SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMTA5201	COMPLEX ANALYSIS	L	T	P	CREDIT
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UNIT 1 FUNDAMENTAL THEOREMS OF COMPLEX INTEGRATION 10 Hrs

Line integrals, Rectifiable arcs, line integrals as functions of arcs, Cauchy's theorem for a rectangle, Cauchy's theorem in a disk – Cauchy's integral formula –The index of a point with respect to a closed curve, The integral formula, Higher derivatives.

UNIT 2 LOCAL PROPERTIES OF ANALYTICAL FUNCTIONS 10 Hrs

Removable singularities, Taylor's theorem, zeros and poles – The local mapping, the maximum principle – chains and cycles, simple connectivity, Homology.

UNIT 3 THE GENERAL FORM OF CAUCHY'S THEOREM AND CALCULUS OF RESIDUES 10 Hrs

The general statement of Cauchy's theorem, proof of Cauchy's theorem, locally exact differentials. The calculus of residue-The residue theorem, the argument principle, Evaluation of definite integrals.

UNIT 4 HARMONIC FUNCTIONS AND POWER SERIES EXPANSIONS 10 Hrs

Harmonic functions – definition and basic properties, The mean-value property, Poisson's formula, Schwarz's theorem, the reflection principle – Power series expansions – Weierstrass's theorem, The Taylor series, The Laurent series.

UNIT 5 PARTIAL FRACTIONS AND FACTORIZATION 10 Hrs

Partial fractions, infinite products, canonical products, the gamma functions.

REFERENCE BOOKS

1. Lars.V. Ahlfors, Complex Analysis, Third Edition, McGraw-HillInternationalEdition,1979.
2. V.Karunakaran, Complex Analysis, Second Edition, NarosaPublications,2005.
3. S. Ponnusamy, Foundations of Complex Analysis, Second Edition, Narosa Publications, 2010.

Course Outcome: At the end of the course, learners would acquire competency in the following skills.

CO1	To define a line integral, index of a point, removable singularities, chains and cycles, harmonic function, genus, infinite product, canonical product, Gamma function and statements of Cauchy's theorem, Cauchy's Integral formula, Taylor's theorem, Poisson's formula, Schwarz theorem, Weierstrass theorem, Taylor's and Laurent's series.
CO2	Classify the complex integral based on Cauchy's integral formula and formula for derivatives, classify the singularities, summarize the properties of harmonic functions, gamma functions.
CO3	Solve the complex integral using Cauchy's integral formula and Cauchy residue theorem
CO4	Differentiate Taylor's theorem and Taylor's series, illustrate to find the real and complex roots using argument principle, explain the concept of genus, canonical product
CO5	Evaluate complex integral by Cauchy's integral formula and its derivatives, real definite integrals using residue theorem, compare the genus of different functions.
CO6	Construct the proofs of all theorems, develop Taylor's series and Laurent's series for a given function.

UNIT – I – Fundamental Theorems of Complex Integration – SMTA5201

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Line Integral

Let f(t) = u(t) + i v(t) be a complex valued continuous function on (a,b) where u and v are real valued continuous functions on (a,b). Then

$$\int_a^b f(t)dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Result

If f : [a,b] → ℂ , g : [a,b] → ℂ are continuous and c ∈ ℂ then

$\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$ and $\int_a^b (cf)(t)\,dt = c\int_a^b f(t)\,dt$

Lemma

If f : [a,b] → ℂ is a continuous function ,then $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

Definition 1

Let Ω be a region and let γ be a piecewise differentiable arc with the equation $z = z(t), a \leq t \leq b$ in Ω .If the function f(z) is defined and continuous on γ, then f(z(t)) is also continuous and we define the line integral,

$$\int_\gamma f(z)dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

Definition 2

The opposite arc of $z = z(t), a \leq t \leq b$ is the arc $z = z(-t), -b \leq t \leq -a$. Opposite arcs are denoted by $-\gamma$ or γ^{-1} depending on the connection. For the opposite arc,

$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f[z(-t)](-z'(-t))\,dt$ and by a change of variable the last integral can be brought to the form $\int_b^a f[z(t)]z'(t)\,dt$. Thus, $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$

Definition 3

If $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \cdots \gamma_n$ then,

$$\int_{\gamma_1+\gamma_2+\gamma_3+\cdots\gamma_n} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \cdots \int_{\gamma_n} f(z)dz$$

Definition 4 Line Integral with respect to arc length

$$\int_\gamma f(z)ds = \int_\gamma f(z)|dz| = \int_\gamma f(z(t))|z'(t)|\,dt$$

Note: If f = 1 then the integral reduces to $\int_\gamma |dz|$ which is by definition the length of γ .

Theorem

If f is continuous on Ω and γ is a curve in Ω then, $\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)||dz|$ since, $\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t)z'(t))|dt = \int_a^b |f(z)||dz|$

Note

$$\left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b |f(z(t)z'(t))|dt = \int_a^b |f(z)||dz|$$

If $|f(z)| \le M$ and 'L' is the length of γ then $\left|\int_\gamma f(z)\,dz\right| \le ML$.

Thus, for calculating the length of the full circle, the parameter equation is $z = z(t) = a + \rho e^{it}, 0 \le t \le 2\pi$, $z'(t) = i\,\rho e^{it}$ and hence

$\int_\gamma |dz| = \int_0^{2\pi} |z'(t)|dt = \int_0^{2\pi} |i\,\rho e^{it}|dt = \int_0^{2\pi} \rho\,dt = 2\pi\rho$ as expected.

Line Integrals as Function of Arcs

Line integrals in general can be written in the form $\int_\gamma pdx + qdy$ where p and q are functions of 'x' and 'y'.

Line integral independent of path

Line integral independent of path means the line integral depends only on the end points and not on the path joining them. Thus if γ_1 and γ_2 have the same initial point and the same end point, then we require that $\int_{\gamma_1} pdx + qdy = \int_{\gamma_2} pdx + qdy$.

To say that an integral depends only on the endpoints is equivalent to saying that the integral over any closed curve is zero.

Theorem 1

The line integral $\int_\gamma pdx + qdy$,defined in Ω , depends only on the endpoints of γ if and only if there exists a function U(x,y) in Ω with the partial derivatives $\frac{\partial U}{\partial x} = p$, $\frac{\partial U}{\partial y} = q$.

Proof

Sufficient Part:

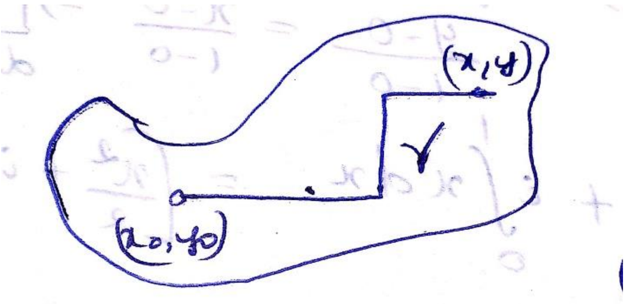
Let there exists a function U(x,y) such that $\frac{\partial U}{\partial x} = p$, $\frac{\partial U}{\partial y} = q$.Now,

$$\int_\gamma pdx + qdy = \int_\gamma \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_\gamma \left[\frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t)\right]dt = \int_a^b \frac{d}{dt}U(x(t), y(t))dt$$

= U(x(b), y(b)) - U(x(a), y(a)) ⇒ The integral depends only on the end points.

Necessary Part:

Choose a fixed point $(x_{0,}y_0) \in \Omega$,join it to (x,y) by a polygon γ contained in Ω whose sides are parallel to the coordinate axes.



Define a function U(x,y) as follows:

U(x,y) = $\int_\gamma \ pdx + qdy$. Since the integral is independent of path, the function is well defined. If we choose the last segment of γ horizontal we can keep 'y' constant. Then

U(x,y) = $\int^x pdx + qdy$ + constant $\Rightarrow \frac{\partial U}{\partial x} = p.$

Similarly, by choosing the last segment vertical $\frac{\partial U}{\partial y} = q.$

Remark

The theorem implies that an integral depends only on the end points if and only if

p dx + q dy is an exact differential.

Corollary to theorem 1

If the integrand is f(z) instead of p dx + q dy, then theorem 1 modifies as follows:

"The integral $\int_\gamma \ f(z)dz$, with continuous f, depends only on the end points of γ iff 'f' is the derivative of an analytic function in Ω".

Cauchy's Theorem for a Rectangle

Let R be a rectangle defined by inequalities $a \le x \le b$, $c \le y \le d$ .The boundary curve or contour of R is denoted by ∂R.

Theorem 2

If the function f(z) is analytic on R, then $\int_{\partial R} f(z)dz = 0.$

Proof:

The proof is based on the method of bisection. Let us introduce the notation $\eta(R) = \int_{\partial R} f(z)dz$ Join the midpoints of the sides of the R to obtain four congruent rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $R^{(4)}$ as shown in the figure below

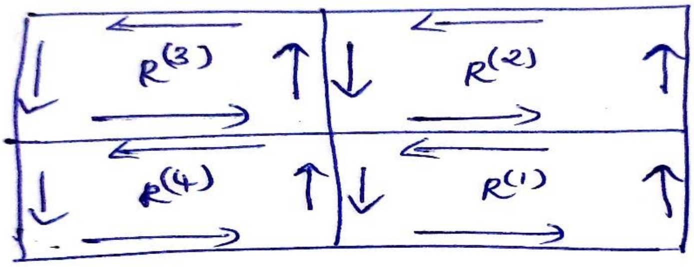


Fig : Bisection of Rectangle .

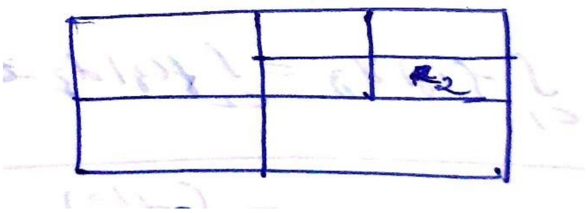
Then $\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$ for the integrals over the common sides cancel each other.

Hence $|\eta(R)| = |\eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})|$

$$|\eta(R)| \leq |\eta(R^{(1)})| + |\eta(R^{(2)})| + |\eta(R^{(3)})| + |\eta(R^{(4)})|$$

It follows that at least one of the rectangles $R^{(k)}$ k=1,2,3,4 must satisfy the condition

$|\eta(R^{(k)})| \geq \frac{1}{4}|\eta(R)|$. We denote this rectangle by R_1; if several $R^{(k)}$ have this property, the choice is made according to some definite rule. Next join the midpoints of the sides of the rectangle R_1 to obtain four congruent rectangles as shown in the figure below.



We denote one of the four rectangles as R_2 satisfying the inequality $|\eta(R_2)| \geq \frac{1}{4}|\eta(R_1)|$

Repeating this process, we obtain a sequence of rectangles $R_1 \supset R_2 \supset R_3 \supset \cdots \supset R_n \supset \cdots$

with the property $|\eta(R_n)| \geq \frac{1}{4}|\eta(R_{n-1})|$ and hence $|\eta(R_2)| \geq \frac{1}{4^n}|\eta(R)|$

The rectangles R_n converge to a point $z^* \epsilon$ R in the sense that R_n will be contained in a prescribed neighbourhood $|z - z^*| < \delta$ for 'n' sufficiently large.

Since f(z) is analytic on R, $f'(z^*)$ exists. But by the definition of $f'(z^*)$,

$f'(z^*) = \lim_{z \to z*} \frac{f(z)-f(z^*)}{z-z^*}$ i.e, for given $\in > 0$ choose a $\delta > 0$ so that,

$$\left|\frac{f(z)-f(z^*}{z-z^*} - f'(z^*)\right| < \in \quad \text{for} \quad |z - z^*| < \delta \quad \Rightarrow$$

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| < \in |z - z^*| \quad \text{for} \quad |z - z^*| < \delta$$

Since '1' and 'z' are the derivatives of z and $z^2/2$ respectively,

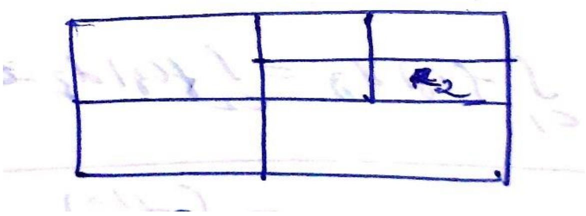
$$\int_{\partial R_n} dz = 0 \text{ and } \int_{\partial R_n} z\, dz = 0 \text{ and hence } |\eta(R_n)| \le \in \int_{\partial R_n} |z - z^*|\, |dz|$$

Now, $|z - z^*|$ is almost equal to the length d_n of the diagonal of R_n. If L_n denotes the length of the perimeter R_n then $|\eta(R_n)| \le \in d_n L_n$ .But if d and L are the corresponding quantities for the original rectangle R then it is clear that $d_n = 2^{-n} d$ and $L_n = 2^{-n} L$. Thus,

$$|\eta(R_n)| \le \in (2^{-n} d)(2^{-n} L) = 4^{-n} dL \in \text{ or}$$

$$|\eta(R)| \le 4^n |\eta(R_n)| = 4^n 4^{-n} dL \in \Rightarrow |\eta(R)| \le dL \in.$$

Since, $\in$ is arbitrary we can only have $\eta(R) = 0$ and the theorem is proved.



We denote one of the four rectangles as R_2 satisfying the inequality $|\eta(R_2)| \ge \frac{1}{4} |\eta(R_1)|$

Repeating this process, we obtain a sequence of rectangles $R_1 \supset R_2 \supset R_3 \supset \cdots \supset R_n \supset \cdots$

with the property $|\eta(R_n)| \ge \frac{1}{4} |\eta(R_{n-1})|$ and hence $|\eta(R_2)| \ge \frac{1}{4^n} |\eta(R)|$

The rectangles R_n converge to a point $z^* \epsilon$ R in the sense that R_n will be contained in a prescribed neighbourhood $|z - z^*| < \delta$ for 'n' sufficiently large.

Since f(z) is analytic on R, $f'(z^*)$ exists. But by the definition of $f'(z^*)$,

$$f'(z^*) = \lim_{z \to z^*} \frac{f(z) - f(z^*)}{z - z^*} \text{ i.e, for given } \in > 0 \text{ choose a } \delta > 0 \text{ so that,}$$

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \in \quad \text{for} \quad |z - z^*| < \delta \quad \Rightarrow$$

$$|f(z) - f(z^*) - (z - z^*) f'(z^*)| < \in |z - z^*| \quad \text{for} \quad |z - z^*| < \delta$$

Since '1' and 'z' are the derivatives of z and $z^2/2$ respectively,

$$\int_{\partial R_n} dz = 0 \text{ and } \int_{\partial R_n} z\, dz = 0 \text{ and hence } |\eta(R_n)| \le \in \int_{\partial R_n} |z - z^*|\, |dz|$$

Now, $|z - z^*|$ is almost equal to the length d_n of the diagonal of R_n. If L_n denotes the length of the perimeter R_n then $|\eta(R_n)| \le \in d_n L_n$ .But if d and L are the corresponding quantities for the original rectangle R then it is clear that $d_n = 2^{-n} d$ and $L_n = 2^{-n} L$. Thus,

$$|\eta(R_n)| \le \in (2^{-n} d)(2^{-n} L) = 4^{-n} dL \in \text{ or}$$

$$|\eta(R)| \le 4^n |\eta(R_n)| = 4^n 4^{-n} dL \in \Rightarrow |\eta(R)| \le dL \in.$$

Since, ∈ is arbitrary we can only have $\eta(R) = 0$ and the theorem is proved.

Cauchy's Theorem in a Disk

If f(z) is analytic in an open disk Δ, then $\int_\gamma f(z)dz = 0$ for every closed curve γ in Δ.

Proof

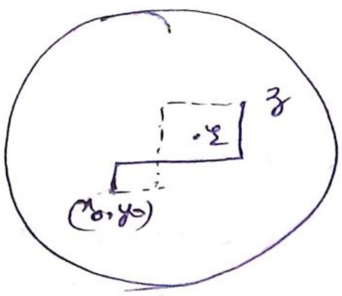
Define a function F(z) by F(z) = $\int_\sigma f(z)dz$ where σ consists of the horizontal line segment from the centre (x_0,y_0) to (x,y_0) and the vertical segment from (x,y_0) to (x,y).It immediately follows that $\frac{\partial F}{\partial y} = if(z)$.Similarly if σ is replaced by a path consisting of a vertical segment followed by a horizontal segment we get $\frac{\partial F}{\partial x} = f(z)$ ⇒ F(z) is analytic in Δ with the derivative f(z).Thus for every closed curve $\int_\gamma f(z)dz = 0$ since the integrand is an exact differential.

Theorem 4

Let f(z) be analytic in the region Δ' obtained by omitting a finite number of points ξ_j from an open disc Δ .If f(z) satisfies the condition $\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0$ for all 'j', then

$\int_\gamma f(z)dz = 0$ holds for any closed curve γ in Δ'.

Proof



The proof must be modified for we cannot let σ pass through the exceptional points. Let us

assume that no ξ_j lies on the lines x=x_0 and y=y_0 . It is then possible to avoid the exceptional points by letting σ consisting of three segments.

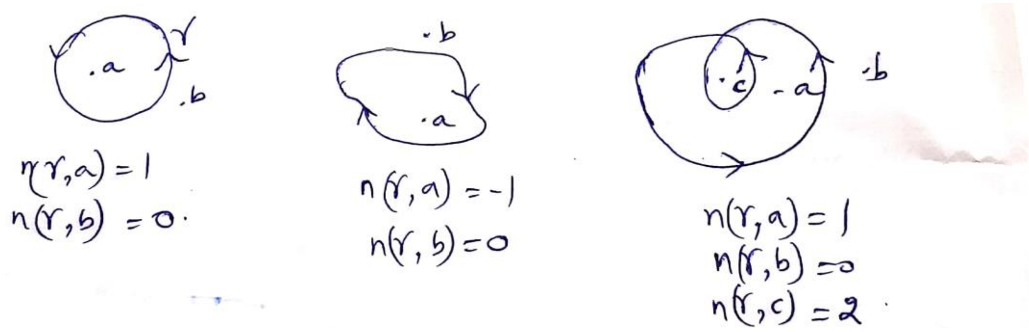
By applying theorem 3, we find that the value of F(z) is independent of the choice of the middle segment. Moreover, the last segment can be either vertical or horizontal. We conclude as before that F(z) is an indefinite integral of f(z) and the theorem follows.

In case there are exceptional points on the lines x=x_0 and y=y_0 , a similar proof can be carried out provided we use four line segments instead of three.

Index of a point with respect to a closed curve

The number of times a closed curve winds around a fixed point not on the curve is known as the Index of a point with respect to the curve. It is also known as the winding number of γ with respect to 'a' and is denoted by $n(\gamma, a)$ and is defined by,

$$n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$$



Theorem

For any closed curve γ and $a \notin \gamma$, $n(\gamma, a)$ is an integer.

Proof

Let the equation of γ be $z = z(t), \alpha \leq t \leq \beta$ .Consider the function

$h(t) = \int_{\alpha}^{t} \frac{z\prime(t)}{z(t)-a}\, dt$. It is defined and continuous on the closed interval $[\alpha, \beta]$ and it has the derivative $h'(t) = \frac{z\prime(t)}{z(t)-a} dt$ whenever $z'(t)$ is continuous. Now,

$\frac{d}{dt}\left[e^{-h(t)}(z(t) - a)\right] = 0 \Rightarrow e^{-h(t)}[z(t) - a] =$ a constant say 'k'

$\Rightarrow z(t) - a = ke^{h(t)}$

Now when $t = \alpha$, $z(\alpha) - a = ke^{h(\alpha)}$ .But $h(\alpha)$ by definition is zero $\Rightarrow z(\alpha) - a = k.$

$\therefore\ e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$. If we put $t = \beta, e^{h(\beta)} = \frac{z(\beta)-a}{z(\alpha)-a} = 1$ since $z(\alpha) = z(\beta)$ as γ is closed

$\Rightarrow h(\beta)$ is an integer multiple of $2\pi i$. But $h(\beta) = \int_{\alpha}^{\beta} \frac{z\prime(t)}{z(t)-a} dt = \int_{\gamma} \frac{dz}{z-a}$ is an integer multiple of $2\pi i$

(or) $n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$ is an integer.

Properties

(i) $n(-\gamma, a) = -n(\gamma, a)$

(ii) If γ lies inside a circle, then $n(\gamma, a) = 0$ for all points 'a' outside of the same circle

(iii) As a function of 'a' the index $n(\gamma, a)$ is constant in each of the regions determined by γ and zero in the unbounded region.

The Integral Formula

Suppose that f(z) is analytic in an open disc Δ and let γ be a closed curve in Δ .For any point 'a' not on γ

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}dz$$

Proof

Consider $F(z) = \frac{f(z)-f(a)}{z-a}$ .This function is analytic everywhere except at z = a. For z = a it is not defined but it satisfies the condition $\lim_{z\to a}(z-a)F(z) = \lim_{z\to a} f(z) - f(a) = 0$ which

is the condition of the theorem 4 and hence,

$\int_{\gamma} \frac{f(z)-f(a)}{z-a}dz = 0 \Rightarrow \int_{\gamma} \frac{f(z)}{z-a}dz = f(a)\int_{\gamma} \frac{dz}{z-a} \Rightarrow \int_{\gamma} \frac{f(z)}{z-a}dz = f(a).2\pi i\, n(\gamma, a).$

Note

If γ is a simple closed curve, then $n(\gamma, a) = 1$ and the integral formula reduces to

$$\boldsymbol{f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}dz}$$

This is known as **Cauchy's Integral Formula**.

Higher Derivatives

In Cauchy's Integral formula, we may let 'a' take different values provided that the order of 'a' with respect to γ remains equal to 1. So we can treat 'a' as a variable and change the notation and rewrite as

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{\xi - z}d\xi$$

We consider a function f(z) which is analytic in an arbitrary region Ω.To a point $a \in \Omega$ we can find a δ-neighbourhood Δ contained in Ω and in Δ a circle C about 'a', Then the Integral formula can be applied to f(z) in Δ.Since $n(C, a) = 1$ we have $n(C, z) = 1$ for all points 'z' inside C. For such 'z' we can write $f(z) = \frac{1}{2\pi i}\int_{C} \frac{f(\xi)}{\xi - z}d\xi$

Differentiating above, provided the integral can be differentiated under the integral sign, we get

$$f'(z) = \frac{1}{2\pi i}\int_{C} \frac{f(\xi)}{(\xi - z)^2}d\xi$$

Differentiating 'n' times, in general, we get

$$\boldsymbol{f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C} \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi}$$

The above equation gives a **"formula for derivatives".**

Lemma

Suppose that $\phi(\xi)$ is continuous on the arc γ .Then the function $F_n(z) = \int_\gamma \frac{\phi(\xi)}{(\xi-z)^2} d\xi$ is analytic in each of the region determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$

Remark

When f(z) is analytic, its derivatives of all orders are also analytic.

This remark is obvious from the above lemma.

Morera's Theorem

If f(z) is defined and continuous in a region Ω ,and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in Ω ,then f(z) is analytic in Ω.

Proof

Given $\int_\gamma f(z)dz = 0$ where γ is a closed curve. Let z_1 and z_2 be two points on γ.Let C_1 be the contour from z_1 to z_2 which is a part of γ and C_2 be another contour from z_1 to z_2 such that $\gamma = C_1 - C_2$. Then, $\int_{C_1-C_2} f(z)dz = 0 \Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

i.e, the integral is independent of path. Therefore, by corollary to theorem 1, there exists an analytic function F(z) such that $F'(z) = f(z)$. Also since whenever f(z) is analytic its derivatives of all orders are also analytic, we have $F'(z)$ is analytic ⇒ f(z) is analytic.

Cauchy's Inequality (or) Cauchy's Estimate

$$\left|f^{(n)}(a)\right| \leq n! \frac{M_r}{r^n}$$

where f(z) is analytic within and on a circle C of radius 'r' and z = a is the centre of the circle C and M_r be the upper bound of f(z) in the circle C.

Proof

The above inequality can be obtained by the simple application of Cauchy's integral formula for derivatives.

Liouville's Theorem

A function which is analytic and bounded in the whole plane must reduce to a constant.

Proof

As the function f(z) is analytic in the whole plane, Cauchy's inequality when n = 1 holds for any choice of 'a' and 'r' i.e, $|f'(a)| \leq \frac{M_r}{r}$

As f(z) is also bounded in the entire complex plane, there exists an M > 0 such that $|f(z)| \leq M$ ∀ 'z' in the complex plane. Thus, $M_r \leq M$ whatever be the value of 'r' may be.

Hence $|f'(a)| \leq \frac{M}{r}$ .This is true for every value of ‘a’. If we select ‘r’ to be as big as possible then $|f'(a)| = 0 \Rightarrow f'(a) = 0 \Rightarrow f'(z) = 0$ for every point ‘z’ in the complex plane (as ‘a’ is arbitrary)

$\Rightarrow$ f(z) is a constant.

Fundamental Theorem of Algebra

Any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ of degree ‘n’ (n≥1) has at least one zero.

Proof

If possible, let us assume there is no point ‘a’ such that P(a) = 0. Let f(z) = 1/ P(z). Then f(z) would be analytic in the whole plane. $P(z) \to \infty$ as $z \to \infty$ and therefore 1/ P(z) $\to 0$ as

$z \to \infty$. This implies boundedness in the whole plane and hence by Liouville’s theorem

1/ P(z) would reduce a constant which is a contradiction as P(z) is not a constant. Hence our assumption is wrong. Therefore, there exists atleast one zero for the equation P(z) = 0.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – Local Properties of Analytical Functions – SMTA5201

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Local Properties of Analytic Functions

Theorem 1

Suppose that f(z) is analytic in the region Ω' obtained by omitting a point 'a' from a region Ω .A necessary and sufficient condition that there exist an analytic function in Ω which coincides with f(z) in Ω' is that $\lim_{z\to a}(z-a)f(z)=0$ .The extended function is uniquely determined.

Taylor's Theorem (Finite Form)

If f(z) is analytic in a region Ω, containing 'a', it is possible to write

$$f(z)=f(a)+\frac{(z-a)}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots+\frac{(z-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+(z-a)^n f_n(z)$$

where $f_n(z)$ is analytic in Ω and the expression for $f_n(z)$ is given by the integral,

$$f_n(z)=\frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-a)^n(\xi-z)}d\xi$$

Proof

Consider the function

$$F(z)=\frac{f(z)-f(a)}{z-a}$$

Then this function is not analytic at z = a but $\lim_{z\to a}(z-a)F(z)=0$ is satisfied. Hence by previous theorem there exists an analytic function which is equal to F(z) for $z\neq a$ and equal to $f'(a)$ for z = a. Let this function be $f_1(z)$.Repeating the process we can define an analytic function $f_2(z)$ which equals $\frac{f_1(z)-f_1(a)}{z-a}$ for $z\neq a$ and equal to $f_1{}'(a)$ for z = a and so on. Proceeding like this we get the following sequence of functions:

$$f(z)=f(a)+(z-a)f_1(z)$$

$$f_1(z)=f_1(a)+(z-a)f_2(z)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_{n-1}(z)=f_{n-1}(a)+(z-a)f_n(z)$$

Substituting the respective values, we prove the theorem.

Zero of a function

A point z = a is said to be a zero of f(z) if f(a)=0. In that case z = a will be a factor of f(z). z = a is said to be a **zero of order 'm'** if $\boldsymbol{f(z)=(z-a)^m\phi(z)}$ where $\phi(a)\neq 0$.

Isolated Zero

A point z = a is said to be an isolated zero of f(z) if there exists a neighbourhood of 'a' in which there is no other zero of f(z).

Result

Zero of an analytic function is isolated.

Singular Point or Singularity

The point z = a is said to be a singular point of a function f(z) if f(z) is not analytic at 'a'.

Isolated singularity

A point z = a is said to be an isolated singularity of f(z) if there exists a neighbourhood of 'a' in which there is no other singularity of f(z).

Types of isolated singularity

(i) Removable Singularity

If $\lim_{z \to a} f(z)$ exists and is finite then z = a is said to be a removable singularity. e.g, $f(z) = \frac{sinz}{z}$. Here z = 0 is a removable singularity.

(ii) Pole

If $\lim_{z \to a} f(z) = \infty$ then z = a is said to be a pole. If z = a is a pole of order 'm' then we can write f(z) as $f(z) = \frac{\phi(z)}{(z-a)^m}$ where $\phi(a) \neq 0$ and $\phi(z)$ is analytic. e.g, $f(z) = \frac{e^z}{z}$, $f(z) = \frac{e^z}{(z-1)^3}$ .Here z = 0 and z = 1 are poles of order '1' and '3'

(iii) Essential Singularity

If $\lim_{z \to a} f(z)$ does not exist then z = a is said to be an essential singularity. e.g, $f(z) = e^{1/z}$. Here z = 0 is an essential singularity.

Note : Consider the conditions

$(i)\ \lim_{z \to a} |z-a|^{\alpha}\, |f(z)| = 0 \quad (ii)\ \lim_{z \to a} |z-a|^{\alpha}\, |f(z)| = \infty$ for real values of α.

If neither (i) nor (ii) holds for any α ,then the point 'a' is called an essential singularity.

Weierstrass Theorem

An analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity.

Proof

Let us prove by assuming the contrary. Then we could find a complex number A > 0 and $\delta > 0$ such that $|f(z) - A| > \delta$ in a deleted neighbourhood of 'a'. Thus, for any $\alpha < 0$, we have $\lim_{z \to a} |z-a|^{\alpha}\, |f(z) - A| = \infty$ .Hence 'a' would not be an essential singularity of f(z) – A

Accordingly, there exists a β with $\lim_{z \to a} |z-a|^{\beta}\, |f(z) - A| = 0$ and we are free to choose $\beta > 0$. Hence in that case , $\lim_{z \to a} |z-a|^{\beta}\, |f(z)| = 0$ which implies that 'a' would not be an essential singularity of f(z) which contradicts that 'a' is an essential singularity. This contradiction proves the theorem.

Theorem 2

Let z_j be the zeros of a function f(z) which is analytic in a disc Δ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = \sum_{j} n(\gamma, z_j)$$

where the sum has only a finite number of terms not equal to zero.

Proof

Let us assume that f(z) has only a finite number of zeros in Δ say $z_1, z_2, \ldots\ z_n$ where each zero is repeated as many times as its order indicates. Then we can write f(z) as

$f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)g(z)$ where g(z) is analytic and not equal to zero in Δ.

Taking 'log' on both sides of the above equation and differentiating, we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

where $z \neq z_j$ and particularly on γ.Integrating the above equation over γ with respect to 'z'

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z - z_1} + \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z - z_2} + \cdots \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z - z_n} + \frac{1}{2\pi i}\int_{\gamma} \frac{g'(z)}{g(z)}dz$$

Since g(z) $\neq 0$ in Δ ,by Cauchy's theorem, $\int_{\gamma} \frac{g'(z)}{g(z)}dz = 0$.Therefore,we get,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = n(\gamma, z_1) + n(\gamma, z_2) + \cdots + n(\gamma, z_n)$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = \sum_{j} n(\gamma, z_j) \cdots\cdots\cdots\cdots\cdots\cdots\cdots (A)$$

where the sum has only a finite number of terms $\neq 0$.

If on the other hand f(z) has infinitely many zeros in Δ,then f(z) will have only finite number of zeros in Δ',a concentric disc smaller than Δ , by Bolzano Weierstrass theorem and hence for the zeros outside Δ', $n(\gamma, z_j) = 0$,thus do not contributing to the summation.

Note If the function $w = f(z)$ maps γ onto a closed curve Γ in the w – plane, then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\Gamma} \frac{dw}{w} = n(\Gamma, 0)$$

By the equation (A), we can conclude that.

$$n(\Gamma, 0) = \sum_j n(\gamma, z_j) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (B)$$

When γ is a circle then $n(\gamma, z_j)$ must be either 0 or 1. Then the equation (A) gives the total number of zeros enclosed by γ.

Note

Let 'a' be any arbitrary complex value. Applying theorem 2 to the function f(z) – a, we find that the zeros of f(z) – a are the roots of the equation f(z) = a and let us denote them by $z_j(a)$.

$$\therefore \; \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}dz = \sum_j n(\gamma, z_j(a)) \quad \& \quad (B)$$

$$\Rightarrow \; n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

Note

If 'a' & 'b' are in the same region determined by Γ then $n(\Gamma, a) = n(\Gamma, b)$ & so we obtain

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b))$$

If γ is a circle it follows that f(z) takes the value 'a' & 'b' equally many times inside of γ.

Local Correspondence Theorem

Suppose that f(z) is analytic at z_0, $f(z_0) = w_0$, and that f(z) - w_0 has a zero of order 'n' at z_0 . If $\epsilon > 0$ is sufficiently small there exists a $\delta > 0$ such that for all 'a' with $|a - w_0| < \delta$, the equation f(z) = a has exactly 'n' roots in the disk $|z - z_0| < \epsilon$.

Proof

Choose ϵ so that f(z) is defined and analytic for $|z - z_0| \le \epsilon$ & so that z_0 is the only zero of f(z) $- w_0$ in this disk. Let γ be the circle $|z - z_0| = \epsilon$ and Γ its image under the mapping w = f(z). Since w_0 belongs to the complement of the closed set Γ there exists a neighbourhood $|w - w_0| < \delta$ which does not intersect Γ.Now, $n(\Gamma, w_0) = n(\Gamma, a)$ for every 'a' in this neighbourhood and since $n(\Gamma, w_0) = \sum_j n(\gamma, z_j(w_0)) = n$ whenever γ is a circle. The equation $f(z) = w_0$ has exactly 'n' coinciding roots inside of γ and hence the equation the equation f(z) = a has exactly 'n' roots in the disk $|z - z_0| < \epsilon$.

Open Mapping Theorem

A non-constant analytic function maps open sets onto open sets.

Proof

Let f(z) be a non-constant analytic function in a region Ω. Let U be an open subset of Ω.Let $z_0 \in U$ such that $f(z_0) = w_0$.Then there exists a positive integer n ≥ 1 such that $f(z) - w_0$ has a zero of order 'n' at z_0 .By local correspondence theorem given $\epsilon > 0$ there exists a $\delta > 0$ such that for all 'a' in the neighbourhood $|w - w_0| < \delta$, the equation f(z) = a has 'n' roots in the disk $|z - z_0| < \epsilon$ & $|z - z_0| < \epsilon$ is a subset of U. Thus $|w - w_0| < \delta$ is completely contained in f(U) i.e, w_0 is interior to f(U).Since w_0 is arbitrary, f(U) is open.

Corollary

If f(z) is analytic at z_0 with $f'(z_0) \neq 0$ it maps a neighbourhood of z_0 conformally and topologically onto a region.

The Maximum Principle

If f(z) is analytic and non-constant in a region Ω , then its absolute value $|f(z)|$ has no maximum in Ω.

Proof

Let z_0 be any point in Ω and let $f(z_0) = w_0$.Then there exists a neighbourhood $|w - w_0| < \epsilon$ contained in the image of Ω. In this neighbourhood, there are points of modulus greater than $|w_0|$ and hence $|f(z_0)|$ is not the maximum of $|f(z)|$.

Another Form of Maximum Principle

If f(z) is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.

Schwarz Lemma

If f(z) is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, f(0) = 0 then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ .If $|f(z)| = |z|$ for some z ≠ 0 , or if $|f'(0)| = 1$,then f(z) = cz with a constant 'c' of absolute value 1.

Proof

Consider the function $F(z) = \frac{f(z)}{z}$ defined by $F(z) = \begin{cases} \frac{f(z)}{z} , & z \neq 0 \\ f'(0) , & z = 0 \end{cases}$

Then F(z) is analytic in $|z| < 1$. On any circle $|z| = r < 1$, $|F(z)| = \left|\frac{f(z)}{z}\right| \leq \left|\frac{1}{r}\right|$. Allowing 'r' to tend to '1', $|F(z)| \leq 1$ i.e, $|f(z)| \leq |z|$ .Also $|F(0)| \leq 1 \Rightarrow |f'(0)| \leq 1$. Suppose $|f(z)| = |z|$ for some z ≠ 0 say z_0 where $|z_0| < 1$ then, we have, $|f(z_0)| = |z_0| \Rightarrow |F(z_0)| = 1$ .This implies that $|F(z)|$ attains its maximum at z_0 and hence by maximum principle F(z) must reduce to a constant i.e, F(z) = c ⇒ f(z) = cz where $|c| = 1$.

Definition of a Chain

Consider a formal sum of arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$ represented by $\gamma_1 + \gamma_2 + \cdots + \gamma_n$ satisfying

$$\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

Such a formal sum of arcs is called a chain.

Definition of a Cycle

A chain is said to be a cycle if it can be represented as a sum of closed curves.

Simple Connectivity

A region is said to be simply connected if its complement with respect to the extended plane is connected.

e.g, A disc, a half plane and a parallel strip are simply connected.

Theorem 3

A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω,and all points 'a' which do not belong to Ω.

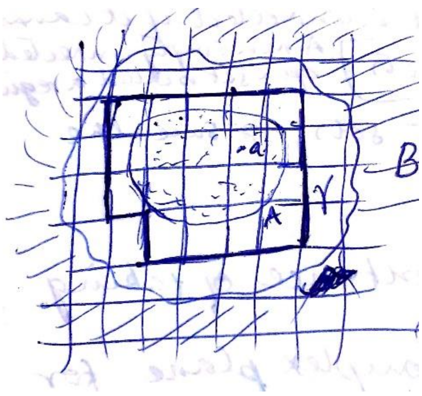
Proof

Necessary Part

Let γ be any cycle in Ω and let Ω be simply connected. Then the complement of Ω is connected. Thus it should be contained in one of the components determined by γ.But ∞ belongs to this complement and hence the complement must be the unbounded component determined by γ.Consequently $n(\gamma, a) = 0 \ \forall\, a \in \Omega^c$.

Sufficient Part

Let $n(\gamma, a) = 0$ for all cycles γ in Ω and all points 'a' which do not belong to Ω.Let us assume the contrary and arrive at a contradiction. Thus, let us assume that Ω^c is not connected then we can write Ω^c as $\Omega^c = A \cup B$ where A and B are two disjoint closed sets. One of these must contain the point at infinity. Let it be B. Then A is closed and bounded. Cover the plane by a net of squares of side δ such that a particular point 'a' $\in$ A falls at the centre of a square in the net.



Let Q denote the squares in the net. Consider the cycle

$$\gamma = \sum_j \partial Q_j$$

where the sum ranges over all the squares in the net which have a point in common with A . Obviously γ does not intersect with B. It also does not intersect with B and hence γ doesn't intersect A and B which implies $\gamma \notin \Omega^c$ i.e, γ lies in Ω .

$$\therefore \quad n(\gamma, a) = \sum_j n(Q_j, a) = 1$$

which is a contradiction with the hypothesis that $n(\gamma, a) = 0$.This contradiction proves the theorem.

Definition of Homology

A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all the points in the complement of Ω.It is written as γ is homologous to zero (mod Ω) i.e, $\gamma \sim 0\ (mod\ \Omega)$. $\gamma_1 \sim \gamma_2$ is equivalent to $\gamma_1 - \gamma_2 \sim 0\ (mod\ \Omega)$.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – The General Form of Cauchy's Theorem and Calculus of Residues – SMTA5201

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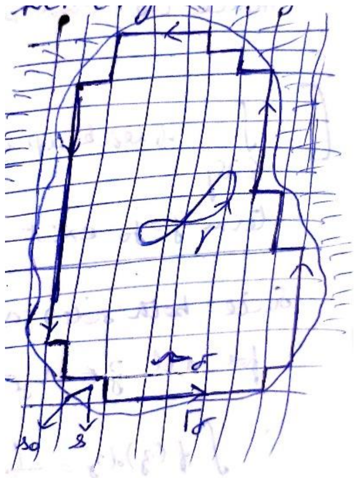
General Form of Cauchy's Theorem

If f(z) is analytic in a region Ω , then $\int_{\gamma} f(z)dz = 0$ $\forall$ cycle γ which is homologous to zero in Ω .

Proof

Case (i)

Assume that Ω is bounded. Given $\delta > 0$ cover the plane by a net of squares and denote the closed squares in the net contained in Ω as Q_j , $j \in J$.



As Ω is bounded, J is finite and non-empty. Define the cycle $\Gamma_\delta = \sum_j \partial Q_j$ and denote Ω_δ as the interior of the union $\bigcup_{j\in J} Q_j$. Let γ be a cycle which is homologous to zero in Ω .Choose δ so small that γ is contained in Ω_δ.

Consider a point $s \in \Omega - \Omega_\delta$. It belongs to at least one Q which is not a Q_j.There is a point s_0 belonging to this Q which is not in Ω .Since $\gamma \subset \Omega_\delta$ it follows that $n(\gamma, s_0) = 0$.In fact,

$n(\gamma, s) = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots (A)$ for all points on Γ_δ. Now,

$$\frac{1}{2\pi i}\int_{Q_j} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} f(z) & for\ j = j_0 \\ 0 & for\ j \neq j_0 \end{cases}$$

Hence, we can say that

$$\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{\sum_j \partial Q_j} \frac{f(\xi)}{\xi - z} d\xi = f(z)$$

$$\therefore \int_{\gamma} f(z)dz = \frac{1}{2\pi i}\int_{\gamma} \int_{\Gamma_\delta} \frac{f(\xi)}{\xi - z} d\xi\, dz = \frac{1}{2\pi i}\int_{\Gamma_\delta} \left[\int_{\gamma} \frac{dz}{\xi - z}\right] f(\xi) d\xi = 0$$

using equation (A). This implies, $\int_{\gamma} f(z)dz = 0$

Case (ii)

Let Ω be unbounded. Let Ω' denote the intersection of Ω with the large disk $|z| = R$ which is large enough to contain the cycle γ.Then Ω' is bounded.

Any point 'a' in the complement of Ω' is either in the complement of Ω or in any case lies outside the disk $|z| = R$ .In either case $n(\gamma, a) = 0$ so that $n(\gamma, a) = 0\ (mod\ \Omega')$.

Hence by case(i) the theorem is applicable to Ω'.Hence the theorem is true for any arbitrary Ω.

Corollary

If f(z) is analytic in a simply connected region Ω, then $\int_\gamma f(z)dz = 0\ \forall$ cycles γ in Ω.

Proof

By the previous theorem, Ω is simply connected ⇔ $n(\gamma, a) = 0\ \forall$ cycles γ in Ω and all the points 'a' which do not belong to Ω which implies γ is homologous to zero in Ω

∴ By Cauchy's theorem, $\int_\gamma f(z)dz = 0$

Calculus of Residues

Consider functions f(z) which are analytic in a region Ω except for isolated singularities. Let there be only a finite number of singular points $a_1, a_2, \cdots, a_n$.The region obtained by excluding the points $a_j{}'s$, j = 1(1)n is denoted by Ω'.

To each a_j there exists a $\delta_j > 0$ such that the deleted neighbourhood $0 < |z - a_j| < \delta_j \subseteq \Omega'$ Let C_j be a circle about a_j with radius less than δ_j and let $P_j = \int_{C_j} f(z)dz$ be the corresponding period. The particular function $\frac{1}{z-a_j}$ has a period given by $P_j = 2\pi i$ `Now define $R_j = \frac{P_j}{2\pi i}$. Then the function f(z) - $\frac{R_j}{z-a_j}$ has a vanishing period i.e, period zero. The constant R_j which produces this result is called the Residue of f(z) at the point a_j

Definition of Residue of a function at a point

The Residue of f(z) at an isolated singularity 'a' is the unique complex number R which makes f(z) - $\frac{R}{z-a}$ the derivative of a single valued analytic function in the annulus $0 < |z - a| < \delta$ .The residue of f(z) at 'a' is denoted by Res (f, a).

Cauchy's Residue Theorem

Let f(z) be analytic except for isolated singularities a_j in a region Ω .Then,

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j)\ Res(f, a_j)$$

for any cycle γ which is homologous to zero in Ω and doesn't pass through any of the points a_j .

Proof

Let us assume that there only a finite number of singular points $a_1, a_2, \cdots, a_n$ for the function f(z) in Ω .The region obtained by excluding the points $a_j's$, j = 1(1)n is denoted by Ω'. To each a_j there exists a $\delta_j > 0$ such that the deleted neighbourhood $0 < |z - a_j| < \delta_j \subseteq \Omega'$ Let C_j be a circle about a_j with radius less than δ_j and let $P_j = \int_{C_j} f(z)dz$ be the corresponding period. The particular function $\frac{1}{z-a_j}$ has a period given by $P_j = 2\pi i$ `Now define $R_j = \frac{P_j}{2\pi i}$. Then the function f(z) - $\frac{R_j}{z-a_j}$ has a vanishing period.

Let γ be any cycle in Ω' which is homologous to zero in Ω. Then it satisfies the homology

$\gamma \sim \sum_j n(\gamma, a_j)C_j$ and hence we obtain

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} n(\gamma, a_j) \int_{C_j} f(z)dz = \sum_{j=1}^{n} n(\gamma, a_j)P_j = 2\pi i \sum_{j=1}^{n} n(\gamma, a_j) R_j$$

$$\therefore \frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{j=1}^{n} n(\gamma, a_j) Res(f, a_j)$$

In the general case when there are infinite number of singularities $a_j's$ it can be shown that $n(\gamma, a_j) = 0$ except for a finite number of points a_j by Bolzano Weierstrass property for otherwise it would contradict the fact that the $a_j's$ are isolated. Hence the theorem on the RHS has only a finite number of terms.

Note

In most applications $n(\gamma, a_j) = 0$ or 1 and hence the residue theorem takes the form,

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j Res(f, a_j)$$

where the sum is extended over all its singularities enclosed by γ.

Argument Principle

If f(z) is meromorphic in Ω with zeros a_j and poles b_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles

Proof

Assuming that the function f(z) has a zero of order ‘h’ at the point z = a and has a pole of order ‘h’ at the point z = b we can show that z = a is a simple pole for the function $\frac{f'(z)}{f(z)}$ with residue ‘h’ and z = b is a simple pole for the function $\frac{f'(z)}{f(z)}$ with residue ‘-h’. Applying Cauchy’s Residue theorem for the function $\frac{f'(z)}{f(z)}$ the theorem is proved.

Rouche’s Theorem

Let γ be homologous to zero in Ω such that $n(\gamma, z) = 0$ is either ‘0’ or ‘1’ for any point ‘z’ not on γ.Suppose that f(z) and g(z) are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ.Then f(z) and g(z) have the same number of zeros enclosed by γ.

Proof

Let $F(z) = \frac{g(z)}{f(z)}$ then the condition $|f(z) - g(z)| < |f(z)| \Rightarrow |F(z) - 1| < 1$ which in turn implies that $n(\Gamma, 0) = 0$ .Hence we get $\frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)} dz = n(\Gamma, 0) = 0$ which on simplification gives, $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)} dz$ .By Argument Principle, Rouche’s theorem is proved.

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UNIT – IV – Harmonic Functions and Power Series Expansions – SMTA5201

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Harmonic Function definition

A real-valued function u(z) or u(x,y) defined and single-valued in a region Ω is said to be harmonic in Ω (or) a potential function if it is continuous together with its partial derivatives of the first order and satisfy the Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

In polar co-ordinates the Laplace equation takes the form $r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.$

u = log r is a harmonic function and hence any linear function of the form 'a log r + b' is harmonic.

Result

Suppose 'u' is harmonic in Ω ,then the function $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in Ω. Then

$f(z)dz = du + i * du$ where $* du$ is called the conjugate differential of du.

Remark 1

$\int_\gamma * du = 0$

Remark 2

$* du = \frac{\partial u}{\partial n}|dz|$ where $\frac{\partial u}{\partial n}$ is the directional derivative of 'u' in the direction of the normal 'n' with respect to the curve γ. If γ is a circle then $\frac{\partial u}{\partial n}$ can be replaced by $\frac{\partial u}{\partial r}$.

Theorem 1

If u_1 and u_2 are harmonic in a region Ω ,then $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ where γ is any cycle which is homologous to zero in Ω.

Proof

Since every cycle γ can be approximated by a polygonal path whose sides are made up of line segments which are parallel to the real and imaginary axis it is sufficient to prove the theorem for $\gamma = \partial R$ where R is a rectangle contained in Ω.In a rectangle u_1 and u_2 have harmonic conjugates v_1 and v_2 .Also $(u_1 + i v_1)(du_2 + i dv_2) = f_1 f_2 dz$

where $f_1 = u_1 + i v_1$ and $f_2 = = \frac{\partial u2}{\partial x} - i\frac{\partial u2}{\partial y}$ are analytic functions in Ω which proves the theorem.

Mean value Property of Harmonic Functions

The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of log r and we have $\frac{1}{2\pi}\int_{|z|=r} u \, d\theta = \alpha \log r + \beta$ and if 'u' is harmonic in a disc $\alpha = 0$ and the arithmetic mean is constant.

Proof

Apply the previous theorem with u_1 = log r and u_2 = a harmonic function 'u' in $|z| < \rho$. Choose γ to be the cycle $C_1 - C_2$ where C_i is a circle $|z| = r_i < \rho$.Then we get,

$$\int_{|z|=r} u\,d\theta - \log r \int_{|z|=r} r\frac{\partial u}{\partial r}\,d\theta = \beta$$

Also, $\int_{\gamma} *du = 0 \Rightarrow \int_{|z|=r} r\frac{\partial u}{\partial r}\,d\theta = \alpha$. Combining the above two equations, we get

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \alpha\log r + \beta$$

If u is harmonic in the whole disk, then $\alpha = 0$ by allowing $r \to 0$.Hence

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \beta$$

Poisson's Formula

Suppose that u(z) is harmonic for $|z| < R$ and continuous for $|z| \leq R$ .Then,

$$u(a) = \frac{1}{2\pi\cdot}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\,d\theta \quad for\ all\ \ |a| < R$$

Proof

The linear transformation $S(s) = z = \frac{R(Rs+a)}{R+\bar{a}S} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$ maps the circle $|s| \leq 1$ onto $|z| \leq R$ with s = 0 corresponding to z = a .By Mean value

theorem, $u(a) = \frac{1}{2\pi\cdot}\int_{|s|=1} u\big(S(s)\big)\,d(\arg s)$. From (1), $s = \frac{R(z-a)}{R^2-\bar{a}z}$

Taking log and differentiating, $\frac{1}{s}ds = \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2-\bar{a}z}\right)dz.$ Also by using the result,

$d\emptyset = \frac{ds}{is}$ $and\ d\theta = \frac{dz}{iz}$ where $\theta = \arg z$ and $\emptyset = \arg s$,

we get, d (arg s) $= \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2-\bar{a}z}\right)d\theta.$ On Simplification, we get, d (arg s)$= \frac{R^2-|a|^2}{|z-a|^2}\ d\theta$

Hence proved.

Weierstrass Theorem on convergence of sequence of Analytic Functions

Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function f(z) in a region Ω, uniformly on every compact subset of Ω.Then f(z) is analytic in Ω. Moreover $f_n{}'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.

Hurwitz Theorem

If the functions $f_n(z)$ are analytic and non-zero in a region Ω ,and if $f_n(z)$ converges to f(z) uniformly on every compact subset of Ω ,then f(z) is either identically zero or never equal to zero in Ω.

Proof

Suppose that f(z) is not identically zero. Then to prove f(z) is never equal to zero in Ω.Then

for any point z_0 we can find a neighbourhood in which $|f(z)|$ has a positive minimum. It follows that $\frac{1}{f_n(z)} \to \frac{1}{f(z)}$ uniformly on a circle C with z_0 as centre and radius 'r'. Also, by Weierstrass theorem, $f_n{}'(z)$ converges uniformly to $f'(z)$ on C. Hence it can be concluded that $\lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n{}'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz$ .By applying Argument Principle, the integral on the LHS of the above equation becomes zero which in turn implies that the integral on the RHS of the above equation also becomes zero which implies f(z) is never equal to zero in Ω.

Taylor's Series Theorem

If f(z) is analytic in the region Ω, containing 'z_0', then the representation

$$f(z) = f(z_0) + \frac{(z - z_0)}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + \cdots$$

is valid in the largest open disk of centre z_0 contained in Ω .

Proof

In the finite form of Taylor's theorem, the remainder term namely,

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^n(\xi - z)} d\xi$$

tends uniformly to zero in any disk about z_0 by arbitrarily choosing the radius which proves the theorem.

Laurent's Series Theorem

Let f(z) be analytic in an annular region $R_1 < |z - a| < R_2$ then for any 'z' belonging to the annulus we can write,

$$f(z) = \sum_{-\infty}^{\infty} A_n (z - a)^n \ \ where\ \ A_n = \frac{1}{2\pi i}\int_{|s-a|=r} \frac{f(s)}{(s - a)^{n+1}}\, ds$$

Remark

Laurent's Series development of any function f(z) is unique.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Partial Fractions and Factorization – SMTA5201

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Mittag-Leffler's Theorem

Let $\{b_\gamma\}$ be a sequence of complex numbers with $\lim_{\gamma\to\infty} b_\gamma = \infty$ and let $P_\gamma(s)$ be polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at the point $b_\gamma$and the corresponding singular parts $P_\gamma\left(\frac{1}{z-b_\gamma}\right)$. Moreover, the most general meromorphic function of this kind can be written in the form,

$$f(z) = \sum_{\gamma}\left[P_\gamma\left(\frac{1}{z-b_\gamma}\right) - p_\gamma(z)\right] + g(z)$$

where the $p_\gamma(z)$ are suitably chosen polynomials and g(z) is analytic in the whole plane.

Infinite Products

Consider a sequence $\{p_n\}$ of complex numbers. A product of the form $p_1p_2\cdots p_n\cdots$ denoted by $\prod_{n=1}^{\infty} p_n$ is called an infinite product. An infinite product is evaluated by taking the limit of the partial product $P_n = p_1p_2\cdots p_n$.The product is said to converge if $\lim_{n\to\infty} P_n = P$ exists and is non-zero.

Theorem 1

The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ with $(1+a_n) \neq 0$ converges simultaneously with the series $\sum_{n=1}^{\infty} \log(1+a_n)$ whose terms represent the values of the principal branch of the algorithm.

Theorem 2

A necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$

Problem

1. Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$

   Proof

   Let $P_m = \prod_{n=2}^{m}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{m}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)$

$$\prod_{n=2}^{m}\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right) = \left(\frac{1}{2}.\frac{3}{2}\right)\left(\frac{2}{3}.\frac{4}{3}\right)\left(\frac{3}{4}.\frac{5}{4}\right)\cdots\left(\frac{m-1}{m}.\frac{m+1}{m}\right) = \left(\frac{1}{2}.\frac{m+1}{m}\right)$$

$$= \frac{1}{2}\ as\ m \to \infty$$

$$\therefore \prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$$

1. Prove that for $|z| < 1$, $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}$

   Try this as an exercise.

Weierstrass Theorem on Entire Function

There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$.Every entire function with these and no other zeros can be written in the form,

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers and g(z) is an entire function.

Corollary

Every function which is meromorphic in the whole plane is the quotient of two entire functions.

Canonical Product, Genus of an Entire Function

By Weierstrass theorem on entire functions we have the expression,

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

Suppose it is possible to choose all the m_n's to be equal to some 'h' say then we get the infinite product as

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\frac{1}{h}\left(\frac{z}{a_n}\right)^{h}} \cdots\cdots\cdots (A)$$

This product converges and represents an entire function provided that the series

$\sum_{n=1}^{\infty} \left(\frac{R}{|a_n|}\right)^{h+1} \frac{1}{h+1}$ converges for all R. But this is the same as saying $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}$ is convergent .Suppose 'h' is the smallest integer for which the series converges then the infinite product (A) is called the **Canonical Product** associated with the sequence $\{a_n\}$ and 'h' is called the **Genus of the Canonical Product**.

Hence we can write,

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\frac{1}{h}\left(\frac{z}{a_n}\right)^{h}} \dots\dots\dots\dots (B)$$

If in this representation g(z) reduces to a polynomial the function f(z) is said to be of **finite genus** and the genus of f(z) is by definition equal to the degree of this polynomial (or) the genus of the canonical product whichever is larger.

Problem

1. Find the form of an entire function of genus zero.

Solution

By definition, genus = max {degree of polynomial, genus of the canonical product}

Genus = 0 ⇒ degree of the polynomial is zero and genus of the canonical product is zero which in turn implies g(z) = c (or) $e^{g(z)} = c_1$ and h = 0

$$\Rightarrow f(z) = z^m C_1 \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$$

2. Find the product representation of sin πz

Solution

The zeros of f(z) = sin πz are z = n, (n = $0, \pm 1, \pm 2, \cdots$) and each is a single zero. Then, by Weierstrass theorem, $\sin \pi z = z e^{g(z)} \prod_{n\neq 0}\left(1 - \frac{z}{n}\right) e^{z/n} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (I)$

Taking log and differentiating, we can prove that g(z) = a constant and therefore, $e^{g(z)} = \pi$

Hence, $\sin \pi z = \pi z \prod_{n\neq 0}\left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) e^{z/n} e^{-z/n}$

(or) $sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$

The Gamma Function

The function G(z) = $\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right) e^{-z/n}$ is a canonical product and has for its zeros the negative integers. Then the function H(z) = $e^{\gamma z} G(z)$ satisfies the functional equation H(z-1) = z H(z) where the constant γ is called the Euler's constant whose approximate value is 0.57722.

We define the Gamma Function $\Gamma(z)$ as $\Gamma(z) = \frac{1}{z\, H(z)}$ so that $\Gamma(z+1) = z\,\Gamma(z)$. The function $\Gamma(z)$ is called the Euler's Gamma Function.

Properties of Gamma Function

1. $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$
2. $\Gamma(z)$ is a meromorphic function with simple poles at z = 0 and all the negative integers.

$\Gamma(z)$ has no zeros.

1. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$
2. $\Gamma(n) = (n-1)!$ where 'n' is positive and Γ can be considered as the generalisation of the factorial function.

Remark: $\Gamma(1) = 1$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4. $\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$
5. **Legendre's Duplication Formula**

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$