalities relating to n-tuples $x = (x_1, x_2, \ldots$, cn) and y —$(y_1, y_2, \ldots$, YR) of scalars. Whenever p occurs alone, and nothing is said to the contrary, we assume that 1 p < oo ; and whenever p and q occur together, we assume that both are greater than 1 and that l/p + l/q$= 1$.

(a) Show that a and b 2 0 a lipb llq a/p 4- b/q. (If a = O or b = O, the conclusion is clear, so assume that both are positive. If lc e (O, 1), define f(t) for t 1 by f(t)

Note that f(l) = O and f' (t) 2 0, and conclude that Ict $(1 -$k). If a b, put t = a/b and lc = l/p; if a < b, **put** t— b/a and lc = l/q; and in each case, draw the required conclusion.)

(b) Prove Holder's inequality: lc,Vil $\leq \|x\|_p\|y\|_q$. (If c = O or y — O, the inequality is obvious, so assume that both are O. Put a.' = (Icil/llcllp) p and bi = and use part (a) to obtain IciYil/llcllpllyllq -l- bi/q. Add these inequalities for i = 1 2 , n, and conclude that

(' ZiYi I)/tI z tIptIyIIq lip + l/q - 1.)

(c) Prove Minkowslci's inequality: llc + yllp + llyllp. (The inequality is evident when p = 1, so assume that p > 1.

Use Hölder's inequality to obtain

jlc + + $^p = \sum_{i=1}^{n} |x_i$ Yilp = $\sum_{1}$ $|x_i + y_i| \, |x_i + y_i|^{p-1}$

$\leq \sum_{i=1}^{n} |x_i|$ | Ici + + $\cdot \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$

$+y\|_p{}^{p/q}$.) $\leq (\|x\|_p + \|y\|_p) \, \|x +$

When p = q — 2, Hölder's inequality becomes Cauchy's inequality as stated and proved in Sec. 15. The Hölder and Minkowski inequalities can easily be extended from finite sums to series. For readers with some knowledge of the theory of measure and integration, we remark that these inequalities can also be stated in the following much more general forms: if f is in LP and g is in Lq, then their pointwise product fg is in Ll and

$$\|fg\|_1 \leq \|f\|_p\|g\|_q;$$

and if f and g are both in Lp, then f + g is also in L, and llf + gllp llflp + l!gllp•

It is to be understood that f and g are measurable functions defined on an arbitrary measure space and that the norms occurring in these inequalities are those defined by formula (7).

2. CONTINUOUS LINEAR TRANSFORMATIONS

Let N and N' be normed linear spaces with the same scalars, and let T be a linear transformation of N into N' When we say that T is continuous, we mean that it is continuous as a mapping of the metric space N into the metric space N'. By Theorem 13-B, this amounts to the condition that cn c in N T (cn) —4 T (c) in N'. Our main purpose in this section is to convert the requirement of continuity into several more useful equivalent forms and to show that the set of all continuous linear transformations of N into N' can itself be made into a normed linear space in a natural way.

Theorem A. Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then the following conditions on T are an equivalent to one another:

(1) T is continuous;

(2) T is continuous at the origin, in the sense that cn O T (cn) O;

[1] In the future, whenever we mention two normed linear spaces with a view to considering linear transformations of one into the other, we shall always assume— without necessarily saying so explicitly—that they have the same scalars.

(3) there ecists a real number K 2 0 with the property that T (c) Il ≤Kllcll for every c E N;

(4) if S = {c: 1} is the closed unit sphere in N, then its image T (S) is a bounded set in N'.

PROOF. (1) (2). If T is continuous, then since T (O) = O it is certainly continuous at the origin. On the other hand, if T is continuous at the origin, then cn —i c cn $- x \to 0 \Rightarrow T(x_n - x) \to 0 \Leftrightarrow T(x_n) - T(x)$

$\to 0$ T(cn) T (c), so T is continuous.

(3). It is obvious that (3) (2), for if such a K exists, then cn —i O clearly implies that T (cn) O. To show that (2) (3), we assume that there is no such K. It follows from this that for each positive integer n we can find a vector cn such that Il T (cn)ll > nllc„ll, or equivalently, such that I l T $x_n/n\|x_n\|)\|$ > 1. If we now put

$y_n =$cn/nllcnE ,

then it is easy to see that yr, O but T(yn) + O, so T is not continuous at the origin.

(3) (4). Since a non-empty subset of a normed linear space is bounded it is contained in a closed sphere centered on the origin, it is evident that (3) (4) ; for if llcll 1, then I l T K. To show that (4) (3), we assume that T (S) is contained in a closed sphere of radius K centered on the origin. If c = O, then T (c) = o, and clearly I l T (c) I l ≤Kllcll ; and if O, then c/llcll e S, and therefore Il T (c/lcll)l K, so again we have I l T Kllcll.

If the linear transformation T in this theorem satisfies condition (3), so that there exists a real number K 2 0 with the property that

$$\|T(x)\| \leq K\|x\|$$

for every c, then K is called a bound for T, and such a T is often referred to as a bounded linear transformation. According to our theorem, T is bounded it is continuous, so these two adjectives can be used interchangeably. We now assume that T is continuous, so that it satisfies condition (4), and we define its norm by

Il T Il = sup { IT (c)l} : llcll 1 }. (1)

When N {O} , this formula can clearly be written in the equivalent form

IT Il = sup { Il T : I$\|x\| = 1\}$. (2)

It is apparent from the proof of Theorem A that the set of all bounds for T equals the set of all radii of closed spheres centered on the origin which contain T (S). This yields yet another expression for the norm of T, namely,

$\|T\|$= inf {K:K 2 0 and $\|T(x)\| \leq K\|x\|$for all ; (3)

and from this we see at once that

Il Il Tll llcll for all c.

We now denote the set of all continuous (or bounded) linear transformations of N into N' by $\mathcal{B}(N,N')$, where the letter "G" is intended to suggest the adjective "bounded." It is a routine matter to verify that this set is a linear space with respect to the pointwise linear operations defined by Eqs. 44-(1) and 44-(2) and to show that formula (1) actually does define a norm on this linear space. We summarize and extend these remarks in

Theorem B. If N and N' are normed linear spaces, then the set $\mathcal{B}(N,N$of an continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations and the norm defined by (1). Further, if N' is a Banach space, then is also a Banach space.

PROOF. We leave to the reader the simple task of showing that $\mathcal{B}(N,N'$is a normed linear space, and we prove that this space is complete when

N' is.

Let f Tn} be a Cauchy in $\mathcal{B}(N,N')$. If x is an arbitrary sequence in vector in N, then Il Tm(c) — = shows that { Tn(c) is a Cauchy sequence in N' ; and since N' is complete, there exists a vector in N'—we denote it by T(z)—such that Tn(c) —+ T (c). This defines a mapping T of N into N', and by the joint continuity of addition and scalar multiplication, T is easily seen to be a linear transformation. To conclude the proof, we have only to show that T is continuous and that Tn T with respect to the norm on $\mathcal{B}(N,N')$. By the inequality 46-(1), the norms of the terms of a Cauchy sequence in a normed linear space form a bounded set of numbers, so

$\|T(x)\|$= Illim = lim Il sup (Il Tn Il llcll) — (sup Il Tn'l)$\|x\|$

shows that T has a bound and is therefore continuous. It remains to be proved that I l Tn — T I l -4 0. Let e > 0 be given, and let no be a positive integer such that m, n 2 no I l Tm Tall < e. If < 1 and m, n 2 no, then

Il Tm(c) - = $\|(T_m - T_n)(x)\| \leq \|T_m - T_n\| \; \|x\| \leq \|T_m - T_n\| < \epsilon.$

We now hold m fixed and allow n to approach 00, and we see that

$\|$Tm(c) —$T_n(x)\| \rightarrow \|T_m(x) - T(x)\|$, from which we conclude that

Il Tm(c) —$T(x)\| \leq \epsilon$ for all m 2 no and all c such that 1. This shows that Il Tm — T Il e for all m 2 no, and the proof is complete.

IRt N be a normed linear space. We call a continuous linear transformation of N into itself an operator on N, and we denote the normed linear space of all operators on N by (B(N) instead of $\mathfrak{B}(N,N$ Theorem B shows that G(N) is a Banach space when N is. Furthermore, if operators are multiplied in accordance with formula 44-(3), then G(N) is an algebra in which multiplication is related to the norm by

$$\|TT'\| \leq \|T\| \, \|T'\|. \qquad (5)$$

This relation is proved by the following computation:

$$\begin{aligned} \|TT'\| &= \sup \{\|(TT')(x)\| : \|x\| \leq 1\} = \sup \{\|T(T'(x))\| : \|x\| \leq 1\} \\ &\leq \sup \{\|T\| \, \|T'(x)\| : \|x\| \leq 1\} = \|T\| \sup \{\|T'(x)\| : \|x\| \leq 1\} \\ &= \|T\| \, \|T'\|. \end{aligned}$$

We know from the previous section that addition and scalar multiplication in (B(N) are jointly continuous, as they are in any normed linear space. Property (5) permits us to conclude that multiplication is also jointly continuous:

Tn — T and T'n— T' TnT'n— TT'.

This follows at once from

$$\|T_nT'_n - TT'\| = \| T_n(T'_n - T') + (T_n - T)T' \| \leq \|T_n\| \, \|T'_n - T'\| + \|T_n - T\| \, \|T'\|.$$

We also remark that when N {O}, then the identity transformation I is an identity for the algebra G(N). In this case, we clearly have

$$\|I\| = 1; \qquad (6)$$

for $\|I\| = \sup$: "c" 1 } = sup : llcll 1}= 1.

We complete this section with some definitions which will often be useful in our later work. Let N and N' be normed linear spaces. An isometric isomorphism of N into N' is a one-to-one linear transformation T of N into N' such that Il $T(x)\| = \|x\|$ for every c in N; and N is said to be isometrically isomorphic to N' if there exists an isometric isomorphism of N onto N', This terminology enables us to give precise meaning to the statement that one normed linear space is essentially the same as another.

Problems

1. If M is a closed linear subspace of a normed linear space N, and if T is the natural mapping of N onto N/M defined by T (c) = c + M, show that T is a continuous linear transformation for which Il T Il≤ 1
2. If T is a continuous linear transformation of a normed linear space N into a normed linear space N', and if M is its null space, show that T induces a natural linear transformation T' of N/M into N' and that Il T'Il = $T\|$.
3. Let N and N' be normed linear spaces with the same scalars. If N is infinite-dimensional and N' show that there exists a linear transformation of N into N' yhich is not continuous. (We shall see in Problem 7 that if N is finite-dimensional, then every linear transformation of N into N' is automatically continuous.)
4. Let a linear space L be made into a normed linear space in two ways, and let the two norms of a vector c be denoted by and llcll'. These norms are said to be equivalent if they generate the same topology on L. Show that this is the case there exist two positive real numbers K I and such that K Il(cll K21!cll for all c. (If L is finite-dimensional, then any two norms defined on it are equivalent. See Problem 7.)

5. If n is a fixed positive integer, the spaces t; (1 p 00) consist of a single underlying linear space with different norms defined on it. Show that these norms are all equivalent to one another. (Hint: show that convergence with respect to each norm amounts to coordinatewise convergence.)
6. If N is an arbitrary normed linear space, show that any linear transformation T of (1 p 00) into N is continuous. (Hint: if ₂, · · ·, enl is the natural basis for C, where e, is the n-tuple with 1 in the ith place and o's elsewhere, then an arbitrary vector c in can be written uniquely in the form

c = arel + -I- + anen,

and from this we get T (c) = alT(el) + a2T(e2) + + anT(en) ; now apply the hint given for Problem 5.)
7. Let N be a finite-dimensional normed linear space with dtmension n > O, and let {el, e, . . . , enl be a basis for N. Each vector z in N can be written uniquely in the form

C = alel + +. . .

If T is the one-to-one linear transformation of N onto h" defined by $(x) =$ (al, a2, • , an), then T—1 is continuous by Problem 6.
   (a) Prove that T is continuous. (Hint: if T is not continuous, then for some e > O there exists a sequence {yn} in N such that yn —» O and l! $T(y_n)\| \geq$ c; if zn = $y_n/\|T(y_n)\|$, then O and $\|T(z_n)\|$= 1; the subset of consisting of all vectors of norm 1 is compact, so {T(zn) } has a subsequence which converges to a vector with norm 1; now use the continuity of T¯1.)
   (b) Show that every linear transformation of N into an arbitrary normed linear space N' is continuous.
   (c) Show that any other norm defined on N is equivalent to the given norm.
   (d) Show that N is complete, and infer from this that every finitedimensional linear subspace of an arbitrary normed linear space is closed.
8. It is a simple consequence of Problem 7 that every finite-dimensional normed linear space is locally compact. Prove the converse, that is, that a locally compact normed linear space N is finite-dimensional. Hint: the closed unit sphere S of N is compact, so there is a finite subset of S, say {$1, $2, . . . , , with the property that each point of S is distant by less than h from one of the cds; let M be the linear subspace of N spanned by the cds; and show that M = N (to do this, assume that there exists a vector y not in M, use the fact that M is closed to infer that d = d(y,M) > O, find mo in M such that d lly — moll 3d/2, and deduce the contradiction that the vector yo in S defined by yo = (y — mo)/lly — moll is distant from M by at least 23)•

3. THE HAHN-BANACH THEOREM

One of the basic principles of strategy in the study of an abstract mathematical system can be stated as follows: consider the set of all structure-preserving mappings of that system into the simplest system of the same type. This principle is richly fruitful in the structure theory (or representation theory) of groups, rings, and algebras, and we shall see in the next section how it works for normed linear spaces.

We have remarked that the spaces R and C are the simplest of all normed linear spaces. If N is an arbitrary normed linear space, the above principle leads us to form the set of all continuous linear transformations of N into R or C, according as N is real or complex. This set— it is or (B(N,C)—is denoted by N* and is called the conjugate space of

N. The elements of N* are called continuous linear functionals, or more briefly, functionals. [1] It follows from our work in the previous section that if these functionals are added and multiplied by scalars

$$\{|f(x)| : \|x\| \leq 1\} \qquad \|f\| — \text{ sup for all c),}$$

$$= \inf \{K : K \geq 0 \text{ and } |f(x)| \leq K\|x\|$$

then N* is a Banach space.

When we consider various specific Banach spaces, the problem arises of determining the concrete nature of the functionals associated with these spaces. It is not our aim in this section to explore the ample body of theory which centers around this problem, and in any case, the machinery necessary for such an enterprise (mostly the theory of measure and integration) is not available to us. Nevertheless, for the reader who may have the required background, we mention some of the main facts without proof.

Let X be a measure space with measure m, and let p be a given real number such that 1 < p < 00. Consider the Banach space LP of all measurable functions f defined on X for which is integrable. If g is a function in Lq, where l/p -l- l/q = 1, we define a function Fg on

$$\text{Fg(f)} = \int f(x)g(x) \text{ dm(c).}$$

The Hölder inequality for integrals mentioned at the end of Problem 46-4 shows that

$$\begin{aligned} |F_g(f)| &= \left|\int f(x)g(x)\, dm(x)\right. \\ &\leq \int |f(x)g(x)|\, dm(x) \\ &\leq \|f\|_p \|g\|_q. \end{aligned}$$

We conclude from this that Fg is a well-defined scalar-valued linear function on LP with the property that llgllq, and is therefore a functional on Lp. It can be shown that equality holds here, so that

$$\|F_g\| = \|g\|_q.$$

It can also be shown that every functional on LP arises in this way, so the mapping g Fg (which is clearly linear) is an isometric isomorphism of Lg onto Lp*. This statement is usually expressed by writing

$$L_p{}^* = L_q, \qquad (1)$$

where the equality sign is to be interpreted in the sense just explained.

If we specialize these considerations to n-tuples of scalars, we see that (1) becomes

$$(l_p^n)^* = l_q^n. \qquad (2)$$

Further, it can be shown that

(3) and that (12) * = . (4)

We sketch proofs of (2), (3), and (4) in the problems. When we consider sequences of scalars, then for 1 < p < [0]0 we have the following special case of (1):

(5) If p — 1, we obtain a natural extension of (3) :

(6)

The corresponding extension of (4) is another matter, for it is false that $= l_1$.Instead, we have

(7)

What is L* ? We saw in Sec. 46 that is a special case of e(X), so this question leads naturally to the problem of determining the nature of the conjugate space e* (X). The classic solution of this problem for a space X which is compact Hausdorff (or even normal) is known as the Riesz representation theorem, and it depends on some of the deeper parts of the theory of measure and integration (see Dunford and Schwartz [8, pp. 261—265]). The situation is somewhat simpler for the case in which X is an interval [a,b] on the real line, but even here an adequate treatment requires a knowledge of Stieltjes integrals (see Riesz and Sz.-Nagy [35, secs. 49—51]).

Most of the theory of conjugate spaces rests on the Hahn-Banach theorem, which asserts that any functional defined on a linear subspace of a normed linear space can be extended linearly and continuously to the whole space without increasing its norm. The proof is rather complicated, so we begin with a lemma which serves to isolate its most diffcult parts.

Lemma. Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If co is a vector not in M, and if

$$M_0 = M + [x_0]$$

is the linear subspace spanned by M and co, then f can be extended to a functional fo defined on MO such that Il foll =$\|f\|$.

PROOF. We first prove the lemma under the assumption that N is a real normed linear space. We may assume, without loss of generality, that "fl! 1. Since co is not in M, each vector y in MO is uniquely expressible in the form y = c + aco with c in M. It is clear that the definition fo(c + acc) = fo(c) + afo(co) = f(c) + aro extends f linearly to MO for every choice of the real number ro = fo(co). Since we are trying to arrange matters so that = 1, our problem is to show that ro can be chosen in such a way that Ifo(c + azo)l + for every c in M and every a 0. Since fo(c 4- acc) = f(c) + aro, this inequality can be written as

$$-\|x + \alpha x_0\| \le f(x) + \alpha r_0 \le \|x + \alpha x_0\|$$

or

$$-\|x + \alpha x_0\| \le \alpha r_0 \le -f(x) + \|x + \alpha x_0\|,$$

which in turn is equivalent to

$$-f\left(\frac{x}{\alpha}\right) - \left\|\frac{x}{\alpha} + x_0\right\| \le r_0 \le -f\left(\frac{x}{\alpha}\right) + \left\|\frac{x}{\alpha} + x_0\right\|. \qquad (8)$$

We now observe that for any two vectors and in M we have

$$f(x_2) - f(x_1) = f(x_2 - x_1) \le |f(x_2 - x_1)| \le \|f\|\,\|x_2 - x_1\|$$
$$= \|x_2 - x_1\| = \|(x_2 + x_0) - (x_1 + x_0)\| \le \|x_2 + x_0\| + \|x_1 + x_0\|,$$

so

$$-f(x_1) - \|x_1 + x_0\| \le -f(x_2) + \|x_2 + x_0\|. \qquad (9)$$

If we define two real numbers a and b by

a = sup { —f(c) $-\|x + x_0\| : x \in M\}$ and b = inf { —f(c) +

then (9) shows that a b. If we now choose ro to be any real number such that a ro b, then the required inequality (8) is satisfied and this part of the proof is complete.

We next use the result of the above paragraph to prove the lemma for the case in which N is complex. Here f is a complex-valued functional defined on M for which I l fl = 1. We begin by remarking that a complex linear space can be regarded as a real linear space by simply restricting the scalars to be real numbers. If g and h are the real and imaginary parts of f, so that f(c) = g(c) + ih(c) for every in M, then both g and h are easily seen to be real-valued functionals on the real space M; and since I f f] = 1, we have "gli 1. The equation

$$f(ic) = if(c),$$

together with f(ic) — g(ic) + ih(ic) and if(c) = -I- ih(c)) = ig(c) — h(c), shows that h(c) = —g(ic), so we can write f(c) = g(c) — ig(ic). By the above paragraph, we can extend g to a real-valued functional go on the real space MO in such a way that [goll = llg}l, and we define fo for c in MO by fo(c) = go(c) — igo(ic). It is easy to see that fo is an extension of f from M to MO, thatfo(c -l- y) = fo(c) -k fo(y), and thatfo(ac) = afo(z) for all real a's. The fact that the property last stated is also valid for all complex a's is a direct consequence of fo(ic) = go(ic) — igo(i^2c) = go(ic) + igo(c) $= i(g_0(x)$ $-$ igo(ic)) = ifo(c),

so fo is linear as a complex-valued function defined on the complex space Mo. All that remains to be proved is that "foll = 1, and we dispose of this by showing that if is a vector in MO for which llcll = 1, then $(x)| \leq$1. If fo(c) is real, this follows from fo($) = go(c) and "goll 1. If fo(c) is complex, then we can write fo(c) = re" with r > O, so

$$|f_0(x)| = r = e^{-i\theta}f_0(x) = f_0(e^{-i\theta}x);$$

and our conclusion now follows from Ile$^{-i\theta}x\| = |$I Il — 1 and the fact that is real.

Theorem A (the Hahn-Banach Theorem). Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. Then f can be ectended to a functional fo defined on the whole space N such that $\|f_0\| = \|f\|$.

PROOF. The set of all extensions of f to functionals g with the same norm defined on subspaces which contain M is clearly a partially ordered set with respect to the following relation: gr means that the domain of gr is contained in the domain of g2, and g2(c) = gr(c) for all c in the domain of gl. It is easy to see that the union of any chain of extensions is also an extension and is therefore an upper bound for the chain. Zorn's lemma now implies that there exists a maximal extension fo. We complete the proof by observing that the domain of fo must be the entire space N, for otherwise it could be extended further by our lemma and would not be maximal.

As we stated in the introduction to this chapter, the main force of the Hahn-Banach theorem lies in the guarantee it provides that any Banach space (or normed linear space) has a rich supply of functionals. This property is to be understood in the sense of the following two theorems, on which most of its applications depend.

Theorem B. If N is a normed linear space and co is a non-zero vector in N, then there exists a functional fo in N* such that fo(co) and llfoll =1. PROOF. Let M = {azo) be the linear subspace of N spanned by co, and define f on M by f(aco) = c oll. It is clear that f is a functional on M such that f(co) and llftl — ‾ 1. By the Hahn-Banach theorem, f can be extended to a functional fo in N* with the required properties.

Among other things, this result shows that N* separates the vectors in N, for if c and y are any two distinct vectors, so that c — y O, then there exists a functional f in N* such that — y) O, or equivalently, f(c) f(y).

Theorem C. If M is a closed linear subspace of a nortned linear space N and co is a vector not in M, then there exists a functional fo in N* such that fo(M) = O and fo(co) O.

PROOF. The natural mapping T of N onto N/M (see Problem 47-1) is a continuous linear transformation such that T (M) = O and

$$T(x_0) = x_0 + M \neq 0.$$

By Theorem B, there exists a functional f in (N/M) * such that f(co -l- M) O.

If we now define fo by fo(c) = f(T(c)), then fo is easily seen to have the desired properties.

These theorems play a critical role in the ideas developed in the following sections, and their significance will emerge quite clearly in the proper context.

Problems

1. Let M be a closed linear subspace of a normed linear space N, and let co be a vector not in M. If d is the distance from co to M, show that there exists a functional fo in N* such thatfo(M) = O, fo(co) = 1,and Il foll =l/d'
2. Prove that a normed linear space N is separable if its conjugate sp*ce N* is. (Hint: let {fn} be a countable dense set in N* and {cn} a corresponding set in N such that 1 and 2 llfnll/2; let M be the set of all linear combinations of the cn's whose coeffcients are rational or—if N is complex—have rational real and imaginary parts; and use Theorem C to show that = N.) We remark that N* need not be separable when N is, for h is easily proved to be separable, h* = L, and is not separable .

REFERENCE BOOKS:

1. G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, International Book company, 1963.

2. Balmohan V. Limaye, Functional Analysis, Second Edition, New Age International Pvt Ltd, 1997.

3. M.Thamban Nair, Functional Analysis, A First Course, PHI, 2002

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT II – Functional Analysis – SMTA5302

Unit-II

Banach spaces (Continuation)

1. THE NATURAL IMBEDDING OF N INN^{**}

Since the conjugate space N* of a normed linear space N is itself a normed linear space, it is possible to form the conjugate space (N*)* of N*. We denote this space by N* * , and we call it the second conjugate space of N.

The importance of N* * rests on the fact that each vector c in N gives rise to a functional in N* *. If we denote a typical element of N* by f, then is defined by

$$F_x(f) = f(x).$$

In other words, we invert the usual practice by regarding the symbol f(c) as specifying a function of f for each fixed c, and we emphasize this point of view by writing f(c) in the form Fz(f). A simple manipulation of the definition shows that F, is linear:

$+ \beta g)$– (af + ßg) (c)

– af(c) 4- ßg(c) aFz(f) + ßFz(g).

If we now compute the norm of Fz, we see that

– sup {IF 1) sup 1} sup fllfll : llfH 1)

Theorem 48-B is exactly what is needed to guarantee that equality holds here, so for each c in N we have

$$\|F_x\| = \|x\|.$$

It follows from these observations that c Fz is a norm-preserving mapping of N into N* * Fz is called the functional on N* induced by the vector c, and we refer to functionals of this kind as induced functionals. We next point out that the mapping c F% is linear and is therefore an isometric isomorphism of N into N* *. To verify this, we must show that Fz+v(f) = (FZ + and Faz(f) = for every f in N*. The first of these relations

$$F_{x+y}(f) = f(x + y)$$

follows from

= f(c) + f(Y)

– Fz(f) + Fv(f)

$$= (F_x + F_y)(f),$$

and the second is proved similarly. The isometric isomorphism c F, is called the natural imbedding of N in N* * , for it allows us to regard N as part of N* * without altering any of its structure as a normed linear space. We write

$$N \subseteq N^{**},$$

where this set inclusion is to be understood in the sense just explained.

A normed linear space N is said to be reflexive if N = N* * The spaces (and L,) for 1 < p < are reflexive, for = and

$$l_p^{**} = l_q^* = l_p.$$

It follows from Problem 48-3 that the spaces for 1 p are also reflexive. Since N* * is complete, N is necessarily complete if it is reflexive. If N is complete, however, it is not necessarily reflexive, as we see from co = h and co* * $= l_1^* = l_\infty$. If X is a compact Hausdorff space, it can be shown that e(X) is reflexive X is a finite set.

There is an interesting criterion for reflexivity, which depends on the concept of the weak topology on a normed linear space N. This is defined to be the weak topology on N generated by the functions in N* in the sense of Sec. 19; that is, it is the weakest topology on N with respect to which all the functions in N* remam continuous. The criterion referred to is the following: if B is a Banach space, and if S = {c: 1} is its closed unit sphere, then B is reflexive S is compact in the weak topology. This fact is something one should know about Banach spaces, but we shall have no need for it ourselves, so we state it without proof.[1] Far more important for our purposes is the weak* topology on N* which is defined to be the weak topology on N* generated by all the induced functionals Fz in N*

* This situation is rather complicated, so we shall try to make clear just what is going on.

First of all, N* (like N) is a normed linear space, and it therefore has a topology derived from its character as a metric space. This is called the strong topology. N* * is the set of all scalar-valued linear functions defined on N* which are continuous with respect to its strong topology. The weak topology on N* (like the weak topology on N) is the weakest topology on N* with respect to which all the functions in N* * are continuous, and clearly this is weaker than its strong topology. So far, as

[1] See Hille and Phillips [20, p. 38] or Dunford and Schwartz [8, p. 425].

we have indicated, these concepts apply equally to N and N*. However, since N* is the conjugate space of N, the natural imbedding enables us to consider N as part of N* * We now form the weakest topology on N* with respect to which all the functions in N—regarded as a subset of N**—remain continuous. This is the weak* topology, and it is evidently weaker than the weak topology. The weak* topology can be given a more explicit description, in which its defining subbasic open sets are displayed. Consider a vector c in N and its induced functional in N ** .The weak* topology on N* is the weakest topology under which all such F*'s are continuous. If fo is an arbitrary element in N* , and if e > O is given, then the set

S(c, fo, e) = {f:fE N* and — < e) = and If(c) <

is an open set (in fact, a neighborhood of fo) in the weak* topology. Furthermore, the class of all sets of this kind, for all c's, fo's, and e's, is the defining open subbase for the weak* topology. All finite intersections of these sets constitute an open base for this topology, and the open sets themselves are all unions of these finite intersections.

We remark at this point that N* is a Hausdorff space with respect to its weak* topology. This follows at once from the fact that if f and g are distinct functionals in N* , then there must exist a vector c in N such that f(c) g(c); for if we put e —— If(c) — then S(c, f, e) and S(c, g, e) are disjoint neighborhoods of f and g in the weak* topology.

Let us now consider the closed unit sphere S* in N* , that is, the set S* = {f:f e N* and I l f" 1}. [1] It is an easy consequence of Problem 2 that S* is compact in the strong topology N is finite-dimensional, so the strong compactness of S* is a very stringent condition. If N is complete, it follows from Problem 3 and our unproved criterion for reflexivity that S* is compact in the weak topology N is reflexive, so the weak compactness of S* is still a fairly substantial restriction. We state these facts to emphasize that the situation is quite different with the weak* topology, for here S* is always compact.

Theorem A. If N is a normed linear space, then the closed unit sphere S* in N* is a compact Hausdorff space in the weak* topology.

PROOF. We already know that S* is a Hausdorff space in this topology, so we confine our attention to proving compactness. With each vector z in N we associate a compact space Cc, where is the closed interval [—%[1] cll, cll] or the closed disc {z: llcll}, according as N is real or complex. By Tychonoff's theorem, the product C of all the Cz's iB also a compact space. For each c, the values f(c) of all f's in S* lie in C_z.This enables us to imbed S* in C by regarding each f in S* as identical with the array of all its values at the vectors c in N. It is clear from the definitions of the topologies concerned that the weak* topology on S* equals its topology as a subspace of C; and since C is compact, it suffces to show that S* is closed as a subspace of C. We show that if g is in S*, then g is in S*. If we consider g to be a function defined on the index set N, then since g is in C we have llcll for every c in N, It therefore suffces to show that g is linear as a function defined on N. Let e > O be given, and let c and y be any two vectors in N. Every basic neighborhood of g intersects S* , so there exists an f in S* such that lg(c) — < e/3, lg(y) — < e/3, and lg(c -i- y) — + < e/3, Since fis linear, -k y) — f(z) — f(y) = O, and we therefore have

lg(c + Y) — g(c) — —
[g(y) —

$$- g(y)| = |[g(x+y) - f(x+y)] - [g(x) - f(x)]$$
$$f(y)]| \leq |g(x+y) - f(x+y)| + |g(x) - f(x)|$$
$$+ |g(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

e/3 + e/3 + /3

The fact that this inequality is true for every e > O now implies that $+ y)$— g(c) + g(y). We can show in the same way that g(az) = ag(c) for every scalar a, so g is linear and the theorem is proved.

We are now in a position to keep the promise made in the last paragraph of Sec. 46, for the following result is an obvious consequence of our preceding work.

[1] When we use the adjective "closed" in referring to S*, we intend only to emphasize the inequality ıı f" 1, as contrasted with llflt < 1.

Theorem B. Let N be a normed linear space, and let S* be the compact Hausdorff space obtained by imposing the weak* topology on the closed unit sphere in N*. Then the mapping c Fz, where Fz(f) = f(c) for each f in S* , is an isometric isomorphism of N into If N is a Banach space, this mapping is an isometric isomorphism of N onto a closed linear subspace *of* $\mathcal{C}(S^*)$.

This theorem shows, in effect, that the most general Banach space is essentially a closed linear subspace of e(X), where X is a compact Hausdorff space. The purpose of representation theorems in abstract mathematics is to reveal the structures of complex systems in terms of simpler ones, and from this point of view, Theorem B is satisfying to a degree. It must be pointed out, however, that we know next to nothing about the closed linear subspaces of though we know a good deal about e(X) itself. Theorem B is therefore somewhat less revealing than aopears at first glance. We shall see in Chaps. 13 and 14 that the corresponding representation theorem for Banach algebras is much more significant and useful.

Problems

1. Let X be a compact Hausdorff space, and justify the assertion that e(X) is reflexive if X is finite.
2. If N is a finite-dimensional normed linear space of dimension n, show that N* also has dimension n. Use this to prove that N is reflexive. 3, If B is a Banach space, prove that B is reflexive B* is reflexive.
4. Prove that if B is a reflexive Banach space, then its closed unit sphere S is weakly compact.
5. Show that a linear subspace of a normed linear space is closed it is weakly closed.

2. THE OPEN MAPPING THEOREM

In this section we have our first encounter with basic theorems which require that the spaces concerned be complete. The following rather technical lemma is the key to these theorems.

Lemma. If B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B', then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'. PROOF. We denote by Sr and the open spheres with radius r centered on the origin in B and B'. It is easy to see that

T(Sr) = T(rSl) - rT(SI),

so it suffices to show that T (SD contains some S'r.

We begin by proving that T (SD contains some S'. Since T is onto, we see that B' = u: I T (Sn). B' is complete, so Baire's theorem implies that some '1'(Sng) has an interior point yo, which may be assumed to lie in T (Sno). The mapping y y — yo is a homeomorphism of B' onto itself, so T(Sn) — yo has the origin as an interior point. Since yo is in T (Sno), we have T (Sno) — yo C T(S2no); and from this we obtain T(Sn,) ‾ Yo = $T(S_{n_0}) - y_0$ C T(S2no), which shows that the origin is an interior point of 'T(S2no). Multiplication by any non-zero scalar is a homeomorphism of B' onto itself, so '1'(S2no) = 2noT(S1) = 2noT(Sl) ; and it follows from this that the origin is also an interior point of I'(SI), so S: C T(SI) for some positive number e.

We conclude the proof by showing that S' C T (SD, which is clearly equivalent to S'e13 T (SD. Let y be a vector in B' such that llyll < c.

Since y is in 765, there exists a vector in B such that < 1 and IIY — < c/2, where = T (CD. We next observe that C T(Sh), so there exists a vector in B such that < h and Il (Y — VI) — < e/4, where = T(C2). Continuing in this way, we obtain a sequence {cn} in B such that < $1/2^{n-1}$ and 'ly — (y: + y, +.+.yn)ll < e/2", where yn = T(cn). If we put

$$s_n = x_1 + x_2 + \cdots + x_n,$$

then it follows from < $1/2^{n-1}$ that {sn} is a Cauchy sequence in B for which llsnll + + . $\cdots + \|x_n\| < 1 + \frac{1}{2} + \cdots$ + $1/2^{n-1}$ < 2.

B is complete, so there exists a vector c in B such that sn -4 c; and

$\|x$Il — Illim s lim llsnll 2 < 3 shows that c is in S3. All that remains is to notice that the continuity of T yields T (c) — T (lim sn) = lim T(sn) = lim (Yl + + $\cdots + y_n) = y,$ from which we see that y is in T (SD.

This makes our main theorem easy to prove.

Theorem A (the Open Mapping Theorem). If B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B', then T is an open mapping.

PROOF. We must show that if G is an open set in B, then T (G) is also an open set in B'. If y is a point in T (G), it suffices to produce an open sphere centered on y and contained in T (G). Let c be a point in G such that T (c) = y. Since G is open, c is the center of an open sphere— which can be written in the form c + Sr—contained in G. Our lemma now implies that T (Sr) contains some S'rt. It is clear that y -f- S'r: is an open sphere centered on y, and the fact that it is contained in T (G) follows at once from y + s'r, c y + T(Sr) = T(c) + T(Sr) = + Sr) C T(G).

Most of the applications of the open mapping theorem depend more directly on the following special case, which we state separately for the sake of emphasis.

Theorem B. A one-to-one continuous linear transformation of one Banach space onto another is a homeomorphism. In particular, if a one-to-one linear transformation T of a Banach space onto itself is continuous, then its inverse is automatically continuous.

As our first application of Theorem B, we give a geometric characterization of the projections on a Banach space. The reader will recall from Sec. 44 that a projection E on a linear space L is simply an idempotent = E) linear transformation of L into itself. He will also recall that projections on L can be described geometrically as follows:

(1) a projection E determines a pair of linear subspaces M and N such that L = M O N, where M = $\{E(x) : x \in L\}$ and N $x : E(x)$= O} are the range and null space of E;

(2) a pair of linear subspaces M and N such that L = M O N determines a projection E whose range and null space are M and N (if z = c + y is the unique representation of a vector in L as a sum of vectors in M and N, then E is defined by

$E(z) = x$).

These facts show that the study of projections on L is equivalent to the study of pairs of linear subspaces which are disjoint and span L.

In the theory of Banach spaces, however, more is required of a projection than mere linearity and idempotence. A projection on a Banach space B is an idempotent operator on B; that is, it is a projection on B in the algebraic sense which is also continuous. Our present task is to assess the effect of the additional requirement of continuity on the geometric descriptions given in (1) and (2) above, The analogue of (1) is easy.

Theorem C. If P is a projection on a Banach space B, and if M and N are its range and null space, then M and N are closed linear subspaces of B such that B = M O N.

PROOF. P is an algebraic projection, so (1) gives everything except the fact that M and N are closed. The null space of any continuous linear transformation is closed, so N is obviously closed; and the fact that M is also closed is a consequence of

$$M = \{P(x) : x \in B\} = \{x : P(x) = x\} = \{x : (I - P)(x) = 0\},$$

which exhibits M as the null space of the operator I — P.

The analogue of (2) is more difficult, for Theorem B is needed in its proof.

Theorem D. Let B be a Banach space, and let M and N be closed linear subspaces of B such that B = M C N. If z = c 4- y is the unique representation of a vector in B as a sum of vectors in M and N, then the mapping P defined by P (z) = c is a projection on B whose range and nun space are M and N.

PROOF. Everything stated is clear from (2) except the fact that P is continuous, and this we prove as follows. By Problem 46-2, if B' denotes the linear space B equipped with the norm defined by

$$\|z\|' = \|x\| + \|y\|,$$

then B' is a Banach space; and since llP(z) Il $= \|x\| \leq \|x\| + \|y\| = \|z\|'$,

P is clearly continuous as a mapping of B' into B. It therefore suffces to prove that B' and B have the same topology. If T denotes the identity mapping of B' onto B, then

$\|T(z)\| = \|z\| =$llc + Yl llcll + $= \|z\|'$

shows that T is continuous as a one-to-one linear transformation of B' onto B. Theorem B now implies that T is a homeomorphism, and the proof is complete.

This theorem raises some interesting and significant questions. Let M be a closed linear subspace of a Banach space B. As we remarked at the end of Sec. 44, there is always at least one algebraic projection defined on B whose range is M, and there may be a great many. However, it might well happen that none of these are continuous, and that consequently none are projections in our present sense. In the light of our theorems, this is equivalent to saying that there might not exist any closed linear subspace N such that B = M N. What sorts of Banach spaces have the property that this awkward situation cannot occur? We shall see in the next chapter that a Hilbert space—which is a special type of Banach space—has this property. We shall also see that this property is closely linked to the satisfying geometric structure which sets Hilbert spaces apart from general Banach spaces.

We now turn to the closed graph theorem. Let B and B' be Banach spaces. If we define a metric on the product B X B' by

(x_1,y_1),($2,Y2)) = max {IICI — $211, — y r*" } ,

then the resulting topology is easily seen to be the same as the product topology, and convergence with respect to this metric is equivalent to coordinatewise convergence. Now let T be a linear transformation of B into B'. We recall that the graph of T is that subset of B X B' which consists of all ordered pairs of the form (c, T(c)). Problem 26-6 shows that if T is continuous, then its graph is closed as a subset of B X B'. In the present context, the converse is also true.

Theorem E (the Closed Graph Theorem), If B and B' are Banach spaces, and if T is a linear transformation of B into B', then T is ccntinuous its graph is closed.

PROOF, In view of the above remarks, we may confine our attention to proving that T is continuous if its graph is closed. We denote by B, the linear space B renormed by llclll — + I l Since

$$\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_1,$$

T is continuous as a mapping of Bl into B'. It therefore suffices to show that B and Bl have the same topology. The identity mapping of Bl onto B is clearly continuous, for llcll + l] T (C) I l If we can show that Bl is complete, then Theorem B will guarantee that this mapping is a homeomorphism, and this will conclude the proof. Let {cn} be a Cauchy sequence in BI. It follows that {cn} and {T (cn)} are also Cauchy sequences in B and B'; and since both of these spaces are complete, there exist vectors c and y in B and B' such that $\|x_n -$ and I l T(cn) — yll O. Our assumption that the graph of T is closed in B X B' implies that (c,y) lies on this graph, so T (c) = y. Th completeness of Bl now follows from

$\|x_n - x\|_1 = \|x_n - x\|$ + $_n - x)\| = \|x_n - x\|$ IIT(c + llT(cn) — $T(x)\|$ + IIT(cn) — $y\| \to 0$

The closed graph theorem has a number of interesting applications to problems in analysis, but since our concern here is mainly with matters of algebra and topology, we do not pause to illustrate its uses in this direction. [1]

Problems

1. Let a Banach space B be made into a Banach space B' by means of a new norm, and show that the topologies generated by these norms are the same if either is stronger than the other,
2. In the text, we used Theorem B to prove the closed graph theorem. Show that Theorem B is a consequence of the closed graph theorem.
3. Let T be a linear transformation of a Banach space B into a Banach space B'. If t f,} is a set of functionals in B'* which separates the vectors in B', and if fiT is continuous for each fi, prove that T is continuous.

3. THE CONJUGATE OF AN OPERATOR

We shall see in this section that each operator T on a normed linear space N induces a corresponding operator, denoted by T* and called the conjugate of T, on the conjugate space N*. ,
Our first task is to define T* and our second is to investigate the properties of the mapping T —i T*. We base our discussion on the following theorem.

Theorem A (the Uniform Boundedness Theorem). Let B be a Banach space and N a normed linear space. If { Ti} is a non-empty set of con[I] see Taylor [41, pp. 181-185].

tinuous linear transformations of B into N with the property that { $T_i(x)$ }is a bounded subset of N for each vector c in B, then { Il Till is a bounded set of numbers; that is, { Ti} is bounded as a subset of G(B,N).
PROOF. For each positive integer n, the set Fn - and $\|T_i(x)\| \leq$ n for all i}
is clearly a closed subset of B, and by our assumption we have

$$B = \cup_{n=1}^{\infty} F_n.$$

Since B is complete, Baire's theorem shows that one of the Fa's, say Fno, has non-empty interior, and thus contains a closed sphere So with center co and radius ro > O. This says, in effect, that each vector in every set Ti(So) has norm less than or equal to no; and for the sake of brevity, we express this fact by writing $\|T_i(S_0)\| \leq$ no. It is clear that So — co is the closed sphere with radius ro centered on the origin, so (So — co)/ro is the closed unit sphere S. Since co is in So, it is evident that Ti(So — co)" 2no. This yields Il 2no/ro, so Il Till ≤2no/ro for every i, and the proof is complete.

This theorem is often called the Banach-Steinhaus theorem, and it has several significant applications to analysis. See, for example, Zygmund [46, vol. 1, pp. 165—168] or Gål [11]. For the purposes we have in view, our main interest is in the following simple consequence of it.

Theorem B. A non-empty subset X of a normed linear space N is bounded ⇔ f(X) is a bounded set of numbers for each f in N*.

PROOF. Since Il fll llcll, it is obvious that if X is bounded, then f(X) is also bounded for each f.

In order to prove the other half of the theorem, it is convenient to exhibit the vectors in X by writing X = {Zi}. We now use the natural imbedding to pass from X to the corresponding subset {F of N* * .Our assumption that f(X) = {f(x,•)} is bounded for each f is clearly equivalent to the assumption that is bounded for each f, and since N* is complete, Theorem A shows that { Fz] is a bounded subset of N* * We know that the natural imbedding preserves norms, so X is evidently a bounded subset of N.

We now turn to the problem of defining the conjugate of an operator on a normed linear space N.

Let L be the linear space of all scalar-valued linear functions defined on N. The conjugate space N* is clearly a linear subspace of L. Let T be a linear transformation of N into itself which is not necessarily continuous. We use T to define a linear transformation T' of L into itself, as follows. If f is in L, then T' (f) is defined by

$$[T'(f)](x) = f(T(x)). \qquad (1)$$

We leave it to the reader to verify that T' (f) actually is linear as a function defined on N, and also that T' is linear as a mapping of L into itself.

The following natural question now presents itself. Under what circumstances does T' map N* into N* ? This question has a simple and elegant answer: T'(N*) c N* T is continuous. If we keep Theorem B in mind, the proof of this statement is very easy; for if S is the closed unit sphere in N, then T is continuous T (S) is bounded is bounded for each f in N* [T' (f)) (S) is bounded for each fin N* T' (f) is in N* for each f in N*.

We now assume that the linear transformation T is continuous and is therefore an operator on N. The preceding developments allow us to consider the restriction of T' to a mapping of N* into itself. We denote this restriction by T* , and we call it the conjugate of T. The action of T* is given by

$$[T^*(f)](x) = f(T(x)), \qquad (2)$$

in which—in contrast to (l)—f is understood to be a functional on N, and not merely a scalar-valued linear function. T* is clearly linear, and the following computation shows that it is continuous:

- sup : Ilfll 1} sup : I l fl and 1) sup : llfll and llcll 1} sup {Il f" Il T Il : Il f] and 1)

Since I l T Il = sup { Il T (c)ll : 1}, we see at once from Theorem 48-B that equality holds here, that is, that

$$\|T^*\| = \|T\|. \qquad (3)$$

The mapping T —i T* is thus a norm-preserving mapping of (B(N) into

We continue in this vein by observing that the mapping T T* also has the following pleasant algebraic properties:

$$(\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^*, \qquad (4)$$

$$(T_1T_2)^* = T_2^*T_1^*,$$

(5) and (6)

$$I^* = I.$$

The proofs of these facts are easy consequences of the definitions. We illustrate the principles involved by proving (5). It must be shown that $T_2)^*(f)$= (T $^*T_1^*)(f)$ for each f in N* , and this means that

$$[(T_1T_2)^*(f)](x) = [(T_2^*T_1^*)(f)](x)$$

for each f in N* and each c in N. A simple computation now shows that

$$\begin{aligned}[(T_1T_2)^*(f)](x) = f((T_1T_2)(x)) &= f(T_1(T_2(x))) = [T_1^*(f)](T_2(x)) \\ &= [T_2^*(T_1^*(f))](x) = [(T_2^*T_1^*)(f)](x).\end{aligned}$$

It may be helpful to the reader to have the following summary of the results of this discussion.

Theorem C. If T is an operator on a normed linear space N, then its conjugate T* defined by Eq. (2) is an operator on N* , and the mapping T T* is an isometric isomorphism of G(N) into which reverses products and preserves the identity transformation.

The general significance of the ideas developed here can be understood only in the light of the theory of operators on Hilbert spaces. Some preliminary comments on these matters are given in the introduction to the next chapter.

Problems

1, Let B be a Banach space and N a normed linear space. If { Tn} is a sequence in such that T (c) = lim Tn(c) exists for each c in B, prove that T is a continuous linear transformation.

2. Let T be an operator on a normed linear space N. If N is considered to be part of N* * by means of the natural imbedding, show that T* * is an extension of T. Observe that if N is reflexive, then T* * = T

3. Let T be an operator on a Banach space B. Show that T has an inverse T—1 T* has an inverse and that in this case

REFERENCE BOOKS:

1. G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, International Book company, 1963.
2. Balmohan V. Limaye, Functional Analysis, Second Edition, New Age International Pvt Ltd, 1997.
3. M.Thamban Nair, Functional Analysis, A First Course, PHI, 2002

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT III – Functional Analysis – SMTA5302

Unit-III

Hilbert Spaces

One of the principal applications of the theory of Banach algebras developed to the study of operators on Hilbert spaces. Our purpose in this chapter is to present enough of the elementary theory of Hilbert spaces and their operators to provide an adequate foundation for the deeper theory discussed in these later chapters.

We shall see from the formal definition given in the next section that a Hilbert space is a special type of Banach space, one which possesses additional structure enabling us to tell when two vectors are orthogonal (or perpendicular). The first part of the chapter is concerned solely with the geometric implications of this additional structure.

As we have said before, the objects of greatest interest in connection with any linear space are the linear transformations on that space. In our treatment of Banach spaces, we took advantage of the metric structure of such a space by focusing our attention on its operators. A Banach space, however, is still a bit too general to yield a really rich theory of operators. One fact that did emerge, which is of great significance for our present work, is that with each operator T on a Banach space B there is associated an operator T* (its conjugate) on the conjugate space B*. We shall see below that one of the central properties of a Hilbert space H is that there is a natural correspondence between H and its conjugate space H*. If T is an operator on H, this correspondence makes it possible to regard the conjugate T* as acting on H itself (instead of H*), where it can be compared with T. These ideas lead to the concept of the adjoint of an operator on a Hilbert space, and they make it easy to understand the importance of operators (such as self adjoint and normal operators) which are related in simple ways to their adjoints.

1. THE DEFINITION AND SOME SIMPLE PROPERTIES

The Banach spaces studied in the previous chapter are little more than linear spaces provided with a reasonable notion of the length of a vector. The main geometric concept missing in an abstract space of this type is that of the angle between two vectors. The theory of Hilbert spaces does not hinge on angles in general, but rather on some means of telling when two vectors are orthogonal.

In order to see how to introduce this concept, we begin by considering the three-dimensional Euclidean space R · A vector in R^3 is of course an ordered triple c = ($1, $2, $3) of real numbers, and its norm is defined by

Il _ (l c 112 + +$|x_3|^2)^{1/2}$.

In elementary vector algebra, the inner product of and another vector Y = (VI, Y2, YD is defined by

(c,y) = .t1Yl + C2Y2 + .t3Y3, [1]and this inner product is related to the norm

by

$$(x,x) = \|x\|^2.$$

We assume that the reader is familiar with the equation

(c y) llyfl cos 9,

where 9 is the angle between c and y, and also with the fact that c and y are orthogonal precisely when (c,y) = O.

Most of these ideas can readily be adapted to the three-dimensional unitary space C^8 . For any two vectors c = ($1, $2, $3) and y — $(y_1, y_2,$ in this space, we define their inner product by

(c,y) = + + C3Y3. (1) Complex conjugates are introduced here to guarantee that the relation

$$(x,x) = \|x\|^2$$

remains true. It is clear that the inner product defined by (1) is linear as a function of c for each fixed y, and is also conjugate-symmetric, in the sense that (c,y) = (y,z). In this case, it is no longer possible to think of (c,y) as representing the product of the norms of c and y and the cosine of the angle between them, for (c,y) is in general a complex number. Nevertheless, if the condition (c,y) = 0 is taken as the definition of orthogonality, then this concept is just as useful here as it is in the real case.

With these ideas as a background, we are now in a position to give our basic definition. A Hilbert space is a complex Banach space whose norm arises from an inner product, that is, in which there is defined a complex function (c,y) of vectors c and y with the following properties:

(1) (ac + ßy, z) = a(z,z) -k ;

$\overline{(x,y)} = (y,x);$
$(x,x) = \|x\|^2.$

It is evident that the further relation

(c, ay 4- ßz) — $\bar{\alpha}(x,y)$

is a direct consequence of properties (1) and (2).

The reader may wonder why we restrict our attention to complex spaces. Why not consider real spaces as well? As a matter of fact, we could easily do so, and many writers adopt this approach. There are a few places in this chapter where complex scalars are necessary, but the theorems involved are not crucial, and we could get along with real scalars without too much diffculty. It is only in the complex case, however, that the theory of operators on a Hilbert space assumes a really satisfactory form. This will appear with particular clarity in the next chapter, where we make essential use of the fact that every polynomial equation of the nth degree with complex coeffcients has exactly n complex roots (some of which, of course, may be repeated). For this and other reasons, we limit ourselves to the complex case throughout the rest of this book.

The following are the main examples of Hilbert spaces. In accordance with the above remarks, the scalars in each example are understood to be the complex numbers.

Example 1. The space G, with the inner product of two vectors

$x = (x_1, x_2, \ldots$, c,) and y $= (y_1, y_2, \ldots, y_n)$ defined by

=.tiYi.

It is obvious that conditions (1) to (3) are satisfied.

Example 2. The space h, with the inner product of the vectors

$$x = \{x_1, x_2, \ldots, x_n, \ldots\} \text{ and } y = \{y_1, y_2, \ldots, y_n, \ldots\}$$

defined by

=cnYn•

The fact that this series converges—and thus defines a complex number— for each c and y in h is an easy consequence of Cauchy's inequality.

Example 3, The space 142 associated with a measure space X with measure m, with the inner product of two functions f and g defined by (f,g) = f dm(c).

This Hilbert space is of course not part of the omcial content of this book, but we mention it anyway in case the reader has some knowledge of these matters.

As our first theorem, we prove a fundamental relation known as the Schwarz inequality.

\ Theorem A. If c and y are any two vectors in a Hilbert space, then llyll.

PROOF. When y = O, the result is clear, for both sides vanish. When y O, the inequality is equivalent to "Cll. We may therefore confine our attention to proving that if llyll = 1, then we have llcll for all x. This is a direct consequence of the fact that

Fig. 36. Schwarz's inequality.

= (c,c) − - 11311 ² -

An inspection of Fig. 36 will reveal the geometric motivation for this computation.

It follows easily from Schwarz's inequality that the inner product in a Hilbert space is jointly continuous:

x_n-4 C and y (cn,yn) (x,y).

To prove this, it sumces to observe that

$$|(x_n,y_n) - (x,y)| = |(x_n,y_n) - (x_n,y) + (x_n,y) - (x,y)| \leq |(x_n,y_n) - (x_n,y)| + |(x_n,y) - (x,y)| = |(x_n, y_n - y)| + |(x_n - x, y)| \leq \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|.$$

A well-known theorem of elementary geometry states that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. This fact has an analogue in the present context, for in any Hilbert space the so-called parallelogram law holds: $\|x + y\|^2 + \|x - y\|$2 = + 211 VI' 2.

This is readily proved by writing out the expression on the left in terms of inner products:

$$\|x + y\|^2 + \|x - y\|^2 = (x + y, x + y) + (x - y, x - y)$$

= (c,c) + (c,y) + (y,c) + (y,y) + (c,c) — (c,y) — (y,c) + (WY)

= 2(c,c) -F 2(y,y) — 211c112 + 211y112.

The parallelogram law has the following important consequence for our work in the next section.

Theorem B. A closed convec subset C of a Hilbert space H contains a unique vector of smallest norm.

PROOF. We recall from the definition in Problem 32-5 that since C is convex, it is non-empty and contains (c + y)/2 whenever it contains c and y. Let d = inf e C}. There clearly exists a sequence {cn} of vectors in C such that $\|x_n\|$-9 d. By the convexity of C, (cm -i- cn)/2 is in C and I l (cm + cn)/211 ≥, so + 2 2d. Using the parallelogram law, we obtain

$$\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$$
$$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2;$$

and since + 211cn11 ^{2} — 4d ^{2} 2d^2 + 2d^2 — 4d^2 = O, it follows that $\{x_n\}$is a Cauchy sequence in C. Since H is complete and C is closed, C is complete, and there exists a vector c in C such that cn -4 c. It is clear by the fact that I IT. II = illim cnll = lim llcnll = d that c is a vector in C with smallest norm. To see that c is unique, suppose that c' is a vector in C other than c which also has norm d. Then (c + c')/2 is also in C, and another application of the parallelogram law yields

$$\left\|\frac{x + x'}{2}\right\|^2 = \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \left\|\frac{x - x'}{2}\right\|^2$$
$$< \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} = d^2,$$

which contradicts the definition of d.

The parallelogram law has another interesting application, which depends on the fact that in any Hilbert space the inner product is related to the norm by the following identity:

4(c,y) $= \|x + y\|^2 -$ "c— + + iy11 2$- i\|x - iy\|^2.$ (2)

This is easily verified by converting the expression on the right into inner products.

Theorem C. If B is a complex Banach space whose norm obeys the paralZelogram law, and if an inner product is defined on B by (2), then B is a Hilbert space.

PROOF : All that is necessary is to make sure that the inner product defined by (2) has the three properties required by the definition of a Hilbert space. This is easy in the case of properties (2) and (3). Property (1) is best treated by splitting it into two parts:

$$(x + y, z) = (x,z) + (y,z),$$

and (ac,y) = a(z,y). The first requires the parallelogram law, and the second follows from the first. We ask the reader (in Problem 6) to work out the details.

This result has no implications at all for our future work. However, it does provide a satisfying geometric insight into the place Hilbert spaces occupy among all complex Banach spaces: they are precisely those in which the parallelogram law is true.

Problems

1. Show that the series which defines the inner product in Example 2 is convergent.
2. The Hilbert cube is the subset of consisting of all sequences

$$x = \{x_1, x_2, \ldots, x_n, \ldots\}$$

such that l/n for all n. Show that this set is compact as a subspace of 12.
3. For the special Hilbert space G, use Cauchy's inequality to prove Schwarz's inequality.
4. Show that the parallelogram law is not true in (n > 1).
5. In a Hilbert space, show that if $\|x\| = \|y\|$ = 1, and if > O is given, then there exists ö > O such that "(c + y)/211 > 1$- \delta \Rightarrow$
< e. A Banach space with this property is said to be uniformly convex. Sec ʳraylor [41, p. 231].
6. Give a detailed proof of Theorem C.

2. ORTHOGONAL COMPLEMENTS

Two vectors and y in a Hilbert space H are said to be orthogonal (written L y) if (c,y) = O. The symbol _L is often pronounced "perp." Since = (y,c), we have c -L y — y -L c. It is also clear that c L 0 for every c, and (c c) shows that O is the only vector orthogonal to itself. One of the simplest geometric facts about orthogonal vectors is the Pythagorean theorem:

$$y \Rightarrow \|x_{+\ yt1}\ ^2 = \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

A vector c is said to be orthogonal to a non-empty set S (written c _L S) if c -L y for every y in S, and the orthogonal complement of S—denoted by S [I] —is the set of all vectors orthogonal to S. The following statements are easy consequences of the definition:

$$\{0\}^\perp = H; H^\perp = \{0\};$$
$$S \cap S^\perp \subseteq \{0\};$$
$$S_1 \subseteq S_2 \Rightarrow S_1{}^\perp \supseteq S_2{}^\perp;$$

S [L] is a closed linear subspace of H.

It is customary to write (S -L) -L in the form S [XL] . Clearly, S C S [XL] .

Let M be a closed linear subspace of H. We know that M [1] is also a closed linear subspace, and that M and M [I] are disjoint in the sense that they have only the zero vector in common. Our aim in this section is to prove that H = M C M [I], and each of our theorems is a step in this direction.

Theorem A. Let M be a closed linear subspace of a Hilbert space H, let c be a vector not in M, and let d be the distance from c to M. Then there exists a unique vector yo in M such that '(c — yo] = d.

PROOF. The set C = c + M is a closed convex set, and d is the distance from the origin to C (see Fig. 37). By Theorem 52-B, there exists a unique vector zo in C such that = d. The vector yo = c — zo is easily seen to be in M, and (lc — yol = llzoll = d. The uniqueness of yo follows from the fact that if is a vector in M such that yo and llc — yr] = d, then = c — yr is a vector in C such that Zl zo and Il Il — d, which contradicts the uniqueness of zo.

We use this result to prove

Theorem B. If M is a proper closed linear subspace of a Hilbert space H, then there exists a non-zero vector zo in H such that zo I M.

PROOF. Let c be a vector not in M, and let d be the distance from x to M.

By Theorem A, there exists a vector yo in M such that llc — yol $= d$.We define zo by zo = $ — yo (see Fig. 37), and we observe that since d > O, zo is a non-zero vector. We conclude the proof by showing that if y is an arbitrary vector in M, then zo _L y. For any scalar a, we have

$$\|z_0 - \alpha y\| = \|x - (y_0 + \alpha y)\| \geq d = \|z_0\|,$$
$$\|z_0 - \alpha y\|^2 - \|z_0\|^2 \geq 0$$
$$-\bar{\alpha}(z_0,y) - \alpha\overline{(z_0,y)} + |\alpha|^2\|y\|^2 \geq 0.$$

2

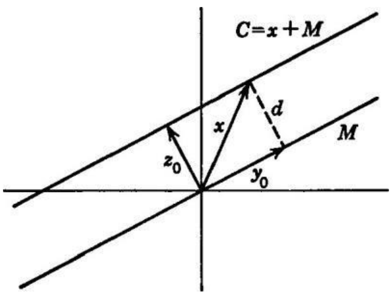
so and

If we put a = $\beta(z_0,y)$ for an arbitrary real number ß, then (l) becomes —2ßl + $\beta^2|(z_0,y)|^2\|y\|^2 \geq 0$.

If we now put a = $|(z_0,$ and b = llY112, we obtain

$-2\beta a$+ ß [2]ab 0, so ßa(ßb — 2) 0 (2)

for all real p. However, if a > O, then (2) is obviously false for al suffciently small positive p. We see from this that a = O, which means that zo -L y.



This proof of Theorem B may strike the reader as being excessively dependent on ingenious computations. If so, he will be pleased to learn that the ideas developed in the next section can be used to provide another proof which is free of computation.

mg. 37 In order to state our next theorem, we need the following additional concept. Two non-empty subsets Sl and S2 of a Hilbert space are said to be orthogonal (written Sl -L S2) if c _L y for all in Sl and y in S2.

Theorem C. If M and N are closed linear subspaces of a Hilbert space H such that M -L N, then the linear subspace M + N is also closed.

PROOF. Let z be a limit point of M + N. It sumces to show that z is in M + N. There certainly exists a sequence {zn} in M + N such that z. z. By the assumption that M L N, we see that M and N are disjoint, so each zn can be written uniquely in the form zn = c -l- y where is in M and y is in N. The Pythagorean theorem shows that — yn t1 [2], so {cn} and {yn} are Cauchy sequences in M and N. M and N are closed, and therefore complete, so there exist vectors c and y in M and N such that cn c and yn y. Since c -l- y is in M + N, our conclusion follows from the. fact that Z = lim zn = lim (cn + yn) = lim + lim yr, $= x + y$.

The way is now clear for the proof of our principal theorem.

Theorem D. If M is a closed linear subspace of a Hilbert space H, then

$$H = M \oplus M^{\perp}.$$

PROOF. Since M and M i are orthogonal closed linear subspaces of H, Theorem C shows that M + M I is also a closed linear subspace of H. We prove that M 4- M -L equals H. If this is not so, then by Theorem B there exists a vector zo O such that zo _L (M -F M I). This non-zero vector must evidently lie in M L M L ; and since this is impossible, we infer that H = M + M I . To conclude the proof, it suffces to observe that since M and M I are disjoint, the statement that H = M + M I can be strengthened to H = M M l .

The main effect of this theorem is to guarantee that a Hilbert space is always rich in projections. In fact, if M is an arbitrary closed linear subspace of a Hilbert space H, then Theorem 50-D shows that there exists a projection defined on H whose range is M and whose null space is M i . This satisfactory state of affairs is to be contrasted with the situation in a general Banach space, as explained in the remarks following Theorem 50-D.

Problems

1. If S is a non-empty subset of a Hilbert space, show that S I = S ILL.

2, If M is a linear subspace of a Hilbert space, show that M is closed

$$\Leftrightarrow M = M^{\perp\perp}.$$

3. If S is a non-empty subset of a Hilbert space H, show that the set of all linear combinations of vectors in S is dense in H S I —04. If S is a non-empty subset of a Hilbert space H, show that S -L -L is the closure of the set of all linear combinations of vectors in S. This is usually expressed by saying that S IX is the smallest closed linear subspace of H which contains S.

3. ORTHONORMAL SETS

An orthonormal set in a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vectors; that is, it is a nonempty subset tet•} of H with the following properties:

(1) $i \neq j \Rightarrow e_i \perp e_j$;

(2) $\|e_i\| = 1$ for every i.

If H contains only the zero vector, then it has no orthonormal sets. If H contains a non-zero vector c, and if we normalize c by considering e = C/ llcll, then the single-element set {e} is clearly an orthonormal set. More generally, if {cd is a non-empty set of mutually orthogonal nonzero vectors in H, and if the cds are normalized by replacing each of them by ei = then the resulting set fed is an orthonormal set.

Example 1. The subset {el, Q, . . . , eni of G, where ei is the n-tuple with 1 in the ith place and o's elsewhere, is evidently an orthonormal set in this space.

Example 2. Similarly, if en is the sequence with 1 in the nth place and o's elsewhere, then {el, 0, , • , , , . } is an orthonormal set in h.

At the end of this section, we give some additional examples taken from the field of analysis.

Every aspect of the theory of orthonormal sets depends in one way or another on our first theorem.

Theorem A. Let {el, e2, . . . , en} be a finite orthonormat set in a Hilbert space H. If is any vector in H, then

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2; \quad (1)$$

further, $x - \sum_{i=1}^{n} (x,e_i)e_i \perp e_j$ (2) for each j.

PROOF. The inequality (1) follows from a computation similar to that used in proving Schwarz's inequality:

$$\begin{aligned} O &\leq \| x - \sum_{i=1}^{n} (x,e_i)e_i \|^2 \\ &= \left(x - \sum_{i=1}^{n} (x,e_i)e_i,\ x - \sum_{j=1}^{n} (x,e_j)e_j\right) \\ &= (x,x) - \sum_{i=1}^{n} (x,e_i)\overline{(x,e_i)} - \sum_{j=1}^{n} (x,e_j)\overline{(x,e_j)} + \sum_{i=1}^{n} \sum_{j=1}^{n} (x,e_i)\overline{(x,e_j)} \quad (ei,e,.) \\ &= \|x\|^2 - \sum_{i=1}^{n} |(x,e_i)|^2. \end{aligned}$$

To conclude the proof, we observe that

$$\left(x - \sum_{i=1}^{n} (x,e_i)e_i, e_j\right) = (x,e_j) - \sum_{i=1}^{n} (x,e_i)(e_i,e_j) = (x,e_j) - (x,e_j) = 0,$$

from which statement (2) follows at once.

The reader should note that the inequality (1) can be given the following loose but illuminating geometric interpretation: the sum of the squares of the components of a vector in various perpendicular directions does not exceed the square of the length of the vector itself. This is usually called Bessel's inequality, though, as we shall see below, it is only a special case of a more general inequality with the same name. In a similar vein, relation (2) says that if we subtract from a vector its components in several perpendicular directions, then the result has no component left in any of these directions.

Our next task is to prove that both parts of Theorem A generalize to the case of an arbitrary orthonormal set. The main problem here is to show that the sums in (1) and (2) can be defined in a reasonable way when no restriction is placed on the number of under consideration. The key to this problem lies in the following theorem.

Theorem B. If is an orthonormal set in a Hilbert space H, and if c is any vector in H, then the set S = : (c,e,•) 0} is either empty or countable.

PROOF. For each positive integer n, consider the set

Sn = $\{e_i : |(x,e_i)|^2$ > llclt ^{2}/n}.

By Bessel's inequality, Sn contains at most n — 1 vectors. The conclusion now follows from the fact that S = un_l^m Sn.

As our first application of this result, we prove the general form of Bessel's inequality.

Theorem C (Bessel's Inequality). If {ed is an orthonormal set in a Hilbert space H, then

$\Sigma|(x,e_i)|^2 \leq \|x\|^2$ (3) for every vector c in H.

PROOF. Our basic obligation here is to explain what is meant by the sum on the left of (3). Once this is clearly understood, the proof is easy. As in the preceding theorem, we write S = {eg•: (c,e,) O}. If S is empty, we define 21 to be the number O; and in this case, (3) is obviously true. We now assume that S is non-empty, and we see from Theorem B that it must be finite or countably infinite. If S is finite, it can be written in the form S = fen, o, • , e n} for some positive integer n. In this case, we define 21 to be $\Sigma_{i=1}^{n} |(x,e_i)|^2$, which is clearly independent of the order in which the elements of S are arranged. The inequality (3) now reduces to (1), which has already been proved. All that remains is to consider the case in which S is countably infinite. Let the vectors in S be arranged in a definite order:

$$S = \{e_1, e_2, \ldots, e_n, \ldots\}.$$

By the theory of absolutely convergent series, if $\Sigma_{n=1}^{\infty} |(x,e_n)|^2$ converges, then every series obtained from this by rearranging its terms also converges, and all such series have the same sum. We therefore define 21 to be $\Sigma_{n=1}^{\infty} |(x,e_n)|^2$, and it follows from the above remark that $\Sigma|(x,e_i)|^2$ is a non-negative extended real number which depends only on S, and not on the arrangement of its vectors. We conclude the proof by observing that in this case, (3) reduces to the assertion that

$$1\, x,e_n)|^2 \leq \text{Il \$11 2,} \quad (4)$$

and since it follows from (1) that no partial sum of the series on the left of (4) can exceed lic11 ^{2}, it is clear that (4) itself is true.

The second part of Theorem A is generalized in essentially the same way.

Theorem D. If fed is an orthonormal set in a Hilbert space H, and if is an arbitrary vector in H, then

$$x - \Sigma(x,e_i)e_i \perp e_j \quad (5) \text{ for each j.}$$

PROOF. As in the above proof, we define for each of the various cases, and we prove (5) as we go along. We again write

S = {a: (c,e,•) 01.

When S is empty, we define to be the vector O, and we observe that (5) reduces to the statement thatc — O = is orthogonal to each e_i which is precisely what is meant by saying that S is empty. When S is non-empty and finite, and can be written in the form

$$S = \{e_1, e_2, \ldots ,\text{enl},$$

we define to be $\Sigma_{i=1}^{n}(x,e_i)e_i$; and in this case, (5) reduces to (2), which has already been proved.

By Theorem B, we may assume for the remainder of the proof that S is countably infinite. Let the vectors in S be listed in a definite order:

$S = \{e_1, e_2, \ldots, e_n, \ldots$0. we put = 2i_1 n and we note that for m > n we have

$$\|s_m - s_n\|^2 = \|\sum_{i=n+1}^{m} (x,e_i)e_i\|^2 = \sum_{i=n+1}^{m} |(x,e_i)|^2.$$

Bessel's inequality shows that the series 1 converges, so {sni is a Cauchy sequence in H; and since H is complete, this sequence converges to a vector s, which we write in the form s = $\Sigma_{n=1}^{\infty} (x$We now define to be 2n_1 and—deferring for a moment the question of what happens when the vectors in S are rearranged—we observe that (5) follows from (2) and the continuity of the inner product:

$(x - \Sigma(x,e_i)e_i, e_j) = (x - s, e_j) = (x,e_j) - (s,e_j)$ = (c,e,•) — (lim sn, e,)

$= (x,e_j) - \lim (s,$

$= ' \mathrm{j}) - (x,$

All that remains is to show that this definition of is valid, in the sense that it does not depend on the arrangement of the vectors in S. IÆt the vectors in S be rearranged in any manner:

$$S = \{f_1, f_2, \ . \ . \ . \ , f_n, \ . \ . \ .\}.$$

We put s'n = $\Sigma_{i=1}^{n} (x,f_i)f_i,$ and we see—as above—that the sequence {s'n} converges to a limit s', which we write in the form s' =$\Sigma_{n=1}^{\infty} (x$

We conclude the proof by showing that s' equals s. Let > O be given, and let no be a positive integer so large that if n 2 no, then llsn $- s\|$tis'n — S' Il < e, and I < €[2] . For some positive integer mo > no, all terms of sno occur among those of s' so — sno is a finite sum of terms of the form for i = no + 1, no + 2, ... This yields 'Is'm, no$\|^2 \leq \Sigma_{i=n_0+1}^{\infty}$ I < €[2], so lis'mo — sno ll < c and

$\|s' - s\| \leq \|s' - s'_{m_\epsilon}\| + \|s'_{n}$ma no no $- s\| < \epsilon + \epsilon + \epsilon = 3\epsilon.$

Since is arbitrary, this shows that s' = s.

Let H be a non-zero Hilbert space, so that the class of all its orthonormal sets is non-empty. This class is clearly a partially ordered set with respect to set inclusion. An orthonormal set {ed in H is said to be complete if it is maximal in this partially ordered set, that is, if it is impossible to adjoin a vector e to {ed in such a way that is an orthonormal set which properly contains fet[e] } .

Theorem E. Every non-zero Hilbert space contains a complete orthonorrnal set.

PROOF. The statement follows at once from Zorn's lemma, since the union of any chain of orthonormal sets is clearly an upper bound for the chain in the partially ordered set of all orthonormal sets.

Orthonormal sets are truly interesting only when they are complete. The reasons for this are presented in our next theorem.

Theorem F. Let H be a Hilbert space, and let {ed be an orthonormal set in H. Then the following conditions are au equivalent to one another:

(1) { es•} is complete;

(3) if is an arbitrary vector in H, then c —$\Sigma(x,e_i)$

(4) if c is an arbitrary vector in H, then = 2 2 PROOF. We prove that each of the conditions (1), (2), and (3) implies the one following it and that (4) implies (1).

(1)(2). If (2) is not true, there exists a vector c O such that x -L {ed. We now define e by e = z/llcll, and we observe that $\{e_i,e\}$is an orthonormal set which properly contains {ed. This contradicts the completeness of {ed .

(2)(3). By Theorem D, c — is orthogonal to {e so

(2) implies that c — = O, or equivalently, that c =$\Sigma(x,e_i)$

(3) (4). By the joint continuity of the inner product, the expression in (3) yields

$$\|x\|^2 = (x,x) = (\Sigma(x,e_i)e_i, \Sigma(x,e_j)e_j) = \Sigma(x,e_i)\overline{(x,e_i)} = \Sigma|(x,e_i)|^2.$$

(1)·If {ed is not complete, it is a proper subset of an orthonormal set Since e is orthogonal to all the (4) yields $\Sigma|(e,e_i)|^2$ = O, and this contradicts the fact that e is a unit vector.

There is some standard terminology which is often used in connection with this theorem. Let {ed be a complete orthonormal set in a Hilbert space H, and let c be an arbitrary vector in H. The numbers (c,e,•) are called the Fourier coefficients of c, the expression c = is called the Fourier $\Sigma(x,e_i)e_i$ expansion of c, and the equation llc11 ^{2} = is called Parseval's equation—all with respect to the particular complete orthonormal set under consideration. These terms come from the classical theory of Fourier series, as indicated in our next example.

Example 3. Consider the Hilbert space 1/2 associated with the measure space [O,2T], where measure is Lebesgue measure and integrals are Lebesgue integrals. [1] This space essentially consists of all complex functions f defined on [O,2T] which are Lebesgue measurable and squareintegrable, in the sense that

$$\int_0^{2\pi} |f(x)|^2 \, dc < 00$$

, Its norm and inner product are defined by

$$\|f\| = \left(\int_0^{2\pi} |f(x)|^2 \, dx\right)^{1/2}$$

and (f,g) = f02r dc.

A simple computation shows that the functions e^{inz}, for

$$n = 0, \pm 1, \pm 2, \ldots,$$

[1] In order to understand this and the next example, the reader should have some knowledge of the modern theory of measure and integration, We wish to emphasize once again that these examples are in no way essential to the structure of the book, and may be skipped by any reader without the necessary background. We advise such a reader to ignore these examples and to proceed at once to the discussion of the Gram-Schmidt process.

are mutually orthogonal in 122 :

$$\int_0^{2\pi} e^{imx}e^{-inx} \, dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n. \end{cases}$$

It follows from this that the functions e_n $(n = 0, \pm 1, \pm 2, \ldots$ ·that the functions en (n .) defined by en(c) = e^{inz}/V2T form an orthonormal set in 122. For any function f in 122, the numbers

$$c_n = (f,e_n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx} \, dc \qquad (6)$$

are its classical Fourier coejicients, and Bessel's inequality takes the form

$$\text{Icn1 2} \leq \int_0^{2\pi} |f(x)|^2 \, dx.$$

It is a fact of very great importance in the theory of Fourier series that the orthonormal set {enl is complete in L2. As we have seen in Theorem F, the completeness of {eni is equivalent to the assertion that for every f in 142, Bessel's inequality can be strengthened to Parseval's equation:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_0^{2\pi} |f(x)|^2 \, dx.$$

Theorem F also tells us that the completeness of ten} is equivalent to the statement that each f in 1.12 has a Fourier expansion:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \qquad (7)$$

It must be emphasized that this expansion is not to be interpreted as saying that the series converges pointwise to the function. The meaning of (7) is that the partial sums of the series, that is, the vectors fn in 1.0 defined by

$$f_n(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} c_k e^{ikz} \quad (8)$$

converge to the vector f in the sense of la:

$$\|f_n - f\| \to 0.$$

This situation is often expressed by saying that f is the limit in the mean of the fn's. We add one final remark to our description of this portion of the theory of Fourier series. If f is an arbitrary function in 142 with Fourier coemcients cn defined by (6), then Bessel's inequality tells us that the series %_lcn1 ^{2} converges. The celebrated Riesz-Fischer theorem asserts the converse: if cn (n = O, ± 1, ± 2, . .) are given complex numbers for which lcn1 ^{2} converges, then there exists a function fin L2 whose Fourier coeffcients are the cn's. If we grant the completeness of 122 as a metric space, this is very easy to prove. All that is necessary is to use the cn's to define a sequence of fn's in accordance with (8). The f,unctions e^{in}%/x./2r form an orthonormal set, so for m > n we have

$$- f_n\|^2 = \sum_{|k|=n+1}^{m} |c_k|^2. \quad (9)$$

By the convergence of ICal ^{2} , the sum on the right of (9) can be made as small as we please for all sufficiently large n and all m > n. This tells us that the fn's form a Cauchy sequence in 10', and since L2 is complete, there exis.ts a function f in 142 such that % This function f is given by (7), and the cn's are clearly its Fourier coemcients. It is apparent from these remarks that the essence of the Riesz-Fischer theorem lies in the completeness of L2 as a metric space.

We shall have use for one further item in the general theory of orthonormal sets, namely, the Gram-Schmidt orthogonalization process. Suppose that {$1, $2, . . . , cn, . . } is a linearly independent set in a Hilbert space H. The problem is to exhibit a constructive procedure for converting this set into a corresponding orthonormal set {el, e, . . . , e_n, . ·. } with the property that for each n the linear subspace of H spanned by {el, 0, • , enl is the same as that spanned by {$1, $2, cn}; Our first step is to normalize cl—which is necessarily nonzero—by putting

el =——.

The next step is to subtract from its component in the direction of el to obtain the vector — orthogonal to el, and then to normalize this by putting

$$e2 = \frac{x_2 - (\$2,e_1)e_1}{\|x_2 - (x_2,e_1)e_1\|}.$$

We observe that since is not a scalar multiple of $1, the vector — $_2,e_1)e_1$ is not zero, so the definition of e2 is valid. Also, it is clear that e2 is a linear combination of and $2, and that is a linear combination of er and e. The next step is to subtract from its components in the directions of er and e2 to obtain a vector orthogonal to e: and Q, and then to normalize this by putting

$$e_3 = \frac{x_3 - (x_3,e_1)e_1 - (x_3,e_2)e_2}{\|x_3 - (x_3,e_1)e_1 - (x_3,e_2)e_2\|}.$$

If this process is continued in the same way, it clearly produces an orthonormal set {el, Q, • • , n, . . } with the required property.

Example 4. Many orthonormal sets of great interest and importance in analysis can be obtained conveniently by applying the Gram-Schmidt process to sequences of simple functions.

(a) In the space 142 associated with the interval [—1, 1], the functions c^n (n — O, 1, 2, . .) are linearly independent. If we take these functions to be the cn's in the Gram-Schmidt process, then the en's are the normalized Legendre polynomials.

(b) Consider the space 112 over the entire real line. If the cn's here are taken to be the functions (n = O, 1, 2, O, then the corresponding en's are the normalized Hermite functions.

(c) Consider the space L2 associated with the interval [0,+ m). If the cn's are the functions (n = O, 1, 2, .), then the en's are the normalized Laguerre functions.

Each of the orthonormal sets described in the above example can be shown to be complete in its corresponding Hilbert space. The analysis involved in a detailed study of these matters is quite complicated and has no proper place in the present book. The reader should recognize, however—and this is our only reason for mentioning the material in Examples 3 and 4—that the theory of Hilbert spaces does have significant contacts with many solid topics in analysis.

Problems

1. Let {el, e, • , en} be a finite orthonormal set in a Hilbert space H, and let be a vector in H. If al, a2, . . . , an are arbitrary scalars, show that — ac•eill attains its minimum value ⇔

$$\alpha_i = (x,e_i)$$

for each i. (Hint: expand [lc — $\Sigma_{i=1}^{n} \alpha_i e_i \|^2$, add and subtract $\Sigma_{i=1}^{n}$I and obtain an expression of the form 1 (c,e,•) — in the result.)

2. Show that the orthonormal sets described in Examples 1 and 2 are complete.
3. Show that every orthonormal set in a Hilbert space is contained in some complete orthonormal set, and use this fact to give an alternative proof of Theorem 53-B.
4. Prove that a Hilbert space H is separable every orthonormal set in H is countable.
5. Show that an orthonormal set in a Hilbert space is linearly independent, and use this to prove that a Hilbert space is finite-dimensional every complete orthonormal set is a basis.
6. Prove that any two complete orthonormal sets in a Hilbert space H have the same cardinal number. This cardinal number is called the orthogonal dimension of H (if H has no complete orthonormal sets, ite orthogonal dimension is said to be O),
7. If H and H' are Hilbert spaces, prove that H is isometrically isomorphic to H' they have the same orthogonal dimension. (Hint: by Eq. 52-(2), an isometric isomorphism T preserves inner products, in the sense that (T(c),T(y)) — (x,y).)
8. Let S be a non-empty set, and let 12(S) be the set of all complex functions f defined on S with the following two properties:

   (1) {s:f(s) O) is empty or countable;

   (2) Elf(s)!' < 00.

   These functions clearly form a complex linear space with respect to pointwise addition and scalar multiplication. Show that 12(S) becomes a Hilbert space if the norm and inner product are defined by $\|f\|$— $(\Sigma|f(s)|^2)^{1/2}$ and (f,g) = $\Sigma f(s)g(s)$. Show also that the set of all functions defined on S which have the value 1 at a single point and are O elsewhere is a complete orthonormal set in 12(S). We shall see in the next problem that Hilbert spaces of the type described here are universal models for all non-zero Hilbert spaces.

t S ={ed be a complete orthonormal set in a Hilbert space H. Each vector c in H determines a function f defined on S by

$$f(e_i) = (x,e_i),$$

and Theorems B and C tell us that f is in b(S). Show that the mapping c -+ f is an isometric isomorphism of H onto h(S).

3. THE CONJUGATE SPACE H*

We pointed out in the introduction to this chapter that one of the fundamental properties of a Hilbert space H is the fact that there is a natural correspondence between the vectors in H and the functionals in H*. Our purpose in this section is to develop the features of this correspondence which are relevant to our work with operators in the rest of the chapter.

Let V be a fixed vector in H, and consider the function f, defined on H by fy(c) = (z,y), It is easy to see that is linear, for

fv(cr + $2)= $(x_1$

= (cr,y) + ($2,y)

= fv(C1) + fv(C2) and fv(ac) = (ac,y) =afv(c).

Further, % is continuous and is therefore a functional, for Schwarz's inequality gives

$$|f_y(x)| = |(x,y)| \leq \|x\| \|y\|,$$

which shows that I l full llyll. Even more, equality is attained here, that is, llfvll — ¯ llyll. This is clear if y = O; and if y O, it follows from

$$\|f_y\| = \sup \{|f_y(x)| : \|x\| = 1\} \geq \left| f_y\left(\frac{y}{\|y\|}\right) \right| = \left| \left(\frac{y}{\|y\|}, y\right) \right| = \|y\|.$$

To summarize, we have seen that y fy is a norm-preserving mapping of H into H* , This observation would be of no more than passing interest if it were not for the fact that every functional in H* arises in just this way.

Theorem A. Let H be a Hilbert space, and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that

$f(x)$— (c,y) (1) for every z in H.

PROOF. It is easy to see that if such a y exists, then it is necessarily unique. For if we also have f(c) = (z,y') for all z, then (c,y') = $(x,y)$and (c, y' — y) = O for all c; and since O is the only vector orthogonal to every vector, this implies that y' — or y' = y.

We now turn to the problem of showing that y does exist. If f = O, then it clearly suffces to choose y = O. We may therefore assume that f O. The null space M of fis thus a proper closed linear subspace of H, and by Theorem 53-B, there exists a non-zero vector yo which is orthogonal to M. We show that if a is a suitably chosen scalar, then the vector y = ayo meets our requirements. We first observe that no matter what a may be, (1) is true for every c in M; for f(c) = O for such an c, and since c is orthogonal to yo, we also have (c,y) = O. This allows us to focus our attention on choosing a in such a way that (1) is true for c = yo. The condition this imposes on a is that f(yo) = (yo,ayo) = 2

We therefore choose a to be $\overline{f(y_0)}/\|y_0\|^2$, and it follows that (1) is true for every c in M and for c = YO' It is easily seen that each c in H can be written in the form c = m + ßyo with m in M: all that is necessary is to choose ß in such a way that f(c — ßyo) = f(c) — ßf(yo) = O, and this is accomplished by putting ß = $f(x)/f(y_0)$. Our conclusion that (1) is true for every c in H now follows at once from f(c) — + ßyo) = f(m) + ßf(yo) — (m,y) + $\beta(y_0,y)$

= (m + ßyo, y) = (x,y).

This result tella us that the norm-preserving mapping of H into H* defined by y fy, where fv(c) = (c,y), (2)

is actually a mapping of H onto H*. It would be pleasant if (2) were also a linear mapping... This is. not quite true, however, for

$$f_{y_1+y_2} = f_{y_1} + f_{y_2} \text{and} \qquad f_{\alpha y} = \bar{\alpha} f_y. \qquad (3)$$

It is an easy consequence of (3) that the mapping (2) is an isometry, for Ilfz - = c — y]. We state several interesting additional facts about this mapping (and what it enables us to do) in the problems, and we leave their verification to the reader. It should be remembered, however, that the real significance of this entire circle of ideas lies in its influence on the theory of the operators on H. We begin the treatment of these matters in the next section.

Problems

1. Verify relations (3).
2. Let H be a Hilbert space, and show that H* is also a Hilbert space with respect to the inner product defined by — In just the same way, the fact that H* is a Hilbert space implies that H* * is a Hilbert space whose inner product is given by (Ff,Fg) = (g,f).
3. Let H be a Hilbert space. We have two natural mappings of H into H* * the second of which is onto: the Banach space natural imbedding Fz, where F$(f) = f(c), and the product mapping $\rightarrow f_x \rightarrow$F'%, where fz(y) = (y,c) and = Show that these mappings are equal, and conclude that H is reflexive. Show also that (Fz,Fv) = (c,y).

REFERENCE BOOKS:

1. G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, International Book company, 1963.
2. Balmohan V. Limaye, Functional Analysis, Second Edition, New Age International Pvt Ltd, 1997.
3. M.Thamban Nair, Functional Analysis, A First Course, PHI, 2002

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT IV – Functional Analysis – SMTA5302

Unit-IV

Hilbert Spaces (Continued)

1. THE ADJOINT OF AN OPERATOR

Throughout the rest of this chapter, we focus our attention on a fixed but arbitrary Hilbert space H, and unless we specifically state otherwise, it is to be understood that H is the context for all our discussions and theorems. Let T be an operator on H. We saw in Sec. 51 that T gives rise to an operator 1' • (its conjugate) on H* , where T* is defined by $(T^*f)x = f(Tx)$.[1]

We also saw that the mapping T T* is an isometric isomorphism of G(H) into (B(H*) which reverses products and preserves the identity transformation. In the same way, T* gives rise to an operator T* * on H* * • and since H is reflexive, it follows that T* * — T when H* * is identified with H by means of the natural imbedding.

These statements depend only on the fact that H is a reflexive Banach space. We now bring its Hilbert space character into the picture, and we use the natural correspondence between H and H* discussed in the previous section to pull T* down to H. The details of this procedure are as follows (see Fig. 38). Let y be a vector in H, and % its corresponding functional in H* ; operate with T* on Fig. 38. The conjugate and the fy to obtain a functional f* = T*fv; and adjoint of T. return to its corresponding vector z in H. There are three mappings under consideration here, and we are forming their product:

$$y \rightarrow f_y \rightarrow T^*f_y = f_z \rightarrow z. \qquad (1)$$

We write z = T *y, and we call this new mapping T* of H into itself the adjoint of T. The same symbol is used for the adjoint of T as for its conjugate because these two mappings are actually the same if H and H* are identified by means of the natural correspondence. It is easy to keep track of whether T* signifies the conjugate or the adjoint of T by noticing whether it operates on functionals or on vectors. The action of the adjoint can be linked more closely to the structure of H by observing that for every vector c we have = fv(Tæ) = (Tz,y) and

$= f_z(x)$= (c,z) = (c, T*y), so that

$$(Tc,y) - (c, T^*y) \qquad (2)$$

for all c and y. Equation (2) is much more than merely a property of theadjoint of T, for it uniquely determines this adjoint. 'Ihe proof is simple: if T' is any mapping of H into itself such that (Tc,y) = $(x,T'y)$for all c and y, then (z, T'y) = (z,T*y) for all c, so T'y and since the latter is true for all y, T' = T*.

Our remarks in the above paragraph have shown that to each operator T on H there corresponds a unique mapping T* of H into itself (called the adjoint of T) which satisfies relation • (2) for all c and y. There is a more direct but less natural approach to these ideas, one which avoids any reference to the conjugate of T. If y is fixed, it is clear that the expression (Tc,y) is a scalar-valued continuous linear function of c. By Theorem 55-A, there exists a unique vector z such that (Tc,y) = (c,z) for all c. We now write z = T *y, and since y is arbitrary, we again have relation (2) for all c and y. The fact that T* is uniquely determined by (2) follows just as before.

The principal value of our approach to the definition of the adjoint (as opposed to that just mentioned) lies in the motivation it provides for considering adjoints at all. We can express this by emphasizing that an operator on a Banach space always has a conjugate which operates on the conjugate space; and when the Banach space happens to be a Hilbert space, then, as we have seen, the natural correspondence discussed in the previous section makes it almost inevitable that we regard the conjugate as an operator on the space itself. Once the definition of the adjoint is fully understood, however, there is no further need to mention conjugates. All our future work with adjoints will be based on Eq, C), and from this point on, the symbol T* will always signify the adjoint of T (and never its conjugate).

As our first step in exploring the properties of adjoints, we verify that T* actually is an operator on H (all we know so far is that it maps H into itself). For any y and z, and for all c, we have

$+ z))$= (TX, y 4- z) = (Tc,y) -l- (Tc,z) $= (x,T^*y) + (x,T^*z) = (x, T^*y + T^*z),$

so

$$T^*(y + z) = T^*y + T^*z.$$

The relation

$$T^*(\alpha y) = \alpha T^*y$$

is proved similarly, so T* is linear. It remains to be seen that T* is continuous; and to prove this, we note that

$\|T^*y\|^2 = (T^*y,T^*y) = (TT^*y,y) \leq$ NTT*yII Il TO IIT*yII$\|y\|$

[1] The reasoning here depends on the fact that if yr and are vectors such that (z,Vk) (z,ya) for all z, then (z, y, — Y2) = O for all x, so yr — Y2 — O**or** $y_1 =$

implies that Il T $y\|$ $\leq$Il T Il llyll for all y, so

$\|T^*\| \leq \|T\|.$

These facts tell us that T T* is a mapping of 8(H) into itself. This mapping is called the adjoint operation on G(H).

Theorem A. The adjoint operation T T* on G (H) has the following properties:

$$(3)\quad \begin{aligned} &(T_1 + T_2)^* = T_1^* + T_2^*;\\ &(\alpha T)^* = \bar{\alpha} T^*;\\ &(T_1T_2)^* = T_2^*T_1^*;\\ &T^{**} = T;\\ &\|T^*\| = \|T\|;\\ &\|T^*T\| = \|T\|^2. \end{aligned}$$

PROOF. The arguments used in proving (1) to (4) are all essentially the same. As an illustration of the method, we observe that (3) follows from the fact that for all z and y we have

$_1T_2)^*y)$= (TIT2c,y) =$(T_2x, T_1{}^*y) = (x, T_2{}^*T_1{}^*y).$

To prove (5), we note that we already have n ll ; and if we apply this to T* instead of T and use (4), we obtain T Il = $\|T^{**}\|$

Half of (6) follows from (5) and the inequality 47-(5), for

$$\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\| \, \|T\| = \|T\|^2;$$

and the fact that IT 11 [2] IIT*TII is an immediate consequence of

$\|Tx\|^2$= (Tc,Tc) = (T^*Tx,x) ≤ llT*Tcll IIT*TII IIc11 [2].

The presence of the adjoint operation is what distinguishes the theory of the operators on H from the more general theory of the operators on a reflexive Banach space. [1] In the next three sections, we use this operation as a tool by means of which we single out for special study certain types of operators on H whose theory is particularly complete and satisfying.

Problems

1. Prove parts (1), (2), and (4) of Theorem A.
2. Show that the adjoint operation is one-to-one onto as a mapping of G(H) into itself.
3. Show that O* = O and I* = I. Use the latter to show that if T is non-singular, then T* is also non-singular, and that in this case
4. Show that Il TT*II =

2. SELF-ADJOINT OPERATORS

There is an interesting analogy between the set (B(H) of all operators on our Hilbert space H and the set C of all complex numbers. This can be summarized by observing that each is a complex algebra together with a mapping of the algebra onto itself (T T* and z Z) and that these mappings have similar properties. We shall see that this analogy is quite useful as an intuitive guide to the study of the operators on H. The most significant difference between these systems is that multiplication in the algebra G(H) is in general non-commutative, and it will become clear as we proceed that this is the primary source of the much greater structural complexity of (B(H).

The most important subsystem of the complex plane is the real line, which is characterized by the relation z = Z. By analogy, we consider those operators A on H which equal their adjoints, that is, which satisfy the condition A = A*. Such an operator is said to be self-adjoint. The self-adjoint operators on H are evidently those which are related in the simplest possible way to their adjoints.

We know that O* = O and I* = I, so O and I are self-adjoint. If Al and A 2 are self-adjoint, and if a and ß are real numbers, then (aAl + M 2) * = äA1* + BA2* = a Al + PA 2

shows that an 1 -l- ßA2 is also self-adjoint. Further, if {An} is a sequence of self-adjoint operators which converges to an operator A, then it is easy to see that A is also self-adjoint; for

$$\|A - A^*\| \leq \|A - A_n\| + \|A_n - A_n^*\| + \|A_n^* - A^*\| = \|A - A_n\| + \|(A_n - A)^*\| = \|A - A_n\| + \|A_n - A\| = 2\|A_n - A\| \to 0$$

shows that A — A* = O, so A = A*. These remarks yield our first theorem.

Theorem A, The self-adjoint operators in (B(H) form a closed real linear subspace of (B(H)—and therefore a real Banach space—which contains the identity transformation.

The reader will notice that we have said nothing here about the product of two self-adjoint operators. Very little is known about such products, and the following simple result represents almost the extent of our information.

Theorem B. If Al and are self-adjoint operators on H, then their product AlA2 is self-adjoint AIA2 = AAI.
PROOF. This is an obvious consequence of

$$(A_1A_2)^* = A_2^*A_1^* = A_2A_1.$$

The order properties of self-adjoint operators are more interesting, and we devote the remainder of the section to establishing some of the simpler facts in this direction.

If T is an arbitrary operator on H, it is easy to see that

T = (Tc,y) = O

for all c and y. It is also clear that T = (Tc,c) = O for all c. We shall need the converse of this implication.

Theorem C. If T is an operator on H for which (Tc,c) = O for au c, then T = O.
PROOF. It sumces to show that (Tz,y) = O for any c and y, and the proof of this depends on the following easily verified identity:

(T (ac + ßy), ac + $- |\alpha|^2(Tx,x) - |\beta|^2(Ty,y)$ ßy) (1)

We first observe that by our hypothesis, the left side of (l)—and therefore the right side as well—equals O for all a and p. If we put a 1 and ß = 1, then (1) becomes

(Tc,y) + (Ty,c) - O; (2)

and if we put a = i and ß = 1, we get

$$i(Tx,y) - i(Ty,x) = 0. \quad (3)$$

Dividing (3) by i and adding the result to (2) yields 2(Tc,y) = 0, so (Tc,y) = O and the proof is complete.

It is worth emphasizing that this proof makes essential use of the fact that the scalars are the complex numbers (and not merely the real num bers).

We now apply this result to proving our next theorem, which indicates that self-adjoint operators are linked to real numbers by stronger ties than might be suspected from the loose analogy that led to their definition. Theorem D. An operator T on H is self-adjoint (Tc,c) is real for an z. PROOF. If T is self-adjoint, then

(Tz,c) = (z, TX) $= (x,T^*x)$ - (Tc,c)

shows that (Tc,c) is real for all c. On the other hand, if (Tc,$) is real for all c, then (Tc,c) (Tc,c) $= (x,T^*x)$ = or

$$([T - T^*]x, x) = 0$$

for all c. By Theorem C, this implies that T — T* = O, so T = T*

This theorem enables us to define a respectable and useful order relation on the set of all self-adjoint operators. If Al and .42 are selfadjoint, we write Al A 2 if (Arc,c) (A 2z,z) for all c. The main elementary facts about this relation are summarized in

Theorem E. The real Banach space of ou self-adjoint operators on H is a partially ordered set whose linear structure and order structure are related by the following properties:

(l) if Al A 2, then A 1 + A A 2 A for every A;

(2) if Al A2 and a 2 0, then a Al a 442.

PROOF. The relation in question is obviously reflexive and transitive (see Sec. 8). To show that it is also antisymmetric, we assume that Al A2 and A 2 Al. This implies at once that (LAI — A2]c, c) = O for all c, so by Theorem C, Al — A2 = O and Al = A2. The proofs of properties (1) and (2) are easy. For instance, if Al A 2, so that (Alt,c) (A2$,c) for all c, then (Arc,c) + (Az,c) (Ag,z) 4- (Ac,z) or + AIC, c) ([A 2 + c) for all x, so Al A A 2 + A. The proof of (2) is similar.

A self-adjoint operator A is said to be positive if A > O, that is, if (Ac,c) 2 0 for all c. It is clear that O and I are positive, as are T* T and T T* for an arbitrary operator T.

Theorem F. If A is a positive operator on H, then I -f- A is non-singular. In particular, I -k T* T and I + T T* are non-singular for an arbitrary operator T on H.

PROOF. We must show that I + A is one-to-one onto as a mapping of H into itself. First, it is one-to-one, for

$$(I + A)x = 0 \Rightarrow Ax = -x \Rightarrow (Ax,x) = (-x,x) = -\|x\|^2 \geq 0 \Rightarrow x = 0.$$

We next show that the range M of I + A is closed. It follows from Il(l + A)z11 ^{2} = 4- Il Ac11 ^{2} 4- 2(Ac,c)— and the assumption that A is positive—that Il (I + A)cll. By this inequality and the completeness of H, M is complete and therefore closed. We conclude the proof by observing that M = H; for otherwise there would exist a nonzero vector co orthogonal to M, and this would contradict the fact that

$$(x_0, [I + A]x_0) = 0 \Rightarrow \|x_0\|^2 = -(Ax_0, x_0) \leq 0 \Rightarrow x_0 = 0.$$

If the reader wonders why we fail to show that the partially ordered set of all self-adjoint operators is a lattice, the reason is simple: it isn't true. As a matter of fact, this system is about as far from being a lattice as a partially ordered set can be, for it can be shown that two operators in the set have a greatest lower bound they are comparable. This whole situation is intimately related to questions of commutativity for algebras of operators and is too complicated for us to explore here. For further details, see Kadison [22].

Problems

1, Define a new operation of "multiplication" for self-adjoint operators by Al º = (AIA2 + AAI)/2, and note that Al º 442 is always self-adjoint and that it equals AIA* whenever Al and commute, Show that this operation has the following properties:

$$A_1 \circ A_2 = A_2 \circ A_1,$$
$$A_1 \circ (A_2 + A_3) = A_1 \circ A_2 + A_1 \circ A_3,$$
$$\alpha(A_1 \circ A_2) = (\alpha A_1) \circ A_2 = A_1 \circ (\alpha A_2),$$

and A º 1 = I º A = A. Show also that Al º oA3) — (Ai º A2) º A, whenever Al and A3 commute.

2. If T is any operator on H, it is clear · that $|(Tx,x)| \leq \|Tx\| \, \|x\| \leq$ that

IITII llc11 ^{2} ; so if H we have sup $\{|(Tx,x)|/\|x\|^2 : x \neq 0\} \leq \|T\|.$

Prove that if T is self-adjoint, then equality holds here. (Hint: write a = sup O} = sup {l : llcll show that I t Tcll a whenever = 1 by putting b — TCI! "—if Tc O—and considering

$4\|Tx\|^2$- (T(bc + b-l Tc), bc + b-l Tx)

- (T(b$ — b-l Tc), — b-l Tc) + b-l Tz11 ^{2}

+ llbc — b—1Tc11 2] = 4allTcll.)

3. NORMAL AND UNITARY OPERATORS

An operator N on H is said to be normal if it commutes with its adjoint, that is, if NN* = N*N. The reason for the importance of normal operators will not become clear until the next chapter. We shall see that they are the most general operators on H for which a simple and revealing structure theory is possible. Our purpose in this section is to present a few of their more elementary properties which are necesury for our later work.

It is obvious that every self-adjoint operator is normal, and that if N is normal and a is any scalar, then aN is also normal. Further, the limit N of any convergent sequence {NE} of normal operators is normal; for we know that N* * —i N* , so

IINN* — N*NII INN* — + · $\|N_kN_k^* - N_k^*N_k\|$

\+ — N*NII - IINN* — — $N_kN_k^*\| + \|N_k^*N_k$ · N*NII —i O,

which implies that NN* — N*N = O. These remarks prove

Theorem A. The set of au normal operators on H is a closed subset of G(H) which contains the set of all self-adjoint operators and is closed under scalar multiplication.

It is natural to wonder whether the sum and product of two normal operators are necessarily normal. They are not, but nevertheless, we can say a little in this direction.

Theorem B. If NI and are normal operators on H with the property that either commutes with the adjoint of the other, then NI + N2 and NIN* are normal

PROOF. It is clear by taking adjoints that

NIN2* - N2*NI N2Nl* =$N_1^*N_2$

so the assumption implies that each commutes with the adjoint of the other. To show that NI + N 2 is normal under the stated conditions, we have only to compare the results of the following computations:

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*) = N_1N_1^* + N_1N_2^* + N_2N_1^* + N_2N_2^*$$

and (NI + N + N2) = (N + N + N2)

=NI*NI + NI*N2 + + N N2.

The fact that N IN' is normal follows similarly from

$$N_1N_2(N_1N_2)^* = N_1N_2N_2^*N_1^* = N_1N_2^*N_2N_1^* = N_2^*N_1N_1^*N_2 = N_2^*N_1^*N_1N_2 = (N_1N_2)^*N_1N_2.$$

By definition, a self-adjoint operator A is one which satisfies the identity A *c = Ac. Many properties of self-adjoint operators do not depend on this, but only on the weaker identity llA*cll = llAcll. Our next theorem shows that all such properties are shared by normal operators.

Theorem C. An operator T on H is normal T — Il TCI(for every c. PROOF. In view of Theorem 57-0, this is implied by the fact that

Il $\|Tx\| \Leftrightarrow \|T^*x\|^2 = \|Tx\|^2 \Leftrightarrow (T^*x,T^*x)$ T*cll =

— $= (Tx,Tx) \Leftrightarrow (TT^*x,x) = (T^*Tx,x) \Leftrightarrow$ (ITT* – $T^*T]x,x)$ – O.

The following consequence of this result will be useful in our later work.

Theorem D, If N is a normal operator on H, then {IN^2 11 — IIN11 ^{2}. PROOF. The preceding theorem shows that

$\|N^2x\|$= IINNcIl = IIN*NcIl

for every c, and this implies that (IN ^{2} 11 = IIN*NII. By Theorem 56-A, we have IIN*Ntl = IIN11 ^{2}, so the proof is complete.

We know that any complex number z can be expressed uniquely in the form z = a + ib where a and b are real numbers, and that these real numbers are called the real and imaginary parts of z and are given by a = (z + E)/2 and b — (z — ä)/2i. The analogy between general operators and complex numbers, and between self-adjoint operators and real numbers, suggests that for an arbitrary operator T on H we form Al = (T + and A2

= (T — Al and A2 are clearly selfadjoint, and they have the property that T = Al + iA2. The uniqueness of this expression for T follows at once from the fact that

$$T^* = A_1 - iA_2.$$

The self-adjoint operators Al and A 2 are called the real part and the imaginary part of T.

We emphasized earlier that the complicated structure of G(H) is due in large part to the fact that operator multiplication is in general non-commutative. Since our future work will be focused mainly on normal operators, it is of interest to see—as the following theorem shows— that the existence of non-normal operators can be traced directly to the non-commutativity of self-adjoint operators.

Theorem E. If T is an operator on H, then T is normal its real and imaginary parts commute.

PROOF. If Al and A 2 are the real and imaginary parts of T, so that T = Al + iA2 and T* = Al — iA2, then

$TT^* = (A_1 + iA_2)(A_1 - iA_2$= A 1 ² + A 2 ²+ $i(A_2A_1$ − A) and

T^*T =(Al — (Al + 'iA2) ¯ Al 2 + A 22 + $i(A_1A_2$— A Al).

It is clear that if A A_2 TT^*- T*T, then $A_1A_2$and AlA2 = A2A1.

—- AAI, then T ᴿI'* = T*T. Conversely, if

- A 2A1 = A 2441 — A IA 2, so IA 2 = 2A2A1

Perhaps the most important subsystem of the complex plane after the real line is the unit circle, which is characterized by either of the equivalent identities 'zl = 1 or zi = Zz = 1. An operator U on H which satisfies the equation U U* = U *U = I is said to be unitary. Unitary operators—which are obviously normal— are thus the natural analogues of complex numbers of absolute value 1. It is clear from the definition that the unitary operators on H are precisely the non-singular operators whose inverses equal their adjoints. The geometric significance of these operators is best understood in the light of our next theorem,

Theorem F. If T is an operator on H, then the following conditions are au equivalent to one another:

$T^*T = I$;

(2) (Tc,Ty) = (c,y) for au and y;

(3) llTcII — - llcll for au x.

PROOF. If (1) is true, then (T*Tc,y) = (c,y) or (Tc,Ty) = (c,y) for all c and y, so (2) is true; and if (2) is true, then by taking y = c we obtain (TX, TX) = (c,c) or ll Tc11 ² = llc11 ² for all c, so (3) is true. The fact that (3) implies (1) is a consequence of Theorem 57-0 and the following chain of implications:

$$\|Tx\| = \|x\| \Rightarrow \|Tx\|^2 = \|x\|^2 \Rightarrow \text{(Tc,Tc)}$$

= (c,c) ([T*T — llc,c) = O.

An operator on H with property (3) of this theorem is simply an isometric isomorphism of H into itself. That an operator of this kind need not be unitary is easily seen by considering the operator on h defined by

T{CI, $2, = $\{0, x_1, x_2, \dots\}$,

which preserves norms but has no inverse. These ideas lead at once to

Theorem G. An operator T on H is unitary it is an isometric isomorphism of H onto itself.

PROOF. If T is unitary, then we know from the definition that it is onto; and since by Theorem F it preserves norms, it is an isometric isomorphism of H onto itself. Conversely, if T is an isometric isomorphism of H onto itself, then T—1 exists, and by Theorem F we have

T^*T= 1. It now follows that $(T^*T)T^{-1}$ = IT-I, so T* = T-1 and T T* = T* T = I, which shows that T is unitary. This theorem makes quite clear the nature of unitary operators: they are precisely those one-to-one mappings of H onto itself which preserve all structure—the linear operations, the norm, and the inner product.

Problems

1. If T is an arbitrary operator on H, and if a and ß are scalars such that la! = lßl, show that aT 4- ßT* is normal.
2, If H is finite-dimensional, show that every isometric isomorphism of H into itself is unitary.
3. Show that an operator T on H is unitary ⇔ $T(\{e_i\})$ is a complete orthonormal set whenever {ed is.
4. Show that the unitary operators on H form a group.

59. PROJECTIONS

According to the definition given in Sec. 50, a projection on a Banach space B is an idempotent operator on B, that is, an operator P with the property that P^2 = P. It was proved in that section that each projection P determines a pair of closed linear subspaces M and N—the range and null space of P—such that B = M N, and also, conversely, that each such pair of closed linear subspaces M and N determines a projection P with range M and null space N. In this way, there is established a one-to-one correspondence between projections on B and pairs of closed linear subspaces of B which span the whole space and have only the zero vector in common.

The context of our preynt work, however, is the Hilbert space H, and not a general Banach space, and the structure which H enjoys in addition to being a Banach space enables us to single out for special attention those projections whose range and null space are orthogonal. Our first theorem gives a convenient characterization of these projections.

Theorem A. If P is a projection on H with range M and null space N, then M L N P is self-adjoint; and in this case, N = M L.

PROOF, Each vector z in H can be written uniquely in the form z = c + y with c and win M and N. If M -L N, so that c -L y, then P* = P will follow by Theorem 57-0 from (P*z,z) = (Pz,z); and this is a consequence of

(P^*z,z) - (z,Pz) = (z,c)= $(x + y, x)$= (c,c) + (w c) = (c,c) and (Pz,z) = (c,z) = $(x, x + y)$ — (C,C) + = (z,c). If, conversely, P* = P, then the conclusion that M _L N follows from the fact that for any c and y in M and N we have

(c,y) = (Pc,y) = (x, P^*y) = (z,Py) = $(x,0)$

All that remains is to see that if M -L N, then N = M x . It is clear that N C M I ; and if N is a proper subset of M l , and therefore a proper closed linear subspace of the Hilbert space M l , then Theorem 53-B implies that there exists a non-zero vector zo in M L such that zo -L N. Since zo _L M and zo L N, and since H = M N, it follows that zo _L H. This is impossible, so we conclude that N = M L .

A projection on H whose range and null space are orthogonal is sometimes called a perpendicular projection. The only projections considered in the theory of Hilbert spaces are those which are perpendicular, so it is customary to omit the adjective and to refer to them simply as projections. In the light of this agreement and Theorem A, a projection on H can be defined as an operator P which satisfies the conditions P^2 = P and P* = P. The operators O and I are projections, and they are distinct — I-I$\neq$ $\{0\}$.

The great importance of the projections on H rests mainly on Theorem 53-D, which allows us to set up a natural one-to-one correspondence between projections and closed linear subspaces. To each projection P there corresponds its range M = {Pc:c e H}, which is a closed linear subspace; and conversely, to each closed linear subspace M there corresponds the projection P with range M defined by 4- y) = c, where c and y are in M and M L . Either way, we speak of P as the projection on M.

It is clear that P is the projection on M I — P is the projection on M L . Also, if P is the projection on M, then

cc M —Pc = IIPclI = $\|x\|$.

The first equivalence here was proved in Problem 44-11; and since for every c in H we have

$\|x\|^2$- + (1 — P)c11 2 - + Il(l - P)c11 2, (1)

the non-trivial part of the second is given by the following chain of implications :

$\|Px\|$= Il pc112 — "(I — P)c11 2 = O = c.

Relation (1) also shows that llpcll llcll for every c, so IIPII 1. If c is an arbitrary vector in H, it is easy to see that

(Pc,c) = (PPc,c) - (Px,P^*x) - (Pz,P$) = 2 0, (2)

so P is a positive operator (O P) in the sense of Sec. 57. Since I — P is also a projection, we also have O I — P or P < I, so O P I. let T be an operator on H. A closed linear subspace M of H is said to be invariant under T if T (M) C M. When this happens, the restriction of T to M can be regarded as an operator on M alone, and the action of T on vectors outside of M can be ignored. If both M and M I are invariant under T, we say that M reduces T, or that T is reduced by M. This situation is much more interesting, for it allows us to replace the study of T as a whole by the study of its restrictions to M and M I , and it invites the hope that these restrictions will turn out to be operators of some particularly simple type. In the following four theorems, we translate these concepts into relations between T and the projection on M.

Theorem B. A closed linear subspace M of H is invariant under an operator T M ^{I} is invariant under T*.

PROOF. Since M ^{IL} = M and T* * = T, it suffces by symmetry to prove that if M is invariant under T, then M -L is invariant under T*. If y is a vector in M ^{L} , our conclusion will follow from (c,T*y) = O for all .c in M. But this is an easy consequence of (c, T *y) — (Tc,y), for the invariance of M under T implies that (Tc,y) = 0.

Theorem C. A closed linear subspace M of H reduces an operator T M is invariant under bcth and T*

PROOF. This is obvious from the definitions and the preceding theorem.

Theorem D. If P is the projection on a closed linear subspace M of H, then M is invariant under an operator T TP = P TP.

PROOF. If M is invariant under T and c is an arbitrary vector in H, then TPc is in M, so PTPc = TPc and PTP = TP. Conversely, if TP = P TP and z is a vector in M, then Tz = TPX = P TPc is also in M, so M is invariant under T.

Theorem E. If P is the projection on a closed linear subspace M of H, then M reduces an operator T TP = PT.

PROOF. M reduces T M is invariant under T and T* TP = P TP and - PT*P TP = PTP and PT = PTP. The last statement in this chain clearly implies that TP = P T; it also follows from it, as we see by multiplying TP = PT on the right and left by P.

Our next theorem shows how projections can be used to express the statement that two closed linear subspaces of H are orthogonal.

Theorem F, If P and Q are the projections on cloged linear subspaces M and N of H, then M N PQ = O QP = O.

PROOF. We first remark that the equivalence of PQ = O and QP = O is clear by taking adjoints. If M _L N, so that N C M^{I} , then the fact that QC is in N for every c implies that PQc = O, so PQ = O. If, conversely, PQ = O, then for every c in N we have PX = PQz = O, so N C M ^{L} and M N.

Motivated by this result, we say that two projections P and Q are orthogonal if PQ = 0.

Our final theorem describes the circumstances under which a sum of projections is also a projection.

Theorem G. If PI, P2, , Pn are the projections on closed linear sub spaces MI, M2, . Mn of H, then P — + + .: · + projection the Pi's are pairwise orthogonal (in the sense that PiPi = O whenever i j) ; and in this case, P is the projection on

M = Ml + .: · +

PROOF, Since P is clearly self-adjoint, it is a projection it is idempotent. If the Pi's are pairwise orthogonal, then a simple computation shows at once that P is idempotent. To prove the converse, we assume that P is idempotent. Let z be a vector in the range of Pi, so that c = Pa. Then

$\|x\|^2 = \|P_ix\|^2 \leq \sum_{j=1}^{n} \|P_jx\|^2$ = (Pic,c) = (Pc,c) = llPc11 ^{2} Ilcll*.

We conclude that equality must hold all along the line here, so

$$\sum_{j=1}^{n} \|P_jx\|^2 = \|P_ix\|^2$$

and $$\|P_jx\| = 0 \qquad \text{for } j \neq i.$$

Thus the range of Pi is contained in the null space of Pi, that is, Mi C Mi [1] for every j i. This means that M,' _L M, whenever i j, and our conclusion that the Pi's are pairwise orthogonal now follows from the preceding theorem. We prove the final statement in two steps. First, we observe that since llPcll = for every z in Mi, each Mi is contained in the range of P, and therefore M is also contained in the range of P. Second, if c is a vector in the range of P, then

Z = Pc = + + + pnc is evidently in M.

There are many other ways in which the algebraic structure of the set of all projections on H can be related to the geometry of its closed R"asaces. and several of these are given in the problems below.

The significance of projections in the general theory of operators on H is the theme of the next chapter. As we shall see, the essence of the matter (the spectral theorem) is that every normal operator is made of projections in a way which clearly reveals the geometric nature of its action on the vectors in H.

Problems

1. If P and Q are the projections on closed linear subspaces M and N of H, prove that PQ is a projection PQ = QP. In this case, show that PQ is the projection on M N.

2. If P and Q are the projections on closed linear subspaces M and N of H, prove that the following statements all equivalent to one another :

$P \leq Q$;
$\|Px\| \leq$ |
$M \subseteq N$;
$PQ = P$;
(b) $QP = P$. IIQ$II for every c;

(Hint: the equivalence of (a) and (b) is easy to prove, as is that of (c), (d), and (e) ; prove that (d) implies (a) by using

(Pc,c) = - $\|PQx\|^2 \leq$ llQcIl* = (Qc,c) ;

and prove that (b) implies (c) by observing that if c IS in M, then Il Il = llPcll IIQzIi ll ^{c} ll.)

3. Show that the projections on H form b complete lattice with respect to their natural ordering as self-adjoint operators. (Compare this situation with that described in the last paragraph of Sec. 57.)

4. If P and Q are the projections on closed linear subspaces M and N of H, prove that Q — P is a projection P Q. In this case. show that Q — P is the projection on N M ^{i}.

REFERENCE BOOKS:

1. G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, International Book company, 1963.
2. Balmohan V. Limaye, Functional Analysis, Second Edition, New Age International Pvt Ltd, 1997.
3. M.Thamban Nair, Functional Analysis, A First Course, PHI, 2002

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT V – Functional Analysis – SMTA5302

Unit-V

Finite dimensional Spectral theory

If T is an operator on a Hilbert space H, then the simplest thing T can do to a vector c is to transform it into a scalar multiple of itself:

$$Tx = \lambda x. \qquad (1)$$

A non-zero vector c such that Eq. (1) is true for some scalar X is called an eigenvector of T, and a scalar X such that (1) holds for some non-zero c is called an eigenvalue of T. Each eigenvalue has one or more eigenvectors associated with it, and to each eigenvector there corresponds precisely one eigenvalue. If H has no non-zero vectors at all, then T certainly has no eigenvectors. In this case the whole theory collapses into triviality, so we assume throughout the present chapter that

$H \neq \{0\}$.

Let X be an eigenvalue of T, and consider the set M of all its corresponding eigenvectors together with the vector O (note that O is not an eigenvector). M is thus the set of all vectors c which satisfy the equation

$$(T - \lambda I)x = 0,$$

and it is clearly a non-zero closed linear subspace of H. We call M the eigenspace of T corresponding to X. It is evident that M is invariant under T and that the restriction of T to M is a very simple operator, namely, scalar multiplication by X.

In order to place the ideas of this chapter in their proper framework, we lay down several rather sweeping hypotheses, whose validity we examine later:

(a) T actually has eigenvalues, and there are finitely many of them,
ay λ_1, λ_2, , km—which are understood to be distinct— with corresponding eigenspaces Ml, M 2, . . . ,

(b) the M/S are pairwise orthogonal, that is, i j Mi -L Mj; (c) the Mi's span H.

Putting aside for a moment the question of whether these statements are true or not, we investigate their implications. By (b) and (c), every vector c in H can be expressed uniquely in the form $x = x_1 + x_2 + \cdots$+ cm, (2)

here x_i is in Mi for each i and the are pairwise orthogonal. It now follows from (a) that

$$\begin{aligned} Tx &= Tx_1 + Tx_2 + \cdots + \text{Tcm} \\ &= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \text{Xmcm}. \end{aligned} \qquad (3)$$

This relation exhibits the action of T over all of H in a manner which renders its structure perfectly clear from the geometric point of view. It will be convenient to express this result in terms of the projections Pi on the eigenspaces Mi. By Theorem 59-F, (b) is equivalent to the following statement:

the Pi's are pairwise orthogonal. (4)

Also, since for each i and for every j i we have Mi C Mi [1], Eq. (2) yields

Pta — cc, and it follows at once from this that

Ic = C = + + + cm

$$= P_1x + P_2x + \cdots + P_mx$$
$$= (P_1 + P_2 + \cdots + P_m)x$$

for every in H, so

I = PI + p2 + . . . (5)

Relation (3) now tells us that

$$= \lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_mx_m$$
$$= \lambda_1P_1x + \lambda_2P_2x + \cdots + \lambda_mP_mx$$
$$= (\lambda_1P_1 + \lambda_2P_2 + \cdots + \lambda_mP_m)x$$

for every z, so

$$T = \lambda_1P_1 + \lambda_2P_2 + \cdots \quad (6)$$

The expression for T given by (6)—when it exists—is called the spectral resolution of T. Whenever this term is used, it is to be understood that the Xi's are distinct and that the Pi's are non-zero projections which satisfy conditions (4) and (5). We shall see later that the spectral resolution of T is unique when it exists.

All our inferences from (a), (b), and (c) are perfectly rigorous, but the status of these three hypotheses remains entirely up in the air. First of all, with reference to (a), does an arbitrary operator T on H necessarily have an eigenvalue? The answer to this is no, as the reader will easily verify by considering the operator T on defined by

$$T\{x_1, x_2, \ldots\} = \{0, x_1, x_2, \ldots\}.$$

On the other hand, if H is finite-dimensional, then we shall see in Sec. 61 that every operator has an eigenvalue. For this reason, we assume for the remainder of the chapter—unless we specifically state otherwise— that H is finite-dimensional with dimension n.

We have seen that if T satisfies conditions (a), (b), and (c), then it has the spectral resolution (6). It is too much to hope that every operator on H meets these requirements, so the question arises as to what restrictions they impose on T. This question is easy to answer: T must be normal. For it follows from (6) that

$$T^* = \overline{\lambda_1}P_1 + \overline{\lambda_2}P_2 + \cdots + \overline{\lambda_m}P_m,$$

and by using (4) we readily obtain

$$TT^* = (\lambda_1P_1 + \lambda_2P_2 + \cdots + \lambda_mP_m)(\overline{\lambda_1}P_1 + \overline{\lambda_2}P_2 + \cdots + \overline{\lambda_m}P_m)$$
$$= |\lambda_1|^2P_1 + |\lambda_2|^2P_2 + \cdots + |\lambda_m|^2P_m$$

and, similarly,

$$T^*T = |\lambda_1|^2P_1 + |\lambda_2|^2P_2 + \cdots + |\lambda_m|^2P_m.$$

This entire circle of ideas will be completed in the neatest possible way if we can show that every normal operator on H satisfies conditions (a), (b), and (c), and therefore has a spectral resolution. Our aim in the present chapter is to prove this assertion, which is known as the spectral theorem, and the machinery treated in the following sections is directed exclusively toward this end. We emphasize once again that H is understood to be finite-dimensional with dimension n > O.

1. MATRICES

Our first goal is to prove that every operator on H has an eigenvalue, and in pursuing this we make use of certain elementary portions of the theory of matrices. We adopt the view that the reader is probably familiar with this theory to some degree and that it suffces here to give a brief sketch of its basic ideas. Our discussion in this section is entirely independent of the Hilbert space character of H and applies equally well to any non-trivial finite-dimensional linear space.

Let B = {el, e, . , e n be an ordered basis for H, so that each vector in H is uniquely expressible as a linear combination of the eds. If T is an operator on H, then for each we have

$$Te_j = \sum_{i=1}^{n} \alpha_{ij}e_i. \qquad (1)$$

The n^2 scalars which are determined in this way by T form the matrix of T relative to the ordered basis B. We symbolize this matrix by [T], or if it seems desirable to indicate the ordered basis under consideration, by [T]B. It iscustomary to write out a matrix as a square array:

$$[T] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}. \qquad (2)$$

The array of scalars (an, an, . · · , α_{in}) is the ith row of the matrix [T], and (al,•, , . . , ani) is its jth column. As this terminology shows, the first subscript on the entry ati always indicates the row to which it belongs, and the second the column. In our work, we generally write (2) more concisely in the form

$$[T] = [\alpha_{ij}]. \qquad (3)$$

The reader should make sure that he has a perfectly clear understanding of the rule according to which the matrix of T is constructed: write T% as a linear combination of el, Q, . , . , en, and use the resulting coeffcients to form the jth column of [T].

We offer several comments on the above paragraph. First, the term matric has not been defined at all, but only "the matrix of an operator relative to an ordered basis." A matrix—defined simply as a square array of scalars—is sometimes regarded as an object worthy of interest in its own right. For the most part, however,[1] we shall consider a matrix to be associated with a definite operator relative to a particular ordered basis, and we shall regard matrices as little more than computational devices which are occasionally useful in handling operators. Next, the matrices we work with are all square matrices. Rectangular matrices occur in connection with linear transformations of one linear space into another and are of no interest to us here. Finally, we took B to be an ordered basis rather than merely a basis, because the appearance of the array (2) clearly depends on the arrangement of the et•'s as well as on the ed's themselves. In most theoretical considerations, however, the order of the rows and columns of a matrix is as irrelevant as the order of the vectors in a basis. For this reason, we usually omit the adjective and speak of "the matrix of an operator relative to a basis."

By using the fixed basis B = {ed, we have assigned a matrix [T] = to each operator T on H, and the mapping T --4 [T] from operators to matrices is described by Tei = The importance of matrices is based primarily on two facts: T [T] is a one-to-one mapping of the set of all operators on H onto the set of all matrices; and algebraic operations can be defined on the set of all matrices in such a manner that the mapping T* [T] preserves the algebraic structure of G(H).

The first of these statements is easy to prove. If we know that $[\alpha_{ij}]$ is the matrix of T, then this information fully determines Tc for every c; for if c = then

$$\begin{aligned} Tx &= \sum_{j=1}^{n} \beta_j T e_j \\ &= \sum_{j=1}^{n} \beta_j \Big(\sum_{i=1}^{n} \alpha_{ij} e_i \Big) \\ &= \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \alpha_{ij} \beta_j \Big) e_i. \end{aligned}$$

This shows that T —i [T] is one-to-one. We see that this mapping is onto by means of the following reasoning: if is any matrix, then Te,• = defines T for the vectors in B, and when T is extended by linearity to all of H, it is clear that the resulting operator has as its matrix.

To establish the second statement, it suffces to discover how to add and multiply two matrices and how to multiply a matrix by a scalar, in such a way that the following matrix equations are true for all operators TI and on H: [TI +T2] = [Tll + [T2], [aTl] = a[Tl], and $[T_1T_2] = [T_1][T_2]$.

Let [aji] and [ßii] be the matrices of Tl and T 2. The computation

$$\begin{aligned} (T_1 + T_2)e_j &= T_1 e_j + T_2 e_j \\ &= \sum_{i=1}^{n} \alpha_{ij} e_i + \sum_{i=1}^{n} \beta_{ij} e_i \\ &= \sum_{i=1}^{n} (\alpha_{ij} + \beta_{ij}) e_i \end{aligned}$$

ßißi

shows that if we define addition for matrices by

(4) then we obtain $$[\alpha_{ij}] + [\beta_{ij}] = [\alpha_{ij} + \beta_{ij}],$$ $$[T_1 + T_2] = [T_1] + [T_2].$$

Similarly, if we multiply a matrix by a scalar in accordance with

(5) then $$\alpha[\alpha_{ij}] = [\alpha\alpha_{ij}],$$ $$[\alpha T_1] = \alpha[T_1].$$

Finally, the computation

$$\begin{aligned} (T_1T_2)e_j &= T_1(T_2e_j) = T_1 \Big(\sum_{k=1}^{n} \beta_{kj} e_k \Big) \\ &= \sum_{k=1}^{n} \beta_{kj} T_1 e_k \\ &= \sum_{k=1}^{n} \beta_{kj} \Big(\sum_{i=1}^{n} \alpha_{ik} e_i \Big) \\ &= \sum^{n} \Big(\sum^{n} \alpha_{ik} \beta_{kj} \Big) e_i \end{aligned}$$

shows that if we define multiplication for matrices by aikßki]' (6)

then we get $$[\alpha_{ij}][\beta_{ij}] = \Big[\sum_{k=1} \alpha_{ik}\beta$$ $$[T_1T_2] = [T_1][T_2].$$

The operations defined by (4), (5), and (6) are the standard algebraic operations for matrices. In words, we add two matrices by adding corresponding entries, and we multiply a matrix by a scalar by multiplying each of its entries by that scalar. The verbal description of (6) is more complicated, and is often called the row-by-column rule: to find the entry in the ith row and ith column of the product take the ith row (ail, an, • of the first factor and the jth column (ßli, ß2,•, , ßnj) of the second, multiply corresponding entries, and add

It is worth noting that the image of the zero operator under the mapping T* [T] is the zero matrix, all of whose entries are O. Further, it is equally clear that the image of the identity operator is the identity matrix, which has I's down the main diagonal (where i = j) and o's elsewhere. If we introduce the standard Kronecker delta, which is defined by

1 if i = j, then the identity matrix can be written $[\delta_{ij}]$. We now reverse our point of view for a moment (but only a moment) and consider the set An of all n X n matrices as an algebraic system in its own right, with addition, scalar multiplication, and multiplication defined by (4), (5), and (6). It can be verified directly from these definitions that An is a complex algebra with identity (the identity matrix), called the total matric algebra of degree n. If we ignore the ideas leading to (4), (5), and (6), then the structure of An is defined, and can be studied, without any reference to its origin as a representing system for the operators on H. This approach would make very little sense, however, because the primary reason for considering matrices in the first place is that they provide a computational tool which is useful in treating certain aspects of the theory of these operators.

Let us return to our original position and observe two facts: that G(H) is an algebra; and that the structure of An is defined in just such a way as to guarantee that the one-to-one mapping T [T] of (B(H) onto An preserves addition, scalar multiplication, and multiplication. It now follows at once that An is an algebra, and that T [T] is an isomorphism (see Problem 45-4) of G(H) onto An.

We give the following formal summary of our work so far.

Theorem A. If B = {ed is a basis for H, then the mapping T —i [T], which assigns to each operator T its matric relative to B, is an isomorphism of the algebra ß(H) onto the total matric algebra An.

If T is a non-singular operator whose matrix relative to B is $[\alpha_{ij}]$, then T—1 clearly has a matrix whose entries are determined in some way by the at•/s. The formulas involved here are rather clumsy and complicated, and since they have no importance for us, we shall say nothing further about them.

It is necessary, however, to know what is meant by the inverse of a matrix, when it is considered purely as an element of An and without reference to any operator which it may represent. We first remark that the identity matrix is easily seen by direct matrix multiplication to be an identity element for the algebra An, in the sense that we have

$$[\alpha_{ij}][\delta_{ij}] = [\delta_{ij}][\alpha_{ij}] = [\alpha_{ij}]$$

for every matrix and by the theory of rings, this identity is unique. A matrix is said to be non-singular if there exists a matrix [ßii] such that

$$[\alpha_{ij}][\beta_{ij}] = [\beta_{ij}][\alpha_{ij}] = [\delta_{ij}];$$

and, again by the theory of rings, if such a matrix exists, then it is unique, it is denoted by and it is called the inverse of $[\alpha_{ij}]$.

These ideas arc connected with operators by the following considerations. Suppose that is the matrix of an operator T relative to B.

We know that the non-singularity of T is equivalent to the existence of an operator T—1 such that

$$TT^{-1} = T^{-1}T = I.$$

The isomorphism of Theorem A transforms this operator equation into the matrix equation

$$[T][T^{-1}] = [T^{-1}][T] = [I],$$

which is equivalent to

= $T^{-1}][\alpha_{ij}]$= [Öiil.

We therefore have

Theorem B. Let B be a basis for H, and T an operator whose matric relative to B is Then T is non-singular is non-singular, and in this case$\alpha_{ij}]^{-1}$

There is one further issue which requires discussion, If T is a fixed operator on H, then its matrix [T]B relative to B obviously depends on the choice of B. If B changes, how does [TIB change? More specifically, if B' = ffl, f2, • , fn} is also a basis for H, what is the relation between [T]B and [T]B,? The answer to this question is best given in terms of the non-singular operator A defined by Aeg• — fi. Let $[\alpha_{ij}]$and [ßi,•] be the matrices of T relative to B and B', so that

$$Te_j = \sum_{i=1}^{n} \alpha_{ij} e_i$$

and Tf,• = ßiifi. Let be the matrix of A relative to B, so that Ae,• = By Theorem B, is non-singular. We now compute Tf,• in two different ways:

$$\begin{aligned} Tf_j &= \sum_{k=1}^{n} \beta_{kj} f_k = \sum_{k=1}^{n} \beta_{kj} A e_k \\ &= \sum_{k=1}^{n} \beta_{kj} \Big(\sum_{i=1}^{n} \gamma_{ik} e_i \Big) \\ &= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \gamma_{ik} \beta_{kj} \Big) e_i; \end{aligned}$$

$$\begin{aligned} Tf_j &= TAe_j = T\Big(\sum_{k=1}^{n} \gamma_{kj} e_k \Big) \\ &= \sum_{k=1}^{n} \gamma_{kj} T e_k \\ &= \sum_{k=1}^{n} \gamma_{kj} \Big(\sum_{i=1}^{n} \alpha_{ik} e_i \Big) \\ &= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} \Big) e_i. \end{aligned}$$

and

A comparison of these results shows that

Yikß%, aikYkj

for all i and j, so

$$[\gamma_{ij}][\beta_{ij}] = [\alpha_{ij}][\gamma_{ij}]$$

or $$[\beta_{ij}] = [\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}].$$ (7) If we now write this in the form

$$[T]_{B'} = [A]_B^{-1}[T]_B[A]_B,$$

then it becomes quite clear how the matrix of T changes when B is replaced by B'.

Two matrices and [ßi,•] are said to be similar if there exists a non-singular matrix such that (7) is true. The analysis given above proves half of the following theorem (we leave the proof of the other half to the reader).

Theorem C. Two matrices in An are similar they are the matrices of a single operator on H relative to (possibly) different bases,

We are now in a position to formulate the fundamental problem of the classical theory of matrices. A given operator on H may have many different matrices relative to different bases, and Theorem C shows in purely matrix terms how these matrices are related to one another. The question arises as to whether it is possible to find, for each operator (or for each operator of a special kind), a basis relative to which its matrix assumes some particularly simple form. This is the canonical form problem of matrix theory, and the most important theorem in this direction is the spectral theorem, which we state in the language of matrices in Sec. 62. In the classical approach to these ideas, it was customary to work exclusively with matrices. However, the great advances in the understanding of algebra which have taken place in recent years have made it plain that problems of this kind are best treated intrinsically, that is, directly in terms of the linear spaces and linear transformations involved. As matters now stand, it is possible—and preferable—to state the main canonical form theorems of matrix theory without mentioning matrices at all. Nevertheless, matrices remain useful for some purposes, notably (from our point of view) in the problem of proving that an arbitrary operator on H has an eigenvalue.

Problems

1. Show that the dimension of (B(H) is n'.
2. A scalar matric in An is one which has the same scalar in every position on the main diagonal and o's elsewhere. Show that a scalar matrix commutes with every matrix, and that a matrix which commutes with every matrix is necessarily scalar. What does this imply about G(H)? (See Problem 45-3.)

3, A diagonal matric in An is one which has arbitrary scalars on the main diagonal and o's elsewhere. Show that all diagonal matrices commute with one another, and that a matrix is necessarily diagonal if it commutes with all diagonal matrices.

4. Complete the proof of Theorem C.

2 . DETERMINANTS AND THE SPECTRUM OF AN OPERATOR

Determinants are often advertised to students of elementary mathematics as a computational device of great value and effciency for solving numerical problems involving systems of linear equations. This is somewhat misleading, for their value in problems of this kind is very limited. On the other hand, they do have definite importance as a theoretical tool. Briefly, they provide a numerical means of distinguishing between singular and non-singular matrices (and operators).

This is not the place for developing the theory of determinants in any detail. Instead, we assume that the reader already knows something about them, and we confine ourselves to listing a few of their simpler properties which are relevant to our present interests.

The determinant function det is thus a scalar-valued function of matrices which has certain properties. In elementary work, the determinant of a matrix is usually written out with vertical bars, as follows,

$$\det([\alpha_{ij}]) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix},$$

and is evaluated by complicated procedures which are of no concern to us here.

We now consider an operator T on H. If B and B' are bases for H, then the matrices and [ßi,•] of T relative to B and B' may be entirely different, but nevertheless they have the same determinant. For we know from the previous section that there exists a non-singular matrix Wii] such that

$$[\beta_{ij}] = [\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}];$$

and therefore, by properties (1), (2), and (3), we have

$$\begin{aligned}\det([\beta_{ij}]) &= \det([\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}]) \\ &= \det([\gamma_{ij}]^{-1}) \det([\alpha_{ij}]) \det([\gamma_{ij}]) \\ &= \det([\gamma_{ij}]^{-1}) \det([\gamma_{ij}]) \det([\alpha_{ij}]) \\ &= \det([\gamma_{ij}]^{-1}[\gamma_{ij}]) \det([\alpha_{ij}]) \\ &= \det([\delta_{ij}]) \det([\alpha_{ij}]) \\ &= \det([\alpha_{ij}]).\end{aligned}$$

This result allows us to speak of the determinant of the operator T, meaning, of course, the determinant of its matrix relative to any basis; and from this point on, we shall regard the determinant function primarily as a scalar-valued function of the operators on H. We at once obtain the following four properties for this function, which are simply translations of those stated above:

(1') det(l) = 1;

(2') det(TIT2) = det(TI) det(T2);

(3') det(T) O — T is non-singular; and

(4') det(T — XI) is a polynomial, with complex coeffcients, of degree n in the variable X.

We are now in a position to take up once again, and to settle, the problem of the existence of eigenvalues.

Let T be an operator on H. If we recall Problem 44-6, it is clear that a scalar X is an eigenvalue of T there exists a non-zero vector c such that (T — Xl)c - O — T — is singular det(T - RI) The eigenvalues of T are therefore precisely the distinct roots of the equation det(T — RI) = O, (1)

which is called the characteristic equation of T. It may illuminate matters somewhat if we choose a basis B for H, find the matrix of T relative to B, and write the characteristic equation in the extended form

$$\begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} - \lambda \end{vmatrix} = 0.$$

Our search for eigenvalues of T is reduced in this way to a search for roots of Eq. (1). Property (4') tells us that this is a polynomial equation, with complex coeffcients, of degree n in the complex variable X. We now appeal to the fundamental theorem of algebra, which guarantees that an equation of this kind always has exactly n complex roots. Some of these roots may of course be repeated, in which case there are fewer than n distinct roots. In summary, we have

Theorem A. If T is an arbitrary operator on H, then the eigenvalues of T constitute a non-empty finite subset of the complec plane. Furthermore, the number of points in this set does not exceed the dimension n of the space H.

The set of eigenvalues of T is called its spectrum, and is denoted by c(T). For future reference, we observe that T(T) is a compact subspace of the complex plane.

It should now be reasonably clear why we required in the definition of a Hilbert space that its scalars be the complex numbers. The reader will easily convince himself that in the Euclidean plane the operation of rotation about the origin through 90 degrees is an operator on this real Banach space which has no eigenvalues at all, for no non-zero vector is transformed into a real multiple of itself. The existence of eigenvalues is therefore linked in an essential way to properties of the complex numbers which are not enjoyed by the real numbers, and the most significant of these properties is that stated in the fundamental theorem of algebra. The mechanism of matrices and determinants turns out to be simply a device for making effective use of this theorem in our basic problem of proving that eigenvalues exist. We also remark that Theorem A and its proof remain valid in the case of an arbitrary linear transformation on any complex linear space of finite dimension n > O.

Problems

1. Let T be an operator on H, and prove the following statements:
   (a) T is singular O e C(T) ;
   (b) if T is non-singular, then X e c(T) $^{-1} \varepsilon \sigma(T^{-1})$;
   (c) if A is non-singular, then TA¯I) =
   (d) if X e T(T), and if p is any polynomial, then $\varepsilon \sigma(p($ p(X)
   (e) if T^k = O for some positive integer k, then c(T) $= \{0\}$.

2. Let the dimension n of H be 2, let B = {er,e2} be a basis for H, and assume that the determinant of a 2 X 2 matrix is given by ana22 awa21.
   (a) Find the spectrum of the operator T on H defined by Tel = e2 and To = -el.
   (b) If T is an arbitrary operator on H whose matrix relative to B is $[\alpha_{ij}]$, Show that T^2 — (all + an) T + — a12a21)I = O. Give a verbal statement of this result.

3. THE SPECTRAL THEOREM

We now return to the central purpose of this chapter, namely, the statement and proof of the spectral theorem.

Let T be an arbitrary operator on H. We know by Theorem 61-A that the distinct eigenvalues of T form a non-empty finite set of complex numbers. Let XI, X2, . , Xm be these eigenvalues; let Ml, M2, . . . ,
Mm be their corresponding eigenspaces; and let PI, P2, . , Pm be the projections on these eigenspaces. We consider the following three statements.

I. The are pairwise orthogonal and span H.

Il. The Pi's are pairwise orthogonal, I = Z.^{m}_l Pi, and T = XiPi. 111. T is normal.

We take the spectral theorem to be the assertion that these statements are all equivalent to one another. It was proved in the introduction to this chapter that I Il Ill. We now complete the cycle by showing that

The hypothesis that T is normal plays its most critical role in our first theorem.

Theorem A. If T is normal, then c is an eigenvector of T with eigenvalue X c is an eigenvector of T* with eigenvalue X.

PROOF. Since T is normal, it is easy to see that the operator T — XI
(whose adjoint is T* — RI) is also normal for any scalar X. By Theorem 58-0, we have

$$\|Tx - \lambda x\| = \|T^*x - \bar{\lambda} x\|$$

for every vector c, and the statements, of the theorem follow at once from

The way is now clear for

Theorem B. If T is normal, then the Mi's are pairwise orthogonal.

PROOF. Let c,' and c,' be vectors in Mi and Mi for i j, so that TCi — and Tc,• = XiC,•. The preceding theorem shows that

$$\begin{aligned} \lambda_i(x_i,x_j) = (\lambda_i x_i,x_j) = (Tx_i,x_j) = (x_i,T^*x_j) \\ = (x_i,\bar{\lambda}_j x_j) = \lambda_j(x_i,x_j); \end{aligned}$$

and since Xj, it is clear that we must have = O.

Our next step is to prove that the Mi's span H when T is normal, and for this we need the following preliminary fact.

Theorem C. If T is normal, then each Mi reduces T.

PROOF. It is obvious that each Mi is invariant under T, so it suffces, by Theorem 59-0, to show that each Mi is also invariant under T*. This is an immediate consequence of Theorem A, for if is a vector in Mi, so that Tc,• = then T = is also in Mi.

Finally, we have

Theorem D. If T is normal, then the Mi's span H.

PROOF. The fact that the Mi's are pairwise orthogonal implies, by Theorems 59-F and 59-G, that M - Ml + + . . . closed linear subspace of H, and that its associated projection is

$$P = P_1 + P_2 + \cdots + P_m.$$

Since each Mi reduces T, we see by Theorem 59-E that TPi PiT for each Pi. It follows from this that TP = P T, so M also reduces T, and consequently M -L is invariant under T. If M I then, since all the eigenvectors of T are contained in M, the restriction of T to M -L is an operator on a non-trivial finite-dimensional Hilbert space which has no eigenvectors, and hence no eigenvalues. Theorem 61-A shows that this is impossible. We therefore conclude that M L = so M = H and the span H.

This completes the proof of the spectral theorem and, in particular, of the fact that if T is normal, then it has a spectral resolution

T = RIP 1 + X2P2 + $\cdots + \lambda_m P_m.$ (1)

We now make several observations which will be useful in carrying out our promise to show that this expression for T is unique. Since the Pi's are pairwise orthogonal, if we square both sides of (1) we obtain

$$T^2 = \sum_{i=1}^{m} \lambda_i^2 P_i.$$

More generally, if n is any positive integer, then

$$T^n = \sum_{i=1}^{m} \lambda_i^n P_i. \quad (2)$$

If we make the customary agreement that T^O = I, then the fact that I = Pi shows that (2) is also valid for the case n = O. Next, let p(z) be any polynomial, with complex coeffcients, in the complex variable z. By taking linear combinations, (2) can evidently be extended to

$$p(T) = \sum_{i=1}^{m} p(\lambda_i) P_i. \quad (3)$$

We would like to find a polynomial p such that the right side of (3) collapses to a specified one of the Pi's, say Pi. What is needed is a polynomial p,• with the property that Pi(Xi) = O if i j and $p_j(\lambda_j)$We define Pi as follows:

$$\text{Pi}(z) = \frac{(z - \lambda_1) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_m)}{(\lambda_j - \lambda_1) \cdots (\lambda_j - \lambda_{j-1})(\lambda_j - \lambda_{j+1}) \cdots (\lambda_j - \lambda_m)}.$$

Since Pi is a polynomial, and since = öij, (3) yields

$$Pi = Pi(T). \qquad (4)$$

In order to interpret these remarks to our advantage, we point out that only three facts about (1) have been used in obtaining (4): the hi's are distinct complex numbers; the Pi's are pairwise orthogonal projections; and I = Pi. By using these properties of (1), and these alone, we have shown that the Pi's are uniquely determined as specific polynomials in T.

We now assume that we have another expression for T similar to (1),

$$T = \alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_k Q_k, \qquad (5)$$

and 'that this is also a spectral resolution of T, in the sense that the α's are distinct complex numbers, the Qi's are non- zero pairwise orthogonal projections, and I = 2f_1 Q. We wish to show that (5) is actually identical with (1), except for notation and order of terms. We begin by proving, in two steps, that the ads are precisely the eigenvalues of T. First, since Qi O, there exists a non-zero vector c in the range of Q', and since = c and = O for j i, we see from (5) that Tc = so each ai is an eigenvalue of T. Next, if X is an eigenvalue of T, so that Tc = Xc for some non-zero x, then

$$Tx = \lambda x = \lambda I x = \lambda \sum_{i=1}^{k} Q_i x \qquad = \lambda Q_i$$

and

$$Tx = \sum_{i=1}^{k} \alpha_i Q_i x,$$

BO — ai)QiZ - O.

Since the Q,a's are pairwise orthogonal, the non-zero vectors among them—there is at least one, for c O—are linearly independent, and this implies that X = a, for some i. These arguments show that the set of a/s equals the set of Xi's, and therefore, by changing notation if necessary, we can write (5) in the form

T = XIQI + + . • + XmQm• (6)

The discussion in the preceding paragraph now applies to (6) and gives

Qi - pj(T) (7)

for every j. On comparing (7) with (4), we see that the Qi's equal the Pi's. This shows that (5) is exactly the same as (l)—except for notation and the order of terms—and completes our proof of the fact that the spectral resolution of T is unique.

We conclude with a brief look at the matrix interpretation of statements I and Il at the beginning of this section. Assume that I is true, that is, that the eigenspaces Ml, M 2, . Mm of T are pairwise orthogonal and span H. For each choose a basis which consists of mutually orthogonal unit vectors. This can always be done, for a basis of this kind—called an orthonormal basis—is precisely a complete orthonormal set for Mi. It is easy to see that the union of these little bases is an orthonormal basis for all of H; and relative to this, the matrix of T has the following diagonal form (all entries off the main diagonal are understood to be 0) :

$$\begin{bmatrix} \lambda_1 & & & & & & & & & & \\ & \cdot & & & & & & & & & \\ & & \cdot & & & & & & & & \\ & & & \cdot & & & & & & & \\ & & & & \lambda_1 & & & & & & \\ & & & & & \lambda_2 & & & & & \\ & & & & & & \cdot & & & & \\ & & & & & & & \cdot & & & \\ & & & & & & & & \cdot & & \\ & & & & & & & & & \lambda_2 & \\ & & & & & & & & & & \cdot \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & \lambda_m \\ & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & & & \lambda_m \end{bmatrix} \quad (8)$$

We next assume that H has an orthonormal basis relative to which the matrix of T is diagonal. If we rearrange the basis vectors in such a way that equal matrix entries adjoin one another on the main diagonal, then the matrix of T relative to this new orthonormal basis will have the form (8). It is easy to see from this that T can be written in the form

$$T = \sum_{i=1}^{m} \lambda_i P_i,$$

where the are distinct complex numbers, the Pi's are non-zero pairwise orthogonal projections, and I = Pi. The uniqueness of the spectral resolution now guarantees that the Xls are the distinct eigenvalues of T and that the Pi's are the projections on the corresponding eigenspaces. The spectral theorem tells us that statements I, Il, and Ill are equivalent to one another. The above remarks carry us a bit further, for they constitute a proof of the fact that these statements are also equivalent to IV. There exists an orthonormal basis for H relative to which the matrix of T is diagonal.

It is interesting to realize that the implication Ill IV, which we proved by showing that Ill I and I IV, can be made to depend more directly on matrix computations. This proof is outlined in the last three problems below.

Problems

1. Show that an operator T on H is normal its adjoint T* is a polynomial in T.
2. Let T be an arbitrary operator on H, and N a normal operator,

   Show that if T commutes with N, then T also commutes with N*. 3. Let T be a normal operator on H with spectrum {Xl, X2, . . . , λ

   and use the spectral resolution of T to prove the following statements: (a) T is self-adjoint each Xi is real; (b) T is positive Xi 2 0 for each i; (c) T is unitary = 1 for each i.
4. Show that a positive operator T on H has a unique positive square root; that is, show that there exists a unique positive operator A on H such that - T.
5. Let B - {el, o, , e n} be an orthonormal basis for H. If T is an operator on H whose matrix relative to B is [as)], show that the matrix of T* relative to B is where ßii = a [ßi,•] is often called the conjugate transpose of $[\alpha_{ij}]$.
6. Let T be an arbitrary operator on H, and prove that there exist n closed linear subspaces Ml, M 2, . , Mn such that

$$\{0\} \subset M_1 \subset M_2 \subset \cdots \subset M_n = H,$$

the dimension of each M_i is i. and each M_i is invariant under T (Hint: if $n = 1$, the statement is clear; and if $n > 1$, assume it for all Hilbert spaces of dimension $n — 1$, and prove it for H by using Theorem 59-B and the fact that T^* has an eigenvector.)

REFERENCE BOOKS:

1. G.F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill, International Book company, 1963.
2. Balmohan V. Limaye, Functional Analysis, Second Edition, New Age International Pvt Ltd, 1997.
3. M.Thamban Nair, Functional Analysis, A First Course, PHI, 2002