SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – TOPOLOGICAL SPACES – SMTA5202

UNIT-I Topological Spaces and Continuous Functions

The concept of topological space grew out of the study of the real line and Euclidean space and the study of continuous functions on lese spaces. In this chapter, we de- fine what a topological space is, and we study a number of ways of constructing a topology on a set X to make it into a topological space. We also consider some of the elementary concept associated with topological spaces. Open and closed sets, limit points, and continuous functions are introduced as natural generalizations of the corresponding ideas for the real line and Euclidean space.

Topological Spaces

The definition of a topological space that is now standard was a long time in being formulated. Various mathematicians—Fróchet, Hausdorff, and others—proposed different definitions over a period of years durng the first decades of the twentieth century, but it took quite a while before mathematicians settled on the one that most suitable. They wanted, of course, a definition that was as broad as possible, so that it would include as special cases all the various examples that were useful in mathematics— Euclidean space, infinite-dimensional Euclidean space, and function spaces among the but they also wanted the definition to be narrow enough that the standard theorems about these family of spaces would hold for topological spaces in

general. This is always the problem when one is trying to fomulate a new mathe- matical concept, to decide how general its definition should be. The detinitioi finally settled on may seem a bit abstract, but as you work through the various ways of con- sFucting topological spaces, you will get a better feeling for what the concept means.

Definition. A *topology* on a set X is a collection $\mathcal{J}$ of subsets of X having the following properties:
  (1)   X and $\varnothing$ are in $\mathcal{J}$.
  (2)   The union of the elements of any subcollection of $\mathcal{J}$ is in $\mathcal{J}$.
  (3)   The intersection of the elements of any finite subcollection of $\mathcal{J}$ is in $\mathcal{J}$.
A set X for which a topology $\mathcal{J}$ has been specified is called a topological *space.*

   EXAMPLE 1 Let X be a three-element set, X = {a, b, c}. There are many possible topologies on X, some of which are indicated schematically in Figure 12.1. The diagram in the upper right-hand comer indicates the topology in which the open sets are X, B, {a, b}, {b}, and {b, c} The

topology in the upper left-hand corner contains only X and B, while the topology in the lower right-hand corner contains every subset of X. You can get other topologies on X by permuting a, *b,* and r
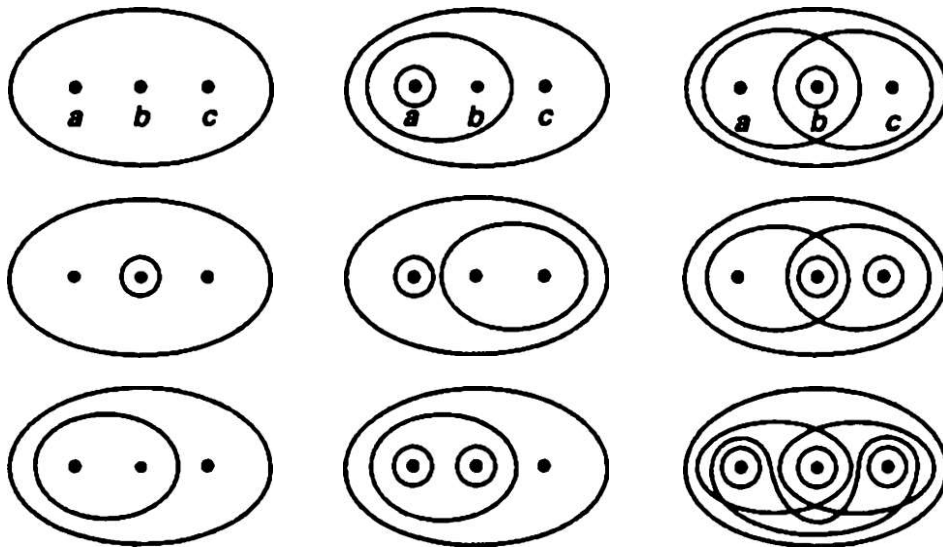
From this example, you can see that even a three-element set has many different topologies. But not every collection of subsets of X is a topology on X Neither of the collections indicated in Figure 12 2 is a topology, for instance.

Figure 12.2

EXAMPLE 2.   If X is any set, the collection of *all* subsets of X is a topology on X, it is called the *discrete topology* The collection consisting of X and $\varnothing$ only is also a topology on X; we shall call it the *indiscrete topology*, or the *trivial topology*

EXAMPLE 3.   Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or is all of X Then $\mathcal{T}_f$ is a topology on X, called the *finite complement topology*. Both X and $\varnothing$ are in $\mathcal{T}_f$, since $X - X$ is finite and $X - \varnothing$ is all of X If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$, to show that $\bigcup U_\alpha$ is in $\mathcal{T}_f$, we compute

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha).$$

The latter set is finite because each set $X - U_\alpha$ is finite If $U_1, \ldots, U_n$ are nonempty elements of $\mathcal{T}_f$, to show that $\bigcap U_i$ is in $\mathcal{T}_f$, we compute

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i).$$

The latter set is a finite union of finite sets and, therefore, finite

EXAMPLE 4   Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X, as you can check

Definition.   Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *strictly coarser*, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

This terminology is suggested by thinking of a topological space as being something like a truckload full of gravel—the pebbles and all unions of collections of pebbles being the open sets. If now we smash the pebbles into smaller ones, the collection of open sets has been enlarged, and the topology, like the gravel, is said to have been made finer by the operation.

Two topologies on X need not be comparable, of course. In Figure 12 1 preceding, the topology in the upper right-hand corner is strictly finer than each of the three topologies in the first column and strictly coarser than each of the other topologies in the third column. But it is not comparable with any of the topologies in the second column.

Other terminology is sometimes used for this concept. If $\mathcal{T}' \supset \mathcal{T}$, some mathematicians would say that $\mathcal{T}'$ is *larger* than $\mathcal{T}$, and $\mathcal{T}$ is *smaller* than $\mathcal{T}'$. This is certainly acceptable terminology, if not as vivid as the words "finer" and "coarser."

Many mathematicians use the words "weaker" and "stronger" in this context. Unfortunately, some of them (particularly analysts) are apt to say that $\mathcal{T}'$ is stronger than $\mathcal{T}$ if $\mathcal{T}' \supset \mathcal{T}$, while others (particularly topologists) are apt to say that $\mathcal{T}'$ is weaker than $\mathcal{T}$ in the same situation! If you run across the terms "strong topology" or "weak topology" in some book, you will have to decide from the context which inclusion is meant. We shall not use these terms in this book.

§13   Basis for a Topology

For each of the examples in the preceding section, we were able to specify the topology by describing the entire collection $\mathcal{T}$ of open sets. Usually this is too difficult. In most cases, one specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition.   If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology $\mathcal{T}$ generated by $\mathcal{B}$* as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

We will check shortly that the collection $\mathcal{T}$ is indeed a topology on X. But first let us consider some examples.

EXAMPLE 1   Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathcal{B}$ satisfies both conditions for a basis The second condition is illustrated in Figure 13 1. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U

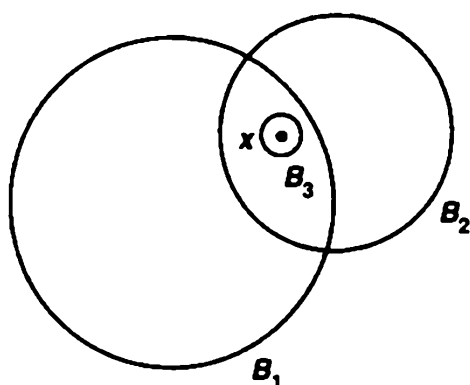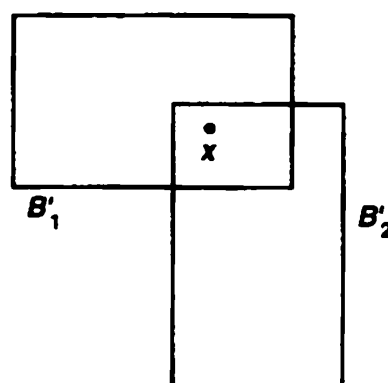*Figure 13.1*   *Figure 13.2*

EXAMPLE 2.   Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes   Then $\mathcal{B}'$ satisfies both conditions for a basis. The second condition is illustrated in Figure 13 2; in this case, the condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty)   As we shall see later, the basis $\mathcal{B}'$ generates the same topology on the plane as the basis $\mathcal{B}$ given in the preceding example

EXAMPLE 3   If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X

Let us check now that the collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is, in fact, a topology on X. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in $\mathcal{T}$, since for each $x \in X$ there is some basis element B containing x and contained in X Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$, of elements of $\mathcal{T}$ and show that

$$U = \bigcup_{\alpha \in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in U$, there is an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition.

Now let us take *two* elements U_1 and U_2 of $\mathcal{T}$ and show that $U_1 \cap U_2$ belongs to $\mathcal{T}$. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis element B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. See Figure 13.3. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to $\mathcal{T}$, by definition.
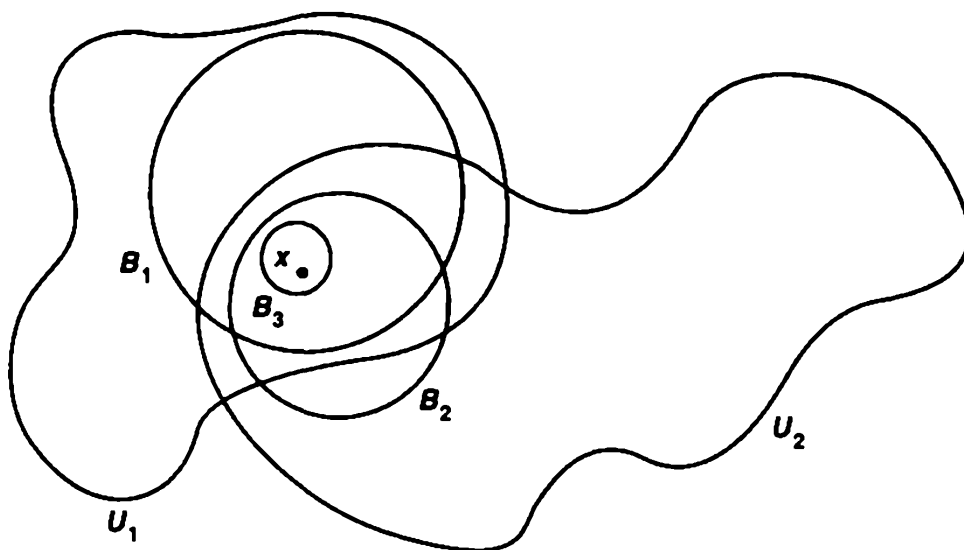

Figure 13.3

Finally, we show by induction that any finite intersection $U_1 \cap \cdots \cap U_n$ of elements of $\mathcal{T}$ is in $\mathcal{T}$. This fact is trivial for $n = 1$; we suppose it true for $n - 1$ and prove it for n. Now

$$(U_1 \cap \cdots \cap U_n) = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$

By hypothesis, $U_1 \cap \cdots \cap U_{n-1}$ belongs to $\mathcal{T}$; by the result just proved, the intersection of $U_1 \cap \cdots \cap U_{n-1}$ and U_n also belongs to $\mathcal{T}$

Thus we have checked that collection of open sets generated by a basis $\mathcal{B}$ is, in fact, a topology.

Another way of describing the topology generated by a basis is given in the following lemma:

Lemma 13.1. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. ∎

This lemma states that every open set U in X can be expressed as a union of basis elements. This expression for U is not, however, unique. Thus the use of the term "basis" in topology differs drastically from its use in linear algebra, where the equation expressing a given vector as a linear combination of basis vectors *is* unique.

We have described in two different ways how to go from a basis to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a basis generating it. Here is one way of obtaining a basis for a given topology; we shall use it frequently.

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. We must show that $\mathcal{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let $\mathcal{T}$ be the collection of open sets of X; we must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$. First, note that if U belongs to $\mathcal{T}$ and if $x \in U$, then there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology $\mathcal{T}'$, by definition. Conversely, if W belongs to the topology $\mathcal{T}'$, then W equals a union of elements of $\mathcal{C}$, by the preceding lemma. Since each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology, W also belongs to $\mathcal{T}$. ∎

When topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than another. One such criterion is the following.

Lemma 13.3. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:*

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. (2) $\Rightarrow$ (1). Given an element U of $\mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition (2) tells us there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}'$, by definition.

(1) $\Rightarrow$ (2). We are given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore, $B \in \mathcal{T}'$ Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. ∎

Some students find this condition hard to remember. "Which way does the inclusion go?" they ask. It may be easier to remember if you recall the analogy between a topological space and a truckload full of gravel. Think of the pebbles as the basis elements of the topology; after the pebbles are smashed to dust, the dust particles are the basis elements of the new topology. The new topology is finer than the old one, and each dust particle was contained inside a pebble, as the criterion states.

EXAMPLE 4.   One can now see that the collection $\mathcal{B}$ of all circular regions in the plane generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions, Figure 13 4 illustrates the proof   We shall treat this example more formally when we study metric spaces
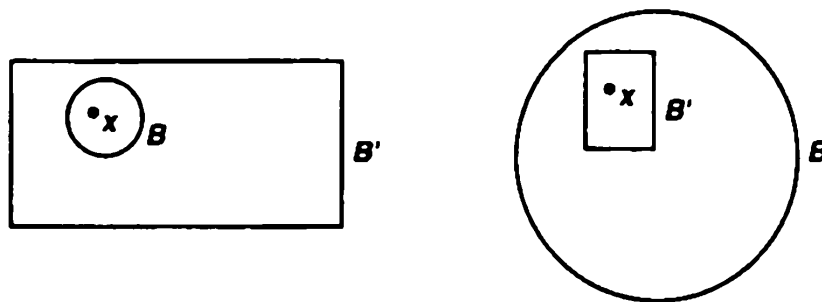

Figure 13.4

We now define three topologies on the real line $\mathbb{R}$, all of which are of interest.

Definition.   If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the *standard topology* on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise. If $\mathcal{B}'$ is the collection of all half-open intervals of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_\ell$. Finally let K denote the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ will be called the **K-topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

It is easy to see that all three of these collections are bases; in each case, the intersection of two basis elements is either another basis element or is empty. The relation between these topologies is the following:

Lemma 13.4. *The topologies of* $\mathbb{R}_\ell$ *and* $\mathbb{R}_K$ *are strictly finer than the standard topology on* $\mathbb{R}$, *but are not comparable with one another.*

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), the basis element $[x, b)$ for $\mathcal{T}'$ contains x and lies in (a, b). On the other hand, given the basis element $[x, d)$ for $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$

A similar argument applies to $\mathbb{R}_K$. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), this same interval is a basis element for $\mathcal{T}''$ that contains x. On the other hand, given the basis element $B = (-1, 1) - K$ for $\mathcal{T}''$ and the point 0 of B, there is no open interval that contains 0 and lies in B.

We leave it to you to show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable. $\blacksquare$

A question may occur to you at this point. Since the topology generated by a basis $\mathcal{B}$ may be described as the collection of arbitrary unions of elements of $\mathcal{B}$, what happens if you start with a given collection of sets and take finite intersections of them as well as arbitrary unions? This question leads to the notion of a subbasis for a topology

Definition. A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

We must of course check that $\mathcal{T}$ is a topology. For this purpose it will suffice to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis, for then the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ is a topology, by Lemma 13.1. Given $x \in X$, it belongs to an element of $\mathcal{S}$ and hence to an element of $\mathcal{B}$; this is the first condition for a basis. To check the second condition, let

$$B_1 = S_1 \cap \cdots \cap S_m \quad \text{and} \quad B_2 = S_1' \cap \cdots \cap S_n'$$

be two elements of $\mathcal{B}$. Their intersection

$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n')$$

is also a finite intersection of elements of S, so it belongs to $\mathcal{B}$.

Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

   (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

7. Consider the following topologies on $\mathbb{R}$:

   $\mathcal{T}_1 = $ the standard topology,

   $\mathcal{T}_2 = $ the topology of $\mathbb{R}_K$,

   $\mathcal{T}_3 = $ the finite complement topology,

   $\mathcal{T}_4 = $ the upper limit topology, having all sets $(a, b]$ as basis,

   $\mathcal{T}_5 = $ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

   Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, \, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

§14 The Order Topology

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the *order topology*; in this section, we consider it and study some of its properties.

Suppose that X is a set having a simple order relation $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the *intervals* determined by a and b. They are the following :

$$(a, b) = \{x \mid a < x < b\},$$
$$(a, b] = \{x \mid a < x \leq b\},$$
$$[a, b) = \{x \mid a \leq x < b\},$$
$$[a, b] = \{x \mid a \leq x \leq b\}.$$

The notation used here is familiar to you already in the case where X is the real line, but these are intervals in an arbitrary ordered set. A set of the first type is called an *open interval* in X, a set of the last type is called a *closed interval* in X, and sets of the second and third types are called *half-open intervals*. The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X. And so they will.

Definition.   Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the *order topology*.

If X has no smallest element, there are no sets of type (2), and if X has no largest element, there are no sets of type (3).

One has to check that $\mathcal{B}$ satisfies the requirements for a basis. First, note that every element x of X lies in at least one element of $\mathcal{B}$: The smallest element (if any) lies in all sets of type (2), the largest element (if any) lies in all sets of type (3), and every other element lies in a set of type (1). Second, note that the intersection of any two sets of the preceding types is again a set of one of these types, or is empty. Several cases need to be checked; we leave it to you.

EXAMPLE 1   The standard topology on $\mathbb{R}$, as defined in the preceding section, is just the order topology derived from the usual order on $\mathbb{R}$.

EXAMPLE 2.   Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order; we shall denote the general element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$, to avoid difficulty with notation The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest element, so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$, and for $a = c$ and $b < d$. These two types of intervals are indicated in Figure 14.1. The subcollection consisting of only intervals of the second type is also a basis for the order topology on $\mathbb{R} \times \mathbb{R}$, as you can check
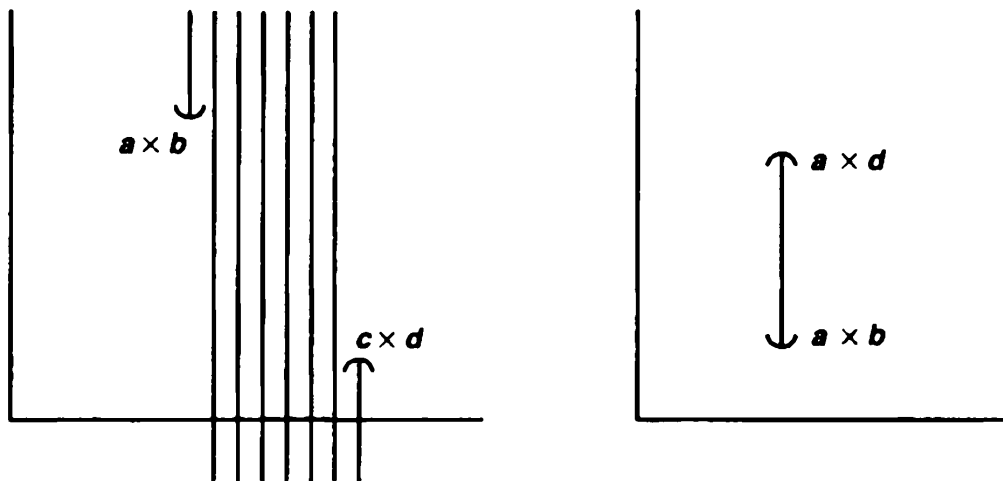

Figure 14.1

EXAMPLE 3   The positive integers $\mathbb{Z}_+$ form an ordered set with a smallest element The order topology on $\mathbb{Z}_+$ is the discrete topology, for every one-point set is open If $n > 1$, then the one-point set $\{n\} = (n - 1, n + 1)$ is a basis element; and if $n = 1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

EXAMPLE 4   The set $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order is another example of an ordered set with a smallest element Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent X by

$$a_1, a_2, \ldots ; b_1, b_2, \ldots$$

The order topology on X is *not* the discrete topology. Most one-point sets are open, but there is an exception—the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the a_i sequence.

Definition.   If X is an ordered set, and a is an element of X, there are four subsets of X that are called the *rays* determined by a They are the following·

$$(a, +\infty) = \{x \mid x > a\},$$
$$(-\infty, a) = \{x \mid x < a\},$$
$$[a, +\infty) = \{x \mid x \geq a\},$$
$$(-\infty, a] = \{x \mid x \leq a\}.$$

Sets of the first two types are called *open rays*, and sets of the last two types are called *closed rays*.

The use of the term "open" suggests that open rays in X are open sets in the order topology. And so they are. Consider, for example, the ray $(a, +\infty)$. If X has a largest element b_0, then $(a, +\infty)$ equals the basis element $(a, b_0]$. If X has no largest element, then $(a, +\infty)$ equals the union of all basis elements of the form (a, x), for $x > a$. In either case, $(a, +\infty)$ is open. A similar argument applies to the ray $(-\infty, a)$.

The open rays, in fact, form a subbasis for the order topology on X, as we now show Because the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology equals a finite intersection of open rays; the interval (a, b) equals the intersection of $(-\infty, b)$ and $(a, +\infty)$, while $[a_0, b)$ and $(a, b_0]$, if they exist, are themselves open rays. Hence the topology generated by the open rays contains the order topology

§15   The Product Topology on $X \times Y$

If X and Y are topological spaces, there is a standard way of defining a topology on the cartesian product $X \times Y$. We consider this topology now and study some of its properties.

Definition.   Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Let us check that $\mathcal{B}$ is a basis. The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition is almost as easy, since the intersection of any two basis elements $U_1 \times V_1$ and $U_2 \times V_2$ is another basis element. For

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

and the latter set is a basis element because $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively. See Figure 15.1.

Note that the collection $\mathcal{B}$ is not a topology on $X \times Y$. The union of the two rectangles pictured in Figure 15.1, for instance, is not a product of two sets, so it cannot belong to $\mathcal{B}$; however, it is open in $X \times Y$.

Each time we introduce a new concept, we shall try to relate it to the concepts that have been previously introduced. In the present case, we ask: What can one say if the topologies on X and Y are given by bases? The answer is as follows:

Theorem 15.1.   *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

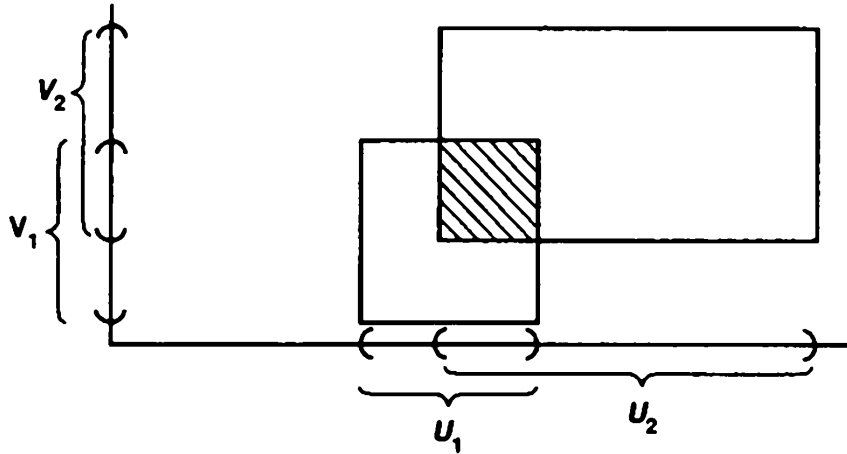*is a basis for the topology of $X \times Y$*

Figure 15.1

Proof. We apply Lemma 13.2. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$, and an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$. Thus the collection $\mathcal{D}$ meets the criterion of Lemma 13.2, so $\mathcal{D}$ is a basis for $X \times Y$.   ■

EXAMPLE 1.   We have a standard topology on $\mathbb{R}$: the order topology The product of this topology with itself is called the *standard topology* on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of $\mathbb{R}$, but the theorem just proved tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$ Each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$. Thus the standard topology on $\mathbb{R}^2$ is just the one we considered in Example 2 of §13

It is sometimes useful to express the product topology in terms of a subbasis. To do this, we first define certain functions called projections.

Definition. Let $\pi_1 . X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

We use the word "onto" because π_1 and π_2 are surjective (unless one of the spaces X or Y happens to be empty, in which case $X \times Y$ is empty and our whole discussion is empty as well!).

If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then

$$\pi_2^{-1}(V) = X \times V,$$

which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$, as indicated in Figure 15.2. This fact leads to the following theorem:

Theorem 15.2.   *The collection*

$$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

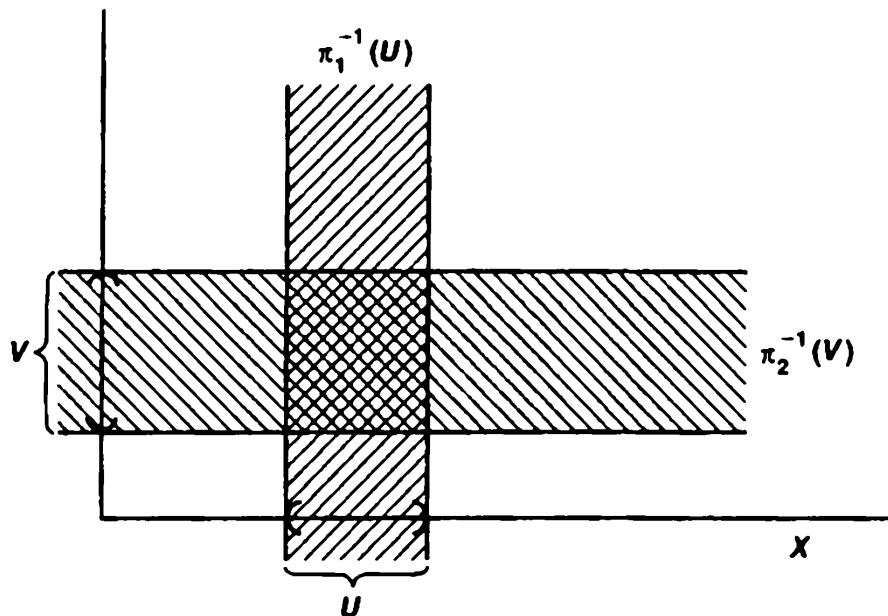*is a subbasis for the product topology on $X \times Y$.*

Figure 15.2

Proof.   Let $\mathcal{T}$ denote the product topology on $X \times Y$, let $\mathcal{T}'$ be the topology generated by S. Because every element of S belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of S. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of S, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Therefore, $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well   ∎

§16   The Subspace Topology

Definition.   Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

It is easy to see that $\mathcal{T}_Y$ is a topology. It contains $\varnothing$ and Y because

$$\varnothing = Y \cap \varnothing \quad \text{and} \quad Y = Y \cap X,$$

where $\varnothing$ and X are elements of $\mathcal{T}$. The fact that it is closed under finite intersections and arbitrary unions follows from the equations

$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y,$$

$$\bigcup_{\alpha \in J}(U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y.$$

Lemma 16.1. *If $\mathcal{B}$ is a basis for the topology of X then the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Given U open in X and given $y \in U \cap Y$, we can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 13.2 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.   ∎

When dealing with a space X and a subspace Y, one needs to be careful when one uses the term "open set". Does one mean an element of the topology of Y or an element of the topology of X? We make the following definition : If Y is a subspace of X, we say that a set U is *open in* Y (or open *relative to* Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is *open in X* if it belongs to the topology of X

There is a special situation in which every set open in Y is also open in X.

Lemma 16.2. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$ ∎

Now let us explore the relation between the subspace topology and the order and product topologies For product topologies, the result is what one might expect; for order topologies, it is not.

Theorem 16.3. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Therefore, $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

The conclusion we draw is that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same. ∎

Now let X be an ordered set in the order topology, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However, *the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X*. We give one example where the subspace and order topologies on Y agree, and two examples where they do not.

EXAMPLE 1   Consider the subset $Y = [0, 1]$ of the real line $\mathbb{R}$, in the *subspace* topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in $\mathbb{R}$. Such a set is of one of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & \text{if } a \text{ and } b \text{ are in } Y, \\ [0, b) & \text{if only } b \text{ is in } Y, \\ (a, 1] & \text{if only } a \text{ is in } Y, \\ Y \text{ or } \varnothing & \text{if neither } a \text{ nor } b \text{ is in } Y. \end{cases}$$

By definition, each of these sets is open in Y. But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note that these sets form a basis for the *order* topology on Y. Thus, we see that in the case of the set $Y = [0, 1]$, its subspace topology (as a subspace of $\mathbb{R}$) and its order topology are the same.

EXAMPLE 2   Let Y be the subset $[0, 1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology on Y the one-point set $\{2\}$ is open, because it is the intersection of the open set $(\frac{3}{2}, \frac{5}{2})$ with Y. But in the order topology on Y, the set $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form

$$\{x \mid x \in Y \text{ and } a < x \le 2\}$$

for some $a \in Y$; such a set necessarily contains points of Y less than 2.

EXAMPLE 3   Let $I = [0, 1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$! For example, the set $\{1/2\} \times (1/2, 1]$ is open in $I \times I$ in the subspace topology, but not in the order topology, as you can check. See Figure 16.1.

The set $I \times I$ in the dictionary order topology will be called the *ordered square*, and denoted by I_o^2.

The anomaly illustrated in Examples 2 and 3 does not occur for intervals or rays in an ordered set X. This we now prove.

Given an ordered set X, let us say that a subset Y of X is *convex* in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X.

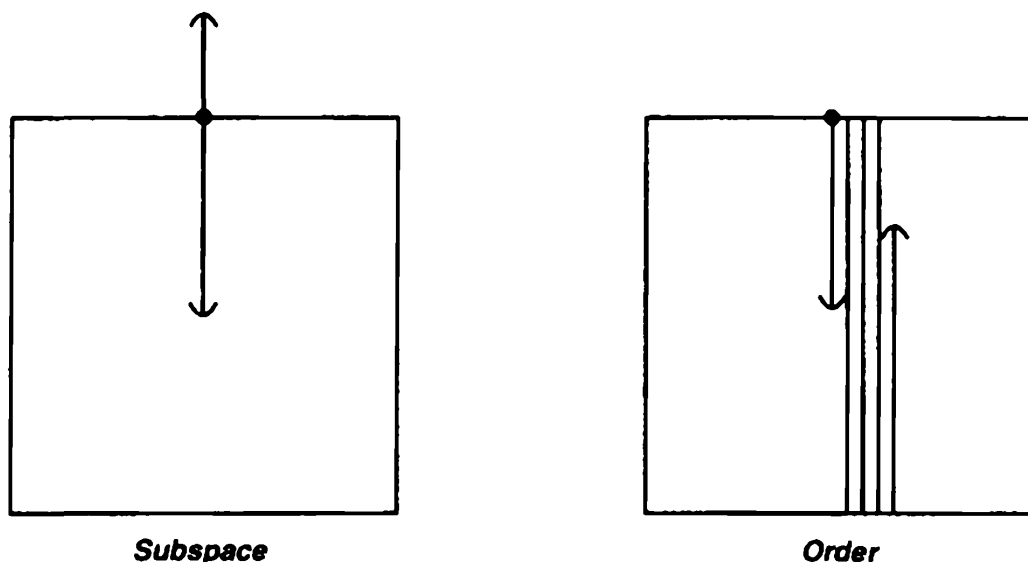*Subspace*   *Order*

Figure 16.1

Theorem 16.4. *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Consider the ray $(a, +\infty)$ in X. What is its intersection with Y? If $a \in Y$, then

$$(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\};$$

this is an open ray of the ordered set Y. If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex. In the former case, the set $(a, +\infty) \cap Y$ equals all of Y; in the latter case, it is empty.

A similar remark shows that the intersection of the ray $(-\infty, a)$ with Y is either an open ray of Y, or Y itself, or empty. Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology on Y, this topology is contained in the subspace topology. ∎

To avoid ambiguity, let us agree that whenever X is an ordered set in the order topology and Y is a subset of X, *we shall assume that Y is given the subspace topology* unless we specifically state otherwise. If Y is convex in X, this is the same as the order topology on Y; otherwise, it may not be.

Exercises

1. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A

inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \{x \mid \tfrac{1}{2} < |x| < 1\},$$

$$B = \{x \mid \tfrac{1}{2} < |x| \le 1\},$$

$$C = \{x \mid \tfrac{1}{2} \le |x| < 1\},$$

$$D = \{x \mid \tfrac{1}{2} \le |x| \le 1\},$$

$$E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.$$

4. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

5. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.
   (a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
   (b) Does the converse of (a) hold? Justify your answer.

6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

7. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

§17 Closed Sets and Limit Points

Now that we have a few examples at hand, we can introduce some of the basic concepts associated with topological spaces. In this section, we treat the notions of *closed set*,

closure of a set, and *limit point*. These lead naturally to consideration of a certain axiom for topological spaces called the *Hausdorff axiom*.

Closed Sets

A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

> EXAMPLE 1.   The subset $[a, b]$ of $\mathbb{R}$ is closed because its complement
>
> $$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty),$$
>
> is open. Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open. These facts justify our use of the terms "closed interval" and "closed ray" The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

> EXAMPLE 2.   In the plane $\mathbb{R}^2$, the set
>
> $$\{x \times y \mid x \geq 0 \text{ and } y \geq 0\}$$
>
> is closed, because its complement is the union of the two sets
>
> $$(-\infty, 0) \times \mathbb{R} \quad \text{and} \quad \mathbb{R} \times (-\infty, 0),$$
>
> each of which is a product of open sets of $\mathbb{R}$ and is, therefore, open in $\mathbb{R}^2$

> EXAMPLE 3   In the finite complement topology on a set X, the closed sets consist of X itself and all finite subsets of X

> EXAMPLE 4   In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

> EXAMPLE 5   Consider the following subset of the real line:
>
> $$Y = [0, 1] \cup (2, 3),$$
>
> in the subspace topology. In this space, the set $[0, 1]$ is open, since it is the intersection of the open set $(-\frac{1}{2}, \frac{3}{2})$ of $\mathbb{R}$ with Y Similarly, $(2, 3)$ is open as a subset of Y; it is even open as a subset of $\mathbb{R}$. Since $[0, 1]$ and $(2, 3)$ are complements in Y of each other, we conclude that both $[0, 1]$ and $(2, 3)$ are closed as subsets of Y

These examples suggest that an answer to the mathematician's riddle: "How is a set different from a door?" should be: "A door must be either open or closed, and cannot be both, while a set can be open, or closed, or both, or neither!"

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X:

Theorem 17.1. *Let X be a topological space. Then the following conditions hold:*

(1) *$\varnothing$ and X are closed.*

(2) *Arbitrary intersections of closed sets are closed.*

(3) *Finite unions of closed sets are closed.*

Proof. (1) $\varnothing$ and X are closed because they are the complements of the open sets X and $\varnothing$, respectively.

(2) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply DeMorgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha).$$

Since the sets $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, \ldots, n$, consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\bigcup A_i$ is closed. ∎

Instead of using open sets, one could just as well specify a topology on a space by giving a collection of sets (to be called "closed sets") satisfying the three properties of this theorem. One could then define open sets as the complements of closed sets and proceed just as before. This procedure has no particular advantage over the one we have adopted, and most mathematicians prefer to use open sets to define topologies.

Now when dealing with subspaces, one needs to be careful in using the term "closed set." If Y is a subspace of X, we say that a set A is *closed in* Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y - A$ is open in Y). We have the following theorem:

Theorem 17.2. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. (See Figure 17.1.) Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y, by definition of the subspace topology. But $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. (See Figure 17.2.) Then $Y - A$ is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X, and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y, as desired. ∎

A set A that is closed in the subspace Y may or may not be closed in the larger space X. As was the case with open sets, there is a criterion for A to be closed in X; we leave the proof to you:
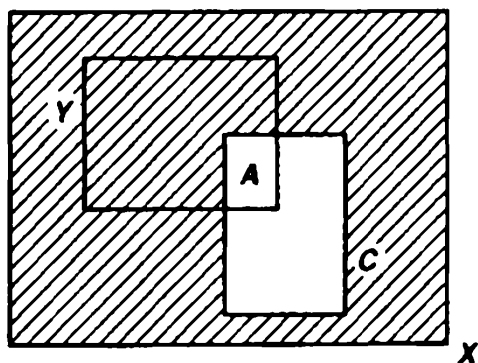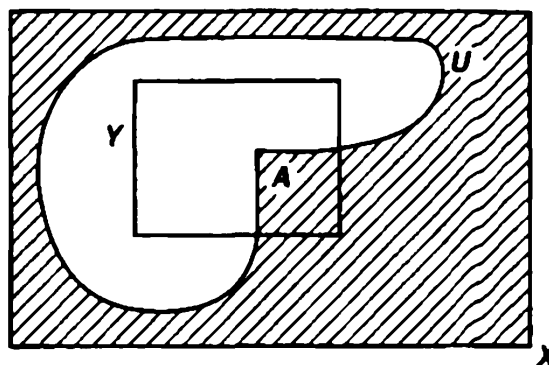
Figure 17.1

Figure 17.2

Theorem 17.3. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Closure and Interior of a Set

Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A and the closure of A is denoted by Cl A or by $\bar{A}$. Obviously Int A is an open set and $\bar{A}$ is a closed set; furthermore,

$$\text{Int } A \subset A \subset \bar{A}.$$

If A is open, $A = \text{Int } A$; while if A is closed, $A = \bar{A}$.

We shall not make much use of the interior of a set, but the closure of a set will be quite important.

When dealing with a topological space X and a subspace Y, one needs to exercise care in taking closures of sets If A is a subset of Y, the closure of A in Y and the closure of A in X will in general be different *In such a situation, we reserve the notation $\bar{A}$ to stand for the closure of A in X*. The closure of A in Y can be expressed in terms of $\bar{A}$, as the following theorem shows:

Theorem 17.4. *Let Y be a subspace of X, let A be a subset of Y, let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y by Theorem 17.2. Since $\bar{A} \cap Y$ contains A, and since by definition B equals the intersection of *all* closed subsets of Y containing A, we must have $B \subset (\bar{A} \cap Y)$.

On the other hand, we know that B is closed in Y. Hence by Theorem 17.2, $B = C \cap Y$ for some set C closed in X. Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of *all* such closed sets, we conclude that $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. ∎

The definition of the closure of a set does not give us a convenient way for actually finding the closures of specific sets, since the collection of all closed sets in X, like the collection of all open sets, is usually much too big to work with. Another way of describing the closure of a set, useful because it involves only a basis for the topology of X, is given in the following theorem.

First let us introduce some convenient terminology. We shall say that a set A *intersects* a set B if the intersection $A \cap B$ is not empty.

Theorem 17.5.   *Let A be a subset of the topological space X.*

(a) *Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.*

(b) *Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.*

Proof. Consider the statement in (a). It is a statement of the form $P \Leftrightarrow Q$. Let us transform each implication to its contrapositive, thereby obtaining the logically equivalent statement (not P) $\Leftrightarrow$ (not Q). Written out, it is the following:

$$x \notin \bar{A} \iff \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A.$$

In this form, our theorem is easy to prove. If x is not in $\bar{A}$, the set $U = X - \bar{A}$ is an open set containing x that does not intersect A, as desired   Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A By definition of the closure $\bar{A}$, the set $X - U$ must contain $\bar{A}$, therefore, x cannot be in $\bar{A}$.

Statement (b) follows readily   If every open set containing x intersects A, so does every basis element B containing x, because B is an open set. Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.   ∎

Mathematicians often use some special terminology here. They shorten the statement "U is an open set containing x" to the phrase

"U is a *neighborhood* of x."

Using this terminology, one can write the first half of the preceding theorem as follows:

If A is a subset of the topological space X, then $x \in \bar{A}$ if and only if every neighborhood of x intersects A.

EXAMPLE 6   Let X be the real line $\mathbb{R}$. If $A = (0, 1]$, then $\bar{A} = [0, 1]$, for every neighborhood of 0 intersects A, while every point outside $[0, 1]$ has a neighborhood disjoint from A Similar arguments apply to the following subsets of X

If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then $\bar{B} = \{0\} \cup B$ If $C = \{0\} \cup (1, 2)$, then $\bar{C} = \{0\} \cup [1, 2]$ If $\mathbb{Q}$ is the set of rational numbers, then $\bar{\mathbb{Q}} = \mathbb{R}$ If $\mathbb{Z}_+$ is the set of positive integers, then $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+$. If $\mathbb{R}_+$ is the set of positive reals, then the closure of $\mathbb{R}_+$ is the set $\mathbb{R}_+ \cup \{0\}$. (This is the reason we introduced the notation $\bar{\mathbb{R}}_+$ for the set $\mathbb{R}_+ \cup \{0\}$, back in §2)

EXAMPLE 7   Consider the subspace $Y = (0, 1]$ of the real line $\mathbb{R}$. The set $A = (0, \frac{1}{2})$ is a subset of Y, its closure in $\mathbb{R}$ is the set $[0, \frac{1}{2}]$, and its closure in Y is the set $[0, \frac{1}{2}] \cap Y = (0, \frac{1}{2}]$

Some mathematicians use the term "neighborhood" differently. They say that A is a neighborhood of x if A merely *contains* an open set containing x. We shall not follow this practice.

Limit Points

There is yet another way of describing the closure of a set, a way that involves the important concept of limit point, which we consider now.

If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point* (or "cluster point," or "point of accumulation") of A if every neighborhood of x intersects A in some point *other than x itself*. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$ The point x may lie in A or not; for this definition it does not matter.

EXAMPLE 8   Consider the real line $\mathbb{R}$. If $A = (0, 1]$, then the point 0 is a limit point of A and so is the point $\frac{1}{2}$ In fact, every point of the interval $[0, 1]$ is a limit point of A, but no other point of $\mathbb{R}$ is a limit point of A

If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B. Every other point x of $\mathbb{R}$ has a neighborhood that either does not intersect B at all, or it intersects B only in the point x itself. If $C = \{0\} \cup (1, 2)$, then the limit points of C are the points of the interval $[1, 2]$. If $\mathbb{Q}$ is the set of rational numbers, every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$. If $\mathbb{Z}_+$ is the set of positive integers, no point of $\mathbb{R}$ is a limit point of $\mathbb{Z}_+$ If $\mathbb{R}_+$ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}_+$ is a limit point of $\mathbb{R}_+$

Comparison of Examples 6 and 8 suggests a relationship between the closure of a set and the limit points of a set. That relationship is given in the following theorem:

Theorem 17.6.   *Let A be a subset of the topological space X, let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'.$$

Proof.   If x is in A', every neighborhood of x intersects A (in a point different from x). Therefore, by Theorem 17.5, x belongs to $\bar{A}$ Hence $A' \subset \bar{A}$. Since by definition $A \subset \bar{A}$, it follows that $A \cup A' \subset \bar{A}$.

To demonstrate the reverse inclusion, we let x be a point of $\bar{A}$ and show that $x \in A \cup A'$. If x happens to lie in A, it is trivial that $x \in A \cup A'$; suppose that x does not lie in A. Since $x \in \bar{A}$, we know that every neighborhood U of x intersects A; because $x \notin A$, the set U must intersect A in a point different from x. Then $x \in A'$, so that $x \in A \cup A'$, as desired.   ∎

Corollary 17.7. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed if and only if $A = \bar{A}$, and the latter holds if and only if $A' \subset A$. ∎

Hausdorff Spaces

One's experience with open and closed sets and limit points in the real line and the plane can be misleading when one considers more general topological spaces. For example, in the spaces $\mathbb{R}$ and $\mathbb{R}^2$, each one-point set $\{x_0\}$ is closed. This fact is easily proved; every point different from x_0 has a neighborhood not intersecting $\{x_0\}$, so that $\{x_0\}$ is its own closure. But this fact is not true for arbitrary topological spaces. Consider the topology on the three-point set $\{a, b, c\}$ indicated in Figure 17.3. In this space, the one-point set $\{b\}$ is not closed, for its complement is *not* open.
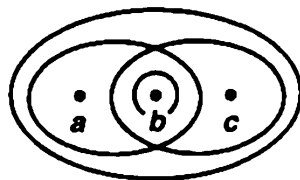

Figure 17.3

Similarly, one's experience with the properties of convergent sequences in $\mathbb{R}$ and $\mathbb{R}^2$ can be misleading when one deals with more general topological spaces. In an arbitrary topological space, one says that a sequence $x_1, x_2, \ldots$ of points of the space X *converges* to the point x of X provided that, corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$. In $\mathbb{R}$ and $\mathbb{R}^2$, a sequence cannot converge to more than one point, but in an arbitrary space, it can. In the space indicated in Figure 17.3, for example, the sequence defined by setting $x_n = b$ for all n converges not only to the point b, but also to the point a and to the point c!

Topologies in which one-point sets are not closed, or in which sequences can converge to more than one point, are considered by many mathematicians to be somewhat strange. They are not really very interesting, for they seldom occur in other branches of mathematics And the theorems that one can prove about topological spaces are rather limited if such examples are allowed. Therefore, one often imposes an additional condition that will rule out examples like this one, bringing the class of spaces under consideration closer to those to which one's geometric intuition applies. The condition was suggested by the mathematician Felix Hausdorff, so mathematicians have come to call it by his name.

Definition. A topological space X is called a *Hausdorff space* if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 17.8. *Every finite point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed.   ∎

The condition that finite point sets be closed is in fact weaker than the Hausdorff condition For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed The condition that finite point sets be closed has been given a name of its own: it is called the T_1 *axiom*. (We shall explain the reason for this strange terminology in Chapter 4.) The T_1 axiom will appear in this book in a few exercises, and in just one theorem, which is the following:

Theorem 17.9. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A

Conversely, suppose that x is a limit point of A, and suppose some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A - \{x\}$ in finitely many points; let $\{x_1, . \quad , x_m\}$ be the points of $U \cap (A - \{x\})$. The set $X - \{x_1, \ldots, x_m\}$ is an open set of X, since the finite point set $\{x_1, \ldots, x_m\}$ is closed; then

$$U \cap (X - \{x_1, \quad .., x_m\})$$

is a neighborhood of x that intersects the set $A - \{x\}$ not at all. This contradicts the assumption that x is a limit point of A.   ∎

One reason for our lack of interest in the T_1 axiom is the fact that many of the interesting theorems of topology require not just that axiom, but the full strength of the Hausdorff axiom. Furthermore, most of the spaces that are important to mathematicians are Hausdorff spaces. The following two theorems give some substance to these remarks.

Theorem 17.10. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X*

Proof. Suppose that x_n is a sequence of points of X that converges to x If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot Therefore, x_n cannot converge to y.   ∎

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$, and we say that x is the *limit* of the sequence x_n.

The proof of the following result is left to the exercises.

Theorem 17.11. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

The Hausdorff condition is generally considered to be a very mild extra condition to impose on a topological space. Indeed, in a first course in topology some mathematicians go so far as to impose this condition at the outset, refusing to consider spaces that are not Hausdorff spaces. We shall not go this far, but we shall certainly assume the Hausdorff condition whenever it is needed in a proof without having any qualms about limiting seriously the range of applications of the results.

The Hausdorff condition is one of a number of extra conditions one can impose on a topological space. Each time one imposes such a condition, one can prove stronger theorems, but one limits the class of spaces to which the theorems apply. Much of the research that has been done in topology since its beginnings has centered on the problem of finding conditions that will be strong enough to enable one to prove interesting theorems about spaces satisfying those conditions, and yet not so strong that they limit severely the range of applications of the results.

We shall study a number of such conditions in the next two chapters. The Hausdorff condition and the T_1 axiom are but two of a collection of conditions similar to one another that are called collectively the *separation axioms*. Other conditions include the *countability axioms*, and various *compactness* and *connectedness* conditions. Some of these are quite stringent requirements, as you will see.

Exercises

1. Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\varnothing$ and X are in $\mathcal{C}$, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology on X.

2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

4. Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

5. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subset [a, b]$. Under what conditions does equality hold?

6. Let A, B, and A_α denote subsets of a space X. Prove the following:
   (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   (c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

7. Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

8. Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.
   (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
   (b) $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$.
   (c) $\overline{A - B} = \bar{A} - \bar{B}$.

9. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

10. Show that every order topology is Hausdorff.

11. Show that the product of two Hausdorff spaces is Hausdorff.

12. Show that a subspace of a Hausdorff space is Hausdorff.

13. Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

14. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

15. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

16. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.
    (a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.
    (b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\},$$
$$B = \{(1 - 1/n) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\},$$
$$C = \{x \times 0 \mid 0 < x < 1\},$$
$$D = \{x \times \tfrac{1}{2} \mid 0 < x < 1\},$$
$$E = \{\tfrac{1}{2} \times y \mid 0 < y < 1\}.$$

19. If $A \subset X$, we define the **boundary** of A by the equation

$$\text{Bd } A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that Int A and Bd A are disjoint, and $\bar{A} = \text{Int } A \cup \text{Bd } A$.

(b) Show that Bd $A = \varnothing \Leftrightarrow A$ is both open and closed.

(c) Show that U is open $\Leftrightarrow$ Bd $U = \bar{U} - U$.

(d) If U is open, is it true that $U = \text{Int}(\bar{U})$? Justify your answer.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$.

(a) $A = \{x \times y \mid y = 0\}$

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y \mid x \text{ is rational}\}$

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

*21. (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained

§18 Continuous Functions

The concept of continuous function is basic to much of mathematics. Continuous functions on the real line appear in the first pages of any calculus book, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes further in mathematics. In this section, we shall formulate a definition of continuity that will include all these as special cases, and we shall study various properties of continuous functions. Many of these properties are direct generalizations of things you learned about continuous functions in calculus and analysis.

Continuity of a Function

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Recall that $f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.

Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range. If we wish to emphasize this fact, we can say that f is continuous *relative* to specific topologies on X and Y.

Let us note that if the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is open. The arbitrary open set V of Y can be written as a union of basis elements

$$V = \bigcup_{\alpha \in J} B_\alpha.$$

Then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha),$$

so that $f^{-1}(V)$ is open if each set $f^{-1}(B_\alpha)$ is open.

If the topology on Y is given by a subbasis S, to prove continuity of f it will even suffice to show that the inverse image of each *subbasis* element is open. The arbitrary basis element B for Y can be written as a finite intersection $S_1 \cap \cdots \cap S_n$ of subbasis elements; it follows from the equation

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

that the inverse image of every basis element is open.

EXAMPLE 1   Let us consider a function like those studied in analysis, a "real-valued function of a real variable,"

$$f \quad \mathbb{R} \longrightarrow \mathbb{R}.$$

In analysis, one defines continuity of f via the "ϵ-δ definition," a bugaboo over the years for every student of mathematics. As one would expect, the ϵ-δ definition and ours are equivalent To prove that our definition implies the ϵ-δ definition, for instance, we proceed as follows:

Given x_0 in $\mathbb{R}$, and given $\epsilon > 0$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$ Therefore, $f^{-1}(V)$ is an open set in the domain space $\mathbb{R}$. Because $f^{-1}(V)$ contains the point x_0, it contains some basis element (a, b) about x_0 We choose δ to be the smaller of the two numbers $x_0 - a$ and $b - x_0$ Then if $|x - x_0| < \delta$, the point x must be in (a, b), so that $f(x) \in V$, and $|f(x) - f(x_0)| < \epsilon$, as desired.

Proving that the ϵ-δ definition implies our definition is no harder; we leave it to you. We shall return to this example when we study metric spaces

EXAMPLE 2.   In calculus one considers the property of continuity for many kinds of functions. For example, one studies functions of the following types:

$f . \mathbb{R} \longrightarrow \mathbb{R}^2$ (curves in the plane)

$f . \mathbb{R} \longrightarrow \mathbb{R}^3$ (curves in space)

$f \quad \mathbb{R}^2 \longrightarrow \mathbb{R}$ (functions $f(x, y)$ of two real variables)

$f . \mathbb{R}^3 \longrightarrow \mathbb{R}$ (functions $f(x, y, z)$ of three real variables)

$f . \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (vector fields $\mathbf{v}(x, y)$ in the plane).

Each of them has a notion of continuity defined for it. Our general definition of continuity includes all these as special cases; this fact will be a consequence of general theorems we shall prove concerning continuous functions on product spaces and on metric spaces.

EXAMPLE 3   Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_\ell$ denote the same set in the lower limit topology. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}_\ell$$

be the identity function; $f(x) = x$ for every real number x. Then f is not a continuous function; the inverse image of the open set $[a, b)$ of $\mathbb{R}_\ell$ equals itself, which is not open in $\mathbb{R}$. On the other hand, the identity function

$$g : \mathbb{R}_\ell \longrightarrow \mathbb{R}$$

is continuous, because the inverse image of (a, b) is itself, which is open in $\mathbb{R}_\ell$.

In analysis, one studies several different but equivalent ways of formulating the definition of continuity. Some of these generalize to arbitrary spaces, and they are considered in the theorems that follow. The familiar "ϵ-δ" definition and the "convergent sequence definition" do not generalize to arbitrary spaces; they will be treated when we study metric spaces.

Theorem 18.1.   *Let X and Y be topological spaces; let $f : X \rightarrow Y$. Then the following are equivalent:*

(1) *f is continuous.*

(2) *For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.*

(3) *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X*

(4) *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

If the condition in (4) holds for the point x of X, we say that f is *continuous at the point x.*

Proof. We show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and that $(1) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Assume that f is continuous. Let A be a subset of X. We show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$, as desired.

$(2) \Rightarrow (3)$. Let B be closed in Y and let $A = f^{-1}(B)$. We wish to prove that A is closed in X; we show that $\bar{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \bar{A}$,

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B,$$

so that $x \in f^{-1}(B) = A$. Thus $\bar{A} \subset A$, so that $\bar{A} = A$, as desired.

$(3) \Rightarrow (1)$. Let V be an open set of Y. Set $B = Y - V$. Then

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Now B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.

(1) ⟹ (4). Let $x \in X$ and let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

(4) ⟹ (1). Let V be an open set of Y; let x be a point of $f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open.   ∎

Homeomorphisms

Let X and Y be topological spaces; let $f . X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a *homeomorphism*.

The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \to X$ is open in Y. But the *inverse image* of U under the map f^{-1} is the same as the *image* of U under the map f. See Figure 18.1. So another way to define a homeomorphism is to say that it is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open.

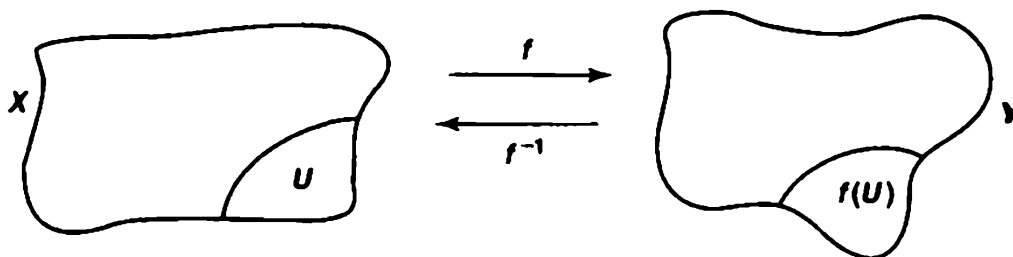*Figure 18.1*

This remark shows that a homeomorphism $f : X \to Y$ gives us a bijective correspondence not only between X and Y but between the collections of open sets of X and of Y. As a result, any property of X that is entirely expressed in terms of the topology of X (that is, in terms of the open sets of X) yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a *topological property* of X.

You may have studied in modern algebra the notion of an *isomorphism* between algebraic objects such as groups or rings. An isomorphism is a bijective correspondence that preserves the algebraic structure involved. The analogous concept in topology is that of *homeomorphism*; it is a bijective correspondence that preserves the topological structure involved.

Now suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a *topological imbedding*, or simply an *imbedding*, of X in Y.

EXAMPLE 4. The function $f . \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism See Figure 18 2. If we define $g \cdot \mathbb{R} \to \mathbb{R}$ by the equation

$$g(y) = \frac{1}{3}(y - 1)$$

then one can check easily that $f(g(y)) = y$ and $g(f(x)) = x$ for all real numbers x and y It follows that f is bijective and that $g = f^{-1}$, the continuity of f and g is a familiar result from calculus.

EXAMPLE 5. The function $F . (-1, 1) \to \mathbb{R}$ defined by

$$F(x) = \frac{x}{1 - x^2}$$

is a homeomorphism See Figure 18.3 We have already noted in Example 9 of §3 that F is a bijective order-preserving correspondence; its inverse is the function G defined by

$$G(y) = \frac{2y}{1 + (1 + 4y^2)^{1/2}}.$$

The fact that F is a homeomorphism can be proved in two ways One way is to note that because F is order preserving and bijective, F carries a basis element for the order topology in $(-1, 1)$ onto a basis element for the order topology in $\mathbb{R}$ and vice versa As a result, F is automatically a homeomorphism of $(-1, 1)$ with $\mathbb{R}$ (both in the order topology) Since the order topology on $(-1, 1)$ and the usual (subspace) topology agree, F is a homeomorphism of $(-1, 1)$ with $\mathbb{R}$

$f(x) = 3x + 1$

$F(x) = \dfrac{x}{1 - x^2}$
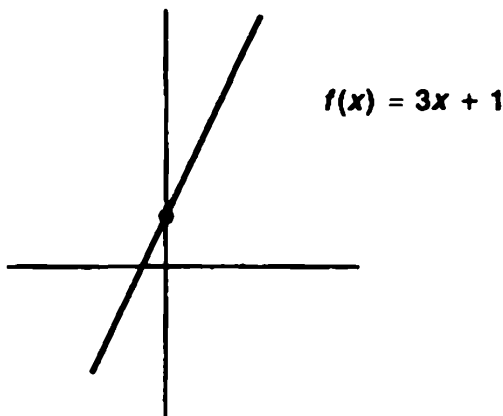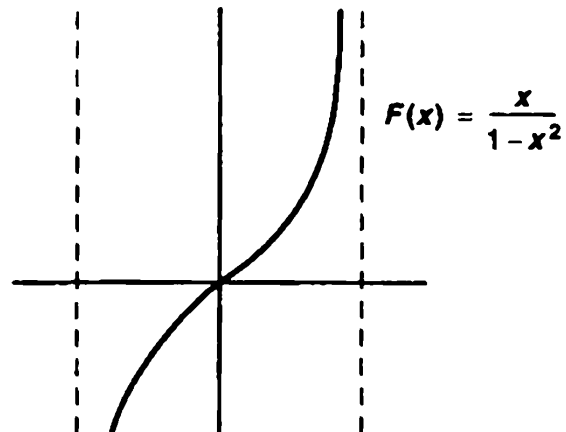
Figure 18.2   *Figure 18.3*

A second way to show F a homeomorphism is to use the continuity of the algebraic functions and the square-root function to show that both F and G are continuous These are familiar facts from calculus

EXAMPLE 6 A bijective function $f . X \to Y$ can be continuous without being a homeomorphism One such function is the identity map $g \ \mathbb{R}_\ell \to \mathbb{R}$ considered in Example 3 Another is the following Let S^1 denote the *unit circle*,

$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$

considered as a subspace of the plane $\mathbb{R}^2$, and let

$$F \cdot [0, 1) \longrightarrow S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. The fact that f is bijective and continuous follows from familiar properties of the trigonometric functions. But the function f^{-1} is not continuous The image under f of the open set $U = [0, \frac{1}{4})$ of the domain, for instance, is not open in S^1, for the point $p = f(0)$ lies in no open set V of $\mathbb{R}^2$ such that $V \cap S^1 \subset f(U)$. See Figure 18.4.
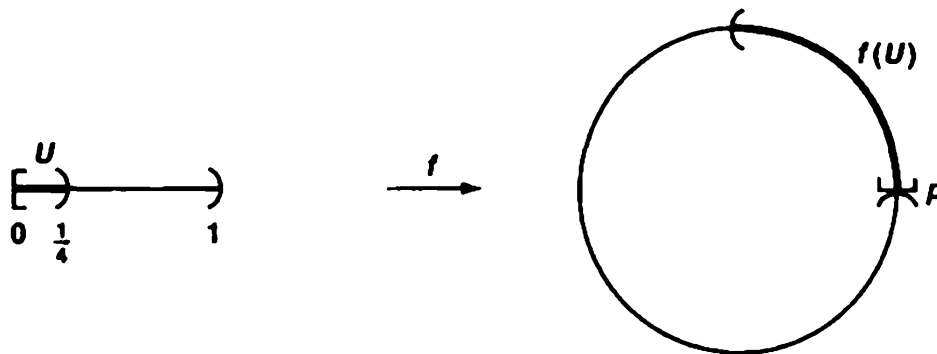
Figure 18.4

EXAMPLE 7.   Consider the function

$$g : [0, 1) \longrightarrow \mathbb{R}^2$$

obtained from the function f of the preceding example by expanding the range. The map g is an example of a continuous injective map that is not an imbedding

Constructing Continuous Functions

How does one go about constructing continuous functions from one topological space to another? There are a number of methods used in analysis, of which some generalize to arbitrary topological spaces and others do not. We study first some constructions that do hold for general topological spaces, deferring consideration of the others until later.

Theorem 18.2 (Rules for constructing continuous functions). *Let X, Y, and Z be topological spaces.*

(a) *(Constant function) If $f . X \rightarrow Y$ maps all of X into the single point y_0 of Y, then f is continuous.*

(b) *(Inclusion) If A is a subspace of X, the inclusion function $j : A \rightarrow X$ is continuous.*

(c) *(Composites) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.*

(d) *(Restricting the domain)* If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|A \cdot A \to Y$ is continuous.

(e) *(Restricting or expanding the range)* Let $f \cdot X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) *(Local formulation of continuity)* The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.

Proof. (a) Let $f(x) = y_0$ for every x in X. Let V be open in Y. The set $f^{-1}(V)$ equals X or $\varnothing$, depending on whether V contains y_0 or not. In either case, it is open.

(b) If U is open in X, then $j^{-1}(U) = U \cap A$, which is open in A by definition of the subspace topology.

(c) If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U),$$

by elementary set theory.

(d) The function $f|A$ equals the composite of the inclusion map $j : A \to X$ and the map $f : X \to Y$, both of which are continuous.

(e) Let $f : X \to Y$ be continuous. If $f(X) \subset Z \subset Y$, we show that the function $g : X \to Z$ obtained from f is continuous. Let B be open in Z. Then $B = Z \cap U$ for some open set U of Y. Because Z contains the entire image set $f(X)$,

$$f^{-1}(U) = g^{-1}(B),$$

by elementary set theory. Since $f^{-1}(U)$ is open, so is $g^{-1}(B)$.

To show $h : X \to Z$ is continuous if Z has Y as a subspace, note that h is the composite of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$.

(f) By hypothesis, we can write X as a union of open sets U_α, such that $f|U_\alpha$ is continuous for each α. Let V be an open set in Y. Then

$$f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V),$$

because both expressions represent the set of those points x lying in U_α for which $f(x) \in V$. Since $f|U_\alpha$ is continuous, this set is open in U_α, and hence open in X But

$$f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha),$$

so that $f^{-1}(V)$ is also open in X. ∎

Theorem 18.3 (The pasting lemma). *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.*

Proof.   Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C),$$

by elementary set theory. Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore, closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X.   ■

This theorem also holds if A and B are open sets in X; this is just a special case of the "local formulation of continuity" rule given in preceding theorem.

EXAMPLE 8   Let us define a function $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(x) = \begin{cases} x & \text{for } x \le 0, \\ x/2 & \text{for } x \ge 0 \end{cases}$$

Each of the "pieces" of this definition is a continuous function, and they agree on the overlapping part of their domains, which is the one-point set $\{0\}$. Since their domains are closed in $\mathbb{R}$, the function h is continuous. One needs the "pieces" of the function to agree on the overlapping part of their domains in order to have a function at all. The equations

$$k(x) = \begin{cases} x - 2 & \text{for } x \le 0, \\ x + 2 & \text{for } x \ge 0, \end{cases}$$

for instance, do not define a function   On the other hand, one needs some limitations on the sets A and B to guarantee continuity. The equations

$$l(x) = \begin{cases} x - 2 & \text{for } x < 0, \\ x + 2 & \text{for } x \ge 0, \end{cases}$$

for instance, do define a function l mapping $\mathbb{R}$ into $\mathbb{R}$, and both of the pieces are continuous. But l is not continuous; the inverse image of the open set $(1, 3)$, for instance, is the nonopen set $[0, 1)$ See Figure 18.5
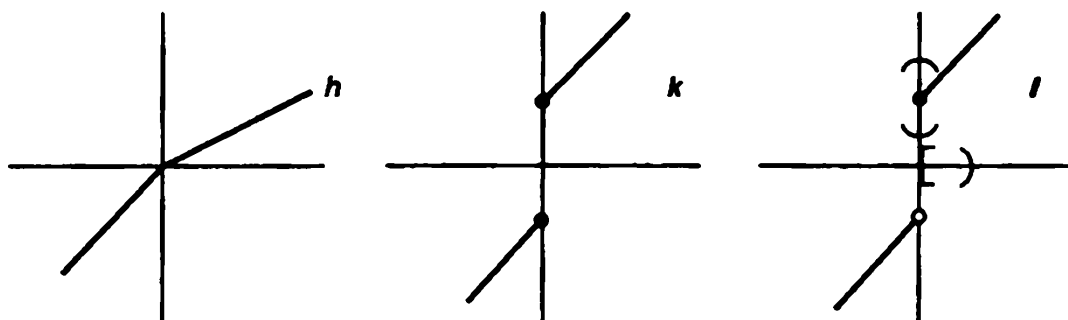

Figure 18.5

Theorem 18.4 (Maps into products). *Let* $f : A \to X \times Y$ *be given by the equation*

$$f(a) = (f_1(a), f_2(a)).$$

Then f *is continuous if and only if the functions*

$$f_1 . A \longrightarrow X \quad \text{and} \quad f_2 : A \longrightarrow Y$$

are continuous.

The maps f_1 and f_2 are called the **coordinate functions** of f.

Proof. Let $\pi_1 . X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto the first and second factors, respectively. These maps are continuous. For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, and these sets are open if U and V are open. Note that for each $a \in A$,

$$f_1(a) = \pi_1(f(a)) \quad \text{and} \quad f_2(a) = \pi_2(f(a)).$$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions and therefore continuous. Conversely, suppose that f_1 and f_2 are continuous. We show that for each basis element $U \times V$ for the topology of $X \times Y$, its inverse image $f^{-1}(U \times V)$ is open. A point a is in $f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, that is, if and only if $f_1(a) \in U$ and $f_2(a) \in V$. Therefore,

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection.   ∎

There is no useful criterion for the continuity of a map $f : A \times B \to X$ whose *domain* is a product space. One might conjecture that f is continuous if it is continuous "in each variable separately," but this conjecture is not true. (See Exercise 12.)

EXAMPLE 9   In calculus, a *parametrized curve* in the plane is defined to be a continuous map $f \quad [a, b] \to \mathbb{R}^2$ It is often expressed in the form $f(t) = (x(t), y(t))$; and one frequently uses the fact that f is a continuous function of t if both x and y are Similarly, a *vector field* in the plane

$$v(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$
$$= (P(x, y), Q(x, y))$$

is said to be continuous if both P and Q are continuous functions, or equivalently, if v is continuous as a map of $\mathbb{R}^2$ into $\mathbb{R}^2$. Both of these statements are simply special cases of the preceding theorem.

One way of forming continuous functions that is used a great deal in analysis is to take sums, differences, products, or quotients of continuous real-valued functions. It is a standard theorem that if $f, g . X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous, and f/g is continuous if $g(x) \neq 0$ for all x. We shall consider this theorem in §21.

Yet another method for constructing continuous functions that is familiar from analysis is to take the limit of an infinite sequence of functions. There is a theorem to the effect that if a sequence of continuous real-valued functions of a real variable converges uniformly to a limit function, then the limit function is necessarily continuous. This theorem is called the *Uniform Limit Theorem*. It is used, for instance, to demonstrate the continuity of the trigonometric functions, when one defines these functions rigorously using the infinite series definitions of the sine and cosine. This theorem generalizes to a theorem about maps of an arbitrary topological space X into a metric space Y. We shall prove it in §21.

Exercises

1. Prove that for functions $f . \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
   (a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
   (b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

7. (a) Suppose that $f . \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

   (b) Can you conjecture what functions $f \cdot \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
   (a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – CONNECTED SPACES – SMTA5202

UNIT-II THE METRIC TOPOLOGY
AND CONNECTED SPACES

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in Us way lie at the heart of modern analysis, for example. 1n this section, we shall define the metric topology and shall give a number of examples. In next section, we shall consider some of the properties that metric topologies satisfy.

Definition. A *metric* on a set d is a function

$$d: \text{X x X} \to \text{R}^+$$

having the following properties:
  (1) $d(x, y) \geq 0$ for all *x, y in X* and the equality holds if and only if x = y.
  (2) $d(x, y) = d(y, x)$ for all *x,* y in X.
  (3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all *x, y, z in* X.

§20 The Metric Topology

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, we shall define the metric topology and shall give a number of examples. In the next section, we shall consider some of the properties that metric topologies satisfy.

Definition. A *metric* on a set X is a function

$$d : X \times X \longrightarrow R$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the *distance* between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-*ball centered at* x. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Definition. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the *metric topology* induced by d.

The first condition for a basis is trivial, since $x \in B(x, \epsilon)$ for any $\epsilon > 0$. Before checking the second condition for a basis, we show that if y is a point of the basis element $B(x, \epsilon)$, then there is a basis element $B(y, \delta)$ *centered* at y that is contained in $B(x, \epsilon)$. Define δ to be the positive number $\epsilon - d(x, y)$. Then $B(y, \delta) \subset B(x, \epsilon)$, for if $z \in B(y, \delta)$, then $d(y, z) < \epsilon - d(x, y)$, from which we conclude that

$$d(x, z) \leq d(x, y) + d(y, z) < \epsilon.$$

See Figure 20.1.

Now to check the second condition for a basis, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Letting δ be the smaller of δ_1 and δ_2, we conclude that $B(y, \delta) \subset B_1 \cap B_2$.

Using what we have just proved, we can rephrase the definition of the metric topology as follows:
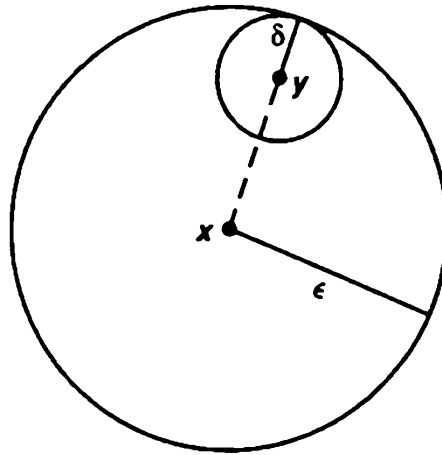
Figure 20.1

*A set U is open in the metric topology induced by d if and only if for each
$y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.*

Clearly this condition implies that U is open. Conversely, if U is open, it contains
a basis element $B = B_d(x, \epsilon)$ containing y, and B in turn contains a basis element
$B_d(y, \delta)$ centered at y

EXAMPLE 1   Given a set X, define

$$d(x, y) = 1 \quad \text{if } x \neq y,$$
$$d(x, y) = 0 \quad \text{if } x = y$$

It is trivial to check that d is a metric. The topology it induces is the discrete topology; the
basis element $B(x, 1)$, for example, consists of the point x alone.

EXAMPLE 2.   The standard metric on the real numbers $\mathbb{R}$ is defined by the equation

$$d(x, y) = |x - y|$$

It is easy to check that d is a metric. The topology it induces is the same as the order
topology: Each basis element (a, b) for the order topology is a basis element for the metric
topology, indeed,

$$(a, b) = B(x, \epsilon),$$

where $x = (a + b)/2$ and $\epsilon = (b - a)/2$. And conversely, each ϵ-ball $B(x, \epsilon)$ equals an
open interval the interval $(x - \epsilon, x + \epsilon)$.

Definition.   If X is a topological space, X is said to be *metrizable* if there exists a
metric d on the set X that induces the topology of X. A *metric space* is a metrizable
space X together with a specific metric d that gives the topology of X.

Many of the spaces important for mathematics are metrizable, but some are not.
Metrizability is always a highly desirable attribute for a space to possess, for the exis-
tence of a metric gives one a valuable tool for proving theorems about the space.

It is, therefore, a problem of fundamental importance in topology to find conditions on a topological space that will guarantee it is metrizable. One of our goals in Chapter 4 will be to find such conditions; they are expressed there in the famous theorem called *Urysohn's metrization theorem*. Further metrization theorems appear in Chapter 6. In the present section we shall content ourselves with proving merely that $\mathbb{R}^n$ and $\mathbb{R}^\omega$ are metrizable.

Although the metrizability problem is an important problem in topology, the study of metric spaces as such does not properly belong to topology as much as it does to analysis Metrizability of a space depends only on the topology of the space in question, but properties that involve a specific metric for X in general do not. For instance, one can make the following definition in a metric space.

Definition. Let X be a metric space with metric d. A subset A of X is said to be *bounded* if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the *diameter* of A is defined to be the number

$$\text{diam } A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Boundedness of a set is not a topological property, for it depends on the particular metric d that is used for X. For instance, if X is a metric space with metric d, then there exists a metric $\bar{d}$ that gives the topology of X, relative to which *every* subset of X is bounded. It is defined as follows:

Theorem 20.1. *Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation*

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the *standard bounded metric* corresponding to d.

Proof. Checking the first two conditions for a metric is trivial. Let us check the triangle inequality:

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

Now if either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then the right side of this inequality is at least 1, since the left side is (by definition) at most 1, the inequality holds. It remains to consider the case in which $d(x, y) < 1$ and $d(y, z) < 1$. In this case, we have

$$d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z).$$

Since $\bar{d}(x, z) \leq d(x, z)$ by definition, the triangle inequality holds for $\bar{d}$.

Now we note that in any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for the metric topology, for every basis element containing x contains such an ϵ-ball centered at x. It follows that d and $\bar{d}$ induce the same topology on X, because the collections of ϵ-balls with $\epsilon < 1$ under these two metrics are the same collection. ∎

Now we consider some familiar spaces and show they are metrizable.

Definition. Given $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the *norm* of $\mathbf{x}$ by the equation

$$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2};$$

and we define the *euclidean metric* d on $\mathbb{R}^n$ by the equation

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

We define the *square metric* ρ by the equation

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The proof that d is a metric requires some work; it is probably already familiar to you. If not, a proof is outlined in the exercises. We shall seldom have occasion to use this metric on $\mathbb{R}^n$.

To show that ρ is a metric is easier. Only the triangle inequality is nontrivial. From the triangle inequality for $\mathbb{R}$ it follows that for each positive integer i,

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

Then by definition of ρ,

$$|x_i - z_i| \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}).$$

As a result

$$\rho(\mathbf{x}, \mathbf{z}) = \max\{|x_i - z_i|\} \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}),$$

as desired.

On the real line $\mathbb{R} = \mathbb{R}^1$, these two metrics coincide with the standard metric for $\mathbb{R}$. In the plane $\mathbb{R}^2$, the basis elements under d can be pictured as circular regions, while the basis elements under ρ can be pictured as square regions.

We now show that each of these metrics induces the usual topology on $\mathbb{R}^n$. We need the following lemma:

Lemma 20.2. *Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$$

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$ Given the basis element $B_d(x, \epsilon)$ for $\mathcal{T}$, there is by Lemma 13.3 a basis element B' for the topology $\mathcal{T}'$ such that $x \in B' \subset B_d(x, \epsilon)$. Within B' we can find a ball $B_{d'}(x, \delta)$ centered at x.

Conversely, suppose the δ-ϵ condition holds Given a basis element B for $\mathcal{T}$ containing x, we can find within B a ball $B_d(x, \epsilon)$ centered at x. By the given condition, there is a δ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$. Then Lemma 13.3 applies to show $\mathcal{T}'$ is finer than $\mathcal{T}$. ∎

Theorem 20.3. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. It is simple algebra to check that

$$\rho(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{n}\rho(\mathbf{x}, \mathbf{y})$$

The first inequality shows that

$$B_d(\mathbf{x}, \epsilon) \subset B_\rho(\mathbf{x}, \epsilon)$$

for all $\mathbf{x}$ and ϵ, since if $d(\mathbf{x}, \mathbf{y}) < \epsilon$, then $\rho(\mathbf{x}, \mathbf{y}) < \epsilon$ also. Similarly, the second inequality shows that

$$B_\rho(\mathbf{x}, \epsilon/\sqrt{n}) \subset B_d(\mathbf{x}, \epsilon)$$

for all $\mathbf{x}$ and ϵ. It follows from the preceding lemma that the two metric topologies are the same.

Now we show that the product topology is the same as that given by the metric ρ. First, let

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

be a basis element for the product topology, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be an element of B. For each i, there is an ϵ_i such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i),$$

choose $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $B_\rho(\mathbf{x}, \epsilon) \subset B$, as you can readily check. As a result, the ρ-topology is finer than the product topology.

Conversely, let $B_\rho(\mathbf{x}, \epsilon)$ be a basis element for the ρ-topology. Given the element $\mathbf{y} \in B_\rho(\mathbf{x}, \epsilon)$, we need to find a basis element B for the product topology such that

$$\mathbf{y} \in B \subset B_\rho(\mathbf{x}, \epsilon).$$

But this is trivial, for

$$B_\rho(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$$

is itself a basis element for the product topology. ∎

Now we consider the infinite cartesian product $\mathbb{R}^\omega$. It is natural to try to generalize the metrics d and ρ to this space. For instance, one can attempt to define a metric d on $\mathbb{R}^\omega$ by the equation

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}.$$

But this equation does not always make sense, for the series in question need not converge. (This equation does define a metric on a certain important subset of $\mathbb{R}^\omega$, however; see the exercises.)

Similarly, one can attempt to generalize the square metric ρ to $\mathbb{R}^\omega$ by defining

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_n - y_n|\}.$$

Again, this formula does not always make sense. If however we replace the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$ by its bounded counterpart $\bar{d}(x, y) = \min\{|x - y|, 1\}$, then this definition *does* make sense; it gives a metric on $\mathbb{R}^\omega$ called the *uniform metric*.

The uniform metric can be defined more generally on the cartesian product $\mathbb{R}^J$ for arbitrary J, as follows:

Definition.   Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is indeed a metric; it is called the *uniform metric* on $\mathbb{R}^J$, and the topology it induces is called the *uniform topology*.

The relation between this topology and the product and box topologies is the following:

Theorem 20.4.   *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.*

Proof.   Suppose that we are given a point $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and a product topology basis element $\prod U_\alpha$ about $\mathbf{x}$. Let $\alpha_1, \ldots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$. Then for each i, choose $\epsilon_i > 0$ so that the ϵ_i-ball centered at x_{α_i} in the $\bar{d}$ metric is contained in U_{α_i}; this we can do because U_{α_i} is open in $\mathbb{R}$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$; then the ϵ-ball centered at $\mathbf{x}$ in the $\bar{\rho}$ metric is contained in $\prod U_\alpha$. For if $\mathbf{z}$ is a point of $\mathbb{R}^J$ such that $\bar{\rho}(\mathbf{x}, \mathbf{z}) < \epsilon$, then $\bar{d}(x_\alpha, z_\alpha) < \epsilon$ for all α, so that $\mathbf{z} \in \prod U_\alpha$. It follows that the uniform topology is finer than the product topology.

On the other hand, let B be the ϵ-ball centered at $\mathbf{x}$ in the $\bar{\rho}$ metric. Then the box neighborhood

$$U = \prod (x_\alpha - \tfrac{1}{2}\epsilon, x_\alpha + \tfrac{1}{2}\epsilon)$$

of x is contained in B. For if $y \in U$, then $\bar{d}(x_\alpha, y_\alpha) < \frac{1}{2}\epsilon$ for all α, so that $\bar{\rho}(x, y) \leq \frac{1}{2}\epsilon$.

Showing these three topologies are different if J is infinite is a task we leave to the exercises.   ∎

In the case where J is infinite, we still have not determined whether $\mathbb{R}^J$ is metrizable in either the box or the product topology. It turns out that the only one of these cases where $\mathbb{R}^J$ *is* metrizable is the case where J is countable and $\mathbb{R}^J$ has the product topology. As we shall see.

Theorem 20.5. *Let* $\bar{d}(a, b) = \min\{|a - b|, 1\}$ *be the standard bounded metric on* $\mathbb{R}$. *If* x *and* y *are two points of* $\mathbb{R}^\omega$, *define*

$$D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}.$$

Then D *is a metric that induces the product topology on* $\mathbb{R}^\omega$.

Proof. The properties of a metric are satisfied trivially except for the triangle inequality, which is proved by noting that for all i,

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \leq D(x, y) + D(y, z),$$

so that

$$\sup\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(x, y) + D(y, z).$$

The fact that D gives the product topology requires a little more work. First, let U be open in the metric topology and let $x \in U$; we find an open set V in the product topology such that $x \in V \subset U$. Choose an ϵ-ball $B_D(x, \epsilon)$ lying in U. Then choose N large enough that $1/N < \epsilon$. Finally, let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots.$$

We assert that $V \subset B_D(x, \epsilon)$: Given any y in $\mathbb{R}^\omega$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \quad \text{for } i \geq N.$$

Therefore,

$$D(x, y) \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \cdots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}.$$

If y is in V, this expression is less than ϵ, so that $V \subset B_D(x, \epsilon)$, as desired.

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $\mathbf{x} \in U$, we find an open set V of the metric topology such that $\mathbf{x} \in V \subset U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \ldots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \ldots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if $i = \alpha_1, \ldots, \alpha_n$, then $\epsilon \leq \epsilon_i / i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore, $\mathbf{y} \in \prod U_i$, as desired. ∎

Exercises

1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

3. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

4. Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.
   (a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \ldots),$$
$$g(t) = (t, t, t, \ldots),$$
$$h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots).$$

   (b) In which topologies do the following sequences converge?

$$\mathbf{w}_1 = (1, 1, 1, 1, \ldots), \qquad \mathbf{x}_1 = (1, 1, 1, 1, \ldots),$$
$$\mathbf{w}_2 = (0, 2, 2, 2, \ldots), \qquad \mathbf{x}_2 = (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots),$$
$$\mathbf{w}_3 = (0, 0, 3, 3, \ldots), \qquad \mathbf{x}_3 = (0, 0, \tfrac{1}{3}, \tfrac{1}{3} \ldots),$$

$$\cdots \qquad\qquad\qquad \cdots$$

$$\mathbf{y}_1 = (1, 0, 0, 0, \ldots), \qquad \mathbf{z}_1 = (1, 1, 0, 0, \ldots),$$
$$\mathbf{y}_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), \qquad \mathbf{z}_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots),$$
$$\mathbf{y}_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots), \qquad \mathbf{z}_3 = (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots),$$

$$\cdots \qquad\qquad\qquad \cdots$$

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

6. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

   (a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
   (b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
   (c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

7. Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

8. Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the ℓ^2-*topology*. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the **Hilbert cube**. Compare the four topologies that H inherits as a subspace of X.

9. Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n),$$

$$c\mathbf{x} = (cx_1, \ldots, cx_n),$$

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [*Hint:* If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [*Hint:* Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

10. Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges. [*Hint:* Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on X.

*11. Show that if d is a metric for X, then

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

is a bounded metric that gives the topology of X. [*Hint:* If $f(x) = x/(1+x)$ for $x > 0$, use the mean-value theorem to show that $f(a + b) - f(b) \leq f(a)$.]

§21   The Metric Topology (continued)

In this section, we discuss the relation of the metric topology to the concepts we have previously introduced.

Subspaces of metric spaces behave the way one would wish them to; if A is a subspace of the topological space X and d is a metric for X, then the restriction of d to $A \times A$ is a metric for the topology of A. This we leave to you to check.

About *order topologies* there is nothing to be said; some are metrizable (for instance, $\mathbb{Z}_+$ and $\mathbb{R}$), and others are not, as we shall see.

The *Hausdorff axiom* is satisfied by every metric topology. If x and y are distinct points of the metric space (X, d), we let $\epsilon = \frac{1}{2}d(x, y)$; then the triangle inequality implies that $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are disjoint.

The *product topology* we have already considered in special cases; we have proved that the products $\mathbb{R}^n$ and $\mathbb{R}^\omega$ are metrizable. It is true in general that countable products of metrizable spaces are metrizable; the proof follows a pattern similar to the proof for $\mathbb{R}^\omega$, so we leave it to the exercises.

About *continuous functions* there is a good deal to be said. Consideration of this topic will occupy the remainder of the section.

When we study continuous functions on metric spaces, we are about as close to the study of calculus and analysis as we shall come in this book. There are two things we want to do at this point.

First, we want to show that the familiar "ϵ-δ definition" of continuity carries over to general metric spaces, and so does the "convergent sequence definition" of continuity.

Second, we want to consider two additional methods for constructing continuous functions, besides those discussed in §18. One is the process of taking sums, differences, products, and quotients of continuous real-valued functions. The other is the process of taking limits of uniformly convergent sequences of continuous functions.

Theorem 21.1. *Let* $f : X \to Y$; *let* X *and* Y *be metrizable with metrics* d_X *and* d_Y, *respectively. Then continuity of* f *is equivalent to the requirement that given* $x \in X$ *and given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Proof. Suppose that f is continuous. Given x and ϵ, consider the set

$$f^{-1}(B(f(x), \epsilon)),$$

which is open in X and contains the point x. It contains some δ-ball $B(x, \delta)$ centered at x. If y is in this δ-ball, then $f(y)$ is in the ϵ-ball centered at $f(x)$, as desired.

Conversely, suppose that the ϵ-δ condition is satisfied. Let V be open in Y; we show that $f^{-1}(V)$ is open in X. Let x be a point of the set $f^{-1}(V)$. Since $f(x) \in V$, there is an ϵ-ball $B(f(x), \epsilon)$ centered at $f(x)$ and contained in V. By the ϵ-δ condition, there is a δ-ball $B(x, \delta)$ centered at x such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Then $B(x, \delta)$ is a neighborhood of x contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open, as desired.   ∎

Now we turn to the convergent sequence definition of continuity. We begin by considering the relation between convergent sequences and closures of sets. It is certainly believable, from one's experience in analysis, that if x lies in the closure of a subset A of the space X, then there should exist a sequence of points of A converging to x. This is not true in general, but it is true for metrizable spaces.

Lemma 21.2 (The sequence lemma).   *Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.*

Proof. Suppose that $x_n \to x$, where $x_n \in A$. Then every neighborhood U of x contains a point of A, so $x \in \bar{A}$ by Theorem 17.5. Conversely, suppose that X is metrizable and $x \in \bar{A}$. Let d be a metric for the topology of X. For each positive integer n, take the neighborhood $B_d(x, 1/n)$ of radius $1/n$ of x, and choose x_n to be a point of its intersection with A. We assert that the sequence x_n converges to x: Any open set U containing x contains an ϵ-ball $B_d(x, \epsilon)$ centered at x; if we choose N so that $1/N < \epsilon$, then U contains x_i for all $i \geq N$.   ∎

Theorem 21.3.   *Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Proof. Assume that f is continuous. Given $x_n \to x$, we wish to show that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, and so there is an N such that $x_n \in f^{-1}(V)$ for $n \geq N$. Then $f(x_n) \in V$ for $n \geq N$.

To prove the converse, assume that the convergent sequence condition is satisfied. Let A be a subset of X; we show that $f(\bar{A}) \subset \overline{f(A)}$. If $x \in \bar{A}$, then there is a sequence x_n of points of A converging to x (by the preceding lemma). By assumption, the sequence $f(x_n)$ converges to $f(x)$. Since $f(x_n) \in f(A)$, the preceding lemma implies that $f(x) \in \overline{f(A)}$. (Note that metrizability of Y is not needed.) Hence $f(\bar{A}) \subset \overline{f(A)}$, as desired.   ∎

Incidentally, in proving Lemma 21.2 and Theorem 21.3 we did not use the full strength of the hypothesis that the space X is metrizable. All we really needed was the countable collection $B_d(x, 1/n)$ of balls about x. This fact leads us to make a new definition.

A space X is said to have a **countable basis at the point** x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at

least one of the sets U_n. A space X that has a countable basis at each of its points is said to satisfy the *first countability axiom*.

If X has a countable basis $\{U_n\}$ at x, then the proof of Lemma 21.2 goes through; one simply replaces the ball $B_d(x, 1/n)$ throughout by the set

$$B_n = U_1 \cap U_2 \cap \cdots \cap U_n.$$

The proof of Theorem 21.3 goes through unchanged.

A metrizable space always satisfies the first countability axiom, but the converse is not true, as we shall see. Like the Hausdorff axiom, the first countability axiom is a requirement that we sometimes impose on a topological space in order to prove stronger theorems about the space. We shall study it in more detail in Chapter 4.

Now we consider additional methods for constructing continuous functions. We need the following lemma:

Lemma 21.4. *The addition, subtraction, and multiplication operations are continuous functions from* $\mathbb{R} \times \mathbb{R}$ *into* $\mathbb{R}$; *and the quotient operation is a continuous function from* $\mathbb{R} \times (\mathbb{R} - \{0\})$ *into* $\mathbb{R}$.

You have probably seen this lemma proved before; it is a standard "ϵ-δ argument." If not, a proof is outlined in Exercise 12 below; you should have no trouble filling in the details.

Theorem 21.5. *If* X *is a topological space, and if* $f, g : X \to \mathbb{R}$ *are continuous functions, then* $f + g$, $f - g$, *and* $f \cdot g$ *are continuous. If* $g(x) \neq 0$ *for all* x, *then* f/g *is continuous.*

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = f(x) \times g(x)$$

is continuous, by Theorem 18.4. The function $f + g$ equals the composite of h and the addition operation

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R};$$

therefore $f + g$ is continuous. Similar arguments apply to $f - g$, $f \cdot g$, and f/g. $\blacksquare$

Finally, we come to the notion of uniform convergence.

Definition. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) *converges uniformly* to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Uniformity of convergence depends not only on the topology of Y but also on its metric. We have the following theorem about uniformly convergent sequences:

Theorem 21.6 (**Uniform limit theorem**). *Let* $f_n : X \to Y$ *be a sequence of continuous functions from the topological space* X *to the metric space* Y. *If* (f_n) *converges uniformly to* f, *then* f *is continuous.*

Proof. Let V be open in Y; let x_0 be a point of $f^{-1}(V)$. We wish to find a neighborhood U of x_0 such that $f(U) \subset V$.

Let $y_0 = f(x_0)$. First choose ϵ so that the ϵ-ball $B(y_0, \epsilon)$ is contained in V. Then, using uniform convergence, choose N so that for all $n \geq N$ and all $x \in X$,

$$d(f_n(x), f(x)) < \epsilon/3.$$

Finally, using continuity of f_N, choose a neighborhood U of x_0 such that f_N carries U into the $\epsilon/3$ ball in Y centered at $f_N(x_0)$.

We claim that f carries U into $B(y_0, \epsilon)$ and hence into V, as desired. For this purpose, note that if $x \in U$, then

$$d(f(x), f_N(x)) < \epsilon/3 \quad \text{(by choice of } N),$$
$$d(f_N(x), f_N(x_0)) < \epsilon/3 \quad \text{(by choice of } U),$$
$$d(f_N(x_0), f(x_0)) < \epsilon/3 \quad \text{(by choice of } N).$$

Adding and using the triangle inequality, we see that $d(f(x), f(x_0)) < \epsilon$, as desired. ∎

Let us remark that the notion of uniform convergence is related to the definition of the uniform metric, which we gave in the preceding section. Consider, for example, the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$, in the uniform metric $\bar{\rho}$. It is not difficult to see that a sequence of functions $f_n : X \to \mathbb{R}$ converges uniformly to f if and only if the sequence (f_n) converges to f when they are considered as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$. We leave the proof to the exercises.

We conclude the section with some examples of spaces that are not metrizable.

EXAMPLE 1. $\mathbb{R}^\omega$ *in the box topology is not metrizable.*

We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$. Let A be the subset of $\mathbb{R}^\omega$ consisting of those points all of whose coordinates are positive:

$$A = \{(x_1, x_2, \ldots) \mid x_i > 0 \text{ for all } i \in \mathbb{Z}_+\}.$$

Let $\mathbf{0}$ be the "origin" in $\mathbb{R}^\omega$, that is, the point $(0, 0, \ldots)$ each of whose coordinates is zero. In the box topology, $\mathbf{0}$ belongs to $\bar{A}$; for if

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

is any basis element containing $\mathbf{0}$, then B intersects A. For instance, the point

$$(\tfrac{1}{2}b_1, \tfrac{1}{2}b_2 \ldots)$$

belongs to $B \cap A$.

But we assert that there is no sequence of points of A converging to $\mathbf{0}$. For let $(\mathbf{a}_n)$ be a sequence of points of A, where

$$\mathbf{a}_n = (x_{1n}, x_{2n}, \ldots, x_{in}, \ldots).$$

Every coordinate x_{in} is positive, so we can construct a basis element B' for the box topology on $\mathbb{R}$ by setting

$$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots .$$

Then B' contains the origin 0, but it contains no member of the sequence (a_n); the point a_n cannot belong to B' because its nth coordinate x_{nn} does not belong to the interval $(-x_{nn}, x_{nn})$. Hence the sequence (a_n) cannot converge to 0 in the box topology.

EXAMPLE 2.   *An uncountable product of $\mathbb{R}$ with itself is not metrizable.*

Let J be an uncountable index set; we show that $\mathbb{R}^J$ does not satisfy the sequence lemma (in the product topology)

Let A be the subset of $\mathbb{R}^J$ consisting of all points (x_α) such that $x_\alpha = 1$ for all but finitely many values of α. Let 0 be the "origin" in $\mathbb{R}^J$, the point each of whose coordinates is 0.

We assert that 0 belongs to the closure of A. Let $\prod U_\alpha$ be a basis element containing 0. Then $U_\alpha \neq \mathbb{R}$ for only finitely many values of α, say for $\alpha = \alpha_1, \ldots, \alpha_n$. Let (x_α) be the point of A defined by letting $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for all other values of α; then $(x_\alpha) \in A \cap \prod U_\alpha$, as desired.

But there is no sequence of points of A converging to 0. For let a_n be a sequence of points of A. Given n, let J_n denote the subset of J consisting of those indices α for which the αth coordinate of a_n is different from 1. The union of all the sets J_n is a countable union of finite sets and therefore countable. Because J itself is uncountable, there is an index in J, say β, that does not lie in any of the sets J_n. This means that for *each* of the points a_n, its βth coordinate equals 1.

Now let U_β be the open interval $(-1, 1)$ in $\mathbb{R}$, and let U be the open set $\pi_\beta^{-1}(U_\beta)$ in $\mathbb{R}^J$. The set U is a neighborhood of 0 that contains none of the points a_n; therefore, the sequence a_n cannot converge to 0.

Exercises

1. Let $A \subset X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology on A.

2. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f :$ $X \rightarrow Y$ have the property that for every pair of points x_1, x_2 of X,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an *isometric imbedding* of X in Y.

3. Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.
   (a) Show that

$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$

is a metric for the product space $X_1 \times \cdots \times X_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $\prod X_i$.

4. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)

5. *Theorem. Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then*

$$x_n + y_n \to x + y,$$

$$x_n - y_n \to x - y,$$

$$x_n y_n \to xy,$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n/y_n \to x/y.$$

[*Hint:* Apply Lemma 21.4; recall from the exercises of §19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$.]

6. Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) does not converge uniformly.

7. Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

8. Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

9. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(b) Show that f_n does not converge uniformly to f. (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform.)

10. Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:

$$A = \{x \times y \mid xy = 1\},$$

$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$

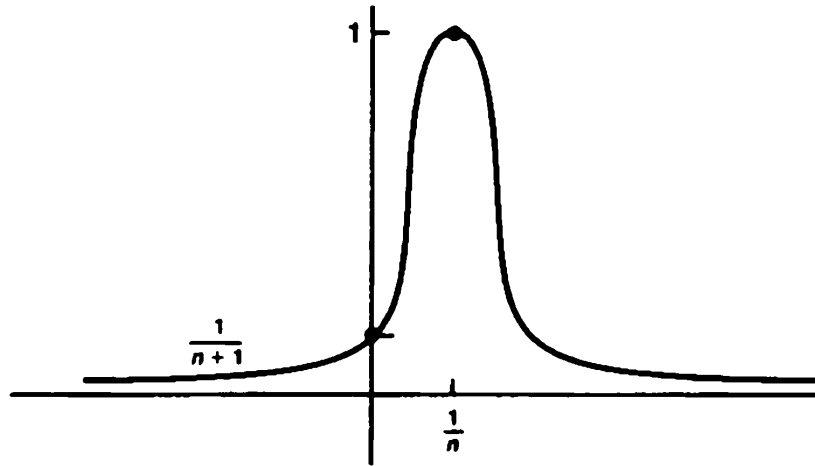$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}.$$

Figure 21.1

The set B^2 is called the (closed) *unit ball* in $\mathbb{R}^2$.

11. Prove the following standard facts about infinite series:

   (a) Show that if (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then (s_n) converges.

   (b) Let (a_n) be a sequence of real numbers; define

   $$s_n = \sum_{i=1}^{n} a_i.$$

   If $s_n \rightarrow s$, we say that the *infinite series*

   $$\sum_{i=1}^{\infty} a_i$$

   converges to s also. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum (ca_i + b_i)$ converges to $cs + t$.

   (c) Prove the *comparison test* for infinite series: If $|a_i| \leq b_i$ for each i, and if the series $\sum b_i$ converges, then the series $\sum a_i$ converges. [*Hint:* Show that the series $\sum |a_i|$ and $\sum c_i$ converge, where $c_i = |a_i| + a_i$.]

   (d) Given a sequence of functions $f_n : X \rightarrow \mathbb{R}$, let

   $$s_n(x) = \sum_{i=1}^{n} f_i(x).$$

   Prove the *Weierstrass M-test* for uniform convergence: If $|f_i(x)| \leq M_i$ for all $x \in X$ and all i, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s. [*Hint:* Let $r_n = \sum_{i=n+1}^{\infty} M_i$. Show that if $k > n$, then $|s_k(x) - s_n(x)| \leq r_n$; conclude that $|s(x) - s_n(x)| \leq r_n$.]

12. Prove continuity of the algebraic operations on $\mathbb{R}$, as follows: Use the metric $d(a, b) = |a - b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ given by the equation

   $$\rho((x, y), (x_0, y_0)) = \max\{|x - x_0|, |y - y_0|\}.$$

Chapter 3

Connectedness
and Compactness

In the study of calculus, there are three basic theorems about continuous functions, and on these theorems the rest of calculus depends. They are the following:

Intermediate value theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and if r is a real number between $f(a)$ and $f(b)$, then there exists an element $c \in [a, b]$ such that $f(c) = r$.

Maximum value theorem. If $f : [a, b] \to R$ is continuous, then there exists an element $c \in [a, b]$ such that $f(x) \leq f(c)$ for every $x \in [a, b]$.

Uniform continuity theorem. If $f : [a, b] \to \mathbb{R}$ is continuous, then given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair of numbers x_1, x_2 of $[a, b]$ for which $|x_1 - x_2| < \delta$.

These theorems are used in a number of places. The intermediate value theorem is used for instance in constructing inverse functions, such as $\sqrt[3]{x}$ and $\arcsin x$; and the maximum value theorem is used for proving the mean value theorem for derivatives, upon which the two *fundamental theorems of calculus* depend. The uniform continuity theorem is used, among other things, for proving that every continuous function is integrable.

We have spoken of these three theorems as theorems about continuous functions. But they can also be considered as theorems about the closed interval $[a, b]$ of real numbers. The theorems depend not only on the continuity of f but also on properties of the topological space $[a, b]$.

The property of the space $[a, b]$ on which the intermediate value theorem depends

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is the property called *connectedness*, and the property on which the other two depend is the property called *compactness*. In this chapter, we shall define these properties for arbitrary topological spaces, and shall prove the appropriate generalized versions of these theorems.

As the three quoted theorems are fundamental for the theory of calculus, so are the notions of connectedness and compactness fundamental in higher analysis, geometry, and topology—indeed, in almost any subject for which the notion of topological space itself is relevant.

§23   Connected Spaces

The definition of connectedness for a topological space is a quite natural one. One says that a space can be "separated" if it can be broken up into two "globs"—disjoint open sets. Otherwise, one says that it is connected. From this simple idea much follows.

Definition.   Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X. Said differently, if X is connected, so is any space homeomorphic to X.

Another way of formulating the definition of connectedness is the following:

A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

For if A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ constitute a separation of X, for they are open, disjoint, and nonempty, and their union is X. Conversely, if U and V form a separation of X, then U is nonempty and different from X, and it is both open and closed in X.

For a subspace Y of a topological space X, there is another useful way of formulating the definition of connectedness:

Lemma 23.1.   *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof.   Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ as usual denotes the closure of A in X). Since A is closed in Y, $A = \bar{A} \cap Y$; or to say the same thing, $\bar{A} \cap B = \varnothing$. Since $\bar{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$;

therefore, we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. Thus both A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. ■

EXAMPLE 1. Let X denote a two-point space in the indiscrete topology. Obviously there is no separation of X, so X is connected.

EXAMPLE 2. Let Y denote the subspace $[-1, 0) \cup (0, 1]$ of the real line $\mathbb{R}$. Each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y (although not in $\mathbb{R}$); therefore, they form a separation of Y. Alternatively, note that neither of these sets contains a limit point of the other. (They do have a limit point 0 in common, but that does not matter.)

EXAMPLE 3. Let X be the subspace $[-1, 1]$ of the real line. The sets $[-1, 0]$ and $(0, 1]$ are disjoint and nonempty, but they do not form a separation of X, because the first set is not open in X. Alternatively, note that the first set contains a limit point, 0, of the second. Indeed, there exists *no* separation of the space $[-1, 1]$. We shall prove this fact shortly.

EXAMPLE 4. The rationals $\mathbb{Q}$ are not connected. Indeed, the only connected subspaces of $\mathbb{Q}$ are the one-point sets: If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number a lying between p and q, and write Y as the union of the open sets

$$Y \cap (-\infty, a) \quad \text{and} \quad Y \cap (a, +\infty).$$

EXAMPLE 5. Consider the following subset of the plane $\mathbb{R}^2$:

$$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = 1/x\}.$$

Then X is not connected; indeed, the two indicated sets form a separation of X because neither contains a limit point of the other. See Figure 23.1.
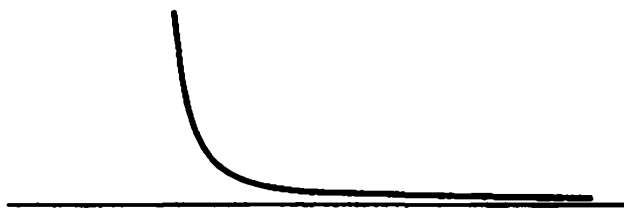
Figure 23.1

We have given several examples of spaces that are not connected. How can one construct spaces that *are* connected? We shall now prove several theorems that tell how to form new connected spaces from given ones. In the next section we shall apply these theorems to show that some specific spaces, such as intervals in $\mathbb{R}$, and balls and cubes in $\mathbb{R}^n$, are connected. First, a lemma:

Lemma 23.2. *If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.*

Proof. Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. These two sets are disjoint and their union is Y; if they were both nonempty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D. ■

Theorem 23.3. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of a space X; let p be a point of $\bigcap A_\alpha$. We prove that the space $Y = \bigcup A_\alpha$ is connected. Suppose that $Y = C \cup D$ is a separation of Y. The point p is in one of the sets C or D; suppose $p \in C$. Since A_α is connected, it must lie entirely in either C or D, and it cannot lie in D because it contains the point p of C. Hence $A_\alpha \subset C$ for every α, so that $\bigcup A_\alpha \subset C$, contradicting the fact that D is nonempty. ∎

Theorem 23.4. *Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.*

Said differently: If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Proof. Let A be connected and let $A \subset B \subset \bar{A}$. Suppose that $B = C \cup D$ is a separation of B. By Lemma 23.2, the set A must lie entirely in C or in D; suppose that $A \subset C$. Then $\bar{A} \subset \bar{C}$; since $\bar{C}$ and D are disjoint, B cannot intersect D. This contradicts the fact that D is a nonempty subset of B. ∎

Theorem 23.5. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \to Y$ be a continuous map; let X be connected. We wish to prove the image space $Z = f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g : X \to Z.$$

Suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z. Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X; they are open in X because g is continuous, and nonempty because g is surjective. Therefore, they form a separation of X, contradicting the assumption that X is connected. ∎

Theorem 23.6. *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two connected spaces X and Y. This proof is easy to visualize. Choose a "base point" $a \times b$ in the product $X \times Y$. Note that the "horizontal slice" $X \times b$ is connected, being homeomorphic with X, and each "vertical slice" $x \times Y$ is connected, being homeomorphic with Y. As a result, each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common. See Figure 23.2. Now form the union $\bigcup_{x \in X} T_x$ of all these T-shaped spaces.

This union is connected because it is the union of a collection of connected spaces that have the point $a \times b$ in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected.
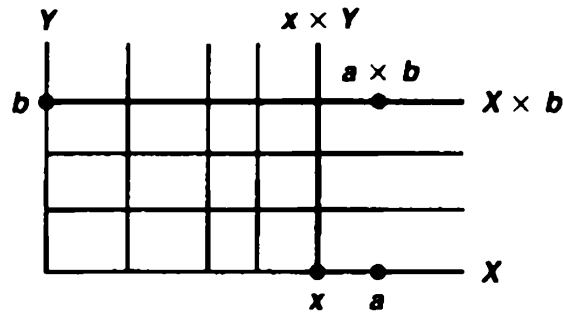

Figure 23.2

The proof for any finite product of connected spaces follows by induction, using the fact (easily proved) that $X_1 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times \cdots \times X_{n-1}) \times X_n$. ∎

It is natural to ask whether this theorem extends to arbitrary products of connected spaces. The answer depends on which topology is used for the product, as the following examples show.

EXAMPLE 6.   Consider the cartesian product $\mathbb{R}^\omega$ in the box topology. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of all bounded sequences of real numbers, and the set B of all unbounded sequences. These sets are disjoint, and each is open in the box topology. For if a is a point of $\mathbb{R}^\omega$, the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$$

consists entirely of bounded sequences if a is bounded, and of unbounded sequences if a if unbounded. Thus, even though $\mathbb{R}$ is connected (as we shall prove in the next section), $\mathbb{R}^\omega$ is not connected in the box topology.

EXAMPLE 7.   Now consider $\mathbb{R}^\omega$ in the product topology. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connected. Let $\mathbb{R}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $x = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. The space $\mathbb{R}^n$ is clearly homeomorphic to $\mathbb{R}^n$, so that it is connected, by the preceding theorem. It follows that the space $\mathbb{R}^\infty$ that is the union of the spaces $\mathbb{R}^n$ is connected, for these spaces have the point $0 = (0, 0, \ldots)$ in common. We show that the closure of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is connected as well.

Let $a = (a_1, a_2, \ldots)$ be a point of $\mathbb{R}^\omega$. Let $U = \prod U_i$ be a basis element for the product topology that contains a. We show that U intersects $\mathbb{R}^\infty$. There is an integer N such that $U_i = \mathbb{R}$ for $i > N$. Then the point

$$x = (a_1, \ldots, a_n, 0, 0, \ldots)$$

of $\mathbb{R}^\infty$ belongs to U, since $a_i \in U_i$ for all i, and $0 \in U_i$ for $i > N$.

The argument just given generalizes to show that an arbitrary product of connected spaces is connected in the product topology. Since we shall not need this result, we leave the proof to the exercises.

Exercises

1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

3. Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

4. Show that if X is an infinite set, it is connected in the finite complement topology.

5. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects Bd A.

7. Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

8. Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

9. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) - (A \times B)$$

   is connected.

10. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of connected spaces; let X be the product space

$$X = \prod_{\alpha \in J} X_\alpha.$$

Let $\mathbf{a} = (a_\alpha)$ be a fixed point of X.

   (a) Given any finite subset K of J, let X_K denote the subspace of X consisting of all points $\mathbf{x} = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show that X_K is connected.

   (b) Show that the union Y of the spaces X_K is connected.

   (c) Show that X equals the closure of Y; conclude that X is connected.

11. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

12. Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

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It is not as natural or intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

Definition. A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an *open covering* of X if its elements are open subsets of X.

Definition. A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

EXAMPLE 1. The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

EXAMPLE 2. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing 0. The set U contains all but finitely many of the points $1/n$; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite subcollection of $\mathcal{A}$ that covers X.

EXAMPLE 3. Any space X containing only finitely many points is necessarily compact, because in this case every open covering of X is finite.

EXAMPLE 4. The interval $(0, 1]$ is not compact; the open covering

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$$

contains no finite subcollection covering $(0, 1]$. Nor is the interval $(0, 1)$ compact; the same argument applies. On the other hand, the interval $[0, 1]$ *is* compact; you are probably already familiar with this fact from analysis. In any case, we shall prove it shortly.

In general, it takes some effort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones. Then in the next section we shall show certain specific spaces are compact. These spaces include all closed intervals in the real line, and all closed and bounded subsets of $\mathbb{R}^n$.

Let us first prove some facts about subspaces. If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to *cover* Y if the union of its elements *contains* Y.

Lemma 26.1. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcollection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y. ∎

Theorem 26.2. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X - Y$, that is,

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - Y$, discard $X - Y$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y. ∎

Theorem 26.3. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X - Y$ is open, so that Y is closed.

Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, \ldots, V_{y_n}$ cover Y. The open set

$$V = V_{y_1} \cup \cdots \cup V_{y_n}$$

contains Y, and it is disjoint from the open set

$$U = U_{y_1} \cap \cdots \cap U_{y_n}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i}$ and so $z \notin U$. See Figure 26.1.

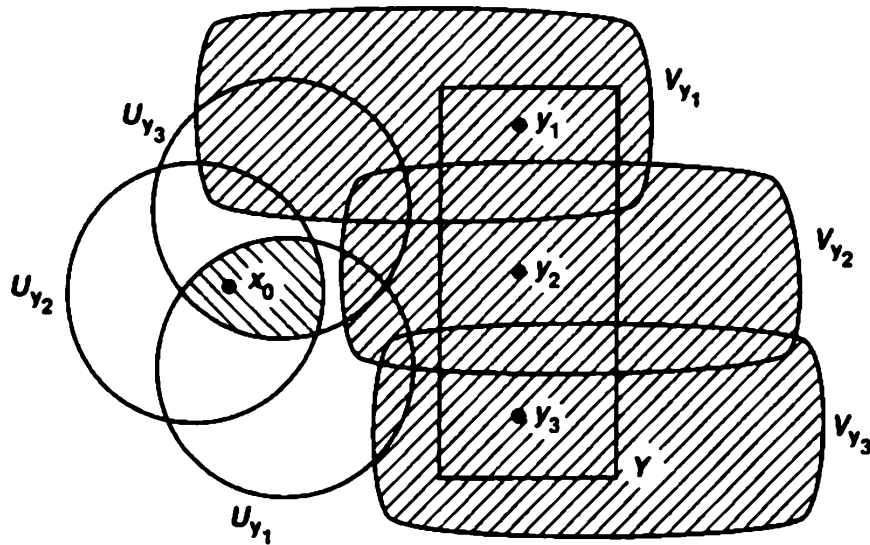Then U is a neighborhood of x_0 disjoint from Y, as desired. ∎

Figure 26.1

The statement we proved in the course of the preceding proof will be useful to us later, so we repeat it here for reference purposes:

Lemma 26.4. *If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.*

EXAMPLE 5. Once we prove that the interval $[a, b]$ in $\mathbb{R}$ is compact, it follows from Theorem 26 2 that any closed subspace of $[a, b]$ is compact. On the other hand, it follows from Theorem 26.3 that the intervals $(a, b]$ and (a, b) in $\mathbb{R}$ cannot be compact (which we knew already) because they are not closed in the Hausdorff space $\mathbb{R}$

EXAMPLE 6. One needs the Hausdorff condition in the hypothesis of Theorem 26 3 Consider, for example, the finite complement topology on the real line The only proper subsets of $\mathbb{R}$ that are closed in this topology are the finite sets. But *every* subset of $\mathbb{R}$ is compact in this topology, as you can check.

Theorem 26.5. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \rightarrow Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them, say

$$f^{-1}(A_1), \ldots, f^{-1}(A_n),$$

cover X. Then the sets $A_1, \ldots, A_n$ cover $f(X)$.   ∎

One important use of the preceding theorem is as a tool for verifying that a map is a homeomorphism:

Theorem 26.6. *Let* $f : X \to Y$ *be a bijective continuous function. If* X *is compact and* Y *is Hausdorff, then* f *is a homeomorphism*

Proof. We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact, by Theorem 26.2. Therefore, by the theorem just proved, $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by Theorem 26.3.   ∎

Theorem 26.7. *The product of finitely many compact spaces is compact.*

Proof. We shall prove that the product of two compact spaces is compact; the theorem follows by induction for any finite product.

Step 1. Suppose that we are given spaces X and Y, with Y compact. Suppose that x_0 is a point of X, and N is an open set of $X \times Y$ containing the "slice" $x_0 \times Y$ of $X \times Y$ We prove the following·

> *There is a neighborhood* W *of* x_0 *in* X *such that* N *contains the entire set* $W \times Y$

The set $W \times Y$ is often called a *tube* about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$) lying in N. The space $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we can cover $x_0 \times Y$ by finitely many such basis elements

$$U_1 \times V_1, \ldots, U_n \times V_n$$

(We assume that each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise that basis element would be superfluous; we could discard it from the finite collection and still have a covering of $x_0 \times Y$.) Define

$$W = U_1 \cap \cdots \cap U_n$$

The set W is open, and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$.

We assert that the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$. Let $x \times y$ be a point of $W \times Y$. Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y$ belongs to $U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for *every* j (because $x \in W$). Therefore, we have $x \times y \in U_i \times V_i$, as desired.

Since all the sets $U_i \times V_i$ lie in N, and since they cover $W \times Y$, the tube $W \times Y$ lies in N also. See Figure 26.2.

Step 2. Now we prove the theorem. Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Their union $N = A_1 \cup \cdots \cup A_m$ is an open set containing $x_0 \times Y$; by Step 1, the open set N contains
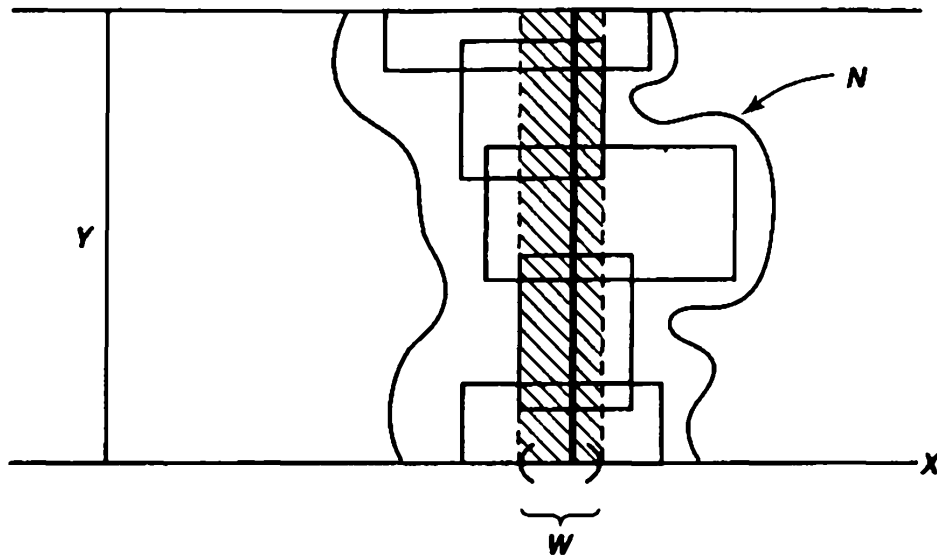
Figure 26.2

a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$.

Thus, for each x in X, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X, there exists a finite subcollection

$$\{W_1, \ldots, W_k\}$$

covering X. The union of the tubes

$$W_1 \times Y, \quad \ldots, W_k \times Y$$

is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$, so may $X \times Y$ be covered.   ∎

The statement proved in Step 1 of the preceding proof will be useful to us later, so we repeat it here as a lemma, for reference purposes:

Lemma 26.8 (The tube lemma).   *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

EXAMPLE 7   The tube lemma is certainly not true if Y is not compact For example, let Y be the y-axis in $\mathbb{R}^2$, and let

$$N = \{x \times y, |x| < 1/(y^2 + 1)\}.$$

Then N is an open set containing the set $0 \times \mathbb{R}$, but it contains no tube about $0 \times \mathbb{R}$ It is illustrated in Figure 26 3
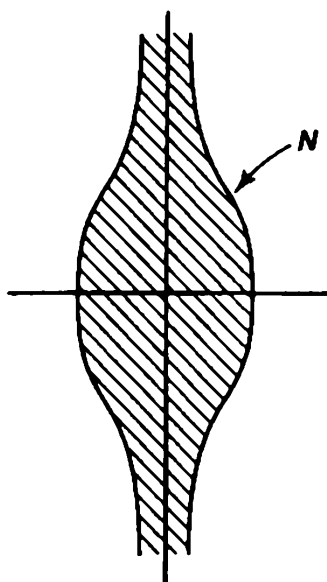
Figure 26.3

There is an obvious question to ask at this point. *Is the product of infinitely many compact spaces compact?* One would hope that the answer is "yes," and in fact it is. The result is important (and difficult) enough to be called by the name of the man who proved it; it is called the *Tychonoff theorem*

In proving the fact that a cartesian product of connected spaces is connected, one proves it first for finite products and derives the general case from that. In proving that cartesian products of compact spaces are compact, however, there is no way to go directly from finite products to infinite ones. The infinite case demands a new approach, and the proof is a difficult one. Because of its difficulty, and also to avoid losing the main thread of our discussion in this chapter, we have decided to postpone it until later. However, you can study it now if you wish; the section in which it is proved (§37) can be studied immediately after this section without causing any disruption in continuity.

There is one final criterion for a space to be compact, a criterion that is formulated in terms of closed sets rather than open sets It does not look very natural nor very useful at first glance, but it in fact proves to be useful on a number of occasions. First we make a definition.

Definition.   A collection C of subsets of X is said to have the *finite intersection property* if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of C, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9. *Let X be a topological space. Then X is compact if and only if for every collection C of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in C} C$ of all the elements of C is nonempty.*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty

(3) The finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty.

The first statement is trivial, while the second and third follow from DeMorgan's law:

$$X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J} (X - A_\alpha).$$

The proof of the theorem now proceeds in two easy steps: taking the *contrapositive* (of the theorem), and then the *complement* (of the sets)!

The statement that X is compact is equivalent to saying: "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X." This statement is equivalent to its contrapositive, which is the following: "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X." Letting $\mathcal{C}$ be, as earlier, the collection $\{X - A \mid A \in \mathcal{A}\}$ and applying (1)–(3), we see that this statement is in turn equivalent to the following: "Given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty." This is just the condition of our theorem. ∎

A special case of this theorem occurs when we have a *nested sequence* $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \ldots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\mathcal{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ automatically has the finite intersection property. Then the intersection

$$\bigcap_{n \in \mathbb{Z}_+} C_n$$

is nonempty.

We shall use the closed set criterion for compactness in the next section to prove the uncountability of the set of real numbers, in Chapter 5 when we prove the Tychonoff theorem, and again in Chapter 8 when we prove the Baire category theorem.

Exercises

1. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

   (b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

(b) Show that $\mathbb{R}_K$ is connected. [*Hint* $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]

(c) Show that $\mathbb{R}_K$ is not path connected.

4. Show that a connected metric space having more than one point is uncountable.

5. Let X be a compact Hausdorff space, let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

6. Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1 + 3k}{3^n}, \frac{2 + 3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

is called the *Cantor set*; it is a subspace of $[0, 1]$

(a) Show that C is totally disconnected.

(b) Show that C is compact.

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.

(d) Show that C has no isolated points.

(e) Conclude that C is uncountable.

§28   Limit Point Compactness

As indicated when we first mentioned compact sets, there are other formulations of the notion of compactness that are frequently useful. In this section we introduce one of them. Weaker in general than compactness, it coincides with compactness for metrizable spaces.

Definition.   A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

In some ways this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name "compactness," while the open covering formulation was called "bicompactness." Later, the word "compact" was shifted to apply to the open covering definition, leaving this one to search for a new

name It still has not found a name on which everyone agrees On historical grounds, some call it "Fréchet compactness", others call it the "Bolzano-Weierstrass property" We have invented the term "limit point compactness" It seems as good a term as any; at least it describes what the property is about.

Theorem 28.1.   *Compactness implies limit point compactness, but not conversely.*

Proof.   Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contrapositive—if A has no limit point, then A must be finite.

So suppose A has no limit point. Then A contains all its limit points, so that A is closed. Furthermore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since $X - A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite.   ∎

EXAMPLE 1   Let Y consist of two points, give Y the topology consisting of Y and the empty set Then the space $X = \mathbb{Z}_+ \times Y$ is limit point compact, for *every* nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X

EXAMPLE 2   Here is a less trivial example Consider the minimal uncountable well-ordered set S_Ω, in the order topology The space S_Ω is not compact, since it has no largest element However, it is limit point compact: Let A be an infinite subset of S_Ω. Choose a subset B of A that is countably infinite Being countable, the set B has an upper bound b in S_Ω; then B is a subset of the interval $[a_0, b]$ of S_Ω, where a_0 is the smallest element of S_Ω. Since S_Ω has the least upper bound property, the interval $[a_0, b]$ is compact By the preceding theorem, B has a limit point x in $[a_0, b]$. The point x is also a limit point of A Thus S_Ω is limit point compact

We now show these two versions of compactness coincide for metrizable spaces; for this purpose, we introduce yet another version of compactness called *sequential compactness*. This result will be used in Chapter 7.

Definition.   Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a *subsequence* of the sequence (x_n). The space X is said to be *sequentially compact* if every sequence of points of X has a convergent subsequence.

*Theorem 28.2.   *Let X be a metrizable space. Then the following are equivalent:*

(1)   *X is compact.*

(2)   *X is limit point compact.*

(3)   *X is sequentially compact.*

Proof. We have already proved that (1) $\Rightarrow$ (2). To show that (2) $\Rightarrow$ (3), assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \in \mathbb{Z}_+\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially. On the other hand, if A is infinite, then A has a limit point x. We define a subsequence of (x_n) converging to x as follows: First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer n_{i-1} is given. Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

Finally, we show that (3) $\Rightarrow$ (1). This is the hardest part of the proof.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X. (This would follow from compactness, but compactness is what we are trying to prove!) Let $\mathcal{A}$ be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$; let C_n be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a. Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. If i is large enough that $1/n_i < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$-neighborhood of x_{n_i}; if i is also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$, then C_{n_i} lies in the ϵ-neighborhood of a. But this means that $C_{n_i} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls. Once again, we proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$. In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon),$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \ldots, n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\epsilon/2$ can contain x_n for at most *one* value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open covering of X. Because X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number δ. Let $\epsilon = \delta/3$; use sequential compactness of X to find a finite

covering of X by open ϵ-balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these ϵ-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X.   ■

EXAMPLE 3.   Recall that $\bar{S}_\Omega$ denotes the minimal uncountable well-ordered set S_Ω with the point Ω adjoined. (In the order topology, Ω is a limit point of S_Ω, which is why we introduced the notation $\bar{S}_\Omega$ for $S_\Omega \cup \{\Omega\}$, back in §10) It is easy to see that the space $\bar{S}_\Omega$ is not metrizable, for it does not satisfy the sequence lemma: The point Ω is a limit point of S_Ω, but it is not the limit of a sequence of points of S_Ω, for any sequence of points of S_Ω has an upper bound in S_Ω The space S_Ω, on the other hand, does satisfy the sequence lemma, as you can readily check Nevertheless, S_Ω is not metrizable, for it is limit point compact but not compact.

Exercises

1. Give $[0, 1]^\omega$ the uniform topology. Find an infinite subset of this space that has no limit point

2. Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

3. Let X be limit point compact.
   (a) If $f \cdot X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?
   (b) If A is a closed subset of X, does it follow that A is limit point compact?
   (c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

   We comment that it is not in general true that the product of two limit point compact spaces is limit point compact, even if the Hausdorff condition is assumed. But the examples are fairly sophisticated. See [S-S], Example 112.

4. A space X is said to be **countably compact** if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness. [*Hint:* If no finite subcollection of U_n covers X, choose $x_n \notin U_1 \cup \cdots \cup U_n$, for each n.]

5. Show that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdot$ of closed nonempty sets of X has a nonempty intersection.

6. Let (X, d) be a metric space. If $f : X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an **isometry** of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism. [*Hint:* If $a \notin f(X)$, choose ϵ so that the ϵ-neighborhood of a is disjoint from $f(X)$ Set $x_1 = a$, and $x_{n+1} = f(x_n)$ in general. Show that $d(x_n, x_m) \geq \epsilon$ for $n \neq m$.]

7. Let (X, d) be a metric space. If f satisfies the condition

$$d(f(x), f(y)) < d(x, y)$$

Chapter 5

The Tychonoff Theorem

We now return to a problem we left unresolved in Chapter 3. We shall prove the Tychonoff theorem, to the effect that arbitrary products of compact spaces are compact. The proof makes use of Zorn's Lemma (see §11). An alternate proof, which relies instead on the well-ordering theorem, is outlined in the exercises.

The Tychonoff theorem is of great usefulness to analysts (less so to geometers). We apply it in §38 to construct the Stone-Čech compactification of a completely regular space, and in §47 in proving the general version of Ascoli's theorem.

§37 The Tychonoff Theorem

Like the Urysohn lemma, the Tychonoff theorem is what we call a "deep" theorem. Its proof involves not one but several original ideas; it is anything but straightforward. We shall discuss the crucial ideas of the proof in some detail before turning to the proof itself.

In Chapter 3, we proved the product $X \times Y$ of two compact spaces to be compact. For that proof the open covering formulation of compactness was quite satisfactory. Given an open covering of $X \times Y$ by basis elements, we covered each slice $x \times Y$ by finitely many of them, and proceeded from that to construct a finite covering of $X \times Y$.

It is quite tricky to make this approach work for an arbitrary product of compact spaces; one must well-order the index set and use transfinite induction. (See

Exercise 5.) An alternate approach is to abandon open coverings and to approach the problem by applying the closed set formulation of compactness, using Zorn's lemma.

To see how this idea might work, let us consider first the simplest possible case: the product of two compact spaces $X_1 \times X_2$. Suppose that $\mathcal{A}$ is a collection of closed subsets of $X_1 \times X_2$ that has the finite intersection property. Consider the projection map $\pi_1 : X_1 \times X_2 \to X_1$. The collection

$$\{\pi_1(A) \mid A \in \mathcal{A}\}$$

of subsets of X_1 also has the finite intersection property, and so does the collection of their closures $\overline{\pi_1(A)}$. Compactness of X_1 guarantees that the intersection of all the sets $\overline{\pi_1(A)}$ is nonempty. Let us choose a point x_1 belonging to this intersection. Similarly, let us choose a point x_2 belonging to all the sets $\overline{\pi_2(A)}$. The obvious conclusion we would like to draw is that the point $x_1 \times x_2$ lies in $\bigcap_{A \in \mathcal{A}} A$, for then our theorem would be proved.

But that is unfortunately not true. Consider the following example, in which $X_1 = X_2 = [0, 1]$ and the collection $\mathcal{A}$ consists of all closed elliptical regions bounded by ellipses that have the points $p = (\frac{1}{3}, \frac{1}{3})$ and $q = (\frac{1}{2}, \frac{2}{3})$ as their foci. See Figure 37.1. Certainly $\mathcal{A}$ has the finite intersection property. Now let us pick a point x_1 in the intersection of the sets $\{\overline{\pi_1(A)} \mid A \in \mathcal{A}\}$. Any point of the interval $[\frac{1}{3}, \frac{1}{2}]$ will do; suppose we choose $x_1 = \frac{1}{2}$. Similarly, choose a point x_2 in the intersection of the sets $\{\overline{\pi_2(A)} \mid A \in \mathcal{A}\}$. Any point of the interval $[\frac{1}{3}, \frac{2}{3}]$ will do; suppose we pick $x_2 = \frac{1}{2}$. This proves to be an unfortunate choice, for the point

$$x_1 \times x_2 = \frac{1}{2} \times \frac{1}{2}$$
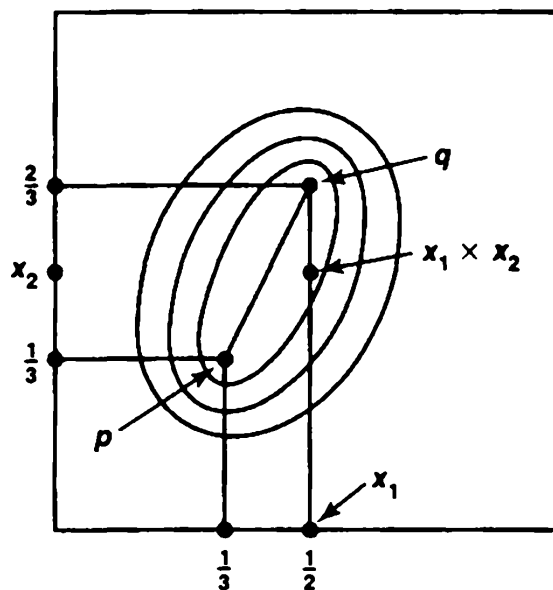
does not lie in the intersection of the sets A.

Figure 37.1

"Aha!" you say, "you made a bad choice. If after choosing $x_1 = \frac{1}{2}$ you had chosen $x_2 = \frac{2}{3}$, then you would have found a point in $\bigcap_{A \in \mathcal{A}} A$." The difficulty with our

tentative proof is that it gave us too much freedom in picking x_1 and x_2; it allowed us to make a "bad" choice instead of a "good" choice.

How can we alter the proof so as to avoid this difficulty?

This question leads to the second idea of the proof: Perhaps if we *expand* the collection $\mathcal{A}$ (retaining the finite intersection property, of course), that expansion will restrict the choices of x_1 and x_2 sufficiently that we will be forced to make the "right" choice. To illustrate, suppose that in the previous example we expand the collection $\mathcal{A}$ to the collection $\mathcal{D}$ consisting of all closed elliptical regions bounded by ellipses that have $p = (\frac{1}{3}, \frac{1}{3})$ as one focus and any point of the line segment pq as the other focus. This collection is illustrated in Figure 37.2. The new collection $\mathcal{D}$ still has the finite intersection property. But if you try to choose a point x_1 in

$$\bigcap_{D \in \mathcal{D}} \overline{\pi_1(D)},$$

the only possible choice for x_1 is $\frac{1}{3}$. Similarly, the only possible choice for x_2 is $\frac{1}{3}$. And $\frac{1}{3} \times \frac{1}{3}$ *does* belong to every set D, and hence to every set A. In other words, expanding the collection $\mathcal{A}$ to the collection $\mathcal{D}$ forces the proper choice on us.
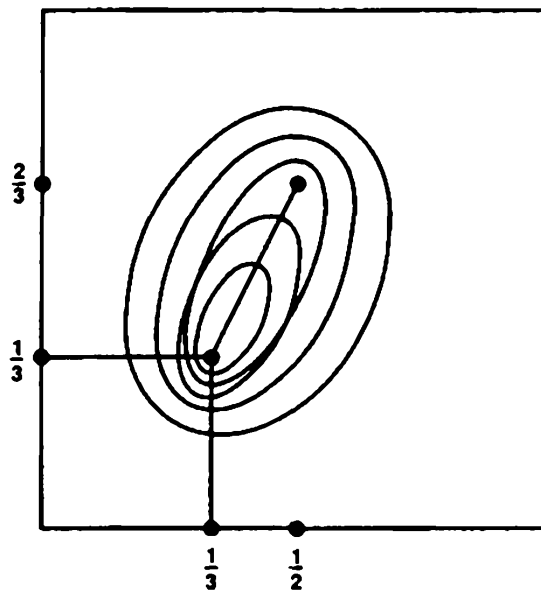
Figure 37.2

Now of course in this example we chose $\mathcal{D}$ carefully so that the proof would work. What hope can we have for choosing $\mathcal{D}$ correctly in general? Here is the third idea of the proof: Why not simply choose $\mathcal{D}$ to be a collection that is "as large as possible"— so that no larger collection has the finite intersection property—and see whether such a $\mathcal{D}$ will work? It is not at all obvious that such a collection $\mathcal{D}$ exists; to prove it, we must appeal to Zorn's lemma. But after we prove that $\mathcal{D}$ exists, we shall in fact be able to show that $\mathcal{D}$ is large enough to force the proper choices on us.

A final remark. The assumption that the elements of the collection $\mathcal{A}$ were closed sets was irrelevant in this discussion. For even if the set $A \in \mathcal{A}$ is closed, the set $\pi_1(A)$ need not be closed, so we had to take its closure in order to apply the closed set formu-

lation of compactness. Therefore, we may as well begin with an arbitrary collection of subsets of X having the finite intersection property, and prove that the intersection of their *closures* is nonempty. This approach actually proves to be more convenient.

Lemma 37.1. *Let X be a set; let A be a collection of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains A, and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property.*

We often say that a collection $\mathcal{D}$ satisfying the conclusion of this theorem is *maximal* with respect to the finite intersection property

Proof. As you might expect, we construct $\mathcal{D}$ by using Zorn's lemma. It states that, given a set A that is strictly partially ordered, in which every simply ordered subset has an upper bound, A itself has a maximal element.

The set A to which we shall apply Zorn's lemma is *not* a subset of X, nor even a collection of subsets of X, but a set whose *elements* are collections of subsets of X. For purposes of this proof, we shall call a set whose elements are collections of subsets of X a "superset" and shall denote it by an outline letter. To summarize the notation:

c is an element of X.

C is a subset of X

$\mathcal{C}$ is a collection of subsets of X

$\mathbb{C}$ is a superset whose elements are collections of subsets of X.

Now by hypothesis, we have a collection A of subsets of X that has the finite intersection property. Let A denote the superset consisting of *all* collections $\mathcal{B}$ of subsets of X such that $\mathcal{B} \supset A$ and $\mathcal{B}$ has the finite intersection property. We use proper inclusion $\subsetneq$ as our strict partial order on A. To prove our lemma, we need to show that A has a maximal element $\mathcal{D}$.

In order to apply Zorn's lemma, we must show that if $\mathbb{B}$ is a "subsuperset" of A that is simply ordered by proper inclusion, then $\mathbb{B}$ has an upper bound in A. We shall show in fact that the collection

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B},$$

which is the union of the collections belonging to $\mathbb{B}$, is an element of A; then it is the required upper bound on $\mathbb{B}$.

To show that $\mathcal{C}$ is an element of A, we must show that $\mathcal{C} \supset A$ and that $\mathcal{C}$ has the finite intersection property. Certainly $\mathcal{C}$ contains A, since each element of $\mathbb{B}$ contains A. To show that $\mathcal{C}$ has the finite intersection property, let $C_1, \ldots, C_n$ be elements of $\mathcal{C}$. Because $\mathcal{C}$ is the union of the elements of $\mathbb{B}$, there is, for each i, an element $\mathcal{B}_i$ of $\mathbb{B}$ such that $C_i \in \mathcal{B}_i$. The superset $\{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is contained in $\mathbb{B}$, so it is simply ordered by the relation of proper inclusion. Being finite, it has a largest element; that is, there is an index k such that $\mathcal{B}_i \subset \mathcal{B}_k$ for $i = 1, \ldots, n$. Then all the sets $C_1, \ldots, C_n$ are elements of $\mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property, the intersection of the sets $C_1, \ldots, C_n$ is nonempty, as desired.   ∎

Lemma 37.2. *Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:*

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Proof. (a) Let B equal the intersection of finitely many elements of $\mathcal{D}$. Define a collection $\mathcal{E}$ by adjoining B to $\mathcal{D}$, so that $\mathcal{E} = \mathcal{D} \cup \{B\}$. We show that $\mathcal{E}$ has the finite intersection property; then maximality of $\mathcal{D}$ implies that $\mathcal{E} = \mathcal{D}$, so that $B \in \mathcal{D}$ as desired

Take finitely many elements of $\mathcal{E}$. If none of them is the set B, then their intersection is nonempty because $\mathcal{D}$ has the finite intersection property. If one of them is the set B, then their intersection is of the form

$$D_1 \cap \cdots \cap D_m \cap B$$

Since B equals a finite intersection of elements of $\mathcal{D}$, this set is nonempty.

(b) Given A, define $\mathcal{E} = \mathcal{D} \cup \{A\}$ We show that $\mathcal{E}$ has the finite intersection property, from which we conclude that A belongs to $\mathcal{D}$. Take finitely many elements of $\mathcal{E}$. If none of them is the set A, their intersection is automatically nonempty. Otherwise, it is of the form

$$D_1 \cap \cdots \cap D_n \cap A.$$

Now $D_1 \cap \cdots \cap D_n$ belongs to $\mathcal{D}$, by (a); therefore, this intersection is nonempty, by hypothesis.   ∎

Theorem 37.3 (Tychonoff theorem). *An arbitrary product of compact spaces is compact in the product topology*

Proof. Let

$$X = \prod_{\alpha \in J} X_\alpha,$$

where each space X_α is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection

$$\bigcap_{A \in \mathcal{A}} \bar{A}$$

is nonempty. Compactness of X follows.

Applying Lemma 37.1, choose a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal with respect to the finite intersection property It will suffice to show that the intersection $\bigcap_{D \in \mathcal{D}} \bar{D}$ is nonempty.

Given $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map, as usual   Consider the collection

$$\{\pi_\alpha(D) \mid D \in \mathcal{D}\}$$

of subsets of X_α. This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α, we can for each α choose a point x_α of X_α such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}.$$

Let x be the point $(x_\alpha)_{\alpha \in J}$ of X. We shall show that $x \in \bar{D}$ for every $D \in \mathcal{D}$; then our proof will be finished.

First we show that if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element (for the product topology on X) containing x, then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathcal{D}$. The set U_β is a neighborhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta(D)$ in some point $\pi_\beta(y)$, where $y \in D$ Then it follows that $y \in \pi_\beta^{-1}(U_\beta) \cap D$.

It follows from (b) of Lemma 37.2 that every subbasis element containing x *belongs* to $\mathcal{D}$. And then it follows from (a) of the same lemma that every *basis* element containing x belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, this means that every basis element containing x intersects every element of $\mathcal{D}$; hence $x \in \bar{D}$ for every $D \in \mathcal{D}$ as desired.   ∎

Exercises

1. Let X be a space. Let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property
   (a) Show that $x \in \bar{D}$ for every $D \in \mathcal{D}$ if and only if every neighborhood of x belongs to $\mathcal{D}$. Which implication uses maximality of $\mathcal{D}$?
   (b) Let $D \in \mathcal{D}$. Show that if $A \supset D$, then $A \in \mathcal{D}$.
   (c) Show that if X satisfies the T_1 axiom, there is at most one point belonging to $\bigcap_{D \in \mathcal{D}} \bar{D}$.

2. A collection $\mathcal{A}$ of subsets of X has the *countable intersection property* if every countable intersection of elements of $\mathcal{A}$ is nonempty. Show that X is a Lindelöf space if and only if for every collection $\mathcal{A}$ of subsets of X having the countable intersection property,

$$\bigcap_{A \in \mathcal{A}} \bar{A}$$

is nonempty.

3. Consider the three statements:
   (i) If X is a set and $\mathcal{A}$ is a collection of subsets of X having the countable intersection property, then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal with respect to the countable intersection property

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – SEPARATION AXIOMS – SMTA5202

Chapter 4

Countability and Separation Axioms

The concepts we are going to introduce now, unlike compactness and connectedness, do not arise naturally from the study of calculus and analysis. They arise instead from a deeper study of topology itself. Such problems as imbedding a given space in a metric space or in a compact Hausdorff space are basically problems of topology rather than analysis. These particular problems have solutions that involve the countability and separation axioms.

We have already introduced the first countability axiom; it arose in connection with our study of convergent sequences in §21. We have also studied one of the separation axioms—the Hausdorff axiom, and mentioned another—the T_1 axiom. In this chapter we shall introduce other, and stronger, axioms like these and explore some of their consequences. Our basic goal is to prove the *Urysohn metrization theorem*. It says that if a topological space X satisfies a certain countability axiom (the second) and a certain separation axiom (the regularity axiom), then X can be imbedded in a metric space and is thus metrizable.

Another imbedding theorem, important to geometers, appears in the last section of the chapter. Given a space that is a compact manifold (the higher-dimensional analogue of a surface), we show that it can be imbedded in some finite-dimensional euclidean space.

§30   The Countability Axioms

Recall the definition we gave in §21.

Definition.   A space X is said to have a *countable basis at x* if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first-countable*.

We have already noted that every metrizable space satisfies this axiom; see §21.

The most useful fact concerning spaces that satisfy this axiom is the fact that in such a space, convergent sequences are adequate to detect limit points of sets and to check continuity of functions. We have noted this before; now we state it formally as a theorem:

Theorem 30.1.   *Let X be a topological space.*

(a) *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.*

(b) *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.*

The proof is a direct generalization of the proof given in §21 under the hypothesis of metrizability, so it will not be repeated here.

Of much greater importance than the first countability axiom is the following:

Definition.   If a space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom*, or to be *second-countable*.

Obviously, the second axiom implies the first: if $\mathcal{B}$ is a countable basis for the topology of X, then the subset of $\mathcal{B}$ consisting of those basis elements containing the point x is a countable basis at x. The second axiom is, in fact, much stronger than the first; it is so strong that not even every metric space satisfies it.

Why then is this second axiom interesting? Well, for one thing, many familiar spaces do satisfy it. For another, it is a crucial hypothesis used in proving such theorems as the Urysohn metrization theorem, as we shall see.

EXAMPLE 1   The real line $\mathbb{R}$ has a countable basis—the collection of all open intervals (a, b) with rational end points. Likewise, $\mathbb{R}^n$ has a countable basis—the collection of all products of intervals having rational end points. Even $\mathbb{R}^\omega$ has a countable basis—the collection of all products $\prod_{n \in \mathbb{Z}_+} U_n$, where U_n is an open interval with rational end points for finitely many values of n, and $U_n = \mathbb{R}$ for all other values of n.

EXAMPLE 2   In the uniform topology, $\mathbb{R}^\omega$ satisfies the first countability axiom (being metrizable). However, it does not satisfy the second. To verify this fact, we first show that if X is a space having a countable basis $\mathcal{B}$, then any discrete subspace A of X must be countable Choose, for each $a \in A$, a basis element B_a that intersects A in the point a

alone. If a and b are distinct points of A, the sets B_a and B_b are different, since the first contains a and the second does not. It follows that the map $a \rightarrow B_a$ is an injection of A into $\mathcal{B}$, so A must be countable.

Now we note that the subspace A of $\mathbb{R}^\omega$ consisting of all sequences of 0's and 1's is uncountable; and it has the discrete topology because $\bar{\rho}(a, b) = 1$ for any two distinct points a and b of A. Therefore, in the uniform topology $\mathbb{R}^\omega$ does not have a countable basis.

Both countability axioms are well behaved with respect to the operations of taking subspaces or countable products:

Theorem 30.2. *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Proof. Consider the second countability axiom. If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A of X If $\mathcal{B}_i$ is a countable basis for the space X_i, then the collection of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$.

The proof for the first countability axiom is similar. ∎

Two consequences of the second countability axiom that will be useful to us later are given in the following theorem. First, a definition:

Definition. A subset A of a space X is said to be *dense* in X if $\bar{A} = X$.

Theorem 30.3. *Suppose that X has a countable basis. Then:*

(a) Every open covering of X contains a countable subcollection covering X.

(b) There exists a countable subset of X that is dense in X.

Proof. Let $\{B_n\}$ be a countable basis for X.

(a) Let $\mathcal{A}$ be an open covering of X. For each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n. The collection $\mathcal{A}'$ of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X: Given a point $x \in X$, we can choose an element A of $\mathcal{A}$ containing x. Since A is open, there is a basis element B_n such that $x \in B_n \subset A$. Because B_n lies in an element of $\mathcal{A}$, the index n belongs to the set J, so A_n is defined; since A_n contains B_n, it contains x. Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

(b) From each nonempty basis element B_n, choose a point x_n. Let D be the set consisting of the points x_n. Then D is dense in X: Given any point x of X, every basis element containing x intersects D, so x belongs to $\bar{D}$. ∎

The two properties listed in Theorem 30.3 are sometimes taken as alternative countability axioms. A space for which every open covering contains a countable subcovering is called a *Lindelöf space* A space having a countable dense subset is often said to be *separable* (an unfortunate choice of terminology).[†] Weaker in general than the second countability axiom, each of these properties is equivalent to the second countability axiom when the space is metrizable (see Exercise 5). They are less important than the second countability axiom, but you should be aware of their existence, for they are sometimes useful. It is often easier, for instance, to show that a space X has a countable dense subset than it is to show that X has a countable basis. If the space is metrizable (as it usually is in analysis), it follows that X is second-countable as well.

We shall not use these properties to prove any theorems, but one of them—the Lindelöf condition—will be useful in dealing with some examples. They are not as well behaved as one might wish under the operations of taking subspaces and cartesian products, as we shall see in the examples and exercises that follow.

EXAMPLE 3.   *The space $\mathbb{R}_\ell$ satisfies all the countability axioms but the second.*

Given $x \in \mathbb{R}_\ell$, the set of all basis elements of the form $[x, x + 1/n)$ is a countable basis at x. And it is easy to see that the rational numbers are dense in $\mathbb{R}_\ell$.

To see that $\mathbb{R}_\ell$ has no countable basis, let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Choose for each x, an element B_x of $\mathcal{B}$ such that $x \in B_x \subset [x, x + 1)$. If $x \neq y$, then $B_x \neq B_y$, since $x = \inf B_x$ and $y = \inf B_y$. Therefore, $\mathcal{B}$ must be uncountable.

To show that $\mathbb{R}_\ell$ is Lindelöf requires more work. It will suffice to show that every open covering of $\mathbb{R}_\ell$ by basis elements contains a countable subcollection covering $\mathbb{R}_\ell$. (You can check this) So let

$$\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$$

be a covering of $\mathbb{R}$ by basis elements for the lower limit topology We wish to find a countable subcollection that covers $\mathbb{R}$.

Let C be the set

$$C = \bigcup_{\alpha \in J}(a_\alpha, b_\alpha),$$

which is a subset of $\mathbb{R}$. We show the set $\mathbb{R} - C$ is countable.

Let x be a point of $\mathbb{R} - C$. We know that x belongs to no open interval (a_α, b_α), therefore $x = a_\beta$ for some index β. Choose such a β and then choose q_x to be a rational number belonging to the interval (a_β, b_β). Because (a_β, b_β) is contained in C, so is the interval $(a_\beta, q_x) = (x, q_x)$. It follows that if x and y are two points of $\mathbb{R} - C$ with $x < y$, then $q_x < q_y$ (For otherwise, we would have $x < y < q_y \leq q_x$, so that y would lie in the interval (x, q_x) and hence in C.) Therefore the map $x \to q_x$ of $\mathbb{R} - C$ into $\mathbb{Q}$ is injective, so that $\mathbb{R} - C$ is countable.

Now we show that some countable subcollection of $\mathcal{A}$ covers $\mathbb{R}$. To begin, choose for each element of $\mathbb{R} - C$ an element of $\mathcal{A}$ containing it; one obtains a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ that covers $\mathbb{R} - C$. Now take the set C and topologize it as a subspace of $\mathbb{R}$; in this topology, C satisfies the second countability axiom. Now C is covered by the sets (a_α, b_α), which are open in $\mathbb{R}$ and hence open in C Then some countable subcollection

[†]This is a good example of how a word can be overused. We have already defined what we mean by a *separation* of a space; and we shall discuss the *separation* axioms shortly

covers C. Suppose this subcollection consists of the elements (a_α, b_α) for $\alpha = \alpha_1, \alpha_2, \ldots$. Then the collection

$$\mathcal{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \ldots\}$$

is a countable subcollection of $\mathcal{A}$ that covers the set C, and $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}_\ell$

EXAMPLE 4   *The product of two Lindelöf spaces need not be Lindelöf.* Although the space $\mathbb{R}_\ell$ is Lindelöf, we shall show that the product space $\mathbb{R}_\ell \times \mathbb{R}_\ell = \mathbb{R}_\ell^2$ is not. The space $\mathbb{R}_\ell^2$ is an extremely useful example in topology called the *Sorgenfrey plane*

The space $\mathbb{R}_\ell^2$ has as basis all sets of the form $[a, b) \times [c, d)$ To show it is not Lindelöf, consider the subspace

$$L = \{x \times (-x) \mid x \in \mathbb{R}_\ell\}$$

It is easy to check that L is closed in $\mathbb{R}_\ell^2$. Let us cover $\mathbb{R}_\ell^2$ by the open set $\mathbb{R}_\ell^2 - L$ and by all basis elements of the form

$$[a, b) \times [-a, d).$$

Each of these open sets intersects L in at most one point. Since L is uncountable, no countable subcollection covers $\mathbb{R}_\ell^2$ See Figure 30.1.

$a \times (-a)$ $[a,b) \times [-a,d)$

L

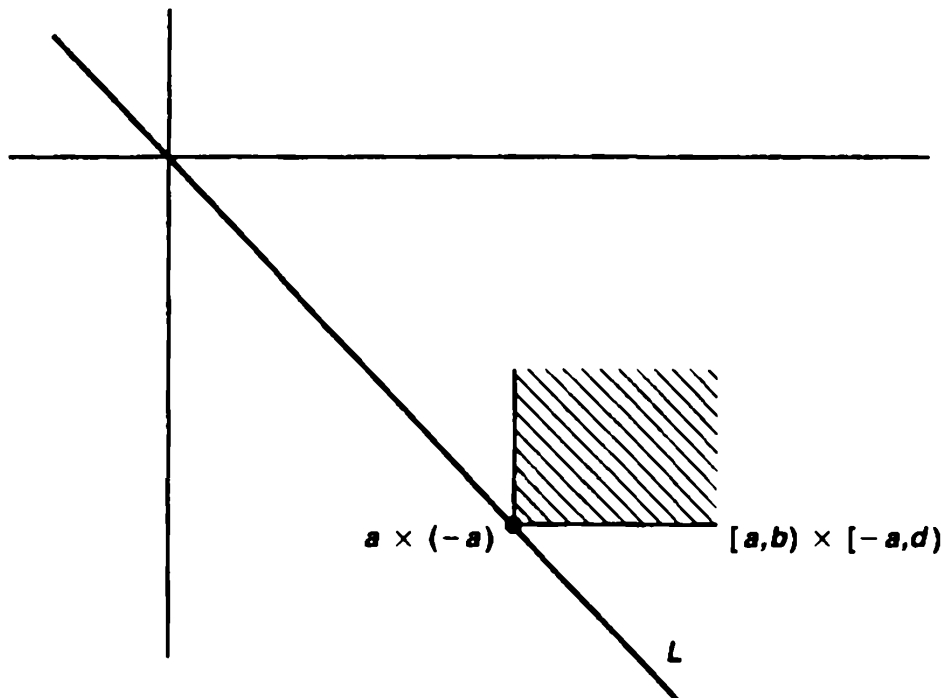*Figure 30.1*

EXAMPLE 5.   *A subspace of a Lindelöf space need not be Lindelöf.* The ordered square I_o^2 is compact; therefore it is Lindelöf, trivially. However, the subspace $A = I \times (0, 1)$ is not Lindelöf. For A is the union of the disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in A. This collection of sets is uncountable, and no proper subcollection covers A.

Exercises

1. (a) A G_δ *set* in a space X is a set A that equals a countable intersection of open sets of X. Show that in a first-countable T_1 space, every one-point set is a G_δ set.

   (b) There is a familiar space in which every one-point set is a G_δ set, which nevertheless does not satisfy the first countability axiom. What is it?

   The terminology here comes from the German. The "G" stands for "Gebiet," which means "open set," and the "δ" for "Durchschnitt," which means "intersection."

2. Show that if X has a countable basis $\{B_n\}$, then every basis $\mathcal{C}$ for X contains a countable basis for X. [*Hint:* For every pair of indices n, m for which it is possible, choose $C_{n,m} \in \mathcal{C}$ such that $B_n \subset C_{n,m} \subset B_m$.]

3. Let X have a countable basis; let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.

4. Show that every compact metrizable space X has a countable basis. [*Hint:* Let $\mathcal{A}_n$ be a finite covering of X by $1/n$-balls.]

5. (a) Show that every metrizable space with a countable dense subset has a countable basis.

   (b) Show that every metrizable Lindelöf space has a countable basis.

6. Show that $\mathbb{R}_\ell$ and I_o^2 are not metrizable.

7. Which of our four countability axioms does S_Ω satisfy? What about $\bar{S}_\Omega$?

8. Which of our four countability axioms does $\mathbb{R}^\omega$ in the uniform topology satisfy?

9. Let A be a closed subspace of X. Show that if X is Lindelöf, then A is Lindelöf. Show by example that if X has a countable dense subset, A need not have a countable dense subset.

10. Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

11. Let $f : X \to Y$ be continuous. Show that if X is Lindelöf, or if X has a countable dense subset, then $f(X)$ satisfies the same condition.

12. Let $f : X \to Y$ be a continuous open map. Show that if X satisfies the first or the second countability axiom, then $f(X)$ satisfies the same axiom.

13. Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

14. Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

15. Give $\mathbb{R}^I$ the uniform metric, where $I = [0, 1]$. Let $\mathcal{C}(I, \mathbb{R})$ be the subspace consisting of continuous functions. Show that $\mathcal{C}(I, \mathbb{R})$ has a countable dense subset, and therefore a countable basis. [*Hint:* Consider those continuous functions whose graphs consist of finitely many line segments with rational end points.]

16. (a) Show that the product space $\mathbb{R}^I$, where $I = [0, 1]$, has a countable dense subset.

(b) Show that if J has cardinality greater than $\mathcal{P}(\mathbb{Z}_+)$, then the product space $\mathbb{R}^J$ does not have a countable dense subset. [*Hint:* If D is dense in $\mathbb{R}^J$, define $f : J \to \mathcal{P}(D)$ by the equation $f(\alpha) = D \cap \pi_\alpha^{-1}((a, b))$, where (a, b) is a fixed interval in $\mathbb{R}$.]

*17. Give $\mathbb{R}^\omega$ the box topology. Let $\mathbb{Q}^\infty$ denote the subspace consisting of sequences of rationals that end in an infinite string of 0's. Which of our four countability axioms does this space satisfy?

*18. Let G be a first-countable topological group. Show that if G has a countable dense subset, or is Lindelöf, then G has a countable basis. [*Hint:* Let $\{B_n\}$ be a countable basis at e. If D is a countable dense subset of G, show the sets dB_n, for $d \in D$, form a basis for G. If G is Lindelöf, choose for each n a countable set C_n such that the sets cB_n, for $c \in C_n$, cover G. Show that as n ranges over $\mathbb{Z}_+$, these sets form a basis for G.]

§31   The Separation Axioms

In this section, we introduce three separation axioms and explore some of their properties. One you have already seen—the Hausdorff axiom. The others are similar but stronger. As always when we introduce new concepts, we shall examine the relationship between these axioms and the concepts introduced earlier in the book.

Recall that a space X is said to be *Hausdorff* if for each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.

Definition. Suppose that one-point sets are closed in X. Then X is said to be *regular* if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be *normal* if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

It is clear that a regular space is Hausdorff, and that a normal space is regular. (We need to include the condition that one-point sets be closed as part of the definition of regularity and normality in order for this to be the case. A two-point space in the indiscrete topology satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff.) For examples showing the regularity axiom stronger than the Hausdorff axiom, and normality stronger than regularity, see Examples 1 and 3.

These axioms are called separation axioms for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets. We have used the word "separation" before, of course, when we studied connected spaces. But in that case, we were trying to find disjoint open sets *whose union was the entire space*.

The present situation is quite different because the open sets need not satisfy this condition.
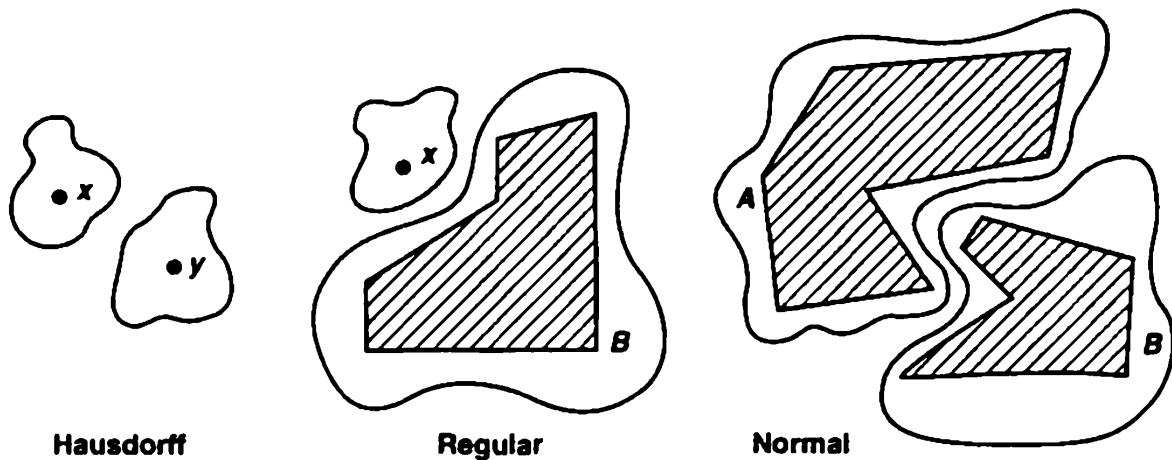

Hausdorff   **Regular**   **Normal**

Figure 31.1

The three separation axioms are illustrated in Figure 31.1.

There are other ways to formulate the separation axioms. One formulation that is sometimes useful is given in the following lemma:

Lemma 31.1. *Let X be a topological space. Let one-point sets in X be closed.*

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof. (a) Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X - U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B, respectively. The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\bar{V} \subset U$, as desired.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X - B$. By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively. Thus X is regular.

(b) This proof uses exactly the same argument; one just replaces the point x by the set A throughout. ∎

Now we relate the separation axioms with the concepts previously introduced.

Theorem 31.2. *(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof. (a) This result was an exercise in §17. We provide a proof here. Let X be Hausdorff. Let x and y be two points of the subspace Y of X. If U and V are disjoint neighborhoods in X of x and y, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y.

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some index β such that $x_\beta \neq y_\beta$. Choose disjoint open sets U and V in X_β containing x_β and y_β, respectively. Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively.

(b) Let Y be a subspace of the regular space X. Then one-point sets are closed in Y. Let x be a point of Y and let B be a closed subset of Y disjoint from x. Now $\bar{B} \cap Y = B$, where $\bar{B}$ denotes the closure of B in X. Therefore, $x \notin \bar{B}$, so, using regularity of X, we can choose disjoint open sets U and V of X containing x and $\bar{B}$, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

Let $\{X_\alpha\}$ be a family of regular spaces; let $X = \prod X_\alpha$. By (a), X is Hausdorff, so that one-point sets are closed in X. We use the preceding lemma to prove regularity of X. Let $\mathbf{x} = (x_\alpha)$ be a point of X and let U be a neighborhood of $\mathbf{x}$ in X. Choose a basis element $\prod U_\alpha$ about $\mathbf{x}$ contained in U. Choose, for each α, a neighborhood V_α of x_α in X_α such that $\bar{V}_\alpha \subset U_\alpha$; if it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of $\mathbf{x}$ in X. Since $\bar{V} = \prod \bar{V}_\alpha$ by Theorem 19.5, it follows at once that $\bar{V} \subset \prod U_\alpha \subset U$, so that X is regular. ∎

There is no analogous theorem for normal spaces, as we shall see shortly, in this section and the next.

EXAMPLE 1   *The space* $\mathbb{R}_K$ *is Hausdorff but not regular.* Recall that $\mathbb{R}_K$ denotes the reals in the topology having as basis all open intervals (a, b) and all sets of the form $(a, b) - K$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$. This space is Hausdorff, because any two distinct points have disjoint open intervals containing them.

But it is not regular. The set K is closed in $\mathbb{R}_K$, and it does not contain the point 0. Suppose that there exist disjoint open sets U and V containing 0 and K, respectively. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) - K$, since each basis element of the form (a, b) containing 0 intersects K. Choose n large enough that $1/n \in (a, b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d). Finally, choose z so that $z < 1/n$ and $z > \max\{c, 1/(n + 1)\}$. Then z belongs to both U and V, so they are not disjoint. See Figure 31.2
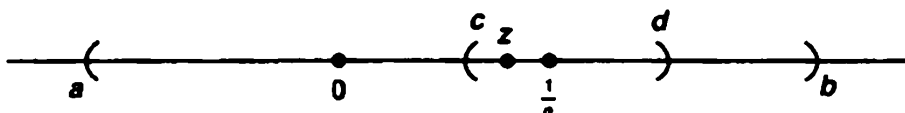

Figure 31.2

EXAMPLE 2.   *The space* $\mathbb{R}_\ell$ *is normal*   It is immediate that one-point sets are closed in $\mathbb{R}_\ell$, since the topology of $\mathbb{R}_\ell$ is finer than that of $\mathbb{R}$. To check normality, suppose that A and B are disjoint closed sets in $\mathbb{R}_\ell$ For each point a of A choose a basis element $[a, x_a)$ not intersecting B, and for each point b of B choose a basis element $[b, x_b)$ not intersecting A. The open sets

$$U = \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x_b)$$

are disjoint open sets about A and B, respectively.

EXAMPLE 3   *The Sorgenfrey plane* $\mathbb{R}_\ell^2$ *is not normal*

The space $\mathbb{R}_\ell$ is regular (in fact, normal), so the product space $\mathbb{R}_\ell^2$ is also regular. Thus this example serves two purposes. It shows that a regular space need not be normal, and it shows that the product of two normal spaces need not be normal

We suppose $\mathbb{R}_\ell^2$ is normal and derive a contradiction   Let L be the subspace of $\mathbb{R}_\ell^2$ consisting of all points of the form $x \times (-x)$. Then L is closed in $\mathbb{R}_\ell^2$, and L has the discrete topology. Hence every subset A of L, being closed in L, is closed in $\mathbb{R}_\ell^2$. Because $L - A$ is also closed in $\mathbb{R}_\ell^2$, this means that for every nonempty proper subset A of L, one can find disjoint open sets U_A and V_A containing A and $L - A$, respectively

Let D denote the set of points of $\mathbb{R}_\ell^2$ having rational coordinates; it is dense in $\mathbb{R}_\ell^2$. We define a map θ that assigns, to each subset of the line L, a subset of the set D, by setting

$$\theta(A) = D \cap U_A \quad \text{if } \varnothing \subsetneq A \subsetneq L,$$
$$\theta(\varnothing) = \varnothing,$$
$$\theta(L) = D.$$

We show that $\theta \cdot \mathscr{P}(L) \to \mathscr{P}(D)$ is injective.

Let A be a proper nonempty subset of L. Then $\theta(A) = D \cap U_A$ is neither empty (since U_A is open and D is dense in $\mathbb{R}_\ell^2$) nor all of D (since $D \cap V_A$ is nonempty). It remains to show that if B is another proper nonempty subset of L, then $\theta(A) \neq \theta(B)$.

One of the sets A, B contains a point not in the other; suppose that $x \in A$ and $x \notin B$. Then $x \in L - B$, so that $x \in U_A \cap V_B$; since the latter set is open and nonempty, it must contain points of D These points belong to U_A and not to U_B, therefore, $D \cap U_A \neq D \cap U_B$, as desired. Thus θ is injective

Now we show there exists an injective map $\phi : \mathscr{P}(D) \to L$. Because D is countably infinite and L has the cardinality of $\mathbb{R}$, it suffices to define an injective map ψ of $\mathscr{P}(\mathbb{Z}_+)$ into $\mathbb{R}$. For that, we let ψ assign to the subset S of $\mathbb{Z}_+$ the infinite decimal $.a_1 a_2 \ldots$, where $a_i = 0$ if $i \in S$ and $a_i = 1$ if $i \notin S$. That is,

$$\psi(S) = \sum_{i=1}^{\infty} a_i / 10^i$$

Now the composite

$$\mathscr{P}(L) \xrightarrow{\ \theta\ } \mathscr{P}(D) \xrightarrow{\ \psi\ } L$$

is an injective map of $\mathscr{P}(L)$ into L. But Theorem 7.8 tells us such a map does not exist! Thus we have reached a contradiction

This proof that $\mathbb{R}_\ell^2$ is not normal is in some ways not very satisfying. We showed only that there must exist some proper nonempty subset A of L such that the sets A and $B = L - A$ are not contained in disjoint open sets of $\mathbb{R}_\ell^2$. But we did not actually find such a set A. In fact, the set A of points of L having rational coordinates is such a set, but the proof is not easy. It is left to the exercises.

Exercises

1. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

2. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

3. Show that every order topology is regular.

4. Let X and X' denote a single set under two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; assume that $\mathcal{T}' \supset \mathcal{T}$. If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?

5. Let $f, g : X \to Y$ be continuous; assume that Y is Hausdorff. Show that $\{x \mid f(x) = g(x)\}$ is closed in X.

6. Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show there is a neighborhood W of y such that $p^{-1}(W) \subset U$.]

7. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a *perfect map*.)
   (a) Show that if X is Hausdorff, then so is Y.
   (b) Show that if X is regular, then so is Y.
   (c) Show that if X is locally compact, then so is Y.
   (d) Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.]

8. Let X be a space; let G be a topological group. An *action* of G on X is a continuous map $\alpha : G \times X \to X$ such that, denoting $\alpha(g \times x)$ by $g \cdot x$, one has:
   (i) $e \cdot x = x$ for all $x \in X$.

   (ii) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Define $x \sim g \cdot x$ for all x and g; the resulting quotient space is denoted X/G and called the *orbit space* of the action α.

Theorem. Let G be a compact topological group; let X be a topological space; let α be an action of G on X. If X is Hausdorff, or regular, or normal, or locally compact, or second-countable, so is X/G.

[*Hint:* See Exercise 13 of §26.]

*9. Let A be the set of all points of $\mathbb{R}_\ell^2$ of the form $x \times (-x)$, for x rational; let B be the set of all points of this form for x irrational. If V is an open set of $\mathbb{R}_\ell^2$ containing B, show there exists no open set U containing A that is disjoint from V, as follows:

(a) Let K_n consist of all irrational numbers x in $[0, 1]$ such that $[x, x + 1/n) \times [-x, -x + 1/n)$ is contained in V. Show $[0, 1]$ is the union of the sets K_n and countably many one-point sets.

(b) Use Exercise 5 of §27 to show that some set $\bar{K}_n$ contains an open interval (a, b) of $\mathbb{R}$.

(c) Show that V contains the open parallelogram consisting of all points of the form $x \times (-x + \epsilon)$ for which $a < x < b$ and $0 < \epsilon < 1/n$.

(d) Conclude that if q is a rational number with $a < q < b$, then the point $q \times (-q)$ of $\mathbb{R}_\ell^2$ is a limit point of V.

§32   Normal Spaces

Now we turn to a more thorough study of spaces satisfying the normality axiom. In one sense, the term "normal" is something of a misnomer, for normal spaces are not as well-behaved as one might wish. On the other hand, most of the spaces with which we are familiar do satisfy this axiom, as we shall see. Its importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results; we shall deal with them later in this chapter.

We begin by proving three theorems that give three important sets of hypotheses under which normality of a space is assured.

Theorem 32.1.   *Every regular space with a countable basis is normal.*

Proof.   Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each point x of A has a neighborhood U not intersecting B. Using regularity, choose a neighborhood V of x whose closure lies in U; finally, choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each x in A, we construct a countable covering of A by open sets whose closures do not intersect B. Since this covering of A is countable, we can index it with the positive integers; let us denote it by $\{U_n\}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B, such that each set $\bar{V}_n$ is disjoint from A. The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B, respectively, but they need not be disjoint. We perform the following simple trick to construct two open sets that *are* disjoint. Given n, define

$$U_n' = U_n - \bigcup_{i=1}^{n} \bar{V}_i \quad \text{and} \quad V_n' = V_n - \bigcup_{i=1}^{n} \bar{U}_i.$$

Note that each set U_n' is open, being the difference of an open set U_n and a closed set $\bigcup_{i=1}^{n} \bar{V}_i$. Similarly, each set V_n' is open. The collection $\{U_n'\}$ covers A, because each x in A belongs to U_n for some n, and x belongs to *none* of the sets $\bar{V}_i$. Similarly, the collection $\{V_n'\}$ covers B. See Figure 32.1.
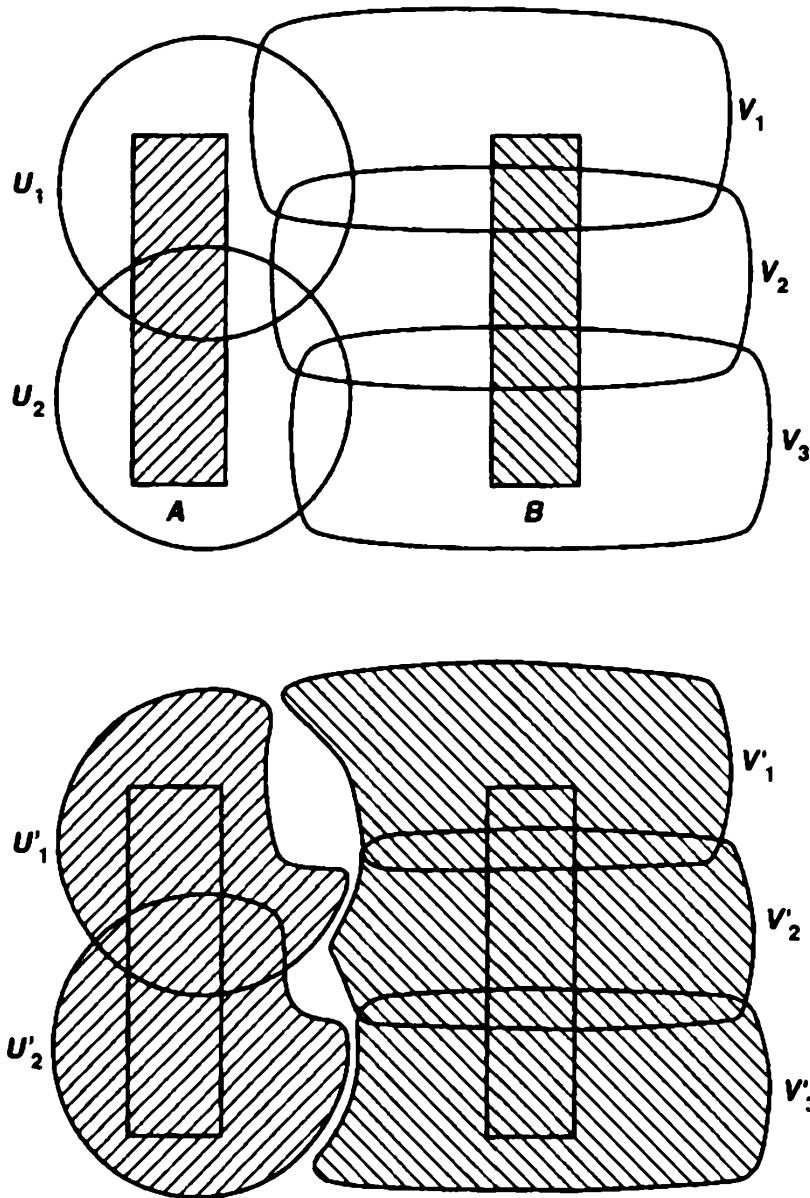
Figure 32.1

Finally, the open sets

$$U' = \bigcup_{n\in Z_+} U_n' \quad \text{and} \quad V' = \bigcup_{n\in Z_+} V_n'$$

are disjoint. For if $x \in U' \cap V'$, then $x \in U_j' \cap V_k'$ for some j and k. Suppose that $j \leq k$. It follows from the definition of U_j' that $x \in U_j$; and since $j \leq k$ it follows from the definition of V_k' that $x \notin \bar{U}_j$. A similar contradiction arises if $j \geq k$. ∎

Theorem 32.2. *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarly, for each b in B, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A. Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and V are open sets containing A and B, respectively; we assert they are disjoint. For if $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and some $b \in B$ The triangle inequality applies to show that $d(a, b) < (\epsilon_a + \epsilon_b)/2$. If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$, so that the ball $B(b, \epsilon_b)$ contains the point a. If $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point b. Neither situation is possible. ∎

Theorem 32.3. *Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. We have already essentially proved that X is regular. For if x is a point of X and B is a closed set in X not containing x, then B is compact, so that Lemma 26.4 applies to show there exist disjoint open sets about x and B, respectively.

Essentially the same argument as given in that lemma can be used to show that X is normal: Given disjoint closed sets A and B in X, choose, for each point a of A, disjoint open sets U_a and V_a containing a and B, respectively. (Here we use regularity of X.) The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \ldots, U_{a_m}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_m} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_m}$$

are disjoint open sets containing A and B, respectively. ∎

Here is a further result about normality that we shall find useful in dealing with some examples.

Theorem 32.4. *Every well-ordered set X is normal in the order topology.*

It is, in fact, true that *every* order topology is normal (see Example 39 of [S-S]); but we shall not have occasion to use this stronger result.

Proof. Let X be a well-ordered set. We assert that every interval of the form $(x, y]$ is open in X. If X has a largest element and y is that element, $(x, y]$ is just a basis element about y. If y is not the largest element of X, then $(x, y]$ equals the open set (x, y'), where y' is the immediate successor of y.

Now let A and B be disjoint closed sets in X; assume for the moment that neither A nor B contains the smallest element a_0 of X. For each $a \in A$, there exists a basis element about a disjoint from B; it contains some interval of the form $(x, a]$. (Here is where we use the fact that a is not the smallest element of X.) Choose, for each $a \in A$, such an interval $(x_a, a]$ disjoint from B. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from A. The sets

$$U = \bigcup_{a \in A} (x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B} (y_b, b]$$

are open sets containing A and B, respectively; we assert they are disjoint. For suppose that $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and some $b \in B$. Assume that $a < b$. Then if $a \leq y_b$, the two intervals are disjoint, while if $a > y_b$, we have $a \in (y_b, b]$, contrary to the fact that $(y_b, b]$ is disjoint from A. A similar contradiction occurs if $b < a$.

Finally, assume that A and B are disjoint closed sets in X, and A contains the smallest element a_0 of X. The set $\{a_0\}$ is both open and closed in X. By the result of the preceding paragraph, there exist disjoint open sets U and V containing the closed sets $A - \{a_0\}$ and B, respectively. Then $U \cup \{a_0\}$ and V are disjoint open sets containing A and B, respectively   ∎

EXAMPLE 1. *If J is uncountable, the product space $\mathbb{R}^J$ is not normal.* The proof is fairly difficult; we leave it as a challenging exercise (see Exercise 9).

This example serves three purposes. It shows that a regular space $\mathbb{R}^J$ need not be normal. It shows that a subspace of a normal space need not be normal, for $\mathbb{R}^J$ is homeomorphic to the subspace $(0, 1)^J$ of $[0, 1]^J$, which (assuming the Tychonoff theorem) is compact Hausdorff and therefore normal And it shows that an uncountable product of normal spaces need not be normal. It leaves unsettled the question as to whether a finite or a countable product of normal spaces might be normal.

EXAMPLE 2. *The product space $S_\Omega \times \bar{S}_\Omega$ is not normal.*[†]

Consider the well-ordered set $\bar{S}_\Omega$, in the order topology, and consider the subset S_Ω, in the subspace topology (which is the same as the order topology). Both spaces are normal, by Theorem 32.4. We shall show that the product space $S_\Omega \times \bar{S}_\Omega$ is not normal.

This example serves three purposes. First, it shows that a regular space need not be normal, for $S_\Omega \times \bar{S}_\Omega$ is a product of regular spaces and therefore regular. Second, it shows that a subspace of a normal space need not be normal, for $S_\Omega \times \bar{S}_\Omega$ is a subspace of $\bar{S}_\Omega \times \bar{S}_\Omega$, which is a compact Hausdorff space and therefore normal Third, it shows that the product of two normal spaces need not be normal.

First, we consider the space $\bar{S}_\Omega \times \bar{S}_\Omega$, and its "diagonal" $\Delta = \{x \times x \mid x \in \bar{S}_\Omega\}$. Because $\bar{S}_\Omega$ is Hausdorff, Δ is closed in $\bar{S}_\Omega \times \bar{S}_\Omega$. If U and V are disjoint neighborhoods of x and y, respectively, then $U \times V$ is a neighborhood of $x \times y$ that does not intersect Δ. Therefore, in the subspace $S_\Omega \times \bar{S}_\Omega$, the set

$$A = \Delta \cap (S_\Omega \times \bar{S}_\Omega) = \Delta - \{\Omega \times \Omega\}$$

[†] Kelley [K] attributes this example to J. Dieudonné and A. P Morse independently
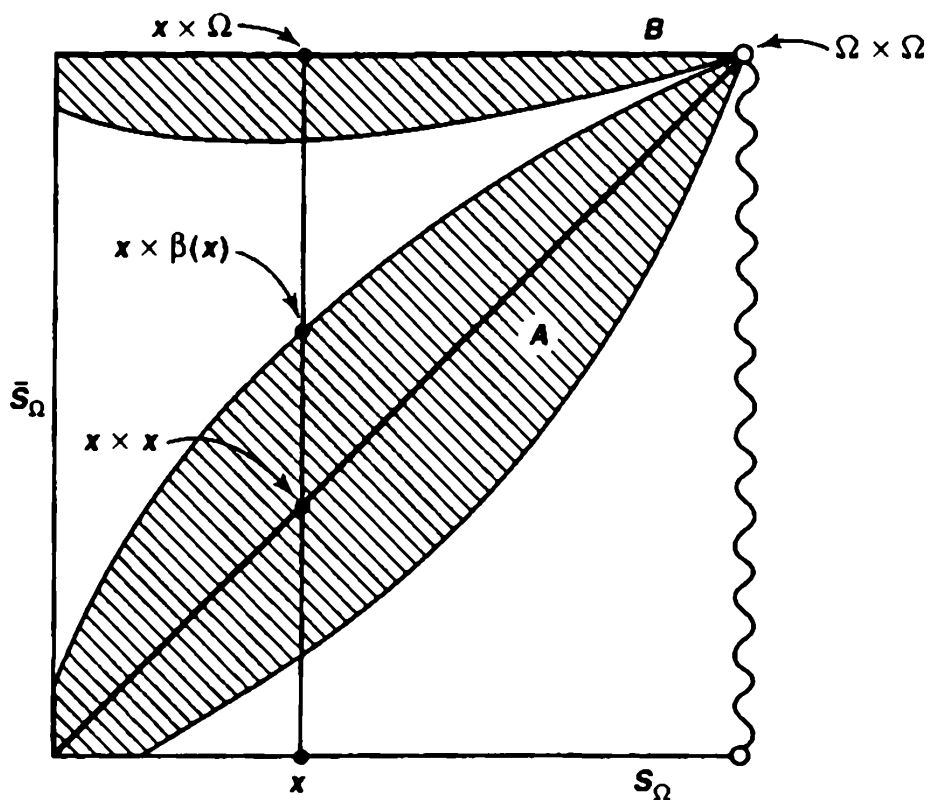
Figure 32.2

is closed. Likewise, the set

$$B = S_\Omega \times \{\Omega\}$$

is closed in $S_\Omega \times \bar{S}_\Omega$, being a "slice" of this product space. The sets A and B are disjoint. We shall assume there exist disjoint open sets U and V of $S_\Omega \times \bar{S}_\Omega$ containing A and B, respectively, and derive a contradiction. See Figure 32.2.

Given $x \in S_\Omega$, consider the vertical slice $x \times \bar{S}_\Omega$. We assert that there is some point β with $x < \beta < \Omega$ such that $x \times \beta$ lies outside U For if U contained all points $x \times \beta$ for $x < \beta < \Omega$, then the top point $x \times \Omega$ of the slice would be a limit point of U, which it is not because V is an open set disjoint from U containing this top point

Choose $\beta(x)$ to be such a point; just to be definite, let $\beta(x)$ be the *smallest* element of S_Ω such that $x < \beta(x) < \Omega$ and $x \times \beta(x)$ lies outside U. Define a sequence of points of S_Ω as follows: Let x_1 be any point of S_Ω Let $x_2 = \beta(x_1)$, and in general, $x_{n+1} = \beta(x_n)$ We have

$$x_1 < x_2 < \ldots,$$

because $\beta(x) > x$ for all x. The set $\{x_n\}$ is countable and therefore has an upper bound in S_Ω; let $b \in S_\Omega$ be its least upper bound. Because the sequence is increasing, it must converge to its least upper bound; thus $x_n \to b$ But $\beta(x_n) = x_{n+1}$, so that $\beta(x_n) \to b$ also. Then

$$x_n \times \beta(x_n) \longrightarrow b \times b$$

in the product space. See Figure 32.3. Now we have a contradiction, for the point $b \times b$ lies in the set A, which is contained in the open set U; and U contains *none* of the points $x_n \times \beta(x_n)$.
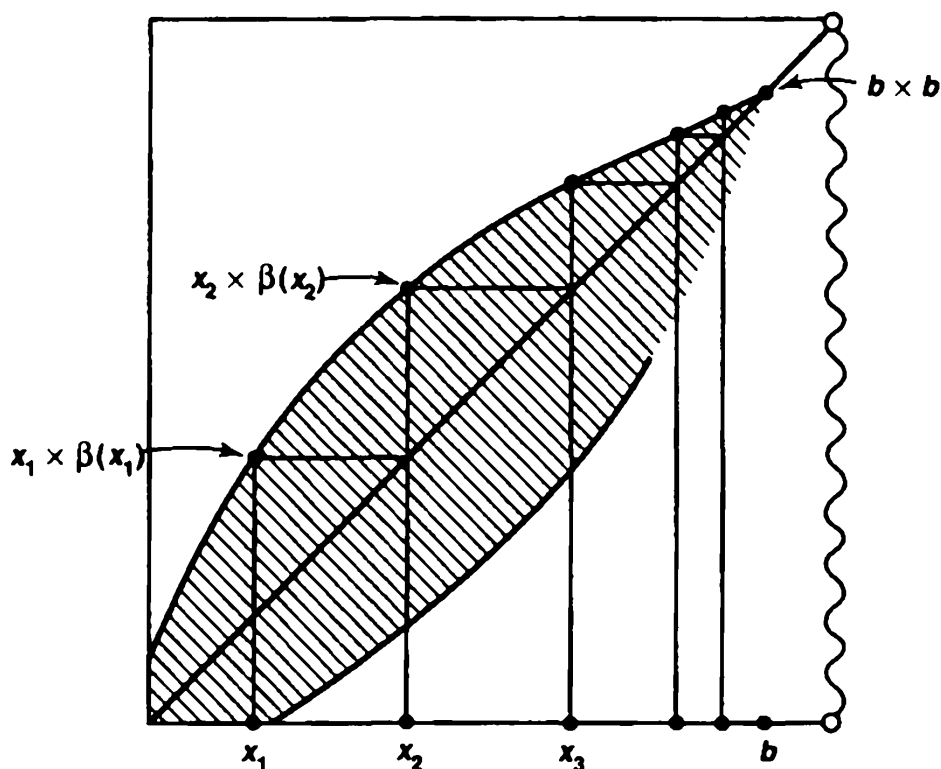
Figure 32.3

Exercises

1. Show that a closed subspace of a normal space is normal.

2. Show that if $\prod X_\alpha$ is Hausdorff, or regular, or normal, then so is X_α. (Assume that each X_α is nonempty.)

3. Show that every locally compact Hausdorff space is regular.

4. Show that every regular Lindelöf space is normal.

5. Is $\mathbb{R}^\omega$ normal in the product topology? In the uniform topology?

   It is not known whether $\mathbb{R}^\omega$ is normal in the box topology. Mary-Ellen Rudin has shown that the answer is affirmative if one assumes the continuum hypothesis [RM]. In fact, she shows it satisfies a stronger condition called *paracompactness*.

6. A space X is said to be *completely normal* if every subspace of X is normal. Show that X is completely normal if and only if for every pair A, B of separated sets in X (that is, sets such that $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$), there exist disjoint open sets containing them. [*Hint:* If X is completely normal, consider $X - (\bar{A} \cap \bar{B})$.]

7. Which of the following spaces are completely normal? Justify your answers.
   (a) A subspace of a completely normal space.
   (b) The product of two completely normal spaces.
   (c) A well-ordered set in the order topology.
   (d) A metrizable space.

(e) A compact Hausdorff space.

(f) A regular space with a countable basis.

(g) The space $\mathbb{R}_\ell$.

*8. Prove the following:

Theorem. Every linear continuum X is normal.

(a) Let C be a nonempty closed subset of X. If U is a component of $X-C$, show that U is a set of the form (c, c') or (c, ∞) or $(-\infty, c)$, where $c, c' \in C$.

(b) Let A and B be closed disjoint subsets of X. For each component W of $X - A \cup B$ that is an open interval with one end point in A and the other in B, choose a point c_W of W. Show that the set C of the points c_W is closed.

(c) Show that if V is a component of $X - C$, then V does not intersect both A and B.

*9. Prove the following:

Theorem. If J is uncountable, then $\mathbb{R}^J$ is not normal.

Proof. (This proof is due to A. H. Stone, as adapted in [S-S].) Let $X = (\mathbb{Z}_+)^J$; it will suffice to show that X is not normal, since X is a closed subspace of $\mathbb{R}^J$. We use functional notation for the elements of X, so that the typical element of X is a function $\mathbf{x} : J \to \mathbb{Z}_+$

(a) If $\mathbf{x} \in X$ and if B is a finite subset of J, let $U(\mathbf{x}, B)$ denote the set consisting of all those elements $\mathbf{y}$ of X such that $\mathbf{y}(\alpha) = \mathbf{x}(\alpha)$ for $\alpha \in B$. Show the sets $U(\mathbf{x}, B)$ are a basis for X.

(b) Define P_n to be the subset of X consisting of those $\mathbf{x}$ such that on the set $J - \mathbf{x}^{-1}(n)$, the map $\mathbf{x}$ is injective. Show that P_1 and P_2 are closed and disjoint.

(c) Suppose U and V are open sets containing P_1 and P_2, respectively. Given a sequence $\alpha_1, \alpha_2, \ldots$ of distinct elements of J, and a sequence

$$0 = n_0 < n_1 < n_2 < \cdots$$

of integers, for each $i \geq 1$ let us set

$$B_i = \{\alpha_1, \cdots, \alpha_{n_i}\}$$

and define $\mathbf{x}_i \in X$ by the equations

$$\mathbf{x}_i(\alpha_j) = j \quad \text{for } 1 \leq j \leq n_{i-1},$$
$$\mathbf{x}_i(\alpha) = 1 \quad \text{for all other values of } \alpha.$$

Show that one can choose the sequences α_j and n_j so that for each i, one has the inclusion

$$U(\mathbf{x}_i, B_i) \subset U.$$

[*Hint:* To begin, note that $\mathbf{x}_1(\alpha) = 1$ for all α; now choose B_1 so that $U(\mathbf{x}_1, B_1) \subset U$.]

(d) Let A be the set $\{\alpha_1, \alpha_2, \ldots \}$ constructed in (c) Define $y \cdot J \to \mathbb{Z}_+$ by the equations

$$y(\alpha_j) = j \quad \text{for } \alpha_j \in A,$$
$$y(\alpha) = 2 \quad \text{for all other values of } \alpha.$$

Choose B so that $U(y, B) \subset V$. Then choose i so that $B \cap A$ is contained in the set B_i. Show that

$$U(x_{i+1}, B_{i+1}) \cap U(y, B)$$

is not empty.

10. Is every topological group normal?

§33 The Urysohn Lemma

Now we come to the first deep theorem of the book, a theorem that is commonly called the "Urysohn lemma." It asserts the existence of certain real-valued continuous functions on a normal space X. It is the crucial tool used in proving a number of important theorems. We shall prove three of them—the Urysohn metrization theorem, the Tietze extension theorem, and an imbedding theorem for manifolds—in succeeding sections of this chapter.

Why do we call the Urysohn lemma a "deep" theorem? Because its proof involves a really original idea, which the previous proofs did not. Perhaps we can explain what we mean this way: By and large, one would expect that if one went through this book and deleted all the proofs we have given up to now and then handed the book to a bright student who had not studied topology, that student ought to be able to go through the book and work out the proofs independently. (It would take a good deal of time and effort, of course; and one would not expect the student to handle the trickier examples.) But the Urysohn lemma is on a different level. It would take considerably more originality than most of us possess to prove this lemma unless we were given copious hints!

Theorem 33.1 (Urysohn lemma). *Let X be a normal space, let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map*

$$f : X \longrightarrow [a, b]$$

such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Proof. We need consider only the case where the interval in question is the interval $[0, 1]$; the general case follows from that one. The first step of the proof is to construct, using normality, a certain family U_p of open sets of X, indexed by the rational numbers. Then one uses these sets to define the continuous function f.

Step 1. Let P be the set of all rational numbers in the interval $[0, 1]$ [†] We shall define, for each p in P, an open set U_p of X, in such a way that whenever $p < q$, we have

$$\bar{U}_p \subset U_q$$

Thus, the sets U_p will be simply ordered by inclusion in the same way their subscripts are ordered by the usual ordering in the real line.

Because P is countable, we can use induction to define the sets U_p (or rather, the principle of recursive definition). Arrange the elements of P in an infinite sequence in some way; for convenience, let us suppose that the numbers 1 and 0 are the first two elements of the sequence.

Now define the sets U_p, as follows. First, define $U_1 = X - B$. Second, because A is a closed set contained in the open set U_1, we may by normality of X choose an open set U_0 such that

$$A \subset U_0 \quad \text{and} \quad \bar{U}_0 \subset U_1$$

In general, let P_n denote the set consisting of the first n rational numbers in the sequence. Suppose that U_p is defined for all rational numbers p belonging to the set P_n, satisfying the condition

$$(*) \qquad\qquad\qquad p < q \Longrightarrow \bar{U}_p \subset U_q.$$

Let r denote the next rational number in the sequence; we wish to define U_r

Consider the set $P_{n+1} = P_n \cup \{r\}$. It is a finite subset of the interval $[0, 1]$, and, as such, it has a simple ordering derived from the usual order relation $<$ on the real line. In a finite simply ordered set, every element (other than the smallest and the largest) has an immediate predecessor and an immediate successor. (See Theorem 10.1) The number 0 is the smallest element, and 1 is the largest element, of the simply ordered set P_{n+1}, and r is neither 0 nor 1 So r has an immediate predecessor p in P_{n+1} and an immediate successor q in P_{n+1} The sets U_p and U_q are already defined, and $\bar{U}_p \subset U_q$ by the induction hypothesis. Using normality of X, we can find an open set U_r of X such that

$$\bar{U}_p \subset U_r \quad \text{and} \quad \bar{U}_r \subset U_q.$$

We assert that $(*)$ now holds for every pair of elements of P_{n+1}. If both elements lie in P_n, $(*)$ holds by the induction hypothesis. If one of them is r and the other is a point s of P_n, then either $s \leq p$, in which case

$$\bar{U}_s \subset \bar{U}_p \subset U_r,$$

or $s \geq q$, in which case

$$\bar{U}_r \subset U_q \subset U_s.$$

[†]Actually, any countable dense subset of $[0, 1]$ will do, providing it contains the points 0 and 1.

Thus, for every pair of elements of P_{n+1}, relation ($*$) holds.

By induction, we have U_p defined for all $p \in P$.

To illustrate, let us suppose we started with the standard way of arranging the elements of P in an infinite sequence

$$P = \{1, 0, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \quad \}$$

After defining U_0 and U_1, we would define $U_{1/2}$ so that $\bar{U}_0 \subset U_{1/2}$ and $\bar{U}_{1/2} \subset U_1$ Then we would fit in $U_{1/3}$ between U_0 and $U_{1/2}$, and $U_{2/3}$ between $U_{1/2}$ and U_1. And so on. At the eighth step of the proof we would have the situation pictured in Figure 33 1 And the ninth step would consist of choosing an open set $U_{2/5}$ to fit in between $U_{1/3}$ and $U_{1/2}$ And so on
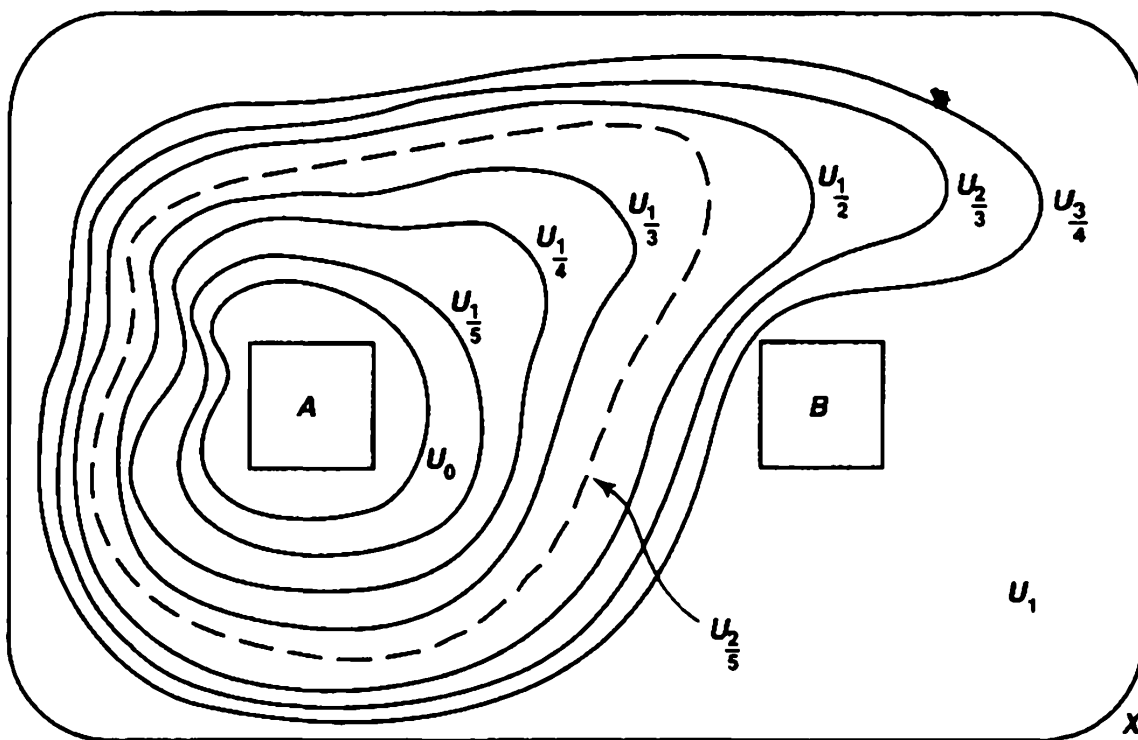
Figure 33.1

Step 2. Now we have defined U_p for all rational numbers p in the interval $[0, 1]$. We extend this definition to all rational numbers p in $\mathbb{R}$ by defining

$$U_p = \varnothing \quad \text{if } p < 0,$$
$$U_p = X \quad \text{if } p > 1$$

It is still true (as you can check) that for any pair of rational numbers p and q,

$$p < q \implies \bar{U}_p \subset U_q$$

Step 3. Given a point x of X, let us define $\mathbb{Q}(x)$ to be the set of those rational numbers p such that the corresponding open sets U_p contain x

$$\mathbb{Q}(x) = \{p \mid x \in U_p\}.$$

This set contains no number less than 0, since no x is in U_p for $p < 0$. And it contains every number greater than 1, since every x is in U_p for $p > 1$. Therefore, $\mathbb{Q}(x)$ is bounded below, and its greatest lower bound is a point of the interval $[0, 1]$. Define

$$f(x) = \inf \mathbb{Q}(x) = \inf\{p \mid x \in U_p\}.$$

Step 4 We show that f is the desired function. If $x \in A$, then $x \in U_p$ for every $p \geq 0$, so that $\mathbb{Q}(x)$ equals the set of all nonnegative rationals, and $f(x) = \inf \mathbb{Q}(x) = 0$. Similarly, if $x \in B$, then $x \in U_p$ for no $p \leq 1$, so that $\mathbb{Q}(x)$ consists of all rational numbers greater than 1, and $f(x) = 1$.

All this is easy. The only hard part is to show that f is continuous. For this purpose, we first prove the following elementary facts:

(1) $x \in \bar{U}_r \Rightarrow f(x) \leq r$

(2) $x \notin U_r \Rightarrow f(x) \geq r$.

To prove (1), note that if $x \in \bar{U}_r$, then $x \in U_s$ for every $s > r$. Therefore, $\mathbb{Q}(x)$ contains all rational numbers greater than r, so that by definition we have

$$f(x) = \inf \mathbb{Q}(x) \leq r$$

To prove (2), note that if $x \notin U_r$, then x is not in U_s for any $s < r$. Therefore, $\mathbb{Q}(x)$ contains no rational numbers less than r, so that

$$f(x) = \inf \mathbb{Q}(x) \geq r.$$

Now we prove continuity of f. Given a point x_0 of X and an open interval (c, d) in $\mathbb{R}$ containing the point $f(x_0)$, we wish to find a neighborhood U of x_0 such that $f(U) \subset (c, d)$. Choose rational numbers p and q such that

$$c < p < f(x_0) < q < d.$$

We assert that the open set

$$U = U_q - \bar{U}_p$$

is the desired neighborhood of x_0. See Figure 33.2.
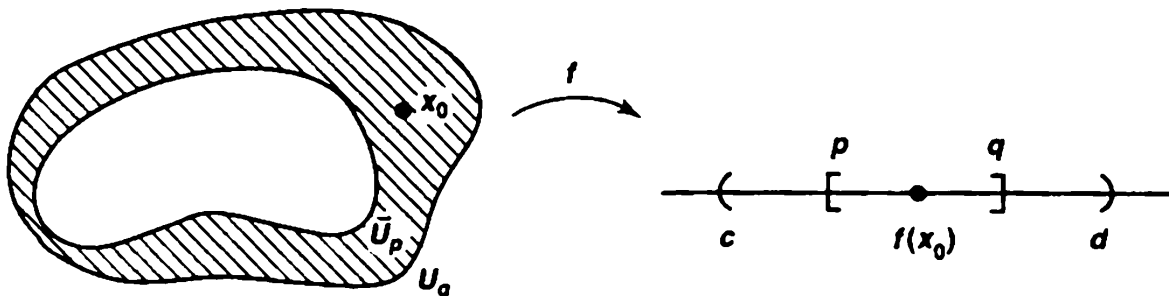
Figure 33.2

First, we note that $x_0 \in U$ For the fact that $f(x_0) < q$ implies by condition (2) that $x_0 \in U_q$, while the fact that $f(x_0) > p$ implies by (1) that $x_0 \notin \bar{U}_p$.

Second, we show that $f(U) \subset (c, d)$. Let $x \in U$. Then $x \in U_q \subset \bar{U}_q$, so that $f(x) \leq q$, by (1). And $x \notin \bar{U}_p$, so that $x \notin U_p$ and $f(x) \geq p$, by (2). Thus, $f(x) \in [p, q] \subset (c, d)$, as desired. ∎

Definition.   If A and B are two subsets of the topological space X, and if there is a continuous function $f . X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B *can be separated by a continuous function.*

The Urysohn lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is trivial, for if $f : X \to [0, 1]$ is the function, then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets containing A and B, respectively

This fact leads to a question that may already have occurred to you: Why cannot the proof of the Urysohn lemma be generalized to show that in a regular space, where you can separate points from closed sets by disjoint open sets, you can also separate points from closed sets by continuous functions?

At first glance, it seems that the proof of the Urysohn lemma should go through. You take a point a and a closed set B not containing a, and you begin the proof just as before by defining $U_1 = X - B$ and choosing U_0 to be an open set about a whose closure is contained in U_1 (using regularity of X). But at the very next step of the proof, you run into difficulty. Suppose that p is the next rational number in the sequence after 0 and 1. You want to find an open set U_p such that $\bar{U}_0 \subset U_p$ and $\bar{U}_p \subset U_1$. For this, regularity is not enough.

Requiring that one be able to separate a point from a closed set by a continuous function is, in fact, a stronger condition than requiring that one can separate them by disjoint open sets. We make this requirement into a new separation axiom:

Definition.   A space X is *completely regular* if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

A normal space is completely regular, by the Urysohn lemma, and a completely regular space is regular, since given f, the sets $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets about A and x_0, respectively As a result, this new axiom fits in between regularity and normality in the list of separation axioms. Note that in the definition one could just as well require the function to map x_0 to 0, and A to $\{1\}$, for $g(x) = 1 - f(x)$ satisfies this condition. But our definition is at times a bit more convenient.

In the early years of topology, the separation axioms, listed in order of increasing strength, were labelled T_1, T_2 (Hausdorff), T_3 (regular), T_4 (normal), and T_5 (completely normal), respectively. The letter "T" comes from the German "Trennungsaxiom," which means "separation axiom." Later, when the notion of complete regularity was introduced, someone suggested facetiously that it should be called the "T-$3\frac{1}{2}$ axiom," since it lies between regularity and normality. This terminology is in fact sometimes used in the literature[1]

Unlike normality, this new separation axiom is nicely behaved with regard to subspaces and products:

Theorem 33.2.   *A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.*

Proof.   Let X be completely regular; let Y be a subspace of X. Let x_0 be a point of Y, and let A be a closed set of Y disjoint from x_0. Now $A = \bar{A} \cap Y$, where $\bar{A}$ denotes the closure of A in X. Therefore, $x_0 \notin \bar{A}$. Since X is completely regular, we can choose a continuous function $f : X \rightarrow [0, 1]$ such that $f(x_0) = 1$ and $f(\bar{A}) = \{0\}$. The restriction of f to Y is the desired continuous function on Y.

Let $X = \prod X_\alpha$ be a product of completely regular spaces. Let $\mathbf{b} = (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from $\mathbf{b}$. Choose a basis element $\prod U_\alpha$ containing $\mathbf{b}$ that does not intersect A; then $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha = \alpha_1, \ldots, \alpha_n$. Given $i = 1, \ldots, n$, choose a continuous function

$$f_i : X_{\alpha_i} \rightarrow [0, 1]$$

such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X - U_{\alpha_i}) = \{0\}$. Let $\phi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$; then ϕ_i maps X continuously into $\mathbb{R}$ and vanishes outside $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$. The product

$$f(\mathbf{x}) = \phi_1(\mathbf{x}) \quad \phi_2(\mathbf{x}) \cdot \quad \cdots \quad \phi_n(\mathbf{x})$$

is the desired continuous function on X, for it equals 1 at $\mathbf{b}$ and vanishes outside $\prod U_\alpha$.   ∎

EXAMPLE 1.   The spaces $\mathbb{R}^2_\ell$ and $S_\Omega \times \bar{S}_\Omega$ are completely regular but not normal. For they are products of spaces that are completely regular (in fact, normal).

A space that is regular but not completely regular is much harder to find. Most of the examples that have been constructed for this purpose are difficult, and require considerable familiarity with cardinal numbers. Fairly recently, however, John Thomas [T] has constructed a much more elementary example, which we outline in Exercise 11.

Exercises

1. Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p > r} U_p - \bigcup_{q < r} U_q,$$

   p, q rational.

2. (a) Show that a connected normal space having more than one point is uncountable.

   (b) Show that a connected regular space having more than one point is uncountable.[†] [*Hint:* Any countable space is Lindelöf.]

3. Give a direct proof of the Urysohn lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

[†]Surprisingly enough, there does exist a connected *Hausdorff* space that is countably infinite. See Example 75 of [S-S].

4. Recall that A is a "G_δ set" in X if A is the intersection of a countable collection of open sets of X.

Theorem. *Let X be normal. There exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.*

A function satisfying the requirements of this theorem is said to **vanish precisely on** A.

5. Prove:

Theorem *(Strong form of the Urysohn lemma). Let X be a normal space. There is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise, if and only if A and B are disjoint closed G_δ sets in X.*

6. A space X is said to be **perfectly normal** if X is normal and if every closed set in X is a G_δ set in X.

(a) Show that every metrizable space is perfectly normal.

(b) Show that a perfectly normal space is completely normal. For this reason the condition of perfect normality is sometimes called the "T_6 axiom." [*Hint:* Let A and B be separated sets in X. Choose continuous functions $f, g : X \to [0, 1]$ that vanish precisely on $\bar{A}$ and $\bar{B}$, respectively. Consider the function $f - g$.]

(c) There is a familiar space that is completely normal but not perfectly normal. What is it?

7. Show that every locally compact Hausdorff space is completely regular.

8. Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

9. Show that $\mathbb{R}^J$ in the box topology is completely regular. [*Hint:* Show that it suffices to consider the case where the box neighborhood $(-1, 1)^J$ is disjoint from A and the point is the origin. Then use the fact that a function continuous in the uniform topology is also continuous in the box topology.]

*10. Prove the following:

Theorem. *Every topological group is completely regular.*

Proof. Let V_0 be a neighborhood of the identity element e, in the topological group G. In general, choose V_n to be a neighborhood of e such that $V_n \cdot V_n \subset V_{n-1}$. Consider the set of all dyadic rationals p, that is, all rational numbers of the form $k/2^n$, with k and n integers. For each dyadic rational p in $(0, 1]$, define an open set $U(p)$ inductively as follows: $U(1) = V_0$ and $U(\frac{1}{2}) = V_1$. Given n, if $U(k/2^n)$ is defined for $0 < k/2^n \leq 1$, define

$$U(1/2^{n+1}) = V_{n+1},$$
$$U((2k + 1)/2^{n+1}) = V_{n+1} \cdot U(k/2^n)$$

for $0 < k < 2^n$. For $p \le 0$, let $U(p) = \varnothing$, and for $p > 1$, let $U(p) = G$. Show that

$$V_n \quad U(k/2^n) \subset U((k+1)/2^n)$$

for all k and n. Proceed as in the Urysohn lemma.

This exercise is adapted from [M-Z], to which the reader is referred for further results on topological groups.

*11. Define a set X as follows: For each even integer m, let L_m denote the line segment $m \times [-1, 0]$ in the plane. For each odd integer n and each integer $k \ge 2$, let $C_{n,k}$ denote the union of the line segments $(n + 1 - 1/k) \times [-1, 0]$ and $(n - 1 + 1/k) \times [-1, 0]$ and the semicircle

$$\{x \times y \mid (x - n)^2 + y^2 = (1 - 1/k)^2 \text{ and } y \ge 0\}$$

in the plane. Let $p_{n,k}$ denote the topmost point $n \times (1 - 1/k)$ of this semicircle. Let X be the union of all the sets L_m and $C_{n,k}$, along with two extra points a and b. Topologize X by taking sets of the following four types as basis elements:

(i) The intersection of X with a horizontal open line segment that contains none of the points $p_{n,k}$.

(ii) A set formed from one of the sets $C_{n,k}$ by deleting finitely many points

(iii) For each even integer m, the union of $\{a\}$ and the set of points $x \times y$ of X for which $x < m$.

(iv) For each even integer m, the union of $\{b\}$ and the set of points $x \times y$ of X for which $x > m$.

(a) Sketch X; show that these sets form a basis for a topology on X.

(b) Let f be a continuous real-valued function on X. Show that for any c, the set $f^{-1}(c)$ is a G_δ set in X. (This is true for any space X.) Conclude that the set $S_{n,k}$ consisting of those points p of $C_{n,k}$ for which $f(p) \ne f(p_{n,k})$ is countable. Choose $d \in [-1, 0]$ so that the line $y = d$ intersects none of the sets $S_{n,k}$. Show that for n odd,

$$f((n - 1) \times d) = \lim_{k \to \infty} f(p_{n,k}) = f((n + 1) \times d).$$

Conclude that $f(a) = f(b)$.

(c) Show that X is regular but not completely regular.

§34   The Urysohn Metrization Theorem

Now we come to the major goal of this chapter, a theorem that gives us conditions under which a topological space is metrizable. The proof weaves together a number of strands from previous parts of the book; it uses results on metric topologies from Chapter 2 as well as facts concerning the countability and separation axioms proved in

the present chapter. The basic construction used in the proof is a simple one, but very useful. You will see it several times more in this book, in various guises.

There are two versions of the proof, and since each has useful generalizations that will appear subsequently, we present both of them here. The first version generalizes to give an imbedding theorem for completely regular spaces. The second version will be generalized in Chapter 6 when we prove the Nagata-Smirnov metrization theorem.

Theorem 34.1 (Urysohn metrization theorem). *Every regular space X with a countable basis is metrizable.*

Proof. We shall prove that X is metrizable by imbedding X in a metrizable space Y, that is, by showing X homeomorphic with a subspace of Y. The two versions of the proof differ in the choice of the metrizable space Y. In the first version, Y is the space $\mathbb{R}^\omega$ in the product topology, a space that we have previously proved to be metrizable (Theorem 20.5) In the second version, the space Y is also $\mathbb{R}^\omega$, but this time in the topology given by the uniform metric $\bar\rho$ (see §20). In each case, it turns out that our construction actually imbeds X in the subspace $[0, 1]^\omega$ of $\mathbb{R}^\omega$

Step 1 We prove the following: *There exists a countable collection of continuous functions $f_n : X \to [0, 1]$ having the property that given any point x_0 of X and any neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes outside U.*

It is a consequence of the Urysohn lemma that, given x_0 and U, there exists such a function. However, if we choose one such function for each pair (x_0, U), the resulting collection will not in general be countable. Our task is to cut the collection down to size. Here is one way to proceed:

Let $\{B_n\}$ be a countable basis for X. For each pair n, m of indices for which $\bar B_n \subset B_m$, apply the Urysohn lemma to choose a continuous function $g_{n,m} . X \to [0, 1]$ such that $g_{n,m}(\bar B_n) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$. Then the collection $\{g_{n,m}\}$ satisfies our requirement: Given x_0 and given a neighborhood U of x_0, one can choose a basis element B_m containing x_0 that is contained in U. Using regularity, one can then choose B_n so that $x_0 \in B_n$ and $\bar B_n \subset B_m$. Then n, m is a pair of indices for which the function $g_{n,m}$ is defined; and it is positive at x_0 and vanishes outside U. Because the collection $\{g_{n,m}\}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it is countable; therefore it can be reindexed with the positive integers, giving us the desired collection $\{f_n\}$.

Step 2 (First version of the proof) Given the functions f_n of Step 1, take $\mathbb{R}^\omega$ in the product topology and define a map $F . X \to \mathbb{R}^\omega$ by the rule

$$F(x) = (f_1(x), f_2(x), . . .).$$

We assert that F is an imbedding.

First, F is continuous because $\mathbb{R}^\omega$ has the product topology and each f_n is continuous. Second, F is injective because given $x \neq y$, we know there is an index n such that $f_n(x) > 0$ and $f_n(y) = 0$; therefore, $F(x) \neq F(y)$

Finally, we must prove that F is a homeomorphism of X onto its image, the subspace $Z = F(X)$ of $\mathbb{R}^\omega$. We know that F defines a continuous bijection of X with Z,

so we need only show that for each open set U in X, the set $F(U)$ is open in Z. Let z_0 be a point of $F(U)$. We shall find an open set W of Z such that

$$z_0 \in W \subset F(U).$$

Let x_0 be the point of U such that $F(x_0) = z_0$. Choose an index N for which $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$. Take the open ray $(0, +\infty)$ in $\mathbb{R}$, and let V be the open set

$$V = \pi_N^{-1}((0, +\infty))$$

of $\mathbb{R}^\omega$. Let $W = V \cap Z$; then W is open in Z, by definition of the subspace topology. See Figure 34.1. We assert that $z_0 \in W \subset F(U)$. First, $z_0 \in W$ because

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0.$$

Second, $W \subset F(U)$. For if $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0, +\infty)$. Since $\pi_N(z) = \pi_N(F(x)) = f_N(x)$, and f_N vanishes outside U, the point x must be in U. Then $z = F(x)$ is in $F(U)$, as desired.

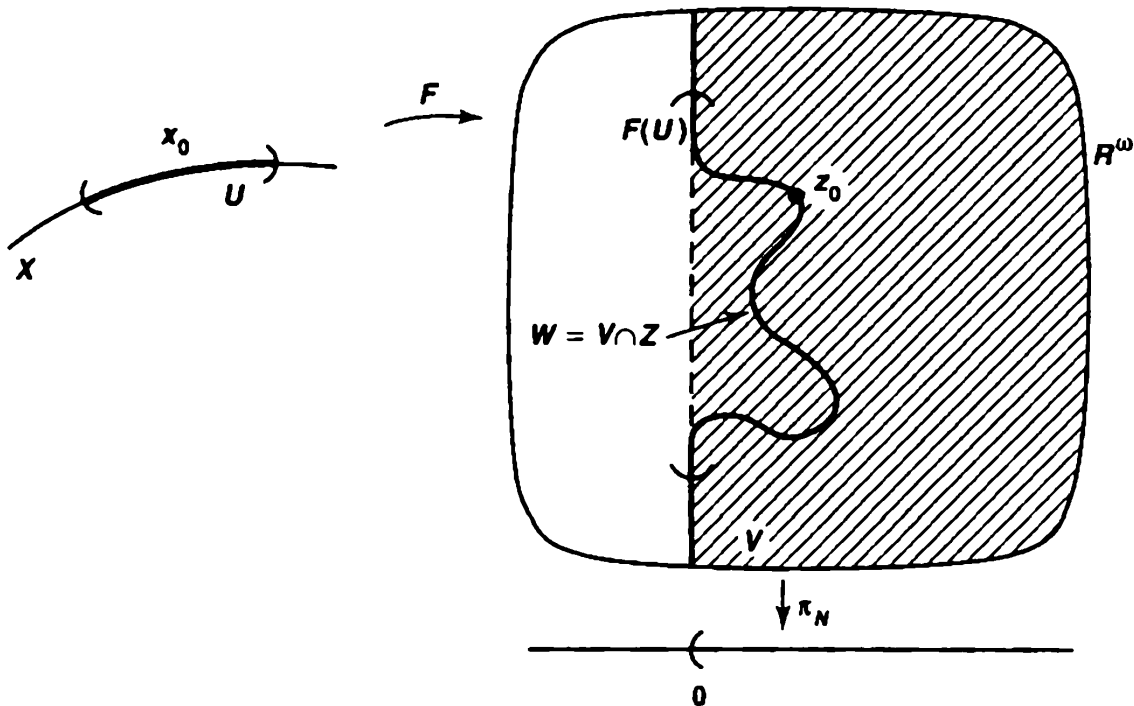Thus F is an imbedding of X in $\mathbb{R}^\omega$.

Figure 34.1

Step 3 (Second version of the proof). In this version, we imbed X in the metric space $(\mathbb{R}^\omega, \bar{\rho})$ Actually, we imbed X in the subspace $[0, 1]^\omega$, on which $\bar{\rho}$ equals the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i|\}.$$

We use the countable collection of functions $f_n : X \to [0, 1]$ constructed in Step 1. But now we impose the additional condition that $f_n(x) \leq 1/n$ for all x (This condition is easy to satisfy; we can just divide each function f_n by n.)

Define $F : X \to [0, 1]^\omega$ by the equation

$$F(x) = (f_1(x), f_2(x), \ldots)$$

as before. We assert that F is now an imbedding relative to the metric ρ on $[0, 1]^\omega$. We know from Step 2 that F is injective. Furthermore, we know that if we use the *product* topology on $[0, 1]^\omega$, the map F carries open sets of X onto open sets of the subspace $Z = F(X)$. This statement remains true if one passes to the finer (larger) topology on $[0, 1]^\omega$ induced by the metric ρ.

The one thing left to do is to prove that F is continuous. This does not follow from the fact that each component function is continuous, for we are not using the product topology on $\mathbb{R}^\omega$ now. Here is where the assumption $f_n(x) \leq 1/n$ comes in.

Let x_0 be a point of X, and let $\epsilon > 0$. To prove continuity, we need to find a neighborhood U of x_0 such that

$$x \in U \implies \rho(F(x), F(x_0)) < \epsilon.$$

First choose N large enough that $1/N \leq \epsilon/2$. Then for each $n = 1, \ldots, N$ use the continuity of f_n to choose a neighborhood U_n of x_0 such that

$$|f_n(x) - f_n(x_0)| \leq \epsilon/2$$

for $x \in U_n$. Let $U = U_1 \cap \cdots \cap U_N$; we show that U is the desired neighborhood of x_0. Let $x \in U$. If $n \leq N$,

$$|f_n(x) - f_n(x_0)| \leq \epsilon/2$$

by choice of U. And if $n > N$, then

$$|f_n(x) - f_n(x_0)| < 1/N \leq \epsilon/2$$

because f_n maps X into $[0, 1/n]$. Therefore for all $x \in U$,

$$\rho(F(x), F(x_0)) \leq \epsilon/2 < \epsilon,$$

as desired.   ∎

In Step 2 of the preceding proof, we actually proved something stronger than the result stated there. For later use, we state it here.

Theorem 34.2 (Imbedding theorem). *Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0, 1]$ for each α, then F imbeds X in $[0, 1]^J$.

The proof is almost a copy of Step 2 of the preceding proof; one merely replaces n by α, and $\mathbb{R}^\omega$ by $\mathbb{R}^J$, throughout. One needs one-point sets in X to be closed in order to be sure that, given $x \neq y$, there is an index α such that $f_\alpha(x) \neq f_\alpha(y)$.

A family of continuous functions that satisfies the hypotheses of this theorem is said to *separate points from closed sets* in X. The existence of such a family is readily seen to be equivalent, for a space X in which one-point sets are closed, to the requirement that X be completely regular. Therefore one has the following immediate corollary:

Theorem 34.3. *A space X is completely regular if and only if it is homeomorphic to a subspace of* $[0, 1]^J$ *for some J.*

Exercises

1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

2. Give an example showing that a space can be completely normal, and satisfy the first countability axiom, the Lindelöf condition, and have a countable dense subset, and still not be metrizable.

3. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

4. Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?

5. Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Is it true that if Y is metrizable, then X has a countable basis?

6. Check the details of the proof of Theorem 34.2.

7. A space X is *locally metrizable* if each point x of X has a neighborhood that is metrizable in the subspace topology. Show that a compact Hausdorff space X is metrizable if it is locally metrizable. [*Hint.* Show that X is a finite union of open subspaces, each of which has a countable basis.]

8. Show that a regular Lindelöf space is metrizable if it is locally metrizable. [*Hint:* A closed subspace of a Lindelöf space is Lindelöf.] Regularity is essential; where do you use it in the proof?

9. Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable. [*Hint* Construct a countable collection $\mathcal{A}$ of open sets of X whose intersections with X_i form a basis for X_i, for $i = 1, 2$. Assume $X_1 - X_2$ and $X_2 - X_1$ belong to $\mathcal{A}$. Let $\mathcal{B}$ consist of finite intersections of elements of $\mathcal{A}$]

basis elements for X that covers the space

$$Z_\beta = \bigcap_{\alpha < \beta} Y_\alpha = \{x \mid \pi_i(x) = p_i \text{ for } i < \beta\},$$

then $\mathcal{A}$ actually covers Y_α for some $\alpha < \beta$. [*Hint.* If β has an immediate predecessor in J, let α be that immediate predecessor Otherwise, for each $A \in \mathcal{A}$, let J_A denote the set of those indices $i < \beta$ for which $\pi_i(A) \neq X_i$; the union of the sets J_A, for $A \in \mathcal{A}$, is finite; let α be the largest element of this union.]

(b) Assume $\mathcal{A}$ is a collection of basis elements for X such that no finite subcollection of $\mathcal{A}$ covers X. Show that one can choose points $p_i \in X_i$ for all i, such that for each α, the space Y_α defined in (a) cannot be finitely covered by $\mathcal{A}$. When α is the largest element of J, one has a contradiction. [*Hint:* If α is the smallest element of J, use the preceding lemma to choose p_α. If p_i is defined for all $i < \beta$, note that (a) implies that the space Z_β cannot be finitely covered by $\mathcal{A}$ and use the lemma to find p_β]

§38 The Stone-Čech Compactification

We have already studied one way of compactifying a topological space X, the one-point compactification (§29); it is in some sense the minimal compactification of X. The Stone-Čech compactification of X, which we study now, is in some sense the maximal compactification of X. It was constructed by M. Stone and E. Čech, independently, in 1937. It has a number of applications in modern analysis, but these lie outside the scope of this book

We recall the following definition:

Definition. *A* **compactification** *of a space X is a compact Hausdorff space Y containing X as a subspace such that $\bar{X} = Y$. Two compactifications Y_1 and Y_2 of X are said to be* **equivalent** *if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for every $x \in X$.*

If X has a compactification Y, then X must be completely regular, being a subspace of the completely regular space Y. Conversely, if X is completely regular, then X has a compactification. For X can be imbedded in the compact Hausdorff space $[0, 1]^J$ for some J, and any such imbedding gives rise to a compactification of X, as the following lemma shows·

Lemma 38.1. *Let X be a space; suppose that $h . X \to Z$ is an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an imbedding $H : Y \to Z$ that equals h on X. The compactification Y is uniquely determined up to equivalence.*

We call Y the compactification *induced* by the imbedding h.

Proof. Given h, let X_0 denote the subspace $h(X)$ of Z, and let Y_0 denote its closure in Z. Then Y_0 is a compact Hausdorff space and $\bar{X}_0 = Y_0$; therefore, Y_0 is a compactification of X_0.

We now construct a space Y containing X such that the pair (X, Y) is homeomorphic to the pair (X_0, Y_0). Let us choose a set A disjoint from X that is in bijective correspondence with the set $Y_0 - X_0$ under some map $k : A \to Y_0 - X_0$. Define $Y = X \cup A$, and define a bijective correspondence $H : Y \to Y_0$ by the rule

$$H(x) = h(x) \quad \text{for } x \in X,$$
$$H(a) = k(a) \quad \text{for } a \in A.$$

Then topologize Y by declaring U to be open in Y if and only if $H(U)$ is open in Y_0. The map H is automatically a homeomorphism; and the space X is a subspace of Y because H equals the homeomorphism h when restricted to the subspace X of Y. By expanding the range of H, we obtain the required imbedding of Y into Z.

Now suppose Y_i is a compactification of X and that $H_i : Y_i \to Z$ is an imbedding that is an extension of h, for $i = 1, 2$. Now H_i maps X onto $h(X) = X_0$ Because H_i is continuous, it must map Y_i into $\bar{X}_0$; because $H_i(Y_i)$ contains X_0 and is closed (being compact), it contains $\bar{X}_0$. Hence $H_i(Y_i) = \bar{X}_0$, and $H_2^{-1} \circ H_1$ defines a homeomorphism of Y_1 with Y_2 that equals the identity on X.   ∎

In general, there are many different ways of compactifying a given space X. Consider for instance the following compactifications of the open interval $X = (0, 1)$:

Example 1   Take the unit circle S^1 in $\mathbb{R}^2$ and let $h . (0, 1) \to S^1$ be the map

$$h(t) = (\cos 2\pi t) \times (\sin 2\pi t).$$

The compactification induced by the imbedding h is equivalent to the one-point compactification of X

Example 2   Let Y be the space $[0, 1]$ Then Y is a compactification of X, it is obtained by "adding one point at each end of $(0, 1)$"

Example 3.   Consider the square $[-1, 1]^2$ in $\mathbb{R}^2$ and let h $(0, 1) \to [-1, 1]^2$ be the map

$$h(x) = x \times \sin(1/x).$$

The space $Y_0 = \overline{h(X)}$ is the topologist's sine curve (see Example 7 of §24). The imbedding h gives rise to a compactification of $(0, 1)$ quite different from the other two. It is obtained by adding one point at the right-hand end of $(0, 1)$, and an entire line segment of points at the left-hand end[1]

A basic problem that occurs in studying compactifications is the following:

If Y is a compactification of X, under what conditions can a continuous real-valued function f defined on X be extended continuously to Y?

The function f will have to be bounded if it is to be extendable, since its extension will carry the compact space Y into $\mathbb{R}$ and will thus be bounded. But boundedness is not enough, in general. Consider the following example·

EXAMPLE 4   Let $X = (0, 1)$ Consider the one-point compactification of X given in Example 1   A bounded continuous function $f : (0, 1) \to \mathbb{R}$ is extendable to this compactification if and only if the limits

$$\lim_{x \to 0+} f(x) \quad \text{and} \quad \lim_{x \to 1-} f(x)$$

exist and are equal.

For the "the two-point compactification" of X considered in Example 2, the function f is extendable if and only if both these limits simply exist

For the compactification of Example 3, extensions exist for a still broader class of functions It is easy to see that f is extendable if both the above limits exist But the function $f(x) = \sin(1/x)$ is also extendable to this compactification· Let H be the imbedding of Y in $\mathbb{R}^2$ that equals h on the subspace X Then the composite map

$$Y \xrightarrow{\ H\ } \mathbb{R} \times \mathbb{R} \xrightarrow{\ \pi_2\ } \mathbb{R}$$

is the desired extension of f. For if $x \in X$, then $H(x) = h(x) = x \times \sin(1/x)$, so that $\pi_2(H(x)) = \sin(1/x)$, as desired

There is something especially interesting about this last compactification. We constructed it by choosing an imbedding

$$h . (0, 1) \longrightarrow \mathbb{R}^2$$

whose component functions were the functions x and $\sin(1/x)$ Then we found that both the functions x and $\sin(1/x)$ could be extended to the compactification This suggests that if we have a whole *collection* of bounded continuous functions defined on $(0, 1)$, we might use them as component functions of an imbedding of $(0, 1)$ into $\mathbb{R}^J$ for some J, and thereby obtain a compactification for which every function in the collection is extendable.

This idea is the basic idea behind the Stone-Čech compactification. It is defined as follows:

Theorem 38.2.   *Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.*

Proof. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of *all* bounded continuous real-valued functions on X, indexed by some index set J For each $\alpha \in J$, choose a closed interval I_α in $\mathbb{R}$ containing $f_\alpha(X)$. To be definite, choose

$$I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)].$$

Then define $h . X \to \prod_{\alpha \in J} I_\alpha$ by the rule

$$h(x) = (f_\alpha(x))_{\alpha \in J}.$$

By the Tychonoff theorem, $\prod I_\alpha$ is compact Because X is completely regular, the collection $\{f_\alpha\}$ separates points from closed sets in X. Therefore, by Theorem 34.2, the map h is an imbedding.

Let Y be the compactification of X induced by the imbedding h. Then there is an imbedding $H : Y \to \prod I_\alpha$ that equals h when restricted to the subspace X of Y. Given a bounded continuous real-valued function f on X, we show it extends to Y. The function f belongs to the collection $\{f_\alpha\}_{\alpha\in J}$, so it equals f_β for some index β. Let $\pi_\beta . \prod I_\alpha \to I_\beta$ be the projection mapping Then the continuous map $\pi_\beta \circ H :$ $Y \to I_\beta$ is the desired extension of f. For if $x \in X$, we have

$$\pi_\beta(H(x)) = \pi_\beta(h(x)) = \pi_\beta((f_\alpha(x))_{\alpha\in J}) = f_\beta(x).$$

Uniqueness of the extension is a consequence of the following lemma.   ∎

Lemma 38.3. *Let $A \subset X$; let $f : A \to Z$ be a continuous map of A into the Hausdorff space Z. There is at most one extension of f to a continuous function $g : \bar{A} \to Z$.*

Proof. This lemma was given as an exercise in §18; we give a proof here. Suppose that $g, g' \cdot \bar{A} \to X$ are two different extensions of f, choose x so that $g(x) \neq g'(x)$. Let U and U' be disjoint neighborhoods of $g(x)$ and $g'(x)$, respectively. Choose a neighborhood V of x so that $g(V) \subset U$ and $g'(V) \subset U'$ Now V intersects A in some point y; then $g(y) \in U$ and $g'(y) \in U'$. But since $y \in A$, we have $g(y) = f(y)$ and $g'(y) = f(y)$. This contradicts the fact that U and U' are disjoint.   ∎

Theorem 38.4. *Let X be a completely regular space; let Y be a compactification of X satisfying the extension property of Theorem 38.2 Given any continuous map $f . X \to C$ of X into a compact Hausdorff space C, the map f extends uniquely to a continuous map $g \cdot Y \to C$.*

Proof. Note that C is completely regular, so that it can be imbedded in $[0, 1]^J$ for some J So we may as well assume that $C \subset [0, 1]^J$. Then each component function f_α of the map f is a bounded continuous real-valued function on X, by hypothesis, f_α can be extended to a continuous map g_α of Y into $\mathbb{R}$. Define $g : Y \to \mathbb{R}^J$ by setting $g(y) = (g_\alpha(y))_{\alpha\in J}$; then g is continuous because $\mathbb{R}^J$ has the product topology. Now in fact g maps Y into the subspace C of $\mathbb{R}^J$. For continuity of g implies that

$$g(Y) = g(\bar{X}) \subset \overline{g(X)} = \overline{f(X)} \subset \bar{C} = C.$$

Thus g is the desired extension of f ∎

Theorem 38.5. *Let X be a completely regular space. If Y_1 and Y_2 are two compactifications of X satisfying the extension property of Theorem 38.2, then Y_1 and Y_2 are equivalent.*

Proof.   Consider the inclusion mapping $j_2 : X \to Y_2$. It is a continuous map of X into the compact Hausdorff space Y_2. Because Y_1 has the extension property, we may, by the preceding theorem, extend j_2 to a continuous map $f_2 : Y_1 \to Y_2$. Similarly, we may extend the inclusion map $j_1 : X \to Y_1$ to a continuous map $f_1 : Y_2 \to Y_1$ (because Y_2 has the extension property and Y_1 is compact Hausdorff).
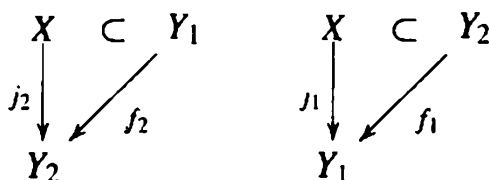
The composite $f_1 \circ f_2 : Y_1 \to Y_1$ has the property that for every $x \in X$, one has $f_1(f_2(x)) = x$ Therefore, $f_1 \circ f_2$ is a continuous extension of the identity map $i_X : X \to X$. But the identity map of Y_1 is also a continuous extension of i_X. By uniqueness of extensions (Lemma 38.3), $f_1 \circ f_2$ must equal the identity map of Y_1. Similarly, $f_2 \circ f_1$ must equal the identity map of Y_2 Thus f_1 and f_2 are homeomorphisms.   ∎

Definition.   For each completely regular space X, let us choose, once and for all, a compactification of X satisfying the extension condition of Theorem 38.2. We will denote this compactification of X by $\beta(X)$ and call it the ***Stone-Čech compactification*** of X. It is characterized by the fact that any continuous map $f . X \to C$ of X into a compact Hausdorff space C extends uniquely to a continuous map g $\beta(X) \to C$.

Exercises

1. Verify the statements made in Example 4.

2. Show that the bounded continuous function g $(0, 1) \to \mathbb{R}$ defined by $g(x) = \cos(1/x)$ cannot be extended to the compactification of Example 3. Define an imbedding $h : (0, 1) \to [0, 1]^3$ such that the functions x, $\sin(1/x)$, and $\cos(1/x)$ are all extendable to the compactification induced by h.

3. Under what conditions does a metrizable space have a metrizable compactification?

4. Let Y be an arbitrary compactification of X; let $\beta(X)$ be the Stone-Čech compactification. Show there is a continuous surjective closed map $g : \beta(X) \to Y$ that equals the identity on X

    [This exercise makes precise what we mean by saying that $\beta(X)$ is the "maximal" compactification of X. It shows that every compactification of X is equivalent to a quotient space of $\beta(X)$.]

5. (a) Show that every continuous real-valued function defined on S_Ω is "eventually constant." [*Hint:* First prove that for each ϵ, there is an element α of S_Ω

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§43 Complete Metric Spaces

In this section we define the notion of completeness and show that if Y is a complete metric space, then the function space $\mathcal{C}(X, Y)$ is complete in the uniform metric. We also show that every metric space can be imbedded isometrically in a complete metric space.

Definition. Let (X, d) be a metric space A sequence (x_n) of points of X is said to be a *Cauchy sequence* in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever } n, m \geq N$$

The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges.

Any convergent sequence in X is necessarily a Cauchy sequence, of course; completeness requires that the converse hold

Note that a closed subset A of a complete metric space (X, d) is necessarily complete in the restricted metric. For a Cauchy sequence in A is also a Cauchy sequence in X, hence it converges in X. Because A is a closed subset of X, the limit must lie in A.

Note also that if X is complete under the metric d, then X is complete under the standard bounded metric

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

corresponding to d, and conversely. For a sequence (x_n) is a Cauchy sequence under $\bar{d}$ if and only if it is a Cauchy sequence under d And a sequence converges under $\bar{d}$ if and only if it converges under d.

A useful criterion for a metric space to be complete is the following:

Lemma 43.1. *A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.*

Proof. Let (x_n) be a Cauchy sequence in (X, d) We show that if (x_n) has a subsequence (x_{n_i}) that converges to a point x, then the sequence (x_n) itself converges to x.

Given $\epsilon > 0$, first choose N large enough that

$$d(x_n, x_m) < \epsilon/2$$

for all $n, m \geq N$ [using the fact that (x_n) is a Cauchy sequence]. Then choose an integer i large enough that $n_i \geq N$ and

$$d(x_{n_i}, x) < \epsilon/2$$

[using the fact that $n_1 < n_2 <$ is an increasing sequence of integers and x_{n_i} converges to x]. Putting these facts together, we have the desired result that for $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \epsilon. \qquad \blacksquare$$

Theorem 43.2. *Euclidean space* $\mathbb{R}^k$ *is complete in either of its usual metrics, the euclidean metric* d *or the square metric* ρ.

Proof. To show the metric space $(\mathbb{R}^k, \rho)$ is complete, let (x_n) be a Cauchy sequence in $(\mathbb{R}^k, \rho)$. Then the set $\{x_n\}$ is a bounded subset of $(\mathbb{R}^k, \rho)$. For if we choose N so that

$$\rho(x_n, x_m) \leq 1$$

for all $n, m \geq N$, then the number

$$M = \max\{\rho(x_1, 0), \ldots, \rho(x_{N-1}, 0), \rho(x_N, 0) + 1\}$$

is an upper bound for $\rho(x_n, 0)$. Thus the points of the sequence (x_n) all lie in the cube $[-M, M]^k$. Since this cube is compact, the sequence (x_n) has a convergent subsequence, by Theorem 28.2. Then $(\mathbb{R}^k, \rho)$ is complete.

To show that $(\mathbb{R}^k, d)$ is complete, note that a sequence is a Cauchy sequence relative to d if and only if it is a Cauchy sequence relative to ρ, and a sequence converges relative to d if and only if it converges relative to ρ. $\qquad \blacksquare$

Now we deal with the product space $\mathbb{R}^\omega$. We need a lemma about sequences in a product space.

Lemma 43.3. *Let* X *be the product space* $X = \prod X_\alpha$; *let* $\mathbf{x}_n$ *be a sequence of points of* X. *Then* $\mathbf{x}_n \to \mathbf{x}$ *if and only if* $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ *for each* α.

Proof. This result was given as an exercise in §19; we give a proof here. Because the projection mapping $\pi_\alpha . X \to X_\alpha$ is continuous, it preserves convergent sequences; the "only if" part of the lemma follows. To prove the converse, suppose $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for each $\alpha \in J$. Let $U = \prod U_\alpha$ be a basis element for X that contains $\mathbf{x}$. For each α for which U_α does *not* equal the entire space X_α, choose N_α so that $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ for $n \geq N_\alpha$. Let N be the largest of the numbers N_α, then for all $n \geq N$, we have $\mathbf{x}_n \in U$ $\qquad \blacksquare$

Theorem 43.4. *There is a metric for the product space* $\mathbb{R}^\omega$ *relative to which* $\mathbb{R}^\omega$ *is complete.*

Proof. Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. Let D be the metric on $\mathbb{R}^\omega$ defined by

$$D(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i)/i\}.$$

Then D induces the product topology on $\mathbb{R}^\omega$; we verify that $\mathbb{R}^\omega$ is complete under D. Let $\mathbf{x}_n$ be a Cauchy sequence in $(\mathbb{R}^\omega, D)$. Because

$$\bar{d}(\pi_i(\mathbf{x}), \pi_i(\mathbf{y})) \leq i D(\mathbf{x}, \mathbf{y}),$$

we see that for fixed i the sequence $\pi_i(\mathbf{x}_n)$ is a Cauchy sequence in $\mathbb{R}$, so it converges, say to a_i. Then the sequence $\mathbf{x}_n$ converges to the point $\mathbf{a} = (a_1, a_2, \dots)$ of $\mathbb{R}^\omega$. ∎

EXAMPLE 1.   An example of a noncomplete metric space is the space $\mathbb{Q}$ of rational numbers in the usual metric $d(x, y) = |x - y|$. For instance, the sequence

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \dots$$

of finite decimals converging (in $\mathbb{R}$) to $\sqrt{2}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge (in $\mathbb{Q}$).

EXAMPLE 2.   Another noncomplete space is the open interval $(-1, 1)$ in $\mathbb{R}$, in the metric $d(x, y) = |x - y|$. In this space the sequence (x_n) defined by

$$x_n = 1 - 1/n$$

is a Cauchy sequence that does not converge. This example shows that completeness is not a topological property, that is, it is not preserved by homeomorphisms For $(-1, 1)$ is homeomorphic to the real line $\mathbb{R}$, and $\mathbb{R}$ is complete in its usual metric.

Although both the product spaces $\mathbb{R}^n$ and $\mathbb{R}^\omega$ have metrics relative to which they are complete, one cannot hope to prove the same result for the product space $\mathbb{R}^J$ in general, because $\mathbb{R}^J$ is not even metrizable if J is uncountable (see §21). There is, however, another topology on the set $\mathbb{R}^J$, the one given by the uniform metric. Relative to this metric, $\mathbb{R}^J$ *is* complete, as we shall see.

We define the uniform metric in general as follows:

Definition.   Let (Y, d) be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the standard bounded metric on Y derived from d. If $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ are points of the cartesian product Y^J, let

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}.$$

It is easy to check that ρ is a metric; it is called the **uniform metric** on Y^J corresponding to the metric d on Y.

Here we have used the standard "tuple" notation for the elements of the cartesian product Y^J. Since the elements of Y^J are simply functions from J to Y, we could also use functional notation for them. In this chapter, functional notation will be more convenient than tuple notation, so we shall use it throughout. In this notation, the definition of the uniform metric takes the following form: If $f, g : J \to Y$, then

$$\bar{\rho}(f, g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) \mid \alpha \in J\}.$$

Theorem 43.5.   *If the space Y is complete in the metric d, then the space Y^J is complete in the uniform metric $\bar{\rho}$ corresponding to d.*

Proof.   Recall that if (Y, d) is complete, so is $(Y, \bar{d})$, where $\bar{d}$ is the bounded metric corresponding to d. Now suppose that $f_1, f_2, \ldots$ is a sequence of points of Y^J that is a Cauchy sequence relative to $\bar{\rho}$. Given α in J, the fact that

$$\bar{d}(f_n(\alpha), f_m(\alpha)) \le \bar{\rho}(f_n, f_m)$$

for all n, m means that the sequence $f_1(\alpha), f_2(\alpha), \ldots$ is a Cauchy sequence in $(Y, \bar{d})$. Hence this sequence converges, say to a point y_α. Let $f : J \to Y$ be the function defined by $f(\alpha) = y_\alpha$. We assert that the sequence (f_n) converges to f in the metric $\bar{\rho}$.

Given $\epsilon > 0$, first choose N large enough that $\bar{\rho}(f_n, f_m) < \epsilon/2$ whenever $n, m \ge N$. Then, in particular,

$$\bar{d}(f_n(\alpha), f_m(\alpha)) < \epsilon/2$$

for $n, m \ge N$ and $\alpha \in J$. Letting n and α be fixed, and letting m become arbitrarily large, we see that

$$\bar{d}(f_n(\alpha), f(\alpha)) \le \epsilon/2.$$

This inequality holds for all α in J, provided merely that $n \ge N$. Therefore,

$$\bar{\rho}(f_n, f) \le \epsilon/2 < \epsilon$$

for $n \ge N$, as desired.   ∎

Now let us specialize somewhat, and consider the set Y^X where X is a *topological space* rather than merely a set. Of course, this has no effect on what has gone before; the topology of X is irrelevant when considering the set of *all* functions $f : X \to Y$. But suppose that we consider the subset $\mathcal{C}(X, Y)$ of Y^X consisting of all *continuous* functions $f : X \to Y$. It turns out that if Y is complete, this subset is also complete in the uniform metric. The same holds for the set $\mathcal{B}(X, Y)$ of all bounded functions $f : X \to Y$. (A function f is said to be *bounded* if its image $f(X)$ is a bounded subset of the metric space (Y, d).)

Theorem 43.6.   *Let X be a topological space and let (Y, d) be a metric space. The set $\mathcal{C}(X, Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $\mathcal{B}(X, Y)$ of bounded functions. Therefore, if Y is complete, these spaces are complete in the uniform metric.*

Proof.   The first part of this theorem is just the uniform limit theorem (Theorem 21.6) in a new guise. First, we show that if a sequence of elements f_n of Y^X converges to the element f of Y^X relative to the metric $\bar{\rho}$ on Y^X, then it converges to f uniformly in the sense defined in §21, relative to the metric d on Y. Given $\epsilon > 0$, choose an integer N such that

$$\bar{\rho}(f, f_n) < \epsilon$$

for all $n > N$. Then for all $x \in X$ and all $n \geq N$,

$$\bar{d}(f_n(x), f(x)) \leq \bar{\rho}(f_n, f) < \epsilon.$$

Thus (f_n) converges uniformly to f.

Now we show that $\mathcal{C}(X, Y)$ is closed in Y^X relative to the metric $\bar{\rho}$. Let f be an element of Y^X that is a limit point of $\mathcal{C}(X, Y)$. Then there is a sequence (f_n) of elements of $\mathcal{C}(X, Y)$ converging to f in the metric $\bar{\rho}$. By the uniform limit theorem, f is continuous, so that $f \in \mathcal{C}(X, Y)$.

Finally, we show that $\mathcal{B}(X, Y)$ is closed in Y^X. If f is a limit point of $\mathcal{B}(X, Y)$, there is a sequence of elements f_n of $\mathcal{B}(X, Y)$ converging to f. Choose N so large that $\bar{\rho}(f_N, f) < 1/2$. Then for $x \in X$, we have $\bar{d}(f_N(x), f(x)) < 1/2$, which implies that $d(f_N(x), f(x)) < 1/2$. It follows that if M is the diameter of the set $f_N(X)$, then $f(X)$ has diameter at most $M + 1$. Hence $f \in \mathcal{B}(X, Y)$.

We conclude that $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are complete in the metric $\bar{\rho}$ if Y is complete in d. ∎

Definition. If (Y, d) is a metric space, one can define another metric on the set $\mathcal{B}(X, Y)$ of bounded functions from X to Y by the equation

$$\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

It is easy to see that ρ is well-defined, for the set $f(X) \cup g(X)$ is bounded if both $f(X)$ and $g(X)$ are. The metric ρ is called the *sup metric*.

There is a simple relation between the sup metric and the uniform metric. Indeed, if $f, g \in \mathcal{B}(X, Y)$, then

$$\bar{\rho}(f, g) = \min\{\rho(f, g), 1\}.$$

For if $\rho(f, g) > 1$, then $d(f(x_0), g(x_0)) > 1$ for at least one $x_0 \in X$, so that $\bar{d}(f(x_0), g(x_0)) = 1$ and $\bar{\rho}(f, g) = 1$ by definition. On the other hand, if $\rho(f, g) \leq 1$, then $\bar{d}(f(x), g(x)) = d(f(x), g(x)) \leq 1$ for all x, so that $\bar{\rho}(f, g) = \rho(f, g)$. Thus on $\mathcal{B}(X, Y)$, the metric $\bar{\rho}$ is just the standard bounded metric derived from the metric ρ. That is the reason we introduced the notation $\bar{\rho}$ for the uniform metric, back in §20!

If X is a compact space, then every continuous function $f : X \to Y$ is bounded; hence the sup metric is defined on $\mathcal{C}(X, Y)$. If Y is complete under d, then $\mathcal{C}(X, Y)$ is complete under the corresponding uniform metric $\bar{\rho}$, so it is also complete under the sup metric ρ. We often use the sup metric rather than the uniform metric in this situation.

We now prove a classical theorem, to the effect that every metric space can be imbedded isometrically in a complete metric space. (A different proof, somewhat more direct, is outlined in Exercise 9.) Although we shall not need this theorem, it is useful in other parts of mathematics.

*Theorem 43.7. *Let* (X, d) *be a metric space. There is an isometric imbedding of* X *into a complete metric space.*

Proof. Let $\mathcal{B}(X, \mathbb{R})$ be the set of all bounded functions mapping X into $\mathbb{R}$. Let x_0 be a fixed point of X. Given $a \in X$, define $\phi_a : X \to \mathbb{R}$ by the equation

$$\phi_a(x) = d(x, a) - d(x, x_0).$$

We assert that ϕ_a is bounded. For it follows, from the inequalities

$$d(x, a) \le d(x, b) + d(a, b),$$
$$d(x, b) \le d(x, a) + d(a, b),$$

that

$$|d(x, a) - d(x, b)| \le d(a, b).$$

Setting $b = x_0$, we conclude that $|\phi_a(x)| \le d(a, x_0)$ for all x.

Define $\Phi : X \to \mathcal{B}(X, \mathbb{R})$ by setting

$$\Phi(a) = \phi_a.$$

We show that Φ is an isometric imbedding of (X, d) into the complete metric space $(\mathcal{B}(X, \mathbb{R}), \rho)$. That is, we show that for every pair of points $a, b \in X$,

$$\rho(\phi_a, \phi_b) = d(a, b).$$

By definition,

$$\rho(\phi_a, \phi_b) = \sup\{|\phi_a(x) - \phi_b(x)| ; \ x \in X\}$$
$$= \sup\{|d(x, a) - d(x, b)| ; \ x \in X\}.$$

We conclude that

$$\rho(\phi_a, \phi_b) \le d(a, b).$$

On the other hand, this inequality cannot be strict, for when $x = a$,

$$|d(x, a) - d(x, b)| = d(a, b). \qquad \blacksquare$$

Definition. Let X be a metric space. If $h : X \to Y$ is an isometric imbedding of X into a complete metric space Y, then the subspace $\overline{h(X)}$ of Y is a complete metric space. It is called the *completion* of X.

The completion of X is uniquely determined up to an isometry. See Exercise 10.

3. (a) If $\mathbb{R}^\omega$ is given the product topology, show there is no continuous surjective map $f : \mathbb{R} \to \mathbb{R}^\omega$. [*Hint:* Show that $\mathbb{R}^\omega$ is not a countable union of compact subspaces.]

   (b) If $\mathbb{R}^\omega$ is given the product topology, determine whether or not there is a continuous surjective map of $\mathbb{R}$ onto the subspace $\mathbb{R}^\infty$.

   (c) What happens to the statements in (a) and (b) if $\mathbb{R}^\omega$ is given the uniform topology or the box topology?

4. (a) Let X be a Hausdorff space. Show that if there is a continuous surjective map $f : I \to X$, then X is compact, connected, weakly locally connected, and metrizable. [*Hint:* Show f is a perfect map.]

   (b) The converse of the result in (a) is a famous theorem of point-set topology called the *Hahn-Mazurkiewicz theorem* (see [H-Y], p. 129). Assuming this theorem, show there is a continuous surjective map $f : I \to I^\omega$.

   A Hausdorff space that is the continuous image of the closed unit interval is often called a *Peano space*.

§45   Compactness in Metric Spaces

We have already shown that compactness, limit point compactness, and sequential compactness are equivalent for metric spaces. There is still another formulation of compactness for metric spaces, one that involves the notion of completeness. We study it in this section. As an application, we shall prove a theorem characterizing those subspaces of $\mathcal{C}(X, \mathbb{R}^n)$ that are compact in the uniform topology.

How is compactness of a metric space X related to completeness of X? It follows from Lemma 43.1 that every compact metric space is complete. The converse does not hold—a complete metric space need not be compact. It is reasonable to ask what extra condition one needs to impose on a complete space to be assured of its compactness. Such a condition is the one called *total boundedness*.

Definition. A metric space (X, d) is said to be *totally bounded* if for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

EXAMPLE 1.   Total boundedness clearly implies boundedness. For if $B(x_1, 1/2), \ldots ,$ $B(x_n, 1/2)$ is a finite covering of X by open balls of radius $1/2$, then X has diameter at most $1 + \max\{d(x_i, x_j)\}$. The converse does not hold, however. For example, in the metric $\bar{d}(a, b) = \min\{1, |a - b|\}$, the real line $\mathbb{R}$ is bounded but not totally bounded.

EXAMPLE 2.   Under the metric $d(a, b) = |a - b|$, the real line $\mathbb{R}$ is complete but not totally bounded, while the subspace $(-1, 1)$ is totally bounded but not complete. The subspace $[-1, 1]$ is both complete and totally bounded.

Theorem 45.1. *A metric space* (X, d) *is compact if and only if it is complete and totally bounded.*

Proof. If X is a compact metric space, then X is complete, as noted above. The fact that X is totally bounded is a consequence of the fact that the covering of X by all open ϵ-balls must contain a finite subcovering.

Conversely, let X be complete and totally bounded. We shall prove that X is sequentially compact. This will suffice.

Let (x_n) be a sequence of points of X. We shall construct a subsequence of (x_n) that is a Cauchy sequence, so that it necessarily converges. First cover X by finitely many balls of radius 1. At least one of these balls, say B_1, contains x_n for infinitely many values of n. Let J_1 be the subset of $\mathbb{Z}_+$ consisting of those indices n for which $x_n \in B_1$.

Next, cover X by finitely many balls of radius $1/2$. Because J_1 is infinite, at least one of these balls, say B_2, must contain x_n for infinitely many values of n in J_1. Choose J_2 to be the set of those indices n for which $n \in J_1$ and $x_n \in B_2$. In general, given an infinite set J_k of positive integers, choose J_{k+1} to be an infinite subset of J_k such that there is a ball B_{k+1} of radius $1/(k+1)$ that contains x_n for all $n \in J_{k+1}$.

Choose $n_1 \in J_1$. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$; this we can do because J_{k+1} is an infinite set. Now for $i, j \geq k$, the indices n_i and n_j both belong to J_k (because $J_1 \supset J_2 \supset \cdots$ is a nested sequence of sets). Therefore, for all $i, j \geq k$, the points x_{n_i} and x_{n_j} are contained in a ball B_k of radius $1/k$. It follows that the sequence (x_{n_i}) is a Cauchy sequence, as desired. ∎

We now apply this result to find the compact subspaces of the space $\mathcal{C}(X, \mathbb{R}^n)$, in the uniform topology. We know that a subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. But it does not, even if X is compact. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies an additional condition, called *equicontinuity*. We consider that notion now.

Definition. Let (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of the function space $\mathcal{C}(X, Y)$. If $x_0 \in X$, the set $\mathcal{F}$ of functions is said to be *equicontinuous at x_0* if given $\epsilon > 0$, there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \epsilon.$$

If the set $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is said simply to be *equicontinuous*.

Continuity of the function f at x_0 means that given f and given $\epsilon > 0$, there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$. Equicontinuity of $\mathcal{F}$ means that a single neighborhood U can be chosen that will work for all the functions f in the collection $\mathcal{F}$.

Note that equicontinuity depends on the specific metric d rather than merely on the topology of Y.

Lemma 45.2. *Let X be a space; let (Y, d) be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is totally bounded under the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.*

Proof. Assume $\mathcal{F}$ is totally bounded. Given $0 < \epsilon < 1$, and given x_0, we find a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$ and $f \in \mathcal{F}$.

Set $\delta = \epsilon/3$; cover $\mathcal{F}$ by finitely many open δ-balls

$$B(f_1, \delta), \ldots, B(f_n, \delta)$$

in $\mathcal{C}(X, Y)$. Each function f_i is continuous; therefore, we can choose a neighborhood U of x_0 such that for $i = 1, \ldots, n$,

$$d(f_i(x), f_i(x_0)) < \delta$$

whenever $x \in U$.

Let f be an arbitrary element of $\mathcal{F}$. Then f belongs to at least one of the above δ-balls, say to $B(f_i, \delta)$. Then for $x \in U$, we have

$$\bar{d}(f(x), f_i(x)) < \delta,$$
$$d(f_i(x), f_i(x_0)) < \delta,$$
$$\bar{d}(f_i(x_0), f(x_0)) < \delta.$$

The first and third inequalities hold because $\bar{\rho}(f, f_i) < \delta$, and the second holds because $x \in U$. Since $\delta < 1$, the first and third also hold if $\bar{d}$ is replaced by d; then the triangle inequality implies that for all $x \in U$, we have $d(f(x), f(x_0)) < \epsilon$, as desired. ∎

Now we prove the classical version of Ascoli's theorem, which concerns compact subspaces of the function space $\mathcal{C}(X, \mathbb{R}^n)$. A more general version, whose proof does not depend on this one, is given in §47. The general version, however, relies on the Tychonoff theorem, whereas this one does not.

We begin by proving a partial converse to the preceding lemma, which holds when X and Y are compact.

*Lemma 45.3. *Let X be a space; let (Y, d) be a metric space; assume X and Y are compact. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is equicontinuous under d, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.*

Proof. Since X is compact, the sup metric ρ is defined on $\mathcal{C}(X, Y)$. Total boundedness under ρ is equivalent to total boundedness under $\bar{\rho}$, for whenever $\epsilon < 1$, every ϵ-ball under ρ is also an ϵ-ball under $\bar{\rho}$, and conversely. Therefore, we may as well use the metric ρ throughout.

Assume $\mathcal{F}$ is equicontinuous. Given $\epsilon > 0$, we cover $\mathcal{F}$ by finitely many sets that are open ϵ-balls in the metric ρ.

Set $\delta = \epsilon/3$. Given any $a \in X$, there is a corresponding neighborhood U_a of a such that $d(f(x), f(a)) < \delta$ for all $x \in U_a$ and all $f \in \mathcal{F}$. Cover X by finitely many such neighborhoods U_a, for $a = a_1, \ldots, a_k$; denote U_{a_i} by U_i. Then cover Y by finitely many open sets $V_1, \ldots, V_m$ of diameter less than δ.

Let J be the collection of all functions $\alpha : \{1, \ldots, k\} \to \{1, \ldots, m\}$. Given $\alpha \in J$, if there exists a function f of $\mathcal{F}$ such that $f(a_i) \in V_{\alpha(i)}$ for each $i = 1, \ldots, k$, choose one such function and label it f_α. The collection $\{f_\alpha\}$ is indexed by a subset J' of the set J and is thus finite. We assert that the open balls $B_\rho(f_\alpha, \epsilon)$, for $\alpha \in J'$, cover $\mathcal{F}$.

Let f be an element of $\mathcal{F}$. For each $i = 1, \ldots, k$, choose an integer $\alpha(i)$ such that $f(a_i) \in V_{\alpha(i)}$. Then the function α is in J'. We assert that f belongs to the ball $B_\rho(f_\alpha, \epsilon)$.

Let x be a point of X. Choose i so that $x \in U_i$. Then

$$d(f(x), f(a_i)) < \delta,$$

$$d(f(a_i), f_\alpha(a_i)) < \delta,$$

$$d(f_\alpha(a_i), f_\alpha(x)) < \delta.$$

The first and third inequalities hold because $x \in U_i$, and the second holds because $f(a_i)$ and $f_\alpha(a_i)$ are in $V_{\alpha(i)}$. We conclude that $d(f(x), f_\alpha(x)) < \epsilon$. Because this inequality holds for every $x \in X$,

$$\rho(f, f_\alpha) = \max\{d(f(x), f_\alpha(x))\} < \epsilon.$$

Thus f belongs to $B_\rho(f_\alpha, \epsilon)$, as asserted. ∎

Definition. If (Y, d) is a metric space, a subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is said to be *pointwise bounded* under d if for each $x \in X$, the subset

$$\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$$

of Y is bounded under d.

*Theorem 45.4 (Ascoli's theorem, classical version).** *Let X be a compact space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under d.*

Proof. Since X is compact, the sup metric ρ is defined on $\mathcal{C}(X, \mathbb{R}^n)$ and gives the uniform topology on $\mathcal{C}(X, \mathbb{R}^n)$. Throughout, let $\mathcal{G}$ denote the closure of $\mathcal{F}$ in $\mathcal{C}(X, \mathbb{R}^n)$.

Step 1. We show that if $\mathcal{G}$ is compact, then $\mathcal{G}$ is equicontinuous and pointwise bounded under d. Since $\mathcal{F} \subset \mathcal{G}$, it follows that $\mathcal{F}$ is also equicontinuous and pointwise bounded under d. This proves the "only if" part of the theorem.

Compactness of $\mathcal{G}$ implies that $\mathcal{G}$ is totally bounded under ρ and $\bar{\rho}$ by Theorem 45.1; this in turn implies that $\mathcal{G}$ is equicontinuous under d, by Lemma 45.2. Compactness of $\mathcal{G}$ also implies that $\mathcal{G}$ is bounded under ρ; this in turn implies that $\mathcal{G}$ is

pointwise bounded under d. For if $\rho(f, g) \leq M$ for all $f, g \in \mathcal{G}$, then in particular $d(f(a), g(a)) \leq M$ for $f, g \in \mathcal{G}$, so that $\mathcal{G}_a$ has diameter at most M.

Step 2. We show that if $\mathcal{F}$ is equicontinuous and pointwise bounded under d, then so is $\mathcal{G}$.

First, we check equicontinuity. Given $x_0 \in X$ and given $\epsilon > 0$, choose a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon/3$ for all $x \in U$ and $f \in \mathcal{F}$. Given $g \in \mathcal{G}$, choose $f \in \mathcal{F}$ so that $\rho(f, g) < \epsilon/3$. The triangle inequality implies that $d(g(x), g(x_0)) < \epsilon$ for all $x \in U$. Since g is arbitrary, equicontinuity of $\mathcal{G}$ at x_0 follows.

Second, we verify pointwise boundedness. Given a, choose M so that $\operatorname{diam} \mathcal{F}_a \leq M$. Then, given $g, g' \in \mathcal{G}$, choose $f, f' \in \mathcal{F}$ such that $\rho(f, g) < 1$ and $\rho(f', g') < 1$. Since $d(f(a), f'(a)) \leq M$, it follows that $d(g(a), g'(a)) \leq M + 2$. Then since g and g' are arbitrary, it follows that $\operatorname{diam} \mathcal{G}_a \leq M + 2$.

Step 3. We show that if $\mathcal{G}$ is equicontinuous and pointwise bounded, then there is a compact subspace Y of $\mathbb{R}^n$ that contains the union of the sets $g(X)$, for $g \in \mathcal{G}$.

Choose, for each $a \in X$, a neighborhood U_a of a such that $d(g(x), g(a)) < 1$ for $x \in U_a$ and $g \in \mathcal{G}$. Since X is compact, we can cover X by finitely many such neighborhoods, say for $a = a_1, \ldots, a_k$. Because the sets $\mathcal{G}_{a_i}$ are bounded, their union is also bounded; suppose it lies in the ball of radius N in $\mathbb{R}^n$ centered at the origin. Then for all $g \in \mathcal{G}$, the set $g(X)$ is contained in the ball of radius $N + 1$ centered at the origin. Let Y be the closure of this ball.

Step 4. We prove the "if" part of the theorem. Assume that $\mathcal{F}$ is equicontinuous and pointwise bounded under d. We show that $\mathcal{G}$ is complete and totally bounded under ρ; then Theorem 45.1 implies that $\mathcal{G}$ is compact.

Completeness is easy, for $\mathcal{G}$ is a closed subspace of the complete metric space $(\mathcal{C}(X, \mathbb{R}^n), \rho)$.

We verify total boundedness. First, Step 2 implies that $\mathcal{G}$ is equicontinuous and pointwise bounded under d; then Step 3 tells us that there is a compact subspace Y of $\mathbb{R}^n$ such that $\mathcal{G} \subset \mathcal{C}(X, Y)$. Equicontinuity of $\mathcal{G}$ now implies, by Lemma 45.3, that $\mathcal{G}$ is totally bounded under ρ, as desired.   ∎

*Corollary 45.5.   *Let X be compact; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric ρ, and equicontinuous under d.*

Proof. If $\mathcal{F}$ is compact, it must be closed and bounded; the preceding theorem implies that it is also equicontinuous. Conversely, if $\mathcal{F}$ is closed, it equals its closure $\mathcal{G}$; if it is bounded under ρ, it is pointwise bounded under d; and if it is also equicontinuous, the preceding theorem implies that it is compact.   ∎

Exercises

1. If X_n is metrizable with metric d_n, then

$$D(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $X = \prod X_n$. Show that X is totally bounded under D if each X_n is totally bounded under d_n. Conclude without using the Ty-chonoff theorem that a countable product of compact metrizable spaces is compact.

2. Let (Y, d) be a metric space; let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$.
   (a) Show that if $\mathcal{F}$ is finite, then $\mathcal{F}$ is equicontinuous.
   (b) Show that if f_n is a sequence of elements of $\mathcal{C}(X, Y)$ that converges uniformly, then the collection $\{f_n\}$ is equicontinuous.
   (c) Suppose that $\mathcal{F}$ is a collection of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that each $x \in \mathbb{R}$ lies in a neighborhood U on which the derivatives of the functions in $\mathcal{F}$ are uniformly bounded. [This means that there is an M such that $|f'(x)| \le M$ for all f in $\mathcal{F}$ and all $x \in U$.] Show that $\mathcal{F}$ is equicontinuous.

3. Prove the following:
   Theorem (Arzela's theorem). *Let X be compact; let $f_n \in \mathcal{C}(X, \mathbb{R}^k)$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a uniformly convergent subsequence.*

4. (a) Let $f_n : I \to \mathbb{R}$ be the function $f_n(x) = x^n$. The collection $\mathcal{F} = \{f_n\}$ is pointwise bounded but the sequence (f_n) has no uniformly convergent subsequence; at what point or points does $\mathcal{F}$ fail to be equicontinuous?
   (b) Repeat (a) for the functions f_n of Exercise 9 of §21.

5. Let X be a space. A subset $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R})$ is said to **vanish uniformly at infinity** if given $\epsilon > 0$, there is a compact subspace C of X such that $|f(x)| < \epsilon$ for $x \in X - C$ and $f \in \mathcal{F}$. If $\mathcal{F}$ consists of a single function f, we say simply that f **vanishes at infinity**. Let $\mathcal{C}_0(X, \mathbb{R})$ denote the set of continuous functions $f : X \to \mathbb{R}$ that vanish at infinity.
   Theorem. *Let X be locally compact Hausdorff; give $\mathcal{C}_0(X, \mathbb{R})$ the uniform topology. A subset $\mathcal{F}$ of $\mathcal{C}_0(X, \mathbb{R})$ has compact closure if and only if it is pointwise bounded, equicontinuous, and vanishes uniformly at infinity.*
   [*Hint:* Let Y denote the one-point compactification of X. Show that $\mathcal{C}_0(X, \mathbb{R})$ is isometric with a closed subspace of $\mathcal{C}(Y, \mathbb{R})$ if both are given the sup metric.]

6. Show that our proof of Ascoli's theorem goes through if $\mathbb{R}^n$ is replaced by any metric space in which all closed bounded subspaces are compact.

*7. Let (X, d) be a metric space. If $A \subset X$ and $\epsilon > 0$, let $U(A, \epsilon)$ be the ϵ-neighborhood of A. Let $\mathcal{H}$ be the collection of all (nonempty) closed, bounded subsets of X. If $A, B \in \mathcal{H}$, define

$$D(A, B) = \inf\{\epsilon \mid A \subset U(B, \epsilon) \text{ and } B \subset U(A, \epsilon)\}.$$

§48 Baire Spaces

The defining condition for a Baire space is probably as "unnatural looking" as any condition we have yet introduced in this book. But bear with us awhile.

In this section, we shall define Baire spaces and shall show that two important classes of spaces—the complete metric spaces and the compact Hausdorff spaces—are contained in the class of Baire spaces. Then we shall give some applications, which, even if they do not make the Baire condition seem any more natural, will at least show what a useful tool it can be. In fact, it turns out to be a very useful and fairly sophisticated tool in both analysis and topology.

Definition. Recall that if A is a subset of a space X, the *interior* of A is defined as the union of all open sets of X that are contained in A. To say that A has *empty interior* is to say then that A contains no open set of X other than the empty set. Equivalently, A has empty interior if every point of A is a limit point of the complement of A, that is, if the complement of A is dense in X.

> EXAMPLE 1   The set Q of rationals has empty interior as a subset of $\mathbb{R}$, but the interval $[0, 1]$ has nonempty interior The interval $[0, 1] \times 0$ has empty interior as a subset of the plane $\mathbb{R}^2$, and so does the subset $Q \times \mathbb{R}$.

Definition. A space X is said to be a *Baire space* if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

> EXAMPLE 2   The space Q of rationals is not a Baire space. For each one-point set in Q is closed and has empty interior in Q; and Q is the countable union of its one-point subsets.
>
> The space $\mathbb{Z}_+$, on the other hand, does form a Baire space Every subset of $\mathbb{Z}_+$ is open, so that there exist no subsets of $\mathbb{Z}_+$ having empty interior, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.
>
> More generally, every closed subspace of $\mathbb{R}$, being a complete metric space, is a Baire space. Somewhat surprising is the fact that the irrationals in $\mathbb{R}$ also form a Baire space; see Exercise 6.

The terminology originally used by R. Baire for this concept involved the word "category." A subset A of a space X was said to be of the *first category* in X if it was contained in the union of a countable collection of closed sets of X having empty interiors in X; otherwise, it was said to be of the *second category* in X. Using this terminology, we can say the following:

> *A space X is a Baire space if and only if every nonempty open set in X is of the second category*

We shall not use the terms "first category" and "second category" in this book.

The preceding definition is the "closed set definition" of a Baire space. There is also a formulation involving open sets that is frequently useful. It is given in the following lemma.

Lemma 48.1.   *X is a Baire space if and only if given any countable collection $\{U_n\}$ of open sets in X, each of which is dense in X, their intersection $\bigcap U_n$ is also dense in X.*

Proof. Recall that a set C is dense in X if $\bar{C} = X$. The theorem now follows at once from the two remarks:

(1) A is closed in X if and only if $X - A$ is open in X.

(2) B has empty interior in X if and only if $X - B$ is dense in X.   ∎

There are a number of theorems giving conditions under which a space is a Baire space. The most important is the following:

Theorem 48.2 (Baire category theorem).   *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Proof. Given a countable collection $\{A_n\}$ of closed set of X having empty interiors, we want to show that their union $\bigcup A_n$ also has empty interior in X. So, given the nonempty open set U_0 of X, we must find a point x of U_0 that does not lie in any of the sets A_n.

Consider the first set A_1. By hypothesis, A_1 does not contain U_0. Therefore, we may choose a point y of U_0 that is not in A_1. Regularity of X, along with the fact that A_1 is closed, enables us to choose a neighborhood U_1 of y such that

$$\bar{U}_1 \cap A_1 = \varnothing,$$
$$\bar{U}_1 \subset U_0.$$

If X is metric, we also choose U_1 small enough that its diameter is less than 1.

In general, given the nonempty open set U_{n-1}, we choose a point of U_{n-1} that is not in the closed set A_n, and then we choose U_n to be a neighborhood of this point such that

$$\bar{U}_n \cap A_n = \varnothing,$$
$$\bar{U}_n \subset U_{n-1},$$
$$\text{diam } U_n < 1/n \quad \text{in the metric case.}$$

We assert that the intersection $\bigcap \bar{U}_n$ is nonempty. From this fact, our theorem will follow. For if x is a point of $\bigcap \bar{U}_n$, then x is in U_0 because $\bar{U}_1 \subset U_0$. And for each n, the point x is not in A_n because $\bar{U}_n$ is disjoint from A_n.

The proof that $\bigcap \bar{U}_n$ is nonempty splits into two parts, depending on whether X is compact Hausdorff or complete metric. If X is compact Hausdorff, we consider the nested sequence $\bar{U}_1 \supset \bar{U}_2 \supset \cdots$ of nonempty subsets of X. The collection $\{\bar{U}_n\}$ has the finite intersection property; since X is compact, the intersection $\bigcap \bar{U}_n$ must be nonempty.

If X is complete metric, we apply the following lemma.   ∎

Lemma 48.3. *Let* $C_1 \supset C_2 \supset \cdot$ · *be a nested sequence of nonempty closed sets in the complete metric space* X. *If* $\operatorname{diam} C_n \to 0$, *then* $\bigcap C_n \neq \varnothing$.

Proof. We gave this as an exercise in §43. Here is a proof: Choose $x_n \in C_n$ for each n. Because $x_n, x_m \in C_N$ for $n, m \geq N$, and because $\operatorname{diam} C_N$ can be made less than any given ϵ by choosing N large enough, the sequence (x_n) is a Cauchy sequence. Suppose that it converges to x. Then for given k, the subsequence $x_k, x_{k+1}, \ldots$ also converges to x. Thus x necessarily belongs to $\bar{C}_k = C_k$. Then $x \in \bigcap C_k$, as desired.
∎

Here is one application of the theory of Baire spaces; we shall give further applications in the sections that follow. This application is perhaps more amusing than profound. It concerns a question that a student might ask concerning convergent sequences of continuous functions.

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for each $x \in [0, 1]$. There are examples that show the limit function f need not be continuous. But one might wonder just how discontinuous f can be Could it be discontinuous everywhere, for instance? The answer is "no." We shall show that f must be continuous at infinitely many points of $[0, 1]$. In fact, the set of points at which f is continuous is dense in $[0, 1]$!

To prove this result, we need the following lemma:

*Lemma 48.4. *Any open subspace* Y *of a Baire space* X *is itself a Baire space.*

Proof. Let A_n be a countable collection of closed sets of Y that have empty interiors in Y. We show that $\bigcup A_n$ has empty interior in Y.

Let $\bar{A}_n$ be the closure of A_n in X; then $\bar{A}_n \cap Y = A_n$. The set $\bar{A}_n$ has empty interior in X. For if U is a nonempty open set of X contained in $\bar{A}_n$, then U must intersect A_n. Then $U \cap Y$ is a nonempty open set of Y contained in A_n, contrary to hypothesis

If the union of the sets A_n contains the nonempty open set W of Y, then the union of the sets $\bar{A}_n$ also contains the set W, which is open in X because Y is open in X. But each set $\bar{A}_n$ has empty interior in X, contradicting the fact that X is a Baire space. ∎

*Theorem 48.5. *Let* X *be a space; let* (Y, d) *be a metric space. Let* $f_n : X \to Y$ *be a sequence of continuous functions such that* $f_n(x) \to f(x)$ *for all* $x \in X$, *where* $f : X \to Y$. *If* X *is a Baire space, the set of points at which* f *is continuous is dense in* X.

Proof. Given a positive integer N and given $\epsilon > 0$, define

$$A_N(\epsilon) = \{x \mid d(f_n(x), f_m(x)) \leq \epsilon \text{ for all } n, m \geq N\}.$$

Note that $A_N(\epsilon)$ is closed in X. For the set of those x for which $d(f_n(x), f_m(x)) \leq \epsilon$ is closed in X, by continuity of f_n and f_m, and $A_N(\epsilon)$ is the intersection of these sets for all $n, m \geq N$.

For fixed ϵ, consider the sets $A_1(\epsilon) \subset A_2(\epsilon) \subset \cdots$. The union of these sets is all of X. For, given $x_0 \in X$, the fact that $f_n(x_0) \to f(x_0)$ implies that the sequence $f_n(x_0)$ is a Cauchy sequence; hence $x_0 \in A_N(\epsilon)$ for some N.

Now let

$$U(\epsilon) = \bigcup_{N \in \mathbb{Z}_+} \text{Int } A_N(\epsilon).$$

We shall prove two things:

(1) $U(\epsilon)$ is open and dense in X.

(2) The function f is continuous at each point of the set

$$C = U(1) \cap U(1/2) \cap U(1/3) \cap \cdots .$$

Our theorem then follows from the fact that X is a Baire space.

To show that $U(\epsilon)$ is dense in X, it suffices to show that for any nonempty open set V of X, there is an N such that the set $V \cap \text{Int } A_N(\epsilon)$ is nonempty. For this purpose, we note first that for each N, the set $V \cap A_N(\epsilon)$ is closed in V. Because V is a Baire space by the preceding lemma, at least one of these sets, say $V \cap A_M(\epsilon)$, must contain a nonempty open set W of V. Because V is open in X, the set W is open in X; therefore, it is contained in $\text{Int } A_M(\epsilon)$.

Now we show that if $x_0 \in C$, then f is continuous at x_0. Given $\epsilon > 0$, we shall find a neighborhood W of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in W$.

First, choose k so that $1/k < \epsilon/3$. Since $x_0 \in C$, we have $x_0 \in U(1/k)$; therefore, there is an N such that $x_0 \in \text{Int } A_N(1/k)$. Finally, continuity of the function f_N enables us to choose a neighborhood W of x_0, contained in $A_N(1/k)$, such that

$$(*) \qquad\qquad d(f_N(x), f_N(x_0)) < \epsilon/3 \qquad \text{for } x \in W.$$

The fact that $W \subset A_N(1/k)$ implies that

$$d(f_n(x), f_N(x)) \leq 1/k \qquad \text{for } n \geq N \text{ and } x \in W.$$

Letting $n \to \infty$, we obtain the inequality

$$(**) \qquad\qquad d(f(x), f_N(x)) \leq 1/k < \epsilon/3 \qquad \text{for } x \in W.$$

In particular, since $x_0 \in W$, we have

$$(***) \qquad\qquad d(f(x_0), f_N(x_0)) < \epsilon/3.$$

Applying the triangle inequality to $(*)$, $(**)$, and $(***)$ gives us our desired result. ∎

Exercises

1. Let X equal the countable union $\bigcup B_n$. Show that if X is a nonempty Baire space, at least one of the sets $\bar{B}_n$ has a nonempty interior.