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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Fourier Series –SMTA1405

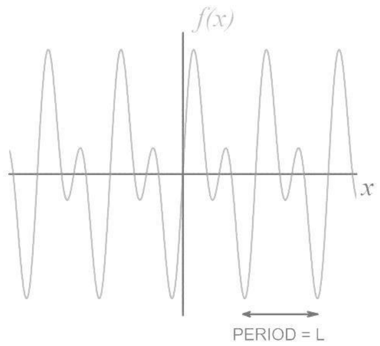
I. Introduction

Contents - Definition- Dirichlets conditions- coefficients- Fourier series for the function defined in [c, c+2π],[c, c+2l] – odd and even functions in fourier series-Parseval's identity (without proof)..

Periodic Functions

A function (x) is said to be periodic, if and only if $(x + L) = (x)$ is true for some value of L and for all values of x. The smallest value of L for which this equation is true for every value of x will be called the period of the function.

A graph of periodic function (x) that has period L exhibits the same pattern every L units along the $x -$ axis, so that $(x + L) = (x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods). For example, $sinx$ and $cosx$ are periodic with period 2π and $tanx$ has period π.



Dirichlet's Conditions

(i) $f(x)$ is single valued and finite in $(c, c + 2\pi)$

(ii) $f(x)$ is continuous or piecewise continuous with finite number of finitediscontinuities in $(c, c + 2\pi)$

(iii) $f(x)$ has a finite number of maxima and minima in $(c, c + 2\pi)$

Note 1: These conditions are not necessary but only sufficient for the existence of Fourierseries.

Note 2: If (x) satisfies Dirichlet's conditions and (x) is defined in $(-\infty, \infty)$, then (x) has to be periodic of periodicity 2π for the existence of Fourier series of period 2π.

Note 3: If (x) satisfies Dirichlet's conditions and (x) is defined in $(c, c + 2\pi)$, then (x) need not be periodic for the existence of Fourier series of period 2π.

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1),$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_\pi^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^\pi\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}\big(1-(-1)^n\big) \\ &= \quad -\frac{1}{n}(-1)^n + \qquad 0 \qquad -\frac{1}{n}\big(1-(-1)^n\big) \end{aligned}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

where $a_0 = \frac{3\pi}{2}$, $\quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $\quad b_n = -\frac{1}{n}$

Example 2

Expand in Fourier series of periodicity 2π $f(x) = xsinx$, for $0 < x < 2\pi$

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} xsinx\, dx$$

$$= \tfrac{1}{\pi}[x(-cosx) - 1.(-sinx)]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi cos2\pi + sin2\pi]$$

$$= \frac{1}{\pi}[-2\pi.1 + 0]$$

$$= \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} xsinxcosnx\, dx$$

$$= \tfrac{1}{2\pi}\int_0^{2\pi} x[sin(n+1)x - \sin(n-1)\,x]\, dx$$

$$= \tfrac{1}{2\pi}\left[x\left(\tfrac{-\cos\,(n+1)x}{n+1}\right) - 1.\left(\tfrac{-\sin(n+1)x}{(n+1)^2}\right) - [x\left(\tfrac{-\cos(n-1)x}{n-1}\right) - 1.\left(\tfrac{-\sin(n-1)x}{(n-1)^2}\right)]\right]_0^{2\pi}$$

$$= \tfrac{1}{2\pi}\left[2\pi\left(\tfrac{-\cos\,(n+1)2\pi}{n+1}\right) - 1.\left(\tfrac{-\sin(n+1)2\pi}{(n+1)^2}\right) - [2\pi\left(\tfrac{-\cos(n-1)2\pi}{n-1}\right) - 1.\left(\tfrac{-\sin(n-1)2\pi}{(n-1)^2}\right)]\right]$$

$$= \tfrac{1}{2\pi}\left[\left(\tfrac{-2\pi}{n+1}\right) - 1.(0) - [\left(\tfrac{-2\pi}{n-1}\right) - 1.(0)]\right]$$

$$= \left(\tfrac{-1}{n+1}\right) + \left(\tfrac{1}{n-1}\right)$$

$a_n = \frac{1}{n^2-1}$ provided $n \neq 1$.

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxcosx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} xsin2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2\,x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-\cos 2(2\pi)}{2}\right) - 1.\left(\frac{-\sin 2(\,2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-1}{2}\right) - 1.(0)\right]$$

$$a_1 = \frac{-1}{2}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} xsinxsinnx\ dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[cos(n-1)x - \cos(n+1)\,x]\ dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin\,(n-1)x}{n-1}\right) - 1.\left(\frac{-\cos(n-1)x}{(n-1)^2}\right) - [x\left(\frac{s\,in(n+1)x}{n+1}\right) - 1.\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{\sin\,(n-1)2\pi}{n-1}\right) - 1.\left(\frac{-\cos(n-1)2\pi}{(n-1)^2}\right) - [2\pi\left(\frac{\sin(n+1)2\pi}{n+1}\right) - 1.\left(\frac{-\cos(n+1)2\pi}{(n+1)^2}\right)]\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{(n-1)^2}\right) - \left[\left(\frac{-1}{(n+1)^2}\right)\right] - \left(\frac{1}{(n-1)^2}\right) + \left[\left(\frac{-1}{(n+1)^2}\right)\right]\right]$$

$b_n = 0$ provided $n \neq 1$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxsinx\ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xsin^2x\ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1 - cos2x}{2}\right)\ dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{sin2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{cos2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\left(2\pi - \frac{sin2(2\pi)}{2}\right) - 1.\left(\frac{(2\pi)^2}{2} + \frac{cos2(2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2 - 2\pi^2 + \frac{1}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{\pi}[2\pi^2]$$

$$b_1 = \pi$$

Therefore, the Fourier series expansion of the function $xsinx$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n\, sinnx)$$

$$xsinx = 1 - \frac{1}{2}cosx + 2\sum_{2}^{\infty}\frac{cosnx}{n^2 - 1} + \pi sinx$$

Example 3

Obtain all the Fourier coefficients of $f(x) = k$ where k is a constant, the periodicity being 2π.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} k\, dx$$

$$= \frac{k}{\pi}\int_0^{2\pi} dx$$

$$= \frac{k}{\pi}\,[x]_0^{2\pi}$$

$$= \frac{k}{\pi}\,[2\pi]$$

$$a_0 = 2k$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} k\, cosnxdx$$

$$= \frac{k}{\pi}\int_0^{2\pi} cosnxdx$$

$$= \frac{k}{\pi}\left[\frac{sinnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{sin2n\pi - sin0}{n}\right]$$

$$a_n = 0$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\ sinnxdx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} k\ sinnxdx$$

$$= \frac{k}{\pi}\int_0^{2\pi} sinnxdx$$

$$= \frac{k}{\pi}\left[\frac{-cosnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{cos2n\pi - cos0}{n}\right]$$

$$= \frac{k}{\pi}\left[\frac{1-1}{n}\right]$$

$$b_n = 0$$

Even and Odd Functions

The function f(x) is said to be even, if f(-x) = f(x).

The function f(x) is said to be odd, if f(-x) = -f(x).

If $f(x)$ is an even function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier cosine series**:**

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\ cosnx}$$

where the Fourier coefficients $\boldsymbol{a_0}$ and $\boldsymbol{a_n}$ are calculated by

(1) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

(2) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ cosnxdx$

If $f(x)$ is an odd function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier sine series**:**

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n\ sinnx}$$

where the Fourier coefficient $\boldsymbol{b_n}$ is calculated by $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ sinnxdx$

Example 4

Find the Fourier series for $f(x) = |cosx|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution.

Since $f(x) = |cosx|$ is an even function, $f(x)$ will contain only cosine terms.

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx$

STEP ONE

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosx\; dx + \int_{\frac{\pi}{2}}^{\pi} (-cosx)dx\right]$$

(Since in $(0, \frac{\pi}{2})$, $cosx$ is positive and in $(\frac{\pi}{2}, \pi)$ $cosx$ is negative)

$$= \frac{2}{\pi}\left[(sinx)_0^{\frac{\pi}{2}} - (sinx)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[sin\frac{\pi}{2} - sin0 - sin\pi + sin\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}[1 - 0 - 0 + 1]$$

$$a_0 = \frac{4}{\pi}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|cosnxdx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosxcosnx\; dx + \int_{\frac{\pi}{2}}^{\pi} (-cosxcosnx)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos(n+1)\,x + \cos(n-1)x\; dx - \int_{\frac{\pi}{2}}^{\pi} \cos(n+1)\,x + \cos\,(n-1)xdx\right]$$

$$= \frac{1}{\pi}\left[\left\{ \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right\}_0^{\pi/2} \right.$$

$$\left. - \left\{ \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1} + \frac{\sin (n+1)\pi/2}{n+1} \right.$$

$$\left. + \frac{\sin (n-1)\pi/2}{n-1} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\left\{ \sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos \frac{n\pi}{2} \sin \frac{\pi}{2} \right\} + \frac{1}{n-1} \right.$$

$$\left. \times \left\{ \sin \frac{n\pi}{2} \cos \frac{\pi}{2} - \cos \frac{n\pi}{2} \sin \frac{\pi}{2} \right\} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} \right] \cos \frac{n\pi}{2}$$

$$= -\frac{4}{\pi (n^2-1)} \cos \frac{n\pi}{2} \text{ if } n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} |\cos x| \cos x \right]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= 0.$$

$$\therefore \quad |\cos x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \cos\frac{n\pi}{2} \cos nx$$

Example 5.

Find the Fourier series of $f(x) = e^x$ in $(-\pi, \pi)$ of periodicity 2π.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\,dx$

$$= \frac{1}{\pi}\,(e^x)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,(e^{\pi} - e^{-\pi})$$

$$= \frac{2}{\pi}\,\sinh \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-1)^n + e^{-\pi}(-1)^n]$$

$$= \frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\sin nx - n \cos nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-n)(-1)^n + e^{-\pi} n (-1)^n]$$

$$= \frac{-2(-1)^n \cdot n}{\pi(1+n^2)} \sinh \pi$$

$$e^x = \frac{\sinh \pi}{\pi} \left[1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} (\cos nx - n \sin nx) \right]$$

Example 6

Derive the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π and hence deduce $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\ [\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{(-\pi)^2}{2} + \frac{(-\ \pi)^3}{3}\right)]$$

$$= \frac{1}{\pi}\ [\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}]$$

$$a_0 = \frac{2\ \pi^2}{3}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)cosnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{sinnx}{n}\right) - (1 + 2x)\left(\frac{-cosnx}{n^2}\right) + (2)\left(\frac{-sinnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (1 - 2\pi)\left(\frac{cosn\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{(-1)^n}{n^2}\right) + 2\pi\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{4}{n^2}(-1)^n$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)sinnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{-cosnx}{n}\right) - (1 + 2x)\left(\frac{-sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) + 2\left(\frac{cosn\pi}{n^3}\right) - (-\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) - 2\left(\frac{cosn\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{-(-1)^n}{n}\right)\right]$$

$$b_n = \frac{2}{n}(-1)^{n+1}$$

Therefore, the Fourier series is of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\, cosnx + b_n\, sinnx)$$

$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} (\frac{4}{n^2} (-1)^n\, cosnx + \frac{2}{n} (-1)^{n+1}\, sinnx)$ ________________ (1)

STEP FOUR

Deduction:

The end points of the range are $x = \pi$ and $x = -\pi$. Therefore, the value of Fourier series at $x = \pi$ is the average value of $f(x)$ at the points $x = \pi$ and $x = -\pi$. Hence put $x = \pi$ in (1),

$$\Rightarrow \frac{f(-\pi)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n\, cosn\pi$$

$$\Rightarrow \frac{(\pi + \pi^2) + (-\pi + \pi^2)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{2n}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 7.

Expand f(x)=x², when $-\pi < x < \pi$ in a Fourier series of periodicity 2π. Hence deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{to } \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{to } \infty = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{to } \infty = \frac{\pi^2}{8}$

$f(x)$ is an even function of x in $-\pi < x < \pi$. Hence bn = 0 and only cosine terms will be present. Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{....}(i)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4(-1)^n}{n^2}.$$

Substituting these values in (i),

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \text{....}(ii)$$

i.e., $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$, in $-\pi < x < \pi$.

The function $f(x) = x^2$ is continuous at $x = 0$. Hence the sum of the Fourier series equals the value of the function at $x = 0$. Putting $x = 0$, in (ii),

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \qquad \text{....}(iii)$$

$x = \pi$ is an end point. Hence the sum of the Fourier series at $x = \pi$ equals $\frac{1}{2}\{f(-\pi + 0) + f(\pi - 0)\}$

Putting $x = \pi$ in the series of (ii),

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\pi}{n^2} = \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)]$$

$$= \frac{1}{2}[\pi^2 + \pi^2] = \pi^2$$

$$\therefore 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2}{3}\pi^2$$

i.e., $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad \text{....}(iv)$$

Adding (*iii*) and (*iv*),

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{4}$$

i.e., $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{ to } \infty = \frac{\pi^2}{8}.$$

Example 8 *Find the Fourier series of periodicity* 2π

$$\text{for} \quad f(x) = \begin{cases} x \text{ when } -\pi < x < 0 \\ 0 \text{ when } 0 < x < \frac{\pi}{2} \\ x - \frac{\pi}{2} \text{ when } \frac{\pi}{2} < x < \pi \end{cases}$$

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...}(1)$$

where $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Taking $c = -\pi$ in the Euler formulas we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x))\, dx + \int_{0}^{\pi} f(x)\, dx \right\}.$$

Now using the hypothesis for the value of $f(x)$, we get

$$a_0 = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-k)\, dx + \int_{0}^{\pi} k dx \right\} = \frac{1}{\pi} \left\{ \left(-kx\right)_{-\pi}^{0} + \left(kx\right)_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi} \{(0 - k\pi) + (k\pi - 0)\}$$

Thus $a_0 = 0$. Again for $n = 1, 2, 3, \ldots.$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx \right\}.$$

Substituting the values supplied for $f(x)$, we have

$$a_n = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-k) \cos nx\, dx + \int_{0}^{\pi} k \cos nx\, dx \right\}$$

$$=\frac{1}{\pi}\left\{\left(-k\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(k\frac{\sin nx}{n}\right)_{0}^{\pi}\right\}$$

Since sin 0, sin $(-n\pi)$ and sin $n\pi$ are all zero, we get $a_n = 0$.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx+\int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-k)\sin nx\,dx+\int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0}+\left[-k\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$=\frac{1}{\pi}\left[\left\{\frac{k}{n}\cos 0-\frac{k}{n}\cos(-n\pi)\right\}+\left\{-\frac{k}{n}\cos n\pi+\frac{k}{n}\cos 0\right\}\right]$$

But $\cos(-\alpha)=\cos\alpha$, giving $\cos(-n\pi)=\cos n\pi$; further, $\cos 0 = 1$.

Hence $b_n=\frac{k}{n\pi}[\{1-\cos n\pi\}+\{-\cos n\pi+1\}]=\frac{k}{n\pi}(2-2\cos n\pi)$

$\therefore\ b_n=\frac{2k}{n\pi}(1-\cos n\pi)$. Now $\cos n\pi=\begin{cases}-1, & \text{for odd } n\\ +1, & \text{for even } n\\ (-1)^n, & \text{for any integer } n\end{cases}$

Hence $b_1 = \frac{4k}{\pi}$; $b_2 = 0$; $b_3 = \frac{4k}{3\pi}$; $b_4 = 0$; $b_5 = \frac{4k}{5\pi}$; $b_6 = 0$;

$b_7 = \frac{4k}{7\pi}$.....

Using the values of a_n and b_n in (i) we obtain

$$f(x) = \frac{4k}{\pi}\{\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x + \cdots \text{ to } \infty\}$$

In the above equation putting $x = \pi/2$, we get

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\}$$

But, by hypothesis, $f\left(\frac{\pi}{2}\right) = k$.

Hence $k = \frac{4k}{\pi}\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\}$

Multiplying both the sides by $\frac{\pi}{4k}$, we have

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty.$$

Note. Functions of the type given in this example occur as external force acting on mechanical systems, electromotive forces in electric circuits etc.

Root Mean Square (RMS)Value

The root-mean-square value of a function $y = f(x)$ over a given (a, b) is defined as

$$\bar{y} = \sqrt{\left\{\frac{\int_a^b y^2\,dx}{b-a}\right\}}$$

If the interval is taken as $(c, c + 2\pi)$, then

$$\bar{y}^2 = \frac{1}{2\pi}\int_c^{c+2\pi} y^2\,dx$$

Suppose that $y = f(x)$ is expressed as a Fourier-series of periodicity 2π in $(c, c+2\pi)$. then,

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...(ii)}$$

$$\left.\begin{aligned} \text{where} \quad a_0 &= \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx \\ \text{and} \quad b_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx. \end{aligned}\right\} \qquad \text{...(iii)}$$

Multiply (*ii*) by $f(x)$ and integrate term by term with respect to x over the given range. Thus,

$$\int_c^{c+2\pi} [f(x)]^2\, dx = \frac{a_0}{2} \int_c^{c+2\pi} f(x)\, dx + \sum_{n=1}^{\infty} \left[a_n \int_c^{c+2\pi} f(x) \cos nx\, dx + b_n \int_c^{c+2\pi} f(x) \sin nx\, dx \right]$$

$$= \frac{a_0}{2} (\pi a_0) + \sum_{n=1}^{\infty} [a_n (\pi a_n) + b_n (\pi b_n)] \text{ using (iii)}$$

$$\int_c^{c+2\pi} [f(x)]^2\, dx = 2\pi \left[\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$= (\text{Range}) \left[\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Ex 9. *Find the Fourier series of periodicity 2π for $f(x) = x^2$, in $-\pi < x < \pi$. Hence show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \text{to } \infty = \frac{\pi^4}{90}.$$

In example 7, we have proved

$f(x) = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx$, which is the first part of this problem. The coefficients a_0, a_n, b_n were seen to be

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0.$$

Hence using the root-mean-square value in series,

$$2\pi\left[\frac{a_0^{\,2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})\right] = \int_{-\pi}^{\pi} [(f(x)^2] \, dx = \int_{-\pi}^{\pi} x^4 dx$$

$$2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right] = \frac{2}{5}\pi^5$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 2 – FOURIER SINE AND COSINE SERIES–SMTA1405

I. Introduction

Contents - Half range cosine series and sine series of f(x) defined in [0,π], [0,l]- Parseval's Identity (without proof) - simple problems – Complex form of Fourier series-Harmonic Analysis.

Can we find a Fourier series expansion of a function defined over a finite interval? Of course we recognize that such a function could not be periodic (as periodicity demands an infinite interval). The answer to this question is yes but we must first convert the given non-periodic function into a periodic function. There are many ways of doing this. We shall concentrate on the most useful extension to produce a so-called half-range Fourier series.

Half-range Fourier series

Suppose that instead of specifying a periodic function we begin with a function f(t) defined on over a limited range of values of t, say $0 < t < \pi$. Suppose further that we wish to represent this function, over $0 < t < \pi$, by a Fourier series. (This situation may seem a little artificial at this point,but this is precisely the situation that will arise in solving differential equations.)

Even: $f(-x) = f(x)$

Odd: $f(-x) = -f(x)$

Properties of even functions

(i) The graph of an even function is always symmetrized about the y-axis.

(ii) f(x) contains only even power of x and may contains only cosx,sec x.

(iii)Sum of two even function is even.

(iv) Product of two even function is even.

Odd Function

(i) Sum of two odd function is odd function.

(ii) Product of an odd function and even function is an odd function.

(iii) Product of two odd function is an even function

b. Find the Fourier series for f (x) where $f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0<x<1\end{cases}$.

Solution:

$$f(x)=\frac{a_0}{2}+\sum_{1}^{\infty}\left\{a_n\cos\frac{\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\}$$

$$a_0=\frac{1}{l}f(x)dx$$

$$=\int_{-1}^{0}0\,dx+\int_{0}^{1}dx$$

$$=(x)_0^1=1$$

$$a_n=\left\{\frac{1}{1}\int_{-1}^{0}0\;dx+\int_{0}^{1}\cos n\pi x\;dx\right\}$$

$$=\left[\sin\frac{n\pi x}{n\pi}\right]_0^1$$

$$=\left[\sin\frac{n\pi}{n\pi}-\sin\frac{n0}{n\pi}\right]=0$$

$$b_n=\frac{1}{1}\int_{0}^{1}1\sin n\pi x\;dx$$

$$=\left[-\frac{\cos\pi n}{n\pi}\right]_0^1=\left[-\frac{\cos n\pi}{n\pi}+\frac{1}{n\pi}\right]=\frac{1-(-1)^n}{n\pi}$$

$$f(x)=\frac{1}{2}+\sum_{1,3}^{\infty}\frac{2}{n\pi}\sin n\pi x.$$

Problem 25 Find the half – range cosine series for f (x) = (x – 1)^2 in (0, 1). Hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$.

Solution:

Here $l=1$

$$f(x)=\frac{a_0}{2}+\sum_{1}^{\infty}a_n\cos n\pi x$$

$$a_0=2\int_{0}^{1}(x-1)^2dx=2\left[\frac{(x-1)^2}{-3}\right]_0^1$$

$$a_0=\frac{2}{3}$$

$$a_n=2\int_{0}^{1}(x-1)^2\cos n\pi x dx$$

$$=2\left\{(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right)-\left(2(x-1)-\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right)+\left((2)+\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right)\right\}_0^1$$

$$= 2\left\{+\frac{2}{n^2\pi^2}\right\} = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos n\pi x \qquad -(1)$$

Put $x = 0$ *in* (1)

$$f(0) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2} \qquad -(2)$$

Here 0 is o pt of discontinuity

$$f(0) = \frac{f\left(0^-\right) + f\left(0^+\right)}{2} = 1$$

$$\therefore 1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\left\{\frac{1}{n^2}\right\}$$

$$\frac{2}{3} = \frac{4}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \ldots\right\}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots\ldots\ldots\infty = \frac{\pi^2}{6}.$$

Problem 26 **a**. Express $f(x) = \begin{cases} 1, & 0 \le x \le \frac{a}{2} \\ -1, & \frac{a}{2} < x \le a \end{cases}$ as a cosine series

Solution: $f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\frac{n\pi x}{a}$

$$a_0 = \frac{2}{a}\left\{\int_0^{\frac{a}{2}} dx + \int_{\frac{a}{2}}^{a}(-1)\,dx\right\} = \frac{2}{a}\left\{\frac{a}{2} - 0 - a + \frac{a}{2}\right\} = 0$$

$$a_n = \frac{2}{a}\left\{\int_0^{\frac{a}{2}}\cos\frac{\pi n x}{a}\,dx - \int_{\frac{a}{2}}^{a}\cos\frac{\pi n x}{a}\,dx\right\}$$

$$= \frac{2}{a}\left\{\left(\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^{\frac{a}{2}} - \left(\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_{\frac{a}{2}}^{a}\right\}$$

$$= \frac{2}{a}\left\{\frac{a}{n\pi}\sin\frac{n\pi}{2} + \frac{a}{n\pi}\sin\frac{n\pi}{2}\right\} = \frac{4}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

$$\therefore f(x) = \sum_{1}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi}{2}\cos\frac{\pi n}{a}.$$

b. Express $f(x)$ as a Fourier sine series where $f(x)=\begin{cases}\frac{1}{4}-x, & \left(0,\frac{1}{2}\right)\\ x-\frac{3}{4}, & \left(\frac{1}{2},1\right)\end{cases}$.

Solution:

We know that $f(x)=\sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$b_n=\frac{2}{1}\left\{\int_0^{\frac{1}{2}}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx+\int_{\frac{1}{2}}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right\}$$

$$\therefore b_n=2\left\{\left(\frac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right)-\left((-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right)\Bigg|_0^{\frac{1}{2}}+\left(\left(x-\frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right)-(1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right)\Bigg|_{\frac{1}{2}}^{1}-\right\}$$

$$=-2\left[\left(\frac{1}{4}-\frac{1}{2}\right)\frac{\cos\frac{n\pi}{2}}{n\pi}+\left(\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)-\left(\frac{1}{4}-0\right)\left(\frac{1}{n\pi}\right)\right]$$

$$-2\left[\left(1-\frac{3}{4}\right)\left(\frac{\cos n\pi}{n\pi}\right)-\left(\frac{\sin n\pi}{n^2\pi^2}\right)-\left(\frac{1}{2}-\frac{3}{4}\right)\left(\frac{\cos n\frac{\pi}{2}}{n\pi}\right)-\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$=-2\left[\frac{\left(\sin\frac{n\pi}{2}\right)}{n^2\pi^2}-\left(\frac{1}{4n\pi}\right)\right]-2\left[\left(\frac{1}{4}\right)\left(\frac{\cos n\pi}{n\pi}\right)-(0)-\left(\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right]$$

$$=\frac{-4\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{1}{2n\pi}-\frac{(-1)^n}{2n\pi}$$

$$b_n=\frac{-4\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{1}{n\pi}\ \textit{if}\ \ n\ \ \text{is}\ \ \textit{odd}$$

$= 0$ if even

$$\therefore f(x)=\sum_{1/3}^{\infty}\left(\frac{-4\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{1}{n\pi}\right)\sin n\pi x.$$

Problem 27 **a.** Find the Fourier cosine series for $x(\pi-x)$ in $0<x<\pi$.

Solution:

$$f(x)=\frac{a_0}{2}\sum_{1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\,dx$$

$$=\frac{2}{\pi}\left\{\frac{\pi x^2}{2}-\frac{x^2}{3}\right\}_0^{\pi}$$

$$=\frac{2}{\pi}\left\{\frac{\pi^3}{2}-\frac{\pi^3}{3}\right\}=\frac{\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\cos nx\;dx$$

$$=\frac{2}{\pi}\left\{x(\pi-x)\frac{\sin nx}{n}-\left((\pi-2x)\left(\frac{-\cos nx}{n^2}\right)\right)+\left((-2)\left(\frac{-\sin nx}{n^3}\right)\right)\right\}_0^{\pi}$$

$$=\frac{2}{\pi}\left\{-\frac{\pi(-1)^n}{n^2}-\frac{\pi}{n^2}\right\}$$

$$=-\frac{2}{n^2}\left\{1+(-1)^n\right\}$$

$$=a_n=-\frac{4}{n^2}\text{ If n is even}$$

$$a_n=0\text{ If n is odd.}$$

$$\therefore f(x)=\frac{\pi^2}{6}+\sum_{2,4}^{\infty}-\frac{4}{n^2}\cos nx.$$

b. Prove that complex form of the Fourier series of the function $f(x)=e^{-x},-1<x<1$ is

$$f(x)=\sum_{-\infty}^{\infty}(-1)^n\frac{1-in\,\pi}{1+n^2\pi^2}\sin h1.e^{in\pi x}\ .$$

Solution:

Here $2l=2,\quad l=1$

$$\therefore f(x)=\sum_{-\infty}^{\infty}C_ne^{in\pi x}$$

$$C_n=\frac{1}{2}\int_{-1}^{1}e^{-x}e^{-in\pi x}dx$$

$$=\frac{1}{2}\int_{-1}^{1}e^{-(1+in\pi x)}dx$$

$$=\frac{1}{2}\left\{\frac{e^{-(1+in\pi x)}}{-(1+in\pi)}\right\}_{-1}^{1}$$

$$=-\frac{-1}{2(1+in\,\pi)}\left\{e^{-(1+in\pi)}-e^{(1+in\pi x)}\right\}$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)}\left\{e^{-1}(\cos n\pi - i\sin n\pi) - e^{1}(\cos n\pi + i\sin n\pi)\right\}$$

$$= -\frac{(1-in\pi)}{2(1+n^2\pi^2)}\cos n\pi\left(e^{-1}-e^{1}\right)$$

$$= \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n\left(e-e^{-1}\right)$$

$$= \frac{(1-in\ \pi)}{2(1+n^2\pi^2)}(-1)^n\sinh(1)$$

$$\therefore f(x) = \sum_{-\infty}^{\infty}\frac{(1-in\ \pi)}{2(1+n^2\pi^2)}(-1)^n\sinh(1)\,e^{in\pi x}.$$

Problem 28 Find the cosine series for f (x) = x in (0, π) and then using Parseval's theorem, show that $\frac{1}{1^4}+\frac{1}{3^4}+\ldots. = \frac{\pi^4}{96}$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left\{\frac{x^2}{2}\right\}_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\frac{x\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{\cos n\pi - 1}{n^2}\right\}$$

$$= \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$a_n = \frac{-4}{n^2\pi}$ if n is odd

$a_n = 0$ if n is even

$$f(x) = \frac{\pi^2}{2} + \sum_{1,3}^{\infty}\frac{-4}{\pi n^2}\cos nx$$

By Parseval's theorem

$$\frac{1}{\pi}\int_0^{\pi}\left[f(x)\right]^2 dx = \frac{a_0{}^2}{4} + \frac{1}{2}\sum a_n{}^2$$

$$\frac{1}{\pi}\int_0^{\pi} x^2\, dx = \frac{\pi^2}{4} + \frac{1}{2}\sum_{1,3}^{\infty}\left(\frac{4}{\pi n^2}\right)^2$$

$$\frac{1}{\pi}\left(\frac{x^2}{3}\right)_0^{\pi} = \frac{\pi^2}{4} + \frac{1}{2}\sum_{1,3}^{\infty}\frac{16}{\pi^2 n^4}$$

$$\frac{1}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{\pi^2}{4} + \frac{8}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \ldots.\right\}$$

$$\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{8}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \ldots.\right\}$$

$$\frac{\pi^2}{12} \times \frac{\pi^2}{8} = \frac{1}{1^4} + \frac{1}{3^4} + \ldots..\infty$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \ldots..\infty$$

Problem 29 **a.** Find the complex form of Fourier series of $f(x)$ if $f(x) = \sin ax \ in \ -\pi < x < \pi$.

Solution: $f(x) = \sum_{-\infty}^{\infty} C_n \ e^{inx} dx$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin ax\, e^{-inx} dx$$

$$= \frac{1}{2\pi}\left\{\frac{e^{-inx}}{\left(a^2 - n^2\right)}\left(-in\sin ax - a\cos ax\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi\left(a^2 - n^2\right)}\left[e^{-in\pi}\left(-in\sin\ a\pi - a\cos a\pi\right) - e^{in\pi}\left(in\sin\ a\pi - a\cos a\pi\right)\right]$$

$$= \frac{1}{2\pi\left(a^2 - n^2\right)}\left[-in\sin a\pi\, 2\cos n\pi + a\cos a\pi\, 2\mathrm{i}\sin n\pi\right]$$

$$= \frac{-2in(-1)^n \sin a\pi}{2\pi\left(a^2 - n^2\right)} = \frac{(-1)^{n+1}\, in\sin a\pi}{\pi\left(a^2 - n^2\right)}$$

$$\therefore f(x) = \frac{\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{in(-1)^{n+1}}{\left(a^2 - n^2\right)} e^{inx}.$$

b. Find the first two harmonic of the Fourier series of f (x) given by

x	0	1	2	3	4	5
f (x)	9	18	24	28	26	20

Solution:
Here the length of the in level is $2l = 6,\ l = 3$

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3} \right)$$

x	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	0	0	9	9	0	9	0
1	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	18	9	15.7	-9	15.6
2	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	24	-12	20.9	-12	-20.8
3	π	2π	28	-28	0	28	0
4	$\frac{4\pi}{3}$	$\frac{8\pi}{3}$	26	-13	-22.6	-13	22.6
5	$\frac{5\pi}{3}$	$\frac{10\pi}{3}$	20	10	17.6	-10	-17.4
			125	-25	-3.4	-7	0

$$a_0 = 2\frac{\sum y}{6} = \frac{2(125)}{6} = 41.66$$

$$a_1 = \frac{2}{6}\left\{ \sum y \cos\frac{\pi x}{3} \right\} = -8.33$$

$$b_1 = \frac{2}{6}\sum y \sin\frac{\pi x}{3} = -1.15$$

$$a_2 = \frac{2}{6}\left\{ \sum y \quad \cos\frac{2\pi x}{3} \right\} = -2.33$$

$$b_2 = \frac{2}{6}\sum y \quad \sin\frac{2\pi x}{3} = 0$$

$$\therefore f(x) = \frac{41.66}{2} - 8.33\cos\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 1.15\sin\frac{\pi x}{3}.$$

Problem 30 **a.** Find the first two harmonic of the Fourier series of f (x). Given by

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)	1	1.4	1.9	1.7	1.5	1.2	1.0

Solution:

$\because$ The last value of y is a repetition of the first; only the first six values will be used

The values of $y\cos x,\ y\cos 2x,\ y\sin x,\ y\sin 2x$ as tabulated

x	$f(x)$	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$
0	1.0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	-0.5	0.866
$\frac{2\pi}{3}$	1.9	-0.5	0.866	-0.5	0.866
π	1.7	-1	0	1	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.866	-0.5	-0.866
$\frac{5\pi}{3}$	1.2	0.5	-0.866	-0.5	-0.866

$$a_0 = 2\frac{\sum y}{6} = 2.9$$
$$a_1 = 2\frac{\sum y\cos x}{6} = -0.37$$
$$a_2 = 2\frac{\sum y\cos 2x}{6} = -0.1$$
$$b_1 = 2\frac{\sum y\sin x}{6} = 0.17$$
$$b_2 = 2\frac{\sum y\sin 2x}{6} = -0.06$$

b. Find the first harmonic of Fourier series of $f(x)$ given by

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
f (x)	1.98	1.30	1.05	1.30	-0.88	-0.35	1.98

Solution:

First and last valve are same Hence we omit the last valve

x	$\theta = \frac{2\pi x}{T}$	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1.0	0	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	1.30	0.5	0.866	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.525	0.9093
$\frac{T}{2}$	π	1.30	-1	0	-1.3	0

$\frac{2T}{3}$	$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	-0.25	0.5	-0.866	-0.125	0.2165
		4.6			1.12	3.013

$$a_0 = \frac{2}{6}\sum y = \frac{4.6}{3} = 1.5$$

$$a_1 = \frac{2\sum y\cos\theta}{6} = \frac{2}{6}(1.12) = 0.37$$

$$b_1 = \frac{2}{6}(3.013) = 1.005$$

$$\therefore\ f(x) = 0.75 + 0.37\cos\theta + 1.005\sin\theta$$

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DEPARTMENT OF MATHEMATICS

UNIT – 3 –APPLICATIONS OF ODE–SMTA1405

I. Introduction

Contents - Particle dynamics: Simple Harmonic motion – Projectiles: – horizontal plane - trajectory – velocity of projection – angle of projection – Range - time of flight – greatest height - projectiles on inclined plane. Central orbit and Central forces: – differential equation of a path – pedal equation of a differential equation – velocity at any point of a central orbit – areal velocity – Kepler's laws of planetary.

Simple Harmonic Motion (S.H.M) is an interesting special type of motion in nature, having forward and backward oscillation (or) to and fro oscillation about a fixed point. The fixed point is known as the mean position or equilibrium position. When the oscillation is very small we prove the motion is simple harmonic. In this section we study about the resultant of two S.H.M'S of the same period in the same straight line and in two perpendicular lines. Also we find the periodic time of oscillation of a simple pendulum.

Examples

Small oscillation of a cradle, simple pendulum, seconds pendulum, simple equivalent pendulum, transverse vibrations of a plucked violin string etc.

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.

Equation (1) is the fundamental differential equation representing a S.H.M.

If v is the velocity of the particle at time t (1) can be written as

$v\frac{dv}{dx} = -\mu x$ i.e. $\quad vdv = -\mu x\, dx \quad$ …………(2)

Integrating (2), we have $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c$ ………..(3)

Initially let the particle starts from rest at the point A where OA = a

Hence when x=a, $v = 0 = \frac{dx}{dt}$

Putting these in (3), $0 = -\frac{\mu a^2}{2}$ +c or $c = \frac{\mu a^2}{2}$

Putting these in (3), $0 = -\frac{\mu a^2}{2}$ +c or $c = \frac{\mu a^2}{2}$

$\therefore v^2 = -\mu\ x^2 + \mu a^2 = \mu\left(a^2 - x^2\right)$

$\therefore$ $v = \pm\ \sqrt{\mu\left(a^2 - x^2\right)}$ ………. (4)

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence we take the negative sign in (4),

$$\frac{dx}{dt} = \mathbf{v} = -\sqrt{\mu\left(a^2 - x^2\right)} \qquad \text{……..(5)}$$

$$-\frac{dx}{\sqrt{\left(a^2 - x^2\right)}} = \sqrt{\mu}\ dt$$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t + A$

Initially when t = 0, x = a, $\cos^{-1}1 = 0 + A \Rightarrow$ $\boxed{A=0}$

$\therefore$ $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t$ or $x = a\cos\sqrt{\mu}\ t$ ………. (6)

To get the time from A to A^1, put $x = -a$ in (6)

We have $\cos\sqrt{\mu}\,t = -1 = \cos\ \pi$, $t = \frac{\pi}{\sqrt{\mu}}$

$\therefore$ The time from A to A' and back $= \dfrac{2\pi}{\sqrt{\mu}}$.

Equation (6) can be written as

$$x = a\cos\sqrt{\mu}\,t = a\cos(\sqrt{\mu}\,t + 2\pi) = a\cos(\sqrt{\mu}\,t + 4\pi) \text{ etc}$$

$$= a\cos\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right) = a\cos\sqrt{\mu}\left(t + \frac{4\pi}{\sqrt{\mu}}\right) \text{ etc.}$$

Differentiating (6),

$$\frac{dx}{dt} = -a\sqrt{\mu}\,.\sin\sqrt{\mu}\,t$$

$$= -a\sqrt{\mu}\sin(\sqrt{\mu}\,t + 2\pi) = -a\sqrt{\mu}\sin(\sqrt{\mu}\,t + 4\pi) \text{ etc.}$$

$$= -a\sqrt{\mu}\sin\sqrt{\mu}\,(t + \frac{2\pi}{\sqrt{\mu}}) = -a\sqrt{\mu}\sin\sqrt{\mu}\,(t + \frac{4\pi}{\sqrt{\mu}}) \text{ etc.}$$

The values of $\frac{dx}{dt}$ are the same if t is increased by $\frac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\frac{2\pi}{\sqrt{\mu}}$. Hence after a time $\frac{2\pi}{\sqrt{\mu}}$ the particle is again at the same point moving with the same velocity in the same direction. Hence the particle has the period $\frac{2\pi}{\sqrt{\mu}}$.

$T = \dfrac{2\pi}{\sqrt{\mu}}$; frequency $= \dfrac{1}{T} = \dfrac{2\pi}{\sqrt{\mu}}$

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Amplitude = OA = OA' = a.

The periodic time $= \frac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Deductions : 1) Maximum acceleration $= \mu.a = \mu$. (amplitude)

2) Since $v = \sqrt{\mu(a^2 - x^2)}$, the greatest value of v is at $x = 0$ and its

Maximum velocity = a $\sqrt{\mu} = \sqrt{\mu}$. (amplitude) at the centre

General solution of the S.H.M. equation

The S.H.M. equation is $\dfrac{d^2x}{d\,t^2} = -\mu\,x$

i.e. $\dfrac{d^2x}{d\,t^2} + \mu\,x = 0$ …….(1)

(1) is a differential equation of the second order with constant coefficients. Its general solution is of the form

$x = A\cos\sqrt{\mu}\;t + B\sin\sqrt{\mu}\;t$ ……..(2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

$x = C\cos(\sqrt{\mu}\,t + \varepsilon)$…. (3) and $x = D\sin(\sqrt{\mu}\;t + \alpha)$ ………(4)

- ❖ If the solution of the S.H.M. equation is $x = a\cos(\sqrt{\mu}\;t + \varepsilon)$, the quantity ε is called the **epoch**.

the epoch.

Definition

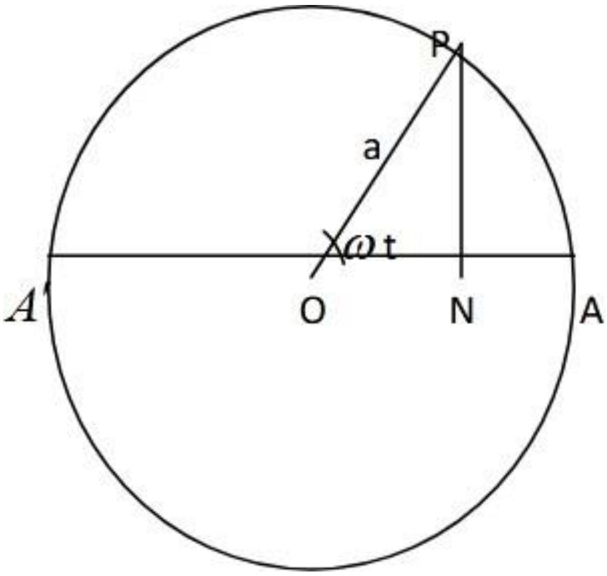
If two simple harmonic motions of the same period can be represented by

$$x_1 = a_1 \cos\left(\sqrt{\mu}\ t + \varepsilon_1\right) \text{and } x_2 = a_2 \cos\left(\sqrt{\mu}\ t + \varepsilon_2\right)$$

- The difference in phase = $\dfrac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$
- If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.
- If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phase.

4.2 Geometrical Representation of S.H.M

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H.M.



Let AA' be the diameter of the circle with centre O and P be the position of the particle at time $t\,\text{secs}$. Let N be the foot of the perpendicular drawn from P on the diameter AA'. P moves along the circumference of the circle with uniform speed and describes equal arcs in equal times. Let ω – be the angular velocity. $\therefore \angle AOP = \omega t$

If ON = x, Op = a, then, $x = a\cos(\omega t)$ ……………… (1)

$$\frac{dx}{dt} = -a\omega\sin(\omega t) \quad \text{……………….(2)}$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos(\omega t) = -\omega^2 x \quad \text{……………… (3)}$$

(3) shows that the motion of N is simple harmonic. When P moves along the circumference of the circle starting from A, N oscillates from A to A' and A' to A.

Problem 1

A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are $x_1, x_2, x_3.$ Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$

Solution:

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that x = a cos $\sqrt{\mu}$ t ….. (1)

Let $x_1, x_2, x_3.$ be the displacements at three consecutive seconds $t_1,\ t_1+1,\ t_1+2$.

Then $x_1 = a\cos\sqrt{\mu}\,t_1$ …… (2)

$x_2 = a\cos\sqrt{\mu}(t_1+1) = a\cos\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)$ …….(3)

$x_3 = a\cos\sqrt{\mu}(t_1+2) = a\cos\left(\sqrt{\mu}t_1 + 2\sqrt{\mu}\right)$ …….(4)

$$\therefore x_1 + x_3 = a\,[\cos\left(\sqrt{\mu}\, t_1 + 2\sqrt{\mu}\right) + \cos\left(\sqrt{\mu}\, t_1\right)]$$

$$= a.2 \cos \frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2} \,.\, \cos \frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$$

$$= 2\,a \cos\left(\sqrt{\mu}\; t_1 + \sqrt{\mu}\right).\cos\sqrt{\mu} = 2x_2 \,.\, \cos\sqrt{\mu}$$

$$\therefore \frac{x_1 + x_3}{2x_2} = \cos\sqrt{\mu}\,, \qquad \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$$

Problem 2

If the displacement of a moving point at any time be given by an equation of the form $x = a \cos \omega t + b \sin \omega t$, show that the motion is a simple harmonic motion.

If $a = 3$, $b=4$, $\omega = 2$ determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

Given $x = a\cos\omega t + b\sin\varpi t$ .... (1)

Differentiating (1) with respect to t,

$$\frac{dx}{dt} = -a\omega\sin\omega t + b\omega\cos\omega t \text{ ....(2)}$$

$$\frac{d^2x}{dt^2} = -\omega^2\cos\omega t - b\,\omega^2\sin\omega t$$

$$= -\omega^2(a\cos\omega t + b\sin\omega t) = -\omega^2 x \text{ ....(3)}$$

$\therefore$ The motion is simple harmonic.

The constant μ of the S,H.M. = ω^2.

$\therefore$ Period = $\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ secs.

Amplitude is the greatest value of x.

When x is maximum, $\frac{dx}{dt} = 0$.

$-a\omega\sin\omega t + b\omega\cos\omega t = 0$ i.e. $a\sin\omega t = b\cos\omega t$ or $\tan\omega t = \frac{b}{a} = \frac{4}{3}$

When $\tan\omega t = \frac{4}{3}$, $\sin\omega t = \frac{4}{5}$ and $\cos\omega t = \frac{3}{5}$

Greatest value of $x = a \times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a+4b}{5} = \frac{3.3+4.4}{5} = 5$

Hence amplitude = 5.

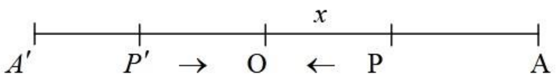
Max. acceleration = μ . Amplitude = 4 x 5 = 20

Max. velocity = $\sqrt{\mu}$. Amplitude = 2 x 5 =10

Problem 3

Show that the energy of a system executing S.H.M. is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration at a distance x from O = μx.

Force = mass $\times$acceleration = m μx

If the particle is given displacement dx from P,

work done against the force = m μ x. dx

Total work done in displacing the particle to a distance x

$$= \int_0^x m\ \mu x dx = m\ \mu \frac{x^2}{2} \qquad \text{.........(1)}$$

Work done = potential energy at P.

If v is the velocity at P. we know that $v^2 = \mu(a^2 - x^2)$,

$$\therefore \text{ Kinetic energy at P} = \frac{1}{2}\ mv^2 = \frac{1}{2} m\mu(a^2 - x^2) \qquad \text{.......... (2)}$$

The total energy at P = Potential energy + Kinetic energy

$$= \frac{m\mu x^2}{2} + \frac{m\mu}{2}(a^2 - x^2) = \frac{m\mu a^2}{2} \text{ .... (3)}$$

Total energy at P α a^2

If n is the frequency, we know that

$$n = \frac{1}{Period} = \frac{1}{\left(\frac{2\pi}{\sqrt{\mu}}\right)} = \frac{\sqrt{\mu}}{2\pi}$$

$$\therefore \sqrt{\mu} = 2\pi\ n \quad \text{or} \quad \mu = 4\pi^2 n^2$$

$$\text{Total energy} = \frac{1}{2} m.\ 4\pi^2 n^2 a^2 = 2\pi^2 m a^2 n^2\ \ \alpha\ n^2$$

Problem 4

A mass of 1 gm. Vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity,

Solution:

Maximum velocity $\quad v = \sqrt{\mu}\,.\,a$

Given, frequency $= \dfrac{1}{T} \quad = 256 = \dfrac{\sqrt{\mu}}{2\pi}.$

$\therefore\ \sqrt{\mu} \quad = 2 \times 256 \times \pi.$

Given, amplitude $= a = 1$ millimeter $= 1 \times 10^{-1}$ c.m.

$\therefore$ Maximum velocity, $V = 2 \times 256 \times \pi \times \dfrac{1}{10} = \dfrac{256\ \pi}{5}$ cm/ sec

Problem 5

In a S.H.M. if f be the acceleration and v the velocity at any time and T is the periodic time. Prove that $f^2T^2 + 4\pi^2v^2$ is constant.

Solution:

Velocity at any time, v $\quad = \sqrt{\mu\left(a^2 - x^2\right)}$

Periodic time $\qquad T = \dfrac{2\pi}{\sqrt{\mu}},\ \dfrac{d^2x}{dt^2} = f.$

For, S.H.M, $\dfrac{d^2x}{dt^2} \ = -\mu.x$

$\therefore\ f \ = -\mu.x$

$$\therefore f^2T^2 + 4\pi^2v^2 = \mu^2x^2.\frac{4\pi^2}{\mu} + 4\pi\ \mu^2\left(a^2 - x^2\right)$$

$$= 4\,\pi^2\mu x^2 + 4\pi^2\mu a^2 - 4\pi^2\mu x^2$$

$$= 4\,\pi^2\mu a^2 \text{ (constant)}$$

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UNIT – 4 –APPLICATION OF PDE–SMTA1405

Introduction

Contents – One dimensional wave equation – Transverse vibration of finite elastic string with fixed ends – boundary and initial value problems – Fourier series solution – one dimension heat equation – steady and unsteady state – boundary and initial value problems – Fourier series solution.

Recall that a partial differential equation or *PDE* is an equation containing the partial derivatives with respect to *several* independent variables. Solving PDEs will be our main application of Fourier series.

I. One-dimensional wave equation

Let us start with the wave equation. Imagine we have a tensioned guitar string of length L. Let us only consider vibrations in one direction. Let x denote the position along the string, let t denote time, and let y denote the displacement of the string from the rest position. See Fig. 1.

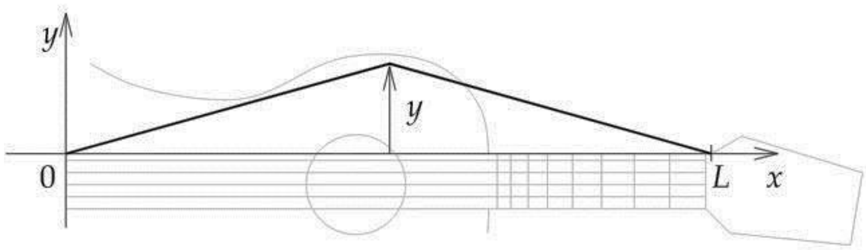


Figure 1: Vibrating string of length L, x is the position, y is displacement

Let (x, t) denote the displacement at point x at time t. The equation governing this setup is the so-called one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{.......} (1)$$

We generally use a more convenient notation for partial derivatives. We write y_t instead of $\frac{\partial y}{\partial t}$, and we write y_{xx} instead of $\frac{\partial^2 y}{\partial x^2}$.

With this notation the equation that governs this setup is the so-called one-dimensional waveequation, becomes $y_{tt} = a^2 y_x$

for some constant $a > 0$. The intuition is similar to the heat equation, replacing velocity with acceleration: the acceleration at a specific point is proportional to the second derivative of the shape of the string. The wave equation is an example of a hyperbolic PDE.

The following assumptions are made while deriving the 1-D wave equation:

1. The motion takes place entirely in one plane. This plane is chosen as the xy-plane.
2. In this plane, each particle of the string moves in a direction perpendicular to theequilibrium position of the string.
3. The tension T caused by the string before fixing it at the end points is constant at alltimes and at all points of the deflected string.

4. The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.
5. The effect of friction is negligible.
6. The string is perfectly flexible. It can transmit only tension but not bending or shearing forces.
7. The slope of the deflection curve is small at all points and at all times.

Solution of the Wave Equation (by the method of separation of variables)

Let $y = X(x).T(t)$ be a solution of (1), where X(x) is a function of x only $T(t)$ is a function t only.

$$\frac{\partial^2 y}{\partial t^2} = XT'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X''T,$$

where $$X'' = \frac{d^2X}{dx^2} \text{ and } T'' = \frac{d^2T}{dt^2}.$$

Hence (1) becomes, $XT'' = a^2 X'' T$

i.e., $$\frac{X''}{X} = \frac{T''}{a^2 T} \qquad ...(2)$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

$\therefore \quad \dfrac{X''}{X} = \dfrac{T''}{a^2 T} = k$ (say) where k is any constant.

Hence $X'' - kX = 0$ and $T'' - a^2 kT = 0$ ...(3)

Solutions of these equations depend upon the nature of the value of k.

Case 1. Let $k = \lambda^2$, a positive value .

Now the equation (3) are $X'' - \lambda^2 X = 0$ and $T'' - a^2\lambda^2 T = 0$.

Solving the ordinary differential equations we get,

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$$

and $$T = C_1 e^{\lambda at} + D_1 e^{-\lambda at}$$

Case 2. Let $k = -\lambda^2$, a negative number.

Then the equations (3) are $X'' + \lambda^2 X = 0$ and $T'' + a^2\lambda^2 T = 0$.

Solving, we get,

$$X = A_2 \cos \lambda x + B_2 \sin \lambda x$$

and $$T = C_2 \cos \lambda\, at + D_2 \sin \lambda\, at.$$

Case 3. Let $k = 0$.

Now the equations (3) are $X'' = 0$ and $T'' = 0$.

Then integrating, $X = A_3 x + B_3$

and $$T = C_3 t + D_3$$

Thus the various possible solutions of the wave equation are

$$y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda at} + D_1 e^{-\lambda at}) \quad ...(I)$$

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at) \quad ...(II)$$

$$y = (A_3 x + B_3)(C_3 t + D_3) \quad ...(III)$$

Example 1

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the position $y = f(x)$. It is set vibrating by giving to each of its points a velocity $\frac{\partial y}{\partial t} = g(x)$ at $t = 0$. Find $y(x, t)$ in the form of Fourier series.

Solution.

The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The boundary conditions under which (1) is to be solved are

(*i*) $y(0, t) = 0$ for $t \geq 0$

(*ii*) $y(l, t) = 0$ for $t \geq 0$

(*iii*) $y(x, 0) = f(x)$, for $0 < x < l$

(*iv*) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$, for $0 < x < l$.

Solving (1) by the method of separation of variables, we get,

$$y(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda at} + D_1 e^{-\lambda at}) \quad ...(I)$$

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at) \quad ...(II)$$

$$y = (A_3 x + B_3)(C_3 t + D_3). \quad ...(III)$$

Since the solution should be periodic in t, we reject solutions (I) and (III) and select (II) to suit the boundary conditions (*i*), (*ii*), (*iii*) and (*iv*).

$$\therefore\ y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \qquad ...(2),$$

where A, B, C, D are arbitrary constants.

Using boundary condition (*i*) in (*ii*),

$A(C\cos\lambda at + D\sin\lambda at) = 0$ for all $t \geq 0$.

$\therefore A = 0$.

Applying the boundary condition (*ii*) in (2),

$B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$, for all $t \geq 0$.

If $B = 0$, the solution becomes $y = 0$ which is not true.

$\sin\lambda l = 0, B \neq 0$.

i.e., $\lambda l = n\pi$, where n is any integer.

$$\therefore\ \lambda = \frac{n\pi}{l}$$

$$\therefore\ y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$$

i.e., $$y(x,t) = \sin\frac{n\pi x}{l}\left(C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right) \qquad ...(3),$$

where $BC = C_n$ and $BD = D_n$.

Since the wave equation is linear and homogeneous, the most general solution of it is

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \qquad ...(4).$$

This satisfies boundary condition (*i*) and (*ii*). To find C_n and D_n we make use of the initial conditions (*iii*) and (*iv*).

$$y(x,0) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} = f(x) \qquad ...(5)$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum\frac{n\pi a}{l}D_n\sin\frac{n\pi x}{l} = g(x) \qquad ...(6)$$

The left-hand sides of (5) and (6) are Fourier series of the right-hand side functions.

$$\text{Hence } C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(7)$$

and $$\frac{n\pi a}{l}D_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(8)$$

Example 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the position $y(x, 0) = y_0 sin^3(\frac{\pi x}{l}) = f(x)$. If it released from rest from this position, find the displacement $y(x, t)$ at any time t and at any distance from the end $x = 0$.

Solution.

The displacement y of the particle at a distance x from the end x=0 and time t is governed by $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are:

$$y(0, t) = 0, \quad \text{for all } t \geq 0 \quad (i)$$

$$y(l, t) = 0, \quad \text{for all } t \geq 0, \quad (ii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad \text{for } 0 \leq x \leq l \quad (iii)$$

$$y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right), \quad \text{for } 0 \leq x \leq l \quad (iv)$$

Now solving (1) and selecting the proper solution to suit the physical nature of the problem and making use of the boundary conditions (i) and (ii) as in the previous problem, we get

$$y(x, t) = B \sin\frac{n\pi x}{l}\left(C \cos\frac{n\pi at}{l} + D \sin\frac{n\pi at}{l}\right) \quad ...(2)$$

Again using the boundary condition (iii),

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = B \sin\frac{n\pi x}{l}\left(D \cdot \frac{n\pi a}{l}\right).$$

If $B = 0$, (2) takes the form $y(x, t) = 0$. Hence B cannot be zero.

$\therefore\ D = 0.$

Hence (2) becomes,

$y(x, t) = B_n \frac{n\pi x}{l} \cos\frac{n\pi at}{l}$, where n is any integer and B_n is any constant.

The most general solution satisfying (1) and the boundary conditions (i), (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad ...(3).$$

To find B_n use the boundary condition (*iv*).

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$= \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

This is true only if $B_1 = \frac{3y_0}{4}$, $B_3 = -\frac{y_0}{4}$ and $B_n = 0$, for $n \neq 1, 3$.

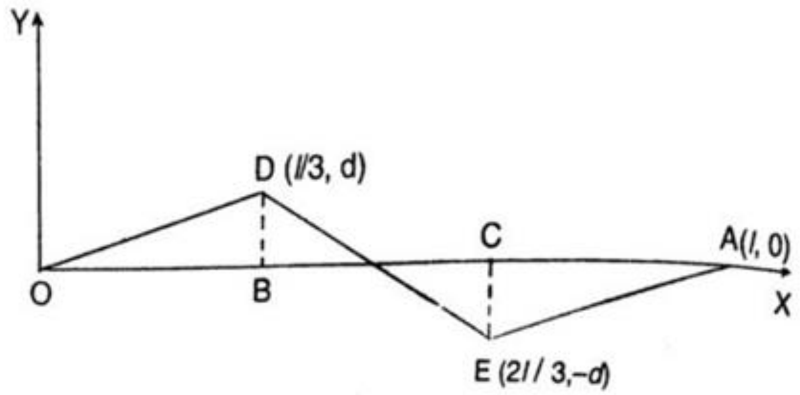
Using these values in (3), the solution of the equation is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$

Example 3

The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string is always remains at rest.

Solution.



$BD = CE = d.$

The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The boundary conditions here are

$$y(0, t) = 0 \qquad \text{for } t \ge 0 \qquad \ldots(i)$$

$$y(l, t) = 0 \qquad \text{for } t \ge 0 \qquad \ldots(ii)$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \qquad \text{for } 0 \le x \le l \qquad \ldots(iii)$$

To find the initial position of the string, we require the equation of $ODEA$.

The equation of OD is $\quad y = \frac{d}{l/3}x = \frac{3dx}{l}$.

The equation of DE is $y - d = -\frac{d}{(l/6)}(x - l/3)$

i.e., $$y = \frac{3d}{l}(l - 2x).$$

The equation of EA is $\quad y = \frac{3d}{l}(x - l)$.

The fourth initial condition is

$$y(x, 0) = \begin{cases} \frac{3dx}{l} & \text{for } 0 \le x \le l/3 \\ \frac{3d}{l}(l - 2x) & \text{for } \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3d}{l}(x - l) & \text{for } \frac{2l}{3} \le x \le l \end{cases} \qquad \text{———— (iv)}$$

Solving (1) and selecting the suitable solution and using the boundary conditions (i), (ii) and (iii) as in example 2, we get

$$y(x, t) = \sum_{n=1} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Using the initial condition (iv) we get,

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y(x, 0) = \frac{3dx}{l} \text{ for } 0 \le x \le l/3$$

$$= \frac{3d}{l}(l - 2x), \text{ for } \frac{l}{3} \le x \le \frac{2l}{3}$$

$$= \frac{3d}{l}(x - l), \text{ for } \frac{2l}{3} \le x \le l.$$

Finding Fourier sine series of $y(x, 0)$ in $(0, l)$ we get in the usual

way $y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

$$\therefore \qquad B_n = b_n = \frac{2}{l}\int_0^1 y(x, 0) \sin\frac{n\pi x}{l}\, dx$$

$$\therefore \qquad B_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3dx}{l}\sin\frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3d}{l}(l - 2x)\sin\frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3d}{l}(x - l)\sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6d}{l^2}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/3}$$

$$+ \frac{6d}{l^2}\left[(l - 2x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-2)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_l^{2l/3}$$

$$+\frac{6d}{l^2}\left[(x-l)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{2l/3}^{l}$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}+\cos n\pi\cdot\sin\frac{n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1+(-1)^n\right]$$

$= 0$ if n is odd.

$$=\frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3}\text{ if } n \text{ is even.}$$

Hence,

$$y(x,t)=\frac{36d}{\pi^2}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

i.e., $$y(x,t)=\frac{9d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cdot\cos\frac{2n\pi at}{l}.$$

By putting $x = l/2$, we get the displacement of the midpoint.

$\therefore\; y\left(\frac{l}{2},t\right)=0$, since $\sin\frac{2n\pi x}{l}$ becomes $\sin n\pi = 0$ when $x = l/2$.

Example 4

A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v, where $v=\begin{cases}\frac{cx}{l}, in\ 0<x<l\\ \frac{c(2l-x)}{l}, in\ l<x<2l\end{cases}$**, x being the distance from an end point. Find the displacement of any point at a distance x from the origin.**

Solution.

The boundary conditions are

$$y(0, t) = 0, \text{ for } t \geq 0 \quad \ldots(i)$$

$$y(2l, t) = 0, \text{ for } t \geq 0 \quad \ldots(ii)$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq 2l \quad \ldots(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{cx}{l}, \text{ in } 0 < x < l$$

$$= \frac{c}{l}(2l - x), \text{ in } l < x < 2l \quad \ldots(iv)$$

As in the previous examples, using boundary conditions (*i*) and (*ii*), we get

$$y(x, t) = \sin\frac{n\pi x}{2l}\left[C_n \cos\frac{n\pi at}{2l} + D_n \sin\frac{n\pi at}{2l}\right]$$

Using (*iii*), $C_n = 0$.

$$\therefore \quad y(x, t) = D_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

The most general solution of the equation (1) is

$$y(x, t) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \quad \ldots(2)$$

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} D_n\left(\frac{n\pi a}{2l}\right)\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}.$$

Using (*iv*),

$$\sum_{n=1}^{\infty} D_n\left(\frac{n\pi a}{2l}\right)\sin\frac{n\pi x}{2l} = v = \frac{cx}{l}, \text{ in } 0 < x < 1$$

$$= \frac{c}{l}(2l - x), \text{ in } l < x < 2l.$$

Expanding v in Fourier sine series, we get

$$D_n \cdot \frac{n\pi a}{2l} = \frac{2}{2l}\left[\frac{c}{l}\int_0^l x \sin\frac{n\pi x}{2l}\,dx + \frac{c}{l}\int_l^{2l}(2l - x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$\therefore\ D_n = \frac{2c}{n\pi al}\left[\left[\left\{x\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}\right]_0^l\right.$$

$$\left. + \left\{(2l-x)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{2c}{n\pi al}\left[\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2}\right.$$

$$\left. + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2c}{n\pi al}\cdot\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{16lc}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

Substituting this value of D_n in (2),

$$y(x, t) = \frac{16cl}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

Example 5

If a string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\frac{\partial y}{\partial t} = v_0 sin^3\left(\frac{\pi x}{l}\right), 0 < x < l$. Determine the transverse displacement $y(x, t)$.

Solution.

The boundary conditions are

$$y(0, t) = 0, \text{ for } t \geq 0 \quad \ldots(i)$$

$$y(l, t) = 0, \text{ for } t \geq 0 \quad \ldots(ii)$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq l \quad \ldots(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \text{ for } 0 \leq x \leq l \quad \ldots(iv)$$

Selecting the solution II, and using boundary conditions (*i*) and (*ii*)

we get $y(x, t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right)$

using (*iii*), $C = 0$

Therefore $y(x, t) = B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$, n any integer

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \quad \ldots(3)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad \ldots(IV)$$

Using in (3),

$$y(x, t) = \frac{3lv_0}{4\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi at}{l} - \frac{v_0 l}{12\pi a} \sin \frac{3\pi x}{l} \sin \frac{3\pi at}{l}$$

Example 6

A string is stretched and fastened to two points l apart. Motion is started by displacing the string in to the form $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement of any point of the string at a distance x from one end at any time t.

Solution.

The boundary conditions are:

$$y(0, t) = 0, \qquad t > 0$$

$$y(l, t) = 0, \qquad t > 0$$

$$\frac{\partial y}{\partial t} = 0, \qquad 0 < x < l$$

$$y(x,0) = k(lx - x^2), \qquad 0 < x < l$$

Using boundary condition (iv),

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = k\,(lx - x^2)$$

This shows that this is the half range Fourier sine series of $k(lx - x^2)$. Using the formula for Fourier coefficients,

$$B_n = b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}\{(-1)^n - 1\}\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$= 0 \text{ if } n \text{ is even}$$

$$= \frac{8kl^2}{n^3\pi^3} \text{ if } n \text{ is odd}$$

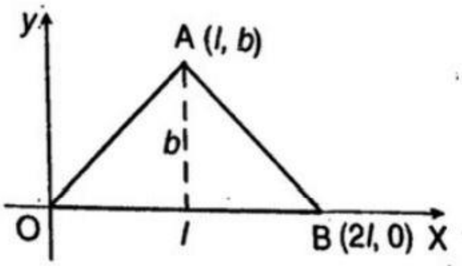
Substituting in IV,

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi at}{l}$$

Example 7

A taut string of length 2l is fastened at both ends. The midpoint of the string is taken to a height b and then released from rest in that position. Derive an expression for the displacement of the string.

Solution.



The boundary conditions are:

$y(0, t) = 0, \ t \geq 0$ ...(i)

$y(2l, t) = 0, \ t \geq 0$ ...(ii)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \leq x \leq 2l$ ...(iii)

$y(x, 0) = \frac{b}{l}x, 0 \leq x \leq l$

$= -\frac{b}{l}(x - 2l), l \leq x \leq 2l$

$\left[\text{since, equation of } OA \text{ is } y = \frac{b}{l}x \text{ and equation of } AB \text{ is } \frac{y-0}{x-2l} = \frac{b-0}{l-2l}\right]$

Starting with the solution

$y(x, t) = (A \cos \lambda x + B \cos \lambda x)(C \cos \lambda at + D \sin \lambda at)$

using the first boundary condition,

$y(o, t) = A (C \cos \lambda at + D \sin \lambda ab) = 0$

$\therefore \quad A = 0,$

using $\quad y(2l, t) = 0$ we get

$B \sin 2l\lambda (C \cos \lambda at + D \sin \lambda at) = 0$

$B \neq 0; 2l\lambda = n\pi ; \lambda = \frac{n\pi}{2l}$

Using $\left(\frac{\partial y}{\partial t}\right)_{t=0}; D = 0.$

$\therefore \quad y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}$ ...IV

Using boundary condition (*iv*) in IV,

$$y(x, 0) = \sum_{1}^{\infty} B_n \sin \frac{n\pi x}{2l} = \frac{b}{l} x, \ 0 \le x \le l$$

$$= -\frac{b}{l}(x - 2l), \ l \le x \le 2l$$

This is half-range Fourier sine series

$$\therefore \ B_n = \frac{2}{2l} \int_0^{2l} f(x) \sin \frac{n\pi x}{2l} \, dx$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{b}{l} x \sin \frac{n\pi x}{2l} \, dx - \frac{b}{l}\int_l^{2l} (x - 2l) \sin \frac{n\pi x}{2l} \, dx\right]$$

$$= \frac{b}{l^2}\left[\left\{(x)\left(-\frac{\cos \frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - \left(-\frac{\sin \frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_0^l \right.$$

$$\left. - \left\{(x - 2l)\left(-\frac{\cos \frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - \left(-\frac{\sin \frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{b}{l^2}\left[-\frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\left(\sin \frac{n\pi}{2}\right) + \frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right]$$

$$= \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$= 0$ for n even

$$= \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2} \text{ for odd } n.$$

Substituting in IV,

$$y(x, t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin (2n-1) \frac{\pi}{2} \cdot \sin \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi a t}{2l}$$

$$= \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi a t}{2l}$$

Exercise

A tightly stretched string with fixed end points x=0 and x=l is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$, find the displacement.

Heat on an insulated wire

Now let us consider with the heat equation. Consider a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and let t denote time. See Figure 2.

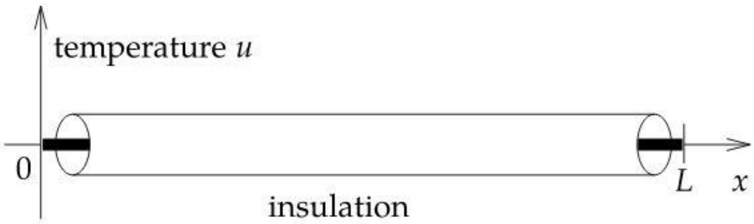


Figure 2: Insulated wire

Let $u(x,t)$ denote the temperature at point x at time t. The equation governing this setup is the so-called one-dimensional heat equation:

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},}$$

where $k > 0$ is a constant (the thermal conductivity of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum and vice-versa.

Therefore, the heat equation is $u_t = k u_{xx}$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. If the ends of the wire are kept at temperature 0, then the conditions are:

(i) $u(0,t) = 0$ and $u(L,t) = 0$.

If, on the other hand, the ends are also insulated, the conditions are:

(ii) $u_x(0,t) = 0$ and $u_x(L,t) = 0$.

Let us see why that is so. If u_x is positive at some point x0, then at a particular time, u is smaller to the left of x0, and higher to the right of x0. Heat is flowing from high heat to low heat, that is to the left. On the other hand if ux is negative then heat is again flowing from high heat to low heat, that is to the right. So when ux is zero, that is a point through which heat is not flowing. In other words, ux(0,t)=0 means no heat is flowing in or out of the wire at the point x=0.

We have two conditions along the x-axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e., u or a derivative of u is set to zero). We also need an initial condition—the temperature distribution at time t=0. That is, u(x,0)=f(x), for some known function f(x).

Solution of heat equation by method of separation of variables

We have to solve the equation

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \qquad \text{---------(1)}$$

where $k = \alpha^2$ is called the diffusivity of the substance.

Assume a solution of the form $u(x,t) = X(x).T(t)$ where X is a function of x and T is a function of t.

Then (1) becomes,

$$XT' = \alpha^2 X''T,$$

where $X'' = \frac{d^2X}{dx^2}$ and $T' = \frac{dT}{dt}$

i.e., $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \qquad \text{-----(2)}$$

The LHS is a function of x alone and the RHS is the function of t alone when x and t are independent variables. Equation (2) can be true only if each expression is equal to a constant.

$\therefore$ Let $\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k$ (constant)

$\therefore$ $X'' - kX = 0$, and $T' - \alpha^2 kT = 0$ ...(3)

The nature of solutions of (3) depends upon the values of k.

Case 1. Let $k = \lambda^2$, a positive number.

Then (3) becomes,

$$X'' - \lambda^2 X = 0, \text{ and } T' - \alpha^2\lambda^2 T = 0.$$

Solving, we get

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x} \text{ and } T = C_1 e^{\alpha^2\lambda^2 t}.$$

Case 2. Let $k = -\lambda^2$, a negative number. Then (3) becomes $X'' + \lambda^2 X = 0$, and $T' + \alpha^2\lambda^2 T = 0$.

Solving, we obtain

$$X = A_2 \cos \lambda x + B_2 \sin \lambda x, \text{ and } T = C_2 e^{-\alpha^2\lambda^2 t}.$$

Case 3. Let $k = 0$.

Then $X'' = 0$ and $T' = 0$.

Solving, we arrive at ,

$$X = A_3 x + B_3 \text{ and } T = C_3.$$

Hence the possible solutions of (1) are

$$u(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})\, C_1 e^{\alpha^2\lambda^2 t} \quad \text{...(I)}$$

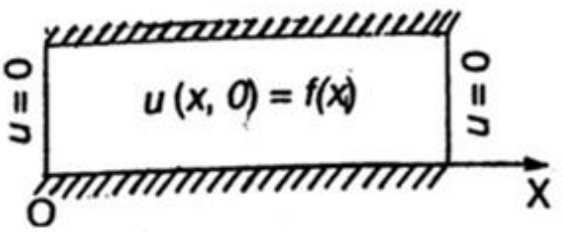
$$u(x, t) = (A_2 \cos \lambda x + B_2 \sin \lambda x)\, C_2\, e^{-\alpha^2\lambda^2 t} \quad \text{...(II)}$$

$$\mathbf{u(x, t)=(A_3x+B_3)C_3}$$

Example 8

A rod l cm with insulated lateral surface is initially at temperature $f(x)$ at an inner point distant x cm from one end. If both the ends are kept at zero temperature, find the temperature at any point of the rod at any subsequent time.

Solution.



Let $u(x, t)$ be the temperature at any point distant x from one end at any time t seconds. Then u satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

The boundary conditions, here, are

$u(0, t) = 0$ for all $t \geq 0$

$u(l, t) = 0$ for all $t \geq 0$

and the initial condition is

$u(x, 0) = f(x)$, for $0 < x < l$ ...(1)

Solving the equation (1) by the method of separation of variables and selecting the suitable solution to suit the physical nature of the problem as explained in the method § 3·6, we get

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x)\, e^{-\alpha^2 \lambda^2 t}$$

Substituting the boundary condition (i) in (2), we get,

$$u(0, t) = A e^{-\alpha^2 \lambda^2 t} = 0, \text{ for all } t \geq 0$$

$\therefore \; A = 0$

Employing the boundary condition (ii) in (2), we obtain,

$$u(l, t) = B \sin \lambda l\, e^{-\alpha^2 \lambda^2 t} = 0, \text{ for all } t \geq 0$$

i.e., $B \sin \lambda l = 0$.

If $B = 0$, (2) will be a trivial solution. Hence

$\sin \lambda l = 0$

$\therefore \; \lambda l = n\pi$, where n is any integer.

$\lambda = \frac{n\pi}{l}$, where n is any integer.

Then (2) reduces to

$$u(x, t) = B_n \sin \frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}} \quad ...(3),$$

where B_n is any constant.

Since the equation (1) is linear, its most general solution is obtained by a linear combination of solutions given by (3).

Hence the most general solution is

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \qquad \ldots(4).$$

(4) should satisfy the initial condition (*iii*).

Using (*iii*) in (4),

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}=f(x),\ \text{ for } 0<x<l \text{ (given)} \qquad \ldots(5).$$

If $u(x, 0)$, for $0 < x < l$, is expressed in a half-range Fourier sine series in $0 < x < l$, we know that

$$u(x,0)=f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \text{ where}$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Comparing this with (5), we get

$$B_n=b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Therefore the temperature function $u(x, t)$, is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right)\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}.$$

Two-Dimensional Heat Flow

When the heat flow is along curves instead of along straight lines, all the curves lying in parallel planes, then the flow is called two-dimensional. Let us consider now the flow of heat in a metal plate in the XOY plane. Let the plate be of uniform thickness h, density ρ, thermal conductivity k and the specific heat c. Since the flow is two dimensional, the temperature at any point of the plate is independent of the z-co-ordinate. The heat flow lies in the XOY plane and is zero along the direction normal to the XOY plane.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 5 –APPLICATIONS OF PDE–SMTA1405

Introduction

Contents – Two dimensional heat equation - steady state heat flow in two dimensions- Laplace equation in Cartesian and polar co-ordinates (excluding annulus) – Fourier series solution.

Recall that a partial differential equation or *PDE* is an equation containing the partial derivatives with respect to *several* independent variables. Solving PDEs will be our main application of Fourier series.

Two-Dimensional Heat Flow

When the heat flow is along curves instead of along straight lines, all the curves lying in parallel planes, then the flow is called two-dimensional. Let us consider now the flow of heat in a metal plate in the XOY plane. Let the plate be of uniform thickness h, density ρ, thermal conductivity k and the specific heat c. Since the flow is two dimensional, the temperature at any point of the plate is independent of the z-co-ordinate. The heat flow lies in the XOY plane and is zero along the direction normal to the XOY plane.

In mathematics and physics, the **heat equation** is a certain partial differential equation. Solutions of the heat equation are sometimes known as **caloric functions**. The theory of the heat equation was first developed by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region.

As the prototypical parabolic partial differential equation, the heat equation is among the most widely studied topics in pure mathematics, and its analysis is regarded as fundamental to the broader field of partial differential equations. The heat equation can also be considered on Riemannian manifolds, leading to many geometric applications. Following work of Subbaramiah Minakshisundaram and Åke Pleijel, the heat equation is closely related with spectral geometry. A seminal nonlinear variant of the heat equation was introduced to differential geometry by James Eells and Joseph Sampson in 1964, inspiring the introduction of the Ricci flow by Richard Hamilton in 1982 and culminating in the proof of the Poincaré conjecture by Grigori Perelman in 2003.

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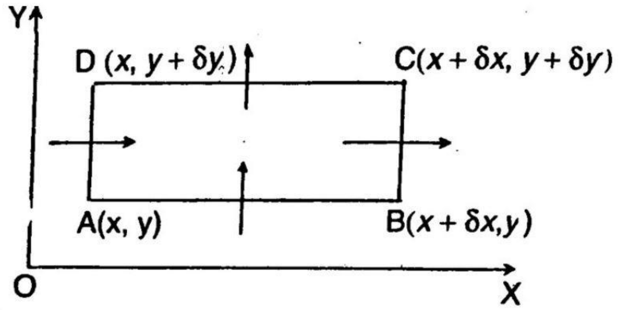


Now, consider a rectangular element $ABCD$ of the plate with sides δx and δy, the edges being parallel to the coordinates axes, as shown in the figure. Then the quantity of heat *entering* the element $ABCD$ per sec. through the surface AB is

$$= -k\left(\frac{\partial u}{\partial y}\right)_y \delta x \cdot h.$$

Similarly the quantity of heat entering the element $ABCD$ per sec. through the surface AD is

$$= -k\left(\frac{\partial u}{\partial x}\right)_x \delta y \cdot h.$$

The amount of heat which flows *out* through the surfaces BC and CD are

$$-k\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \cdot \delta y \cdot h \text{ and } -k\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \cdot \delta x \cdot h \text{ respectiveiy.}$$

Therefore the total gain of heat by the rectangular element $ABCD$ per sec. = inflow–outflow

$$= kh\left[\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\}\delta y + \left\{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y \delta x\right\}\right]$$

$$= kh\delta x \cdot \delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \qquad \text{....(1)}$$

The rate of gain of heat by the element $ABCD$ is also given by

$$\rho\delta x \cdot \delta y \cdot h \cdot c \cdot \frac{\partial u}{\partial t} \qquad \text{....(2)}$$

Equating the two-expressions for gain of heat per sec. from (1) and (2), we have, Equating the two-expressions for gain of heat per sec. from (1) and (2), we have,

$$\rho\,\delta x\cdot\delta y\cdot h\cdot c\cdot\frac{\partial u}{\partial t}=h\,k\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

$$i.e.,\quad \frac{\partial u}{\partial t}=\frac{k}{\rho c}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

Taking the limit as $\delta x\to 0,\ \delta y\to 0$, the above reduces to

$$\frac{\partial u}{\partial t}=\frac{k}{\rho c}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Putting $\alpha^2=\frac{k}{\rho c}$ as before, the equation becomes,

$$\frac{\partial u}{\partial t}=\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{....(3)}$$

The equation (3) gives the temperature distribution of the plate in the transient state.

In the steady-state, u is independent of t, *so that* $\frac{\partial u}{\partial t}=0$. *Hence the temperature distribution of the platc in the steady-state is* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

i.e., $\nabla^2 u=0$, which is known as *Laplace's Equation* in two-dimensions.

Corollary. If the stream lines are parallel to the x-axis, then the rate of change $\frac{\partial u}{\partial y}$ of the temperature in the direction of the y-axis will be zero. Then the heat-flow equation reduces to $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ which is the heat-flow equation in one-dimension.

Solution of the Equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

The equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$ ....(1)

Assume the solution $u(x, y)=X(x)\cdot Y(y)$,
where X is a function of x alone and Y a function of y alone.

$\therefore \quad \frac{\partial^2 u}{\partial x^2}=X'' Y$, and $\frac{\partial^2 u}{\partial y^2}=XY''$.

The Laplace equation $\nabla^2 u=0$ becomes $X''Y+Y''X=0$

i.e., $\quad \frac{X''}{X}=\frac{-Y''}{Y}$ ....(2)

The left hand side of (2) is a function of x alone and the right hand side is a function of y alone. Also x and y are independent variables. Hence, this is possible only if each quantity is equal to a constant k.

$\therefore \quad$ Let $\frac{X''}{X}=\frac{-Y''}{Y}=k$ ....(3)

i.e., $\quad X''-kX=0$, and $Y''+kY=0$. ...(4)

Case 1. Let $k=\lambda^2$, a positive number.
Then $X''-\lambda^2 X=0$, and $Y''+\lambda^2 Y=0$.
Solving, $X=A_1 e^{\lambda x}+B_1 e^{-\lambda x}$ and $Y=C_1 \cos \lambda y+D_1 \sin \lambda y$.

Case 2. Let $k=-\lambda^2$, a negative number.
Then (4) becomes $X''+\lambda^2 X=0$ and $Y''-\lambda^2 Y=0$.
Solving these equations, we have,

$X=A_2 \cos \lambda x+B_2 \sin \lambda x$ and $Y=C_2 e^{\lambda y}+D_2 e^{-\lambda y}$.

Case 3. Let $k=0$. Then (4) reduces to
$X''=0$ and $Y''=0$.
On solving these equations,
$X=A_3 x+B_3$ and $Y=C_3 y+D_3$.

Therefore, the possible solutions of (1) are

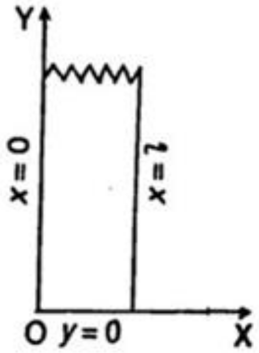
$u(x, y)=(A_1 e^{\lambda x}+B_1 e^{-\lambda x})(C_1 \cos \lambda y+D_1 \sin \lambda y)$ ...(I)

$u(x, y)=(A_2 \cos \lambda x+B_2 \sin \lambda x)(C_2 e^{\lambda y}+D_2 e^{-\lambda y})$ ...(II)

$u(x, y)=(A_3 x+B_3)(C_3 y+D_3)$ ...(III)

In problems where the boundary conditions are given, we have to select a suitable solution or a linear combination of solutions to satisfy (1) and the boundary conditions.

Example 9. *An infinitely long plane uniform plate is bounded by two parallel edges $x = 0$ and $x = l$, and an end at right angles to them. The breadth of this edge $y = 0$ is l and is maintained at a temperature $f(x)$. All the other three edges are at temperature zero. Find the steady-state temperature at any interior point of the plate.*



Let $u(x, y)$ be the temperature at any point (x, y) of the plate.

Then u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{....(1)}$$

The boundary conditions are

$$\begin{aligned} u(0, y) &= 0, \quad \text{for } 0 \le y \le \infty & \text{....}(i) \\ u(l, y) &= 0, \quad \text{for } 0 \le y \le \infty & \text{....}(ii) \\ u(x, \infty) &= 0, \quad \text{for } 0 \le x \le l & \text{....}(iii) \\ u(x, 0) &= f(x), \quad \text{for } 0 < x < l & \text{....}(iv) \end{aligned}$$

Solving (1), we get,

$$u(x, y) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos \lambda y + D_1 \sin \lambda y) \qquad \text{....(I)}$$

$$u(x, y) = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) \qquad \text{....(II)}$$

$$u(x, y) = (A_3 x + B_3)(C_3 y + D_3) \qquad \text{....(III)}$$

Of these solutions, we have to select a solution to suit the boundary conditions.

Since $u = 0$ as $y \to \infty$, we select the solution (II) as a possible solution (rejecting the other two).

$$\therefore \; u(x, y) = (A \cos \lambda x + B \sin \lambda x)\; (Ce^{\lambda y} + De^{-\lambda y}) \qquad \text{....(2)}$$

Using the boundary condition (1),

$$u(0, y) = A\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y \le \infty. \;\therefore\; A = 0$$

Using the boundary condition (ii) in (2),

$$u(l, y) = B \sin \lambda l\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y \le \infty.$$

Since $B \neq 0$, $\sin \lambda l = 0$. Hence $\lambda l = n\pi$

i.e., $\lambda = \frac{n\pi}{l}$, where n is any integer.

As $y \to \infty$, $u \to 0$, from (iii). $\therefore\; C = 0$.

Hence $u(x, y) = B_n \sin \frac{n\pi x}{l} e^{-\frac{n\pi y}{l}}$, where $BD = B_n$.

Therefore the most general solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{n\pi y}{l}} \quad \text{....(3)}$$

Using the boundary condition (*iv*) in (3),

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x), \text{ in } 0 < x < l \quad \text{....(4)}$$

Expressing $f(x)$ as a half-range Fourier sine series in $(0, l)$, we have

$$f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{....(5),}$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$.

Comparing (4) and (5), $B_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$.

Therefore the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \right) \sin \frac{n\pi x}{l} \cdot e^{-\frac{n\pi y}{l}}$$

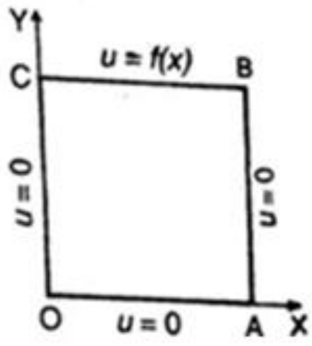
Note. If $f(x)$ is given explicitly in any problem, evaluate the value of B_n from the integral and substitute.

Example 10. The vertices of a thin square plate are $(0,0), (l,0), (0,l), (l,l)$. The upper edge of the square is maintained at an arbitrary temperature given by $u(x,l) = f(x)$. The other three edges are kept at zero temperature. Find the steady state temperature at any point on the plate.

Solution.

Suppose that $u(x, y)$ is the temperature at any point (x, y) of the plate in steady-state.

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ....(1)



The boundary conditions are

$u(0, y) = 0$, for $0 \le y < l$ (*i*)

$u(l, y) = 0$, for $0 \le y < l$ (*ii*)

$u(x, 0) = 0$, for $0 \le x \le l$ (*iii*)

$u(x, l) = f(x)$, for $0 < x < l$ (*iv*)

Solving (1), we get the three possible solutions,

$$u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C \cos \lambda y + D \sin \lambda y) \quad \text{....(I)}$$

$$u(x, y) = (A \cos \lambda x + B \sin \lambda x)(C e^{\lambda y} + D e^{-\lambda y}) \quad \text{....(II)}$$

$$u(x, y) = (Ax + B)(Cy + D) \quad \text{....(III)}$$

where A, B, C, D are different arbitary constants in each solution.

Now we shall select the solution II.

i.e., $u(x, y) = (A \cos \lambda x + B \sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ ....(II)

Using the boundary condition (*i*) in (*II*),

$$A\ (Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y < l.\ \therefore A = 0$$

Using the condition (*ii*) in (*II*)

$$u(l, y) = B \sin \lambda l\ (Ce^{\lambda y} + De^{-\lambda y}) = 0. \text{ But } B \neq 0; \sin \lambda l = 0$$

i.e., $\lambda l = n\pi$

i.e., $\lambda = \frac{n\pi}{l}$ where *nn* is any integer.

Using (*iii*) in *II*,

$u(x, 0) = (C + D)(B \sin \lambda x) = 0$, for $0 \le x \le l$.

$B \neq 0$ Hence $C + D = 0.\ \therefore D = -C$.

Hence (*II*) reduces to,

$$u(x, y) = BC \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$