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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Fourier Series – SMTA1401

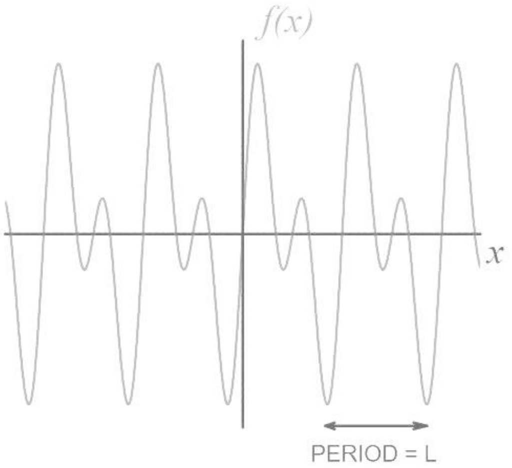
I. Introduction

Contents - Fourier series – Euler's formula – Dirichlet's conditions – Fourier series for a periodic function – Parseval's identity (without proof) – Half range cosine series and sine series – simple problems – Harmonic Analysis.

Periodic Functions

A function $f(x)$ is said to be periodic, if and only if $f(x+L)=f(x)$ is true for some value of L and for all values of x. The smallest value of L for which this equation is true for every value of x will be called the period of the function.

A graph of periodic function $f(x)$ that has period L exhibits the same pattern every L units along the $x-$ axis, so that $f(x+L) \ = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods). For example, $sinx$ and $cosx$ are periodic with period 2π and $tanx$ has period π.



Dirichlet's Conditions

(i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

(ii) $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c+2\pi)$

(iii) $f(x)$ has a finite number of maxima and minima in $(c, c+2\pi)$

Note 1: These conditions are not necessary but only sufficient for the existence of Fourier series.

Note 2: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(-\infty, \infty)$, then $f(x)$ has to be periodic of periodicity 2π for the existence of Fourier series of period 2π.

Note 3: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(c, c+2\pi)$, then $f(x)$ need not be periodic for the existence of Fourier series of period 2π.

Note 4: If $x = a$ is a point of continuity of $f(x)$, then the value of Fourier series at $x = a$ is $f(a)$. If $x = a$ is a point of discontinuity of $f(x)$, then the value of Fourier series at $x = a$ is $\frac{1}{2}[f(a+) + f(a-)]$. In other words, specifying a particular value of $x = a$ in a Fourier series, gives a series of constants that should equal $f(a)$. However, if $f(x)$ is discontinuous at this value of x, then the series converges to a value that is half-way between the two possible function values.

Fourier Series

Periodic functions occur frequently in engineering problems. Such periodic functions are often complicated. Therefore, it is desirable to represent these in terms of the simple periodic functions of sine and cosine. A development of a given periodic function into a series of sines and cosines was studied by the French physicist and mathematician Joseph Fourier (1768-1830). The series of sines and cosines was named after him.

If $f(x)$ is a periodic function with period 2π defined in $(c, c + 2\pi)$ and the Dirichlet's conditions are satisfied, then $f(x)$ can be expanded as a **Fourier series** of the form

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n\, sinnx)}$$

where the Fourier coefficients $\boldsymbol{a_0, a_n\ and\ b_n}$ are calculate using Euler's formula.

Euler's Formula

(1) $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

(2) $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, cosnxdx$

(3) $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, sinnxdx$

Standard Integrals

1. $\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}[asinbx - bcosbx]$
2. $\int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}[acosbx + bsinbx]$
3. Bernoulli's generalized formula of integration by parts

$\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Trigonometric results

1. $sinn\pi = 0,$ if n is an integer
2. $cosn\pi = (-1)^n,$ if n is an integer

Example 1

Obtain the Fourier series of the following function defined in $(0, 2\pi)$.

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \quad \text{and has period } 2\pi$$

Solution.

STEP ONE

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\ &= \frac{\pi}{2} + \pi \\ \text{i.e. } a_0 &= \frac{3\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{1}{n}\Big(\pi\sin n\pi - 0\cdot\sin n0\Big) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\ &\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi) \end{aligned}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1),$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}\big(1-(-1)^n\big) \\ &= -\frac{1}{n}(-1)^n + 0 - \frac{1}{n}\big(1-(-1)^n\big) \end{aligned}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]$$

where $a_0 = \frac{3\pi}{2}$, $\quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $\quad b_n = -\frac{1}{n}$

Example 2

Expand in Fourier series of periodicity 2π $f(x) = xsinx$, for $0 < x < 2\pi$

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} xsinx\, dx$$

$$= \frac{1}{\pi}[x(-cosx) - 1.(-sinx)]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi cos2\pi + sin2\pi]$$

$$= \frac{1}{\pi}[-2\pi.1 + 0]$$

$$= \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} xsinxcosnx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[sin(n+1)x - \sin(n-1)\,x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos\,(n+1)x}{n+1}\right) - 1.\left(\frac{-\sin(n+1)x}{(n+1)^2}\right) - [x\left(\frac{-\cos(n-1)x}{n-1}\right) - 1.\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-\cos\,(n+1)2\pi}{n+1}\right) - 1.\left(\frac{-\sin(n+1)2\pi}{(n+1)^2}\right) - [2\pi\left(\frac{-\cos(n-1)2\pi}{n-1}\right) - 1.\left(\frac{-\sin(n-1)2\pi}{(n-1)^2}\right)]\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi}{n+1}\right) - 1.(0) - [\left(\frac{-2\pi}{n-1}\right) - 1.(0)]\right]$$

$$= \left(\frac{-1}{n+1}\right) + \left(\frac{1}{n-1}\right)$$

$a_n = \frac{1}{n^2-1}$ provided $n \neq 1$.

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxcosx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} xsin2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2\,x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-\cos 2(2\pi)}{2}\right) - 1.\left(\frac{-\sin 2(\,2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[2\pi\left(\frac{-1}{2}\right) - 1.(0)]$$

$$a_1 = \frac{-1}{2}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} xsinxsinnx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[cos(n-1)x - \cos(n+1)\,x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin\,(n-1)x}{n-1}\right) - 1.\left(\frac{-\cos(n-1)x}{(n-1)^2}\right) - [x\left(\frac{s\,in(n+1)x}{n+1}\right) - 1.\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{\sin\,(n-1)2\pi}{n-1}\right) - 1.\left(\frac{-\cos(n-1)2\pi}{(n-1)^2}\right) - [2\pi\left(\frac{\sin(n+1)2\pi}{n+1}\right) - 1.\left(\frac{-\cos(n+1)2\pi}{(n+1)^2}\right)]\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{(n-1)^2}\right) - \left[\left(\frac{-1}{(n+1)^2}\right)\right] - \left(\frac{1}{(n-1)^2}\right) + \left[\left(\frac{-1}{(n+1)^2}\right)\right]\right]$$

$b_n = 0$ provided $n \neq 1$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxsinx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xsin^2x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1 - cos2x}{2}\right)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{sin2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{cos2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\left(2\pi - \frac{sin2(2\pi)}{2}\right) - 1.\left(\frac{(2\pi)^2}{2} + \frac{cos2(2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2 - 2\pi^2 + \frac{1}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{\pi}[2\pi^2]$$

$$b_1 = \pi$$

Therefore, the Fourier series expansion of the function $xsinx$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n\, sinnx)$$

$$xsinx = 1 - \frac{1}{2}cosx + 2\sum_{2}^{\infty}\frac{cosnx}{n^2 - 1} + \pi sinx$$

Example 3

Obtain all the Fourier coefficients of $f(x) = k$ where k is a constant, the periodicity being 2π.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} k\, dx$$

$$= \frac{k}{\pi}\int_{0}^{2\pi} dx$$

$$= \frac{k}{\pi}\,[x]_0^{2\pi}$$

$$= \frac{k}{\pi}\,[2\pi]$$

$$a_0 = 2k$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} k\, cosnxdx$$

$$= \frac{k}{\pi}\int_{0}^{2\pi} cosnxdx$$

$$= \frac{k}{\pi}\left[\frac{sinnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{sin2n\pi - sin0}{n}\right]$$

$$a_n = 0$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\ sinnxdx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} k\ sinnxdx$$

$$= \frac{k}{\pi}\int_0^{2\pi} sinnxdx$$

$$= \frac{k}{\pi}\left[\frac{-cosnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{cos2n\pi - cos0}{n}\right]$$

$$= \frac{k}{\pi}\left[\frac{1-1}{n}\right]$$

$$b_n = 0$$

Even and Odd Functions

The function f(x) is said to be even, if f(-x) = f(x).

The function f(x) is said to be odd, if f(-x) = -f(x).

If $f(x)$ is an even function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier cosine series**:**

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\ cosnx}$$

where the Fourier coefficients $\boldsymbol{a_0}$ and $\boldsymbol{a_n}$ are calculated by

(1) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

(2) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ cosnxdx$

If $f(x)$ is an odd function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier sine series**:**

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n\ sinnx}$$

where the Fourier coefficient $\boldsymbol{b_n}$ is calculated by $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ sinnxdx$

Example 4

Find the Fourier series for $f(x) = |cosx|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution.

Since $f(x) = |cosx|$ is an even function, $f(x)$ will contain only cosine terms.

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx$

STEP ONE

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosx\ dx + \int_{\frac{\pi}{2}}^{\pi} (-cosx)dx\right]$$

(Since in $(0, \frac{\pi}{2})$, $cosx$ is positive and in $(\frac{\pi}{2}, \pi)$ $cosx$ is negative)

$$= \frac{2}{\pi}\left[(sinx)_0^{\frac{\pi}{2}} - (sinx)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[sin\frac{\pi}{2} - sin0 - sin\pi + sin\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}[1 - 0 - 0 + 1]$$

$$a_0 = \frac{4}{\pi}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|cosnxdx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosxcosnx\ dx + \int_{\frac{\pi}{2}}^{\pi} (-cosxcosnx)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos(n+1)\,x + \cos(n-1)x\ dx - \int_{\frac{\pi}{2}}^{\pi} \cos(n+1)\,x + \cos\,(n-1)xdx\right]$$

$$= \frac{1}{\pi}\left[\left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_0^{\pi/2} \right.$$

$$\left. - \left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} \right.$$

$$\left. + \frac{\sin(n-1)\pi/2}{n-1} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\left\{ \sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2} \right\} + \frac{1}{n-1} \right.$$

$$\left. \times \left\{ \sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2} \right\} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} \right]\cos\frac{n\pi}{2}$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2} \text{ if } n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} |\cos x| \cos x \right]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= 0.$$

$$\therefore \quad |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

Example 5.

Find the Fourier series of $f(x) = e^x$ in $(-\pi, \pi)$ of periodicity 2π.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$

$$= \frac{1}{\pi}\,(e^x)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,(e^{\pi} - e^{-\pi})$$

$$= \frac{2}{\pi}\,\sinh\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} \left(\cos nx + n \sin nx \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-1)^n + e^{-\pi}(-1)^n]$$

$$= \frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} \left(\sin nx - n \cos nx \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-n)(-1)^n + e^{-\pi} n (-1)^n]$$

$$= \frac{-2(-1)^n \cdot n}{\pi(1+n^2)} \sinh \pi$$

$$e^x = \frac{\sinh \pi}{\pi} \left[1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \left(\cos nx - n \sin nx \right) \right]$$

Example 6

Derive the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π and hence deduce $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\ [\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{(-\pi)^2}{2} + \frac{(-\ \pi)^3}{3}\right)]$$

$$= \frac{1}{\pi}\ [\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}]$$

$$a_0 = \frac{2\ \pi^2}{3}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)cosnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{sinnx}{n}\right) - (1 + 2x)\left(\frac{-cosnx}{n^2}\right) + (2)\left(\frac{-sinnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (1 - 2\pi)\left(\frac{cosn\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{(-1)^n}{n^2}\right) + 2\pi\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{4}{n^2}(-1)^n$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)sinnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{-cosnx}{n}\right) - (1 + 2x)\left(\frac{-sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) + 2\left(\frac{cosn\pi}{n^3}\right) - (-\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) - 2\left(\frac{cosn\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{-(-1)^n}{n}\right)\right]$$

$$b_n = \frac{2}{n}(-1)^{n+1}$$

Therefore, the Fourier series is of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \, cosnx + b_n \, sinnx)$$

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} (\frac{4}{n^2} (-1)^n \, cosnx + \frac{2}{n} (-1)^{n+1} \, sinnx) \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)$$

STEP FOUR

Deduction:

The end points of the range are $x = \pi$ and $x = -\pi$. Therefore, the value of Fourier series at $x = \pi$ is the average value of $f(x)$ at the points $x = \pi$ and $x = -\pi$. Hence put $x = \pi$ in (1),

$$\Rightarrow \frac{f(-\pi)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \, cosn\pi$$

$$\Rightarrow \frac{(\pi + \pi^2) + (-\pi + \pi^2)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{2n}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 7.

Expand f(x)=x², when $-\pi < x < \pi$ in a Fourier series of periodicity 2π. Hence deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{to } \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{to } \infty = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{to } \infty = \frac{\pi^2}{8}$

$f(x)$ is an even function of x in $-\pi < x < \pi$. Hence bn = 0 and only cosine terms will be present. Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{....}(i)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \left[(x^2) \left(\frac{\sin nx}{n} \right) - (2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{2\pi}{n^2} \cos n\pi \right] = \frac{4(-1)^n}{n^2}.$$

Substituting these values in (i),

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad \text{....}(ii)$$

i.e., $x^2 = \frac{\pi^2}{3} - 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \right]$, in $-\pi < x < \pi$.

The function $f(x) = x^2$ is continuous at $x = 0$. Hence the sum of the Fourier series equals the value of the function at $x = 0$. Putting $x = 0$, in (ii),

$$0 = \frac{\pi^2}{3} - 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \qquad \text{....}(iii)$$

$x = \pi$ is an end point. Hence the sum of the Fourier series at $x = \pi$ equals $\frac{1}{2} \{ f(-\pi + 0) + f(\pi - 0) \}$

Putting $x = \pi$ in the series of (ii),

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2} = \frac{1}{2} [f(-\pi + 0) + f(\pi - 0)]$$

$$= \frac{1}{2} [\pi^2 + \pi^2] = \pi^2$$

$$\therefore 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2}{3}\pi^2$$

i.e., $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad \text{....}(iv)$$

Adding (*iii*) and (*iv*),

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{4}$$

i.e., $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{ to } \infty = \frac{\pi^2}{8}.$$

Example 8 *Find the Fourier series of periodicity* 2π

$$\text{for} \quad f(x) = \begin{cases} x \text{ when } -\pi < x < 0 \\ 0 \text{ when } 0 < x < \frac{\pi}{2} \\ x - \frac{\pi}{2} \text{ when } \frac{\pi}{2} < x < \pi \end{cases}$$

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...}(1)$$

where $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Taking $c = -\pi$ in the Euler formulas we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x))\, dx + \int_{0}^{\pi} f(x)\, dx \right\}.$$

Now using the hypothesis for the value of $f(x)$, we get

$$a_0 = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-k)\, dx + \int_{0}^{\pi} k dx \right\} = \frac{1}{\pi} \left\{ \left(-kx \right)_{-\pi}^{0} + \left(kx \right)_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi} \{(0 - k\pi) + (k\pi - 0)\}$$

Thus $a_0 = 0$. Again for $n = 1, 2, 3, \ldots$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx \right\}.$$

Substituting the values supplied for $f(x)$, we have

$$a_n = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-k) \cos nx\, dx + \int_{0}^{\pi} k \cos nx\, dx \right\}$$

$$= \frac{1}{\pi}\left\{\left(-k\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(k\frac{\sin nx}{n}\right)_{0}^{\pi}\right\}$$

Since sin 0, sin $(-n\pi)$ and sin $n\pi$ are all zero, we get $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-k\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{k}{n}\cos 0 - \frac{k}{n}\cos(-n\pi)\right\} + \left\{-\frac{k}{n}\cos n\pi + \frac{k}{n}\cos 0\right\}\right]$$

But $\cos(-\alpha) = \cos\alpha$, giving $\cos(-n\pi) = \cos n\pi$; further, $\cos 0 = 1$.

Hence $b_n = \frac{k}{n\pi}[\{1 - \cos n\pi\} + \{-\cos n\pi + 1\}] = \frac{k}{n\pi}(2 - 2\cos n\pi)$

$\therefore\ b_n = \frac{2k}{n\pi}(1 - \cos n\pi)$. Now $\cos n\pi = \begin{cases} -1, & \text{for odd } n \\ +1, & \text{for even } n \\ (-1)^n, & \text{for any integer } n \end{cases}$

Hence $b_1 = \frac{4k}{\pi}$; $b_2 = 0$; $b_3 = \frac{4k}{3\pi}$; $b_4 = 0$; $b_5 = \frac{4k}{5\pi}$; $b_6 = 0$;

$$b_7 = \frac{4k}{7\pi} \ldots\ldots$$

Using the values of a_n and b_n in (i) we obtain

$$f(x) = \frac{4k}{\pi}\left\{\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x + \cdots \text{ to } \infty\right\}$$

In the above equation putting $x = \pi/2$, we get

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\left\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\right\}$$

But, by hypothesis, $f\left(\frac{\pi}{2}\right) = k.$

Hence $k = \frac{4k}{\pi}\left\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\right\}$

Multiplying both the sides by $\frac{\pi}{4k}$, we have

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty.$$

Note. Functions of the type given in this example occur as external force acting on mechanical systems, electromotive forces in electric circuits etc.

Root Mean Square (RMS)Value

The root-mean-square value of a function $y = f(x)$ over a given (a, b) is defined as

$$\bar{y} = \sqrt{\left\{\frac{\int_a^b y^2\,dx}{b-a}\right\}}$$

If the interval is taken as $(c, c + 2\pi)$, then

$$\bar{y}^2 = \frac{1}{2\pi}\int_c^{c+2\pi} y^2\,dx$$

Suppose that $y = f(x)$ is expressed as a Fourier-series of periodicity 2π in $(c, c+2\pi)$. then,

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(ii)$$

$$\left.\begin{aligned} \text{where} \quad a_0 &= \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx \\ \text{and} \quad b_n &= \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx. \end{aligned}\right\} \qquad \ldots(iii)$$

Multiply (*ii*) by $f(x)$ and integrate term by term with respect to x over the given range. Thus,

$$\int_c^{c+2\pi} [f(x)]^2\, dx = \frac{a_0}{2} \int_c^{c+2\pi} f(x)\, dx + \sum_{n=1}^{\infty} \left[a_n \int_c^{c+2\pi} f(x) \cos nx\, dx + b_n \int_c^{c+2\pi} f(x) \sin nx\, dx \right]$$

$$= \frac{a_0}{2} (\pi a_0) + \sum_{n=1}^{\infty} [a_n (\pi a_n) + b_n (\pi b_n)] \text{ using } (iii)$$

$$\int_c^{c+2\pi} [f(x)]^2\, dx = 2\pi \left[\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$= (\text{Range}) \left[\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Ex 9. *Find the Fourier series of periodicity 2π for $f(x)=x^2$, in $-\pi<x<\pi$. Hence show that*

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\text{to }\infty=\frac{\pi^4}{90}.$$

In example 7, we have proved

$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$, which is the first part of this problem. The coefficients a_0, a_n, b_n were seen to be

$$a_0=\frac{2\pi^2}{3},\quad a_n=\frac{4(-1)^n}{n^2},\quad b_n=0.$$

Hence using the root-mean-square value in series,

$$2\pi\left[\frac{a_0{}^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2+b_n{}^2)\right]=\int_{-\pi}^{\pi}[(f(x)^2]\,dx=\int_{-\pi}^{\pi}x^4dx$$

$$2\pi\left[\frac{\pi^4}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right]=\frac{2}{5}\pi^5$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{5}-\frac{\pi^4}{9}=\frac{4\pi^4}{45}$$

$$\therefore\ \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

Change of Interval

Example 10 *Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$, in $0 < x < 3$*

Here the range and the period are same (equal to 3)

It is a full range series.

$\therefore\ 2l = 3;\ l = \frac{3}{2}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$

where $$a_0 = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = 0$$

$$a_n = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2 - 2x)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} \right) - (2 - 2x)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}} \right) + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}} \right) \right]_0^3$$

$$= \frac{3}{n\pi}$$

$$\therefore \quad f(x) = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{3}\right)$$

Half-Range Fourier Series

Example 11

Express $f(x) = x(\pi - x), 0 < x < \pi$ as a Fourier series of periodicity 2π containing (i) sine terms only and (ii) cosine terms only. Hence deduce, $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$ and $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Solution.

(i) sine series:

Let $f(x) = \sum_{1}^{\infty} b_n \sin nx$

where $\quad b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[\{\pi x - x^2\}\left(-\frac{\cos nx}{n} \right) - (\pi - 2x)\left(\frac{\sin nx}{n^2} \right) + (-2)\left(\frac{\cos nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\left\{ (-1)^n - 1 \right\} \right]$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n \right]$$

$= 0$ if n is even

$= \frac{8}{\pi n^3}$ if n is odd

$$\therefore \quad f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin (2n-1)\, x.$$

Setting $x = \pi/2$ which is a point of continuity we get first deduction .

(ii) cosine series:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$$

where $\quad a_0 = \frac{2}{\pi} \int_0^{\pi} x (\pi - x)\, dx$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n} \right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2} \right) + (-2)\left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n^2}(-1)^n - \frac{\pi}{n^2} \right] = -\frac{2}{n^2}[\, 1 + (-1)^n \,]$$

$= 0$ for n odd

$= -\frac{4}{n^2}$ for n even

$$x(\pi - x) = \frac{\pi^2}{6} - 4 \sum_{n = 2, 4, 6 \ldots}^{\infty} \frac{1}{n^2} \cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{1}^{\infty} \frac{1}{n^2} \cos 2nx.$$

Setting $x = \pi/2$ which is a point of continuity,

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - \sum \frac{1}{n^2}(-1)^n$$

$$\sum \frac{1}{n^2}(-1)^n = \frac{\pi^2}{6} - \frac{\pi^2}{4}$$

Harmonic Analysis

Example 12

Compute the first three harmonics of the Fourier series of f(x) given by the following table.

x	0	π/3	2π/3	π	4π/3	5π/3	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution.

We will form the table for the convenience of work.

We exclude the last point x = 2π.

x	f(x)	cos x	sin x	cos 2x	sin 2x	cos 3x	sin 3x
0	1.0	1	0	1	0	1	0
π/3	1.4	0.5	0.866	-0.5	0.866	-1	0
2π/3	1.9	-0.5	0.866	-0.5	-0.866	1	0
π	1.7	-1	0	1	0	-1	0
4π/3	1.5	-0.5	-0.866	-0.5	0.866	1	0
5π/3	1.2	0.5	-0.866	-0.5	-0.866	-1	0

$a_0 = 2/6 \sum f(x) = 1/3(1.0 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2) = 2.9$

$a_1 = 2/6 \sum f(x) \cos x = 1/6(1 + 0.7 – 0.95 – 1.7 – 0.75 + 0.6) = -0.37$

$a_2 = 2/6 \sum f(x) \cos 2x = -0.1$

$a_3 = 2/6 \sum f(x) \cos 3x = 0.03$

$b_1 = 2/6 \sum f(x) \sin x = 0.17$

$b_2 = 2/6 \sum f(x) \sin 2x = -0.06$

$b_3 = 2/6 \sum f(x) \sin 3x = 0$

$f(x) = 1.45 – 0.33 \cos x – 0.1 \cos 2x + 0.03 \cos 3x + 0.17 \sin x – 0.06 \sin 2x$

Example 13

The values of x and the corresponding values of f(x) over a period T are given below. Show that $f(x) = 0.75 + 0.37 \cos \theta + 1.004 \sin \theta$ where $\theta = 2\pi x/T$.

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution.

We omit the last values since f(x) at x = 0 is known. $\theta = 2\pi x/T$.

When x varies from 0 to T, θ varies from 0 to 2π with an increase of 2π/6.

Let $f(x) = F(\theta) = a_0/2 + a_1 \cos \theta + b_1 \sin \theta$

θ	y	cos θ	sin θ	y cos θ	y sin θ
0	1.98	1.0	0	1.98	0
π/3	1.30	0.5	0.866	0.65	1.1258
2π/3	1.05	-0.5	0.866	-0.525	0.9093
π	1.30	-1	0	-1.3	0
4π/3	-0.88	-0.5	-0.866	0.44	0.762
5π/3	-0.25	0.5	-.866	-0.125	0.2165
∑	4.6			1.12	3.013

$a_0 = 2/6 \sum f(x) = 4.6/3 = 1.5$

a_1 = 2(1.12)/6 = 0.37

b_1 = 2/6(3.013) = 1.004

Therefore, f(x) = 0.75 + 0.37 cos θ + 1.004 sin θ

Example 14

Find the first three harmonics of Fourier series of y = f(x) from the following data.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	298	356	373	337	254	155	80	51	60	93	147	221

Solution.

The table can be formulated in the usual way.

Let y = $a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Here a_0 = 2/12 ∑y = 1/6(2425) ≈ 404

a_1 = 2/12 ∑y cos x = 107.048 ≈ 107

a_2= 2/12 ∑y cos 2x ≈ -13

a_3 = 2/12 ∑y cos 3x ≈ 2.0

b_1 = 2/12 ∑y sin x ≈ 121

b_2 = 2/12 ∑y sin 2x ≈ 9

b_3 = 2/12 ∑y sin 3x ≈ -1

Therefore, y ≈ 202 + 107 cos x – 13 cos 2x + 2 cos 3x + 121 sin x + 9 sin 2x – sin 3x.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION – SMTA1401

I. Introduction

Contents - One dimensional wave equation – Transverse vibrating of finite elastic string with fixed ends – Boundary and initial value problems – One dimensional heat equation – Steady state problems with zero boundary conditions – Two dimensional heat equation – Steady state heat flow in two dimensions- Laplace equation in Cartesian form (No derivations required).

Recall that a partial differential equation or *PDE* is an equation containing the partial derivatives with respect to *several* independent variables. Solving PDEs will be our main application of Fourier series.

II. One-dimensional wave equation

Let us start with the wave equation. Imagine we have a tensioned guitar string of length L. Let us only consider vibrations in one direction. Let x denote the position along the string, let t denote time, and let y denote the displacement of the string from the rest position. See Fig. 1.

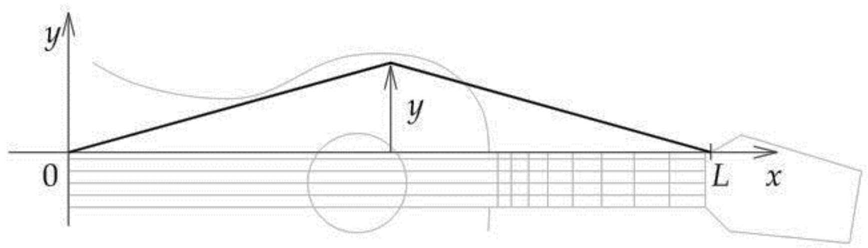


Figure 1: Vibrating string of length L, x is the position, y is displacement

Let $y(x,t)$ denote the displacement at point x at time t. The equation governing this setup is the so-called one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots\ldots. (1)$$

We generally use a more convenient notation for partial derivatives. We write y_t instead of $\frac{\partial y}{\partial t}$, and we write y_{xx} instead of $\frac{\partial^2 y}{\partial x^2}$.

With this notation the equation that governs this setup is the so-called one-dimensional wave equation, becomes $y_{tt} = a^2 y_{xx}$

for some constant $a > 0$. The intuition is similar to the heat equation, replacing velocity with acceleration: the acceleration at a specific point is proportional to the second derivative of the shape of the string. The wave equation is an example of a hyperbolic PDE.

The following assumptions are made while deriving the 1-D wave equation:

1. The motion takes place entirely in one plane. This plane is chosen as the xy-plane.
2. In this plane, each particle of the string moves in a direction perpendicular to the equilibrium position of the string.
3. The tension T caused by the string before fixing it at the end points is constant at all times and at all points of the deflected string.

4. The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.
5. The effect of friction is negligible.
6. The string is perfectly flexible. It can transmit only tension but not bending or shearing forces.
7. The slope of the deflection curve is small at all points and at all times.

Solution of the Wave Equation (by the method of separation of variables)

Let $y = X(x).T(t)$ be a solution of (1), where X(x) is a function of x only $T(t)$ is a function t only.

$$\frac{\partial^2 y}{\partial t^2} = XT'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X''T,$$

where $$X'' = \frac{d^2X}{dx^2} \text{ and } T'' = \frac{d^2T}{dt^2}.$$

Hence (1) becomes, $XT'' = a^2 X'' T$

i.e., $$\frac{X''}{X} = \frac{T''}{a^2T} \qquad ...(2)$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

$\therefore \quad \dfrac{X''}{X} = \dfrac{T''}{a^2T} = k$ (say) where k is any constant.

Hence $X'' - kX = 0$ and $T'' - a^2 kT = 0$ ...(3)

Solutions of these equations depend upon the nature of the value of k.

Case 1. Let $k = \lambda^2$, a positive value .

Now the equation (3) are $X'' - \lambda^2 X = 0$ and $T'' - a^2\lambda^2 T = 0$.

Solving the ordinary differential equations we get,

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$$

and $$T = C_1 e^{\lambda at} + D_1 e^{-\lambda at}$$

Case 2. Let $k = -\lambda^2$, a negative number.

Then the equations (3) are $X'' + \lambda^2 X = 0$ and $T'' + a^2\lambda^2 T = 0$.

Solving, we get,

$$X = A_2 \cos \lambda x + B_2 \sin \lambda x$$

and $$T = C_2 \cos \lambda at + D_2 \sin \lambda at.$$

Case 3. Let $k = 0$.

Now the equations (3) are $X'' = 0$ and $T'' = 0$.

Then integrating, $X = A_3 x + B_3$

and $$T = C_3 t + D_3$$

Thus the various possible solutions of the wave equation are

$$y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda at} + D_1 e^{-\lambda at}) \quad \text{...(I)}$$

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at) \quad \text{...(II)}$$

$$y = (A_3 x + B_3)(C_3 t + D_3) \quad \text{...(III)}$$

Example 1

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the position $y = f(x)$. It is set vibrating by giving to each of its points a velocity $\frac{\partial y}{\partial t} = g(x)$ at $t = 0$. Find $y(x, t)$ in the form of Fourier series.

Solution.

The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

The boundary conditions under which (1) is to be solved are

(*i*) $y(0, t) = 0$ for $t \geq 0$

(*ii*) $y(l, t) = 0$ for $t \geq 0$

(*iii*) $y(x, 0) = f(x)$, for $0 < x < l$

(*iv*) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$, for $0 < x < l$.

Solving (1) by the method of separation of variables, we get,

$$y(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda at} + D_1 e^{-\lambda at}) \quad \text{...(I)}$$

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at) \quad \text{...(II)}$$

$$y = (A_3 x + B_3)(C_3 t + D_3). \quad \text{...(III)}$$

Since the solution should be periodic in t, we reject solutions (I) and (III) and select (II) to suit the boundary conditions (*i*), (*ii*), (*iii*) and (*iv*).

$$\therefore\ y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \qquad ...(2),$$

where A, B, C, D are arbitrary constants.

Using boundary condition (*i*) in (*ii*),

$A(C\cos\lambda at + D\sin\lambda at) = 0$ for all $t \geq 0$.

$\therefore A = 0$.

Applying the boundary condition (*ii*) in (2),

$B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$, for all $t \geq 0$.

If $B = 0$, the solution becomes $y = 0$ which is not true.

$\sin\lambda l = 0$, $B \neq 0$.

i.e., $\lambda l = n\pi$, where n is any integer.

$$\therefore\ \lambda = \frac{n\pi}{l}$$

$$\therefore\ y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$$

i.e.,
$$y(x,t) = \sin\frac{n\pi x}{l}\left(C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right) \qquad ...(3),$$

where $BC = C_n$ and $BD = D_n$.

Since the wave equation is linear and homogeneous, the most general solution of it is

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \qquad ...(4).$$

This satisfies boundary condition (*i*) and (*ii*). To find C_n and D_n we make use of the initial conditions (*iii*) and (*iv*).

$$y(x,0) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} = f(x) \qquad ...(5)$$

and
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum\frac{n\pi a}{l}D_n\sin\frac{n\pi x}{l} = g(x) \qquad ...(6)$$

The left-hand sides of (5) and (6) are Fourier series of the right-hand side functions.

$$\text{Hence } C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(7)$$

and
$$\frac{n\pi a}{l}D_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(8)$$

Example 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the position $y(x, 0) = y_0 sin^3(\frac{\pi x}{l}) = f(x)$. If it released from rest from this position, find the displacement $y(x, t)$ at any time t and at any distance from the end $x = 0$.

Solution.

The displacement y of the particle at a distance x from the end x=0 and time t is governed by $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are:

$$y(0, t) = 0, \quad \text{for all } t \geq 0 \quad (i)$$

$$y(l, t) = 0, \quad \text{for all } t \geq 0, \quad (ii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad \text{for } 0 \leq x \leq l \quad (iii)$$

$$y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right), \quad \text{for } 0 \leq x \leq l \quad (iv)$$

Now solving (1) and selecting the proper solution to suit the physical nature of the problem and making use of the boundary conditions (*i*) and (*ii*) as in the previous problem, we get

$$y(x, t) = B \sin\frac{n\pi x}{l}\left(C \cos\frac{n\pi at}{l} + D \sin\frac{n\pi at}{l}\right) \quad ...(2)$$

Again using the boundary condition (*iii*),

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = B \sin\frac{n\pi x}{l}\left(D \cdot \frac{n\pi a}{l}\right).$$

If $B = 0$, (2) takes the form $y(x, t) = 0$. Hence B cannot be zero.

$\therefore\ D = 0.$

Hence (2) becomes,

$y(x, t) = B_n \frac{n\pi x}{l} \cos\frac{n\pi at}{l}$, where n is any integer and B_n is any constant.

The most general solution satisfying (1) and the boundary conditions (*i*), *ii*) and (*iii*) is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad ...(3).$$

To find B_n use the boundary condition (iv).

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$= \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

This is true only if $B_1 = \frac{3y_0}{4}$, $B_3 = -\frac{y_0}{4}$ and $B_n = 0$, for $n \neq 1, 3$.

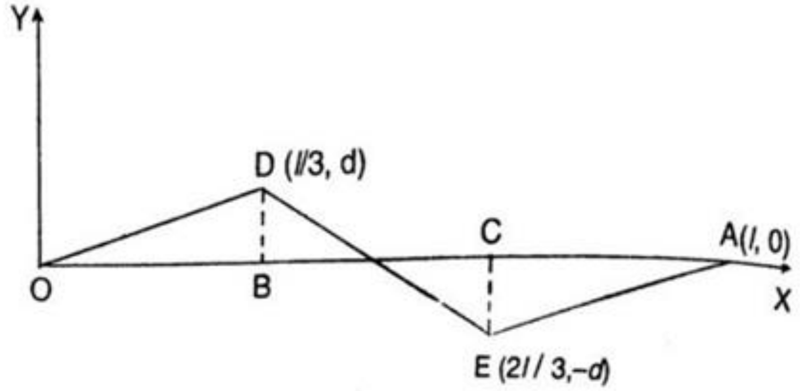
Using these values in (3), the solution of the equation is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l}$$

Example 3

The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string is always remains at rest.

Solution.



$BD = CE = d.$

The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The boundary conditions here are

$$y(0,t)=0 \qquad \text{for } t \geq 0 \qquad \text{...}(i)$$

$$y(l,t)=0 \qquad \text{for } t \geq 0 \qquad \text{...}(ii)$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0}=0, \qquad \text{for } 0 \leq x \leq l \qquad \text{...}(iii)$$

To find the initial position of the string, we require the equation of $ODEA$.

The equation of OD is $\quad y=\frac{d}{l/3}x=\frac{3dx}{l}$.

The equation of DE is $y-d=-\frac{d}{(l/6)}(x-l/3)$

i.e., $$y=\frac{3d}{l}(l-2x).$$

The equation of EA is $\quad y=\frac{3d}{l}(x-l)$.

The fourth initial condition is

$$y(x,0)=\begin{cases} \frac{3dx}{l} & \text{for } 0 \leq x \leq l/3 \\ \frac{3d}{l}(l-2x) & \text{for } \frac{l}{3} \leq x \leq \frac{2l}{3} \\ \frac{3d}{l}(x-l) & \text{for } \frac{2l}{3} \leq x \leq l \end{cases} \qquad \text{——— (iv)}$$

Solving (1) and selecting the suitable solution and using the boundary conditions (i), (ii) and (iii) as in example 2, we get

$$y(x,t)=\sum_{n=1} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Using the initial condition (iv) we get,

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = y(x, 0) = \frac{3dx}{l} \text{ for } 0 \le x \le l/3$$

$$= \frac{3d}{l}(l - 2x), \text{ for } \frac{l}{3} \le x \le \frac{2l}{3}$$

$$= \frac{3d}{l}(x - l), \text{ for } \frac{2l}{3} \le x \le l.$$

Finding Fourier sine series of $y(x, 0)$ in $(0, l)$ we get in the usual

way $y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$.

$$\therefore \qquad B_n = b_n = \frac{2}{l} \int_0^1 y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$\therefore \qquad B_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3dx}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3d}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3d}{l}(x - l) \sin \frac{n\pi x}{l}\, dx \right]$$

$$= \frac{6d}{l^2}\left[x\left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1)\left(-\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^{l/3}$$

$$+ \frac{6d}{l^2}\left[(l - 2x)\left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-2)\left(-\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_{l}^{2l/3}$$

$$+\frac{6d}{l^2}\left[(x-l)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{2l/3}^{l}$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}+\cos n\pi\cdot\sin\frac{n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}[1+(-1)^n]$$

$= 0$ if n is odd.

$$=\frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3}\text{ if } n \text{ is even.}$$

Hence,

$$y(x,t)=\frac{36d}{\pi^2}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

i.e., $$y(x,t)=\frac{9d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cdot\cos\frac{2n\pi at}{l}.$$

By putting $x = l/2$, we get the displacement of the midpoint.

$\therefore\ y\left(\frac{l}{2},t\right)=0$, since $\sin\frac{2n\pi x}{l}$ becomes $\sin n\pi = 0$ when $x = l/2$.

Example 4

A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v, where $v=\begin{cases}\frac{cx}{l}, in\ 0<x<l\\ \frac{c(2l-x)}{l}, in\ l<x<2l\end{cases}$**, x being the distance from an end point. Find the displacement of any point at a distance x from the origin.**

Solution.

The boundary conditions are

$$y(0, t) = 0, \text{ for } t \geq 0 \quad \text{...}(i)$$

$$y(2l, t) = 0, \text{ for } t \geq 0 \quad \text{...}(ii)$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq 2l \quad \text{...(iii)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{cx}{l}, \text{ in } 0 < x < l$$

$$= \frac{c}{l}(2l - x), \text{ in } l < x < 2l \quad \text{...(iv)}$$

As in the previous examples, using boundary conditions (*i*) and (*ii*), we get

$$y(x, t) = \sin\frac{n\pi x}{2l}\left[C_n \cos\frac{n\pi at}{2l} + D_n \sin\frac{n\pi at}{2l}\right]$$

Using (*iii*), $C_n = 0$.

$$\therefore \quad y(x, t) = D_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

The most general solution of the equation (1) is

$$y(x, t) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \quad \text{...(2)}$$

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} D_n\left(\frac{n\pi a}{2l}\right)\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}.$$

Using (*iv*),

$$\sum_{n=1}^{\infty} D_n\left(\frac{n\pi a}{2l}\right)\sin\frac{n\pi x}{2l} = v = \frac{cx}{l}, \text{ in } 0 < x < 1$$

$$= \frac{c}{l}(2l - x), \text{ in } l < x < 2l.$$

Expanding v in Fourier sine series, we get

$$D_n \cdot \frac{n\pi a}{2l} = \frac{2}{2l}\left[\frac{c}{l}\int_0^l x \sin\frac{n\pi x}{2l}\,dx + \frac{c}{l}\int_l^{2l}(2l - x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$\therefore\ D_n = \frac{2c}{n\pi a l}\left[\left[\left\{x\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}\right]_0^l\right.$$

$$\left.+\left\{(2l-x)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)-(-1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{2c}{n\pi a l}\left[\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2}+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}\right.$$

$$\left.+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2c}{n\pi a l}\cdot\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{16lc}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

Substituting this value of D_n in (2),

$$y(x,t) = \frac{16cl}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi a t}{2l}$$

Example 5

If a string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\frac{\partial y}{\partial t} = v_0 sin^3\left(\frac{\pi x}{l}\right), 0 < x < l$. Determine the transverse displacement $y(x,t)$.

Solution.

The boundary conditions are

$$y(0, t) = 0, \text{ for } t \geq 0 \qquad \text{...(i)}$$

$$y(l, t) = 0, \text{ for } t \geq 0 \qquad \text{...(ii)}$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq l \qquad \text{...(iii)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \text{ for } 0 \leq x \leq l \qquad \text{...(iv)}$$

Selecting the solution II, and using boundary conditions (i) and (ii)

we get $y(x, t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right)$

using (iii), $C = 0$

Therefore $y(x, t) = B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$, n any integer

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \qquad \text{...(3)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \qquad \text{...(IV)}$$

Using in (3),

$$y(x, t) = \frac{3lv_0}{4\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi at}{l} - \frac{v_0 l}{12\pi a} \sin \frac{3\pi x}{l} \sin \frac{3\pi at}{l}$$

Example 6

A string is stretched and fastened to two points l apart. Motion is started by displacing the string in to the form $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement of any point of the string at a distance x from one end at any time t.

Solution.

The boundary conditions are:

$$y(0, t) = 0, \qquad t > 0$$

$$y(l, t) = 0, \qquad t > 0$$

$$\frac{\partial y}{\partial t} = 0, \qquad 0 < x < l$$

$$y(x, 0) = k(lx - x^2), \qquad 0 < x < l$$

Using boundary condition (iv),

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = k\,(lx - x^2)$$

This shows that this is the half range Fourier sine series of $k(lx - x^2)$.
Using the formula for Fourier coefficients,

$$B_n = b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}\{(-1)^n - 1\}\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$= 0$ if n is even

$= \dfrac{8kl^2}{n^3\pi^3}$ if n is odd

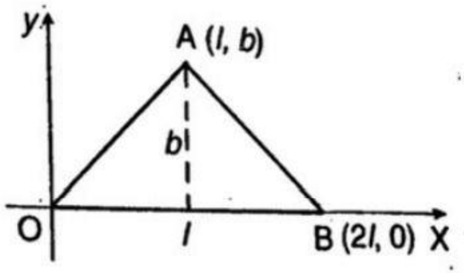
Substituting in IV,

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi at}{l}$$

Example 7

A taut string of length 2l is fastened at both ends. The midpoint of the string is taken to a height b and then released from rest in that position. Derive an expression for the displacement of the string.

Solution.



The boundary conditions are:

$$y(0, t) = 0,\ t \geq 0 \qquad \text{...(i)}$$

$$y(2l, t) = 0,\ t \geq 0 \qquad \text{...(ii)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0,\ 0 \leq x \leq 2l \qquad \text{...(iii)}$$

$$y(x, 0) = \frac{b}{l}x,\ 0 \leq x \leq l$$

$$= -\frac{b}{l}(x - 2l),\ l \leq x \leq 2l$$

$\left[\text{since, equation of } OA \text{ is } y = \frac{b}{l}x \text{ and equation of } AB \text{ is } \frac{y-0}{x-2l} = \frac{b-0}{l-2l}\right]$

Starting with the solution

$$y(x, t) = (A \cos \lambda x + B \cos \lambda x)(C \cos \lambda at + D \sin \lambda at)$$

using the first boundary condition,

$$y(o, t) = A\,(C \cos \lambda at + D \sin \lambda ab) = 0$$

$\therefore \qquad A = 0,$

using $\qquad y(2l, t) = 0$ we get

$$B \sin 2l\lambda\,(C \cos \lambda at + D \sin \lambda at) = 0$$

$$B \neq 0;\ 2l\lambda = n\pi\ ;\ \lambda = \frac{n\pi}{2l}$$

Using $\left(\frac{\partial y}{\partial t}\right)_{t=0}$; $D = 0.$

$$\therefore \quad y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \qquad \text{...IV}$$

Using boundary condition (*iv*) in IV,

$$y(x, 0) = \sum_{1}^{\infty} B_n \sin \frac{n\pi x}{2l} = \frac{b}{l} x, 0 \le x \le l$$

$$= -\frac{b}{l}(x - 2l), l \le x \le 2l$$

This is half-range Fourier sine series

$$\therefore \ B_n = \frac{2}{2l} \int_0^{2l} f(x) \sin \frac{n\pi x}{2l} \, dx$$

$$= \frac{1}{l} \left[\int_0^l \frac{b}{l} x \sin \frac{n\pi x}{2l} \, dx - \frac{b}{l} \int_l^{2l} (x - 2l) \sin \frac{n\pi x}{2l} \, dx \right]$$

$$= \frac{b}{l^2} \left[\left\{ (x) \left(-\frac{\cos \frac{n\pi x}{2l}}{\frac{n\pi}{2l}} \right) - \left(-\frac{\sin \frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}} \right) \right\}_0^l \right.$$

$$\left. - \left\{ (x - 2l) \left(-\frac{\cos \frac{n\pi x}{2l}}{\frac{n\pi}{2l}} \right) - \left(-\frac{\sin \frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}} \right) \right\}_l^{2l} \right]$$

$$= \frac{b}{l^2} \left[-\frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2} \left(\sin \frac{n\pi}{2} \right) + \frac{2l^2}{n\pi} \cos \frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{b}{l^2} \left[\frac{8l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= 0 \text{ for } n \text{ even}$$

$$= \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2} \text{ for odd } n.$$

Substituting in IV,

$$y(x, t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin (2n-1) \frac{\pi}{2} \cdot \sin \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi at}{2l}$$

$$= \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi at}{2l}$$

Exercise

A tightly stretched string with fixed end points x=0 and x=l is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$, find the displacement.

Heat on an insulated wire

Now let us consider with the heat equation. Consider a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and let t denote time. See Figure 2.

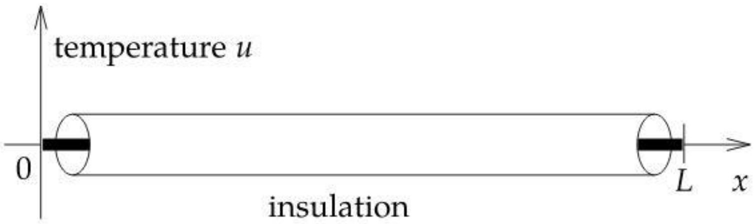


Figure 2: Insulated wire

Let $u(x,t)$ denote the temperature at point x at time t. The equation governing this setup is the so-called one-dimensional heat equation:

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},}$$

where $k > 0$ is a constant (the thermal conductivity of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum and vice-versa.

Therefore, the heat equation is $u_t = ku_{xx}$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. If the ends of the wire are kept at temperature 0, then the conditions are:

(i) $u(0,t) = 0$ and $u(L,t) = 0$.

If, on the other hand, the ends are also insulated, the conditions are:

(ii) $u_x(0,t) = 0$ and $u_x(L,t) = 0$.

Let us see why that is so. If u_x is positive at some point x0, then at a particular time, u is smaller to the left of x0, and higher to the right of x0. Heat is flowing from high heat to low heat, that is to the left. On the other hand if ux is negative then heat is again flowing from high heat to low heat, that is to the right. So when ux is zero, that is a point through which heat is not flowing. In other words, ux(0,t)=0 means no heat is flowing in or out of the wire at the point x=0.

We have two conditions along the x-axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e., u or a derivative of u is set to zero). We also need an initial condition—the temperature distribution at time t=0. That is, u(x,0)=f(x), for some known function f(x).

Solution of heat equation by method of separation of variables

We have to solve the equation

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \quad \textbf{---------(1)}$$

where $k = \alpha^2$ is called the diffusivity of the substance.

Assume a solution of the form $u(x,t) = X(x).T(t)$ where X is a function of x and T is a function of t.

Then (1) becomes,

$$XT' = \alpha^2 X''T,$$

where $X'' = \dfrac{d^2X}{dx^2}$ and $T' = \dfrac{dT}{dt}$

i.e., $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \quad \textbf{-----(2)}$$

The LHS is a function of x alone and the RHS is the function of t alone when x and t are independent variables. Equation (2) can be true only if each expression is equal to a constant.

$\therefore$ Let $\dfrac{X''}{X} = \dfrac{T'}{\alpha^2 T} = k$ (constant)

$\therefore\ X'' - kX = 0$, and $T' - \alpha^2 kT = 0$ ...(3)

The nature of solutions of (3) depends upon the values of k.

Case 1. Let $k = \lambda^2$, a positive number.

Then (3) becomes,

$$X'' - \lambda^2 X = 0, \text{ and } T' - \alpha^2\lambda^2 T = 0.$$

Solving, we get

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x} \text{ and } T = C_1 e^{\alpha^2\lambda^2 t}.$$

Case 2. Let $k = -\lambda^2$, a negative number. Then (3) becomes $X'' + \lambda^2 X = 0$, and $T' + \alpha^2\lambda^2 T = 0$.

Solving, we obtain

$$X = A_2 \cos \lambda x + B_2 \sin \lambda x, \text{ and } T = C_2 e^{-\alpha^2\lambda^2 t}.$$

Case 3. Let $k = 0$.

Then $X'' = 0$ and $T' = 0$.

Solving, we arrive at ,

$$X = A_3 x + B_3 \text{ and } T = C_3.$$

Hence the possible solutions of (1) are

$$u(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})\, C_1 e^{\alpha^2\lambda^2 t} \quad \text{...(I)}$$

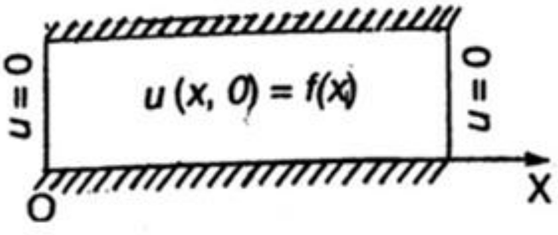
$$u(x, t) = (A_2 \cos \lambda x + B_2 \sin \lambda x)\, C_2\, e^{-\alpha^2\lambda^2 t} \quad \text{...(II)}$$

$$\mathbf{u(x, t) = (A_3 x + B_3) C_3}$$

Example 8

A rod l cm with insulated lateral surface is initially at temperature $f(x)$ at an inner point distant x cm from one end. If both the ends are kept at zero temperature, find the temperature at any point of the rod at any subsequent time.

Solution.



Let $u(x, t)$ be the temperature at any point distant x from one end at any time t seconds. Then u satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

The boundary conditions, here, are

$u(0, t) = 0$ for all $t \geq 0$

$u(l, t) = 0$ for all $t \geq 0$

and the initial condition is

$u(x, 0) = f(x)$, for $0 < x < l$ ...(iii)

Solving the equation (1) by the method of separation of variables and selecting the suitable solution to suit the physical nature of the problem as explained in the method § 3·6, we get

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x)\, e^{-\alpha^2 \lambda^2 t}$$

Substituting the boundary condition (*i*) in (2), we get,

$$u(0, t) = Ae^{-\alpha^2 \lambda^2 t} = 0, \text{ for all } t \geq 0$$

$\therefore\ A = 0$

Employing the boundary condition (*ii*) in (2), we obtain,

$$u(l, t) = B \sin \lambda l\, e^{-\alpha^2 \lambda^2 t} = 0, \text{ for all } t \geq 0$$

i.e., $B \sin \lambda l = 0$.

If $B = 0$, (2) will be a trivial solution. Hence

$\sin \lambda l = 0$

$\therefore\ \lambda l = n\pi$, where n is any integer.

$\lambda = \frac{n\pi}{l}$, where n is any integer.

Then (2) reduces to

$$u(x, t) = B_n \sin \frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}} \quad ...(3),$$

where B_n is any constant.

Since the equation (1) is linear, its most general solution is obtained by a linear combination of solutions given by (3).

Hence the most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad ...(4).$$

(4) should satisfy the initial condition (*iii*).

Using (*iii*) in (4),

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x), \text{ for } 0 < x < l \text{ (given)} \quad ...(5).$$

If $u(x, 0)$, for $0 < x < l$, is expressed in a half-range Fourier sine series in $0 < x < l$, we know that

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

Comparing this with (5), we get

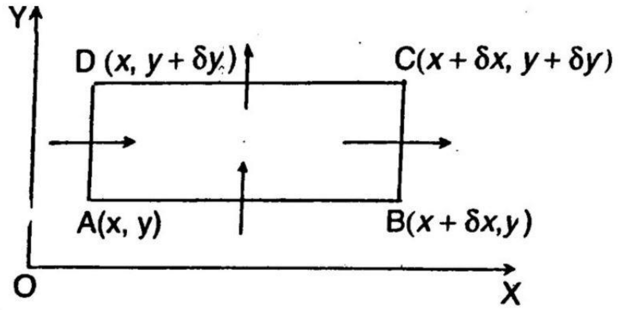
$$B_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

Therefore the temperature function $u(x, t)$, is

$$u(x, t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \right) \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}.$$

Two-Dimensional Heat Flow

When the heat flow is along curves instead of along straight lines, all the curves lying in parallel planes, then the flow is called two-dimensional. Let us consider now the flow of heat in a metal plate in the XOY plane. Let the plate be of uniform thickness h, density ρ, thermal conductivity k and the specific heat c. Since the flow is two dimensional, the temperature at any point of the plate is independent of the z-co-ordinate. The heat flow lies in the XOY plane and is zero along the direction normal to the XOY plane.



Now, consider a rectangular element $ABCD$ of the plate with sides δx and δy, the edges being parallel to the coordinates axes, as shown in the figure. Then the quantity of heat *entering* the element $ABCD$ per sec. through the surface AB is

$$= -k\left(\frac{\partial u}{\partial y}\right)_y \delta x \cdot h.$$

Similarly the quantity of heat entering the element $ABCD$ per sec. through the surface AD is

$$= -k\left(\frac{\partial u}{\partial x}\right)_x \delta y \cdot h.$$

The amount of heat which flows *out* through the surfaces BC and CD are

$$-k\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \cdot \delta y \cdot h \text{ and } -k\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \cdot \delta x \cdot h \text{ respectiveiy.}$$

Therefore the total gain of heat by the rectangular element $ABCD$ per sec. = inflow–outflow

$$= kh\left[\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\}\delta y + \left\{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y \delta x\right\}\right]$$

$$= kh\delta x \cdot \delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad \text{....(1)}$$

The rate of gain of heat by the element $ABCD$ is also given by

$$\rho\delta x \cdot \delta y \cdot h \cdot c \cdot \frac{\partial u}{\partial t} \quad \text{....(2)}$$

Equating the two-expressions for gain of heat per sec. from (1) and (2), we have, Equating the two-expressions for gain of heat per sec. from (1) and (2), we have,

$$\rho\,\delta x\cdot\delta y\cdot h\cdot c\cdot\frac{\partial u}{\partial t}=h\,k\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]$$

$$i.e.,\quad \frac{\partial u}{\partial t}=\frac{k}{\rho c}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]$$

Taking the limit as $\delta x\to 0,\ \delta y\to 0$, the above reduces to

$$\frac{\partial u}{\partial t}=\frac{k}{\rho c}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Putting $\alpha^2=\dfrac{k}{\rho c}$ as before, the equation becomes,

$$\frac{\partial u}{\partial t}=\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\qquad ....(3)$$

The equation (3) gives the temperature distribution of the plate in the transient state.

In the steady-state, u is independent of t, *so that* $\frac{\partial u}{\partial t}=0$. *Hence the temperature distribution of the platc in the steady-state is* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

i.e., $\nabla^2 u=0$, which is known as *Laplace's Equation* in two-dimensions.

Corollary. If the stream lines are parallel to the x-axis, then the rate of change $\frac{\partial u}{\partial y}$ of the temperature in the direction of the y-axis will be zero. Then the heat-flow equation reduces to $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ which is the heat-flow equation in one-dimension.

Solution of the Equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

The equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$ ....(1)

Assume the solution $u(x, y)=X(x)\cdot Y(y)$,

where X is a function of x alone and Y a function of y alone.

$\therefore \frac{\partial^2 u}{\partial x^2}=X'' Y$, and $\frac{\partial^2 u}{\partial y^2}=XY''$.

The Laplace equation $\nabla^2 u=0$ becomes $X''Y+Y''X=0$

i.e., $\frac{X''}{X}=\frac{-Y''}{Y}$ ....(2)

The left hand side of (2) is a function of x alone and the right hand side is a function of y alone. Also x and y are independent variables. Hence, this is possible only if each quantity is equal to a constant k.

$\therefore$ Let $\frac{X''}{X}=\frac{-Y''}{Y}=k$ ....(3)

i.e., $X''-kX=0$, and $Y''+kY=0$. **...(4)**

Case 1. Let $k=\lambda^2$, a positive number.

Then $X''-\lambda^2 X=0$, and $Y''+\lambda^2 Y=0$.

Solving, $X=A_1 e^{\lambda x}+B_1 e^{-\lambda x}$ and $Y=C_1 \cos \lambda y+D_1 \sin \lambda y$.

Case 2. Let $k=-\lambda^2$, a negative number.

Then (4) becomes $X''+\lambda^2 X=0$ and $Y''-\lambda^2 Y=0$.

Solving these equations, we have,

$X=A_2 \cos \lambda x+B_2 \sin \lambda x$ and $Y=C_2 e^{\lambda y}+D_2 e^{-\lambda y}$.

Case 3. Let $k=0$. Then (4) reduces to

$X''=0$ and $Y''=0$.

On solving these equations,

$X=A_3 x+B_3$ and $Y=C_3 y+D_3$.

Therefore, the possible solutions of (1) are

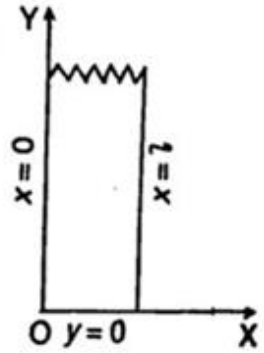
$u(x, y)=(A_1 e^{\lambda x}+B_1 e^{-\lambda x})(C_1 \cos \lambda y+D_1 \sin \lambda y)$ ...(I)

$u(x, y)=(A_2 \cos \lambda x+B_2 \sin \lambda x)(C_2 e^{\lambda y}+D_2 e^{-\lambda y})$ ...(II)

$u(x, y)=(A_3 x+B_3)(C_3 y+D_3)$ ...(III)

In problems where the boundary conditions are given, we have to select a suitable solution or a linear combination of solutions to satisfy (1) and the boundary conditions.

Example 9. *An infinitely long plane uniform plate is bounded by two parallel edges $x = 0$ and $x = l$, and an end at right angles to them. The breadth of this edge $y = 0$ is l and is maintained at a temperature $f(x)$. All the other three edges are at temperature zero. Find the steady-state temperature at any interior point of the plate.*



Let $u(x, y)$ be the temperature at any point (x, y) of the plate.

Then u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{....(1)}$$

The boundary conditions are

$$u(0, y) = 0, \quad \text{for } 0 \le y \le \infty \qquad \text{....(}i\text{)}$$
$$u(l, y) = 0, \quad \text{for } 0 \le y \le \infty \qquad \text{....(}ii\text{)}$$
$$u(x, \infty) = 0, \quad \text{for } 0 \le x \le l \qquad \text{....(}iii\text{)}$$
$$u(x, 0) = f(x), \quad \text{for } 0 < x < l \qquad \text{....(}iv\text{)}$$

Solving (1), we get,

$$u(x, y) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos \lambda y + D_1 \sin \lambda y) \qquad \text{....(I)}$$

$$u(x, y) = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) \qquad \text{....(II)}$$

$$u(x, y) = (A_3 x + B_3)(C_3 y + D_3) \qquad \text{....(III)}$$

Of these solutions, we have to select a solution to suit the boundary conditions.

Since $u = 0$ as $y \to \infty$, we select the solution (*II*) as a possible solution (rejecting the other two).

$$\therefore \; u(x, y) = (A \cos \lambda x + B \sin \lambda x)\;(Ce^{\lambda y} + De^{-\lambda y}) \qquad \text{....(2)}$$

Using the boundary condition (1),

$$u(0, y) = A\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y \le \infty. \; \therefore \; A = 0$$

Using the boundary condition (*ii*) in (2),

$$u(l, y) = B \sin \lambda l\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y \le \infty.$$

Since $B \ne 0$, $\sin \lambda l = 0$. Hence $\lambda l = n\pi$

i.e., $\lambda = \dfrac{n\pi}{l}$, where n is any integer.

As $y \to \infty$, $u \to 0$, from (*iii*). $\therefore \; C = 0$.

Hence $u(x, y) = B_n \sin \dfrac{n\pi x}{l}\, e^{-\frac{n\pi y}{l}}$, where $BD = B_n$.

Therefore the most general solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{n\pi y}{l}} \quad \text{....(3)}$$

Using the boundary condition (*iv*) in (3),

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x), \text{ in } 0 < x < l \quad \text{....(4)}$$

Expressing $f(x)$ as a half-range Fourier sine series in $(0, l)$, we have

$$f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{....(5),}$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$

Comparing (4) and (5), $B_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$

Therefore the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \right) \sin \frac{n\pi x}{l} \cdot e^{-\frac{n\pi y}{l}}$$

Note. If $f(x)$ is given explicitly in any problem, evaluate the value of B_n from the integral and substitute.

Example 10. The vertices of a thin square plate are $(0,0), (l,0), (0,l), (l,l)$. The upper edge of the square is maintained at an arbitrary temperature given by $u(x,l) = f(x)$. The other three edges are kept at zero temperature. Find the steady state temperature at any point on the plate.

Solution.

Suppose that $u(x, y)$ is the temperature at any point (x, y) of the plate in steady-state.

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{....(1)}$

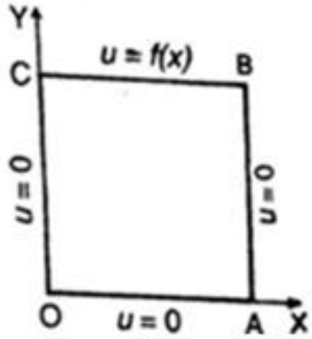
The boundary conditions are

$u(0, y) = 0, \quad \text{for } 0 \le y < l \quad$ (*i*)

$u(l, y) = 0, \quad \text{for } 0 \le y < l \quad$ (*ii*)

$u(x, 0) = 0, \quad \text{for } 0 \le x \le l \quad$ (*iii*)

$u(x, l) = f(x), \quad \text{for } 0 < x < l \quad$ (*iv*)



Solving (1), we get the three possible solutions,

$$u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C \cos \lambda y + D \sin \lambda y) \quad \text{....(I)}$$

$$u(x, y) = (A \cos \lambda x + B \sin \lambda x)(C e^{\lambda y} + D e^{-\lambda y}) \quad \text{....(II)}$$

$$u(x, y) = (Ax + B)(Cy + D) \quad \text{....(III)}$$

where A, B, C, D are different arbitary constants in each solution.

Now we shall select the solution II.

i.e., $u(x, y) = (A \cos \lambda x + B \sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ ....(II)

Using the boundary condition (i) in (II),

$$A\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y < l. \therefore A = 0$$

Using the condition (ii) in (II)

$$u(l, y) = B \sin \lambda l\ (Ce^{\lambda y} + De^{-\lambda y}) = 0. \text{ But } B \ne 0; \sin \lambda l = 0$$

i.e., $\lambda l = n\pi$

i.e., $\lambda = \dfrac{n\pi}{l}$ where nn is any integer.

Using (iii) in II,

$u(x, 0) = (C + D)(B \sin \lambda x) = 0$, for $0 \le x \le l$.

$B \ne 0$ Hence $C + D = 0$. $\therefore D = -C$.

Hence (II) reduces to,

$$u(x, y) = BC \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – NUMERICAL METHODS FOR SOLVING EQUATIONS – SMTA1401

I. Introduction

Contents - Solution of algebraic equation and transcendental equation: Regula Falsi Method, Newton Raphson Method Solution of simultaneous linear algebraic equations: Gauss Elimination Method, Gauss Jacobi & Gauss Seidel Method.

In the field of Science and Engineering, the solution of equations of the form $f(x) = 0$ occurs in many applications. If $f(x)$ is a polynomial of degree two or three or four, exact formulae are available. But, if $f(x)$ is transcendental function like $a + be^x + csinx + dlogx$ etc., the solution is not exact and we do not have formulae to get the solution. When the coefficients are numerical values, we adopt various numerical approximate methods to solve such algebraic and transcendental equations. We will see below methods of solving such numerical equations. From the theory of equations, we have the following theorem:

If $f(x)$ is continuous in the interval in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x) = 0$ will have at least one real root between a and b.

Regula Falsi method (or the method of false position)

Consider the equation $f(x)=0$ and let $f(a)$ and $f(b)$ be of opposite signs. Also, let $a<b$. The curve $y=f(x)$ will meet the x-axis at some point between $A\,(a, f(a))$ and $B\,(b, f(b))$. The equation of the chord joining the two points $A\,(a, f(a))$ and $B\,(b, f(b))$ is $\dfrac{y-f(a)}{x-a} = \dfrac{f(a)-f(b)}{a-b}$. The x-coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the root of $f(x)=0$. Setting $y=0$ in the chord equation, we get

$$\frac{-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

$$x\,[f(a)-f(b)] - af(a) + af(b) = -af(a) + bf(a)$$

$$x\,[f(a)-f(b)] = bf(a) - af(b)$$

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

This value of x_1 gives an approximate value of the root of $f(x)=0$. $(a<x_1<b)$

Now $f(x_1)$ and $f(a)$ are of opposite signs or $f(x_1)$ and $f(b)$ are of opposite signs.

If $f(x_1).f(a)<0$, then x_2 lies between x_1 and a.

Hence $x_2 = \dfrac{af(x_1)-x_1f(a)}{f(x_1)-f(a)}$

In the same way, we get $x_3, x_4, \ldots$

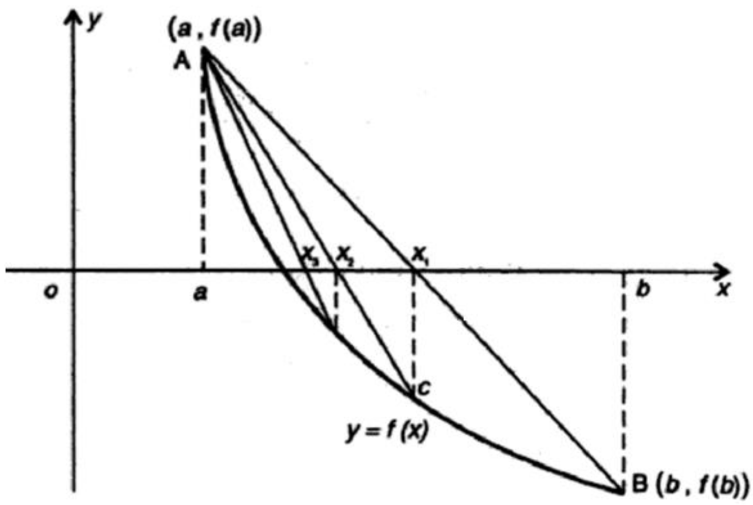
This sequence $x_1, x_2, x_3, \ldots$ will converge to the required root. In practice, we get x_i and x_{i+1} such that $|x_i - x_{i+1}| < \varepsilon$, the required accuracy.

Geometrical interpretation

If $A(a, f(a))$ and $B(b, f(b))$ are two points on $y = f(x)$ such that $f(a)$ and $f(b)$ are opposite in sign, then the chord AB meets x-axis at $x = x_1$. This x_1 is the approximate root of $f(x) = 0$. Now $c(x_1, f(x_1))$ is on the curve.

If $f(a).f(x_1) < 0$, join the chord AC which cuts x-axis at $x = x_2$. Then x_2 is the second approximate root of $f(x) = 0$. This process is continued until we get the root to the desired accuracy.

The order of convergence of Regula Falsi method is 1·618. (This may be assumed.)



Example 1. *Solve for a positive root of $x^3 - 4x + 1 = 0$ by Regula Falsi method.*

Solution. Let $\quad f(x) = x^3 - 4x + 1 = 0$

$f(1) = -2 = -ve$; $f(2) = 1 = +ve$, $f(0) = 1 = +ve$

$\therefore$ a root lies between 0 and 1

Another root lies between 1 and 2

We shall find the root that lies between 0 and 1

Here $a = 0,\ b = 1$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = \frac{-1}{-2-1} = 0{\cdot}333333$$

$$f(x_1) = f\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{4}{3} + 1 = -0{\cdot}2963$$

Now $f(0)$ and $f\left(\frac{1}{3}\right)$ are opposite in sign.

Hence the root lies between 0 and 1/3.

Hence $$x_2 = \frac{0 f\left(\frac{1}{3}\right) - \frac{1}{3} f(0)}{f\left(\frac{1}{3}\right) - f(0)}$$

$$x_2 = \frac{-\frac{1}{3}}{-1{\cdot}2963} = 0{\cdot}25714$$

Now $f(x_2) = f(0{\cdot}25714) = -0{\cdot}011558 = -ve$

$\therefore$ The root lies between 0 and $0{\cdot}25714$

$$x_3 = \frac{0 \times f(0{\cdot}25714) - 0{\cdot}25714 f(0)}{f(0{\cdot}25714) - f(0)}$$

$$= \frac{-0{\cdot}25714}{-1{\cdot}011558} = 0{\cdot}25420$$

$f(x_3) = f(0{\cdot}25420) = -0{\cdot}0003742$

$\therefore$ The root lies between 0 and $0{\cdot}25420$

$\therefore$ $$x_4 = \frac{0 \times f(0{\cdot}25420) - 0{\cdot}25420 \times f(0)}{f(0{\cdot}25420) - f(0)}$$

$$= \frac{-0{\cdot}25420}{-1{\cdot}0003742} = 0{\cdot}25410$$

$$f(x_4) = f(0{\cdot}25410) = -0{\cdot}000012936$$

The root lies between 0 and $0{\cdot}25410$

$$x_5 = \frac{0 \times f(0{\cdot}25410) - 0{\cdot}25410 \times f(0)}{f(0{\cdot}25410) - f(0)}$$

$$= \frac{-0{\cdot}25410}{-1{\cdot}000012936} = 0{\cdot}25410$$

Hence the root is 0·25410.

Example 2. *Find an approximate root of x $\log_{10} x - 1{\cdot}2 = 0$ by False position method.*

Solution. Let $f(x) = x \log_{10} x - 1{\cdot}2$

$$f(1) = -1{\cdot}2 = -ve; f(2) = 2 \times 0{\cdot}30103 - 1{\cdot}2 = -0{\cdot}59794$$

$$f(3) = 3 \times 0{\cdot}47712 - 1{\cdot}2 = 0{\cdot}231364 = +ve$$

Hence a root lies between 2 and 3.

$$\therefore \quad x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2 \times 0{\cdot}23136 - 3 \times (-0{\cdot}59794)}{0{\cdot}23136 + 0{\cdot}59794} = 2{\cdot}721014$$

$$f(x_1) = f(2{\cdot}7210) = -0{\cdot}017104$$

The root lies between x_1 and 3.

$$x_2 = \frac{x_1 \times f(3) - 3 \times f(x_1)}{f(3) - f(x_1)}$$

$$= \frac{2{\cdot}721014 \times 0{\cdot}231364 - 3 \times (-0{\cdot}017104)}{0{\cdot}23136 + 0{\cdot}017104}$$

$$= \frac{0{\cdot}68084}{0{\cdot}24846} = 2{\cdot}740211$$

$$f(x_2) = f(2{\cdot}7402) = 2{\cdot}7402 \times \log(2{\cdot}7402) - 1{\cdot}2$$

$$= -0{\cdot}00038905$$

$\therefore$ The root lies between 2·740211 and 3

$$\therefore \quad x_3 = \frac{2{\cdot}7402 \times f(3) - 3 \times f(2{\cdot}7402)}{f(3) - f(2{\cdot}7402)}$$

$$= \frac{2{\cdot}7402 \times 0{\cdot}23136 + 3 \times 0{\cdot}00038905}{0{\cdot}23136 + 0{\cdot}00038905}$$

$$= \frac{0{\cdot}63514}{0{\cdot}23175} = 2{\cdot}740627$$

$$f(2.7406) = 0.00011998$$

$\therefore$ The root lies between 2.740211 and 2.740627

$$x_4 = \frac{2.7402 \times f(2.7406) - 2.7406 \times f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.00011998 + 2.7406 \times 0.00038905}{0.00011998 + 0.00038905}$$

$$= \frac{0.0013950}{0.00050903} = 2.7405$$

Hence the root is 2.7405.

Example 3. *Find the positive root of $x^3 = 2x + 5$ by False Position method.*

Solution. Let $f(x) = x^3 - 2x - 5 = 0$

There is only one positive root by Descarte's rule of signs.

$f(2) = 8 - 9 = -1 = -ve$; $f(3) = 16 = +ve$

$\therefore$ the positive root lies between 2 and 3. It is closer to 2 also.

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2 \times f(3) - 3 \times f(2)}{f(3) - f(2)}$$

$$= \frac{32 + 3}{17} = 2.058824$$

$$f(x_1) = f(2.058824) = -0.390795$$

$\therefore$ The root lies between 2.058824 and 3

$$x_2 = \frac{2.058824 \times f(3) - 3 \times f(2.058824)}{f(3) - f(2.058824)}$$

$$= \frac{34.113569}{16.390795} = 2.081264$$

$$f(x_2) = f(2.081264) = -0.147200$$

$\therefore$ The root lies between 2.081264 and 3

$$x_3 = \frac{2.081264 \times 16 - 3 \times (-0.147200)}{16 + 0.147200} = 2.089639$$

$$f(x_3) = f(2.089639) = -0.054679$$

The root lies between 2·089639 and 3

$$\therefore \quad x_4 = \frac{2{\cdot}089639 \times f(3) - 3 \times f(2{\cdot}089639)}{f(3) - f(2{\cdot}089639)} = 2{\cdot}092740$$

$$f(x_4) = f(2{\cdot}09274) = -0{\cdot}020198$$

$\therefore$ The root lies between 2·09274 and 3

$$x_5 = \frac{2{\cdot}09274 \times 16 + 3 \times (0{\cdot}020198)}{16{\cdot}020198} = 2{\cdot}093884$$

$$f(x_5) = f(2{\cdot}093884) = -0{\cdot}007447$$

The root lies between 2·093884 and 3

$$x_6 = \frac{2{\cdot}093884 \times 16 + 3 \times 0{\cdot}007447}{16{\cdot}007447} = 2{\cdot}094306$$

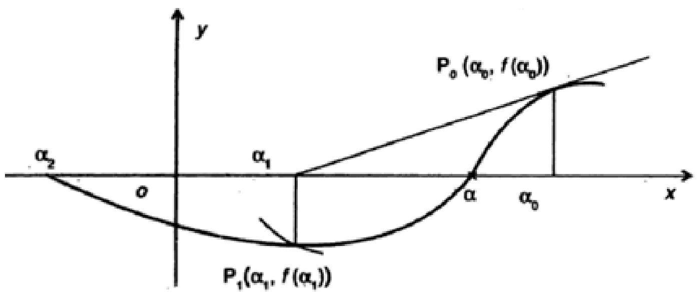
$$f(x_6) = f(2{\cdot}094306) = -0{\cdot}002740$$

$\therefore$ The root lies between 2·094306 and 3

$$x_7 = \frac{2{\cdot}094306 \times 16 - 3 \times (-0{\cdot}002740)}{16{\cdot}002740} = 2{\cdot}094461$$

Similarly, $x_8 = 2{\cdot}0945$ correct to 4 decimal places.

Newton-Raphson Method



The iterative formula in Newton's method is:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This is really an iteration method where

$$x_{i+1} = \phi(x_i) \text{ and } \phi(x_i) = x_i - \frac{f(x_i)}{f'(x_i)}$$

Hence the equation is

$$x = \phi(x) \text{ where } \phi(x) = x - \frac{f(x)}{f'(x)}$$

The sequence $x_1, x_2, x_3, \ldots$ converges to the exact value if $|\phi'(x)| < 1$

i.e., if $\left| 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} \right| < 1$

i.e., if $\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| < 1$

Therefore, the condition for convergence is:

$|f(x)f''(x)| < [f'(x)]^2$.

Order of convergence of Newton's method:

The convergence is quadratic and is of order 2.

Example 1. *Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to five decimal places.*

Solution. Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(1) = 2 - 3 - 6 = -7 = -ve$ and $f(2) = 16 - 6 - 6 = 4 = +ve$

$\therefore$ a root lies between 1 and 2

By Descarte's rule of sign, we can prove that there is only one positive root.

Take $\alpha_0 = 2$

$$\therefore \quad \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} = \alpha_0 - \frac{2\alpha_0^3 - 3\alpha_0 - 6}{6\alpha_0^2 - 3} = \frac{4\alpha_0^3 + 6}{6\alpha_0^2 - 3}$$

$$\alpha_{i+1} = \frac{4\alpha_i^3 + 6}{6\alpha_i^2 - 3}$$

$$\alpha_1 = \frac{4(2)^2 + 6}{6(2)^2 - 3} = \frac{38}{21} = 1{\cdot}809524$$

$$\alpha_2 = \frac{4(1{\cdot}809524)^3 + 6}{6(1{\cdot}809524)^2 - 3} = \frac{29{\cdot}700256}{16{\cdot}646263} = 1{\cdot}784200$$

$$\alpha_3 = \frac{4(1{\cdot}784200)^3 + 6}{6(1{\cdot}784200)^2 - 3} = \frac{28{\cdot}719072}{16{\cdot}100218} = 1{\cdot}783769$$

Solution of Simultaneous Linear Equations

Gauss-Elimination Method (Direct method). This is a *direct* method based on the elimination of the unknowns by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which could be solved by back substitution.

Consider the n linear equations in n unknowns, *viz.*

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \ldots(1) \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{aligned}
$$

where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to

$$AX = B \qquad \ldots(2)$$

where $A = \begin{pmatrix} a_{11}\, a_{12} \cdots a_{1n} \\ a_{21}\, a_{22} \cdots a_{2n} \\ \cdots\cdots\cdots \\ a_{n1}\, a_{n2} \cdots a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix (A, B) to upper triangular matrix.

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \qquad \ldots(3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\dfrac{a_{i1}}{a_{11}}$ and add to the ith row of (A, B), where $i = 2, 3, \ldots, n$. By this, all elements in the first column of (A, B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right) \qquad \ldots(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\dfrac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row $(i = 3, 4, \ldots, n)$. Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} \ldots a_{in} & b_1 \\ 0 & b_{22} & b_{23} \ldots b_{2n} & c_2 \\ 0 & 0 & c_{33} \ldots c_{3n} & d_3 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ 0 & 0 & c_{n3} \ldots c_{nn} & d_n \end{array}\right) \qquad \ldots(5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A, B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \cdot\cdot & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdot\cdot & & b_{2n} & c_2 \\ 0 & 0 & c_{33} & c_{34} & \cdots & c_{3n} & d_3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \alpha_{nn} & K_n \end{array}\right) \quad \ldots(6)$$

From, (6), the given system of linear equations is equivalent to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n = c_2$$

$$c_{33}x_3 + \cdots c_{3n}\, x_n = d_3$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{nn}x_n = K_n$$

Going from the bottom of these equation, we solve for $x_n = \dfrac{K_n}{\alpha_{nn}}$.

Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method, we solve for

$x_n, x_{n-1}, x_{n-2}, \cdots x_2, x_1.$

Note. This method of making the matrix A as upper triangular matrix had been taught in lower classes while finding the rank of the matrix A.

Example 1. *Solve the system by Gauss-Elimination method*
$2x + 3y - z = 5;\quad 4x + 4y - 3z = 3$ *and* $2x - 3y + 2z = 2.$

Solution. The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \qquad X = B$$

$$\therefore \qquad (A, B)=\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

Step 1. Taking $a_{11}=2$ as the pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad R_{21}(-2), R_{31}(-1)$$

Step 2. Taking the element -2 in the position (2, 2) as pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad R_{32}(-3)$$

Hence $\qquad 2x+3y-z=5$

$$-2y-z=-7$$

$$6z=18$$

$\therefore \quad z=3, y=2, x=1.$ by back substitution.

Gauss-Jacobi Method

Let us consider this method in the case of three equations in three unknowns.

Consider the 3 linear equations in 3 unknowns,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

This method is applied only when diagonal elements are exceeding all other elements in the respective equations i.e.,

$$|a_1| > |b_1| + |c_1| = d_1$$
$$|a_2| > |b_2| + |c_2| = d_2$$
$$|a_3| > |b_3| + |c_3| = d_3$$

Let the above condition is true we apply this method or we have to rearrange the equations in the above form to fulfill the above condition.

We start with initial values of x,y and z as zero. Solve x, y ,z in terms of other.

1 . Solve the system of equation by Gauss-Jacobi method

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + 6y + 54z = 110$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

i.e., $27 > 6+1$; $15 > 6+2$; $54 > 1+1$. So iteration method can be applied

$$x = 1/27\,(85 - 6y + z)$$
$$y = 1/15\,(72 - 6x - 2z)$$
$$z = 1/54\,(110 - x - y)$$

First iteration : From the above equations, we start with $x = y = z = 0$

$$x^{(1)} = 85/27 = 3.14815 \qquad \ldots\ldots\ldots\ldots(1)$$
$$y^{(1)} = 72/15 = 4.8 \qquad \ldots\ldots\ldots\ldots(2)$$
$$z^{(1)} = 110/54 = 2.03704 \qquad \ldots\ldots\ldots\ldots(3)$$

Second iteration :Consider the new values of $y^{(1)} = 4.8$ *and* $z^{(1)} = 2.03704$ *in the first equation*

$$x^{(2)} = 1/27(85 \mid 6 \times 4.8 + 2.03704) = 2.15693$$

$$y^{(2)} = 1/15\,(72 - 6 \times 3.14815 - 2 \times 2.03704) = 3.26913$$

$$z^{(2)} = 1/54\,(110 \mid 3.14815 \mid 4.8) = -0.515$$

Fourth iteration : Consider the new values of $x^{(2)} = 2.15693$, $y^{(2)} = 3.26913$ *and* $z^{(2)} = -0.515$ *in the first equation*

$$x^{(3)} = 1/27(85 \mid 6 \times 3.26913 + -0.515) = 2.49167$$

$$y^{(3)} = 1/15\,(72 - 6 \times 2.15693 - 2 \times 2.15693) = 3.68525$$

$$z^{(3)} = 1/54\,(110 \mid 2.15693 \mid 3.26913) = 1.93655$$

Thus, we continue the iteration and result is noted below

Iteration No.	*x*	*y*	*z*
4	*2.40093*	*3.54513*	*1.92265*
5	*2.43155*	*3.58327*	*1.92692*
6	*2.42323*	*3.57046*	*1.92565*
7	*2.42603*	*3.57395*	*1.92604*
8	*2.42527*	*3.57278*	*1.92593*
9	*2.42552*	*3.57310*	*1.92596*
10	*2.42546*	*3.57300*	*1.92595*

From the above table 9 th and 10 th iterations are equal by considering the four decimal places. Hence the solution of the equation is

$x = 2.4255 \qquad y = 3.5730 \qquad z = 1.9260.$

Illustration 2 . *Solve the system of equation by Gauss-Jacobi method*

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = 3$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

i.e., $10 > 5+2$; $10 > 4+3$; $10 > 1+6$. So iteration method can be applied

$$x = 1/10\,(3 + 5y + 2z\ )$$
$$y = 1/10\,(3 + 4x + 3z\ )$$
$$z = 1/10\,(-3 - x - 6y\ )$$

First iteration : From the above equations, we start with $x = y = z = 0$

$$x^{(1)} = 3/10 = 0.3 \qquad \ldots\ldots\ldots\ldots(1)$$
$$y^{(1)} = 3/10 = 0.3 \qquad \ldots\ldots\ldots\ldots(2)$$
$$z^{(1)} = -3/10 = -0.3 \qquad \ldots\ldots\ldots\ldots(3)$$

Second iteration :Consider the new values of $y^{(1)} = 0.3$ and $z^{(1)} = -0.3$ in the first equation

$$x^{(2)} = 1/10(\ 3 + 5\,x.3 + (-0.3)) = 0.39$$

$$y^{(2)} = 1/10\,(\ 3 + 4\,x0.3 + 3\,x(-0.3)\ ) = 0.33$$

$$z^{(2)} = 1/10\,[\,-3 \mid (0.3) \mid 6(0.3)\,] = -0.51$$

Third iteration : Consider the new values of $x^{(2)} = 0.39$, $y^{(2)} = 0.33$ and $z^{(2)} = -0.51$ in the first equation

$$x^{(3)} = \quad 1/10\,[\ 3 \ | \quad 5\,x\,0.33 \ +(-0.51)] \qquad = 0.363$$

$$y^{(3)} = \quad 1/10\,(\ 3 \ + \ 4\,x\,0.39 \ + 3\,x\,(-0.51)\,) \ = 0.303$$

$$z^{(3)} = \quad 1/10[\ -3 \ | \ \ 0.39 \ \ | \ \ 6\,x\,(0.33)\ ] \qquad = -0.537$$

Thus, we continue the iteration and result is noted below

Iteration No.	*X*	*y*	*z*
4	*0.3441*	*0.2841*	*-0.5181*
5	*0.33843*	*0.2822*	*-0.50487*
6	*0.340126*	*0.283911*	*0.503163*
7	*0.3413229*	*0.2851015*	*-0.5043592*
8	*0.34167891*	*0.2852214*	*-0.50519319*
9	*0.341572062*	*0.285113607*	*-0.505300731*

From the above table 8 th and 9 th iterations are equal by considering the 3 decimal places. Hence the solution of the equation is

$x = 0.342$, $y = 0.285$, $z = -0.505$.

Answers:

1. (3.017, 1.986, 0.912)
2. (0.994, 1.507, 1.849)
3. (-1.0, 0.999, 3)
4. (0.83, 0.32, 1.07)

Gauss-Seidel Method

This method is only an enhancement of Gauss-Jaobi Method.

Consider the 3 linear equations in 3 unknowns,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

This method is applied only when diagonal elements are exceeding all other elements in the respective equations i.e.,

$$|\ a_1\ | \ > |\ b_1\ | + |c_1| \ = d_1$$
$$|\ a_2\ | \ > |\ b_2\ | + |c_2| \ = d_2$$
$$|\ a_3\ | \ > |\ b_3\ | + |c_3| \ = d_3$$

We start with initial values of x,y and z as zero.

Note : 1. For all systems of equation, this method will not work

2.Iteration method is self correcting method. Any error made in computation is corrected automatically in subsequent iterations

3. Iteration is stopped when any two successive iteration values are equal

Illustration : 1 . *Solve the system of equation by Gauss-Seidel method*

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = 3$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

ie., 10 > 5+ 2 ; 10 > 4+ 3 ; 10 > 1+6 . So iteration method can be applied

$$x = 1/10\,(3 + 5y + 2z)$$
$$y = 1/10\,(3 + 4x + 3z)$$
$$z = 1/10\,(-3 - x - 6y)$$

First iteration :

From the above equations, we start with $x = y = z = 0$

$$x^{(1)} = 3/10 = 0.3 \quad \ldots\ldots\ldots\ldots(1)$$

New value of x is used for further calculation ie., $x = 0.3$

$$y^{(1)} = 1/10\,(3 + 4x\,0.3 + 3(0)] = 0.42 \quad \ldots\ldots\ldots\ldots(2)$$

New values of x and y is used for further calculation ie., $x = 0.3$ *and* $y = 0.42$

$$z^{(1)} = 1/10\,(-3 - 0.3 - 6(0.42) = -0.582 \quad \ldots\ldots\ldots\ldots(3)$$

Second iteration :

Consider the new values of $y^{(1)} = 0.42$ *and* $z^{(1)} = -0.582$ *in the first equation*

$$x^{(2)} = 1/10(3 + 5 \times 0.42 + (-0.582)) = 0.3936$$

$$y^{(2)} = 1/10\,(3 + 4 \times 0.3936 + 3 \times (-0.582)) = 0.28284$$

$$z^{(2)} = 1/10\,[-3 - (0.3936) - 6(0.28284)] = -0.509064$$

2. *Solve the system of equation by Gauss-Seidel method*

$$28x + 4y - z = 32$$
$$4x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

To apply this method , first we have to rewrite the equation in such way that to fulfill diagonal elements are dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$4x + 3y + 10z = 24$$

ie., 28 > 4+ 1 ; 17 > 2+ 4 ; 10 > 4+3 . So iteration method can be applied

$$x = 1/28\,(32 - 4y + z)$$
$$y = 1/17\,(35 - 2x - 4z)$$
$$z = 1/10\,(24 - x - 3y)$$

First iteration :

From the above equations, we start with $y = z = 0$, we get

$$x^{(1)} = 32/28 = 1.1429$$

New value of x is used for further calculation ie., $x = 1.1429$

$$y^{(1)} = 1/17\,(35 + 1.1429 + 3(0)] = 1.9244$$

New values of x and y is used for further calculation ie., $x = 1.1429$

and $y = 1.9244$

$$z^{(1)} = 1/10\,[24 - 1.1429 - 3(1.9244)] = 1.8084)$$

Second iteration :

Consider the new values of $y^{(1)} = 1.9244$ and $z^{(1)} = 1.8084$

$$x^{(2)} = 1/28[\,32 - 4\,(1.9244) + (1.8084)] = 0.9325$$

$$y^{(2)} = 1/17\,[\,35 - 2\,(0.9325) - 4\,(1.8084)\,] = 1.5236$$

$$z^{(2)} = 1/10\,[\,24 \mid (0.9325) \mid 3(1.5236)\,] = 1.8497$$

Third iteration :

Consider the new values of $x^{(2)} = 0.9325$, $y^{(2)} = 1.5236$ and $z^{(2)} = 1.8497$

$$x^{(3)} = 1/28[\,32 - 4\,(1.5236) + (1.8497)] = 0.9913$$

$$y^{(3)} = 1/17\,[\,35 - 2\,(0.9913) - 4\,(1.8497)\,) = 1.5070$$

$$z^{(3)} = 1/10\,[\,24 \mid (0.9913) \mid 3(1.5070)\,] = 1.8488$$

Thus, we continue the iteration and result is noted below

Iteration No.	*X*	*y*	*Z*
4	*0.9936*	*1.5069*	*1.8486*
5	*0.9936*	*1.5069*	*1.8486*

Therefore $x = 0.9936$, $y = 1.5069$, $z = 1.8486$

Practise Problems

Solve the following system of linear equations using Gauss -Seidel Method.

1. $8x - 6y + z = 13.67$; $3x + y - 2z = 17.59$; $2x - 6y + 9z = 29.29$
2. $30x - 2y + 3z = 75$; $2x + 2y + 18z = 30$; $x + 17y - 2z = 48$
3. $y - x + 10z = 35.61$; $x + z + 10y = 20.08$; $y - z + 10x = 11.19$
4. $10x - 2y + z = 12$; $x + 9y - z = 10$; $2x - y + 11z = 20$
5. $8x - y + z = 18$; $2x + 5y - 2z = 3$; $x + y - 3z = -16$
6. $2x + y + z = 4$; $x + 2y - z = 4$; $x + y + 2z = 4$

Answers

1. 0.83, 0.32,1.07
2. 2.5796, 2.7976, 1.0693
3. 1.321, 1.522, 3.541
4. 1.2624, 1.1591, 1.694
5. 2, 0.9998, 2.9999
6. 1, 1, 1

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – INTERPOLATION, NUMERICAL DIFFERENTIATION AND INTEGRATION – SMTA1401

I. Introduction

Contents - Interpolation: Newton forward and backward interpolation formula, Lagrange's formula for unequal intervals – Numerical differentiation: Newton's forward and backward differences to compute first and second derivatives – Numerical integration: Trapezoidal rule, Simpson's 1/3rd rule and Simpson's 3/8th rule.

1. Interpolation

Interpolation

The process of computing intermediate values of (x_0, x_n) for a function $y(x)$ from a given set of values of a function

Gregory-Newton's forward interpolation formula

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + --- (a)$$

$$where\ u = \frac{1}{h}(x - x_0)$$

Gregory-Newton's backward interpolation formula

$$y(x) = y_n + \frac{\nabla y_n}{1}v + \frac{\nabla^2 y_n}{2}v(v+1) + \frac{\nabla^3 y_n}{6}v(v+1)(v+2) + \frac{\nabla^4 y_n}{24}v(v+1)(v+2)(v+3) + --- (b)$$

$$where\ v = \frac{1}{h}(x - x_n)$$

Remark:

(i) The process of finding the values of $y(x_i)$ outside the interval (x_0, x_n) is called *extrapolation*

(ii) The *interpolating polynomial* is a function $p_n(x)$ through the data points $y_i = f(x_i) = P_n(x_i)$ i=0,12,..n

(iii) Gregory-Newton's forward interpolation formula (a) can be applicable if the interval difference h is constant and used to interpolate the value of $\mathrm{y(x_i)}$ nearer to beginning value $\mathrm{x_0}$ of the data set

(iv) If $y = f(x)$is the exact curve and $y = p_n(x)$ is the interpolating polynomial then the *Error in polynomial interpolation* is $y(x) - p_n(x)$ given by

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)--(x-x_n): x_0 < x < x_n, x_0 < c < x_n --- (c)$$

(v) *Error in Newton's forward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}u(u-1)(u-2)--(u-n): x_0 < x < x_n, x_0 < c < x_n ---- (d)$$

Problem1: Estimate θ at $x = 43$ & $x = 84$ from the following table .also find $y(x)$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution: Here all the intervals are equal with $h=x_1-x_0=10$ we apply Newton interpolation

Solution: Here all the intervals are equal with $h=x_1-x_0=10$ we apply Newton interpolation

Difference Table:

x	$\theta = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	$184 = y_0$	$y_1 - y_0 = 20 = \Delta y_0$				
50	$204 = y_1$	$y_2 - y_1 = 22 = \Delta y_1$	$2 = \Delta^2 y_0$	$0 = \Delta^3 y_0$		
60	$226 = y_2$	$y_3 - y_2 = 24 = \Delta y_2$	$2 = \Delta^2 y_1$	$0 = \Delta^3 y_1$	$0 = \Delta^4 y_0$	$0 = \nabla^5 y_n$
70	$250 = y_3$	$y_4 - y_3 = 26 = \Delta y_3$	$2 = \Delta^2 y_2$	$0 = \nabla^3 y_n$	$0 = \nabla^4 y_n$	
80	$276 = y_4$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$2 = \nabla^2 y_n$			
90	$304 = y_n$					

Case (i): to find the value of θ at $x = 43$

Since $x = 43$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(43-40) = \frac{3}{10} = 0.3 \Rightarrow u-1 = -0.7, u-2 = -1.7, u-3 = -2.7 --- (2)$$

Substituting (2) in (1), we get $y(x=43) = 184 + \frac{20}{1}(\frac{3}{10}) + \frac{2}{2}(\frac{3}{10})(\frac{-7}{10}) + 0 = \frac{18979}{10} = 189.79$

Case (ii): to find the value of θ at $x = 84$

Since $x = 84$ is nearer to x_n we apply Newton's backward Interpolation

$$y(x) = y_n + \frac{\nabla y_n}{1} v + \frac{\nabla^2 y_n}{2} v(v+1) + \frac{\nabla^3 y_n}{6} v(v+1)(v+2) + \frac{\nabla^4 y_n}{24} v(v+1)(v+2)(v+3) + ---(3)$$

$$where\ v = \frac{1}{h}(x - x_n) = \frac{1}{10}(84-90) = \frac{-6}{10} \Rightarrow v+1 = \frac{4}{10}, v+2 = \frac{14}{10}, v+3 = \frac{24}{10} ---(4)$$

Substituting (4) in (3), we get $y(x=84) = 304 + \frac{28}{1}(\frac{-6}{10}) + \frac{2}{2}(\frac{-6}{10})(\frac{4}{10}) + 0 = \frac{7174}{25} = 286.96$

To find polynomial $y(x)$, from (1) we get

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(x-40) \Rightarrow u-1 = \frac{1}{10}(x-50), u-2 = \frac{1}{10}(x-60), u-3 = \frac{1}{10}(x-60) ---(2)^1$$

Substituting (4) in (3), we get

$$y(x) = 184 + \frac{20}{1}\frac{1}{10}(x-40) + \frac{2}{2}\frac{1}{10}(x-40)\frac{1}{10}(x-50) + 0 = 184 + 2x - 80 + \frac{1}{100}(x^2 - 90x + 2000)$$

$$\Rightarrow y(x) = \frac{1}{100}(x^2 + 110x + 12400) ----------(5)$$

To Estimate θ at $x = 43$ & $x = 84$, put $x = 43$ & $x = 84$ in (5), we get

$$y(43) = \frac{1}{100}(18979) = 189.79\ and\ \ y(84) = \frac{1}{100}(28696) = 286.96$$

Problem2: Estimate the number of students whose weight is between 60 lbs and 70 lbs from the following data

Weight(lbs)	0-40	40-60	60-80	80-100	100-120
No.Students	250	120	100	70	50

Solution: let x-Weight less than 40 lbs, y-Number of Students, $\Rightarrow x_0 = 40, x_1 = 60, x_2 = 80, x_3 = 100, x_n = 120$, Here all the intervals are equal with h=x_1-x_0=20 we apply Newton interpolation

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$250 = y_0$	$y_1 - y_0 = 120 = \Delta y_0$			
60	$370 = y_1$	$y_2 - y_1 = 100 = \Delta y_1$	$-20 = \Delta^2 y_0$	$-10 = \Delta^3 y_0$	
80	$470 = y_2$	$y_3 - y_2 = 70 = \Delta y_2$	$-30 = \Delta^2 y_1$	$10 = \nabla^2 y_n$	$20 = \Delta^4 y_0 = \nabla^4 y_n$
100	$540 = y_3$	$y_n - y_{n-1} = 50 = \nabla y_n$	$-20 = \nabla^2 y_n$		
120	$590 = y_n$				

Case (i): to find the number of students y whose weight less than 60 lbs ($x = 60$)

From the difference table the number of students y whose weight less than 60 lbs ($x = 60) = 370$

Case (ii): to find the number of students y whose weight less than 70 lbs ($x = 70$)

Since $x = 70$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + ----(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{20}(70-40) = \frac{3}{2} \Rightarrow u-1 = \frac{3}{2}, u-2 = \frac{2}{2}, u-2 = \frac{-1}{2}, u-3 = \frac{-3}{2} -----(2)$$

Substituting (2) in (1), we get

$$y(x=70) = 250 + \frac{120}{1}(\frac{3}{2}) + \frac{-20}{2}(\frac{3}{2})(\frac{1}{2}) + \frac{-10}{6}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2}) + \frac{20}{24}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2}) = 423.59$$

The number of students y whose weight less than 70 lbs ($x = 70$) =424

Number of students whose weight is between 60 lbs and 70 lbs =

$$\begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 70 lbs} \end{Bmatrix} - \begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 60 lbs} \end{Bmatrix} = 424\text{-}370 = 54$$

Lagrange's interpolation formula Unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)--(x-x_n)}{(x_0-x_1)(x_0-x_2)--(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)--(x-x_n)}{(x_1-x_0)(x_1-x_2)--(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)--(x-x_n)}{(x_2-x_0)(x_2-x_1)--(x_2-x_n)}y_2+---+\frac{(x-x_0)(x-x_1)--(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)--(x_n-x_{n-1})}y_n$$

Problem 3: Determine the value of $y(1)$ from the following data using Lagrange's Interpolation

x	-1	0	2	3
y	-8	3	1	12

Solution: given

x	$x_0=-1$	$x_1=0$	$x_2=3$	$x_n=3$
y	$y_0=-8$	$y_1=3$	$y_2=1$	$y_n=12$

Since the intervals ere not uniform we cannot apply Newton's interpolation.

Hence by Lagrange's interpolation for unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}y_2+\frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}y_n$$

$$y(x)=\frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)----(1)$$

To compute $y(1)$ put $x=1$ in (1), we get

$$y(x=1)=\frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(1+1)(1-2)(1-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(1+1)(1-0)(1-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(1+1)(1-0)(1-2)}{(3+1)(3-0)(3-2)}(12)$$
$$\Rightarrow y(x=1)=2$$

To find polynomial $y(x)$, from (1) we get

$$y(x)=\frac{2}{3}(x^3-5x^2+6x)+\frac{1}{2}(x^3-4x^2+x+6)$$
$$-\frac{1}{6}(x^3-2x^2-3x)+\frac{1}{1}(x^3-x^2-2x)----(1)$$
$$y(x)=x^3\left(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1\right)+x^2\left(\frac{-10}{3}+\frac{-4}{2}+\frac{2}{6}-1\right)+x\left(\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2\right)+\left(\frac{6}{2}\right)$$
$$\Rightarrow y(x)=2x^3-6x^2+3x+3-----(2)$$

To compute $y(1)$ put $x=1$ in (2), we get $y(x=1)=2-6+3+3=2$

2. Numerical Differentiation and Integration

Engineers and scientists are frequently faced with the problem of differentiation or Integration of some functions. If the functions have a closed form representation and are amenable for standard calculus methods, then differentiation and integration can be carried out. However, in many situations, we may not know the exact functions. We will be knowing only, the values of the functions at a discrete set of points. In some instances, the functions are known but they are so complicated that analytic differentiation, integration is difficult. In both these situations, we seek the help of numerical techniques to obtain the estimates of derivatives or integrals. The method of obtaining the derivative of a function using a numerical technique is known as numerical differentiation.

The method of finding the value of an integral of the form $\int_a^b f(x)dx$ using numerical techniques is called "Numerical Integration". In this unit, we discuss various numerical differentiation and numerical integration methods. We have to understand that while analytical methods give exact answers, the numerical techniques provide only approximate answers.

Definition (Numerical differentiation):

Numerical differentiation is the process by which we can find the derivative or derivatives of a function at some values of the independent variable when we are given a set of values of that function.

Uses of Numerical differentiation:

The numerical differentiation techniques can be used in the following situations:

1. The function values corresponding to distinct values of the argument are known but the function is unknown. For example, we may knowing the values of $f(x)$ at various values of x, say x_i, $i = 1,2,3, \dots n$ in a tabulated form.

2. The function to be differentiated is complicated, and so, it is difficult to differentiate by usual procedures.

Numerical differentiation is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of data points (x_i, y_i = f(x_i)), i = 0,1,2,..., n which correspond to the values of an unknown function y = f(x). To find $\frac{dy}{dx}$, we first replace the exact relation y = f(x) by the best interpolating polynomial y = ϕ(x) as we know earlier and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the points are equally spaced and $\frac{dy}{dx}$ is required near the beginning of the table, we use **Newton-Gregory's Forward Interpolation Formula**.

If we require the derivative at the end of the table, we employ **Newton-Gregory's Backward Interpolation Formula**.

If the value of the derivative is required near the middle of the table, we use one of the **Central Difference Interpolation Formula**.

If the values of x are not Equi-spaced, we use **Newton's Divided difference Interpolation Formula** or $\frac{dy}{dx}$ to get the derivative value.

Formulae for Derivatives:

Consider the function y = f (x) which is tabulated for the values x_i (= x_0 + ih), i = 0, 1, 2,...., n.

Derivatives using Newton's Forward Difference Formula:

Suppose that we are given a set of values (x_i, y_i), i = 0,1,2,...., n.

We want to find the derivative of y = f (x) passing through the (n + 1)points, at a point nearer to the starting value at x = x_0.

Newton's Forward Difference Interpolation Formula is

$$y = y0 + p\,\Delta y0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots\dots\dots \qquad \text{------------------ (1)}$$

Where $p = \frac{x - x_0}{h}$ ------------------ (2)

On differentiation (1) w.r.t., p we have

On differentiation (2) w.r.t. x we have, $\frac{dp}{dx} \approx \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{\mathrm{h}} \left[\begin{array}{l} \Delta \mathrm{y}0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6} \Delta^3 y_0 \\ + \frac{4p^3 - 18p^2 + 22p - 6}{24} \Delta^4 y_0 + \ldots \end{array} \right] \qquad \text{......................(3)}$$

Equation (3) gives the value of $\frac{dy}{dx}$ at any point x which may be anywhere in the interval.

At $x = x_0$ and $p = 0$, hence putting $p = 0$, equation (3) gives

$$\left(\frac{dy}{dx}\right)_{x \approx x_1} = \left(\frac{dy}{dp}\right)_{p \approx 1} = \frac{1}{\mathrm{h}} \left[\begin{array}{l} \Delta \mathrm{y}0 + \frac{1}{2} \Delta^2 y_0 + \frac{1}{6} \Delta^3 y_0 \\ + \frac{4p^3 - 18p^2 + 22p - 6}{24} \Delta^4 y_0 + \ldots \end{array} \right] \qquad \text{......................(3)}$$

Again on differentiation (3) we get

$$\frac{d^2 y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(p-1)}{} \Delta^3 y_0 + \frac{6p^2 - 18p + 11}{12} \Delta^4 y_0 + \ldots\right]$$

From which we obtain

$$\left(\frac{d^2 y}{dx^2}\right) x \approx x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + ..\right] \text{ at } x = x_0 \text{ and } p = 0 \text{ .... (5)}$$

$$\text{Similarly, } \left(\frac{d^3 y}{dx^3}\right) x \approx x_0 = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \ldots\ldots\right] \qquad \text{....... (6)}$$

Derivatives using Newton's Backward Difference Formula:

Newton's Backward Difference Interpolation Formula is

$$y(x) = y_n + p\,\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_n + \ldots\ldots\ldots\ldots \quad \text{------------------ (7)}$$

Where $p = \dfrac{x - x_n}{h}$ ------------------ (8)

On differentiation (7) w.r.t., p we have

$$\frac{dy}{dp} = \left[\Delta y_n + \frac{2p+1}{2}\Delta^2 y_n + \frac{3p^2+6p+2}{6}\Delta^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Delta^4 y_n + \ldots\right].$$

On differentiation (8) w.r.t. x we have, $\dfrac{dp}{dx} \approx \dfrac{1}{h}$ Now

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{\text{h}}\left[\nabla y_n + \frac{2p+1}{2}\Delta^2 y_n + \frac{3p^2+6p+2}{6}\Delta^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Delta^4 y_n + ..\right]$$

......................(9)

Equation (9) gives the value of $\dfrac{dy}{dx}$ at any point x which may be anywhere in the interval.

At x = x_n and p = 0, hence putting p = 0, equation (9) gives

$$\left(\frac{dy}{dx}\right)x \approx x_{n_1} = \left(\frac{dy}{dx}\right)x_n = \frac{1}{\text{h}}\left[\Delta y\text{n} + \frac{1}{2}\Delta^2 y_n + \frac{1}{3}\Delta^3 y_n + \frac{1}{4}\Delta^4 y_n + \ldots\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Again on differentiation (09) we obtain

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}\cdot\frac{dp}{dx} = \frac{\text{d}}{\text{dp}}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{\text{d}}{\text{dp}}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_n + \frac{(p+1)}{}\Delta^3 y_n + \frac{6p^2+18p+11}{12}\Delta^4 y_n + .\right]$$

From which we obtain ...... (11)

$$\left(\frac{d^2y}{dx^2}\right)x \approx x_n = \frac{1}{h^2}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + ..\right] \text{ at x= x}_n \text{ and p = 0}$$

Similarly, $\left(\dfrac{d^3y}{dx^3}\right)x \approx x_n = \dfrac{1}{h^3}\left[\Delta^3 y_n - \dfrac{3}{2}\Delta^4 y_0 + \ldots\ldots\right]$ .... (12)

Maxima and Minima of a tabulated Function:

Given a set of data points (x_i, y_i) , i = 0,1,2,....., n, we can get the interpolating polynomial of degree n. Now we wish to estimate the value of x at which the curve is maximum or minimum.

We know that the maximum and minimum values of a function can be determined by equating the first derivative to zero and solving for the variable. The same procedure can be applied to find the maxima and minima of a tabulated function. Assume that the points are equally spaced with a step size of h.

Consider Newton's forward difference interpolation formula

$y \approx y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-3)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$ On differentiation it w.r.t. p, we get $\frac{dy}{dp} = \left[\Delta y_n + \frac{2p-1}{2}\Delta^2 y_n + \frac{3p^2-6p+2}{6}\Delta^3 y_n + ..\right]$ ....(1)

For maxima and minima $\frac{dy}{dp} \approx 0$. hence equating the RHS of (1) to zero and for simplicity only upto 3rd differences we obtain

$$\left[\Delta y_n + \frac{2p-1}{2}\Delta^2 y_n + \frac{3p^2-6p+2}{6}\Delta^3 y_n + ..\right] = 0$$

Re-arranging this as a quadratic in p, we get

$$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + \left(\Delta^2 y_0 - \Delta^3 y_0\right)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \frac{1}{3}\Delta^3 y_0$ from the difference table, We solve the equation (2) for p. Then the corresponding values of x are given by $x = x_0 + ph$ at which y is maximum or minimum.

Problem #01: Find the first and second derivatives of the function tabulated below at the point x = 1.5.

x:	1.5	2.0	2.5	3.0	3.5	4.0
y:	3.375	7.0	13.625	24.0	38.875	59.0

Solution: the difference diagonal table as below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2.0	7.0		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3.0	24.0		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4.0	59.0					

Here $x_0 = 1.5$, $y_0 = 3.375$, $h = 0.5$

By Newton's Difference Formula, we have

$$\left[\frac{dy}{dx}\right] at\ x \approx x_0 \approx \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - ....\right]$$

$$\therefore \left[\frac{dy}{dx}\right] at\ x \approx 1.5 \approx \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75) - \frac{1}{4}(0) + \frac{1}{5}(0)\right]$$

$$\left[\frac{dy}{dx}\right] at\ x \approx 1.5 \approx \frac{1}{0.5}[3.625 - 1.5 + 0.25]$$

$$\therefore \left[\frac{dy}{dx}\right] at\ x \approx 1.5 \approx \frac{2.375}{0.5} \approx 4.75$$

And

$$\left[\frac{d^2 y}{dx^2}\right] at\ x \approx x_0 \approx \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

$$\therefore \left[\frac{d^2 y}{dx^2}\right] at\ x \approx x_0 \approx \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

$$\therefore \left[\frac{d^2 y}{dx^2}\right] at\ x \approx 1.5 \approx \frac{1}{(0.5)^2}\left[3 - 0.75 + \frac{11}{12}(0) - \frac{5}{6}(0)\right]$$

$$\therefore \left[\frac{d^2 y}{dx^2}\right] at\ x \approx 1.5 \approx \frac{2.25}{(0.25)} \approx 9$$ Hence the solution.

Problem #02: From the following table Find the value of x for which y is maximum and find this value of y.

X	1.2	1.3	1.4	1.5	1.6
Y	0.9320	0.9636	0.9855	0.9975	0.9996

Solution: The difference table is as below

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320			
		0.0316		
1.3	0.9636		- 0.0097	
		0.0219		- 0.0002
1.4	0.9855		- 0.0099	
		0.0120		0
1.5	0.9975		- 0.0099	
		0.0021		
1.6	0.9996			

Here h = 0.1

Taking $x_0 = 1.2$, we have $y_0 = 0.9320$, $\Delta y_0 = 0.0316$, $\Delta^2 y = -0.0097$ and $\Delta^3 y = -0.0002$.

$\therefore$ Newton's Forward Difference Formula, terminated after second differences, gives as y $0.9320 + p(0.0316) + \frac{p(p-1)}{2}(-0.0097)$ .... (1)

$$\therefore \frac{dy}{dp} \approx 0.0316 + \frac{2p-1}{2}(-0.0097)$$

For y to be maximum,

$$\frac{dy}{dp} \approx 0 \quad \Rightarrow \quad \frac{dy}{dp} \approx 0.0316 + \frac{2p-1}{2}(-0.0097) = 0 \quad \Rightarrow 2(0.0316) = (2p-1)(0.0097)$$

$$\Rightarrow 2p-1 = \frac{0.0632}{0.0097} \approx 6.51546 \Rightarrow \boxed{p = 3.7577}$$

Hence $x = x_0 + ph = 1.2 + (3.7577)(0.1) = 1.5758$.

So, y is maximum when x = 1.5758 = 1.58

Putting p = 3.7577 in (1), the maximum value of y

$$= 0.9320 + (3.7577)(0.0316) + \frac{(3.7577)(3.7577-1)}{2}(-0.0097)$$

$= 0.9320 + 0.11874 - 0.0502586 = 1.00048 = 1.00$

$\boxed{\text{Therefore } y=1.0}$

Hence the solution.

Problem #03: Compute the first and second derivatives for the following table of data at x = -3 and x = 0

X	-3	-2	-1	0	1	2	3
Y	-33	-12	-3	0	3	12	33

Solution: the difference table is as below:

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^5 y$	$\Delta^6 y$
-3	-33							
		21						
-2	-12		-12					
		9		6				
-1	-3		-6		0			
		3		6		0	0	
0	0		0		0			0
		3		6		0	0	
1	3		6		0			
		9		6				
2	12		12					
		21						
3	33							

Here x_0 = -3, y_0 = -33 and h = 1; By Newton's Forward Difference Formula, we have

$$\left(\frac{dy}{dx}\right)x = x_0 = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + ...\right]$$

$$\therefore \left(\frac{dy}{dx}\right)x = x_0 = \frac{1}{1}\left[21 - \frac{1}{2}(-12) + \frac{1}{3}(6) - \frac{1}{4}(0) + \frac{1}{5}(0) - \frac{1}{6}(0)\right]$$

$$\left(\frac{dy}{dx}\right)x = x_0 = \frac{1}{1}[21 + 6 + 2 - 0 + 0 - 0] = 29 \quad \Rightarrow \left(\frac{dy}{dx}\right)x = x_0 = 29$$

$$\left(\frac{d^2 y}{dx^2}\right)x = x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

$$\therefore \left(\frac{d^2 y}{dx^2}\right)x = x_0 = \frac{1}{1^2}\left[(-12) - 6 + \frac{11}{12}(0) - \frac{5}{6}(0)\right]$$

$$\Rightarrow \left(\frac{d^2 y}{dx^2}\right)x = x_0 = -18 \text{ is solution.}$$

Problem #04: Compute the first and second derivatives for the following table of data at x = 1.1

X	1.0	1.2	1.4	1.6	1.8	2.0
Y	0.000	0.128	0.544	1.296	2.432	4.000

$$\Rightarrow \left(\frac{dy}{dx}\right) x = 1.1 = 0.128/2 \approx 0.64$$

Solution: the difference table is as below:

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.000					
		0.128				
1.2	0.128		0.288			
		0.416		0.048		
1.4	0.544		0.336		0	
		0.752		0.048		0
1.6	1.296		0.384		0	
		1.136		0.048		
1.8	2.432		0.432			
		1.568				
2.0	4.000					

Here $x_0 = 1$, $y_0 = 0$ and $h = 0.2$

By Newton's Forward Difference Formula, we have

$$\left(\frac{dy}{dx}\right) x = x_0 = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - .....\right]$$

$$\therefore \left(\frac{dy}{dx}\right) x = x_0 = \frac{1}{0.2}\left[0.128 - \frac{1}{2}(0.288) + \frac{1}{3}(0.048) - \frac{1}{4}(0) + \frac{1}{5}(0)\right]$$

$$\left(\frac{dy}{dx}\right) x = x_0 = \frac{1}{0.2}[0.128 - 0.144 + 0.016 - 0 + 0] = 0$$

$$\left(\frac{d^2 y}{dx^2}\right) x = x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

$$\therefore \left(\frac{d^2 y}{dx^2}\right) x = x_0 = \frac{1}{0.2^2}\left[(0.288) - 0.048 + \frac{11}{12}(0) - \frac{5}{6}(0)\right]$$

$$\left(\frac{d^2 y}{dx^2}\right) x = x_0 = \frac{1}{0.04}[0.240] \approx 6$$

$$\Rightarrow \left(\frac{d^2 y}{dx^2}\right) x = 1.1 = 6 \text{ X } 1.1 = 6.6$$

Problem #05: The velocity v of a particle moving in a straight line covers a distance x in time t. They are related as follows: Find f'(15):

X	0	10	20	30	40
Y	45	60	65	54	42

[**Ans.** $f'(15)$=-0.05416]

Problem #06: A rod is rotating in a plane. The following table gives the angle θ in radians through which the rod has turned for various values of the time t sec.

T	0.0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when t = 0.6 sec. [**Ans.** 3.8167, 6.75 rad/sec^2.]

3. Numerical Integration

Introduction:

We know that a definite integral of the form $\int_a^b f(x)\,dx$ represents the area under the curve y = f(x), enclosed between the limits x = a and x = b. This integration is possible only if f(x) is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows:

Given set of (n + 1) data points (x_i, y_i), i = 0,1,2,.......,n of the function y = f(x), where f(x) is not known explicitly, it is required to evaluate $\int_{x_0}^{x_n} f(x)\,dx$.

The problem of numerical integration, like that of numerical differentiation is solved by replacing f(x) with an interpolating polynomial $P_n(x)$ and obtaining $\int_{x_0}^{x_n} P_n(x)\,dx$ which is approximately taken as the value of $\int_{x_0}^{x_n} f(x)\,dx$. Numerical Integration is also known as “Numerical Quadrature”.

Newton-Cote's Quadrature Formula (General Quadrature Formula):

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods known as Newton-Cote's methods.

Derivation of Newton-Cotes formula:

Let the interval [a, b] be divided into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < x_3 \ldots\ldots\ldots < x_n = b$. Then $x_n = x_0 + nh$.

Newton forward difference formula is

$$y(x) = y(x_0 + ph) = P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Where $p = \dfrac{x - x_0}{h}$ .Now, instead of f(x) we will replace it by this interpolating polynomial.

$\therefore \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx$, where $P_n(x)$ is an interpolating polynomial of degree n

$$= \int_{x_0}^{x_0+nh} P_n(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] dx$$

Since $x = x_0 + ph$, $dx = h.dp$ and hence the above integral becomes

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] dp$$

$$= h\left[y_0(p) + \frac{p^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - 3\frac{p^3}{3} + 2\frac{p^2}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

$$= h\left[ny_0 + \frac{n^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - 3\frac{n^3}{3} + 2\frac{n^2}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

$$= nh\left[y_0 + \frac{n\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^2}{3} - \frac{n}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^3}{4} - 3\frac{n^2}{3} + 2\frac{n}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]$$

$$= nh\left[y_0 + \frac{n\Delta y_0}{2} + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - 3\frac{n^3}{2} + 11\frac{n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \ldots\right]$$

This is called **Newton-Cote's Quadrature for)mula.** ....(2)

Definition: The process of finding or evaluating a definite integral

$\boxed{I = \int_a^b f(x)\,dx}$ from a set of numerical values of the integrand f(x). If it is applied to the integration of a function of a single variable the process is known as "**Quadrature**".

The problem of numerical integration is solved by first approximating the integrand by a polynomial with the help of an interpolation formula and then integrating this expression between the desired limits.

The problem of numerical integration is solved by first approximating the integrand by a polynomial with the help of an interpolation formula and then integrating this expression between the desired limits.

Thus, to evaluate the definite integral $\int_a^b f(x)\,dx$ first expression the function f(x) by an interpolation formula say p(x) and then $\boxed{\int_a^b f(x)\,dx \sim \int_a^b p(x)\,dx\ .}$

The error E is such type of approximate given by,

$$\boxed{\int_a^b f(x)\,dx - \int_a^b p(x)\,dx = \int_a^b [f(x) - p(x)]\,dx}\ .$$

Definition: Trapezoidal Rule:

Here the function f(x) is approximated by a first- order polynomial $P_1(x)$ which passes through two points.

Putting n = 1 in the above general formula, all differences higher than the first will become zero (since other differences do not exist if n = 1) and we get

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx = \text{h}\left[y_0 + \frac{1}{2}\Delta y_0\right] = \text{h}\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

and $\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx = \text{h}\left[y_1 + \frac{1}{2}\Delta y_1\right] = \text{h}\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] = \frac{h}{2}(y_1 + y_2)$

$$\int_{x_2}^{x_3} f(x)\,dx = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = \text{h}\left[y_2 + \frac{1}{2}\Delta y_2\right] = \text{h}\left[y_2 + \frac{1}{2}(y_3 - y_2)\right] = \frac{h}{2}(y_2 + y_3)$$

....

Finally,

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Hence,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+3h} f(x)\,dx + \ldots.. + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \ldots\ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}[(y_0 + y_1) - 2(y_1 + y_2 + y_3 + y_4 + \ldots\ldots + y_{n-2} + y_{n-1}]$$

.... (3)

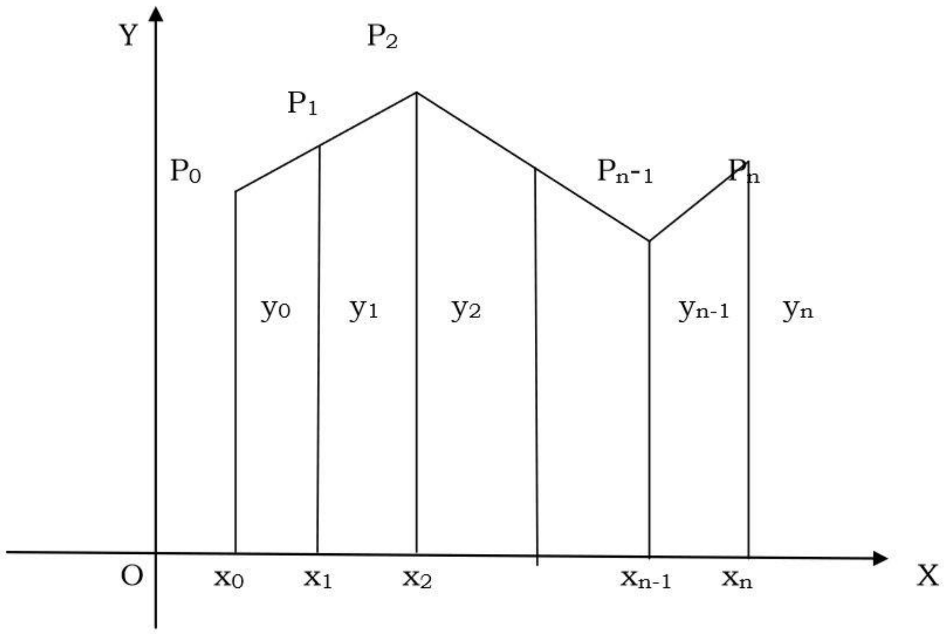
$\int_{x_0}^{x_n} f(x)\,dx \sim$

$\frac{h}{2}[\mathit{sum\ of\ the\ first\ and\ last\ ordnates}) - 2(\mathit{sum\ of\ the\ remaining\ ordinates})]$

This is known as **Trapezoidal Rule**.

Geometrical Interpretation:

Consider the points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$,$P_2(x_2, y_2)$, $P_3(x_3, y_3)$,....., $P_n(x_n, y_n)$. Suppose the curve y = f(x) passing through the above points be approximated by the union of the line segments joining (P_0, P_1), (P_1, P_2), (P_2, P_3),,.....,(P_{n-1}, P_n).



Geometrically, the curve y = f(x) is replaced by n straight line segments joining the points (x_0, y_0) and (x_1, y_1); (x_2, y_2) and (x_3, y_3); ,....., (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve y = f(x), x- axis and the ordinates $x = x_0$ and $x = x_n$ is then approximately equal to the sum of the areas of the n trapezium as shown in the figure above.

The total area given by

$$\frac{h}{2}\left[y_0 + y_1\right] + \frac{h}{2}\left[y_1 + y_2\right] + \frac{h}{2}\left[y_2 + y_3\right] + ... + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[(y_0 + 2(y_1 + y_2 + y_3 + y_4 + ... + y_{n-1}) + y_n\right] = \int_{x_0}^{x_n} f(x)\,dx \text{ approximately.}$$

Note: Throughout this Trapezoidal rule method is very simple for calculation purposes of numerical integration; the error in this case is significant.

Note: The accuracy of the result can be improved by increasing the number of intervals or by decreasing the value of h.

This is another popular and important method. Here, the function f(x) is approximated by a second order polynomial $P_2(x)$ which passes through three successive points.

Putting n = 2 in Newton-Cotes Quadrature formula i.e., by replacing the curve y = f(x) by n/2 parabolas, we have

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0 \right] = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0 \right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0) \right] \quad = 2h\left[\frac{1}{6} y_0 + \frac{2}{3} y_1 + \frac{1}{6} y_2 \right]$$

$$= \frac{2h}{6}\left[y_0 + 4y_1 + y_2 \right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$

Similarly, $\displaystyle\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right]$

....

$\displaystyle\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$ Adding all these integrals, we obtain

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots\dots\dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right] + \dots\dots\dots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \dots\dots\dots\dots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + y_3 + y_5 + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots\dots + y_{n-2})]$$

.... (4)

$$= \frac{h}{3}\begin{bmatrix} sum\ of\ the\ first\ and\ last\ ordnates) + 4(sum\ of\ the\ odd\ ordinates) \\ + 2(sum of\ the\ remaining\ even\ ordinates \end{bmatrix}$$

With the convention that y_0, y_2, y_4,....., y_{2n} are even ordinates and y_1, y_3, y_5,....., y_{2n-1} are odd ordinates.

This is known as **Simpson's 1/3 rule** or simply **Simpson's rule**.

Note: This rule requires the given interval must be divided into even number of equal sub-intervals of width h.

Simpson's 1/3 rule was derived using three points that fit a Quadratic equation. We can extend this approach by incorporating four successive points so that the rule can be exact for a polynomial f(x) of degree 3. Putting n = 3 in Newton-Cote's Quadrature formula, all differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3(3-2)^2}{24}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}\text{h}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3}{8}\text{h}[y_3 + 3y_4 + 3y_5 + y_6] \quad \text{and so on.}$$

Adding all these integrals, from x_0 to x_n, where n is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \dots\dots\dots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \dots\dots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots\dots + y_n)\right]$$

.... (5)

Equation (5) is called **Simpson's 3/8 rule** which is applicable only when n is multiple of 3. This rule is not so accurate as Simpson's 1/3 rule.

Note: while there is no restriction for the number of intervals in Trapezoidal rule, number of sub-intervals n in the case of Simpson's 1/3 rule must be even, for Simpson's 3/8 rule must be multiple of 3.

The error in Simpson's rule is of the order h^4 and the error in Trapezoidal's rule is of the order h^2 .

Example 1. *Evaluate* $\int_{-3}^{3} x^4 dx$ *by using (1) Trapezoidal rule* **(2)** *Simpson's rule. Verify your results by actual integration.*

Solution. Here $y(x) = x^4$. Interval length $(b - a) = 6$. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$. We form below the table

x	:	−3	−2	−1	0	1	2	3
y	:	81	16	1	0	1	16	81

(*i*) By Trapezoidal rule,

$$\int_{-3}^{3} y\,dx \approx \frac{h}{2}\,[(\text{sum of the first and last ordinates} + 2\,(\text{sum of the remaining ordinates})]$$

$$\approx \frac{1}{2}\,[(81 + 81) + 2\,(16 + 1 + 0 + 1 + 16)]$$

$$\approx 115$$

(*ii*) By Simpson's one-third rule (since number of ordinates is odd)

$$\int_{-3}^{3} y\,dx \approx \frac{1}{3}[(81+81)+2(1+1)+4(16+0+16)]$$

$$\approx 98.$$

(*iii*) Since $n=6$, (multiple of three), we can also use Simpson's three-eighths rule. By this rule,

$$\int_{-3}^{3} y\,dx \approx \frac{3}{8}[(81+81)+3(16+1+1+16)+2(0)] \approx 99.$$

(*iv*) By actual integration,

$$\int_{-3}^{3} x^4 dx = 2\times\left(\frac{x^5}{5}\right)_0^3 = \frac{2\times 243}{5} = 97{\cdot}2.$$

From the results obtained by various methods, we see that Simpson's rule gives better result than Trapezoidal rule (It is true in general; but not always).

Example 2. *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$, *using Trapezoidal rule with* $h=0{\cdot}2$. *Hence obtain an approximate value of* π. *Can you use other formulae in this case.*

Solution. Let $y(x) = \frac{1}{1+x^2}$

Interval is $(1-0)=1$ $\because$ The value of y are calculated as points taking $h=0{\cdot}2$

x :	0	0·2	0·4	0·6	0·8	1·0
$y=\frac{1}{1+x^2}$:	1	0·96154	0·86207	0·73529	0·60976	0·50000

(*i*) By trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})]$$

$$= \frac{0{\cdot}2}{2}[(1+0{\cdot}5)+2(0{\cdot}96154+0{\cdot}86207+0{\cdot}73529+0{\cdot}60976)]$$

$$= (0{\cdot}1)[1{\cdot}5+6{\cdot}33732]$$

$$= 0{\cdot}783732$$

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1}x)_0^1 = \pi/4$$

In this case, we cannot use Simpson's rule (since number of intervals is 5).

Example 3. *From the following table, find the area bounded by the curve and the x-axis from $x = 7{\cdot}47$ to $x = 7{\cdot}52$.*

x :	*7·47*	*7·48*	*7·49*	*7·50*	*7·51*	*7·52*
$y = f(x)$:	*1·93*	*1·95*	*1·98*	*2·01*	*2·03*	*2·06*

Solution. Since only 6 ordinates ($n = 5$) are given, we cannot use Simpson's rule . So, we will use Trapezoidal rule.

$$\text{Area} = \int_{7\cdot47}^{7\cdot52} f(x)\,dx = \frac{0{\cdot}01}{2}\,[(1{\cdot}93 + 2{\cdot}06) + 2\,(1{\cdot}95 + 1{\cdot}98 + 2{\cdot}01 + 2{\cdot}03)]$$

$$= 0{\cdot}09965.$$

Example 4. *Evaluate the integral* $I=\int_4^{5.2} \log_e x\, dx$ *using Trapezoidal, Simpson's rules .*

Solution. Here $b-a=5.2-4=1.2$. We shall divide the interval into 6 equal parts.

Hence, $h=\frac{1.2}{6}=0.2$. We form the table.

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)=\log_e x$	1.3862944	1.4350845	1.4816045	1.5260563	1.5686159	1.6094379	1.6486586

(*i*) By Trapezoidal rule,

$$\int_4^{5.2} \log x\, dx=\frac{0.2}{2}[(1.3862944+1.6486586)+2(1.4350845+1.4816045+1.5260563+1.5686159+1.6094379)]$$
$$=\mathbf{1.82765512}$$

(*ii*) Since $n=6$, we can use Simpson's rule ·
By Simpson's one-third rule,

$$I=\frac{0.2}{3}[(1.3862944+1.6486586)+2(1.4816045+1.5686159)+4(1.4350845+1.5260563)]$$
$$=\mathbf{1.82784724}$$

(*iii*) By Simpson, three-eighths rule,

$$I=\frac{3(0.2)}{8}[(1.3862944+1.6486586)+3(1.4350845+1.4816045+1.5686159+1.6094379+2(1.5260563)]$$
$$=\mathbf{1.82784705}$$

Example 5. *Evaluate* $I=\int_0^6 \frac{1}{1+x}dx$ *using (i) Trapezoidal rule (ii) Simpson's* ~~*rule*~~ *(both) Also, check up by direct integration.*

Solution. Take the number of intervals as 6.

$$\therefore \qquad h=\frac{6-0}{6}=1.$$

x :	0	1	2	3	4	5	6
$y=\frac{1}{1+x}$:	1	0.5	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x}=\frac{1}{2}\left[\left(1+\frac{1}{7}\right)+2\left(0.5+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\right]$$
$$=\mathbf{2.02142857}$$

(*ii*) By Simpson's one-third rule,

$$I = \frac{1}{3}\left[\left(1 + \frac{1}{7}\right) + 2\left(\frac{1}{3} + \frac{1}{5}\right) + 4\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)\right]$$

$$= \frac{1}{3}\left(1 + \frac{1}{7} + \frac{16}{15} + \frac{22}{6}\right) = \mathbf{1{\cdot}95873016}$$

(*iii*) By Simpson's three-eighth's rule,

$$I = \frac{3 \times 1}{8}\left[\left(1 + \frac{1}{7}\right) + 3\left(0{\cdot}5 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6}\right) + 2\left(\frac{1}{4}\right)\right]$$

$$= \mathbf{1{\cdot}96607143}$$

By actual integration,

$$\int_0^6 \frac{1}{1+x}\, dx = [\log(1+x)]_0^6 = \log_e 7 = 1{\cdot}94591015.$$

Example 6. *By dividing the range into ten equal parts, evaluate* $\int_0^\pi \sin x \, dx$ *by Trapezoidal and Simpson's rule. Verify your answer with integration.*

Solution. Range $= \pi - 0 = \pi$

Hence $\qquad h = \frac{\pi}{10}$.

We tabulate below the values of y at different x's.

x	:	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$
$y = \sin x$	:	0·0	0·3090	0·5878	0·8090	0·9511	1·0
x	:	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π	
$y = \sin x$	:	0·9511	0·8090	0·5878	0·3090	0	

$\left(\text{Note that the values are symmetrical about } x = \frac{\pi}{2}\right)$

(*i*) By Trapezoidal rule,

$$I = \frac{\pi}{20}[(0+0) + 2(0{\cdot}3090 + 0{\cdot}5878 + 0{\cdot}8090 + 0{\cdot}9511 + 1{\cdot}0$$
$$+ 0{\cdot}9511 + 0{\cdot}8090 + 0{\cdot}5878 + 0{\cdot}3090)]$$

$= 1{\cdot}9843$ nearly.

(*ii*) By Simpson's one-third rule (since three are 11 ordinates),

$$I=\frac{1}{3}\left(\frac{\pi}{10}\right)[(0+0)+2\,(0{\cdot}5878+0{\cdot}9511+0{\cdot}9511+0{\cdot}5878)$$
$$+4\,(0{\cdot}3090+0{\cdot}8090+1+0{\cdot}8090+0{\cdot}3090)]$$
$$=2{\cdot}00091$$

Note: We cannot use Simpson's three-eighth's rule

(*iii*) By actual integration, $I=(-\cos x)_0^{\pi}=2$

Hence, Simpson's rule is more accurate than the Trapezoidal rule.

Example 7. *Evaluate* $\int_0^1 e^x\,dx$ *by Simpson's one-third rule correct to five decimal places, by proper choice of h.*

Solution. Here, interval length $=b-a=1$

Hence we take $h=0{\cdot}1$ to have the accuracy required.

$$\therefore\ \int_0^1 e^x\,dx=\frac{0{\cdot}1}{3}[(1+e)+2\,(e^{0\cdot2}+e^{0\cdot4}+e^{0\cdot6}+e^{0\cdot8})$$
$$+4\,(e^{0\cdot1}+e^{0\cdot3}+e^{0\cdot5}+e^{0\cdot7}+e^{0\cdot9})]$$
$$=1{\cdot}718283$$

By actual integration, $\int_0^1 e^x\,dx=(e^x)_0^1=e-1=1{\cdot}71828183$

Correct to five decimal places, the answer is 1·71828.

Example 8. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by (i) Trapezoidal rule (ii) Simpson's rule*
Also check up the results by actual integration

Solution. Here, $b-a=6-0=6$. Divide into 6 equal parts $h=\frac{6}{6}=1$. Hence, the table is

x	:	0	1	2	3	4	5	6
$\frac{1}{1+x^2}=f(x)$	:	1·00	0·500	0·200	0·100	0·058824	0·038462	0·027027

There are 7 ordinates ($n=6$). We can use all the formula.

(*i*) By Trapezoidal rule,

$$I=\int_0^6 \frac{dx}{1+x^2}=\frac{1}{2}[(1+0{\cdot}027027)+2\,(0{\cdot}5+0{\cdot}2+0{\cdot}1+0{\cdot}058824+0{\cdot}038462)]$$
$$=1{\cdot}41079950$$

(*ii*) By Simpson's one-third rule,

$$I = \frac{1}{3}\left[(1 + 0.027027) + 2(0.2 + 0.058824) + 4(0.5 + 0.1 + 0.038462)\right]$$

$$= \frac{1}{3}(1.027027 + 0.517648 + 2.553848)$$

$$= \mathbf{1.36617433}$$

(*iii*) By Simpson's three-eighths rule,

$$I = \frac{3 \times 1}{8}\left[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.038462) + 2(0.1)\right]$$

$$= \mathbf{1.35708188}$$

By actual integration,

$$I = \int_0^6 \frac{dx}{1 + x^2} = (\tan^{-1} x)_0^6 = \tan^{-1} 6 = \mathbf{1.40564765}$$

Practice Problems

Problem #01 Dividing the range into 10 equal parts, find the approximate value of $\int_0^{\pi} \sin x \, dx$ by (i) Trapezoidal Rule (ii) Simpson's Rule.

[**Ans.** (i) 1.9843 (ii) 2.0009]

Problem #02 A rocket is launched from the ground. Its acceleration measured every 5 sec. Is tabulated below. Find the velocity and the position of the rocket at t = 40 seconds. Use Trapezoidal rule as well as Simpson's rule?

T	0	5	10	15	20	25	30	35	40
a(t)	40.0	42.25	48.50	51.25	54.35	59.48	61.5	64.3	68.7

[**Ans.** velocity 2194.9;position 87796;velocity 2197.5;position 87900]

Problem #03 Evaluate the following integral using Simpson's 1/3 rule for n = 4 $\int_1^2 \frac{e^x}{x} dx$? [Ans. 3.0592]

Problem #04 Evaluate $\int_0^1 \frac{1}{1+x} dx$ (i) by Trapezoidal Rule and Simpson's 1/3 rule (ii) Using Simpson's 3/8 rule? [Ans.0.69485;0.6931;0.6932]

Problem # 05 Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$ taking h = π/6? [**Ans.**3.0815]

Problem # 06 Evaluate $\int_0^1 x^3\,dx$ with five sub-intervals by Trapezoidal rule?

[**Ans.**0.26]

Problem # 07 When a train is moving ar 30 m/sec, steam is shut off and brakes are supplied. The spped of the train per second after t sec. Is given by using Simpson's rule, determine the distance moved by the train in 40 seconds?

Time	0	5	10	15	20	25	30	35	40
speed	30	24	19.5	16	13.6	11.7	10	8.5	7.0

[**Ans.573.4367**]

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS – SMTA1401

I. Introduction

Contents - Ordinary differential equations: Taylor series method, Runge Kutta method for fourth order – Partial differential equations – Finite differences – Laplace equation and its solutions by Liebmann's process – Solution of Poisson equation – Solutions of parabolic equations by Bender Schmidt Method – Solution of hyperbolic equations.

1. Numerical Methods for Solving Ordinary Differential Equations

Introduction to Ordinary Differential Equations

An ordinary differential equation of order n in of the form $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$, where $y^{(n)} = \frac{d^n y}{dx^n}$.

We will discuss the Numerical solution to first order linear ordinary differential equations by Taylor series method, and Runge - Kutta method, given the initial condition $y(x_0) = y_0$.

Taylor Series method

Consider the first order differential equation of the form $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.

The solution of the above initial value problem is obtained in two types

- Power series solution
- Point wise solution

(i) Power series solution

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

(ii) Point wise solution

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

Problems:

1. Using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = y + 1, y(0) = 1$.

Solution:

Given $\frac{dy}{dx} = y + 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

Taylor series formula for $y(0.1)$ is

$$y(x) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots$$

$y'(x) = y + 1$	$y'(0) = y(0) + 1 = 1 + 1 = 2$
$y''(x) = y'$	$y''(0) = y'(0) = 2$
$y'''(x) = y''$	$y'''(0) = y''(0) = 2$

Substituting in the Taylor's series expansion:

$$y(0.1) = y(0) + hy'(0) + \frac{h^2}{2!}y''(0) + \cdots$$

$$= 1 + 0.1 \times 2 + \frac{0.01}{2} \times 2 + \frac{0.001}{6} \times 2 + \cdots$$

$$= 1 + 0.1 \times 2 + \frac{0.01}{2} \times 2 + \frac{0.001}{6} \times 2 + \cdots$$

$$y(0.1) = 1.2103$$

2. Find the Taylor series solution with three terms for the initial value problem $\frac{dy}{dx} = x^2 + y, y(1) = 1$

Solution:

Given $\frac{dy}{dx} = x^2 + y, x_0 = 1, y_0 = 1$

$y'(x) = x^2 + y$	$y'(1) = 1 + 1 = 2$
$y''(x) = 2x + y'$	$y''(1) = 2 + 2 = 4$
$y'''(x) = 2 + y''$	$y'''(1) = 2 + 4 = 6$
$y'^v(x) = y'''$	$y'^v(1) = 6$

The Taylor's series expansion about a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Hence at $x = 1$

$$y(x) = y(1) + \frac{(x - 1)}{1!}y'(1) + \frac{(x - 1)^2}{2!}y''(1) + \frac{(x - 1)^3}{3!}y'''(1) + \cdots$$

$$y(x) = 1 + 2\frac{(x - 1)}{1!} + 4\frac{(x - 1)^2}{2!} + 6\frac{(x - 1)^3}{3!} + \cdots$$

Runge-Kutta method

Runge-kutta methods of solving initial value problem do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formula possesses the advantage of requiring only the function values at some selected points. These methods agree with Taylor series solutions up to the term in h^r where r is called the order of that method.

Fourth-order Runge-Kutta method

Let $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ be given.

Working rule to find $y(x_1)$

The value of $y_n = y(x_n)$ where $x_n = x_{n-1} + h$ where h is the incremental value for x is obtained as below:

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Compute the incremental value for y

$$\Delta y = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

The iterative formula to compute successive value of y is $y_{n+1} = y_n + \Delta y$

Problems

1. Find the value of y at $x = 0.2$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1$, using R-K method of order IV.

Sol:

Here $f(x, y) = x^2 + y, y(0) = 1$

Choosing $h = 0.1, x_0 = 0, y_0 = 1$

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.00525$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.00525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.0110050$$

$y(0.1) = 1.0053$

To find $y(0.2)$ given $x_2 = x_1 + h = 0.2, y_1 = 1.0053$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.0110$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.01727$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.01728$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.02409$$

y(0.2)=1.0227

Problems

Evaluate using Runge-Kutta methods. Unless otherwise mentioned, use fourth order R.K. method.

1. Find $y(0{\cdot}2)$ given $\frac{dy}{dx} = y - x, y(0) = 2$ taking $h = 0{\cdot}1$.

2. Evaluate $y(1{\cdot}4)$ given $\frac{dy}{dx} = x + y, y(1{\cdot}2) = 2$.

3. Obtain the value of y at $x = 0{\cdot}2$ if y satisfies

   $\frac{dy}{dx} - x^2 y = x; y(0) = 1$ taking $h = 0{\cdot}1$.

4. Solve $\frac{dy}{dx} = xy$ for $x = 1{\cdot}4$, taking $y(1) = 2, h = 0{\cdot}2$.

5. Solve: $y' = \frac{y - x}{y + x}$ given $y(0) = 1$, to obtain $y(0{\cdot}2)$.

2. Numerical Methods for Solving Partial Differential Equations

Classification of Partial Differential Equations of the Second Order

The most general linear partial differential equation of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

i.e.,

$$A\,u_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0 \qquad \ldots(1)$$

where A, B, C, D, E, F are in general functions of x and y.

The above equation of second order (linear) (1) is said to

(*i*) *elliptic* at a point (x, y) in the plane if $B^2-4AC<0$

(*ii*) *parabolic* if $B^2-4AC=0$

(*iii*) *hyperbolic if* $B^2-4AC>0$.

Note: The same differential equation may be elliptic in one region, parabolic in another and hyperbolic in some other region. For example, $xu_{xx}+u_{yy}=0$ is elliptic if $x>0$, hyperbolic if $x<0$ and parabolic if $x=0$.

Example 1. *Classify the following equations:*

(*i*) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

(*ii*) $x^2 f_{xx} + (1 - y^2) f_{yy} = 0$

(*i*) Here $A = 1, B = 2, C = 1$

$B^2 - 4AC = 4 - 4 = 0$, for all x, y.

Hence, the equation is parabolic at all points.

(*ii*) $A = x^2, B = 0, C = 1 - y^2$

$$B^2 - 4AC = -4x^2(1 - y^2)$$
$$= 4x^2(y^2 - 1)$$

For all x except $x = 0$, x^2 is +ve.

If $-1 < y < 1$, $y^2 - 1$ is negative.

$\therefore$ $B^2 - 4AC$ is –ve if $-1 < y < 1, x \neq 0$

$\therefore$ For $-\infty < x < \infty$ $(x \neq 0)$, $-1 < y < 1$, the equation is elliptic.

For $-\infty < x < \infty, x \neq 0, y < -1$ or $y > 1$, the equation is hyperbolic.

For $x = 0$ for all y or for all $x, y = \pm 1$ the equation is parabolic.

Example 2. *Classify the following partial differential equations:*

(*i*) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)\, u_{yy} = \sin(x + y)$

(*ii*) $(x + 1)\, u_{xx} - 2\,(x + 2)\, u_{xy} + (x + 3)\, u_{yy} = 0$

(*iii*) $x f_{xx} + y f_{yy} = 0, x > 0, y > 0.$

Solution. (*i*) Here, $A = 1, B = 4, C = (x^2 + 4y^2)$

$$B^2 - 4AC = 16 - 4\,(x^2 + 4y^2)$$
$$= 4\,[4 - x^2 - 4y^2]$$

The equation is elliptic if $4 - x^2 - 4y^2 < 0$

i.e., $$x^2 + 4y^2 > 4$$

i.e., $$\frac{x^2}{4} + \frac{y^2}{1} > 1$$

$\therefore$ It is elliptic in the region *outside the ellipse*

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

It is hyperbolic *inside the ellipse* $\frac{x^2}{4} + \frac{y^2}{1} = 1.$

It is parabolic *on the ellipse* $\frac{x^2}{4} + \frac{y^2}{1} = 1.$

(*ii*) Here, $A = x + 1,\ \ B = -2\,(x + 2),\ \ C = x + 3$

$$B^2 - 4AC = 4\,(x + 2)^2 - 4\,(x + 1)(x + 3)$$
$$= 4\,[1] = 4 > 0$$

$\therefore$ The equation is hyperbolic at all points of the region.

(*iii*) $A = x, B = 0, C = y$

$$B^2 - 4AC = -4xy,\ \ (x > 0, y > 0 \text{ given})$$
$$= -\text{ve}$$

$\therefore$ It is elliptic for all $x > 0,\ y > 0.$

Classify the following equations as elliptic, parabolic, or hyperbolic.

1. $f_{xx} - 2f_{xy} = 0$
2. $f_{xx} + 2f_{xy} + 4f_{yy} = 0$
3. $f_{xx} - 2f_{xy} + f_{yy} = 0$
4. $f_{xy} - f_x = 0$
5. $u_{xx} = u_t$

Elliptic Equations

An important and frequently occurring elliptic equation is Laplace's equation, *i.e.*,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad i.e., \ \nabla^2 u=0 \quad \text{or} \quad u_{xx}+u_{yy}=0: \qquad ...(1)$$

Replacing the derivatives by difference quotients given under Article 12·3, of this chapter, we get,

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}=0$$

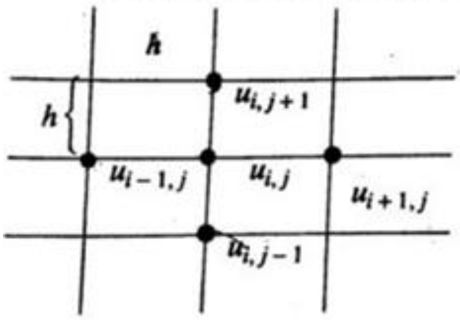
Taking $k=h$, (square mesh) in the above equation,

$$4u_{i,j}=u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}$$

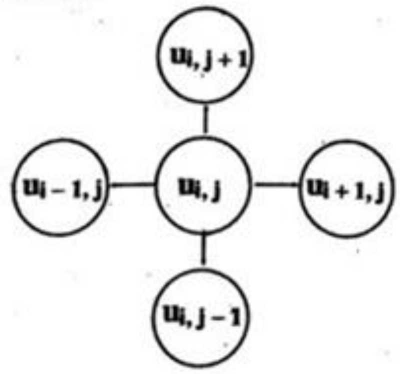
$$\therefore \qquad \boxed{\mathbf{u_{i,j}=\frac{1}{4}\left[u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}\right]}} \qquad ...(1)$$

That is, the value of u at any interior point is the arithmetic mean of the values of u at the four lattice points (Two of them are vertically just above and below and the other two in the horizontal line just after and before this point).

This is called standard five point formula.



or



Schematic diagram.

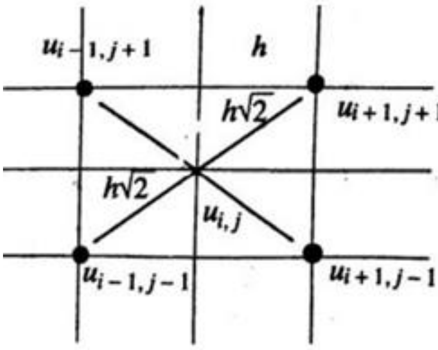
Central value = average of the other four values.

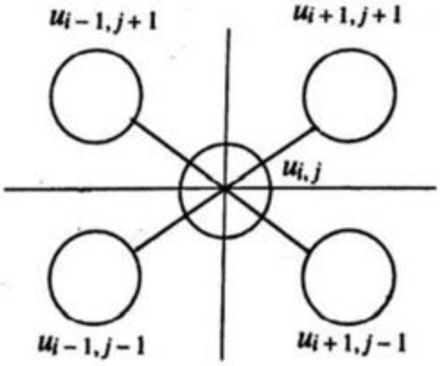
Diagonal five-point formula

Instead of the formula (1), we can also use the formula

$$u_{i,j} = \frac{1}{4}\left(u_{i-1,j-1} + u_{i-1,j+1}, u_{i+1,j-1} + u_{i+1,j+1}\right) \qquad \text{...(2)}$$

which is called the *diagonal five-point formula* since this formula involves the values on the diagonals through $u_{i,j}$. Since the Laplace equation is invariant in any coordinate system, the formula remains same when the coordinate axes are rotated through 45°. But the error in the diagonal formula is four times the error in the standard formula. Therefore, we always prefer the standard formula to the diagonal formula.



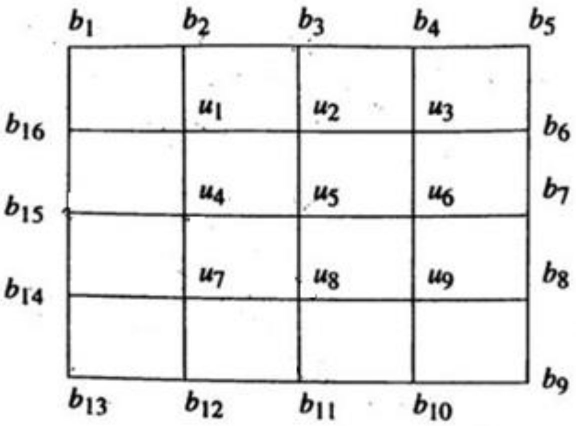


Schematic diagram of diagonal formula.

12·6. Solution of Laplace's Eqation: (By Liebmann's iteration process)

AIM: To solve the Laplace's equation $u_{xx} + u_{yy} = 0$ (i) in bounded square region R with a boundary C when the boundary values of u are given on the boundary (or at least at the grid points in the boundary).

Let us divide the square region into a network of sub-squares of side h (refer to the figure).



The boundary values of u at the grid points are given and noted by $b_1, b_2, \ldots b_{16}$. The values of u at the interior lattice or grid points are assumed to be $u_1, u_2, \ldots u_9$.

To start the iteration process, initially we find rough values at interior points and then we improve them by iterative process mostly using standard five point formula.

Find u_5 first: $u_5 = \frac{1}{4}(b_3 + b_7 + b_{11} + b_{15})$ (by standard five point formula—SFPF)

Knowing u_5, we find u_1, u_3, u_7, u_9, that is the values at the centres of the four larger inner squares by using diagonal five point formula—DFPF.

That is, $u_1 = \frac{1}{4}(b_3 + b_{15} + b_1 + u_5)$

$$u_3 = \frac{1}{4}(b_5 + u_5 + b_3 + b_7)$$

$$u_7 = \frac{1}{4}(u_5 + b_{13} + b_{11} + b_{15})$$

$$u_9 = \frac{1}{4}(b_7 + b_{11} + b_9 + u_5)$$

The remaining 4 values u_2, u_4, u_6, u_8 can be got by using SFPF.

That is, $u_2 = \frac{1}{4}(b_3 + u_5 + u_1 + u_3)$

$$u_4 = \frac{1}{4}(u_1 + u_7 + u_5 + b_{15})$$

$$u_6 = \frac{1}{4}(u_3 + u_9 + u_5 + b_7)$$

$$u_8 = \frac{1}{4}(u_5 + b_{11} + u_7 + u_9)$$

Now we know all the boundary values of u and rough values of u at every grid point in the interior of the region R. Now, we iterate the process and improve the values of u with accuracy. Start with u_5 and proceed to get the values of $u_1, u_3, \ldots u_9$ always using SFPF, taking into account the latest available values of u to use in the formula. The iterative formula is

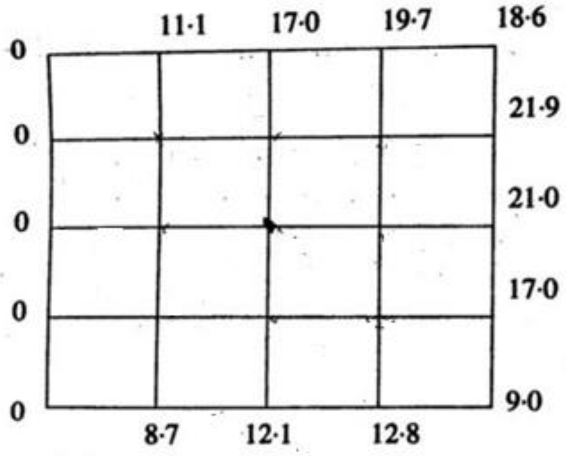
$$u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i+1,j}^{(n)}\, u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right) \qquad \ldots\text{I}$$

where the superscript of u denotes the iteration number.

Equation I is called **LIEBMANN'S** iteration process. The process is stoppd once we get the values with desired accuracy.

Note: To solve the nine unknowns $u_1, u_2, \ldots u_9$ from the nine equations, we can also use Gauss-Seidel method or other method.

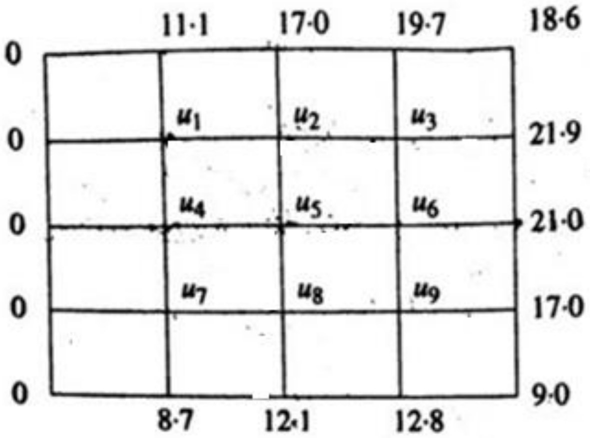
Example 3. *Find by the Liebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown in the following figure.*



Solution. Since u is harmonic, it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ in the square.} \qquad \text{...(1)}$$

Let the interior values of u at the 9 grid points be $u_1, u_2, \ldots, u_9$. We will find the values of u at the interior mesh points as explained in the previous article. We will first find the rough values of u and then proceed to refine them.



Finding rough values:

$$u_5 = \frac{1}{4}(0 + 17.0 + 21.0 + 12.1) = 12.5 \qquad \text{(SFPF)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 17.0) = 7.4 \qquad \text{(DFPF)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17.0 + 1.0) = 17.3 \qquad \text{(DFPF)}$$

$$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.2 \qquad \text{(DFPF)}$$

$$u_9 = \frac{1}{4}(12.5 + 9.0 + 12.1 + 21.0) = 13.7 \qquad \text{(DFPF)}$$

$$u_2 = \frac{1}{4}(17.0 + 12.5 + 7.4 + 17.3) = 13.6 \qquad \text{(SFPF)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.5 \qquad \text{(SFPF)}$$

$$u_6 = \frac{1}{4}(12.5 + 21.0 + 17.3 + 13.7) = 16.1 \qquad \text{(SFPF)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.1 \qquad \text{(SFPF)}$$

Now, we have got the rough values at all interior grid points and already we possess the boundary values at the lattice points. We will now improve the values by using always SFPF.

First iteration: (We obtain all values by SFPF)

$$u_1^{(1)} = \frac{1}{4}(0 + 11.1 + u_2 + u_4) = \frac{1}{4}(0 + 11.1 + 13.6 + 6.5) = 7.8$$

$$u_2^{(1)} = \frac{1}{4}(17.0 + 12.5 + 7.8 + 17.3) = 13.7$$

$$u_3^{(1)} = \frac{1}{4}(13.7 + 21.9 + 19.7 + 16.1) = 17.9$$

$$u_4^{(1)} = \frac{1}{4}(0 + 12.5 + 7.8 + 6.2) = 6.6$$

$$u_5^{(1)} = \frac{1}{4}(13.7 + 11.1 + 6.6 + 16.1) = 11.9$$

$$u_6^{(1)} = \frac{1}{4}(17.9 + 13.7 + 11.9 + 21.0) = 16.1$$

$$u_7^{(1)} = \frac{1}{4}(6.6 + 8.7 + 0 + 11.1) = 6.6$$

$$u_8^{(1)} = \frac{1}{4}(11.9 + 12.1 + 6.6 + 13.7) = 11.1$$

$$u_9^{(1)} = \frac{1}{4}(16.1 + 12.8 + 17.0 + 11.1) = 14.3$$

Now we go for the second iteration.

Second iteration:

$$u_1^{(2)} = \frac{1}{4}(0 + 11.1 + 13.7 + 6.6) = 7.9$$

$$u_2^{(2)} = \frac{1}{4}(17.0 + 17.9 + 7.9 + 11.9) = 13.7$$

$$u_3^{(2)} = \frac{1}{4}(13.7 + 19.7 + 21.9 + 16.1) = 17.9$$

$$u_4^{(2)} = \frac{1}{4}(7.9 + 0 + 11.9 + 6.6) = 6.6$$

$$u_5^{(2)} = \frac{1}{4}(13{\cdot}7 + 6{\cdot}6 + 16{\cdot}1 + 11{\cdot}1) = 11{\cdot}9$$

$$u_6^{(2)} = \frac{1}{4}(11{\cdot}9 + 17{\cdot}9 + 21{\cdot}0 + 14{\cdot}3) = 16{\cdot}3$$

$$u_7^{(2)} = \frac{1}{4}(0 + 6{\cdot}6 + 11{\cdot}1 + 8{\cdot}7) = 6{\cdot}6$$

$$u_8^{(2)} = \frac{1}{4}(6{\cdot}6 + 11{\cdot}9 + 14{\cdot}3 + 12{\cdot}1) = 11{\cdot}2$$

$$u_9^{(2)} = \frac{1}{4}(11{\cdot}2 + 16{\cdot}3 + 17{\cdot}0 + 12{\cdot}8) = 14{\cdot}3$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}(0 + 11{\cdot}1 + 13{\cdot}7 + 6{\cdot}6) = 7{\cdot}9$$

$$u_2^{(3)} = \frac{1}{4}(7{\cdot}9 + 17{\cdot}9 + 17{\cdot}0 + 11{\cdot}9) = 13{\cdot}7$$

$$u_3^{(3)} = \frac{1}{4}(13{\cdot}7 + 21{\cdot}9 + 19{\cdot}7 + 16{\cdot}3) = 17{\cdot}9$$

$$u_4^{(3)} = \frac{1}{4}(6{\cdot}6 + 0 + 7{\cdot}9 + 11{\cdot}9) = 6{\cdot}6$$

$$u_5^{(3)} = \frac{1}{4}(11{\cdot}2 + 6{\cdot}6 + 13{\cdot}7 + 16{\cdot}3) = 11{\cdot}9$$

$$u_6^{(3)} = \frac{1}{4}(11{\cdot}9 + 21{\cdot}0 + 17{\cdot}9 + 14{\cdot}3) = 16{\cdot}3$$

$$u_7^{(3)} = \frac{1}{4}(0 + 6{\cdot}6 + 11{\cdot}2 + 8{\cdot}7) = 6{\cdot}6$$

$$u_8^{(3)} = \frac{1}{4}(6{\cdot}6 + 14{\cdot}3 + 11{\cdot}9 + 12{\cdot}1) = 11{\cdot}2$$

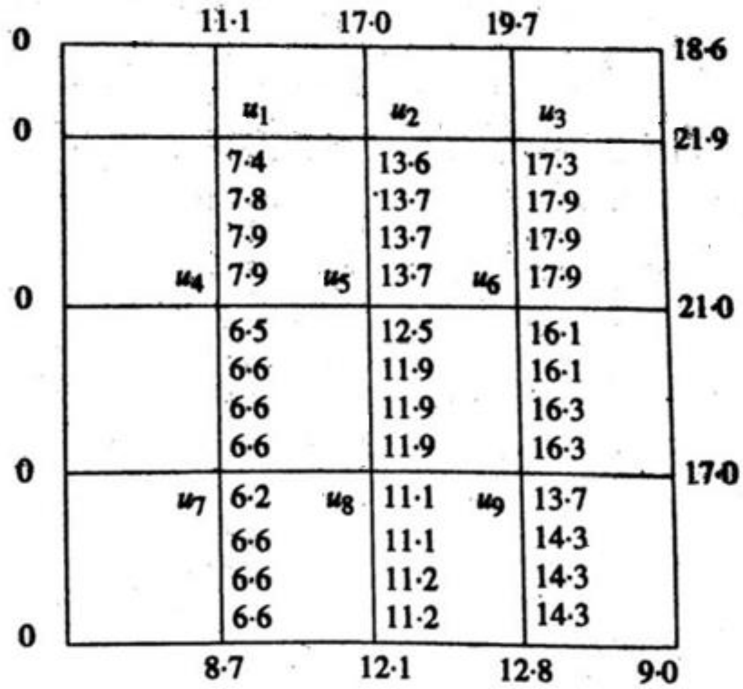
$$u_9^{(3)} = \frac{1}{4}(11{\cdot}2 + 16{\cdot}3 + 17{\cdot}0 + 12{\cdot}8) = 14{\cdot}3$$

Now, all the 9 values of u of the third iteration are same as the corresponding values of the second iteration. Hence we stop the procedure and accept

$u_1 = 7{\cdot}9$, $u_2 = 13{\cdot}7$, $u_3 = 17{\cdot}9$, $u_4 = 6{\cdot}6$, $u_5 = 11{\cdot}9$

$u_6 = 16{\cdot}3$, $u_7 = 6{\cdot}6$, $u_8 = 11{\cdot}2$ and $u_9 = 14{\cdot}3$.

Instead of working out so elaborately, we can write the values of u's at each grid paint and work out the scheme easily. The values of u's are shown below:



Poisson's Equation

An equation of the form $\nabla^2 u = f(x, y)$

i.e., $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad ...(1)$$

is called as POISSON'S equation where $f(x, y)$ is a function of x and y only.

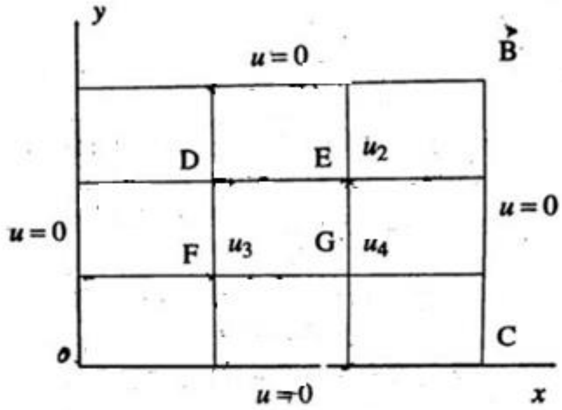
We will solve the above equation numerically at the points of the square mesh, replacing the derivatives by difference quotients. Taking $x = ih$, $y = jk = jh$ (here) the differential equation reduces to

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$$

i.e., $$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \qquad ...(2)$$

Example 9. *Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0$, $y = 0$, $x = 3$, $y = 3$ with $u = 0$ on the boundary and mesh length 1 unit.* (*MS. 1976*)

Solution.



The P.D.E. is $\nabla^2 u = -10(x^2 + y^2 + 10)$ ...(1)

using the theory, (here $h = 1$)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \qquad ...(2)$$

Applying the formula (2) at D $(i = 1, j = 2)$

$$0 + 0 + u_2 + u_3 - 4u_1 = -10(15) = -150$$

$$u_2 + u_3 - 4u_1 = -150 \qquad ...(3)$$

Applying at E $(i = 2, j = 2)$

$$u_1 + u_4 - 4u_2 = -180 \qquad ...(4)$$

Applying (2) at F, $(i = 1, j = 1)$

$$u_1 + u_4 - 4u_3 = -120 \qquad ...(5)$$

Applying (2) at G, $(i = 2, j = 1)$

$$u_2 + u_3 - 4u_4 = -10(2^2 + 1^2 + 10) = -150 \qquad ...(6)$$

We can solve the equation (3), (4), (5), (6) either by direct elimination or by Gauss-Seidel method.

Method 1. (5) – (4) gives, (Eliminate u_1)

$$4(u_2 - u_3) = 60$$

$$u_2 - u_3 = 15 \qquad ...(7)$$

Eliminate u_1 from (3) and (4); (3) + 4(4) gives,

$$-15u_2 + u_3 + 4u_4 = -870 \qquad ...(8)$$

Adding (6) and (8) $-7u_2 + u_3 = -510$ ...(9)

From (7), (9) adding, $u_2 = 82{\cdot}5$

Using (7), $u_3 = u_2 - 15 = 82{\cdot}5 - 15 = 67{\cdot}5$

Put in (3), $4u_1 = 300 \therefore u_1 = 75$

$$4u_4 = 150 + 150; \; u_4 = 75$$

$$\boxed{\therefore \; \mathbf{u_1 = u_4 = 75, \; u_2 = 82{\cdot}5, \; u_3 = 67{\cdot}5}}$$

Note: Since the differential equation is unchanged when x, y are interchanged and boundary conditions are also same after interchange x and y, the result will be symmetrical about the line $y = x$ $\therefore u_4 = u_1$.

If we use this idea the 4 equations would have reduced to 3 equations namely,

$$u_2 + u_3 - 4u_1 = -150, \; 2u_1 - 4u_2 = -180,$$

$$2u_1 - 4u_3 = -120 \text{ and } u_2 + u_3 - 4u_1 = -150.$$

Solving will be easier now.

Method 2. We can use Gauss-Seidel method to solve.

$$u_1 = \frac{1}{4}(150 + u_2 + u_3)$$

$$u_2 = \frac{1}{4}(2u_1 + 180)$$

$$u_3 = \frac{1}{4}(2u_1 + 120)$$

The tabular values are:

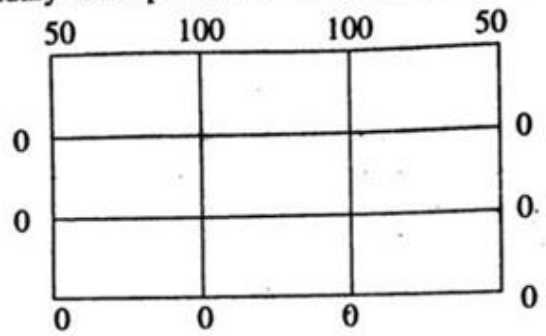
	1	2	3	4	5	6	7	8	9	10
$u_4 = u_1$	–	37·5	65·56	72·64	74·41	74·85	74·96	74·99	75	75
u_2	0	63·75	77·79	81·32	82·21	82·43	82·48	82·5	82·5	82·5
u_3	0	48·75	62·78	66·32	67·21	67·43	67·48	67·5	67·5	67·5

We get the values after 9 iterations as

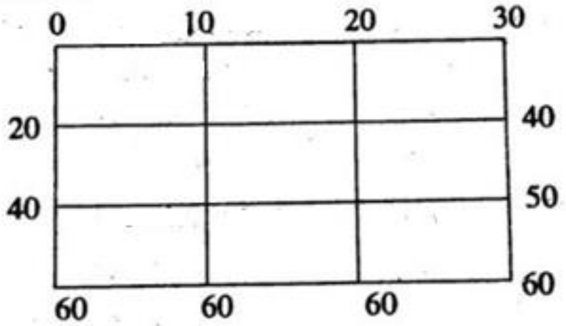
$$\boxed{\mathbf{u_1 = 75 = u_4, \; u_2 = 82{\cdot}5, \; u_3 = 67{\cdot}5}}$$

Problems

1. Sove the boundary value problem $\nabla^2 u = 0$ for the square of sides three units.



2. Sove $\nabla^2 u = 0$ at the nodal points for the following square region given the boundary conditions.



Parabolic Equations

Bender-Schmidt Method

The one dimensional heat equation, namely,

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$ is an example of parabolic equation.

Setting $\alpha^2 = \frac{1}{a}$, the equation becomes, $\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0.$

Here $A = 1$, $B = 0$, $C = 0$ $\therefore$ $B^2 - 4AC = 0$. Therefore, it is parabolic at all points.

AIM : Our aim is to solve this by the method of finite differences.

To solve : $$u_{xx} = au_t \quad \ldots(1)$$

with boundary conditions,

$$u(0, t) = T_0 \quad \ldots(2)$$

$$u(l, t) = T_l \quad \ldots(3)$$

and with initial condition $u(x, 0) = f(x), 0 < x < l$ $\quad \ldots(4)$

We select a spacing h for the variable x and a spacing k for the time variable t.

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and
$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Hence (1) becomes,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{a}{k}(u_{i,j+1} - u_{i,j})$$

$$\therefore \quad u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$= \lambda (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

where
$$\lambda = \frac{k}{ah^2}.$$

i.e.,
$$\mathbf{u}_{i,j+1} = \lambda \mathbf{u}_{i+1,j} + (1 - 2\lambda)\,\mathbf{u}_{i,j} + \lambda \mathbf{u}_{i-1,j} \qquad \text{...(5)}$$

Writing the boundary condition as $u_{0,j} = T_0$ ...(6)

$u_{n,j} = T_l$ ...(7)

where $nh = l$.

and initial condition as

$$u_{i,0} = f(ih),\ i = 1, 2, \ldots \qquad \text{...(8)}$$

u is known at $t = 0$.

Equation (5) facilitates to get the value of u at $x = ih$ and time t_{j+k}

Equation (5) is called EXPLICIT FORMULA. It is valid if $0 < \lambda \le \frac{1}{2}$.

If we take, $\lambda = \frac{1}{2}$, the coefficient of $u_{i,j}$ vanishes.

Hence Equation (5) becomes,

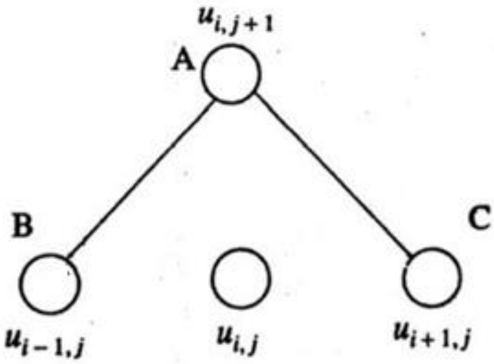
$$\boxed{\mathbf{u}_{i,j+1} = \frac{1}{2}[\mathbf{u}_{i-1,j} + \mathbf{u}_{i+1,j}]} \qquad \text{...(9)}$$

when
$$\lambda = \frac{1}{2} = \frac{k}{ah^2};\ \ i.e.,\ \ k = \frac{a}{2}h^2$$

i.e., the value of u at $x = x_i$ at $t = t_{j+1}$ is equal to the average of the values of u the surrounding points x_{i-1} and x_{i+1} at the previous time t_j.

Equation (9) is called **Bender-Schmidt recurrence equation.**

This is valid only if $k = \frac{a}{2}h^2$ (so, select k like this)



Schematic diagram

Value of u at $A = \frac{1}{2}$ [Value of u at B + value of u at C]

Example 11. *Solve* $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$, *given* $u(0, t) = 0$, $u(4, t) = 0$ $u(x, 0) = x(4 - x)$. *Assume* $h = 1$. *Find the values of* u *upto* $t = 5$

Solution. $u_{xx} = au_t$ $\therefore$ $a = 2$

To use Bender-Schmidt's equation, $k = \frac{a}{2}h^2 = 1$

Step-size in time = k = 1. The values of $u_{i,j}$ are tabulated below.

x-direction $\rightarrow$

	j \ i	0	1	2	3	4	
	0	0	3	4	3	0	$\leftarrow u(x, 0) = x(4 - x)$
$\downarrow$	1	0	2	3	2	0	
t-direction	2	0	1·5	2	1·5	0	
	3	0	1	1·5	1	0	
	4	0	0·75	1	0·75	0	
	5	0	0·5	0·75	0·5	0	

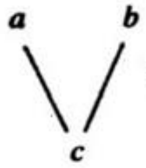
Analysis: Range for x : (0, 4); for t : (0, 5)

$u(x, 0) = x(4 - x)$. This gives $u(0, 0) = 0$, $u(1, 0) = 3$,

$u(2, 0) = 4$, $u(3, 0) = 3$, $u(4, 0) = 0$.

For all t, at $x = 0$, $u = 0$ and for all t at $x = 4$, $u = 0$.

Using these values we fill up column under $x = 0, x = 4$ and row against $t = 0$.



This means $c = \dfrac{a+b}{2}$

The values of u at $t = 1$ are written by seeing the values of u at $t = 0$ and using the average formula.

Example 12. *Solve* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ *given* $u(0, t) = 0$, $u(4, t) = 0$, $u(x, 0) = x(4 - x)$ *assuming* $h = k = 1$.

Find the values of u upto $t = 5$.

Solution. If we want to use Bender-Schmidt formula, we should have $k = \dfrac{a}{2} h^2$.

Here, $k = h = 1, a = 1$. These values do not satisfy the condition. Hence we cannot employ Bender-Schmidt formula.

Hence we go to the basic equation

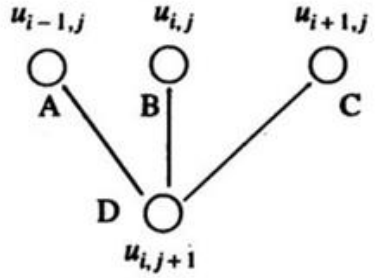
$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i-1,j} \qquad ...(1)$$

Now $\quad \lambda = \dfrac{k}{ah^2} = \dfrac{1}{1 \times 1} = 1$

Hence, (1) reduces to,

$$u_{i,j+1} = u_{i+1,j} - u_{i,j} + u_{i-1,j}$$

That is,



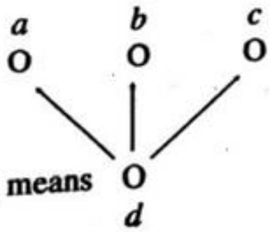
Value of u at D = value of u at A + value of u at C − value of u at B.

$\rightarrow x$ direction

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	0
2	0	1	0	1	0
3	0	− 1	2	− 1	0
4	0	3	− 4	3	0
5	0	− 7	10	− 7	0

$\downarrow$ direction

Here,



$d = a + c - b$

This figure means

Note: Since $\lambda = 1$ is used in the working, it violates the condition for use of *Explicit formula*. So the solution is not stable and it is not a practical problem. Such questions should be avoided, since unstable solutions do not exist.

Example 13. *Solve $u_t = u_{xx}$ subject to $u\ (0, t) = 0$, $u\ (1, t) = 0$ and $u\ (x, 0) = \sin \pi x$, $0 < x < 1$.*

Solution. Since h, and k are not given we will select them properly and use Bender-Schmidt method.

$$k = \frac{a}{2} h^2 = \frac{1}{2} h^2 \quad \because \quad a = 1.$$

Since range of x is (0, 1), take $h = 0{\cdot}2$.

Hence $k = \dfrac{(0{\cdot}2)^2}{2} = 0{\cdot}02$

The formula is $u_{i, j+1} = \dfrac{1}{2}(u_{i-1, j} + u_{i+1, j})$

$$u\ (0, 0) = 0, \ u\ (0{\cdot}2, 0) = \sin \frac{\pi}{5} = 0{\cdot}5878$$

$$u\ (0{\cdot}4, 0) = \sin \frac{2\pi}{5} = 0{\cdot}9511; \sin\ (0{\cdot}6, 0) = \sin \frac{3\pi}{5} = 0{\cdot}9511$$

$$\sin\ (0{\cdot}8, 0) = \sin \frac{4\pi}{5} = 0{\cdot}5878 \qquad \sin\ (1, 0) = 0$$

We form the table.

$x \rightarrow$ direction $h = 0.2$

$j \backslash i$	0	0·2	0·4	0·6	0·8	1
0	0	0·5878	0·9511	0·9511	0·5878	0
0·02	0	0·4756	0·7695	0·7695	0·4756	0
0·04	0	0·3848	0·6225	0·6225	0·3848	0
0·06	0	0·3113	0·5036	0·5036	0·3113	0
0·08	0	0·2518	0·4074	0·4074	0·2518	0
0·1	0	0·2037	0·3296	0·3296	0·2037	0

$\downarrow$ t-direction $k = 0.02$.

Example 14. *Given* $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial f}{\partial t}$, $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$, *find f in the range taking h = 1 and upto 5 seconds.*

Solution. To use Schmidt method, $k = \dfrac{a}{2} h^2$.

Here, $a = 1, h = 1 \quad \therefore \quad k = 1/2$

Step-size of time = 1/2

Step-size of x = 1.

$f(0, 0) = 0, f(1, 0) = 24, f(2, 0) = 84, f(3, 0) = 144, f(4, 0) = 144,$
$f(5, 0) = 0.$

We have, $u_{i, j+1} = \dfrac{1}{2}(u_{i-1, j} + u_{i+1, j})$

$\rightarrow x$ direction

$j \backslash i$	0	1	2	3	4	5
0	0	24	84	144	144	0
½	0	42	84	114	72	0
1	0	42	78	78	57	0
1·5	0	39	60	67·5	39	0
2	0	30	53·25	49·5	33·75	0
2·5	0	26·625	39·75	43·5	24·75	0
3	0	19·875	35·0625	32·25	21·75	0
3·5	0	17·5312	26·0625	28·4062	16·125	0
4	0	13·0312	22·9687	21·0938	14·2031	0
4·5	0	11·4843	17·0625	18·5859	10·5469	0
5	0	8·5312	15·0351	13·8047	9·2929	0

$\downarrow$ t-direction

Example 15. *Solve* $u_{xx} = 32u_t$, *taking* $h = 0{\cdot}25$ *for* $t > 0$, $0 < x < 1$ *and* $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = t$.

Solution. The range for x is $(0, 1)$; $h = 0{\cdot}25$

$$k = \frac{a}{2} h^2 = \frac{32}{2}\left(\frac{1}{4}\right)^2 = 1$$

Step-size of time t is 1;

direction of x

j \ i	0	0·25	0·5	0·75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0·5	2
3	0	0	0·25	1	3
4	0	0·125	0·5	1·625	4
5	0	0·25	0·875	2·25	5

$t \downarrow$

Values of u

Hyperbolic Equations

The wave equation in one dimension (vibration of strings) is

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \; ; \; \textit{i.e.,} \; a^2 u_{xx} - u_{tt} = 0.$$

Here, $A = a^2, B = 0, C = -1 \;\; \therefore \;\; B^2 - 4AC = 4a^2 = +ve.$

Therefore, the equation is hyperbolic.

Let us solve this equation by reducing it to difference equation.

AIM: Solve $a^2 u_{xx} - u_{tt} = 0$ ...(1)

together with the boundary conditions

$$u(0, t) = 0 \quad \text{...(2)}$$

$$u(l, t) = 0 \quad \text{...(3)}$$

and the initial conditions

$$u(x, 0) = f(x) \quad \text{...(4)}$$

$$u_t(x, 0) = 0 \quad \text{...(5)}$$

Assuming $\Delta x = h$, $\Delta t = k$, we have

$$u_{xx} = \frac{u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{tt} = \frac{u_{i,j+1} - 2\,u_{i,j} + u_{i,j-1}}{k^2}$$

Substituting these values in (1),

$$\frac{a^2}{h^2}(u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}) - \frac{1}{k^2}(u_{i,j+1} - 2\,u_{i,j} + u_{i,j-1}) = 0$$

i.e., $$\lambda^2 a^2 (u_{i+1,j} - 2\,u_{i,j} + u_{i-1,j}) - u_{i,j+1} + 2\,u_{i,j} - u_{i,j-1} = 0$$

where $$\lambda = \frac{k}{h}.$$

$$\boxed{\mathbf{u_{i,j+1} = 2\,(1 - \lambda^2 a^2)\,u_{i,j} + \lambda^2 a^2\,(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}}} \quad ...(6)$$

To make the equation simpler, select λ such that

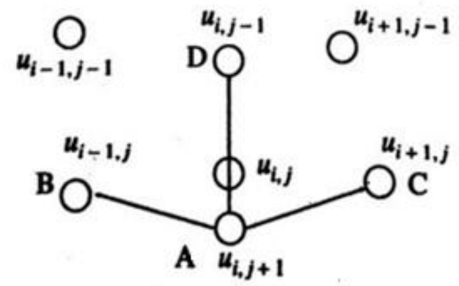
$$1 - \lambda^2 a^2 = 0 \quad i.e., \quad \lambda^2 = \frac{1}{a^2} = \frac{k^2}{h^2} \quad i.e., \quad k = \frac{h}{a}$$

under this selection of $\lambda^2 = \frac{1}{a^2}$; *i.e.*, $k = \frac{h}{a}$, the equation (6) reduces to the simplest form

$$\boxed{\mathbf{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}}} \quad ...(7)$$

Equation (6) is called an **Explicit scheme or Explicit formula** to solve the wave equation.

Equation (7) gives a simpler form under the condition $k = \frac{h}{a}$.



The value of u at A = value of u at B + value of u at C – value of u at D.

Note 1. The boundary condition $u(0, t) = 0$ *i.e.*, $u_{0,j} = 0$ gives the values of u along the line $x = 0$; that all $u = 0$.

The boundary condition $u(l, t) = 0$ or $u_{n,j} = 0$ gives the values of u along the line $x = l$, *i.e.*, all $u = 0$ along this line.

Note 2. Initial condition $u(x, 0) = f(x)$ becomes

$$\boxed{\mathbf{u(i, 0) = f(ih), i = 1, 2, 3, ...}}$$

This gives the value of u along $t = 0$ for various values of i.

$$\boxed{\mathbf{u(i, 0) = f(ih) = f_i}} \quad ...(8)$$

Note 3. The initial condition $u_t(x, 0) = 0$ gives $\dfrac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$ when $j = 0$ (central difference approximation)

$$\therefore \quad \boxed{\mathbf{u_{i,1} = u_{i,-1}}} \quad ...(9)$$

Setting $j = 0$ in (7),

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,1}, \text{ using (9)},$$

$$\boxed{\therefore \quad \mathbf{u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})}} \quad ...(10)$$

Note 4. If $1 - \lambda^2 a^2 < 0$, *i.e.*, $\lambda a > 1$, *i.e.*, $\dfrac{ak}{h} > 1$, the solution is unstable. If $\dfrac{ka}{h} = 1$, it is stable and if $\dfrac{ka}{h} < 1$, it is stable but the accuracy of the solution decreases as $\dfrac{ak}{h}$ decreases.

That is, for $\lambda = \dfrac{1}{a}$ the solution is stable.

Example 19. *Solve numerically,* $4u_{xx} = u_{tt}$ *with the boundary conditions* $u(0, t) = 0$, $u(4, t) = 0$ *and the initial conditions* $u_t(x, 0) = 0$ *and* $u(x, 0) = x\cdot(4 - x)$, *taking* $h = 1$. *(for 4 time steps)*

Solution. Since $a^2 = 4, h = 1, k = \dfrac{h}{a} = \dfrac{1}{2}$ ...(1)

$\therefore$ Taking $k = 1/2$, we use the formula,

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad ...(2)$$

From $u(0, t) = 0 \Rightarrow u$ along $x = 0$ are all zero.

From $u(4, t) = 0 \Rightarrow u$ along $x = 4$ are all zero.

$u(x, 0) = x(4 - x)$ implies that

$u(0, 0) = 0, u(1, 0) = 3, u(2, 0) = 4, u(3, 0) = 3.$

Now, we fill up the row $t = 0$ using the above values,

$$u_t(x, 0) = 0, \text{ implies } u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2} \quad ...(3)$$

Now we draw the table; for that we require

$$u_{1,1} = \frac{u_{2,0} + u_{0,0}}{2} = \frac{4 + 0}{2} = 2$$

$$u_{2,1} = \frac{u_{3,0} + u_{1,0}}{2} = \frac{3+3}{2} = 3$$

$$u_{3,1} = \frac{u_{4,0} + u_{2,0}}{2} = 2$$

$$u_{4,1} = 0.$$

Table

$t \backslash x$	0	1	2	3	4	
0	0	3	4	3	0	$u(x, 0) = x(4-x)$
0·5	0	2	3	2	0	
1	0	0	0	0	0	
		$(3+0-3)$	$(2+2-4)$	$(3+0-3)$		
1·5	0	-2	-3	-2	0	
2	0	-3	-4	-3	0	
2·5	0	-2	-3	-2	0	
3	0	0	0	0	0	
3·5	0	2	3	2	0	
4	0	3	4	3	0	

Period is 4 seconds or $8(k) = 8\left(\frac{1}{2}\right) = 4$ secs.

Example 20. *Solve* $25\,u_{xx} - u_{tt} = 0$ *for* u *at the pivotal points, given* $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ *and* $u(x, 0) = 2x$ *for* $0 \le x \le 2{\cdot}5$

$$= 10 - 2x$$

for $2{\cdot}5 \le x \le 5$.

for one half period of vibration.

Solution. Here, $a^2 = 25 \therefore a = 5$;

Period of vibration $= \frac{2l}{a} = \frac{2 \times 5}{5} = 2$ seconds.

half period = 1 second.

Therefore we want values upto $t = 1$ second

$$k = \frac{h}{a} = \frac{1}{5}, \text{ taking } h = 1$$

Step-size in t-direction $= \frac{1}{5}$.

The Explicit scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \text{...(1)}$$

Boundary conditions are $\left.\begin{aligned} u(0, t) = 0 \text{ or } u_{0,j} = 0 \\ u(5, t) = 0 \text{ or } u_{5,j} = 0 \end{aligned}\right\}$ for all j

$$u_t(x, 0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2} \quad \ldots(2)$$

$$u(x, 0) = 2x \text{ for } 0 \le x \le 2{\cdot}5$$
$$= 10 - 2x \text{ for } 2{\cdot}5 \le x \le 5$$

$\therefore$ $u(0, 0) = 0;\ u(1, 0) = 2,\ u(2, 0) = 4,$
$u(3, 0) = 4,\ u(4, 0) = 2,\ u(5, 0) = 0.$

$$u_{1,1} = \frac{u_{2,0} + u_{0,0}}{2} = \frac{4+0}{2} = 2 \qquad \text{Here, } u(x, 0) = u(i, 0)$$

$$u_{2,1} = \frac{u_{3,0} + u_{1,0}}{2} = 3$$

$$u_{3,1} = \frac{u_{4,0} + u_{2,0}}{2} = 3$$

$$u_{4,1} = \frac{u_{5,0} + u_{3,0}}{2} = 2$$

t \ x	0	1	2	3	4	5
0 ($j = 0$)	0	2	4	4	2	0
$t = \frac{1}{5}$ ($j = 1$)	0	2	3	3	2	0
$t = \frac{2}{5}$ ($j = 2$)	0	1	1	1	1	0
$t = \frac{3}{5}$ ($j = 3$)	0	-1	-1	-1	-1	0
$t = \frac{4}{5}$ ($j = 4$)	0	-2	-3	-3	-2	0
$t = 1$ ($j = 5$)	0	-2	-4	-4	-2	0

Problems

1. Solve $u_{tt} = u_{xx}$, given $u(0, t) = u(4, t) = 0$, $u(x, 0) = \frac{1}{2}x(4 - x)$, and $u_t(x, 0) = 0$. Take $h = 1$. Find the solutions upto 5 steps in t-direction.

2. Solve $u_{xx} = u_{tt}$ upto $t = 0{\cdot}5$ with spacing of $0{\cdot}1$, given $u(0, t) = u(1, t) = 0$, $u(x, 0) = 10x(10 - x)$, $u_t(x, 0) = 0$. (Take $h = 0{\cdot}1 = k$).

3. Solve numerically, $25u_{xx} = u_{tt}$, given, $u_t(x, 0) = 0$, $u(0, t) = u(5, t) = 0$ and $u(x, 0) = 20x$ for $0 \le x \le 1$
$= 5(5 - x)$ for $1 \le x \le 5$.

4. Show by suitable transformation of variables, the equation $u_{xx} - a^2 u_{tt} = 0$ can be transformed into the normalised form of the wave equation $u_{xx} - u_{tt} = 0$.