SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

B.SC-MATHEMATICS

SMTA 1306- TRIGNOMETRY & 2D ANALYTICAL GEOMETRY

Course Material

UNIT I- EXPANSIONS AND HYPERBOLIC FUNCTIONS:

Expansions of cosn θ ,sinn θ in powers of sin θ and cosθ -Expansion of tannθ θ in powers of tanθ -Powers of sines and cosines of θ in terms of functions of multiples of θ-Expansions of sinθ θ and cos θ in a series of ascending powers of θ -Evaluation of limits-Hyperbolic functions Relations between hyperbolic functions-Inverse hyperbolic functions-problems.

UNIT II -SUMMATION OF TRIGONOMETRIC SERIES

Logarithm of a complex number-Summation of trigonometrical series by the method of differences-Sum of series of n angles in A.P.-Summation of series using C+iS form-Gregory's series-Euler's series.

UNIT III CONIC

Geometric definition of Conic – the focus, directrix and eccentricity of a Conic. Classification of Conic into Ellipse, Parabola and Hyperbola based on the value of eccentricity. Paramectric representation of Conics in standard form. Condition for a given straight line to be tangent to a Conic. Parabola-Equation of the tangent and normal to the parabolaEllipse-Equation of the tangent and normal to the ellipse-conjugate diameters-Hyperbola-Equation of the tangent and normal to the hyperbola.

UNIT IV ASYMPTOTES

Asymptotes - Conjugate hyperbola - Conjugate diameters - Rectangular hyperbola - Equation of the tangent and normal to the rectangular hyperbola.

UNIT V POLAR EQUATION

Polar equation of a conic-Equation of the tangent and normal to the conic l/r=1+ ecosθ whose vectorial angle is α-Equation of asymtotes to the conic l/r=1+ ecosθ - Equation of polar to the conic l/r=1+ ecosθ .

Course Outcomes:

CO1	List the formula of sin θ and cos θ in a series of ascending powers of θ . Identify the relations between hyperbolic functions and different forms of conics.
CO2	Explain the methods of Expansions of trigonometric functions in powers of sinθ , cosθ and tanθ . Summarize the trigonometric series by the various methods
CO3	Choose the series formula and expand the trigonometric functions in a series of ascending powers of . Solve the problems on inverse hyperbolic functions and problems on equations of polar to conics.
CO4	Classification of Conics. Compare the tangent and normal equations of standard conics.
CO5	Evaluate the problems in Asymptotes, Conjugate hyperbola and diameters.

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INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

UNIT-I

EXPANSIONS AND HYPERBOLIC FUNCTIONS

Some basic formulae in Trigonometry:

1. Euler's theorem : $e^{in\theta} = \cos n\theta + i\sin n\theta$
2. De-Moivre's theorem: If n is any positive integer

$$\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

$$(\cos\theta + i\sin\theta)^{-n} = \cos(n\theta) - i\sin(n\theta)$$

3. $cos^2\theta + sin^2\theta = 1, \sin(-\theta) = -sin\theta, \cos(-\theta) = cos\theta$
4. $cos^2\theta = \frac{1+cos2\theta}{2}$
5. $sin^2\theta = \frac{1-cos2\theta}{2}$
6. $cos2\theta = \frac{1-tan^2\theta}{1+tan^2\theta} = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = 1 - 2sin^2\theta$
7. $sin2\theta = 2sin\theta cos\theta$
8. $sin2\theta = \frac{2\,tan\theta}{1+tan^2\theta}$
9. $tan2\theta = \frac{2\,tan\theta}{1-tan^2\theta}$
10. $sin3\theta = 3sin\theta - 4sin^3\theta$
11. $cos3\theta = 4cos^3\theta - 3cos\theta$
12. $sec^2\theta = 1 + tan^2\theta$
13. $cosec^2\theta = 1 + cot^2\theta$
14. $sin(A + \mathrm{B}) = sinAcos\mathrm{B} + \mathrm{cosAsinB}$
15. $sin(A - \mathrm{B}) = sinAcos\mathrm{B} - \mathrm{cosAsinB}$
16. $cos(A + \mathrm{B}) = cosAcos\mathrm{B} - \mathrm{sinAsinB}$
17. $cos(A - \mathrm{B}) = cosAcos\mathrm{B} + \mathrm{sinAsinB}$
18. $tan(A + \mathrm{B}) = \frac{tanA + tan\mathrm{B}}{1 - tanAtan\mathrm{B}}$
19. $tan(A - \mathrm{B}) = \frac{tanA - tan\mathrm{B}}{1 + tanAtan\mathrm{B}}$
20. $2sinAcos\mathrm{B} = sin(A + \mathrm{B}) + sin(A - \mathrm{B})$

21. $2cosAsinB = sin(A+B) - sin(A-B)$
22. $2cosAcosB = cos(A+B) + cos(A-B)$
23. $2sinAsinB = cos(A-B) - cos(A+B)$
24. $sinC + sinD = 2sin\left(\frac{C+D}{2}\right)cos\left(\frac{C-D}{2}\right)$
25. $sinC - sinD = 2cos\left(\frac{C+D}{2}\right)sin\left(\frac{C-D}{2}\right)$
26. $cosC + cosD = 2cos\left(\frac{C+D}{2}\right)cos\left(\frac{C-D}{2}\right)$
27. $cosC - cosD = -2sin\left(\frac{C+D}{2}\right)sin\left(\frac{C-D}{2}\right)$

Binomial theorem for positive integer n:

$$(x+a)^n = x^n + nC_1x^{n-1}a + nC_2x^{n-2}a^2 + \cdots + a^n$$

Hence $nC_r = nC_{n-r}$, $nC_n = nC_0 = 1$

28. Exponential series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$$

29. $cosn\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$
30. $sinn\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$

Results to remember

(i) We know that $cos^2\theta + sin^2\theta = 1$

$$cos^2\theta = 1 - sin^2\theta$$

$$cos^4\theta = (1 - sin^2\theta)^2$$

$$cos^6\theta = (1 - sin^2\theta)^3$$

Similarly $sin^2\theta = 1 - cos^2\theta$

$$sin^4\theta = (1 - cos^2\theta)^2$$

$$sin^6\theta = (1 - cos^2\theta)^3$$

Expansions of $cosn\theta$ and sin$n\theta$ in powers of $sin\ \theta$ and cos θ,nbeing a positive integer:

By using Binomial theorem,

$$(\cos\theta + isin\ \theta)^n = cos^n\theta + nC_1cos^{n-1}\theta(isin\ \theta) + nC_2cos^{n-2}\theta(isin\ \theta)^2 + \ldots + i^n sin^n\theta$$

$$= cos^n\theta + inC_1cos^{n-1}\theta sin\ \theta - nC_2cos^{n-2}\theta sin^2\theta - inC_3cos^{n-3}\theta sin^3\theta + \ nC_4cos^{n-4}\theta sin^4\theta + \cdots$$

But $(\cos\theta + isin\ \theta)^n = \cos n\theta + isinn\theta$ by De-Moivre's theorem:

$\cos n\theta + isinn$

$$= cos^n\theta + inC_1cos^{n-1}\theta sin\ \theta - nC_2cos^{n-2}\theta sin^2\theta - inC_3cos^{n-3}\theta sin^3\theta + \quad nC_4cos^{n-4}\theta sin^4\theta + \cdots$$

Equating the real and imaginary parts on both sides we get,

$$\cos n\theta = cos^n\theta - nC_2cos^{n-2}\theta sin^2\theta + \ nC_4cos^{n-4}\theta sin^4\theta + \cdots$$

$$sinn\theta = nC_1cos^{n-1}\theta sin\ \theta - nC_3cos^{n-3}\theta sin^3\theta + \cdots$$

Expansion of $tann\theta$ in powers of $tan\theta$:

We know that $tann\theta = \frac{sinn\theta}{cosn\theta} = \frac{nC_1cos^{n-1}\theta sin\ \theta - nC_3cos^{n-3}\theta sin^3\theta + \cdots}{cos^n\theta - nC_2cos^{n-2}\theta sin^2\theta + nC_4cos^{n-4}\theta sin^4\theta + \cdots}$

Dividing both Numerator and denominator by $cos^n\theta$ we get,

$$tann\theta = \frac{nC_1 tan\,\theta - nC_3 tan^3\theta + nC_5 tan^5\theta - \cdots}{1 - nC_2 tan^2\theta + nC_4 tan^4\theta - \cdots}$$

Solved problems:

1. Expand $sin7\theta$ in powers of $sin\theta$ and $cos\theta$

Solution:

$$sinn\theta = nC_1 cos^{n-1}\theta sin\,\theta - nC_3 cos^{n-3}\theta sin^3\theta + \cdots$$

$$sin7\theta = 7C_1 cos^6\theta sin\,\theta - 7C_3 cos^4\theta sin^3\theta + 7C_5 cos^2\theta sin^5\theta - 7C_7 sin^7$$

$$\therefore sin7\theta = 7cos^6\theta sin\,\theta - 35cos^4\theta sin^3\theta + 21cos^2\theta sin^5\theta - sin^7\theta.$$

2. Expand $cos4\theta$ in powers of $sin\theta$ and $cos\theta$

Solution:

$$\cos n\theta = cos^n\theta - nC_2 cos^{n-2}\theta sin^2\theta + nC_4 cos^{n-4}\theta sin^4\theta + \cdots$$

$$\cos 4\theta = cos^4\theta - 4C_2 cos^2\theta sin^2\theta + 4C_4 sin^4\theta$$

$$\therefore \cos 4\theta = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$$

3. Expand $tan8\theta$ in powers of $tan\theta$

Solution: $tann\theta = \frac{nC_1 tan\,\theta - nC_3 tan^3\theta + nC_5 tan^5\theta - \cdots}{1 - nC_2 tan^2\theta + nC_4 tan^4\theta - \cdots}$

$$tan8\theta = \frac{8C_1 tan\,\theta - 8C_3 tan^3\theta + 8C_5 tan^5\theta - 8C_7 tan^7\theta}{1 - 8C_2 tan^2\theta + 8C_4 tan^4\theta - 8C_6 tan^6\theta + 8C_8 tan^8\theta}$$

$$\therefore tan8\theta = \frac{8tan\,\theta - 56tan^3\theta + 56tan^5\theta - 8tan^7\theta}{1 - 28tan^2\theta + 70tan^4\theta - 28tan^6\theta + tan^8\theta}$$

4. Express $cos6\theta$ in terms of $cos\theta$

Solution:

$$\cos n\theta = cos^n\theta - nC_2 cos^{n-2}\theta sin^2\theta + nC_4 cos^{n-4}\theta sin^4\theta + \cdots$$

$$\cos 6\theta = cos^6\theta - 6C_2cos^4\theta sin^2\theta + \ 6C_4cos^2\theta sin^4\theta - 6C_6sin^6\theta$$

$$= cos^6\theta - 15cos^4\theta sin^2\theta + 15cos^2\theta sin^4\theta - sin^6\theta$$

$$= cos^6\theta - 15cos^4\theta(1 - cos^2\theta) + 15cos^2\theta \quad (1 - cos^2\theta)^2 - (1 - cos^2\theta)^3$$

$$= cos^6\theta - 15cos^4\theta + 15cos^6\theta + 15cos^2\theta \quad - 30cos^4\theta + 15cos^6\theta - 1 + 3cos^2\theta - 3cos^4\theta + cos^6\theta$$

$$\therefore \cos 6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$$

5. Express $sin7\theta$ in terms of $sin\theta$

Solution:

$$sinn\theta = nC_1cos^{n-1}\theta sin\,\theta - nC_3cos^{n-3}\theta sin^3\theta + \cdots$$

$$sin7\theta = 7C_1cos^6\theta sin\,\theta - 7C_3cos^4\theta sin^3\theta + 7C_5cos^2\theta sin^5\theta - 7C_7sin^7$$

$$sin7\theta = 7cos^6\theta sin\,\theta - 35cos^4\theta sin^3\theta + 21cos^2\theta sin^5\theta - sin^7\theta$$

$$= 7(1 - sin^2\theta)^3 sin\,\theta - 35(1 - sin^2\theta)^2 sin^3\theta + 21(1 - sin^2\theta)sin^5\theta - sin^7\theta$$

$$= 7sin\theta - 21sin^3\theta + 21sin^5\theta - 7sin^7\theta - 35sin^3\theta + 70sin^5\theta - 35sin^7\theta + 21sin^5\theta - 21sin^7\theta - sin^7\theta$$

$$\therefore sin7\theta = 7sin\theta - 56sin^3\theta + 112sin^5\theta - 64sin^7\theta.$$

6. Show that $\frac{sin6\theta}{sin\theta} = 32cos^5\theta - 32cos^3\theta + 6cos\theta$

Solution:

$$sinn\theta = nC_1cos^{n-1}\theta sin\,\theta - nC_3cos^{n-3}\theta sin^3\theta + \cdots$$

$$sin6\theta = 6C_1cos^5\theta sin\,\theta - 6C_3cos^3\theta sin^3\theta + 6C_5cos\theta sin^5\theta$$

$$= 6cos^5\theta sin\,\theta - 20cos^3\theta sin^3\theta + 6cos\theta sin^5\theta \ldots$$

$$\frac{sin6\theta}{sin\theta} = 6cos^5\theta - 20cos^3\theta sin^2\theta + 6cos\theta sin^4\theta$$

$$= 6cos^5\theta - 20cos^3\theta(1 - cos^2\theta) + 6cos\theta(1 - cos^2\theta)^2$$

$$= 6cos^5\theta - 20cos^3\theta + 20cos^5\theta + 6cos\theta - 12cos^3\theta + 6cos^5\theta$$

$$\therefore \frac{sin6\theta}{sin\theta} = 32cos^5\theta - 32cos^3\theta + 6cos\theta.$$

7. Prove that $\frac{cos7\theta}{cos\theta} = 64cos^6\theta - 112cos^4\theta + 56cos^2\theta - 7$

Solution: $\cos n\theta = cos^n\theta - nC_2cos^{n-2}\theta sin^2\theta + nC_4cos^{n-4}\theta sin^4\theta + \cdots$

$$\cos 7\theta = cos^7\theta - 7C_2cos^5\theta sin^2\theta + 7C_4cos^3\theta sin^4\theta - 7C_6cos\theta sin^6\theta$$

$$= cos^7\theta - 21cos^5\theta sin^2\theta + 35cos^3\theta sin^4\theta - 7cos\theta sin^6\theta$$

$$= cos^7\theta - 21cos^5\theta(1 - cos^2\theta) + 35cos^3\theta(1 - cos^2\theta)^2 - 7cos\theta(1 - cos^2\theta)^3$$

$$= cos^7\theta - 21cos^5\theta + 21cos^7\theta + 35cos^3\theta - 70cos^5\theta + 35cos^7\theta - 7cos\theta + 21cos^3\theta - 21cos^5\theta + 7cos^7\theta$$

$$= 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7cos\theta$$

$$\therefore \frac{cos7\theta}{cos\theta} = 64cos^6\theta - 112cos^4\theta + 56cos^2\theta - 7.$$

8. Prove that $\frac{1+cos7\theta}{1+cos\theta} = (x^3 - x^2 - 2x + 1)^2$ where $x = 2cos\theta$

Solution: We know that $1 + cos2\theta = 2cos^2\theta$

$$1 + cos7\theta = 2cos^2\left(\frac{7\theta}{2}\right)\left[since\ 1 + cos\theta = 2cos^2\left(\frac{\theta}{2}\right)\right.$$

Consider $\frac{1+cos7\theta}{1+cos\theta} = \frac{2cos^2\left(\frac{7\theta}{2}\right)}{2cos^2\left(\frac{\theta}{2}\right)} = \left[\frac{cos\left(\frac{7\theta}{2}\right)}{cos\left(\frac{\theta}{2}\right)}\right]^2$

Let $\emptyset = \frac{\theta}{2}$

$$\frac{1 + cos7\theta}{1 + cos\theta} = \left[\frac{cos7\emptyset}{cos\emptyset}\right]^2 \dots\dots\dots\dots\dots..(1)$$

Expanding $\frac{cos7\emptyset}{cos\emptyset}$ in terms of $\emptyset$ we get,

$$\frac{cos7\emptyset}{cos\emptyset} = 64cos^6\emptyset - 112cos^4\emptyset + 56cos^2\emptyset - 7$$

Substituting this in (1) we get

$$\frac{1 + cos7\theta}{1 + cos\theta} = [64cos^6\emptyset - 112cos^4\emptyset + 56cos^2\emptyset - 7]^2 \dots\dots..(2)$$

Given $x = 2cos\theta \therefore\ cos\theta = \frac{x}{2}$

$$1 + cos\theta = 1 + \frac{x}{2}$$

$$2cos^2\left(\frac{\theta}{2}\right) = \frac{2 + x}{2}$$

$$4cos^2\emptyset = x + 2$$

$$(2cos\emptyset)^2 = x + 2$$

$$\therefore 4cos^2\emptyset = (2cos\emptyset)^2 = x + 2$$

$$16cos^4\emptyset = (2cos\emptyset)^4 = (x + 2)^2$$

$$64cos^6\emptyset = (2cos\emptyset)^6 = (x + 2)^3$$

Substituting this in (2) we get

$$\frac{1 + cos7\theta}{1 + cos\theta} = [(x + 2)^3 - 7(x + 2)^2 + 14(x + 2) - 7]^2$$

$$= [x^3 + 6x^2 + 12x + 8 - 7x^2 - 28x + 28 + 14x + 28 - 7]^2$$

$$\therefore \frac{1 + cos7\theta}{1 + cos\theta} = [x^3 - x^2 - 2x + 1]^2$$

Expansion of $sin^n\theta$ and $cos^n\theta$ in terms of sines and cosines of multiples of , n being a positive integer.

Let $x = \cos\theta + isin\,\theta \ldots\ldots\ldots\ldots\ldots\ldots.(1)$

$\frac{1}{x} = x^{-1} = (\cos\theta + isin\,\theta)^{-1} = \cos\theta - isin\,\theta \ldots\ldots\ldots\ldots\ldots(2)$

$x^n = (\cos\theta + isin\,\theta)^n = \cos n\theta + isinn\theta \ldots\ldots\ldots\ldots\ldots\ldots.(3)$

$$\frac{1}{x^n} = (\cos\theta - isin\,\theta)^n = \cos n\theta - isinn\theta \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

From (1) and (2) we get

$$x + \frac{1}{x} = 2cos\theta$$

$$x - \frac{1}{x} = 2i\ sin\theta$$

From (3) and (4) we get

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Note:

1. **To get expansion of $cos^n\theta$, we have to consider** $(2cos\theta)^n = \left(x + \frac{1}{x}\right)^n$
2. **To get expansion of $sin^n\theta$, we have to consider** $(2isin\theta)^n = \left(x - \frac{1}{x}\right)^n$

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Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

3. To get expansion of $\boldsymbol{sin^m\theta\ cos^n\theta}$, we have to consider

$$(2isin\theta)^{\boldsymbol{m}}(2cos\theta)^{\boldsymbol{n}} = \left(x-\frac{1}{x}\right)^{\boldsymbol{m}}\left(x+\frac{1}{x}\right)^{\boldsymbol{n}}$$

Solved problems:

1. Prove that $sin^6\theta = -\frac{1}{32}[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10]$

Solution: Let $x = \cos\theta + isin\,\theta\, x^n = (\cos\theta + isin\,\theta)^n$

$$\frac{1}{x} = \cos\theta - isin\,\theta\,\frac{1}{x^n} = \cos n\theta - isinn\theta$$

$$x+\frac{1}{x} = 2cos\theta x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x-\frac{1}{x} = 2i\ sin\theta\ x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\therefore\ (2isin\theta)^{\boldsymbol{6}} = \left(x-\frac{1}{x}\right)^{\boldsymbol{6}}$$

$$= x^6 - 6C_1x^5\frac{1}{x} + 6C_2x^4\frac{1}{x^2} - 6C_3x^3\frac{1}{x^3} + 6C_4x^2\frac{1}{x^4} - 6C_5\ \mathrm{x}\ \frac{1}{x^5} + 6C_6\frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20$$

$$(2)^{\boldsymbol{6}}(i)^{\boldsymbol{6}}sin^6\theta = 2\ cos6\theta - 6(2\ cos4\theta) + 15(2\ cos2\theta) - 20$$

$$\therefore\ sin^6\theta = -\frac{1}{32}[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10].$$

2. Prove that $cos^5\theta = \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$

Solution: Let $x = \cos\theta + isin\,\theta\, x^n = (\cos\theta + isin\,\theta)^n$

$\frac{1}{x} = \cos\theta - isin\,\theta\,\frac{1}{x^n} = \cos n\theta - isinn\theta$

$$x + \frac{1}{x} = 2cos\theta x^n + \frac{1}{x^n} = 2\cos n\theta$$
$$x - \frac{1}{x} = 2i\,sin\theta\; x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\therefore\ (2cos\theta)^{\mathbf{5}} = \left(x + \frac{1}{x}\right)^{\mathbf{5}} = x^5 + 5C_1x^4\frac{1}{x} + 5C_2x^3\frac{1}{x^2} + 5C_3x^2\frac{1}{x^3} + 5C_4x\quad\frac{1}{x^4} + 5C_5\frac{1}{x^5}$$

$$= x^5 + 5x^3 + 10x\quad + 10\frac{1}{x} + 5\frac{1}{x^3} + \frac{1}{x^5}$$

$$2^5cos^5\theta = \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$$

$$2^5cos^5\theta = 2\,cos5\theta + 5(2\,cos3\theta) + 10(2\,cos\theta)$$

$$\therefore\ cos^5\theta = \frac{1}{2^4}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$$
$$= \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta].$$

3. Prove that $cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\,\theta + 35]$

Solution: Let $x = \cos\theta + isin\,\theta\, x^n = (\cos\theta + isin\,\theta)^n$

$\frac{1}{x} = \cos\theta - isin\,\theta\,\frac{1}{x^n} = \cos n\theta - isinn\theta$

$$x + \frac{1}{x} = 2cos\theta x^n + \frac{1}{x^n} = 2\cos n\theta$$
$$x - \frac{1}{x} = 2i\,sin\theta\; x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\therefore\ (2cos\theta)^{\mathbf{8}} = \left(x+\frac{1}{x}\right)^{\mathbf{8}}$$

$$= x^8 + 8C_1x^7\frac{1}{x} + 8C_2x^6\frac{1}{x^2} + 8C_3x^5\frac{1}{x^3} + 8C_4x^4\frac{1}{x^4} + 8C_5x^3\frac{1}{x^5} + 8C_6x^2\frac{1}{x^6}$$

$$+ 8C_7\,\mathrm{x}\frac{1}{x^7} + 8C_8\frac{1}{x^8}$$

$$2^8cos^8\theta \quad = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\frac{1}{x^2} + 28\frac{1}{x^4} + 8\frac{1}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8+\frac{1}{x^8}\right) + 8\left(x^6+\frac{1}{x^6}\right) + 28\left(x^4+\frac{1}{x^4}\right) + 56\left(x^2+\frac{1}{x^2}\right) + 70$$

$$= (2\ cos8\theta) + 8(2\ cos6\theta) + 28(2\ cos4\theta) + 56(2\ cos2\theta) + 70$$

$$\therefore\ cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\,\theta + 35].$$

4. Prove that $cos^5\theta sin^4\theta = \frac{1}{2^8}[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\,\theta + 6\cos\theta]$

Solution: Let $x = \cos\theta + isin\ \theta\ x^n = (\cos\theta + isin\ \theta)^n$

$$\frac{1}{x} = \cos\theta - isin\ \theta\,\frac{1}{x^n} = \cos n\theta - isinn\theta$$

$$x+\frac{1}{x} = 2cos\theta x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x-\frac{1}{x} = 2i\ sin\theta\ x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2cos\theta)^{\mathbf{5}}(2isin\theta)^{\mathbf{4}} = \left(x+\frac{1}{x}\right)^{\mathbf{5}}\left(x-\frac{1}{x}\right)^{\mathbf{4}}$$

$$2^9i^4cos^5\theta sin^4\theta = \left(x+\frac{1}{x}\right)^{\mathbf{4}}\left(x-\frac{1}{x}\right)^{\mathbf{4}}\left(x+\frac{1}{x}\right) = \left(x^2-\frac{1}{x^2}\right)^{\mathbf{4}}\left(x+\frac{1}{x}\right)$$

$$= \left[(x^2)^{\mathbf{4}} - 4C_1(x^2)^{\mathbf{3}}\frac{1}{x^2} + 4C_2(x^2)^{\mathbf{2}}\frac{1}{(x^2)^{\mathbf{2}}} - 4C_3x^2\frac{1}{(x^2)^{\mathbf{3}}} + 4C_4\frac{1}{(x^2)^{\mathbf{4}}}\right]\left(x+\frac{1}{x}\right)$$

$$= \left[x^8 - 4x^4 + 6 - 4\frac{1}{x^4} + \frac{1}{x^8}\right]\left(x + \frac{1}{x}\right)$$

$$= \left[x^9 - 4x^5 + 6x - 4\frac{1}{x^3} + \frac{1}{x^7} + x^7 - 4x^3 + \frac{6}{x} - 4\frac{1}{x^5} + \frac{1}{x^9}\right]$$

$$= \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right) - 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x + \frac{1}{x}\right)$$

$$2^9 cos^5\theta sin^4\theta = 2\cos 9\theta + 2\cos 7\theta - 4(2\cos 5\theta) - 4(2\cos 3\theta) + 6(2\cos\theta)$$

$$\therefore\ cos^5\theta sin^4\theta = \frac{1}{2^8}[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta]$$

5. Prove that $cos^5\theta sin^7\theta = -\frac{1}{2^{11}}[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta]$

Solution: Let $x = \cos\theta + isin\,\theta\, x^n = (\cos\theta + isin\,\theta)^n$

$$\frac{1}{x} = \cos\theta - isin\,\theta\,\frac{1}{x^n} = \cos n\theta - isinn\theta$$

$$x + \frac{1}{x} = 2cos\theta x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\, sin\theta\, x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2cos\theta)^{\mathbf{5}}(2isin\theta)^{\mathbf{7}} = \left(x + \frac{1}{x}\right)^{\mathbf{5}}\left(x - \frac{1}{x}\right)^{\mathbf{7}}$$

$$2^{12} i^7 cos^5\theta sin^7\theta = \left(x + \frac{1}{x}\right)^{\mathbf{5}}\left(x - \frac{1}{x}\right)^{\mathbf{5}}\left(x - \frac{1}{x}\right)^{\mathbf{2}} = \left(x^2 - \frac{1}{x^2}\right)^{\mathbf{5}}\left(x - \frac{1}{x}\right)^{\mathbf{2}}$$

$$= \left[(x^2)^{\mathbf{5}} - 5C_1(x^2)^{\mathbf{4}}\frac{1}{x^2} + 5C_2(x^2)^{\mathbf{3}}\frac{1}{(x^2)^{\mathbf{2}}} - 5C_3(x^2)^{\mathbf{2}}\frac{1}{(x^2)^{\mathbf{3}}} + 5C_4x^2\frac{1}{(x^2)^{\mathbf{4}}} - 5C_5\frac{1}{(x^2)^{\mathbf{5}}}\right]\left(x^2 + \frac{1}{x^2} - 2\right)$$

$$= \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left(x^2 + \frac{1}{x^2} - 2\right)$$

$$= \left[x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8} + x^8 - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}\right]$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$-i2^{12}cos^5\theta sin^7\theta = 2i[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta]$$

$$\therefore\ cos^5\theta sin^7\theta = -\frac{1}{2^{11}}[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta]$$

Expansions of $\cos\theta$ *and* $\sin\theta$ in ascending powers of θ:

1. $sin\,\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\infty$
2. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\infty$
3. $tan\,\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots\infty$

Note: θ should be in radians.

Problems:

1. If $\frac{sin\,\theta}{\theta} = \frac{5045}{5046}$, prove that the angle θ is $1^\circ 58'$ nearly.

Solution: We know that $sin\,\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\infty$

$$= \theta\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta^6}{7!} + \cdots\infty\right]$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta^6}{7!} + \cdots\infty$$

Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046} = 1 - \frac{1}{5046}$ which is nearly equal to 1.

Hence θ must be very small. Therefore omitting θ^4 and higher powers we get ,

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} = 1 - \frac{1}{5046}$$

$1 - \frac{\theta^2}{6} = 1 - \frac{1}{5046} \Rightarrow \frac{\theta^2}{6} = \frac{1}{5046}$

$\theta^2 = \frac{6}{5046} = \frac{1}{841}$

$\therefore\ \theta = \frac{1}{29}$radians $\Rightarrow \theta = \frac{1}{29} \times \frac{180}{\pi}$ $degrees$

$\therefore\ \theta = 1^{\circ}58'$nearly.

2. Solve approximately $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51$

Solution: Here 0.51 is nearly equal to $\frac{1}{2}$ which is of the value of $\sin\frac{\pi}{6}$.

$$\left(Since\ \sin\frac{\pi}{6} = \sin 30^{\circ} = \frac{1}{2}\right)$$

Hence θ is very small .

$$\sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

$$= \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\infty\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\infty\right)$$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta$[omitting$\theta^2$ and higher powers of θ

Hence the given equation becomes $\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$

$$\frac{\sqrt{3}}{2}\theta = 0.51 - 0.5 = 0.01 = \frac{1}{100}$$

$$\theta = \frac{2}{100\sqrt{3}} = \frac{1}{50\sqrt{3}} radian = \frac{\sqrt{3}}{150} radian$$

$$\theta = \frac{\sqrt{3}}{150} \times \frac{180}{\pi} = 39'$$

3. Evaluate $\lim_{x\to 0}\left[\frac{x-sinx}{x^3}\right]$

Solution:$\lim_{x\to 0}\left[\frac{x-sinx}{x^3}\right] = \lim_{x\to 0}\left[\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\infty\right)}{x^3}\right]$

$= \lim_{x\to 0}\left[\frac{\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\infty}{x^3}\right] = \lim_{x\to 0} x^3\left[\frac{\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}+\cdots\infty}{x^3}\right] = \frac{1}{3!} = \frac{1}{6}.$

4. Evaluate $\lim_{\theta\to 0}\left[\frac{\tan\theta - \sin\theta}{\theta^3}\right]$

Solution:$\lim_{\theta\to 0}\left[\frac{\tan\theta-\sin\theta}{\theta^3}\right] = \lim_{\theta\to 0}\left[\frac{\left(\theta+\frac{\theta^3}{3}+\frac{2}{15}\theta^5+\cdots\infty\right)-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\infty\right)}{\theta^3}\right]$

$$= \lim_{\theta\to 0}\left[\frac{\frac{\theta^3}{3}+\frac{2}{15}\theta^5+\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+ higherpowersof\theta}{\theta^3}\right]$$

$$= \lim_{\theta \to 0} \theta^3 \left[\frac{\left(\frac{1}{3}+\frac{1}{3!}\right)+\theta^2\left(\frac{2}{15}-\frac{1}{5!}\right)+higherpowersof\theta}{\theta^3} \right]$$

$$= \lim_{\theta \to 0} \left(\frac{1}{3}+\frac{1}{3!}\right)+\theta^2\left(\frac{2}{15}-\frac{1}{5!}\right)+higherpowersof\theta$$

$$= \frac{1}{3}+\frac{1}{3!} = \frac{1}{3}+\frac{1}{6}$$

$\therefore \lim_{\theta \to 0} \left[\frac{\tan\theta - \sin\theta}{\theta^3}\right] = \frac{1}{2}$.

Hyperbolic Functions:

If x be a real or complex, the expression

(i) $\frac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and it is denoted by sinhx.

$$\therefore sinhx = \frac{e^x - e^{-x}}{2}$$

Similarly $coshx = \frac{e^x + e^{-x}}{2}$

$$tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$$

$$cosechx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}$$

Relation between hyperbolic functions and circular functions:

We know that $sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ $andcos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Put $\theta = ix$ we get $sinix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right)$

$\therefore sinix = isinhx$

Similarly $cosix = coshx$, $tanix = itanhx$

Formulae of hyperbolic functions:

1. $cosh^2x - sinh^2x = 1$
2. $sech^2x + tanh^2x = 1$
3. $coth^2x - cosech^2x = 1$
4. $sinh(x \pm y) = sinhxcoshy \pm coshxsinhy$
5. $cosh(x \pm y) = coshxcoshy \pm sinhxsinhy$
6. $tanh(x \pm y) = \frac{tanhx \pm tanhy}{1 \pm tanhxtanhy}$
7. $sinh2x = 2sinhxcoshx$
8. $cosh2x = cosh^2x - sinh^2x = 2cosh^2x - 1 = 1 + 2sinh^2x$
9. $tanh2x = \frac{2tanhx}{1 + tanh^2x}$
10. $sinh3x = 3\,sinhx + 4sinh^3x$
11. $cosh3x = 4cosh^3x - 3\,coshx$
12. $tanh3x = \frac{3tanhx + tanh^3x}{1 + 3tanh^2x}$
13. $sinhx + sinhy = 2\,sinh\left(\frac{x+y}{2}\right)cosh\left(\frac{x-y}{2}\right)$
14. $sinhx - sinhy = 2\,cosh\left(\frac{x+y}{2}\right)sinh\left(\frac{x-y}{2}\right)$
15. $coshx + coshy = 2\,cosh\left(\frac{x+y}{2}\right)cosh\left(\frac{x-y}{2}\right)$
16. $coshx - coshy = 2\,sinh\left(\frac{x+y}{2}\right)sinh\left(\frac{x-y}{2}\right)$

Exponential function of a complex variable:

When x is real, we know that3

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

Similarly we can define the exponential function of the complex variable z = x+iy is

$$e^{x+iy} = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \infty \ldots\ldots\ldots\ldots\ldots (1)$$

Putting x = 0 in (1) ,we get

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \infty$$

$$= 1 + \frac{iy}{1!} - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} \ldots \infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots \infty\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots \infty\right)$$

$$e^{iy} = cosy + isiny$$

Real part of
$e^{iy} = cosy$ and imaginary part of $e^{iy} = siny$

Similarly $e^{-iy} = cosy - isiny$

$$\therefore e^z = e^{x+iy} = e^x e^{iy} = e^x[cosy + isiny]$$

Solved Problems:

Separate into real and imaginary parts of $(i)sin(x+iy)(ii)tan(x+iy)(iii)sec(x+iy)$

Solution: (i) We know that $sin(x+iy) = sinxcosiy + \cos x \, \text{siniy}$

$$Since\ sin(A + \text{B}) = sinAcos\text{B} + \text{cosAsinB}$$

$$= sinxcoshy + i\cos x \, \text{sinhy}[Sincecosiy = coshyandsiniy = isinhy]$$

Hence Real part = $sinxcoshy$

Imaginary part = $\cos x$ sinhy

(ii) We know that $tanh(x+iy) = \frac{sin(x+iy)}{cos(x+iy)}$

Multiply and divide by the conjugate of $cos(x+iy)$

$$= \frac{2sin(x+iy)cos(x-iy)}{2cos(x+iy)cos(x-iy)}$$

$$= \frac{sin(x+iy+x-iy)+sin(x+iy+x-iy)}{cos(x+iy+x-iy)+cos(x+iy-x+iy)}$$

$$= \frac{sin\ 2x + sin\ 2iy}{cos\ 2x + cos\ 2iy} = \frac{sin\ 2x + isinh2y}{cos\ 2x + cosh2y}$$

$$= \frac{sin\ 2x}{cos\ 2x + cosh2y} + i\frac{sinh2y}{cos\ 2x + cosh2y}$$

Hence Real part = $\frac{sin\ 2x}{cos\ 2x+cosh2y}$ and imaginary part = $\frac{sinh2y}{cos\ 2x+cosh2y}$

(ii) $sec(x+iy) = \frac{1}{cos(x+iy)} = \frac{cos(x-iy)}{cos(x+iy)cos(x-iy)}$

$$= \frac{cos\ x\ cos\ iy + sin\ x\ sin\ iy}{\frac{1}{2}[cos(x+iy+x-iy) + cos\ (x+iy-x+iy)]}$$

$$= \frac{2[cos\ x\ cos\ hy + isin\ x\ sin\ hy]}{cos\ 2x + cos\ 2iy} = \frac{2cos\ x\ cos\ hy}{cos\ 2x + cos\ h2y} + i\frac{2sin\ x\ sin\ hy}{cos\ 2x + cos\ h2y}$$

Hence Real part = $\frac{2cos\ x\ coshy}{cos\ 2x+cosh2y}$ and imaginary part = $\frac{2sin\ x\ sinhy}{cos\ 2x+cosh2y}$

2. Separate into real and imaginary parts of

$$(i)\cosh(x+iy)\ (ii)\coth(x+iy)\ (iii)\ cosech\ (x+iy)$$

Solution: (i) We know that , $coshx = cos\ ix$

$$\therefore\ \cosh(x+iy) = \cos i(x+iy) = cos(ix-y)$$

$$= \cos ix\ cosy + \sin ix\ siny = coshx\ cosy + isinhx\ siny$$

Hence Real part = $coshx\ cosy$ and imaginary part = $sinhx\ siny$

(ii) We know that $\coth(x+iy) = \frac{cosh(x+iy)}{sinh(x+iy)} = \frac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)} = \frac{\mathrm{i}\cos(ix-y)sin(-ix-y)}{\sin(ix-y)sin(-ix-y)}$

$$= \frac{\mathrm{i}2\cos(ix-y)\sin(ix+y)}{2\sin(ix-y)\sin(ix+y)} = i\frac{\boldsymbol{sin(ix+y+ix-y)+sin(ix+y-ix+y)}}{\boldsymbol{cos(ix-y-ix-y)-cos(ix-y+ix+y)}}$$

$$= i\frac{\sin 2ix+\sin 2y}{\cos 2y-\cos 2ix} = i\frac{(i\sinh 2x+\sin 2y)}{(\cos 2y-\cosh 2x)} = \frac{-\sinh 2x+i\sin 2y}{\cos 2y-\cosh 2x}$$

Hence Real part $=\frac{-\sinh 2x}{\cos 2y-\cosh 2x}$ and imaginary part = $\frac{\sin 2y}{\cos 2y-\cosh 2x}$

(iii) We know that $cosech\ (x+iy) = \frac{1}{sinh(x+iy)} = \frac{1}{\frac{1}{i}\sin i(x+iy)}$

$$= i\frac{1}{\sin(ix-y)} \times \frac{\sin(-ix-y)}{\sin(-ix-y)}$$

$$= i\frac{2\sin(ix+y)}{2\sin(ix-y)\sin(ix+y)} = \frac{2i\ (\sin ix\cos y + \cos ix \sin y)}{\boldsymbol{cos(ix-y-ix-y)-cos(ix-y+ix+y)}}$$

$$= \frac{2i\ (i\sinh x\cos y + \cosh x\sin y)}{\cos 2y - \cosh 2x} = \frac{-2\ sinhx\cos y + i\ 2coshx\sin y}{\cos 2y-\cosh 2x}$$

Hence Real part = $\frac{-2\ sinhx\cos y}{\cos 2y-\cosh 2x}$ and imaginary part = $\frac{2coshx\sin y}{\cos 2y-\cosh 2x}$

3. If $\tan\frac{x}{2} = \tanh\left(\frac{y}{2}\right)$, prove that $\cos x\cosh y = 1$

Solution: Consider $\cos x\cosh y = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}} \times \frac{1+tanh^2\frac{y}{2}}{1-tanh^2\frac{y}{2}}$

$$=\frac{1-tanh^2\frac{y}{2}}{1+tanh^2\frac{y}{2}}\times\frac{1+tanh^2\frac{y}{2}}{1-tanh^2\frac{y}{2}}=1$$

4. Separate real and imaginary parts of $\cosh(1+i)$

Solution: $\cosh(1+i)=\cos i(1+i)=cos(i-1)$

$$=\cos i\cos 1+\sin i\sin 1=\cos i1\cos 1+\sin i\ 1\sin 1$$

$$=\cos h1\cos 1+i\sin h\ 1\sin 1$$

Hence Real part $=\cos h1\cos 1$ and imaginary part $=\ \sin h\ 1\sin 1$

5. If $u+iv=\cosh(x+iy)\,, Prove\ that\ \frac{u^2}{cosh^2x}+\frac{v^2}{sinh^2x}=1\ and\ \frac{u^2}{cos^2y}-\frac{v^2}{sin^2y}=1$

Solution: Given $u+iv=\cosh(x+iy)=\cos i(x+iy)=\cos(ix-y)$

$$u+iv=\cos ix\ cosy+\sin ix\ siny=coshx\ cosy+isinhx\ siny$$

Separating the real and imaginary parts we get

$$u=coshx\ cosy\ \Rightarrow u^2=\ cosh^2xcos^2y\ \ldots\ldots\ldots\ldots\ldots..(1)$$

$$v=sinhx\ siny\ \Rightarrow u^2=\ sinh^2xsin^2y\ \ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{u^2}{cosh^2x}+\frac{v^2}{sinh^2x}=\frac{cosh^2xcos^2y}{cosh^2x}+\frac{sinh^2xsin^2y}{sinh^2x}$$

$$=cos^2y+sin^2y=1$$

Similarly $\frac{u^2}{cos^2y}-\frac{v^2}{sin^2y}=\frac{cosh^2xcos^2y}{cos^2y}-\frac{sinh^2xsin^2y}{sin^2y}=cosh^2x-sinh^2x=1$

6. If $\sin(\theta+i\emptyset)=\cos\alpha+i\sin\alpha\,, prove\ that\ cos^2\theta\ =\pm\sin\alpha$

Solution: Given $\sin(\theta + i\emptyset) = \cos\alpha + i\sin\alpha$

$$\sin\theta\cos i\emptyset + \cos\theta\sin i\emptyset = \cos\alpha + i\sin\alpha$$

$$\sin\theta\cos h\emptyset + i\cos\theta\sin h\emptyset = \cos\alpha + i\sin\alpha$$

Equating the real and imaginary parts we get,

$$\sin\theta\cos h\emptyset = \cos\alpha \Rightarrow \cos h\emptyset = \frac{\cos\alpha}{\sin\theta} \ldots\ldots\ldots\ldots\ldots (1)$$

$$\cos\theta\sin h\emptyset = \sin\alpha \quad \Rightarrow \sin h\emptyset = \frac{\sin\alpha}{\cos\theta} \ldots\ldots\ldots\ldots\ldots (2)$$

We know that $cosh^2\emptyset - sinh^2\emptyset = 1$

$$\frac{cos^2\alpha}{sin^2\theta} - \frac{sin^2\alpha}{cos^2\theta} = 1 \;\Rightarrow\; cos^2\alpha cos^2\theta - sin^2\alpha sin^2\theta = sin^2\theta cos^2\theta$$

$$(1 - sin^2\alpha)cos^2\theta - sin^2\alpha(1 - cos^2\theta) = (1 - cos^2\theta)cos^2\theta$$

$$cos^2\theta - sin^2\alpha cos^2\theta - sin^2\alpha + sin^2\alpha cos^2\theta = cos^2\theta - cos^4\theta$$

$$cos^4\theta = sin^2\alpha \;\Rightarrow\; (cos^2\theta)^2 = sin^2\alpha$$

$$\therefore\; cos^2\theta = \pm\sin\alpha$$

7. If $\tan\frac{\theta}{2} = \tanh\left(\frac{u}{2}\right)$, *prove that* $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Solution: Given $\tanh\left(\frac{u}{2}\right) = tan\frac{\theta}{2}$

$$\frac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = tan\frac{\theta}{2} \;\Rightarrow\; \frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{e^{\frac{u}{2}} - e^{-\frac{u}{2}}} = \frac{1}{tan\frac{\theta}{2}}$$

Using componendo and dividendo we get,

$$\frac{e^{\frac{u}{2}}+e^{-\frac{u}{2}}+e^{\frac{u}{2}}-e^{-\frac{u}{2}}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}-e^{\frac{u}{2}}+e^{-\frac{u}{2}}}=\frac{1+tan\frac{\theta}{2}}{1-tan\frac{\theta}{2}}$$

$$\frac{2e^{\frac{u}{2}}}{2e^{-\frac{u}{2}}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \Rightarrow e^{u}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \ u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

8. Express $cosh^5x$ in terms of hyperbolic cosines of multiples of x.

Solution: $cosh^5x=\left(\frac{e^x+e^{-x}}{2}\right)^5$

$$=\frac{1}{2^5}\Bigg[[(e^x)^5]+5C_1(e^x)^4\frac{1}{e^x}+5C_2(e^x)^3\frac{1}{(e^x)^2}+5C_3(e^x)^2\frac{1}{(e^x)^3}+5C_4e^x\frac{1}{(e^x)^4}+5C_5\frac{1}{(e^x)^5}\Bigg]$$

$$=\frac{1}{2^5}[e^{5x}+5e^{3x}+10e^{x}+10e^{-x}+5e^{-3x}+e^{-5x}]$$

$$=\frac{1}{2^5}[(e^{5x}+e^{-5x})+5(e^{3x}+e^{-3x})+10(e^{x}+e^{-x})]$$

$$=\frac{2}{2^5}[\cosh 5x+5cosh3x+10coshx]$$

$$\therefore cosh^5x=\frac{1}{2^4}[\cosh 5x+5cosh3x+10coshx]$$

9. Express $sinh^6x$ in terms of hyperbolic cosines of multiples of x

Solution:$sinh^6x=\left(\frac{e^x-e^{-x}}{2}\right)^6$

$$= \frac{1}{2^6}\left[(e^x)^6 - 6C_1(e^x)^5\frac{1}{e^x} + 6C_2(e^x)^4\frac{1}{(e^x)^2} - 6C_3(e^x)^3\frac{1}{(e^x)^3} + 6C_4(e^x)^2\frac{1}{(e^x)^4} - 6C_5e^x\frac{1}{(e^x)^5}^x + 6C_6\frac{1}{(e^x)^6}\right]$$

$$= \frac{1}{2^6}[e^{6x} - 6e^{4x} + 15e^{2x} - 20 + 15e^{-2x} - 6e^{-4x} + e^{-6x}]$$

$$= \frac{1}{2^5}[(e^{6x} + e^{-6x}) - 6(e^{4x} + e^{-4x}) + 15(e^{2x} + e^{-2x}) - 20]$$

$$= \frac{2}{2^6}[\cosh 6x - 6cosh4x + 15cosh2x - 10]$$

$$\therefore sinh^6x = \frac{1}{2^5}[\cosh 6x - 6cosh4x + 15cosh2x - 10]$$

10. If tan (θ + iϕ) = cosα+i sinα, then prove that (i) $\theta = n\pi/2 + \pi/4$

(ii) $\phi = ½ \, log \, tan\,(\alpha/2 + \pi/4)$

Given tan (θ + iϕ) = cosα+i sinα,

Changing i into –i, we get

tan (θ − iϕ) = cosα-i sinα,

$2\theta = (\theta + i\phi) + (\theta - i\phi)$

Taking tan on both sides

tan 2θ =tan[(θ+iϕ) +(θ-iϕ)]

$$\tan 2\theta = \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{(\cos\alpha+i\sin\alpha)+(\cos\alpha-i\sin\alpha)}{1-(\cos\alpha+i\sin\alpha)(\cos\alpha+i\sin\alpha)} = \frac{2\cos\alpha}{1-1} = \infty$$

$\tan 2\theta = \infty \Rightarrow \tan 2\theta = \tan\pi/2 \Rightarrow \theta = n\pi/2 + \pi/4$

$\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha,$

$$2i\phi = (\theta + i\phi) - (\theta - i\phi)$$

Taking tan on both sides

$\tan 2i\phi = \tan[(\theta+i\phi) - (\theta-i\phi)]$

$\tan 2i\phi = \frac{\tan(\theta+i\phi) - \tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1+(\cos\alpha+i\sin\alpha)(\cos\alpha+i\sin\alpha)} = i\sin\alpha$$

$i\tan h2\phi = i\sin\alpha$

$\tan h2\phi = \sin\alpha$

$$\frac{e^{2\emptyset} - e^{-2\emptyset}}{e^{2\emptyset} + e^{-2\emptyset}} = \frac{\sin\alpha}{1}$$

$$\frac{e^{2\emptyset} + e^{-2\emptyset}}{e^{2\emptyset} - e^{-2\emptyset}} = \frac{1}{\sin\alpha}$$

By componendo and dividend we get,

$$\frac{e^{2\emptyset}+e^{-2\emptyset}+e^{2\emptyset}-e^{-2\emptyset}}{e^{2\emptyset}+e^{-2\emptyset}-e^{2\emptyset}+e^{-2\emptyset}} = \frac{1+\sin\alpha}{1-\sin\alpha} \Rightarrow e^{4\emptyset} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$e^{4\emptyset} = \frac{1+\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{1-\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}} \Rightarrow [e^{2\emptyset}]^2 = \frac{(1+\tan\frac{\alpha}{2})^2}{(1-\tan\frac{\alpha}{2})^2} = \frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$$

$$e^{2\emptyset} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \Rightarrow \phi = ½ \log\tan(\alpha/2 + \pi/4)$$

Hence the proof.

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COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Exercise

1. Express $\dfrac{\sin 6\theta}{\sin\theta}$ as a polynomial in $\cos\theta$.
2. Show that $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.
3. Show that $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$, where $x = 2\cos\theta$.
4. Show that $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$, where $t = \tan\theta$.
5. Show that $2^6\sin^7\theta = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - 7\theta$.
6. Show that $\cos^7\theta = \dfrac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$.
7. Show that $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$
8. Show that $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.
9. Show that $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta)$.

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COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

UNIT-2

SUMMATION OF TRIGONOMETRIC SERIES

Logarithm of a complex number:

In complex analysis, a complex logarithm of the non-zero complex number z, denoted by w = log z, is defined to be any complex number w for which e^w= z.

Important result

Prove that $log(a+ib) = \ log\sqrt{a^2+b^2} \ + \ i \ tan^{-1}\frac{b}{a}$

Proof:

$$\text{Let } a+ib = \ r(\ cos\theta + i\ sin\theta) = re^{i\theta}$$

$$a = \ r\ cos\theta, \qquad b = r\ sin\theta$$

$$a^2+b^2 = r^2cos^2\theta + r^2sin^2\theta = r^2 \Rightarrow r = \sqrt{a^2+b^2}$$

$$\frac{b}{a} = \frac{r\ sin\theta}{r\ cos\theta} = tan\ \theta \Rightarrow \theta = tan^{-1}\frac{b}{a}$$

$$a+ib = \ re^{i\theta}$$

Taking log on both sides

$$log\ (a+ib) = \log\left(re^{i\theta}\right) = \log r + i\theta$$

$$log(a+ib) = \ log\sqrt{a^2+b^2} \ + \ i\ tan^{-1}\frac{b}{a}$$

Note: $Log\ (a+ib) = \ log\ (a+ib) + 2in\pi$

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DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Solved problems

1. Separate the real part and imaginary part of $log(4+3i)$

Sol:

$$log(4+3i) = log\sqrt{4^2+3^2} + i\,tan^{-1}\frac{3}{4}$$

$$log(4+3i) = log\sqrt{25} + i\,tan^{-1}\frac{3}{4}$$

$$log(4+3i) = \log 5 + i\,tan^{-1}\frac{3}{4}$$

Real part $= \log 5$ Imaginary part $= tan^{-1}\frac{3}{4}$

2. Find the expression (i) $Log\ 3$ (ii) $Log\ (-i)$ (iii) $Log(1+i)$

Sol: (i) $Log\ 3 = log\ (3+i0) + 2in\pi$

$$= log\ \sqrt{9} + i\,tan^{-1}\frac{0}{3} + 2in\pi = log\ 3 + 2in\pi$$

(ii) $Log\ (-i) = log\sqrt{0^2+(-i)^2} + i\,tan^{-1}\frac{-1}{0} + 2in\pi = log\ 1 + i\frac{\pi}{2} + 2in\pi$

$$= i\pi\left(\frac{1+4n}{2}\right)$$

(iii) $Log\ (1+i) = log\sqrt{1^2+1^2} + i\,tan^{-1}\frac{1}{1} + 2in\pi = log\sqrt{2} + i\frac{\pi}{4} + 2in\pi$

$$= log\sqrt{2} + i\pi\left(\frac{1+8n}{2}\right)$$

3. Show that $log\ (cos\theta + isin\theta) = i\theta$

Proof: $log(a+ib) = log\sqrt{a^2+b^2} + i\ tan^{-1}\frac{b}{a}$

$$log(cos\theta + isin\theta) = \frac{1}{2}log\sqrt{cos^2\theta + sin^2\theta} + i\ tan^{-1}\left(\frac{sin\theta}{cos\theta}\right)$$

$$= \frac{1}{2}log1 + i\ tan^{-1}tan\theta = 0 + i\theta = i\theta$$

4. If $i^{i^{\cdot^{\cdot^{\infty}}}} = x + iy$, then prove the following (i) y = $\tan\frac{\pi x}{2}$ (ii) $x^2+y^2 = e^{-\pi y}$

Solution: Given $i^{i^{\cdot^{\cdot^{\infty}}}} = A + iB$ means $i^{A+iB} = A + iB$ (since.... $i^{i^{\cdot^{\cdot^{\infty}}}} = A + iB$)

Taking logs on both sides, $(A + iB)\log i = \log(A + iB)$

$$(A+iB)\left[\frac{1}{2}\log 1 + i\tan^{-1}\infty\right] = \frac{1}{2}\log(A^2+B^2) + i\tan^{-1}\frac{B}{A}$$

$$(A+iB)\left[i\frac{\pi}{2}\right] = \frac{1}{2}\log(A^2+B^2) + i\tan^{-1}\frac{B}{A}$$

Now equating real and imaginary parts,

$$\frac{1}{2}\log(A^2+B^2) = \frac{-B\pi}{2} \quad \text{implying} \quad A^2 + B^2 = e^{-B\pi}$$

and $$\tan^{-1}\frac{B}{A} = A\frac{\pi}{2} \quad \text{implying} \quad \frac{B}{A} = \tan\frac{\pi A}{2}$$

5. Prove that the real part of principal value of $i^{\log(1+i)}$ is $e^{\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}log2\right)$

Solution: $\text{L.H.S.} = i^{\log(1+i)} = e^{\log(1+i)\cdot\log i} = e^{\left[\frac{1}{2}\log 2 + i\tan^{-1}1\right]\cdot\left[\log 1 + i\tan^{-1}\infty\right]}$

$$= e^{\left[\frac{1}{2}\log 2 + i\frac{\pi}{4}\right]\left[i\frac{\pi}{2}\right]} = e^{\left[-\frac{\pi^2}{8} + i\frac{\pi\log 2}{4}\right]}$$

$$= e^{-\frac{\pi^2}{8}} e^{i\frac{\pi\log 2}{4}} = e^{-\frac{\pi^2}{8}}\left[\cos\frac{\pi\log 2}{4} + i\sin\frac{\pi\log 2}{4}\right]$$

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INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

6. Find the real and imaginary part of $(\alpha + i\beta)^{x+iy}$

Solution: Substitute $\alpha = r\cos\theta$, $\beta = r\sin\theta$ so that $r = \sqrt{(\alpha^2+\beta^2)}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

Here $(\alpha + i\beta)^{x+iy} = e^{(x+iy)\mathrm{Log}(\alpha+i\beta)}$, (using $a^z = e^{z\log_e a}$)

$$= e^{(x+iy)[2in\pi + \log(\alpha+i\beta)}$$

$$= e^{(x+iy)[2in\pi + re^{i\theta}]}$$

$$= e^{(x+iy)[\log r + i(2n\pi+\theta)]}$$

$$= e^{A+iB} = e^{A}(\cos B + i\sin B)$$

where $A = x\log r - y(2n\pi+\theta)$ and $B = y\log r + x(2n\pi+\theta)$

$\therefore$ Required real part $= e^{A}\cos B$ and the imaginary part $= e^{A}\sin B$.

7. If $(a_1 + ib_1).(a_2 + ib_2).(a_3 + ib_3)\ldots.(a_n + ib_n)$ = A+iB

Prove that

(i) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

(ii) $(a_1^2 + b_1^2)(a_2^2 + b_2^2) + \ldots + (a_n^2 + b_n^2) = A^2 + B^2$

Solution:

Let $a_1 = r_1\cos\alpha_1$, $b_1 = r_1\sin\alpha_1$ with $\alpha_1 = \tan^{-1}\frac{b_1}{a_1}$;

$a_2 = r_2\cos\alpha_2$, $b_2 = r_2\sin\alpha_2$ with $\alpha_2 = \tan^{-1}\frac{b_2}{a_2}$;

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DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

$$a_n = r_n \cos\alpha_n, \qquad b_n = r_n \sin\alpha_n \text{ with } \alpha_n = \tan^{-1}\frac{b_n}{a_n};$$

$$A = r\cos\theta, \qquad B = r\sin\theta \quad \text{with } \theta = \tan^{-1}\frac{b}{a}$$

$$(a_1 + ib_1) \,.\, (a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$$

$$r_1(\cos\alpha_1 + i\sin\alpha_1)\, r_2(\cos\alpha_2 + i\sin\alpha_2) \ldots r_n(\cos\alpha_n + i\sin\alpha_n) = r(\cos\theta + i\sin\theta)$$

$$r_1 r_2 \ldots r_n[\cos(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + i\sin(\alpha_1 + \alpha_2 + \ldots + \alpha_n)] = r(\cos\theta + i\sin\theta)$$

Therefore, we see that modulus and amplitude are

$$r_1 r_2 \ldots r_n = r \text{ or } (a_1^2 + b_1^2)(a_2^2 + b_2^2) + \ldots (a_n^2 + b_n^2) = A^2 + B^2$$

and $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots\ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{b}{a}$

5. If $(a + ib)^p = m^{x+iy}$ Prove that $\frac{y}{x} = \frac{2\tan^{-1}(\frac{b}{a})}{\log(a^2+b^2)}$

Solution: Taking logs on both sides of $(a + ib)^p = m^{x+iy}$, we get

$$p\log(a + ib) = (x + iy)\log m$$

implying $p\left[\frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}\right] = (x + iy)\log m$

Now equating real and imaginary parts,

$$\frac{1}{2}p\log(a^2 + b^2) = x\log m \qquad \dots(i)$$

and $$p\tan^{-1}\frac{b}{a} = y\log m \qquad ..(ii)$$

Dividing the two, $\frac{y}{x} = \frac{p\tan^{-1}\frac{b}{a}}{\frac{p}{2}\log(a^2 + b^2)} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2 + b^2)}$

6. Prove that $tan\left[i\ log\ \left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Solution: Let $a = r\cos\theta$ and $b = r\sin\theta$ then $\tan\theta = \frac{b}{a}$, $r^2 = a^2 + b^2$ … (1)

Now, L.H.S. $= \tan\left[i\log\left(\frac{a - ib}{a + ib}\right)\right]$

$$= \tan i[\log(a - ib) - \log(a + ib)]$$
$$= \tan i[\log(re^{-i\theta}) - \log(re^{i\theta})]$$
$$= \tan i[\log r - i\theta) - (\log r + i\theta)]$$
$$= \tan i(-2i\theta) = \tan(-i^2 2\theta) = \tan 2\theta$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\frac{b}{a}}{1 - \frac{b^2}{a^2}} = \frac{2ab}{a^2 - b^2} \text{ (Using equation (1))}$$

7. If $tan[log\ (x+iy)] = a+ib$, then prove that $tan\ [log(x^2+y^2)] = \frac{2a}{1-a^2-b^2}$, $a^2+b^2 \neq 0$

Exercise

Solution: Given $\tan\log(x+iy) = a+ib$ implies $\log(x+iy) = \tan^{-1}(a+ib)$

Now $\log(x+iy) = \log r \operatorname{cis}\theta = \tan^{-1}(a+ib)$, where $r=\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

or $$\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} = \tan^{-1}(a+ib) \quad \ldots(1)$$

Identically, $$\frac{1}{2}\log(x^2+y^2) - i\tan^{-1}\frac{y}{x} = \tan^{-1}(a-ib) \quad \ldots(2)$$

Adding the two, $\log(x^2+y^2) = \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$

$$= \tan^{-1}\frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}$$

$$= \tan^{-1}\frac{2a}{1-a^2-b^2}$$

or $$\tan[\log(x^2+y^2)] = \frac{2a}{1-a^2-b^2}$$

1. Find the general value of (*i*) $\log(6+8i)$ (*ii*) $\log(-1)$ (*iii*) i^i
2. Show that (*i*) $\log(1+i\tan\alpha) = \log(\sec\alpha) + i\alpha$, where α is an

   (*ii*) $\log_e \frac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\frac{1}{3}\right)$
3. Find the modulus and argument (*i*) $(1+i)^{1-i}$ (*ii*) $\log(1+i)$
4. If $(a_1+ib_1)(a_2+ib_2)\ \ldots\ldots\ (a_n+ib_n) = A+iB$, prove that

   (*i*) $(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots(a_n^2+b_n^2) = A^2+B^2$

   (*ii*) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

5. Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2\tan^{-1}\frac{b}{a}$, hence evaluate $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]$.

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DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Gregory series

$$\tan\theta = \tan\theta - + \frac{tan^3\theta}{3} + \frac{tan^5\theta}{5} - \cdots, \quad (1)$$

Note:

1. Put $x = \ tan\,\theta$, then (1) becomes

$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots,$ …..(2)

2. Put $x = \ 1$, then (2) becomes

$tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots,$ …..(2)

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots,$

Solved problems

1. Prove that $\frac{1}{2} - \frac{1}{3}.\frac{1}{2^3} + \frac{1}{5}.\frac{1}{2^3} = tan^{-1}\frac{1}{2}$

Sol: we know that $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots,$

Put $x = \frac{1}{2}$

$$tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} - \cdots$$

$$\therefore \frac{1}{2} - \frac{1}{3}.\frac{1}{2^3} + \frac{1}{5}.\frac{1}{2^3} = tan^{-1}\frac{1}{2}$$

Hence the proof.

2. Prove that $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3}.\frac{1}{3} + \frac{1}{5}.\frac{1}{3^2} - \cdots$

Sol: we know that $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots,$

R.H.S $= 1 - \frac{1}{3}.\frac{1}{3} + \frac{1}{5}.\frac{1}{3^2} -$

$= 1 - \frac{1}{\sqrt{3}^2}.\frac{1}{3} + \frac{1}{5}.\frac{1}{\sqrt{3}^4} - \cdots$

Multiply and divide by $\sqrt{3}$, we get

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}^3}.\frac{1}{3} + \frac{1}{5}.\frac{1}{\sqrt{3}^5} - \cdots\right)$$

$= \sqrt{3}\left[tan^{-1}\frac{1}{\sqrt{3}}\right] = \sqrt{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}$. Hence the proof

Summation of trigonometric series by difference method

Suppose the r^th term T_r of the series is expressed as $T_r = f(r+1) - f(r)$. Then

$$T_1 = f(2) - f(1), T_2 = f(3) - f(2)$$

$$T_3 = f(4) - f(3) \ldots. \quad \ldots.. \quad \ldots..$$

$$\ldots..T_{n-1} = f(n) - f(n-1), T_n = f(n+1) - f(n)$$

Addding these n relations, we have $S_n = f(n+1) - f(n),\ where$

$$S_n = T_1 + T_2 + \cdots + T_n = sum\ of\ the\ 1st\ n\ terms\ of\ the\ series.$$

Solved problems

1. Create $sec\theta\ sec2\theta + sec2\theta sec3\theta + \cdots$ n terms

Sol:

Let $T_1 = sec\theta sec2\theta = \frac{1}{cos\theta}\frac{1}{cos2\theta}$

$$= \frac{1}{sin\theta}\left[\frac{sin\theta}{cos\theta cos2\theta}\right]$$

$$= \frac{1}{sin\theta}\left[\frac{\sin(2\theta-\theta)}{cos\theta cos2\theta}\right] = \frac{1}{sin\theta}\left[\frac{\sin 2\theta cos\theta - cos2\theta sin\theta}{cos\theta cos2\theta}\right]$$

$$T_1 = \frac{1}{sin\theta}[tan2\theta - tan\theta]$$

$$T_2 = \frac{1}{sin\theta}[tan3\theta - tan2\theta]$$

$$\vdots$$

$$T_n = \frac{1}{sin\theta}[\tan(n+1)\theta - tann\theta]$$

Adding all the terms

$$S_n = \frac{1}{sin\theta}[\tan(n+1)\theta - tan\theta]$$

2. Sum the series $\frac{1}{cos\theta+cos3\theta} + \frac{1}{cos\theta+cos5\theta} + \frac{1}{cos\theta+cos7\theta} + \cdots +$n terms

Sol: Let $T_1 = sec\theta sec2\theta = \frac{1}{cos\theta+cos3\theta}$

$$= \frac{1}{2\cos\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}$$

$$= \frac{1}{2cos\theta cos2\theta}$$

$$T_1 = \frac{1}{2sin\theta}[tan2\theta - tan\theta]$$

$$T_2 = \frac{1}{2sin\theta}[tan3\theta - tan2\theta]$$

$$\vdots$$

$$T_n = \frac{1}{2sin\theta}[\tan(n+1)\theta - tann\theta]$$

Adding all the terms,

$$S_n = \frac{1}{2sin\theta}[\tan{(n+1)\theta} - tan\theta]$$

3. Find $tan^{-1}\frac{x}{1+1.2x^2} + tan^{-1}\frac{x}{1+2.3x^2} + \cdots + tan^{-1}\frac{x}{1+n(n+1)x^2}$

Sol: Let $T_1 = tan^{-1}\frac{x}{1+1.2x^2} = tan^{-1}\frac{2x-x}{1+1.2x^2} = tan^{-1}2x - tan^{-1}x$

$$T_2 = tan^{-1}\frac{x}{1+2.3x^2} = tan^{-1}\frac{3x-2x}{1+2.3x^2} = tan^{-1}3x - tan^{-1}2x$$

$$\vdots$$

$$T_n = tan^{-1}(n+1)x - tan^{-1}nx$$

Adding all the terms, $S_n = tan^{-1}(n+1)x - tan^{-1}x$

$$= tan^{-1}\frac{(n+1)x-x}{1+(n+1)x^2} = tan^{-1}\frac{nx}{1+(n+1)x^2}$$

4. Sum to the series $\frac{sin\theta}{sin2\theta sin3\theta} + \frac{sin\theta}{sin3\theta sin4\theta} + \frac{sin\theta}{sin4\theta sin5\theta} + \cdots n$ terms

Sol: Let $T_1 = \frac{sin\theta}{sin2\theta sin3\theta} = \frac{\sin{(3\theta-2\theta)}}{sin2\theta sin3\theta} = \frac{\sin{3\theta}cos2\theta - cos3\theta sin2\theta}{sin2\theta sin3\theta}$

$$= cot2\theta - cot3\theta$$

Similarly, $T_2 = cot3\theta - cot4\theta$

$$T_3 = cot4\theta - cot3\theta$$

$$\vdots$$

$$T_n = \cot{(n+1)\theta} - \cot{(n+2)\theta}$$

Adding all the terms $S_n = \cot{2\theta} - \cot{(n+2)\theta}$

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COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Important formulae:

1. Exponential Series: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty, \quad -\infty < x < \infty$

2. Geometric Series (i) $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, r < 1$

(ii) $a + ar + ar^2 + \cdots \infty = \frac{a}{1-r},$

3. Trigonometric series: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots \infty$, $\sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots \infty$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \infty$, $\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots \infty$

4. Logarithmic series: (i) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty,$

$(ii) \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots \infty,$

5. Binomial series: $(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots \infty$

6. Gregory series: $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \dots$, $tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} \dots = \frac{1}{2} \log \frac{1+x}{1-x}$

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DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Summation of series by c+is method

Sum the series $x\sin\theta - \frac{1}{2}\cdot x^2\sin 2\theta + \frac{1}{3}\cdot x^3\sin 3\theta - \ldots\ldots\ldots\infty$

Solution: Let $S = x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}\cdot x^3\sin 3\theta - \ldots\ldots\ldots\infty$

$$C = x\cos\theta - \frac{1}{2}x^2\cos 2\theta + \frac{1}{3}\cdot x^3\cos 3\theta - \ldots\ldots\ldots\infty$$

Therefore,

$$C + iS = x(\cos\theta + i\sin\theta) - \frac{x^2}{2}(\cos 2\theta + i\sin 2\theta) + \frac{x^3}{3}(\cos 3\theta + i\sin 3\theta) - \ldots\ldots\infty$$

$$= xe^{i\theta} - \frac{x^2}{2}e^{2i\theta} + \frac{x^3}{3}e^{3i\theta} - \ldots\ldots\ldots\infty, \text{ (a logarithmic series)}$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots\infty = \log(1+z), \text{ where } z = xe^{i\theta}$$

$$= \log[1 + x(\cos\theta + i\sin\theta) = \log r \operatorname{cis}\theta$$

$$= \log\sqrt{\left[(1+x\cos\theta)^2 + (x\sin\theta)^2\right]} + i\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}$$

Equating imaginary parts, $S = \tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}$ [except, when $x\cos\theta = -1$]

6. Sum the series $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots\infty$ by C+iS method

Sol:

Let S= $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots\infty$

C= $\cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \cdots\infty$

Therefore,

C+iS = $[\cos\alpha + \sin\alpha] + x[\cos(\alpha+\beta) + i\,s\,in(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta) + sin2\beta)]+\ldots$

$$= e^{i\alpha} + \frac{xe^{i(\alpha+\beta)}}{1!} + \frac{x^2e^{i(\alpha+2\beta)}}{2!} + \cdots$$

$$= e^{i\alpha} + \frac{xe^{i\alpha}e^{i\beta}}{1!} + \frac{x^2e^{i\alpha}e^{i2\beta}}{2!} + \cdots$$

$$= e^{i\alpha}[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2e^{i2\beta}}{2!} + \cdots]$$

We Know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,\ -\infty < x < \infty$

$$\therefore \quad C + iS = e^{i\alpha}\,[1 + \frac{xe^{i\beta}}{1!} + \frac{(xe^{i\beta})^2}{2!} + \cdots]$$

$$= e^{i\alpha}.\,e^{ix[cos\beta + sin\beta]}$$

$$= e^{i(\alpha + xsin\beta)}.\,e^{xcos\beta}$$

On comparing, real and imaginary,

S= $[\cos(\alpha + x\,sin\,\beta) + i\sin(\alpha + x\,sin\beta)]e^{xcos\beta}$

7. Find the sum of the series (i) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots n$ terms

(ii)$\sin\alpha + \sin(\alpha+\beta) + s\,in(\alpha+2\beta) + \cdots n$

Solution: Given $C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots$ to n terms

$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots$ to n terms

Thus,

$$C + iS = e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)} + \ldots \text{ to } n \text{ terms}$$

$$= e^{i\alpha}(1 + e^{i\beta} + e^{2i\beta} + \ldots \text{ to } n \text{ terms})$$

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$$= e^{i\alpha}\left(\frac{1(1-e^{in\beta})}{1-e^{i\beta}}\right), \qquad \left(\because \text{sum of G.P. with } n \text{ terms} = \frac{a(1-r^n)}{1-r}\right)$$

$$= \frac{e^{i\alpha}(1-e^{in\beta})(1-e^{-i\beta})}{(1-e^{i\beta})(1-e^{-i\beta})}$$

$$= \frac{e^{i\alpha}\left[1-e^{in\beta}-e^{-i\beta}+e^{i(n-1)\beta}\right]}{(1-e^{i\beta}-e^{-i\beta}+1)}$$

$$C + iS = \left[\frac{e^{i\alpha} - e^{i(\alpha-\beta)} - e^{i(\alpha+n\beta)} + e^{i(\alpha+\underline{n-1})\beta}}{2-2\cos\beta}\right]$$

On taking real parts only,

$$C = \frac{\cos\alpha - \cos(\alpha-\beta) + \cos(\alpha+\underline{n-1}\beta) - \cos(\alpha+n\beta)}{2.2\sin^2\frac{\beta}{2}} \qquad \left(\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right)$$

$$= \frac{2\sin\frac{\alpha+(\alpha-\beta)}{2}\sin\frac{(\alpha-\beta)-\alpha}{2} + 2\sin\left(\frac{(\alpha+\underline{n-1}\beta)+(\alpha+n\beta)}{2}\right)\sin\left(\frac{(\alpha+n\beta)-(\alpha+\underline{n-1}\beta)}{2}\right)}{4\sin^2\frac{\beta}{2}}$$

$$= \frac{-2\sin\left(\frac{2\alpha-\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)+2\sin\left(\frac{\overline{2n-1}\beta+2\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}{4\sin^2\frac{\beta}{2}}$$

$$= \frac{\sin\left(\frac{\overline{2n-1}\beta+2\alpha}{2}\right)-\sin\left(\frac{2\alpha-\beta}{2}\right)}{2\sin\frac{\beta}{2}} \quad \text{(Using CD formula)}$$

$$= \frac{2\cos\left(\frac{\overline{2n-1}\beta+2\alpha}{2}+\frac{2\alpha-\beta}{2}\right)\sin\left(\frac{\overline{2n-1}+2\alpha}{2}-\frac{2\alpha-\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$C = \frac{\cos\left(\alpha+\frac{(n-1)}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Equating Imaginary part we get (ii)

(1) sum the series to infinity

$$a\cos\theta+\frac{a^2\cos 2\theta}{2}+\frac{a^3\cos 3\theta}{3}+\dots\dots\dots\infty$$

Let $C = a\cos\theta+\frac{a^2\cos 2\theta}{2}+\frac{a^3\cos 3\theta}{3}+\dots\dots\dots\infty$

$$S = a\sin\theta+\frac{a^2\sin 2\theta}{2}+\frac{a^3\sin 3\theta}{3}+\dots\dots\dots\infty$$

$$C + is = a(\cos\theta + i\sin\theta) + \frac{a^2}{2}(\cos 2\theta + i\sin 2\theta) + \frac{a^3}{3}(\cos 3\theta + i\sin 3\theta) + \ldots\ldots\ldots\ldots\infty$$

$$= ae^{i\theta} + \frac{a^2}{2}ae^{i2\theta} + \frac{a^3}{3}ae^{i3\theta} + \ldots\ldots\ldots\ldots\infty$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\ldots\ldots\infty$$

Where $x = ae^{i\theta}$

$$= -\log(1 - x)$$

$$= -\log(1 - ae^{i\theta})$$

$$= -\log[1 - a(\cos\theta + i\sin\theta)]$$

$$= -\log[1 - a\cos\theta - isain\theta]$$

$$= -\left[\frac{1}{2}\log[1 - a\cos\theta)^2 + a^2\sin^2\theta] + i\tan^{-1}\left(\frac{-a\sin\theta}{1 - a\cos\theta}\right)\right]$$

Equal to real part

$$C = -\frac{1}{2}\log\left[(1 - a\cos\theta)^2 + a^2\sin^2\theta\right]$$

$$= -\frac{1}{2}\log\left[1 + a^2\cos^2\theta + a^2\sin^2\theta - 2a\cos\theta\right]$$

$$= -\frac{1}{2}\log\left[1 + a^2 - 2a\cos\theta\right]$$

Exercise:

1. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots\ldots\ldots\infty$

2. $x\cos\theta - \frac{x^2}{2}\cos 2\theta + \frac{x^3}{3}\cos 3\theta - \ldots\ldots\ldots\infty$

3. $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots\ldots\ldots\infty$

4. $\sin^2\theta - \frac{1}{2}\sin 2\theta\sin^2\theta + \frac{1}{3}\sin 3\theta\sin^3\theta - \frac{1}{4}\sin 4\theta\sin^4\theta + \ldots\ldots\ldots$

5. $\cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \frac{x^3}{3!}\cos(\alpha+3\beta) + \ldots\ldots\ldots\infty$

   $\left[\text{Hint}: C + iS = e^{i\alpha}e^{z}, \text{ where } z = xe^{i\beta}\right]$

6. $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \frac{x^3}{3!}\sin(\alpha+3\beta) + \ldots\ldots\ldots\infty$

   $\left[\text{Hint}: C + iS = e^{i\alpha}e^{z}, z = xe^{i\beta}\right]$

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INSTITUTE OF SCIENCE AND TECHNOLOGY

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COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

UNIT-3

CONIC

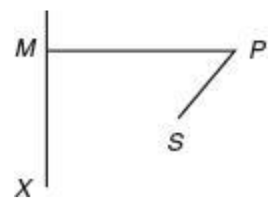
Introduction:

If a point moves in a plane such that its distance from a fixed point bears a constant ratio to its perpendicular distance from a fixed straight line then the path described by the moving point is called a conic. In other words, if *S* is a fixed point, *l* is a fixed straight line and *P* is a moving point and *PM* is the perpendicular distance from *P* on *l*, such that $\frac{SP}{PM} =$ a constant, then the locus of *P* is called a conic. This constant is called the eccentricity of the conic and is denoted by *e*.

If $e = 1$, the conic is called a parabola.

If $e < 1$, the conic is called an ellipse.

If $e > 1$, the conic is called a hyperbola.



The fixed point *S* is called the focus of the conic. The fixed straight line is called the directrix of the conic. The property $\frac{SP}{PM} = e$, is called the focus-directrix property of the conic.

General equation of a conic

We can show that the equation of a conic is a second degree equation in x and y. This is derived from the focus-directrix property of a conic. Let $S(x_1, y_2)$ be the focus and P(x, y) be any point on the conic and lx + my + n = 0 be the equation of the directrix. The focus-directrix property of the conic states

$$\frac{SP}{PM} = e \text{ (i.e) } SP^2 = e^2 PM^2$$

$$(x-x_1)^2 + (y-y_1)^2 = e^2\left(\frac{lx+my+n}{\sqrt{l^2+m^2}}\right)^2$$

This equation can be expressed in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which is a second degree equation in *x* and *y*.

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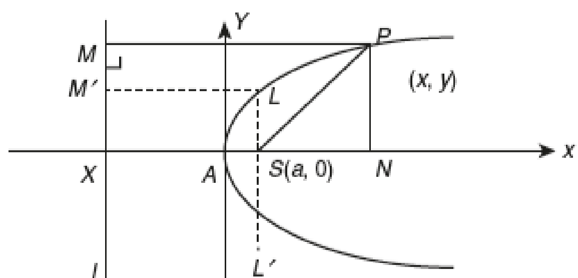
Equation of a parabola

Let S be the focus and the line l be the directrix. We have to find the locus of a point P such that its distance from the focus S is equal to its distance from the fixed line l.

(i.e.) $\frac{SP}{PM} = 1$, where PM is perpendicular to the directrix.

Draw SX perpendicular to the directrix and bisect SX. Let A be the point of bisection and SA = AX = a. Then the point A is a point on the parabola. Since take AS as the x -axis and AY perpendicular to AS as the y-axis. Then the coordinate of S are (a, 0). Let (x, y) be the coordinates of the point P. Draw PN perpendicular to the x-axis.



$$PM = NX = NA + AX = x + a$$

Since $$\frac{SP}{PM} = 1$$

$$SP^2 = PM^2$$

(i.e.) $$(x-a)^2 + y^2 = (x+a)^2$$

$$y^2 = (x+a)^2 - (x-a)^2 \text{ or } y_2 = 4ax.$$

This, being the locus of the point *P*, is the equation of the parabola. This equation is the simplest possible equation to a parabola and is called the standard equation of the parabola.

Note:

- The line AS(x-axis) is called the axis of the parabola.
- The point A is called the vertex of the parabola.
- AY(y-axis) is called the tangent at the vertex.
- The perpendicular through the focus is called the latus rectum.
- The double ordinate through the focus is called the length of the latus rectum.

- The equation of the directrix is x + a = 0.
- The equation of the latus rectum is x – a = 0.

Length of latus rectum

To find the length of the latus rectum, draw *LM′* perpendicular to the directrix. Then

$$\frac{SL}{LM'} = 1$$

$$\therefore SL = LM' = SX = 2a$$

$$\therefore LL' = 2SX = 4a$$

Tracing of the curve $y^2 = 4ax$

- If x < 0, y is imaginary. Therefore, the curve does not pass through the left side of y-axis.
- When y = 0, we get x = 0. Therefore, the curve meets the y-axis at only one point, that is, (0, 0).
- When x = 0, $y^2 = 0$, that is, y = 0. Hence the y-axis meets the curve at two coincident points (0, 0). Hence the y-axis is a tangent to the curve at (0, 0).
- If (x, y) is a point on the parabola $y^2 = 4ax$, (x, –y) is also a point. Therefore, the curve is symmetrical about the x-axis.

As x increases indefinitely, the values of y also increases indefinitely. Therefore the points of the curve lying on the opposite sides of x-axis extend to infinity towards the positive side of x-axis.

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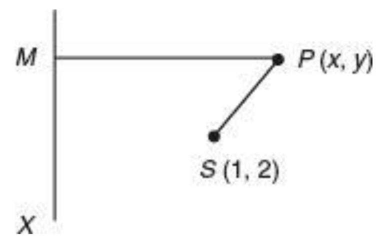
Solved Examples

1. Find the equation of the parabola with the following foci and directrix for (1, 2) : $x + y - 2 = 0$.

Solution

Let $P(x, y)$ be any point on the parabola. Draw PM perpendicular to the directrix.
Then from the definition of the parabola, $\frac{SP}{PM} = 1$



$SP^2 = (x - 1)^2 + (y - 2)^2$

PM = perpendicular distance from (x, y) on $x + y - 2 = 0$

$$= \pm\frac{(x+y-2)}{\sqrt{2}}$$

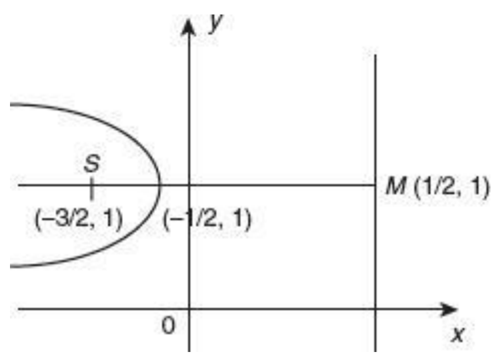
$$\therefore (x-1)^2 + (y-2)^2 = \left(\frac{x+y-2}{\sqrt{2}}\right)^2$$

$$\therefore 2(x^2 - 2x + 1) + 2(y^2 - 4y + 4) = x^2 + y^2 + 4 + 2xy - 4x - 4y$$

$$\text{(i.e.)} \quad x^2 + y^2 - 2xy - 4y + 6 = 0$$

2. Find the foci, latus rectum, vertices and directrices of the following parabola $y^2 + 4x - 2y + 3 = 0$

Solution



$$y^2 + 4x - 2y + 3 = 0$$
$$y^2 - 2y = -4x - 3$$
$$y^2 - 2y + 1 = -4x - 3 + 1$$

$$\Rightarrow (y-1)^2 = -4\left(x + \frac{1}{2}\right)$$

$$\frac{1}{2}$$

Take x + 1/2= X , y – 1 = Y . Shifting the origin to the point (-1/2, 1) the equation of the parabola becomes $Y^2 = -4X$.

Vertex is (-1/2, 1), latus rectum is 4, focus is (-3/2, 1) and foot of the directrix is (1/2, 1). The equation of the directrix is $x = 1/2$ or $2x-1=0$

Condition for tangency

4. Find the condition for the straight line y = mx + c to be a tangent to the parabola $y^2 = 4ax$ and find the point of contact.

Solution

The equation of the parabola is $y^2 = 4ax$ ……(1)

The equation of the line is $y = mx + c$ ……..(2)

Solving equations (1) and (2), we get their points of intersection. The x-coordinates of the points of intersection are given by

$$(mx + c)^2 = 4ax \Rightarrow m^2 x^2 + 2(mc - 2a)x + c^2 = 0 \quad \text{……..(3)}$$

If y = mx + c is a tangent to the parabola, then the roots of this equation are equal. The condition for this is the discriminant is equal to zero.

$$\therefore\ 4(mc - 2a)^2 = 4m^2c^2$$

$$\Rightarrow \cancel{m^2c^2} + 4a^2 - 4mca = \cancel{m^2c^2} \Rightarrow c = a/m$$

Hence, the condition for y = mx + c to be a tangent to the parabola y2 = 4ax is c = a/m.

Substituting c = a/m in equation (3), we get

$$m^2x^2 - 2ax + \frac{a^2}{m^2} = 0 \Rightarrow \left(mx - \frac{a}{m}\right)^2 = 0$$

Therefore, the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Note 6.6.1: Any tangent to the parabola is $y = mx + \frac{a}{m}$.

Number of tangents

5. Show that two tangents can always be drawn from a point to a parabola.

Solution

Let the equation to the parabola be $y^2 = 4ax$. Let (x1, y1) be the given point. Any tangent to the parabola is y=mx+a/m If this tangent passes through (x_1,y_1), then

$$y_1 = mx_1 + \frac{a}{m} \Rightarrow my_1 = m^2x_1 + a \text{ or } m^2x_1 - my_1 + a = 0.$$

This is a quadratic equation in m.

Therefore, there are two values of m and for each value of m there is a tangent. Hence, there are two tangents from a given point to the parabola.

Note: If m1, m2 are the slopes of the two tangents then they are the roots of equation (3).

$$m_1 + m_2 = -\frac{y_1}{x_1} \quad \text{and} \quad m_1m_2 = \frac{a}{x}$$

Perpendicular tangents

6. Show that the locus of the point of intersection of perpendicular tangents to a parabola is the directrix.

Solution

Let the equation of the parabola be $y^2 = 4ax$. Let (x_1, y_1) be the point of intersection of the two tangents to the parabola.

Any tangent to the parabola is $y = mx + \frac{a}{m}$ …….(4)

If this tangent passes through (x_1, y_1), then $y_1 = mx_1 + \frac{a}{m}$.

(i.e.) $m^2x_1 - my_1 + a = 0$ …….(5)

If m_1, m_2 are the slopes of the two tangents from (x_1, y_1), then they are the roots of equation (5). Since the tangents are perpendicular,

$$m_1 \cdot m_2 = -1 \Rightarrow \frac{a}{x_1} = -1 \text{ or } x_1 + a = 0$$

Therefore, the locus of (x_1, y_1) is $x + a = 0$, which is the directrix.

7. Show that the locus of the point of intersection of two tangents to the parabola that make complementary angles with the axis is a line through the focus.

Solution

Let (x_1, y_1) be the point of intersection of tangents to the parabola $y^2 = 4ax$.

Any tangent to the parabola is $y = mx + \frac{a}{m}$.

If this line passes through (x_1, y_1), then

$$y_1 = mx_1 + \frac{a}{m} \Rightarrow m^2x_1 - my_1 + a = 0.$$

If m_1, m_2 are the slopes of the two tangents, then $m_1 + m_2 = \frac{y_1}{x_1}, m_1m_2 = \frac{a}{x_1}$

If the tangents make complementary angles with the axis of the parabola, then $m_1 = \tan\theta$ and $m_2 = \tan(90 - \theta)$.

$$m_1m_2 = \tan\theta \times \tan(90 - \theta) = \tan\theta \times \cot\theta = 1$$

$$\Rightarrow \frac{a}{x_1} = 1 \quad \text{or} \quad x_1 - a = 0.$$

$\therefore$ The locus of the point of intersection of the tangents is $x - a = 0$, which is a straight line through the origin.

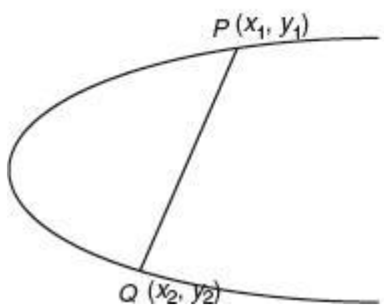
Equation of tangent

8. Find the equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$. Let $P(x_1, y))$ and

$Q(x_2, y_2)$ be two points on the parabola $y^2 = 4ax$. Then

$$y_1^2 = 4ax_1 \qquad \ldots\ldots (6)$$

$$y_2^2 = 4ax_2 \qquad \ldots\ldots. (7)$$



The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

From equations (6) and (7), we get

$$y_1^2-y_2^2=4a(x_1-x_2) \quad \text{or} \quad \frac{y_1-y_2}{x_1-x_2}=\frac{4a}{y_1+y_2}$$

Hence, the equation of the chord PQ is

$$\frac{y-y_1}{x-x_1}=\frac{4a}{y_1+y_2} \quad \text{or} \quad y-y_1=\frac{4a(x-x_1)}{y_1+y_2}$$

When the point $Q(x_2, y_2)$ tends to coincide with $P(x_1, y_1)$, the chord PQ becomes the tangent at P.

Hence, the equation of the tangent at P is

$$y-y_1=\frac{2a}{y_1}(x-x_1) \quad \text{or} \quad yy_1-y_1^2=2ax-2ax_1$$

$$yy_1=2ax-2ax_1+y_1^2=2ax-2ax_2+4ax_1$$

$$\text{(i.e)} \quad yy_1=2a(x+x_1)$$

Aliter: The equation of the parabola is $y^2 = 4ax$. Differentiating this equation with respect to x_1, we get $2y\frac{dy}{dx} = 4a$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\left[\frac{dy}{dx}\right]_{at\ (x_1, y_1)} = \frac{2a}{y_1} = \text{slope of the tangent at } (x_1, y_1)$$

The equation of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1) \Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$
$$\text{or} \quad yy_1 = 2ax - 2ax_1 + 4ax_1 = 2a(x + x_1)$$
$$\therefore \quad yy_1 = 2a(x + x_1)$$

Equation of normal

8. Find the equation of the normal at (x_1, y_1) on the parabola $y^2 = 4ax$.

Solution

The slope of the tangent at (x_1, y_1) is $\frac{2a}{y_1}$.

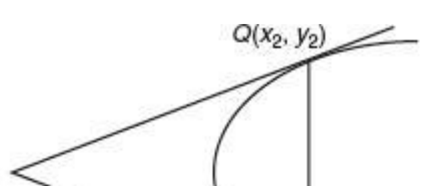
Therefore, the slope of the normal at (x_1, y_1) is $\frac{-y_1}{2a}$.

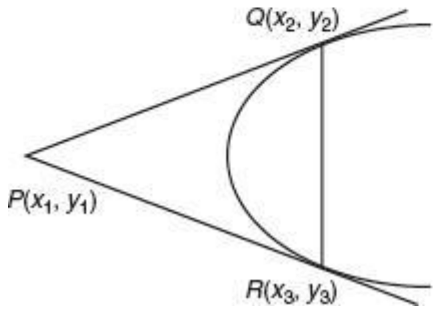
The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \Rightarrow 2ay - 2ay_1 = -xy_1 + x_1y_1$$
$$\text{or} \quad xy_1 + 2ay = 2ay_1 + y_1x_1$$

Equation of chord of contact

9. Find the equation of the chord of contact of tangents from (x_1, y_1) to the parabola $y^2 = 4ax$.





Solution

Let QR be the chord of contact of tangents from $P(x_1, y_1)$. Let Q and R be the points (x_2, y_2) and (x_3,y_3), respectively. Then, the equation of tangents at Q and R are

$$yy_2 = 2a\,(x + x_2) \quad \text{……... (8)}$$

$$yy_3 = 2a\,(x + x_3) \quad \text{……… (9)}$$

These two tangents pass through $P(x_1, y_1)$.

$$\therefore\ y_1 y_2 = 2a\,(x_1 + x_2) \quad \text{………(10)}$$

$$y_1 y_3 = 2a\,(x_1 + x_3) \quad \text{……..(11)}$$

These two equations show that the points (x_2, y_2) and (x_3,y_3) lie on the line $yy_1 = 2a(x+x_1)$.

Therefore, the equation of the chord of contact of tangents from $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

Parametric representation

The point at x= at^2, y = 2at satisfy the equation $y^2 = 4ax$. This means (at^2, 2at) is a point on the parabola. This point is denoted by 't' and t is called a parameter.

Chord joining two points

10. Find the equation of the chord joining the points (at_1^2, $2at_1$) and (at_2^2, $2at_2$) on the parabola $y^2 = 4ax$.

Solution

The equation of the chord joining the points is

$$\frac{y-2at_1}{x-at_1^2} = \frac{2at_1-2at_2}{at_1^2-at_2^2} = \frac{2a(t_1-t_2)}{a(t_1-t_2)(t_1+t_2)} = \frac{2}{t_1+t_2}$$

$$\therefore y(t_1+t_2) - 2at_1(t_1+t_2) = 2x - 2at_1^2$$

(i.e.) $y(t_1+t_2) = 2x - 2at_1^2 + 2at_1^2 + 2at_1t_2$

$$y(t_1+t_2) = 2x + 2t_1t_2$$

Note : The chord becomes the tangent at 't' if $t_1 = t_2 = t$. Therefore, the

equation of the tangent at t is $y(2t) = 2x + 2at^2$ or $yt = x + at^2$

Equations of tangent and normal

11. Find the equation of the tangent and normal at '*t*' on the parabola $y = 4ax$.

Solution

The equation of the parabola is $y2 = 4ax$. Differentiating with respect to *x*,

$$2y\frac{dy}{dx} = 4a \quad \text{(ie)} \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\left(\frac{dy}{dx}\right) \text{ at } (at^2, 2at) = \frac{2a}{2at} = \frac{1}{t} = \text{slope of the tangent at } t$$

The equation of the tangent at *t* is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$yt - 2at^2 = x - at^2 \quad \text{(i.e.)} \quad yt = x + at^2$$

The slope of the normal at t is $-t$. The equation of the normal at 't' is

$$y - 2at = -t(x - at^2)$$

(i.e.) $y - 2at = -xt + at^3$

(i.e.) $y + xy = 2at + at^3$

Point of intersection of tangents

12. Find the point of intersection of tangents at t_1 and t_2 on the parabola $y^2 = 4ax$.

Solution

The equation of tangents at t_1 and t_2 are

$$yt_1 = x + at_1^2$$
$$yt_2 = x + at_2^2$$

Subtrating, $y(t_1 - t_2) = a(t_1^2 - t_2^2)$

(i.e.) $y = a\,(t_1 + t_2)$.

$$\therefore a(t_1 + t_2)\,t_1 = x + at_1^2$$
$$\therefore x = at_1t_2$$

Hence, the point of intersection is $[at_1 t_2, a(t_1 + t_2)]$.

Number of normals from a point

13. Show that three normals can always be drawn from a given point to a parabola.

Solution

Let the equation of the parabola be $y^2 = 4ax$.

The equation of the normal at t is $y + xt = 2at + at^3$

If this passes through (x_1, y_1) then

$$y_1 + x_1 t = 2at + at^3$$

$$\therefore at^3 - t(2a - x_1) - y_1 = 0 \qquad \ldots\ldots(12)$$

This being a cubic equation in t, there are three values for t. For each value of t there is a normal from (x_1, y_1) to the parabola $y^2 = 4ax$.

Note : If t_1, t_2, t_3 are the roots of equation (12), then

$$t_1 + t_2 + t_3 = 0 \qquad \ldots\ldots\ldots(13)$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - x_1}{a} \qquad \ldots\ldots\ldots..(14)$$

$$t_1t_2t_3 = \frac{y_1}{a} \qquad \ldots\ldots\ldots..(15)$$

Note: From (12), $2at_1 + 2at_2 + 2at_3 = 0$

Therefore, the sum of the coordinates of the feet of the normal is always zero.

Solved Examples Based On Tangents And Normals

14. Find the equations of the tangent and normal to the parabola $y^2 = 4(x - 1)$ at (5, 4).

Solution

Given $y^2 = 4(x - 1)$

Differentiating with respect to x,

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\left(\frac{dy}{dx}\right)_{\text{at}(5,\,4)} = \frac{2}{4} = \frac{1}{2} = \text{Slope of the tangent at } (5, 4)$$

The equation of the tangent at (5, 4) is

$$y - 4 = \frac{1}{2}(x - 5).$$

$$2y - 8 = x - 5 \text{ or } x - 2y + 3 = 0.$$

The slope of the normal at (5, 4) is −2.
The equation of normal at (5, 4) is $y - 4 = -2(x - 5)$ or $2x + y = 14$.

15**.** Find the condition that the straight line $lx + my + n = 0$ is a tangent to the parabola.

Solution:

Any straight line tangent to the parabola $y^2 = 4ax$ is of the form $y = mx + c$ if

$c = \frac{a}{m}$. ……(16)

Consider the line $lx + my + n = 0$ (i.e.) $my = -lx - n$

$\Rightarrow y = \frac{-l}{m}x - \frac{n}{m}$

If this is a tangent to the parabola, $y2 = 4ax$ then $\frac{-n}{m} = \frac{-a}{(l/m)}$

$\Rightarrow \frac{-n}{m} = \frac{-am}{l}$ (i.e.) $am^2 = nl$

16. A common tangent is drawn to the circle $x^2 + y^2 = r^2$ and the parabola $y^2 = 4ax$. Show that the angle θ which it makes with the axis of the parabola is given by $\tan^2\theta = \frac{\sqrt{r^2+4a^2}-r}{2r}$.

Solution

Let $y = mx + c$ be a common tangent to the parabola

$y^2 = 4ax$ ……(17)

and the circle

$x^2 + y^2 = r^2$ ……..(18)

If $y = mx + c$ is tangent to the parabola (17) then

$$c = \frac{a}{m}$$

(i.e.) $y = mx + \frac{a}{m}$ …….(19)

If $y = mx + c$ is a tangent to the circle (19) then

$y = mx + r\sqrt{1+m^2}$ ……..(20)

Equations (19) and (20) represent the same straight line. Identifying we get,

$$\frac{a}{m} = r\sqrt{1+m^2} \Rightarrow a^2 = m^2r^2(1+m^2)$$

$$r^2m^4 + r^2m^2 - a^2 = 0$$

$$m^2 = \frac{-r^2 \pm \sqrt{r^4+4a^2r^2}}{2r^2} = \frac{-r^2 \pm r\sqrt{r^2+4a^2}}{2r^2}$$

Since m_2 has to be positive,

$$m^2 = \frac{-r+\sqrt{r^2+4a^2}}{2r} \text{ or } \tan^2\theta = \frac{\sqrt{r^2+4a^2}-r}{2r}$$

17. A straight line touches the circle $x2 + y2 = 2a2$ and the parabola $y2 = 8ax$. Show that its equation is $y= \pm(x + 2a)$.

Solution

The equation of the circle is $x^2 + y^2 = 2a^2$ …….(21)
The equation of the parabola is $y^2 = 4ax$ …….(22)

A tangent to the parabola (22) is $y = mx + \frac{2a}{m}$ …….(23)

A tangent to the circle (23) is $y = mx + \sqrt{2}a\sqrt{1+m^2}$ …….(24)
Equations (23) and (24) represent the same straight line. Identifying we get,

$$\sqrt{2}a\sqrt{1+m^2} = \frac{2a}{m}$$

$$2a^2m^2(1+m^2) = 4a^2$$

$$\text{or } m^4 + m^2 - 2 = 0$$

$$(m^2 - 1)(m^2 + 2) = 0$$

$m_2 = 1$ or -2; $m_2 = -2$ is impossible. So $m_2 = 1$ or $m = \pm 1$. The equation of the common tangent is $y = \pm x \pm 2a$. (i.e) $y = \pm(x + 2a)$

18. Prove that if two tangents to a parabola intersect on the latus rectum produced then they are inclined to the axis of the parabola at complementary angles.

Solution

Let (x_1, y_1) the equation of the parabola be $y^2 = 4ax$. Let $y = mx + \frac{a}{m}$ be any tangent to the parabola. Let the two tangents intersect at (a, y_1), a point on the latus rectum. Then (a_1, y_1) lies, on $y = mx + \frac{a}{m}$.

$\therefore y_1 = ma + \frac{a}{m}$ or $m^2a - my_1 + a = 0$ ……(25)

If m_1 and m_2 are the slopes of the two tangents to the parabola then $m_1 m_2 = 1$.
(i.e.) $\tan\theta \cdot \tan(90 - \theta) = 1$.
(i.e.) The tangent makes complementary angles to the axis of the parabola.

Note: If $\alpha = 90°$, the locus of (x_1, y_1) is $x + 4a = 0$.

19. If the normals at two points t_1, t_2 on the parabola $y^2 = 4ax$ intersect again at point on the curve show that $t_1 + t_2 + t_3 = 0$ and $t_1 t_2 = 2$ and the product of ordinates of the two points is $8a^2$.

Solution: The normals t_1 and t_2 meet at t_3.

$$\therefore t_3 = -t_1 - \frac{2}{t_1} \qquad \ldots\ldots (26)$$

$$t_3 = -t_2 - \frac{2}{t_2} \qquad \ldots\ldots(27)$$

Subtracting

$$0 = -(t_1 - t_2) + 2\frac{(t_1 - t_2)}{t_1 t_2}.$$

Since $t_1 - t_2 \neq 0$, $t_1 t_2 = 2$.

Solving equations (26) and (27), we get

$$2t_3 = -(t_1 + t_2) - 2\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

$$= -(t_1 + t_2) - 2\frac{(t_1 + t_2)}{t_1 t_2} = -(t_1 + t_2) - (t_1 + t_2)$$

$$\therefore t_1 + t_2 + t_3 = 0$$

20. Find the condition that the line $lx + my + n = 0$ is a normal to the parabola is $y^2 = 4ax$.

Solution

Let the line $lx + my + n = 0$ be a normal at 't'. The parabola is $y^2 = 4ax$. The equation of the normal at t is

$$y + xt = 2at + at^3 \qquad \ldots\ldots(28)$$

But the equation of the normal is given as

$$lx + my = -n \qquad \ldots\ldots(29)$$

Identifying equations (28) and (29), we have

$$\frac{t}{l} = \frac{1}{m} = \frac{2at + at^3}{-n} \therefore t = \frac{l}{m} \qquad (2a + at^2)l = 2\left(2a + \frac{al^2}{m^2}\right)l = -n$$

(i.e.) $al^3 + 2alm^2 + m^2 = 0$

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

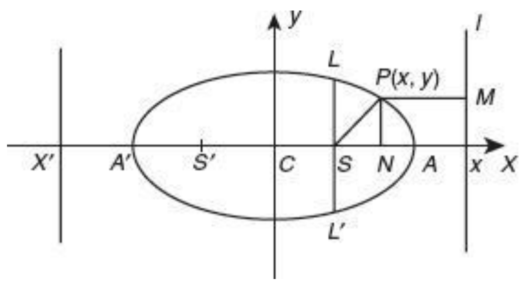
ELLIPSE

STANDARD EQUATION

Introduction

A conic is defined as the locus of a point such that its distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is called the focus and the fixed straight line is called the directrix. The constant ratio is called the eccentricity of the conic. If the eccentricity is less than unity the conic is called an ellipse. Let us now derive the standard equation of an ellipse using the above property called focus-directrix property.

Standard equation of an ellipse



Let S be the focus and line l be the directrix. Draw SX perpendicular to the directrix. Divide SX internally and externally in the ratio e:1 (e < 1).
Let A and A′ be the points of division. Since SA/AX=e and SA′/ A′X=e, from the definition of ellipse, the points A and A′ lie on the ellipse. Let AA′ = 2a and C be its middle point.

SA=eAX ……(1)

SA′=eA′X……(2)

Adding equations (1) and (2), we get SA + SA′ = e(AX + A′X).

(i.e.) $AA' = e(AX + A'X) = e(CX - CA + CX + CA$

$= e \cdot 2CX$ Since $CA = CA'$

$\therefore CX = \frac{a}{e}$ …..(3)

Subtracting equations (1) from (2), we get SA′ − SA = e(CX′ − CX)

$$CS + CA + (CS - CA) = e(AA')$$

$$2CS = e \cdot 2a \Rightarrow CS = ae \quad \text{.....(4)}$$

Take CS as the x-axis and CM perpendicular to CS, as y-axis.

Let P(x, y) be any point on the ellipse. Draw PM perpendicular to the directrix. Then the coordinates of S are (ae, 0). From the focus-directrix property of the ellipse, SP/PM=e

$$\therefore SP^2 = e^2 PM^2 = e^2 NX^2 = e^2(CX - CN)^2$$

$$\text{(i.e.)} \quad (x - ae)^2 + y^2 = e^2 (CX - CN)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

Dividing by $a^2(1 - e^2)$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$\text{(i.e.)} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2)$ ....(5)

This is called the standard equation of an ellipse.

Note : Equation (5) can be written as: $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} = \frac{(a+x)(a-x)}{a^2}$

$$\frac{PN^2}{b^2} = \frac{AN \cdot NX'}{a^2} \Rightarrow \frac{PN^2}{AN \cdot NA'} = \frac{b^2}{a^2} = \frac{BC^2}{AC^2}$$

- AA′ is called the major axis of the ellipse.
- BB′ is called the minor axis of the ellipse.
- C is called the centre of the ellipse.
- The curve meets the x-axis at the point A(a, 0) and A′(−a, 0).

- The curve meets the y-axis at the points B(0, b) and B′(0, −b).
- The curve is symmetrical about both the axes. If (x, y) is a point on the curve, then (x, −y) and (−x, y) are also the points on the curve.
- From the equation of the ellipse, we get
$$x = \pm\frac{a\sqrt{b^2 - y^2}}{b}, y = \pm\frac{b\sqrt{a^2 - x^2}}{a}$$

  Therefore, for any point (x, y) on the curve, −a ≤ x ≤ a and −b ≤ y ≤ b.
- The double ordinate through the focus is called the latus rectum of the ellipse. (i.e.) LSL′ is the latus rectum.
- $$\text{Length of latus rectum} = LL' = 2SL = 2e(CX - CS) = 2e\left(\frac{a}{e} - ae\right)$$
$$= 2a(1 - e^2).$$
$$= 2a \cdot \frac{b^2}{a^2} = \frac{2b^2}{a}$$
- **Second focus and second directrix:**

  On the negative side of the origin, take a point *S′* such that *CS* = *CS′* and another point *X′* such that *CX* = *CX′* = *a*.

  Draw *X′M′* perpendicular to *AA′* and *PM′* perpendicular to *X′M′*.
  Then we can show that $\frac{S'P}{PM'} = e$ gives the locus of *P* as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

  Here *S′* is called the second focus and *X′M′* is the second directrix.
- Shifting the origin to the focus *S*, the equation of the ellipse is $\frac{(x - ae)^2}{a^2} + \frac{y^2}{b^2} = 1.$
- Shifting the origin to *A*, the equation of the ellipse is $\frac{(x - a)^2}{a^2} + \frac{y^2}{b^2} = 1.$
- Shifting the origin to *X*, the equation of the focus is
$$\frac{\left(x - \frac{a}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation of an ellipse is easily determined if we are given the focus and the equation of the directrix

The sum of the focal distances of any point on the ellipse is equal to the length of the major axis.

In the above figure,

$$SP = ePM$$

$$S'P = ePM^2.$$

Adding $SP + S'P = e(PM + PM') = e(NX + NX')$

$$= e[CX - CN + CX' + CN] = e\left[\frac{a}{e} - x_1 + \frac{a}{e} + x_1\right]$$

$$= 2a \text{ where } P \text{ is } (x_1, y_1).$$

$$= \text{length of the major axis}$$

Note:

$$SP = ePM = a - ex_1$$

$$S'P = ePM' = a + ex_1$$

Position of a point:

A point (x1, y1) lies inside, on or outside of the ellipse according as $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ is negative, zero or positive.

Let Q(x1, y1) be a point on the ordinate PN where P is a point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Then,

$$QN < PN \Rightarrow \frac{QN^2}{b^2} < \frac{PN^2}{b^2} \Rightarrow \frac{y'^2}{b^2} < 1 - \frac{x'^2}{a^2}$$

$\therefore \frac{x'^2}{a^2} + \frac{y'^2}{b^2} < 1$

Similarly, if the $Q'(x', y')$ is a point outside the ellipse, $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} > 1.$ Evidently if

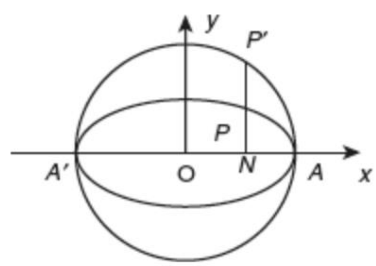
$Q(x', y')$ is a point on the ellipse, $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0.$

Auxiliary circle

The circle described on the major axis as diameter is called the auxiliary circle.

Let P be any point on the ellipse. Let the ordinate through P meet the auxiliary circle at P'. Since $\angle A'P'A = 90°$, we have the geometrical relation, $P'N2 = AN \cdot A'N$.

The point P' where the ordinate PN meets the auxiliary circle is called the corresponding point of P.
Therefore, the ordinate of any point on the ellipse to that of corresponding point on the ellipse are in the ratios of lengths of semi-minor axis and semi-major axis.
This ratio gives another definition to an ellipse.
Consider a circle and from each point on it, draw perpendicular to a diameter.



However, we know that $PN^2 = AN \cdot A'N$.

$\therefore PN^2 : P'N^2 = b^2 : a^2$

or

$\frac{PN}{P'N} = \frac{b}{a}$

The locus of these points dividing these perpendiculars in a given ratio is an ellipse and for this ellipse the given circle is the auxiliary circle.

Examples Based On Focus-Directrix Property

1. Find the equation of the ellipse whose foci, directrix and eccentricity are given below:
Focus is (1, 2), directrix is $2x - 3y + 6 = 0$ and eccentricity is 2/3

Solution

Let P (x_1, y_1) be a point on the ellipse. Then $\frac{SP}{PM} = e \Rightarrow \frac{SP^2}{PM^2} = e^2$

$$\therefore SP^2 = e^2 PM^2$$

$$SP = \sqrt{(x_1-1)^2+(y_1-2)^2},$$

$$PM = \pm\frac{2x_1-3y_1+6}{\sqrt{13}};\ e=\frac{2}{3}.$$

Hence, $$(x_1-1)^2+(y_1-2)^2 = \frac{4}{9}\left(\frac{2x_1-3y_1+6}{\sqrt{13}}\right)^2$$

$$117[x_1^2-2x_1+1+y_1^2-4y_1+4] = 4[4x_1^2+9y_1^2+36-12x_1y_1+24x_1-36y_1]$$

Therefore, the locus of (x_1, y_1) is the ellipse $101x^2 + 81y^2 + 48x - 330x - 324y + 441 = 0$

2. Find the equation of the ellipse whose Foci are (4, 0) and (−4, 0) and e=1/3

Solution

If the foci are $(ae, 0)$ and $(-ae, 0)$ then the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Here, $ae = 4$ and e=1/3

$$a=\frac{4}{e}=4\times 3=12$$

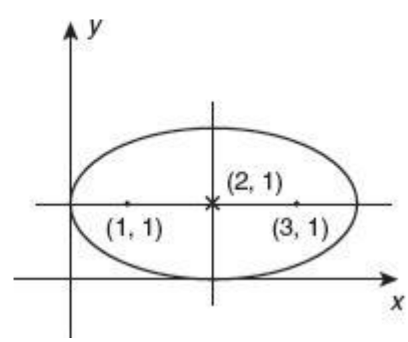
$$b^2=a^2(1-e^2)=144\left(1-\frac{1}{9}\right)=144\times\frac{8}{9}=128$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{144}+\frac{y^2}{128}=1$.

3. Find the eccentricity, foci and the length of the latus rectum of the ellipse $3x^2 + 4y^2 - 12x - 8y + 4 = 0$

Solution

$$3x^2+4y^2-12x-8y+4=0$$
$$(3x^2-12x)+(4y^2-8y)+4=0$$
$$3(x^2-4x)+4(y^2-2y)+4=0$$
$$3(x^2-4x+4)-12+4(y^2-2y+1)-4+4=0$$
$$\Rightarrow 3(x-2)^2+4(y-1)^2=12$$
$$\Rightarrow \frac{(x-2)^2}{4}+\frac{(y-1)^2}{3}=1$$



Shift the origin to the point (2, 1).
Therefore, centre is (2, 1).
Therefore, the equation of the ellipse is

$$\frac{X^2}{4}+\frac{Y^2}{3}=1$$

$$a^2=4,\ b^2=3$$

$$a^2e^2=a^2-b^2=4-3=1$$

$$4e^2=1 \text{ or } e^2=\frac{1}{4} \text{ or } e=\frac{1}{2} \quad \therefore ae=2\times\frac{1}{2}=1$$

Therefore, foci are (3, 1) and (1, 1) with respect to old axes.

Length of the latus rectum $=\frac{2b^2}{a}=2\times\frac{3}{2}=3.$

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INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Condition for tangency

To find the condition that the straight line $y = mx + c$ may be a tangent to the ellipse:

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{.......(6)}$$

Let the equation of the straight line be

$$y = mx + c \qquad \text{........(7)}$$

Solving equations (6) and (7), we get their points of intersection; the x-coordinates of the points of intersection are given by $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1.$

(i.e.) $b^2x^2 + a^2(mx + c)^2 = a^2b^2$

(i.e.) $x^2(b^2 + a^2m^2) + 2cma^2x + a^2(c^2 - b^2) = 0$

If $y = mx + c$ is a tangent to the ellipse then the two values of x of this equation are equal. The condition for that is the discriminant of the quadratic equation is zero.

$$\therefore 4a^4m^2c^2 - 4(a^2m^2 + b^2)\cdot a^2(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$a^2m^2c^2 - (a^2m^2c^2 - a^2m^2b^2 + b^2c^2 - b^4) = 0$$

or

$$b^2c^2 = a^2m^2b^2 + b^4$$

$$\therefore c^2 = a^2m^2 + b^2$$

This is the required condition for the line $y = mx + c$ to be a tangent to the given ellipse.

Note : The equation of any tangent to the ellipse is given by
$y = mx + \sqrt{a^2m^2 + b^2}$.

To show that always two tangents can be drawn from a given point to an ellipse and the locus of point of intersection of perpendicular tangents is a circle:

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad …….(9)$$

Any tangent to this ellipse is

$$y=mx+\sqrt{a^2m^2+b^2} \qquad …….(10)$$

If this tangent passes through the point (x_1, y_1) then $y1 = mx_1+\sqrt{a^2m^2+b^2}$.

(i.e.) $(y_1-mx_1)^2=a^2m^2+b^2$

$$\therefore m^2(x_1^2-a^2)-2mx_1y_1+y_1^2-b^2=0 \qquad …..(11)$$

This is a quadratic equation in *m* and hence there are two values for *m*. For each value of *m*, there is a tangent (real or imaginary) and hence there are two tangents from a given point to an ellipse. If m_1 and m_2 are the roots of the equation (11),

then $m_1+m_2=\frac{-2x_1y_1}{x_1^2-a^2}$, $m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$.

If the two tangents are perpendicular then $m_1m_2=-1$.

$$\therefore \frac{y_1^2-b^2}{x_1^2-a^2}=-1 \text{ or } y_1^2-b^2=-x_1^2+a^2$$

$$\Rightarrow x_1^2+y_1^2=a^2+b^2$$

The locus of (x_1, y_2) is $x^2+y^2=a^2+b^2$ which is a circle, centre at (0, 0) and radius $\sqrt{a^2+b^2}$.

Note : This circle is called the director circle of the ellipse.

Equation of the tangent

To find the equation of the chord joining the points ($x1, y1$) and ($x2, y2$) and find the equation of the tangent at ($x1, y1$) to the ellipse

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the ellipse. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Then $\frac{{x_1}^2}{a^2} + \frac{{y_1}^2}{b^2} = 1$ and $\frac{{x_2}^2}{a^2} + \frac{{y_2}^2}{b^2} = 1$

Subtracting,

$$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} + \frac{(y_1 - y_2)(y_1 + y_2)}{b^2} = 0 \qquad \text{……(15)}$$

From equation (15), we get the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) as:

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b^2}{a^2}\left(\frac{x_1 + x_2}{y_1 + y_2}\right) \qquad \text{……..(16)}$$

This chord becomes the tangent at (x_1, y_1) if Q tends to P and coincides with P. Hence, by putting $x_2 = x_1$ and $y_2 = y_1$ in equation (16), we get the equation of the tangent at (x_1, y_1).

Therefore, the equation of the tangent at (x_1, y_1) is:

$$\frac{y - y_1}{x - x_1} = \frac{-b^2 x_1}{a^2 y_1}$$

or $a^2 y y_1 - a^2 y_1^2 = -b^2 x x_1 + b^2 x_1^2$

$$b^2 x x_1 + a^2 y y_1 = b^2 x_1^2 + a^2 y_1^2$$

Dividing by a^2 b^2, we get

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \qquad \text{………(17)}$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1.$$

since (x_1, y_1) lies on the ellipse.

Equation of tangent and normal

To find the equation of tangent and normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{.....(18)}$$

Differentiating with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}$$

However, $\frac{dy}{dx}$ at (x_1, y_1) = slope of the tangent at (x_1, y_1). Therefore, the equation of the tangent at (x_1, y_1) is, $y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$

$$\text{or } a^2yy_1-a^2y_1^2=-b^2xx_1+b^2x_1^2$$

$$\text{or } a^2yy_1+b^2xx_1=b^2x_1^2+a^2y_1^2$$

Dividing by a^2b^2, we get

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \text{since } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

Slope of the normal at (x_1, y_1) is $\frac{a^2y_1}{b^2x_1}$.

The equation of the normal at (x_1, y_1) to the ellipse is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$$

$$\text{or} \quad b^2x_1y-a^2xy_1=(b^2-a^2)x_1y_1$$

Dividing by x_1y_1, we get,

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

Therefore, the equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

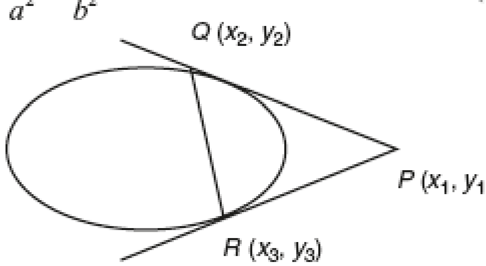
is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$

Equation to the chord of contact

To find the equation to the chord of contact of tangents drawn from $(x1, y1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ (



Let QR be the chord of contact of tangents from $P(x_1, y_1)$. Let Q and R be the points (x_2, y_2) and (x_3, y_3), respectively. Then the equation of tangentsat Q and R are

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \quad ....(20)$$

$$\frac{xx_3}{a^2}+\frac{yy_3}{b^2}=1 \quad ......(21)$$

These two tangents pass through $P(x_1, y_1)$.

Therefore, $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$ and $\frac{x_1x_3}{a^2}+\frac{y_1y_3}{b^2}=1.$
The above two equations show that the points(x_2, y_2) and (x_3,y_3) lie on the line $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. Hence, the equation of the chord of contact is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Solved Examples On Tangent Nd Normal

4. Find the equation of the tangent to the ellipse $x^2 + 2y^2 = 6$ at $(2, -1)$.

Solution

The equation of the ellipse is $x^2 + 2y^2 = 6$.

(i.e.) $\frac{x^2}{6}+\frac{y^2}{3}=1$

The equation of the tangent at $(x1, y1)$ is $\frac{xx_1}{6}+\frac{yy_1}{3}=1.$ Therefore, the equation of the tangent at $(2, -1)$ is $\frac{2x}{6}-\frac{y}{3}=1.$

(i.e.) $2x - 2y = 6 \Rightarrow x - y = 3$

5. Find the equation of the normal to the ellipse $3x^2 + 2y^2 = 5$ at $(-1, 1)$.

Solution

Therefore, the equation of the normal to the ellipse $3x^2 + 2y^2 = 5$ is $2x + 3y = 1$.

The equation of the ellipse is $3x^2 + 2y^2 = 5$.

$$\text{(i.e.)} \quad \frac{x^2}{\frac{5}{3}} + \frac{y^2}{\frac{5}{2}} = 1$$

The equation of the normal at (x_1, y_1) is $\frac{ax^2}{x_1} - \frac{by^2}{y_1} = a^2 - b^2$.

$$\frac{5}{3} \cdot \frac{x}{(-1)} - \frac{5}{2} \cdot \frac{y}{1} = \frac{5}{3} - \frac{5}{2}.$$

$$\frac{-5x}{3} - \frac{5y}{2} = \frac{5}{3} - \frac{5}{2} \Rightarrow \frac{-5x}{3} - \frac{5y}{2} = \frac{10-15}{6}$$

$$\Rightarrow \frac{-5x}{3} - \frac{5y}{2} = \frac{-5}{6} \Rightarrow 2x + 3y = 1$$

Therefore, the equation of the normal to the ellipse $3x^2 + 2y^2 = 5$ is $2x + 3y = 1$.

6. Find the condition that the line lx+my+n=0 is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

Let the line be tangent at the point θ

The equation of the tangent at θ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ …….(1)

But the equation of the tangent is lx+my=-n …….(2)

Equation (1) and (2) represents the same line.

∴ Solving we get $\frac{\cos\theta}{al} = \frac{\sin\theta}{bm} = \frac{-1}{n}$

$$\cos\theta = \frac{-al}{n} \quad …….(3)$$

$$\sin\theta = \frac{-bm}{n} \quad ………(4)$$

Squaring and adding (3) & (4), $\frac{a^2l^2}{n^2} + \frac{b^2m^2}{n^2} = 1$

$a^2l^2 + b^2m^2 = n^2$, which is the required condition.

7. Find the condition that the line lx+my+n=0 is normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

Let the line be normal at the point θ
The equation of the normal at θ is $\frac{ax}{cos\theta} - \frac{by}{sin\theta} = a^2 - b^2$ …….(1)
But the equation of the nomral is lx + my = -n …….(2)

Equation (1) and (2) represents the same line.

∴Solving we get $\frac{a}{lcos\theta} = \frac{-b}{msin\theta} = \frac{a^2-b^2}{-n}$

$$cos\theta = \frac{-an}{l(a^2-b^2)} \quad …….(3)$$

$$sin\theta = \frac{-bn}{m(a^2-b^2)} \quad ………(4)$$

Squaring and adding (3) & (4), $\frac{a^2n^2}{l^2(a^2-b^2)^2} + \frac{b^2m^2}{m^2(a^2-b^2)^2} = 1$

$a^2/l^2 + b^2/m^2 = (a^2 - b^2)^2/n^2$ which is the required condition.

HYPERBOLA

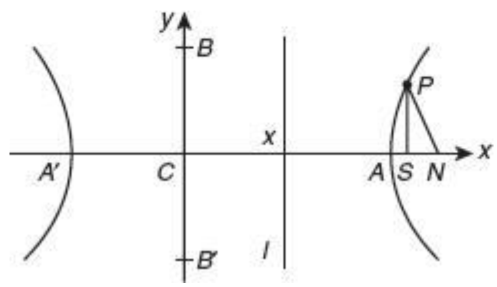
Introduction

A hyperbola is defined as the locus of a point that moves in a plane such that its distance from a fixed point is always *e* times (*e* > 1) its distance from a fixed line. The fixed point is called the focus of the hyperbola. The fixed straight line is called the directrix and the constant *e* is called the eccentricity of the hyperbola.

Standard equation

Let *S* be the focus and the line *l* be the directrix. Draw *SX* perpendicular to the directrix.
Divide *SX*internally and externally in the ratio *e* : 1 (*e* > 1).

Let A and A' be the point of division. Since $\frac{SA}{AX} = e$ and $\frac{SA'}{A'X} = e,$ the points A and A' lie on the curve.



Let $AA' = 2a$ and C be its middle point.

$SA = e \cdot AX$ …….(1)

$SA' = e \cdot A'X$ …….(2)

Adding equations (1) and (2), we get

$$SA + SA' = e(AX + A'X)$$
$$(CS - CA) + (CS + CA') = eAA'$$
$$2CS = e \cdot 2a$$
$$\therefore CS = ae$$

Subtracting equation (1) from equation (2), we get

$$SA' - SA = e(A'X - AX)$$
$$AA' = e \cdot 2CX$$
$$2a = e \cdot 2CX$$
$$\therefore CX = \frac{a}{e}$$

Take CS as the x -axis and CY perpendicular to CX as the y- axis. Then, the coordinates of S are $(ae, 0)$. Let $P(x, y)$ be any point on the curve.
Draw PM perpendicular to the directrix and PN perpendicular to x-axis. From the focus directrix property of hyperbola, $\frac{SP}{PM} = e.$

$$SP^2 = e^2 \cdot PM^2$$
$$\Rightarrow (x - ae)^2 + y^2 = e^2 \cdot NX^2 = e^2(CN - CX)^2$$
$$= e^2\left(x - \frac{a}{e}\right)^2$$
$$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$
$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$$

Dividing by $a^2(e^2 - 1)$, we get

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

(i.e.) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$

This is called the standard equation of hyperbola.

Note :

- The curve meets the *x*-axis at points (a, 0) and ($-a$, 0).
- When $x = 0$, $y2 = -a2$. Therefore, the curve meets the *y*-axis only at imaginary points, that is, there are no real points of intersection of the curve and *y*-axis.
- If (x, y) is a point on the curve, (x, $-y$) and ($-x$, y) are also points on the curve. This shows that the curve is symmetrical about both the axes.
- For any value of *y*, there are two values of *x*; as *y* increases, *x* increases and when $y \to \infty$, x also $\to \infty$. The curve consists of two symmetrical branches, each extending to infinity in both the directions.

- AA' is called the **transverse axis** and its **length** is **2*a***.
- BB' is called the **conjugate axis** and its **length** is **2*b***.
- A hyperbola in which $a = b$ is called a **rectangular hyperbola**. Its equation is

  $x^2 - y^2 = a^2$. Its eccentricity is $e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{2}$.
- The double ordinate through the focus S is called **latus** rectum and its length is $\frac{2b^2}{a}$.
- There is a second focus S' and a second directrix l' to the hyperbola.

Equation of hyperbola in parametric form

($a \sec \theta$, $b \tan \theta$) is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of θ, θ is called a parameter and is denoted by 'θ'. The parametric equations of hyperbola are $x = a \sec \theta$, $y = b \tan \theta$.

Standard results of the hyperbola whose equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **:**

- The equation of the tangent at (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.
- The equation of the normal at (x_1,y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.
- The equation of the chord of contact of tangents from (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.
- The polar of (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.
- The condition that the straight line y = mx + c is a tangent to the hyperbola is $c^2 = a^2 m^2 - b^2$ and $\sqrt{a^2m^2-b^2}$ is the equation of a tangent.
- The equation of the chord of the hyperbola having (x_1,y_1) as the midpoint is $T=S_1$ (i.e.) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ or $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$.
- The equation of the pair of tangents from (x_1,y_1) is $T_2 = SS_1$ (i.e.) $\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$

Parametric representation: $x = a \sec\theta$, $y = b\tan\theta$ is a point on the hyperbola and this point is denoted by θ. Θ is called a parameter of the hyperbola.

- The equation of the tangent at θ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.
- The equation of the normal at θ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$.
- The circle described on the transverse axis as diameter is called the auxiliary circle and its equation is $x^2 + y^2 = a^2$.

- The equation of the director circle (the locus of the point of intersection of perpendicular tangents) is $x^2 + y^2 = a^2 - b^2$

Condition for tangency

To find the condition that the straight line *y* = *mx* + *c* may be a tangent to the hyperbola:

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ……(1)

Let the equation of the straight line be $y = mx + c$ ……(2)

Solving equations (1) and (2), we get their points of intersection; the *x*-coordinates of the points of intersection are given by

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow x^2(b^2 - m^2) - 2a^2mcx - a^2(b^2 + c^2) = 0$$

If $y = mx + c$ is a tangent to the hyperbola then the roots of the equations are normal

$$4a^4m^2c^2 + 4(b^2 - a^2m^2)a^2(b^2 + c^2) = 0$$
$$\therefore c^2 = a^2m^2 - b^2$$

This is the required condition for the line $y = mx + c$ to be a tangent to the given hyperbola .

Note : The equation of any tangent to the hyperbola is given $y = mx + \sqrt{a^2m^2 - b^2}$

To find the equation of the tangent and normal at (x_1,y_1)to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:

The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Differentiating with respect to x,

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

At (x_1,y_1) , $\left(\frac{dy}{dx}\right)=\frac{b^2x_1}{a^2y_1}(x-x_1)$ = slope of the tangent at (x_1,y_1)

The equation of the tangent at (x_1,y_1) is $\frac{b^2x_1}{a^2y_1}(x-x_1)=y-y_1$

$$b^2xx_1-a^2yy_1=b^2{x_1}^2-a^2{y_1}^2$$

Dividing by a^2b^2, $\Rightarrow$ $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{{x_1}^2}{a^2}-\frac{{y_1}^2}{b^2}=1$

The equation of the normal at (x_1,y_1)

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$ [to derive this refer ellipse]

Solved Examples

1. Find the equation of the hyperbola whose focus is (2, 2), eccentricity= 3/2 and directrix $3x-4y=1$

Solution

Focus is (2,2) , e = 3/2 and directrix $3x-4y=1$. Let $P(x, y)$ be any point on the hyperbola.

$$\frac{SP}{PM}=e \quad \text{or} \quad SP^2=e^2PM^2$$

$$(x-2)^2+(y-2)^2=\frac{9}{4}\left(\frac{3x-4y-1}{5}\right)^2$$

$\Rightarrow 100\,(x^2-4x+4+y^2-4y+4)=9\,(9x^2+16y^2+1-24xy-6x+8y)$

Hence, the equation of the hyperbola is $19x^2+216xy-44y^2-346x-472y-791=0$

2. Find the equation of the hyperbola whose foci are (6, 4) and (−4, 4) and eccentricity =2

Solution

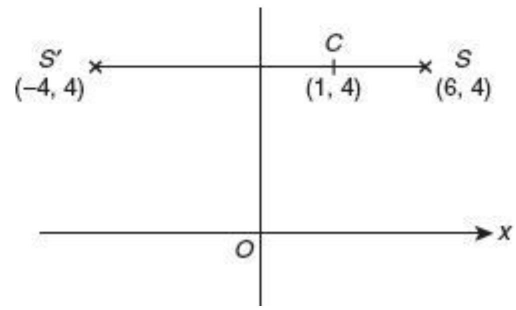
S is (6, 4) and S' (−4, 4), and C is the midpoint of SS'

$\therefore C$ is $(1, 4)$

$CS = 6 - 1 = 5$. But $CS = ae = 2a$.

$\therefore 2a = 5 \quad \text{or} \quad a = \frac{5}{2}$

$$= a^2(e^2 - 1) = \frac{25}{4} \times 3 = \frac{75}{4}$$



Hence, the equation of the hyperbola is

$$\frac{(x-1)^2}{\frac{25}{4}} - \frac{(y-4)^2}{\frac{75}{4}} = 1$$

$$\frac{4(x-1)^2}{25} - \frac{4(y-4)^2}{75} = 1$$

3. Find the equation of the hyperbola whose center is (−3, 2), one end of the transverse axis is (−3, 4) and eccentricity is $\frac{5}{2}$.

Solution

Centre is (−3, 4)

A is (−3, 4) ∴ *A′* is (−3, 6); *a* = 2

$$b^2 = a^2(e^2 - 1) = 4\left(\frac{25}{4} - 1\right) = 21$$

Hence, the equation of the hyperbola is

$$\frac{(y-2)^2}{4} - \frac{(x+3)^2}{21} = 1$$ (since the line parallel to *y*-axis is the transverse axis)

$$\frac{y^2 + 4 - 4y}{4} - \frac{x^2 + 9 + 6x}{21} = 1$$

$$-4x^2 - 36 - 24x + 21y^2 + 84 - 84y = 84$$

$$4x^2 - 21y^2 + 24x + 84y + 36 = 0$$

4. Find the equation of the hyperbola whose centre is (1, 0), one focus is (6, 0), and length of transverse axis is 6.

Solution

$2a = 6 \therefore a = 3$

$b^2 = a^2(e^2 - 1) = 9\left(\frac{25}{9} - 1\right) = 16$

$ae = 5 \quad \therefore e = \frac{5}{3}$

Hence, the equation of the hyperbola is $\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$ (i.e.) $16x2 - 9y2 - 32x - 128 = 0$.

5. Find the equation of the hyperbola whose centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2).

Solution

C is (3, 2), *A* is (4, 2) and *S* is (5, 2).

Hence, *CA* = 1and the transverse axis is parallel to *x*-axis.

$a = 1$

Also $ae = 2$. Since $a = 1$ and $e = 2$, $b2 = a2\,(e2 - 1) = 1\,(4 - 1) = 3$. Hence, the equation of the hyperbola is $\frac{(x-3)^2}{1} - \frac{(y-2)^2}{3} = 1$

$3x^2 - 18x + 27 - y^2 + 4y - 4 = 3$

$3x^2 - y^2 - 18x + 4y + 20 = 0$

6. Find the equation of the hyperbola whose centre is (6, 2), one focus is (4, 2) and $e = 2$.

Solution

Transverse axis is parallel to *x*-axis and *CS* = 2 units in magnitude.

$ae = 2 \quad \therefore a = 1$

$b^2 = a^2\,(e^2 - 1) = 1\,(4 - 1) = 3$

Hence, the equation of the hyperbola is $\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1.$

7. Find the centre, foci and eccentricity of $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

Solution

$$(12x^2 - 24x) - (4y^2 - 32y) - 127 = 0$$

$$12(x^2 - 2x) - 4(y^2 - 8y) - 127 = 0$$

$$12(x-1)^2 - 12 - 4(y-4)^2 + 64 - 127 = 0$$

$$12(x-1)^2 - 4(y-4)^2 = 75$$

$$\frac{(x-1)^2}{\left(\frac{75}{12}\right)} - \frac{(y-4)^2}{\left(\frac{75}{4}\right)} = 1$$

Hence, centre is (1, 4).

$$a^2 = \frac{75}{12}, b^2 = \frac{75}{4}$$

$$b^2 = a^2(e^2 - 1)$$

$$\frac{75}{4} = \frac{75}{12}(e^2 - 1)$$

$$\Rightarrow e^2 - 1 = \frac{12}{4} = 3$$

$$\Rightarrow e^2 = 4 \quad \text{or} \quad e = 2$$

$$CS = ae = \frac{5\sqrt{3}}{2\sqrt{3}} \times 2 = \frac{5}{2} \times 2 = 5$$

Hence, the foci are (6, 4) and (−4, 4).

SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

UNIT-4

ASYMPTOTES

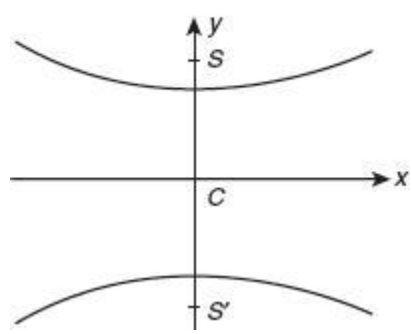
Introduction

An asymptote of a hyperbola is a straight line that touches the hyperbola at infinity but does not lie altogether at infinity.

Conjugate hyperbola

The foci are $S(ae, 0)$ and $S'(-ae, 0)$ and the equations of the directrices are $x = \pm\frac{a}{e}$. By the symmetry of the hyperbola, if we take the transverse axis as the y-axis and the conjugate axis as x-axis, then the equation of the hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (i.e.) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

This hyperbola is called the conjugate hyperbola. Here, the coordinates of the foci are $S(0, be)$ and $S'(0, -be)$. The equations of the directrices are $x = \pm\frac{b}{e}$



The length of the transverse axis is $2b$.

The length of the conjugate axis is $2a$.

The length of the latus rectum is $\frac{2a^2}{b}$.

Equations of asymptotes of the hyperbola

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let $y = mx + c$ be an asymptote of the hyperbola. Solving these two equations, we get their points of intersection. The x coordinates of the points of intersection are given by

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$\text{(i.e.)} \quad b^2x^2 - a^2(mx+c)^2 = a^2b^2$$

$$\text{(i.e.)} \quad x^2(b^2 - a^2m^2) - 2mca^2x - a^2(b^2+c^2) = 0$$

If $y = mx + c$ is an asymptote, then the roots of the above equation are infinite. The conditions for these are the coefficient of $x2 = 0$ and the coefficient of $x = 0$, $b^2 - a^2m^2$ and $mca^2 = 0$.

$$\text{(i.e.)} \quad m = \pm\frac{b}{a} \text{ and } c = 0$$

The equations of the asymptotes are $y = \pm\frac{b}{a}x$

$$\text{(i.e.)} \quad \frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0$$

The combined equation of the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Note:

- The asymptotes of the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are also given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ Therefore, the hyperbola and the conjugate hyperbola have the same asymptotes.
- The equation of the hyperbola is H: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$
- The equation of the asymptotes is $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$
- The equation of the conjugate hyperbola is
- The equation of the asymptotes differs from that of the hyperbola by a constant and the equation of the conjugate hyperbola differs from that of the asymptotes by the same constant term. This result holds good even when the equations of the hyperbola and its asymptotes are in the most general form.
- The asymptotes pass through the centre (0,0) of the hyperbola.
- The slopes of the asymptotes are $\frac{b}{a}$ and $\frac{-b}{a}$.
- Hence, they are equally inclined to the coordinate axes, which are the transverse and conjugate axes.

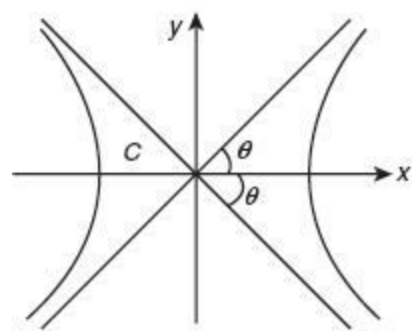
Angle between the asymptotes

Let 2θ be the angle between the asymptotes. Then,

$$m = \tan\theta = \frac{b}{a}$$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2} = e^2$$

$$\therefore \sec\theta = e$$



Hence, the angle between the asymptotes is 2sec−1(e).

Solved problems

1. Find the equation of the asymptotes of the hyperbola $3x2 - 5xy - 2y2 + 17x + y + 14 = 0$

Solution

The combined equation of the asymptotes should differ from that of the hyperbola only by a constant term.

The combined equation of the asymptotes is

$$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$$

$$\begin{aligned} 3x^2 - 5xy - 2y^2 &= 3x^2 - 6xy + xy - 2y^2 \\ &= 3x(x-2y) + y(x-2y) \\ &= (3x+y)(x-2y) \end{aligned}$$

Hence, the asymptotes are $3x + y + l = 0$ and $x - 2y + m = 0$.

$$(3x + y + l)(x - 2y + m) = 3x^2 - 5yx - 2y^2 + 17x + y + k$$

Equating the coefficients of the terms x and y and the constant terms, we get

$$l + 3m = 17$$

$$-2l + m = 1$$

Solving these two equations, we get l = 2 and m = 5.

lm = k.

∴ k = 10. The combined equation of the asymptotes is (3x + y + 2) (x − 2y + 5)=0.

2. Find the equations of the asymptotes and the conjugate hyperbola given that the hyperbola has eccentricity $\sqrt{2}$, focus at the origin and the directrix along $x + y + 1 = 0$.

Solution

From the focus directrix property, the equation of the hyperbola is

$$x^2 + y^2 = 2\left(\frac{x+y+1}{\sqrt{2}}\right)^2$$

$$\text{(i.e.)}\quad 2xy + 2x + 2y + 1 = 0$$

The combined equation of the asymptotes is $2xy + 2x + 2y + k = 0$,
where k is a constant. Let the asymptotes be $2x + l = 0$ and $y + m = 0$. Then,

$$2xy + 2x + 2y + k = (2x + l)(y + m)$$

Equating like terms, we get $2m = 2$. ∴ $m = 1$. Similarly, $l = 2$. As $lm = k$, we get $k = 2$.

Therefore, the asymptotes of the combined equation of the asymptotes is $2xy + 2x + 2y + 2 = 0$.

The equation of the asymptotes of the conjugate hyperbola should differ by the same constant.

The equation of the asymptotes of the conjugate hyperbola is $2xy + 2x + 2y + 1 = 0$

Conjugate diameters

Two diameters are said to be conjugate if each bisects chords parallel to the other. The condition of the diameters $y = mx$ and $y = m'x$ to be conjugate diameters is $mm' = \frac{b^2}{a^2}$.

Note: These diameters are also conjugate diameters of the conjugate hyperbola

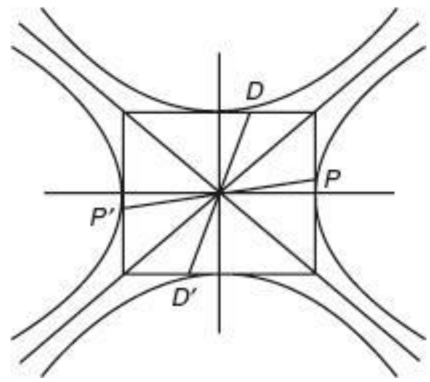
$\frac{x^2}{-a^2} - \frac{y^2}{-b^2} = -1$ since $\frac{-b^2}{-a^2} = \frac{b^2}{a^2}$

3. Prove that the parallelogram formed by the tangents at the extremities of conjugate diameters of hyperbola has its vertices lying on the asymptotes and is of constant area.

Proof

Let P and D be points ($a \sec \theta$, $b \tan \theta$) and ($a \tan \theta$, $b \sec \theta$) on the hyperbola and its conjugate.

Then D' and P' are ($-a \tan \theta$, $-b \sec \theta$) and ($-a \sec \theta$, $-b \tan \theta$), respectively.



The equations of the asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

The equations of the tangents at P, P', D, D' are

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \quad \text{...... (1)}$$

$$-\frac{x\sec\theta}{a} + \frac{y\tan\theta}{b} = 1 \quad \text{.......(2)}$$

$$\frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = -1 \quad \text{.......(3)}$$

$$-\frac{x\tan\theta}{a} + \frac{y\sec\theta}{b} = -1 \quad \text{........(4)}$$

Clearly the tangents at P and P' are parallel and also the tangents at D and D' are parallel. Solving (1) and (3) we get the coordinates of D are [a(sec θ + tan θ), b[sec θ + tan θ)].

This lies on the asymptote $\frac{x}{a} - \frac{y}{b} = 0$.

Similarly the other points of intersection also lie on the asymptotes. The equations of PCP' and DCD' are

$$y = \frac{b\tan\theta}{a\sec\theta}x \quad \text{......(5)}$$

$$y = \frac{b\sec\theta}{a\tan\theta}x \quad \text{.......(6)}$$

Lines are parallel (4), (5), (6) and also the lines (1), (2), (3) are parallel.

Therefore, area of parallelogram *ABCD* = 4 area of parallelogram *CPAD*.

$$= 4CP \text{ (Perpendicular from } C \text{ on } AD)$$

$$= 4\sqrt{a^2 \sec^2\theta + b^2 \tan^2\theta}.\frac{1}{\sqrt{\frac{\tan^2\theta}{a^2}+\frac{\sec^2\theta}{b^2}}}$$

$$= 4\,ab \text{ which is a constant.}$$

4. Find the condition that the pair of lines $Ax^2 + 2Hxy + By^2 = 0$ to be conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Solution

Let the two straight lines represented by $Ax^2 + 2Hxy + By^2 = 0$ be $y = m_1x$ and $y = m_2x$. Then

$$m_1 + m_2 = -\frac{2H}{B} \quad \text{and} \quad m_1m_2 = \frac{A}{B} \qquad \ldots\ldots(1)$$

If these lines are the conjugate diameters of the hyperbola then

$$m_1m_2 = \frac{b^2}{a^2} \qquad \ldots\ldots..(2)$$

From (1) and (2)

$$\frac{A}{B} = \frac{b^2}{a^2}$$

or $$a^2A = b^2B.$$

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INSTITUTE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

RECTANGULAR HYPERBOLA

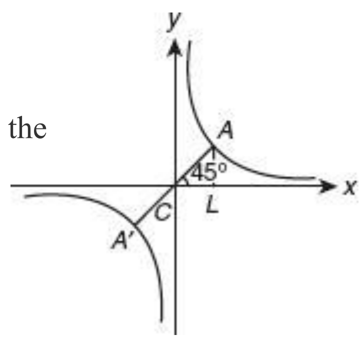
If in a hyperbola the length of the semi-transverse axis is equal to the length of the semi-conjugate axis, then the hyperbola is said to be a rectangle hyperbola.

Equation of rectangular hyperbola with reference to asymptotes as axes

In a rectangular hyperbola, the asymptotes are perpendicular to each other. Since the axes of coordinates are also perpendicular to each other, we can take the asymptotes as the *x*- and *y*-axes.

Then the equations of the asymptotes are $x = 0$ and $y = 0$. The combined equation of the asymptotes is $xy = 0$.

The equation of the hyperbola will differ from that of asymptotes only by a constant. Hence, the equation of the rectangular hyperbola is xy = k where k is a constant to be determined. Let AA′ be the



transverse axis and its length be 2a. Then, AC = CA′ = a.

Draw AL perpendicular to x-axis. Since the asymptotes bisect the angle between the axes, $\underline{|ACL} = 45°$.

$$CL = CA\cos 45° = \frac{a}{\sqrt{2}} \text{ and } AL = CA\sin 45° = \frac{a}{\sqrt{2}}$$

The coordinates of A are $\left[\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right]$. Since it lies on the rectangular hyperbola xy = k, we get $\frac{a^2}{2} = k$. Hence, the equation of the rectangular hyperbola is $xy = \frac{a^2}{2}$ or $xy = c^2$ where $c^2 = \frac{a^2}{2}$

Note: The parametric equations of the rectangular hyperbola $xy = c^2$ are $x = ct$ and $y = \frac{c}{t}$

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Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

Equations Of Tangent And Normal At (x_1, y_1) On the Rectangular Hyperbola xy =c^2

The equation of rectangular hyperbola is $xy = c^2$.
Differentiating with respect to x, we get

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\left|\frac{dy}{dx}\right|_{at(x_1,y_1)} = \frac{-y_1}{x_1} = \text{ slope of the tangent at}(x_1, y_1).$$

The equation of the tangent at (x_1,y_1) is

$$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$$

$$\Rightarrow x_1y - x_1y_1 = -y_1x + x_1y_1$$

$$\Rightarrow y_1x + x_1y = 2x_1y_1$$

$$\text{(i.e.)} \quad y_1x + x_1y = 2c^2$$

Since $x_1y_1 = c^2$

The slope of the normal at (x_1, y_1) is $\frac{x_1}{y_1}$

The equation of the normal at (x_1, y_1) is $y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$$yy_1 - y_1^2 = xx_1 - x_1^2$$

$$\text{(i.e.)} \quad xx_1 - yy_1 = x_1^2 - y_1^2$$

Equation of Tangent and Normal at (ct, c/t) on the Rectangular Hyperbola xy = c^2

The equation of the rectangular hyperbola is xy = c^2. Differentiating with respect to x, we get

$$x\frac{dy}{dx} + y = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y}{x}$$

$\frac{dy}{dx}\left(ct,\frac{c}{t}\right)=\frac{-1}{t^2}=$ slope of the tangent at $\left(ct,\frac{c}{t}\right)$

The equation of the tangent at (ct, c/t) is $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct)$

$$\text{(i.e.)}\quad yt^2-ct=-x+ct \text{ or } x+yt^2=2ct$$

The slope of the normal at '*t*' is −t.

The equation of the normal at '*t*' is

$$y-\frac{c}{t}=-t^2(x-ct)$$
$$y-\frac{c}{t}=-xt^2+ct^3$$

Dividing by *t*, we get

$$\frac{y}{t}-\frac{c}{t^2}=-xt+ct^2 \quad\text{or}\quad xt-\frac{y}{t}=c\left(t^2-\frac{1}{t^2}\right)$$

Prove that any two conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

Proof: Let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. The equation of the asymptotes is $x^2 - y^2 = 0$. Let $y = mx$ and y=(1/m)x be a pair of conjugate diameters of the rectangular hyperbola $x^2 - y^2 = a^2$. Then, the combined equation of the conjugate diameters is $(y-mx)(x-my)=0$

$$\text{(i.e.)}\quad mx^2-(m^2+1)xy+my^2=0$$

The combined equation of the bisectors of the angles between these two lines is

$$\frac{x^2-y^2}{0}=\frac{-xy}{\frac{1}{2}(m^2+1)}\ \text{(i.e.)}\ x^2-y^2=0$$

which is the combined equation of the asymptotes. Therefore, the asymptotes bisect the angle between the conjugate diameter.

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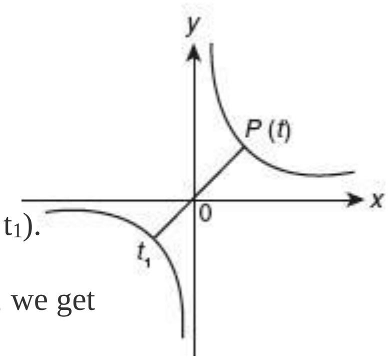
Important results on Rectangular Hyperbola

- The equation of the normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.
- The equation of the pair of tangents from (x_1, y_1) is $(xy_1 + yx_1 - 2c^2)^2 = 4(xy - c^2)(x_1 y_1 - c^2)$.
- The equations of the chord having (x_1, y_1) as its midpoint is $xy1 + yx1 = 2x1y1$.
- The equation of the chord of contact from $(x1, y1)$ is $xy1 + yx1 = 2c2$.
- The equation of the tangent at (x_1, y_1) on the rectangular
- hyperbola $xy = c^2$ is $\frac{1}{2}(xy_1 + yx_1) = c^2$.

Solved problems

1. If the normal to the rectangular hyperbola $xy = c2$ at the point t as it intersect the rectangular hyperbola at $t1$ then show that $t3t1 = -1$.

Solution



The equation of the normal at t is $xt - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$.

The equation of the chord joining the points t and t_1 is $x + y\, t\, t_1 = c(t + t_1)$.

These two equations represent the same straight line. Identifying them, we get

$$\frac{t}{1} = \frac{-1/t}{tt_1} = \frac{c\left(t^2 - \frac{1}{t^2}\right)}{c(t + t_1)}$$

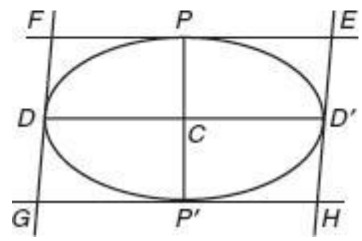
$$\therefore t = \frac{-1}{t^2 t_1} \text{ or } t^3 t_1 = -1$$

2.Prove that tangents at the extremities of a pair of Conjugate Diameters of an ellipse encloses a parallelogram whose area is constant

Proof:

Let PCP' an e diameters. Let P be the point (a cosθ, b sinθ). Then D is the point $\left(a\cos\left(\frac{}{2}+\theta\right), b\sin\left(\frac{}{2}+\theta\right)\right)$

(i.e.) ($-a$ sinθ, b cosθ).



The equation of the tangent at P is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$

The slope of the tangent at P is $\frac{-b\cos\theta}{a\sin\theta}.$ The

slope of CD is $\frac{-b\cos\theta}{a\sin\theta}.$

Since the two slopes are equal, the tangents at P is parallel to DCD′. Similarly, we can show that the tangent at P′ is parallel to DCD′. Therefore, the tangent at P and P′ are parallel. Similarly, the tangent D and D′ are parallel. Hence, the tangents at P, P′, D, D′ from a parallelogram EFGH.

The area of the parallelogram EFGH

$= 4 \times \text{Area of parallelogram } CPFD$

$= 4 \times 2 \text{ area of } \Delta CPD = 8 \text{ area of } \Delta CPD$

$= 8\times\frac{1}{2}[a\cos\theta(b\cos\theta) - a\sin\theta(-b\sin\theta)]$

$= 4ab(\cos^2\theta+\sin^2\theta)$

$= 4ab$ which is a constant.

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COURSE MATERIAL

Subject Name: Trigonometry and 2D Analytical Geometry **Subject Code: SMTA1306**

UNIT-5

POLAR EQUATION OF A CONIC

Introduction

Earlier we defined parabola, ellipse and hyperbola in terms of focus directrix. Now let us show that it is possible to give a more unified treatment of all these three types of conic using polar coordinates. Furthermore, if we place the focus at the origin then a conic section has simple polar equation.

Let S be a fixed point (called the focus) and XM, a fixed straight line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). Then the set of all points P in the plane such that $\frac{SP}{PM} = e$ is called a conic section. The conic is (1) an ellipse if $e < 1$,

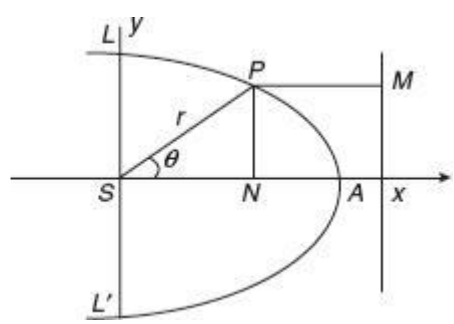
(ii) a parabola if $e = 1$ (iii) a hyperbola if $e > 1$.

Polar equation of a conic

Let S be focus and XM be the directrix. Draw SX perpendicular to the directrix. Let S be the pole and SX be the initial line. Let P(r, θ) be any point on the conic; then SP = r, ∠XSP = θ. Draw PMperpendicular to the directrix and PN perpendicular to the initial line.

Let LSL' be the double ordinate through the focus (latus rectum). The focus directrix property is

$$\frac{SP}{PM} = e$$



$$\text{(i.e.)}\quad SP = ePM$$

$$= eNX$$

$$= e(SX - SN)$$

$$\text{(i.e.)}\quad r = e\left(\frac{l}{e} - r\cos\theta\right)$$

$$r = l - er\cos\theta$$

$$r(1 + e\cos\theta) = l$$

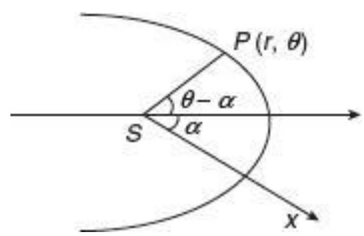
$$\text{or}\quad \frac{l}{r} = 1 + e\cos\theta$$

This is the required polar equation of the conic.

Note : If the axis of the conic is inclined at an angle α to the initial line then the polar equation of conic is $\frac{l}{r} = 1 + e\cos(\theta\text{-}\alpha)$

To trace the conic, $\frac{l}{r} = 1 + e\cos\theta$

$\cos\theta$ is a periodic function of period 2π.



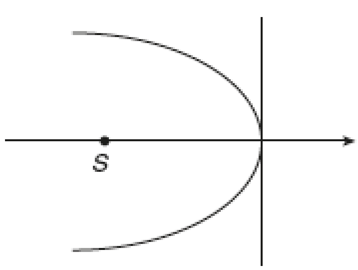
Therefore, to trace the conic it is enough if we consider the variation of θ from $-\pi$ to π. Since $\cos(-\theta) = \cos\theta$ the curve is symmetrical about the initial line. Hence it is enough if we study the variation ofθ from 0 to π. Let us discuss the various cases for different values of θ.

Case 1: Let $e = 0$. In this case, the conic becomes $r = l$ which is a circle of radius l with its centre at the pole.

Case 2: Let $e = 1$. In this case, the equation of the conic becomes $\frac{l}{r} = 1 + e\cos\theta$ or $r = \frac{l}{1+e\cos\theta}$

When θ varies from 0 to π, $1 + \cos\theta$ varies from 2 to 0 and $\frac{l}{1+\cos\theta}$ varies from 1/2 to ∞.

The conic in this case is a parabola and is shown below.



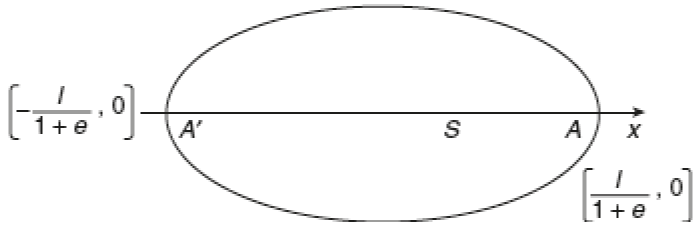
$r = l/1 + e\cos\theta$

Case 3: Let $e < 1$.

As θ varies from 0 to π, $1 + e\cos\theta$ decreases from $1 + e$ to $1 - e$.

r increases from $\frac{l}{1+e}$ to $\frac{l}{1-e}$

The curve is clearly closed and is symmetrical about the initial line.The conic is an ellipse.

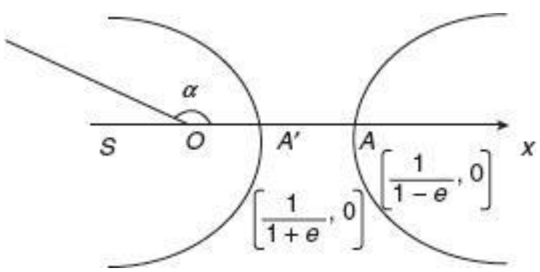


Case 4: Let $e > 1$.

As θ varies from 0 to $\frac{\pi}{2}$, $1 + e\cos\theta$ decreases from $(1 + e)$ to 1 and hence r increases from $1+e$ to l.

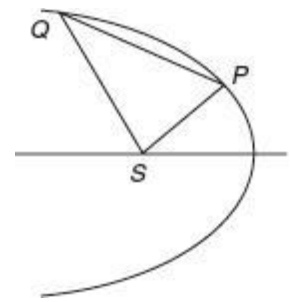
As θ varies from $\frac{\pi}{2}$ to π, $1 + e\cos\theta$ decreases from 1 to $(1 – e)$. Therefore, there exists an angle α such that $\frac{\pi}{2} < \alpha < \pi$ at which $1 + e\cos\theta > 0$. (i.e.) $\cos\alpha > \frac{-1}{e}$

Hence, as θ varies from $\frac{\pi}{2}$ to α, r increases from 1 to ∞. As θ varies from α to π, $1 + e\cos\theta$ remains negative and varies from 0 to $(1 - e)$. r varies from to $-\infty$ to $\frac{1}{1-e}$.



The conic is shown above and is a hyperbola.

Equation of Chord Joining the Points whose Vectorial Angles are α and $\alpha + \beta$ on the Conic



Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$.

Let the equation of the chord *PQ* be $\frac{l}{r} = A\cos\theta + B\cos(\theta - \alpha)$.

This chord passes through the point $(SP, \alpha - \beta)$ and $(SQ, \alpha + \beta)$.

$\frac{l}{SP} = A\cos(\alpha - \beta) + B\cos\beta$ …..(1)

$\frac{l}{SQ} = A\cos(\alpha - \beta) + B\cos\beta$ ……(2)

Also these two points lie on the conic $\frac{l}{r} = 1 + e\cos\theta$,

$\frac{l}{SP} = 1 + e\cos(\alpha - \beta)$ …..(3)

$\frac{l}{SQ} = 1 + e\cos(\alpha + \beta)$…….(4)

From equations 1 and 3, we get

$A\cos(\alpha - \beta) + B\cos\beta = 1 + e\cos(\alpha - \beta)$ …..(5)

From equation 2 and 4, we get

$A\cos(\alpha + \beta) + B\cos\beta = 1 + e\cos(\alpha + \beta)$ …….(6)

Subtracting, we get

$$A[\cos(\alpha-\beta)-\cos(\alpha+\beta)]=e[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$
$$A=e$$

From equation (5) we get

$$e\cos(\alpha-\beta)+B\cos\beta=1+e\cos(\alpha-\beta)$$
$$B\cos\beta=1 \quad \text{(i.e.)} \quad B=\sec\beta$$

The equation of the chord PQ is $\frac{l}{r}=\mathrm{e}\cos\theta+\sec\beta\ \cos(\theta-\alpha)$

1. Find the quation of the tangent at the point whose vectorial angle is α on the conic

$\frac{l}{r}=1+\mathrm{e}\cos\ \theta$

Sol:

The equation of the chord joining the points with vectorial angles α and $\alpha+\beta$ is $\frac{l}{r}=\mathrm{e}\cos\theta+\sec\beta\ \cos(\theta-\alpha)$

This chord becomes the tangent at α if $\beta=0$.

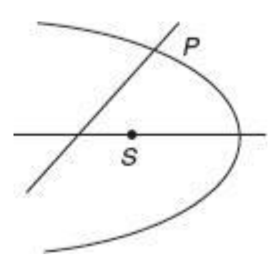
The equation of tangent at α is $\frac{l}{r}=1+\mathrm{e}\cos\theta+\cos(\theta-\alpha)$.

Equation of normal at the point whose vectorial angle is α on the conic

The equation of the conic is $\frac{l}{r}=1+\mathrm{e}\cos\ \theta$

The equation of tangent at α on the

conic $\frac{l}{r}=1+\mathrm{e}\cos\ \theta$ is $\frac{l}{r}=\mathrm{e}\cos\theta+\cos(\theta-\alpha)$



The equation of the line perpendicular to this tangent

is $\frac{k}{r}=e\cos\left(\theta+\frac{\pi}{2}\right)+\cos\left(\theta+\frac{\pi}{2}-\alpha\right)$.

i.e $\frac{k}{r}=-e\sin\theta-\sin(\theta-\alpha)$

If this perpendicular line is normal at *P*, then it passes through the point (*SP, α*).

$$\frac{k}{SP} = -e\sin\alpha \text{ or } k = -SP\cdot e\sin\alpha \quad .....(1)$$

Since the point (*SP, α*) also lies on the conic $\frac{l}{r} = 1 + e\cos\theta$ we have

$$\frac{l}{SP} = 1 + e\cos\alpha$$

$$SP = \frac{l}{1+e\cos\alpha}$$

From equation (1), we get $k = \frac{le\sin\alpha}{1+e\cos\alpha}$.

Hence, the equation of the normal at *α*

$$-\frac{1}{r}\frac{le\sin\alpha}{l+e\cos\alpha} = -e\sin\theta - \sin(\theta-\alpha)$$

$$\text{(i.e.)} \quad \frac{1}{r}\frac{le\sin\alpha}{l+e\cos\alpha} = e\sin\theta + \sin(\theta-\alpha)$$

Asymptotes of the conic is $\frac{l}{r} = 1 + e\cos\theta$ $(e > 1)$

The equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$ $(e > 1)$ …(2)

The equation of the tangent at *α* is $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$ …(3)

This tangent becomes an asymptote if the point of contact is at infinity, that is, the polar coordinates of the point of contact are (∞, *α*). Since this point has to satisfy the equation of the conic (2)we have from equation (2),

$$0=1+e\cos\theta \text{ or } \cos\alpha = 1/e \quad …(4)$$

The equation(3)can be written as $\frac{l}{r} = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta.$

Substituting $\cos\alpha = 1/e$ and $\cos = \pm\sqrt{1 - \frac{1}{e^2}}$, we get the equation of the

asymptotes as $\frac{l}{r} = \left(e - \frac{1}{e}\right)\cos\theta \pm \sqrt{1 - \frac{1}{e^2}}\sin\theta.$

$$\text{(i.e.)} \quad \frac{l}{r} = \frac{e^2 - 1}{e}\cos\theta \pm \frac{\sqrt{e^2 - 1}}{e}\sin\theta$$

$$\therefore \quad \frac{l}{r} = \frac{e^2 - 1}{e}\left[\cos\theta \pm \frac{1}{\sqrt{e^2 - 1}}\sin\theta\right]$$

2. Show that the polar of a point with respect to a conic is defined as the locus of the point of intersection of tangents at the extremities of a variable chord passing through the point P(r_1, θ_1) is given by $\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1).$

Proof:

Let the tangents at *Q* and *R* intersect *T*. Since *QR* is the chord of contact of tangents from *T* (*R*, ϕ), its equation is

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{R} - e\cos\phi\right) = \cos(\theta - \phi) \quad \ldots\ldots(A)$$

Since this passes through the point P(r_1, θ_1) we have

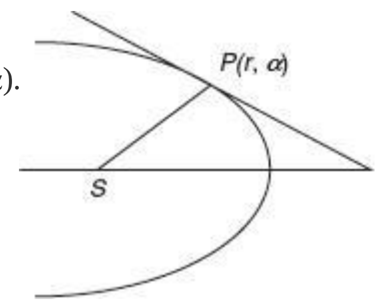
$$\left(\frac{l}{r_1} - e\cos\theta_1\right)\left(\frac{l}{R} - e\cos\phi\right) = \cos(\theta_1 - \phi) \quad \ldots\ldots(B)$$

Now the locus of the point *T*(*R*, ϕ) is polar of the (r_1, θ_1).

The polar of (r_1, θ_1). from equation B is $\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1).$

3. Find the condition that the straight line $\frac{l}{r} = A\cos\theta + B\ \sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$

Solution



Let the line $\frac{l}{r} = A\cos\theta + B\ \sin\theta$ touches the conic at the point (r, α).

Then the equation of tangent at (r, α) is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \quad \ldots\ldots(5)$$

$$\text{(i.e.)}\ \frac{l}{r} = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta \quad \ldots..(6)$$

However, the equation of tangent is given as

$$\frac{l}{r} = A\cos\theta + B\sin\theta \quad \ldots..(7)$$

Equation 6 and 7 represent the same line.

Identifying equations 6 & 7, we get

$A = e + \cos\alpha$ (or) $A - e = \cos\alpha$

$B = \sin\alpha$

Squaring and adding, we get $(A - e)^2 + B^2 = 1$

This is the required condition.

4.Show that the locus of the point of intersection of tangents at the extremities of a variable focal chord is the corresponding directrix.

Solution

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$

The equation of tangent at α is $\frac{l}{r} = 1 + e\cos\theta + \cos(\theta - \alpha)$

The equation of tangent at $\alpha + \pi$ is $\frac{l}{r} = 1 + e\cos\theta + \cos(\theta - \overline{\alpha + \pi})$

$$\therefore \frac{l}{r} = e\cos\theta - \cos(\theta - \alpha)$$

Let (r_1, θ_1) be the point of intersection of these two tangents. Then,

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha)$$

$$\frac{l}{r_1} = e\cos\theta_1 - \cos(\theta_1 - \alpha)$$

Adding these two equations, we get

$$\frac{2l}{r_1} = 2e\cos\theta_1 \text{ or } \frac{l}{r_1} = e\cos\theta_1$$

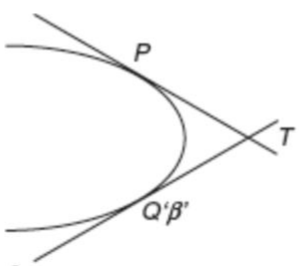
Therefore, the locus is the corresponding directrix $\frac{l}{r} = e\cos\theta$

5. Show that the locus of the point of intersection of perpendicular tangents to a conic is a circle or a straight line.

Solution

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ …….(1)

Let *P* and *Q* be the points on the conic whose vectorial angles are *α* and *β*. The equations of tangents at *P* and *Q* are



$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \quad \text{…..(2)}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta) \quad \text{……(3)}$$

Let $(r1, \theta1)$ be the point of intersection of tangents at *P* and *Q*. Then

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha) \qquad \ldots(4)$$

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \beta) \qquad \ldots(5)$$

From equations 4 and 5 we get

$$\cos\ (\theta_1 - \alpha) = (\theta_1 - \beta)$$
$$\theta_1 - \alpha = \pm(\theta_1 - \beta)$$
$$\alpha = \beta \text{ or } \theta_1 = \frac{\alpha+\beta}{2}$$

But $\alpha = \beta$ is not possible. $\therefore \theta_1 = \frac{\alpha+\beta}{2}$

From equation 4, we get $\cos\frac{\alpha-\beta}{2} = \frac{l - e\cos\theta}{r_1}$

Expanding equations 1 and 2, we get

$$\frac{l}{r} = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta$$
$$\frac{l}{r} = (e+\cos\beta)\cos\theta + \sin\beta\sin\theta$$

Since these two lines are perpendicular, we have

$$(e+\cos\alpha)(e+\cos\beta) + \sin\alpha\sin\beta = 0$$

$$\text{(i.e.) } e^2 + e(\cos\alpha + \cos\beta) + \cos(\alpha - \beta) = 0$$

$$e^2 + e\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right) + 2\cos^2\frac{\alpha-\beta}{2} - 1 = 0$$

Substituting for $\frac{\alpha+\beta}{2}$ and $\frac{\alpha-\beta}{2}$, we get

$$e^2 + 2e\cos\theta_1\left(\frac{l}{r_1} - e\cos\theta_1\right) + 2\left(\frac{l}{r_1} - e\cos\theta_1\right)^2 - 1 = 0$$

$$(e^2 - 1) - \frac{2el}{r_1}\cos\theta_1 + \frac{2l^2}{r_1^2} = 0$$

$$(1-e^2)r_1^2 + 2elr_1\cos\theta_1 - 2l^2 = 0$$

Therefore, the locus of (r_1,θ_1) is $(1 - e^2)r^2 + 2elr\cos\theta - 2l^2 = 0$.

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