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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Ordinary Differential Equation – SMTA1303

COURSE NAME: DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

COURSE CODE : SMTA1303

UNIT – I ORDINARY DIFFERENTIAL EQUATION

Exact differential equation.

A first order differential equation of type $M(x,y)dx + N(x,y)dy = 0$ is called an *exact differential equation* if there exists a function of two variables *u*(*x*,*y*) with continuous partial derivatives such that $du(x,y) = M(x,y)dx + N(x,y)dy$

The general solution of an exact equation is given by $u(x,y) + \int f(y)dy = c$,where *c* is an arbitrary constant

Test for Exactness

Let functions $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in a certain domain *D*.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Algorithm for Solving an Exact Differential Equation

1. First it's necessary to make sure that the differential equation is *exact* using the *test for exactness*:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

2. Integrate M with respect to x keeping y constant ie $\int M dx$
3. Integrate those terms in N not containing x with respect to y.ie $\int \left[N - \frac{\partial}{\partial y}\int M dx \right] dy$
4. The general solution of the exact differential equation is given by $\int M dx + \int \left[N - \frac{\partial}{\partial y}\int M dx \right] dy = c$

Example1. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3 \qquad N = 2x^3y - 3x^2y^2 - 5y^4$

$\Rightarrow \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \qquad and \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2 \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \therefore the\ given\ equation\ is\ exact.$

The required solution is given by $\int M dx + \int [terms\ of\ N\ not\ containing\ x] dy = c$

$\int (5x^4 + 3x^2y^2 - 2xy^3)dx \quad + \quad \int (-5y^4)dy = c$

$x^5 + x^3y^2 - x^2y^3 - y^5 = c$

Equations Reducible to Exact equations.

Rule1. $If\ \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of x alone ,say f (x) $then\ I.F = e^{\int f(x)dx}$

Rule2. $If\ \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of y alone ,say f (y) $then\ I.F = e^{\int f(y)dy}$

Rule3. $If\ M\ is\ of\ the\ form\ M = yf_1(xy)\ N\ is\ of\ the\ form\ N = xf_2(xy), then\ I.F = \frac{1}{Mx - Ny}$

Rule4. $If Mdx + Ndy = 0\ is\ a$ hom$ogeneous\ equation\ in\ x\ and\ y\ then\ I.F = \frac{1}{Mx + Ny}$

Example2. Solve $(2x\log x - xy)dy + 2ydx = 0.$

Solution . Given $(2x\log x - xy)dy + 2ydx = 0.$ (1)

Here M = 2y, N = 2xlogx-xy.

$\Rightarrow \frac{\partial M}{\partial y} = 2 \quad and \quad \frac{\partial N}{\partial x} = 2(1 + \log x) - y \Rightarrow$

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{-2\log x + y}{2x\log x - xy} = -\frac{1}{x} = f(x).$$

$I.F = e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$

$(1) I.F \Rightarrow \frac{2y}{x}dx + (2\log x - y)dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$

The required solution is given by $\int m dx + \int [terms\ of\ n\ not\ containing\ x] dy = c$

$\Rightarrow$ The required solution is given by $\int \frac{2y}{x}dx + \int (-y)dy = 0$.

$\Rightarrow$ The required solution is given by $2y\log x - \frac{y^2}{2} = 0$.

Example3. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

Solution . Given $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ (1)

Here $M = y^4 + 2y \quad N = xy^3 + 2y^4 - 4x$

$\Rightarrow \frac{\partial M}{\partial y} = 4y^3 + 2 \quad and \quad \frac{\partial N}{\partial x} = y^3 - 4 \Rightarrow \quad \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -\frac{(4y^3 + 2) - (y^3 - 4)}{y^4 + 2y} = -\frac{3}{y} = f(y).$

$I.F = e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$

$(1) I.F \Rightarrow (y + \frac{2}{y^2})dx + (x + 2y - \frac{4x}{y^3})dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$

The required solution is given by $\int m dx + \int [terms\ of\ n\ not\ containing\ x] dy = c$

$\Rightarrow$ The required solution is given by $\int (y+\frac{2}{y^2})dx + \int (2y)dy = c$.

The required solution is given by $x(y+\frac{2}{y^2}) + y^2 = c$

Example4. Solve $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$

Solution . Given $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$ (1)

$\Rightarrow y(1+2xy)dx + x(1-xy)dy = 0$

$M = y(1+2xy) = yf_1(xy)$ and $N = x(1-xy) = xf_2(xy)$,

Then I.F= $\frac{1}{Mx-Ny} == \frac{1}{y(1+2xy)x - x(1-xy)y} = \frac{1}{3x^2y^2}$

$(1)I.F \Rightarrow (\frac{1}{3x^2y} + \frac{2}{3x})dx + (\frac{1}{3xy^2} - \frac{1}{3y})dy = 0 \Rightarrow mdx + ndy = 0 \; which \; is \; exact.$

The required solution is given by $\int mdx + \int [terms\, of \; n \; not \; containing \; x]dy = c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{3x^2y} + \frac{2}{3x})dx + \int (-\frac{1}{3y})dy = c$.

$\Rightarrow$ The required solution is given by $-\frac{1}{3xy} + \frac{2\log x}{3} - \frac{\log y}{3} = c.$

Example5. Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}.$

Solution . Given $(x^3+y^3)dx + (xy^2)dy = 0.$ (1)

$Here \; M = (x^3+y^3) \; and \; N = -(xy^2) \; which\, are \; \mathrm{hom}ogeneous\, in\, x\, and \; y.$

$then\, I.F = \frac{1}{Mx+Ny} = \frac{1}{(x^3+y^3)x + (-xy^2)y} = \frac{1}{x^4}$

$(1)I.F \Rightarrow (\frac{1}{x} + \frac{y^3}{x^4})dx - (\frac{y^2}{x^3})dy = 0 \Rightarrow mdx + ndy = 0 \; which \; is \; exact.$

The required solution is given by $\int mdx + \int [terms\, of \; n \; not \; containing \; x]dy = c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{x} + \frac{y^3}{x^4})dx + \int (0)dy = c$.

$\Rightarrow$ The required solution is given by $\log x - \frac{y^3}{3x^3} = c.$

Example6. *Solve* $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Solution . Given $(y^3-2x^2y)dx+(2xy^2-x^3)dy=0.$

$$Here\frac{dy}{dx}=f(x,y)=-\frac{y^3-2x^2y}{2xy^2-x^3} \quad (1)$$ which are homogeneous in x and y.

$$put\ y=vx\ \ in\ \ (1), v+x\frac{dv}{dx}=\frac{-v^3x^3+2x^2vx}{2xv^2x^2-x^3}=\frac{2v-v^3}{2v^2-1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v-v^3}{2v^2-1}-v=\frac{3v-3v^3}{2v^2-1}$$
$$\Rightarrow \frac{2v^2-1}{-3v(v^2-1)}dv=xdx \Rightarrow \frac{2v^2-1}{v(v^2-1)}dv=\left[\frac{A}{v}+\frac{B}{v+1}+\frac{C}{v-1}\right]dv=-3\frac{dx}{x}$$
$$\Rightarrow\int\left[\frac{1}{v}+\frac{1/2}{v+1}+\frac{1/2}{v-1}\right]dv=\int -3\frac{dx}{x}+c$$
$$\Rightarrow\log(v\sqrt{v^2-1})=-\log x^3+\log c$$
$$\Rightarrow x^3(v\sqrt{v^2-1})=c$$
$$\Rightarrow x^2y^2(x^2-y^2)=c$$

LINEAR EQUATIONS OF HIGHER ORDER

A linear equation of n^{th} order with constant coefficients is of the form

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+.....a_ny=X \quad (1)$$

where a_1, a_2,…….a_n are constants and X is a function of x. This equation can also be written in the form $\left(D^n+a_1D^{n-1}+a_2D^{n-2}+.....+a_n\right)y=X$ where $D=\frac{d}{dx}, D^2=\frac{d^2}{dx^2}$,…….. $D^n=\frac{d^n}{dx^n}$

Consider $\left(D^n+a_1D^{n-1}+a_2D^{n-2}+.....+a_n\right)y=0$ (2)

The general solution of equation (2) is given by $Y=c_1y_1+c_2y_2+......c_ny_n$

where $y_1,y_2,......y_n$ are n independent solutions and $c_1,c_2,......c_n$ are arbitrary constants.

Y is called the complementary function (C.F) of equation (1).

Suppose u is a particular solution (particular integral) of equation (1)

Then the general solution of equation (1) is of the form y=Y+u where Y is the complementary function and u is a particular integral (P.I).

Thus y=C.F + P.I

To find Complementary functions

Case (1)

Roots of the A.E are real and distinct say m_1 and m_2

y= $c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case (2)

Roots of the A.E are imaginary then

y= $e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$

Case (3)

Roots of the A.E are real and equal say $m_1 = m_2$ then

$y = e^{m_1 x}\left(c_1 x + c_2\right)$

1.Solve $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

Put $\frac{d}{dx} = D$

$\left(D^2 y - 2Dy + 3y\right) = 0$

$\left(D^2 - 2D + 3\right)y = 0$

The auxiliary equation is $m^2 - 2m + 3 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(1)(3)}}{(2)(1)}$$

$$m = 2 \pm \frac{\sqrt{-8}}{2}$$

$$m = \frac{2 \pm i2\sqrt{2}}{2}$$

$$m = 1 \pm i\sqrt{2}$$

C.F = $e^{x}\left[c_1 \cos\left(\sqrt{2}x\right)+c_2 \sin\left(\sqrt{2}x\right)\right]$

The general solution is y = C.F+P.I

y= $e^{x}\left[c_1 \cos\left(\sqrt{2}x\right)+c_2 \sin\left(\sqrt{2}x\right)\right]$+0

To find Particular integral

When the R.H.S of the given differential equation is a function of x , we have to find particular Integral.

Case (i)

If $f(x)=e^{ax}$, then $P.I=\frac{1}{F(D)}e^{ax}$. Replace D by a in F(D), provided F(D)$\neq$0.

If F(a) = 0 then $P.I=\frac{x}{F'(D)}e^{ax}$ provided $F'(a)\neq 0$

If F'(a) = 0 then $P.I=\frac{x}{F''(D)}e^{ax}$ provided $F''(a)\neq 0$ and so on

Case (ii)

If f(x) = sinax or cosax then $P.I=\frac{1}{F(D)}\sin axor \cos ax$

Replace $D^2 by - a^2$ in F(D), provided F(D)$\neq$0.

If F(D) = 0, when we replace D^2 by $-a^2$ then proceed as case (i)

Case (iii)

If f(x) = a^n then $P.I=\frac{1}{F(D)}x^n$

$P.I=\left[F(D)\right]^{-1}x^n$, Expand $\left[F(D)\right]^{-1}$ by using binomial theorem and then operate on x^n.

Case (iv)

If f(x)=e^{ax}x, where X is sinax (or) cosax (or) x then

$$P.I=\frac{1}{F(D)}e^{ax}X=e^{ax}\frac{1}{F(D+a)}X$$

Here $\frac{1}{F(D+a)}X$ can be evaluated by using anyone of the first three types.

Problems

1.Solve $\left(D^2+6D+9\right)y=5e^{3x}$

$m^2+6m+9=0$

$\left(m+3\right)^2=0$

m=-3,-3

$C.F=\left(c_1x+c_2\right)e^{-3x}$

P.I = $\left(\frac{1}{\left(D^2+6D+9\right)}\right)5e^{3x}$

= $\left(\frac{1}{\left(3\right)^2+6\left(3\right)+9}\right)5e^{3x}$

= $\frac{5}{36}e^{3x}$

The general solution is y = C.F + P.I

y= $\left(c_1x+c_2\right)e^{-3x}+\frac{5}{36}e^{3x}$

2. Solve $\left(D^2+6D+5\right)y=e^{-x}$

$m^2+6m+5=0$

$\left(m+5\right)\left(m+1\right)=0$

m=-1,-5

C.F = $c_1e^{-x}+c_2e^{-5x}$

P.I = $\left(\frac{1}{\left(D^2+6D+5\right)}\right)e^{-x}$

= $\left(\frac{1}{\left(-1\right)^2+6\left(-1\right)+5}\right)e^{-x}$

$$= \frac{x}{2D+6}e^{-x} = \frac{x}{2(-1)+6}e^{-x}$$

$$= \frac{x}{4}e^{-x}$$

The general solution is y = C.F + P.I

y= $c_1e^{-x} + c_2e^{-5x} + \frac{x}{4}e^{-x}$

2.Solve $(D^2 + D + 1)y = \sin 2x$

Solution:

The auxiliary equation is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

C.F= $e^{\frac{-x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

P.I $= \left(\frac{1}{(D^2 + D + 1)}\right)\sin 2x$

$$= \left(\frac{1}{(-4 + D + 1)}\right)\sin 2x$$

$$= \left(\frac{1}{D-3}\right)\sin 2x$$

$$= \left(\frac{D+3}{D^2-9}\right)\sin 2x$$

$$= \left(\frac{D+3}{-13}\right)\sin 2x$$

$$=- \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$$

The general solution is y = C.F + P.I

$$y = e^{\frac{-x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] - \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$$

3.Solve $(D^2 + 3D + 2)y = x^2$

Solution:

The auxiliary equation is $m^2 + 3m + 2 = 0$

(m + 2) (m + 1) = 0

Hence m = -2, -1

C.F = $c_1 e^{-2x} + c_2 e^{-x}$

$$\text{P.I} = \left(\frac{1}{(D^2 + 3D + 2)}\right) x^2$$

$$= \frac{1}{2}\left(1 + \frac{3D + D^2}{2}\right)^{-1} x^2$$

$$= \frac{1}{2}\left(1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{3D + D^2}{2}\right)^2\right) x^2$$

$$= \frac{1}{2}\left(1 - \frac{3D}{2} + \frac{7D^2}{4}\right) x^2$$

$$= \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

The general solution is y = C.F + P.I

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

4. Solve $(D^2 - 4D + 3)y = e^x \cos 2x$

Solution:

The auxiliary equation is $m^2 - 4m + 3 = 0$

(m -1) (m -3) = 0

Hence m = 1, 3

C.F = $c_1e^x + c_2e^{3x}$

P.I = $\left(\frac{1}{(D^2-4D+3)}\right)e^x \cos 2x$

$= \left(\frac{e^x}{(D+1)^2-4(D+1)+3}\right)\cos 2x$

$= \left(\frac{e^x}{D^2-2D}\right)\cos 2x$

$= \left(\frac{e^x}{-4-2D}\right)\cos 2x$

$= -\frac{1}{2}\left(\frac{e^x}{D+2}\right)\cos 2x$

$= -\frac{e^x}{2}\left(\frac{D-2}{D^2-4}\right)\cos 2x$

$= -\frac{e^x}{2}\left[\frac{(D-2)\cos 2x}{-8}\right]$

$= \frac{e^x}{16}(-2\sin 2x - 2\cos 2x)$

$= -\frac{e^x}{8}(\sin 2x + \cos 2x)$

The general solution is y =C.F+ P.I

y = $c_1e^x + c_2e^{3x} - \frac{e^x}{8}(\sin 2x + \cos 2x)$

5. Solve $(D^2-2D+2)y = e^x \sin x$

The auxiliary equation is $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F = $e^x[c_1 \cos x + c_2 \sin x]$

P.I = $\left(\frac{1}{(D^2-2D+2)}\right)e^x \sin x$

= $\left[\frac{e^x}{(D+1)^2-2(D+1)+2}\right]\sin x$

= $\left[\frac{e^x}{D^2+1}\right]\sin x$

= $\left[\frac{e^x}{(D+i)(D-i)}\right]\sin x$

= e^x Imaginary part of $\left[\frac{1}{(D+i)(D-i)}\right]e^{ix}$

= e^x Imaginary part of $\left[\frac{1}{2i}xe^{ix}\right]$

= e^x Imaginary part of $\left[-\frac{1}{2}ix(\cos x + i\sin x)\right]$

= $-\frac{1}{2}xe^x \cos x$

The general solution is y = C.F + P.I

y = $e^x[c_1\cos x + c_2 \sin x]\ -\frac{1}{2}xe^x\cos x$

6. Solve $(D^3-3D^2+3D-1)y = x^2e^x$

The auxiliary equation is $m^3-3m^2+3m-1=0$

$(m-1)^3=0$

m=1 (thrice)

C.F = $e^x(c_1+c_2x+c_3x^2)$

P.I = $\frac{1}{D^3-3D^2+3D-1}x^2e^x$

= $\left[\frac{e^x}{(D+1)^3-3(D+1)^2+3(D+1)-1}\right]x^2$

$$= \quad e^{x}\left(\frac{1}{D^{3}}\right)x^{2}$$

$$= \quad \frac{e^{x}x^{5}}{60}$$ (By integrating x^2 thrice with respect to x)

The general solution is y = C.F + P.I

$$y = e^{x}\left(c_{1} + c_{2}x + c_{3}x^{2}\right) + \frac{e^{x}x^{5}}{60}$$

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UNIT- II PARTIAL DIFFERENTIAL EQUATION

INTRODUCTION

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a partial differential equation contains one dependent variable and more than one independent variable

Notations in PDE

$p = \partial z/\partial x$ $q = \partial z/\partial y$

$r = \partial^2 z/\partial x^2$

$s = \partial^2 z/\partial x\partial y$

$t = \partial^2 z/\partial y^2$

Formation of partial differential equations:

There are two methods to form a partial differential equation.

(i) By elimination of arbitrary constants.

(ii) By elimination of arbitrary functions.

Formation of partial differential equations by elimination of arbitrary constants:

1. Form a p.d.e by eliminating the arbitrary constants a and b from $Z=(x+a)^2+(y-b)^2$

Solution:

Given $Z= (x+a)^2+(y-b)^2$

$$P = \frac{\partial z}{\partial x} = 2(x+a) \quad , \qquad \text{ie)} \; x+a = \frac{p}{2}$$

$$q = \frac{\partial z}{\partial y} = 2(y-b) \quad , \qquad \text{ie)} \; y-b = \frac{q}{2}$$

$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$4z = p^2+q^2$

which is the required p.d.e.

2. Find the p.d.e of all planes having equal intercepts on the X and Y axis.

Solution:

Intercept form of the plane equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Given : a=b. [Equal intercepts on the x and y axis]

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad .. \qquad (1)$$

Here a and c are the two arbitrary constants.

Differentiating (1) p.w.r.to ‘x’ we get

$$\frac{1}{a}+0+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\frac{1}{a}+\frac{1}{c}p=0.$$

$$\frac{1}{a}=-\frac{1}{c}p. \qquad (2)$$

Diff (1) p.w.r.to. ‘y’ we get

$$0+\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0.$$

$$\frac{1}{a}+\frac{1}{c}q=0$$

$$\frac{1}{a}=-\frac{1}{c}q \qquad (3)$$

From (2) and (3) $\Rightarrow -\frac{1}{c}p=-\frac{1}{c}q$

p = q ,which is the required p.d e.

3. Form the p.d.e by eliminating the constants a and b from $z = ax^n+by^n$.

Solution:

Given: $z = ax^n+by^n$. (1)

$$P=\frac{\partial z}{\partial x}=anx^{n-1}$$

$$\frac{p}{n}=ax^{n-1}$$

Multiply ' x' we get, $\frac{px}{n}=ax^n$ (2)

$$q=\frac{\partial z}{\partial y}=bny^{n-1}$$

$$\frac{q}{n}=by^{n-1}$$

Multiply ' y' we get , $\frac{qy}{n}=by^n$ (3)

Substitute (2) and (3) in (1) we get the required p.d.e $z=\frac{px}{n}+\frac{qy}{n}$

zn = px+qy.

Formation of partial differential equations by elimination of arbitrary functions:

1. Eliminate the arbitrary function f from z= $f\left(\frac{y}{x}\right)$ and form a partial differential equation.

Solution:

Given z = $f\left(\frac{y}{x}\right)$ (1)

Differentiating (1) p.w.r.to 'x' we get

$$P = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad (2)$$

Differentiating (1) p.w.r.to y we get

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{-y}{x}$$

$\therefore$ px = -qy

ie) px+qy = 0 is the required p.d.e.

2. Eliminate the arbitrary functions f and g from z = f(x+iy)+g(x-iy) to obtain a partial differential equation involving z,x,y.

Solution:

Given : z = f(x+iy)+g(x-iy) (1)

$$P = \frac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy) \quad (2)$$

$$q = \frac{\partial z}{\partial y} = i f'(x+iy)-ig'(x-iy) \quad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy)+g''(x-iy) \quad (4)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -f''(x+iy)-g''(x-iy) \quad (5)$$

r + t = 0 is the required p.d.e.

3. Form the p.d.e by eliminating arbitrary function ϕ from the relation $\phi(xyz, x^2 + y^2 + z^2) = 0$

Solution:

The pde is obtained from $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$$\begin{vmatrix} yz + xyp & 2x + 2zp \\ xz + xyq & 2y + 2zq \end{vmatrix} = 0$$

(yz+xyp)(2y+2zq)-(xz+xyq)(2x+2zp)=0

SOLUTION OF PDE

Complete solution: A solution which contains as many arbitrary constants as there are independent variables is called a complete integral (or)complete solution.(number of arbitrary constants=number of independent variables)

Particular solution: A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral (or) particular solution.

General solution: A solution of a p.d.e which contains the maximum possible number of arbitrary functions is called a general integral (or) general solution.

1. Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$

Solution:

Given $\frac{\partial^2 z}{\partial y^2} = 0$

ie) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=0$

Integrating w.r.to ‘y’ on both sides

$\frac{\partial z}{\partial y}$ = a (constants)

ie) $\frac{\partial z}{\partial y}$ =f(x)

Again integrating w.r.to ‘y’ on both sides.

z = f(x) y + b which is the required solution.

Lagrange’s linear equations:

The equation of the form Pp + Qq = R is known as Lagrange’s equation, where P, Q and R are functions of x, y and z. To solve this equation it is enough to solve the subsidiary equations.

dx/P = dy/Q = dz/R

If the solution of the subsidiary equation is of the form u(x, y, z) = c_1 and v(x, y, z) = c_2 then the solution of the given Lagrange’s equation is Φ(u, v) = 0.

To solve the subsidiary equations we have two methods:

1 Method of Grouping:

Consider the subsidiary equation dx/P = dy/Q = dz/R..Take any two members say first two or last two or first and last members. Now consider the first two members dx/P = dy/Q. If P and Q contain z (other than x and y) try to eliminate it. Now direct integration gives u(x, y) = c_1. Similarly take another two members dy/Q = dz/R. If Q and R contain x(other than y and z) try to eliminate it. Now direct integration gives v(y, z) = c_2 . Therefore solution of the given Lagrange’s equation is Φ(u, v) = 0.

1. Solve px + qy = z

Solution:

The Lagrange's eqn is Pp + Qq = R

and the auxilliary eqn. is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

ie $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ (1)

Taking the first two ratios,

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating, logx = logy + loga

$$\frac{x}{y}=a \qquad (2)$$

Similarly, taking last two ratios of eqn (1),

$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating, logy = logz + logb

$$\frac{y}{z}=b \qquad (3)$$

Eqns (2) and (3) are independent solns of (1).

Hence the complete soln of the given eqn. is φ(u,v)=0

ie; $\phi\left(\frac{x}{y},\frac{y}{z}\right)=0$

Method of multiplier's

Choose any three multipliers l, m, n may be constants or function of x, y and z such that in $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

the expression lP + mQ + nR = 0. Hence ldx + mdy + ndz = 0

[since each of the above ratios equal to a constant $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{ldx+mdy+ndz}{lP+mQ+nR}=k(say)$

$$ldx+mdy+ndz=k(lP+mQ+nR)$$

If $lP + mQ + nR = 0$ then $ldx + mdy + ndz = 0$]

Now direct integration gives u(x, y, z) = c_1.

similarly choose another set of multipliers l′, m′, n′

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

the expression $l'P + m'Q + n'R = 0$

therefore $l'dx + m'dy + n'dz = 0$ (as explained earlier)

Now direct integration gives v(x, y, z) = c_2.

Therefore solution of the given Lagrange's equation is Φ(u, v) = 0.

1. Solve $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

   Solution:

   The Lagrange's eqn is Pp + Qq = R

   and the auxilliary eqn. is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$$

   Taking multpliers as x,y,z;

$$\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = k(say)$$

$$xdx + ydy + zdz = k(x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2))$$

$$xdx + ydy + zdz = 0$$

   Integrating , $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c}{2}$

   ie; $x^2 + y^2 + z^2 = c$

$u = x^2 + y^2 + z^2$ (1)

Again taking the multipliers as 1/x,-1/y,-1/z,

$$\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = k(say)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = k(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = 0$$

Integrating, log x- log y- log z= log C'

$$\frac{x}{yz} = c'$$

$v = \frac{x}{yz}$ (2)

Solution is $(x^2 + y^2 + z^2, \frac{x}{yz}) = 0$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – Numerical Methods for Solving Equations and Interpolation– SMTA1303

COURSE NAME: DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

COURSE CODE: SMTA1303

UNIT- III NUMERICAL METHODS FOR SOLVING EQUATIONS AND INTERPOLATION

INTRODUCTION

Solution of Algebraic and Transcendental Equations

A polynomial equation of the form

$$f(x) = p_n(x) = a_0 x^{n-1} + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

is called an Algebraic equation. For example,

$x^4 - 4x^2 + 5 = 0$, $4x^2 - 5x + 7 = 0$; $2x^3 - 5x^2 + 7x + 5 = 0$ are algebraic equations.

An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example,

$$\tan x - e^x = 0; \quad \sin x - xe^{2x} = 0; \qquad x\, e^x = \cos x$$

are transcendental equations.

Finding the roots or zeros of an equation of the form $f(x) = 0$ is an important problem in science and engineering. We assume that $f(x)$ is continuous in the required interval. A root of an equation $f(x) = 0$ is the value of x, say $x = \alpha$ for which $f(\alpha) = 0$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x – axis (see Fig. 1)

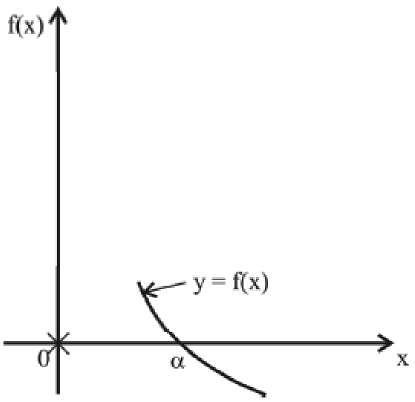


Fig. 1 Geometrical Interpretation of a root of $f(x) = 0$

A number α is a simple root of $f(x) = 0$; if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, we can write

$f(x)$ as, $f(x) = (x - \alpha)\, g(x)$, $g(\alpha) \neq 0$.

A number α is a multiple root of multiplicity m of $f(x) = 0$,

and $$f^{m}(\alpha) = 0.$$

Then, $f(x)$ can be writhen as,

$$f(x) = (x - \alpha)^{m}\, g(x),\ g(\alpha) \neq 0$$

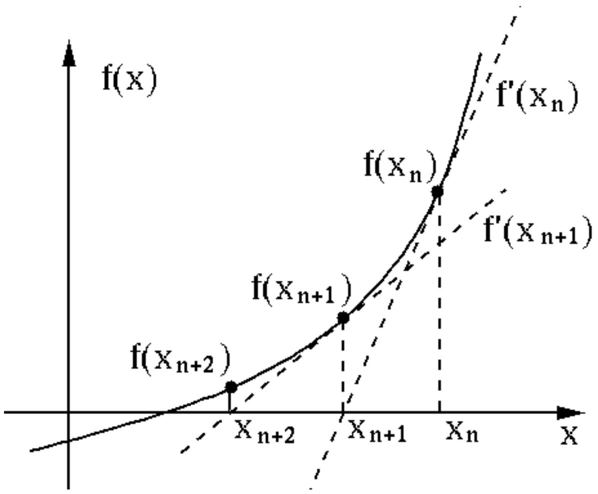
A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of $f(x)$.

The methods of finding the roots of $f(x) = 0$ are classified as,

1. Direct Methods
2. Numerical Methods.

Direct methods give the exact values of all the roots in a finite number of steps. Numerical methods are based on the idea of successive approximations. In these methods, we start with one or two initial approximations to the root and obtain a sequence of approximations x_0, x_1, … x_k which in the limit as $k \to \infty$ converge to the exact root $x = a$. There are no direct methods for solving higher degree algebraic equations or transcendental equations. Such equations can be solved by Numerical methods. In these methods, we first find an interval in which the root lies. If a and b are two numbers such that $f(a)$ and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between a and b. We take a or b or any valve in between a or b as first approximation x_1. This is further improved by numerical methods. Here we discuss few important Numerical methods to find a root of $f(x) = 0$.

NEWTON RAPHSON METHOD



This is another important method. Let x_0 be approximation for the root of $f(x) = 0$. Let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_1) = f(x_0 + h)$ by Taylor series, we get

$$f(x_1) = f(x_1 + h) = f(x_0) + h\ f'(x_0) + \frac{h^2}{2!}\ f''(x_0) + \ldots\ldots = 0 \qquad \ldots..(1)$$

For small valves of h, neglecting the terms with h^2, h^3 ….. etc,. We get

$$\therefore \qquad f(x_0) + h\ f'(x_0) = 0 \qquad \ldots..(2)$$

and $\qquad h = -\dfrac{f(x_0)}{f'(x_0)}$

$$\therefore \qquad x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Proceeding like this, successive approximation $x_2, x_3, \ldots x_{n+1}$ are given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \ldots..(3)$$

For n = 0, 1, 2, ……

***Note*:**

(i) The approximation x_{n+1} given by (3) converges, provided that the initial approximation x_0 is chosen sufficiently close to root of $f(x) = 0$.

(ii) Convergence of Newton-Raphson method: Newton-Raphson method is similar to iteration method

$$\phi(x) = x - \frac{f(x)}{f'(x)} \qquad \ldots..(1)$$

differentiating (1) w.r.t to 'x' and using condition for convergence of iteration method i.e.

$$|\phi'(x)| < 1,$$

We get

$$\left|1-\frac{f'(x).f'(x)-f(x)f''(x)}{[f'(x)]^2}\right|<1$$

Simplifying we get condition for convergence of Newton-Raphson method is

$$|f(x).f''(x)|<[f(x)]^2$$

Example 1

Using Newton-Raphson method (a) Find square root of a number (b) Find a reciprocal of a number.

Solution

(a) Let n be the number and $x = \sqrt{n}$ $x^2 = n$

If $f(x) = x^2 - n = 0$ …..(1)

Then the solution to $f(x) = x^2 - n = 0$ is $x = \sqrt{n}$

$f^1(x) = 2x$

by Newton Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \left(\frac{x_i^2 - n}{2x_i}\right)$$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{x}{x_i}\right)$$

using the above formula the square root of any number 'n' can be found to required accuracy.

(b) To find the reciprocal of a number 'n'

$f(x) = \frac{1}{x} - n = 0$ …..(1)

$\therefore$ solution of (1) is $x = \frac{1}{n}$

$f^1(x) = -\frac{1}{x^2}$

Now by Newton-Raphson method,

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)} \right)$$

$$x_{i+1} = x_i - \left(\frac{\frac{1}{x_i} - N}{-\frac{1}{x_1^2}} \right)$$

$$x_{i+1} = x_i \ (2 - x_i \ n)$$

using the above formula the reciprocal of a number can be found to required accuracy.

Example 2

Find the reciprocal of 18 using Newton–Raphson method

Solution

The Newton-Raphson method

$x_{i+1} = x_i \ (2 - x_i \ n)$ …..(1)

considering the initial approximate value of x as $x_0 = 0.055$ and given $n = 18$

$\therefore\ x_1 = 0.055\ [2 - (0.055)\ (18)]$

$\therefore\ x_1 = 0.0555$

$x_2 = 0.0555\ [2 - 0.0555 \times 18]$

$x_2 = (0.0555)\ (1.001)$

$x_2 = 0.0555$

Hence $x_1 = x_2 = 0.0555$

$\therefore$ The reciprocal of 18 is 0.0555.

Example 3

Find a real root for $x \tan x + 1 = 0$ using Newton–Raphson method

Solution

Given $f(x) = x \tan x + 1 = 0$

$f^1(x) = x \sec2 x + \tan x$

$f(2) = 2 \tan 2 + 1 = -3.370079 < 0$

$f(3) = 2 \tan 3 + 1 = -0.572370 > 0$

$\therefore$ The root lies between 2 and 3

Take $x_0 = \frac{2+3}{2} = 2.5$ (average of 2 and 3), By Newton-Raphson method

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)}\right)$$

$$x_1 = x_0 - \left(\frac{f(x_0)}{f^1(x_0)}\right)$$

$$x_1 = 2.5 - \frac{(-0.86755)}{3.14808}$$

$$x_1 = 2.77558$$

$$x_2 = x_1 - \frac{f(x_i)}{f^1(x_i)} ;$$

$f(x_1) = -0.06383,\quad f^1(x_1) = 2.80004$

$$x_2 = 2.77558 - \frac{(-0.06383)}{2.80004}$$

$$x_2 = 2.798$$

$f(x_2) = -0.001080,\quad f^1(x_2) = 2.7983$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)} = 2.798 - \frac{[-0.001080]}{2.7983}$$

$$x_3 = 2.798.$$

$\therefore \quad x_2 = x_3$

$\therefore$ The real root of $x \tan x + 1 = 0$ is 2.798

Example 4

Find a root of $e^x \sin x = 1$ using Newton–Raphson method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

$f^1(x) = e^x \sin x + e^x \cos x$

Take $x_1 = 0$, $x_2 = 1$

$f(0) = f(x_1) = e^0 \sin 0 - 1 = -1 < 0$

$f(1) = f(x_2) = e^1 \sin(1) - 1 = 1.287 > 0$

The root of the equation lies between 0 and 1.Using Newton Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

Now consider x_0 = average of 0 and 1

$$x_0 = \frac{1+0}{2} = 0.5$$

$$x_0 = 0.5$$

$$f(x_0) = e^{0.5} \sin(0.5) - 1$$

$$f^1(x_0) = e^{0.5} \sin(0.5) + e^{0.5} \cos(0.5) = 2.2373$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.5 - \frac{(-0.20956)}{2.2373}$$

$$x_1 = 0.5936$$

$$f(x_1) = e^{0.5936} \sin(0.5936) - 1 = 0.0128$$

$$f^1(x_1) = e^{0.5936} \sin(0.5936) + e^{0.5936} \cos(0.5936) = 2.5136$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.5936 - \frac{(0.0128)}{2.5136}$$

$\therefore$ $$x_2 = 0.58854$$

similarly $$x_3 = x_2 - \frac{f(x_1)}{f^1(x_1)}$$

$$f(x_2) = e^{0.58854} \sin(0.58854) - 1 = 0.0000181$$

$$f^1(x_2) = e^{0.58854} \sin(0.58854) + e^{0.58854} \cos(0.58854)$$

$$f(x_2) = 2.4983$$

$\therefore$ $$x_3 = 0.58854 - \frac{0.0000181}{2.4983}$$

$$x_3 = 0.5885$$

$\therefore$ $$x_2 - x_3 = 0.5885$$

0.5885 is the root of the equation $e^x \sin x - 1 = 0$

SOLVING A SYSTEM OF LINEAR EQUATIONS

The system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots\ldots, a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots, a_{2n}x_n = b_2 \\ \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots, a_{nn}x_n = b_n \end{cases}$$

can be expressed in the matrix form as AX = B where

where $[A]$ is an $n \times n$ matrix of coefficients

$$\begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

B is the $n \times 1$ column vector of constants, and X is the $n \times 1$ column vector of unknowns:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}$$

Two types of approach, Direct and Indirect (or Iterative) methods are used for solving system of linear equations. In direct methods, the solution is obtained by performing arithmetic operations with the equations using matrix form whereas in iterative methods, an initial approximate solution is assumed and then an iterative process is used for obtaining successively more accurate solutions.

Direct methods for solving system of equations

i) Gauss-Jordan method ii) Crout's method

Indirect methods for solving system of equations

Gauss-Seidel method

In Gauss Jordan method, coefficient matrix is converted into a diagonal matrix.

In the case of Crout's method, "A" matrix is decomposed into LU matrix where L is the lower triangular matrix and U is the unit upper triangular matrix and then the unknown values are obtained.

In Gauss-Seidel method, transform the equations in such a way that the first equation has 'x' coefficient as the largest; the second equation has 'y' coefficient as the largest; the third equation has 'z' coefficient as the largest and so on. Assuming initially the values of the unknowns as 0, refine the values of the unknowns by taking the latest values at each stage.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

Let's now perform row operations on this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 - 7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row echelon form so we can stop. It corresponds to the augmented matrix of the following system.

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

We can express the solutions of this system as

$$x = \tfrac{7}{4}z, \quad y = \tfrac{1}{2} - \tfrac{1}{4}z.$$

Since there is no specific value for z, it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$\boxed{x = \tfrac{7}{4}t, \quad y = \tfrac{1}{2} - \tfrac{1}{4}t, \quad z = t}$$

Any value of the parameter t gives us a solution of the system. For example,

$$t = 4 \quad \text{gives the solution} \quad (x, y, z) = (7, -\tfrac{1}{2}, 4)$$

$$t = -2 \quad \text{gives the solution} \quad (x, y, z) = (-\tfrac{7}{2}, 1, -2).$$

Example 4. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} A+B+2C=1 \\ 2A-B+D=-2 \\ A-B-C-2D=4 \\ 2A-B+2C-D=0 \end{cases}$$

Solution: We will perform row operations on the augmented matrix of the system until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right] \xrightarrow{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1/2 & 4 & -17/2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array}\right] \xrightarrow{-\frac{1}{18}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-8R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-\frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-2R_3,\ R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{A=1, \quad B=2, \quad C=-1, \quad D=-2.}$$

CROUT'S METHOD

1. Solve by Crout's method the system of equations 2x+3y+z = -1,5x+y+z = 9,3x+2y+4z = 11

The given system of equations can be written in matrix form AX=B as follows

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

Let, A=LU such that

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & ag & ah \\ b & bg+c & bh+ci \\ d & dg+e & dh+ei+f \end{bmatrix}$$

$$\therefore a = 2, b = 5, d = 3$$

$$ag = 3 \quad \therefore g = \frac{3}{2}$$

$$bg + c = 1 \quad \therefore c = 1 - 5(3/2) = \frac{-13}{2}$$

$$dg + e = 2 \quad \therefore e = 2 - 3.\frac{3}{2} = \frac{-5}{2}$$

$$ah = 1 \quad \therefore h = 1/2$$

$$bh + ci = -3 \quad \therefore (5)(1/2) + (-13/2)i = 1$$

$$\therefore i = \frac{1 - 5/2}{-13/2} = \frac{-3}{2} \times \frac{-2}{12} = \frac{3}{13}$$

$$dh + ei + f = 4$$

$$\therefore 3(\frac{1}{2}) + \frac{-5}{2} \cdot \frac{3}{13} + f = 4 \quad \therefore f = 4 - \frac{3}{2} + \frac{15}{26} = \frac{104 - 39 + 15}{26} = \frac{80}{26} = \frac{40}{13}$$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & -13/2 & 0 \\ 3 & -5/2 & 40/3 \end{bmatrix} \quad and\, U = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/13 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Now, \;\; LY = B \;\; Where Y = UX$$

$$\therefore \begin{bmatrix} 2 & 0 & 0 \\ 5 & -13/2 & 0 \\ 3 & -5/2 & 40/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$$\therefore 2y_1 = -1 \quad \therefore y_1 = -1/2$$

$$5y_1 - \frac{13}{2}y_2 = 19 \;\therefore -\frac{13}{2}y_2 = 9 - 5(-1/2) = 9 + \frac{5}{2} = \frac{23}{2}$$

$$\therefore y_2 = \frac{23}{2} \cdot \frac{2}{-13} = \frac{-23}{13}$$

$$3y_1 - \frac{5}{2} \cdot \frac{-23}{13} + \frac{40}{13}y_3 = 11$$

$$\therefore \frac{40}{13}y_3 = 11 + \frac{3}{2} - \frac{115}{26} = \frac{210}{26} \quad \therefore y_3 = \frac{210}{26} \times \frac{13}{40} = \frac{21}{8}$$

$$Now\; UX = Y$$

$$\therefore \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/13 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ -23/12 \\ 21/8 \end{bmatrix}$$

$$\therefore z = \frac{21}{8} = 2.625$$

$$y + \frac{3}{13}z = -\frac{23}{13} \quad \therefore y = \frac{-23}{13} - \frac{3}{13} . \frac{21}{8} = \frac{-247}{104} = -2.375$$

$$x - \frac{3}{2}y + \frac{1}{2}z = \frac{-1}{2} \quad x = \frac{-3}{2} . \frac{-247}{104} - \frac{1}{2} . \frac{21}{8} - \frac{1}{2}$$

$$\therefore x = 1.75$$

$$\therefore x = 1.75, y = -2.375, z = 2.625$$

2. Solve by Crout's method the system of equations

$5X_1 + 4X_2 + X_3 = 3.4$

$10X_1 + 9X_2 + 4X_3 = 8.8$

$10X_1 + 13X_2 + 15X_3 = 19.2$

$$A = \begin{pmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{pmatrix} \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad B = \begin{pmatrix} 3.4 \\ 8.8 \\ 19.2 \end{pmatrix}$$

Let A = LU

$$\begin{pmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{pmatrix} = \begin{pmatrix} a & ag & ah \\ b & bg + c & bh + ci \\ d & dg + e & dh + ei + f \end{pmatrix}$$

a = 5 b = 10 d = 10

g = 4/5 c = 1 e = 5

h = 1/5 i = 5 f = 3

Let Ly = B

$$\begin{pmatrix} 5 & 0 & 0 \\ 10 & 1 & 0 \\ 10 & 5 & 3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 8.8 \\ 19.2 \end{pmatrix}$$

$5Y_1 = 3.4; Y_1 = 0.68$

$10Y_1 + Y_2 = 8.8; Y_2 = 2$

$10Y_1 + 5Y_2 + 3Y_3 = 19.2; Y_3 = 0.80$

$Y_1 = 0.68 \quad Y_2 = 2 \quad Y_3 = 0.80$

Let Ux = y

$$\begin{pmatrix} 1 & \frac{4}{5} & \frac{1}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.68 \\ 2 \\ 0.80 \end{pmatrix}$$

$X_3 = 0.80$

$X_2 + 2X_3 = 2; X_2 = 0.40$

$X_1 + 4/5X_2 + 1/5X_3 = 0.68; X_1 = 0.20$

$$X = \begin{pmatrix} 0.20 \\ 0.40 \\ 0.80 \end{pmatrix}$$

Example 1

Solve the following system of equations by Gauss – Seidel method

$28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

Solution

Since the diagonal element in given system are not dominant, we rearrange the equation as follows

$28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

Hence

$x = 1/28[32 - 4y + z]$

$y = 1/17[35 - 2x - 4z]$

$z = 1/10[24 - x - 3y]$

Setting y = 0 and z = 0, we get,

First iteration

$x^{(1)} = 1/28 [32 - 4(0) + (0)] = 1.1429$

$y^{(1)} = 1/17 [35 - 2(1.1429) - 4(0)] = 1.9244$

$z^{(1)} = 1/10 [24 - 1.1429 - 3(1.9244)] = 1.8084$

Second Iteration

$x^{(2)}$ = 1/28 [32- 4(1.9244) +(1.8084)] = 0.9325

$y^{(2)}$ = 1/17 [35 – 2(0.9325) -4(1.8084)] = 1.5236

$z^{(2)}$ = 1/10 [24 – 0.9325 – 3(1.5236)] = 1.8497

Third Iteration

$x^{(3)}$ = 1/28 [32- 4(1.5236) +(1.8497)] = 0.9913

$y^{(3)}$ = 1/17 [35 – 2(0.9913) -4(1.8497)] = 1.5070

$z^{(3)}$ = 1/10 [24 –0.9913– 3(1.5070)] = 1.8488

Fourth Iteration

$x^{(4)}$ = 1/28 [32- 4(1.5070) +(1.8488)] = 0.9936

$y^{(4)}$ = 1/17 [35 – 2(0.9936) -4(1.8488)] = 1.5069

$z^{(4)}$ = 1/10 [24 – 0.9936 – 3(1.5069)] = 1.8486

Fifth Iteration

$x^{(5)}$ = 1/28 [32- 4(1.5069) +(1.8486)] = 0.9936

$y^{(5)}$ = 1/17 [35 – 2(0.9936) -4(1.8486)] = 1.5069

$z^{(5)}$ = 1/10 [24 – 0.9936 – 3(1.5069)] =1.8486

Since the values of x, y, z are same in the 4^{th} and 5^{th} Iteration, we stop the procedure here.

Hence x = 0.9936, y = 1.5069, z = 1.8486.

Interpolation

The process of computing intermediate values of (x_0, x_n) for a function $y(x)$ from a given set of values of a function

Gregory-Newton's forward interpolation formula

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + --- (a)$$

$$where\ u = \frac{1}{h}(x - x_0)$$

Gregory-Newton's backward interpolation formula

$$y(x) = y_n + \frac{\nabla y_n}{1}v + \frac{\nabla^2 y_n}{2}v(v+1) + \frac{\nabla^3 y_n}{6}v(v+1)(v+2) + \frac{\nabla^4 y_n}{24}v(v+1)(v+2)(v+3) + --- (b)$$

$$where\ v = \frac{1}{h}(x - x_n)$$

Remark:

(i) The process of finding the values of $y(x_i)$ outside the interval (x_0, x_n) is called *extrapolation*

(ii) The *interpolating polynomial* is a function $p_n(x)$ through the data points $y_i = f(x_i) = P_n(x_i)$ i=0,12,..n

(iii) Gregory-Newton's forward interpolation formula (a) can be applicable if the interval difference h is constant and used to interpolate the value of $y(x_i)$ nearer to beginning value x_0 of the data set

(iv) If $y = f(x)$is the exact curve and $y = p_n(x)$ is the interpolating polynomial then the *Error in polynomial interpolation* is $y(x) - p_n(x)$ given by

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)--(x-x_n): x_0 < x < x_n, x_0 < c < x_n --- (c)$$

(v) *Error in Newton's forward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}u(u-1)(u-2)--(u-n): x_0 < x < x_n, x_0 < c < x_n ---- (d)$$

(vi) *Error in Newton's backward interpolation is*

$$Error = \frac{h^{n+1} y^{(n+1)}(c)}{(n+1)!} v(v+1)(v+2)--(v+n): x_0 < x < x_n, x_0 < c < x_n ----(e)$$

Problem1: Estimate θ at $x = 43$ & $x = 84$ from the following table .also find $y(x)$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation

Difference Table:

x	$\theta = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	$184 = y_0$	$y_1 - y_0 = 20 = \Delta y_0$				
50	$204 = y_1$	$y_2 - y_1 = 22 = \Delta y_1$	$2 = \Delta^2 y_0$	$0 = \Delta^3 y_0$		
60	$226 = y_2$	$y_3 - y_2 = 24 = \Delta y_2$	$2 = \Delta^2 y_1$	$0 = \Delta^3 y_1$	$0 = \Delta^4 y_0$	$0 = \nabla^5 y_n$
70	$250 = y_3$	$y_4 - y_3 = 26 = \Delta y_3$	$2 = \Delta^2 y_2$	$0 = \nabla^3 y_n$	$0 = \nabla^4 y_n$	
80	$276 = y_4$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$2 = \nabla^2 y_n$			
90	$304 = y_n$					

Case (i): to find the value of θ at $x = 43$

Since $x = 43$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(43-40) = \frac{3}{10} = 0.3 \Rightarrow u-1 = -0.7, u-2 = -1.7, u-3 = -2.7 --- (2)$$

Substituting (2) in (1), we get $y(x=43) = 184 + \frac{20}{1}(\frac{3}{10}) + \frac{2}{2}(\frac{3}{10})(\frac{-7}{10}) + 0 = \frac{18979}{10} = 189.79$

Case (ii): to find the value of θ at $x = 84$

Since $x = 84$ is nearer to x_n we apply Newton's backward Interpolation

$$y(x) = y_n + \frac{\nabla y_n}{1} v + \frac{\nabla^2 y_n}{2} v(v+1) + \frac{\nabla^3 y_n}{6} v(v+1)(v+2) + \frac{\nabla^4 y_n}{24} v(v+1)(v+2)(v+3) + --- (3)$$

$$where\ v = \frac{1}{h}(x - x_n) = \frac{1}{10}(84-90) = \frac{-6}{10} \Rightarrow v+1 = \frac{4}{10}, v+2 = \frac{14}{10}, v+3 = \frac{24}{10} --- (4)$$

Substituting (4) in (3), we get $y(x=84) = 304 + \frac{28}{1}(\frac{-6}{10}) + \frac{2}{2}(\frac{-6}{10})(\frac{4}{10}) + 0 = \frac{7174}{25} = 286.96$

To find polynomial $y(x)$, from (1) we get

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + --- (1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(x-40) \Rightarrow u-1 = \frac{1}{10}(x-50), u-2 = \frac{1}{10}(x-60), u-3 = \frac{1}{10}(x-60) --- (2)^1$$

Substituting (4) in (3), we get

$$y(x) = 184 + \frac{20}{1}\frac{1}{10}(x-40) + \frac{2}{2}\frac{1}{10}(x-40)\frac{1}{10}(x-50) + 0 = 184 + 2x - 80 + \frac{1}{100}(x^2 - 90x + 2000)$$

$$\Rightarrow y(x) = \frac{1}{100}(x^2 + 110x + 12400) ---------- (5)$$

To Estimate θ at $x = 43$ & $x = 84$, put $x = 43$ & $x = 84$ in (5), we get

$$y(43) = \frac{1}{100}(18979) = 189.79\ and\ \ y(84) = \frac{1}{100}(28696) = 286.96$$

Problem2: Estimate the number of students whose weight is between 60 lbs and 70 lbs from the following data

Weight(lbs)	0-40	40-60	60-80	80-100	100-120
No.Students	250	120	100	70	50

Solution: let x-Weight less than 40 lbs, y-Number of Students, $\Rightarrow x_0 = 40, x_1 = 60, x_2 = 80, x_3 = 100, x_n = 120$, Here all the intervals are equal with $h=x_1-x_0=20$ we apply Newton interpolation

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$250 = y_0$	$y_1 - y_0 = 120 = \Delta y_0$			
60	$370 = y_1$	$y_2 - y_1 = 100 = \Delta y_1$	$-20 = \Delta^2 y_0$	$-10 = \Delta^3 y_0$	
80	$470 = y_2$	$y_3 - y_2 = 70 = \Delta y_2$	$-30 = \Delta^2 y_1$	$10 = \nabla^2 y_n$	$20 = \Delta^4 y_0 = \nabla^4 y_n$
100	$540 = y_3$	$y_n - y_{n-1} = 50 = \nabla y_n$	$-20 = \nabla^2 y_n$		
120	$590 = y_n$				

Case (i): to find the number of students y whose weight less than 60 lbs $(x = 60)$

From the difference table the number of students y whose weight less than 60 lbs $(x = 60) = 370$

Case (ii): to find the number of students y whose weight less than 70 lbs $(x = 70)$

Since $x = 70$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ----(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{20}(70-40) = \frac{3}{2} \Rightarrow u-1 = \frac{3}{2}, u-2 = \frac{2}{2}, u-2 = \frac{-1}{2}, u-3 = \frac{-3}{2} -----(2)$$

Substituting (2) in (1), we get

$$y(x=70) = 250 + \frac{120}{1}(\frac{3}{2}) + \frac{-20}{2}(\frac{3}{2})(\frac{1}{2}) + \frac{-10}{6}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2}) + \frac{20}{24}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2}) = 423.59$$

The number of students y whose weight less than 70 lbs $(x = 70)$ =424

Number of students whose weight is between 60 lbs and 70 lbs $=$

$$\begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 70 lbs} \end{Bmatrix} - \begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 60 lbs} \end{Bmatrix} = 424-370 = 54$$

Lagrange's interpolation formula Unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)--(x-x_n)}{(x_0-x_1)(x_0-x_2)--(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)--(x-x_n)}{(x_1-x_0)(x_1-x_2)--(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)--(x-x_n)}{(x_2-x_0)(x_2-x_1)--(x_2-x_n)}y_2+---+\frac{(x-x_0)(x-x_1)--(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)--(x_n-x_{n-1})}y_n$$

Problem 3: Determine the value of $y(1)$ from the following data using Lagrange's Interpolation

x	-1	0	2	3
y	-8	3	1	12

Solution: given

x	$x_0=-1$	$x_1=0$	$x_2=3$	$x_n=3$
y	$y_0=-8$	$y_1=3$	$y_2=1$	$y_n=12$

Since the intervals ere not uniform we cannot apply Newton's interpolation.

Hence by Lagrange's interpolation for unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}y_2+\frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}y_n$$

$$y(x)=\frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)----(1)$$

To compute $y(1)$ put $x=1$ in (1), we get

$$y(x=1)=\frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(1+1)(1-2)(1-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(1+1)(1-0)(1-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(1+1)(1-0)(1-2)}{(3+1)(3-0)(3-2)}(12)$$
$$\Rightarrow y(x=1)=2$$

To find polynomial $y(x)$, from (1) we get

$$y(x)=\frac{2}{3}(x^3-5x^2+6x)+\frac{1}{2}(x^3-4x^2+x+6)$$

$$-\frac{1}{6}(x^3-2x^2-3x)+\frac{1}{1}(x^3-x^2-2x)----(1)$$

$$y(x)=x^3(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1)+x^2(\frac{-10}{3}+\frac{-4}{2}+\frac{2}{6}-1)+x\ (\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2)+(\frac{6}{2})$$

$$\Rightarrow y(x)=2x^3-6x^2+3x\ +3-----(2)$$

To compute $y(1)$ put $x=1$ in (2), we get $y(x=1)=2-6+3+3=2$

Inverse interpolation

For a given set of values of x and y, the process of finding $x(dependent)$ given $y(independent)$ is called Inverse interpolation

$$x(y)=\frac{(y-y_1)(y-y_2)--(y-y_n)}{(y_0-y_1)(y_0-y_2)--(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)--(y-y_n)}{(y_1-y_0)(y_1-y_2)--(y_1-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_1)--(y-y_n)}{(y_2-y_0)(y_2-y_1)--(y_2-y_n)}x_2+---+\frac{(y-y_0)(y-y_1)--(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)--(y_n-y_{n-1})}x_n$$

Problem 4: Estimate the value of x given $y=100$ from the following data, $y(3)=6$ $y(5)=24$, $y(7)=58$, $y(9)=108$, $y(11)=174$

Solution: given

x	$x_0=3$	$x_1=5$	$x_2=7$	$x_3=9$	$x_n=11$
y	$y_0=6$	$y_1=24$	$y_2=58$	$y_3=108$	$y_n=174$

By applying Lagrange's inverse interpolation

$$x(y)=\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_n)}{(y_0-x_1)(y_0-y_2)(y_0-y_3)(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_n)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_n)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_n)}x_3$$

$$+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)(y_n-y_{n-1})}x_n$$

$$\Rightarrow x(100)=\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3)+\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$

$$+\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7)+\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$

$$+\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$

$$\Rightarrow x(100)=0.35344-1.51547+2.88703+7.06759-0.13686=8.65573$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Numerical Solution of Ordinary Differential Equation – SMTA1303

COURSE NAME: DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

COURSE CODE: SMTA1303

UNIT - IV NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Introduction

An ordinary differential equation of order n in of the form $F(x, y, y', y'', \dots, y^{(n)}) = 0$, where $y^{(n)} = \frac{d^n y}{dx^n}$.

We will discuss the Numerical solution to first order linear ordinary differential equations by Taylor series method, and Runge - Kutta method, given the initial condition $y(x_0) = y_0$.

Taylor Series method

Consider the first order differential equation of the form $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

The solution of the above initial value problem is obtained in two types

- Power series solution
- Point wise solution

(i) Power series solution

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

(ii) Point wise solution

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

Problems:

Using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = y + 1, y(0) = 1$.

Solution:

Given $\frac{dy}{dx} = y + 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

Taylor series formula for $y(0.1)$ is

$$y(x) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots$$

$y'(x) = y + 1$	$y'(0) = y(0) + 1 = 1 + 1 = 2$
$y''(x) = y'$	$y''(0) = y'(0) = 2$
$y'''(x) = y''$	$y'''(0) = y''(0) = 2$

Substituting in the Taylor's series expansion:

$$y(0.1) = y(0) + hy'(0) + \frac{h^2}{2!}y''(0) + \cdots$$

$$= 1 + 0.1 \times 2 + \frac{0.01}{2} \times 2 + \frac{0.001}{6} \times 2 + \cdots$$

$$y(0.1) = 1.2103$$

Find the Taylor series solution with three terms for the initial value problem $\frac{dy}{dx} = x^2 + y, y(1) = 1$

Solution:

Given $\frac{dy}{dx} = x^2 + y, x_0 = 1, y_0 = 1$

$y'(x) = x^2 + y$	$y'(1) = 1 + 1 = 2$
$y''(x) = 2x + y'$	$y''(1) = 2 + 2 = 4$
$y'''(x) = 2 + y''$	$y'''(1) = 2 + 4 = 6$

$y^{iv}(x) = y'''$	$y^{iv}(1) = 6$

The Taylor's series expansion about a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Hence at $x = 1$

$$y(x) = y(1) + \frac{(x - 1)}{1!}y'(1) + \frac{(x - 1)^2}{2!}y''(1) + \frac{(x - 1)^3}{3!}y'''(1) + \cdots$$

$$y(x) = 1 + 2\frac{(x - 1)}{1!} + 4\frac{(x - 1)^2}{2!} + 6\frac{(x - 1)^3}{3!} + \cdots$$

Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution:

Differentiating successively, we get

$$y' = x + y \qquad y'(0) = 1 \qquad [\because y(0) = 1]$$

$$y'' = 1 + y' \qquad y''(0) = 2$$

$$y''' = y'' \qquad y'''(0) = 2$$

$$y'''' = y''' \qquad y''''(0) = 2, \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0$, $y_0 = 1$

$$\therefore \qquad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4)\cdots$$

Thus $$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\cdots$$

$$= 1.1103$$

and $$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$

$$= 1.2427$$

Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution:

(a) We have $y' = 2y + 3e^x$; $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0$, $y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$
$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \cdots$$
$$= 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{15}{8}x^4 + \cdots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.2)^4 + \cdots = 0.8110$ $\quad(i)$

(b) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x

Its I.F. being e_{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \cdot e^{-2x}\,dx + c = -3e^{-x} + c \text{ or } \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\qquad \therefore c = 3.$

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ $\quad(ii)$

Comparing (i) and (ii), it is clear that (i) approximates to the exact value up to three decimal places

Solving simultaneous equations by Taylor's series method

Let the simultaneous differential equations be

$$\frac{dy}{dx} = f(x, y, z) \qquad (1)$$

and $$\frac{dz}{dx} = g(x, y, z) \qquad (2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$

Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' \ldots \qquad (3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' \ldots \qquad (4)$$

Differentiating (1) and (2) successively we get y'', z'', etc.

So the values $y_0', y_0'', y_0''' \ldots$ and $z_0', z_0'', z_0''' \ldots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' \ldots \qquad (5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' \ldots \qquad (6)$$

Since y_1 and z_1 are known, we calculate $y_1', y_1'' \ldots$ and $z_1', z_1'' \ldots$ substituting these in (5) and (6) we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

Given $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = -xz - y$ with $y(0) = 1$, $z(0) = 0$, obtain y and z for $x = 0.1, 0.2, 0.3$ by Taylor's series method.

Hint:

We have

$$y' = z \quad \text{and} \quad z' = -xz - y$$

We use Taylor's series method to find y and z.

Runge-Kutta method

Runge-kutta methods of solving initial value problem do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formula possesses the advantage of requiring only the function values at some selected points. These methods agree with Taylor series solutions up to the term in h^r where r is called the order of that method.

Fourth-order Runge-Kutta method

Let $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ be given.

Working rule to find $y(x_1)$

The value of $y_n = y(x_n)$ where $x_n = x_{n-1} + h$ where h is the incremental value for x is obtained as below:

Compute the auxiliary values

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Compute the incremental value for y

$$\Delta y = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

The iterative formula to compute successive value of y is $y_{n+1} = y_n + \Delta y$

Problems

Find the value of y at $x = 0.2$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1$, using R-K method of order IV.

Sol:

Here $f(x, y) = x^2 + y, y(0) = 1$

Choosing $h = 0.1, x_0 = 0, y_0 = 1$

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.00525$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.00525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.0110050$$

$y(0.1) = 1.0053$

To find $y(0.2)$ given $x_2 = x_1 + h = 0.2, y_1 = 1.0053$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.0110$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.01727$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.01728$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.02409$$

y(0.2)=1.0227

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Numerical Solution of Partial Differential Equation – SMTA1303

COURSE NAME: DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

COURSE CODE: SMTA1303

UNIT - V NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Numerical Solution to Partial Differential Equations

Solution of Laplace Equation and Poisson equation

Partial differential equations with boundary conditions can be solved in a region by replacing the partial derivative by their finite difference approximations. The finite difference approximations to partial derivatives at a point (x_i, y_i) are given below:

$$u_x(x_i, y_i) = \frac{u(x_{i+1}, y_i) - u(x_i, y_i)}{h}$$

$$u_y(x_i, y_i) = \frac{u(x_i, y_{i+1}) - u(x_i, y_i)}{k}$$

$$u_{xx}(x_i, y_i) = \frac{u_x(x_{i+1}, y_i) - u_x(x_i, y_i)}{h} = \frac{u(x_{i+1}, y_i) - 2u(x_i, y_i) + u(x_{i-1}, y_i)}{h^2}$$

$$u_{yy}(x_i, y_i) = \frac{u_y(x_i, y_{i+1}) - u_y(x_i, y_i)}{k} = \frac{u(x_i, y_{i+1}) - 2u(x_i, y_i) + u(x_i, y_{i-1})}{k^2}$$

Graphical Representation

The xy-plane is divided into small rectangles of length and breadth by drawing the lines x = ih and y = jk , parallel to the coordinate axes. The points of intersection of these lines are called grid points or mesh points or lattice points. The grid points (x_i ,y_j) is denoted by (i,j) and is surrounded by the neighbouring grid points (i-1, j) to the left, (i+1, j) to the right, (i, j+1) above and (i, j-1) below.

Note

The most general linear P.D.E of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x, y)$$

Where A, B, C, D, E, F are functions of x and y.

A partial differential equation in the above form is said to be

- Elliptic if $B^2 - 4AC < 0$
-
- Parabolic if $B^2 - 4AC = 0$

Hyperbolic if $B^2 - 4AC > 0$

Standard Five Point Formula (SFPF)

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$$

Diagonal Five Point Formula (DFPF)

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}]$$

Solution of Laplace equation uxx+uyy=0

Leibmann's Iteration Process

We compute the initial values of u_1, u_2, ... ,u_9 by using standard five point formula and diagonal five point formula .First we compute u5 by standard five point formula (SFPF).

$$u_5 = \frac{1}{4}[b_7 + b_{15} + b_{11} + b_3]$$

We compute u_1, u_3, u_7, u_9 by using diagonal five point formula (DFPF)

$$u_1 = \frac{1}{4}[b_1 + u_5 + b_3 + b_{15}]$$

$$u_3 = \frac{1}{4}[u_5 + b_5 + b_3 + b_7]$$

$$u_7 = \frac{1}{4}[b_{13} + u_5 + b_{15} + b_{11}]$$

$$u_9 = \frac{1}{4}[b_7 + b_{11} + b_9 + u_5]$$

Finally, we compute u_2, u4, u_6, u_8 by using standard five point formula.

$$u_2 = \frac{1}{4}[u_5 + b_3 + u_1 + u_3]$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + b_{15} + u_7]$$

$$u_6 = \frac{1}{4}[u_3 + u_9 + u_5 + b_7]$$

$$u_8 = \frac{1}{4}[u_7 + b_{11} + u_9 + u_5]$$

Solve the system of simultaneous equations obtained by finite difference method to get the value at the interior mesh points. This process is called Leibmann's method.

Solution of Poisson equation

An equation of the type $\nabla^2 u = f(x, y)$ i.e., is called Poisson's equation where f(x,y) is a function of x and y. Substituting the finite difference approximations to the partial differential coefficients, we get

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$$

Solution of One dimensional heat equation

In this chapter, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit method.

Explicit Method (Bender-Schmidt method)

Consider the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. This equation is an example of parabolic equation.

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

The above expression is called the explicit formula and it is valid for $0 \le \lambda \le \frac{1}{2}$

If $\lambda = \frac{1}{2}$ the equation reduces to $u_{i,j+1} = \frac{1}{2}\{u_{i-1,j} + u_{i+1,j}\}$

This formula is called Bender-Schmidt formula.

Solution of One dimensional wave equation

One Dimensional wave equation $u_{tt} = a^2 u_{xx}$ is of hyperbolic type.

The solution is given by the recurrence relation

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

(for filling the other 'u' values where k = h/a). The above formula is called explicit scheme or explicit formula to solve the wave equation.

When j = 0, $u_{i,1} = \frac{1}{2}\{u_{i-1,0} + u_{i+1,0}\}$ (for filling the I row 'u' values)

Problems

Classify the PDE $u_{xx}+4u_{xy}+(x^2+4y^2)u_{yy}=0$

Solution: Here A=1, B=4, C = x^2+4y^2, $B^2-4AC = 16-4(x^2+4y^2)$,

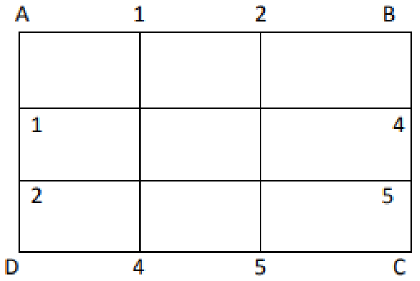
The equation is elliptic, if $4-x^2-4y^2<0$, $x^2+4y^2 > 4$, $\frac{x^2}{4}+\frac{y^2}{1}>1$.

It is elliptic in the region outside the ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$.

It is Hyperbolic inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

It is parabolic on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

. Solve $u_{xx}+u_{yy} = 0$ for the following square mesh with boundary values as shown in the figure below.



Solution: The boundary values are symmetrical about the diagonal AC but not about BD.

Let the values at the interval grid points be u_1, u_2, u_3, u_4.

By Symmetry, $u_2 = u_3; u_1 \neq u_4$.

Assume $u_2 = 3$ $\left(Since\ \ u_2 = 2+\frac{1}{3}(5-2) = 3\right)$.

Rough values: $u_1 = \frac{1}{4}(1+1+2u_2)=2$. (SFPF).

$u_2 = 3$, $u_4 = \frac{1}{4}(5+5+2u_2) = \frac{1}{2}(5+u_2)=4$

First Iteration: $u_1 = \frac{1}{2}(1+u_2)=2$, $u_2 = \frac{1}{4}(6+u_1+u_4)=\frac{1}{4}(6+2+4)=3$, $u_4 = \frac{1}{2}(5+u_2)=4$.

Result $u_1=2$, $u_2 = 3$, $u_4 = 4$.

Solve $u_{xx}-2u_t=0$, given u(0,t)=0,u(4,t)=0, u(x,0)=x(4-x).Assume h=1.Find the values of u up to t=5 by Bender-Schmidt recurrence equation.

Solution: $u_{xx}=au_t$, here a=2.

To use Bender-Schmidt recurrence equation, k=$\frac{a}{2}h^2$=1.

Step -size in time =k=1.

The values of $u_{i,j}$ are tabulated below

i / j	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Solve u_{xx}-32u_t=0, taking h=0.25 for t>0,0<x<1 and u(x,0)=0, u(0,t)=0, u(1,t)=t using Bender – Schmidt method.

Solution: The range of x is (0, 1); h=0.25.

k=$\frac{a}{2}h^2$=$\frac{32}{2}\left(\frac{1}{16}\right)$=1. Step size of t is 1.

J i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5

Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides x = 0, x = 3, y = 0, y = 3 with u = 0 on the boundary and mesh length 1 unit.

Using the theory,

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \dots \dots \dots (2)$

Applying the formula at i=1, j=2, we get, 0+0+ i=1,j=2,

we get , 0+0+u_2+$u_3 - 4u_1 = -10(15)$

$$u_2 + u_3 - 4u_1 = -150 \dots \dots \dots (3)$$

Applying at i=2, j=2, we get, $u_1 + u_4 - 4u_2 = -180 \dots \dots \dots \dots (4)$

Applying at i=1, j=1, we get, $u_1 + u_4 - 4u_3 = -120$......................(5)

Applying at i=2, j=1, we get, $u_2 + u_3 - 4u_4 = -150$......................(6)

Solving equations (3), (4), (5), (6), we get,$u_1 = u_4 = 75, u_2 = 82.5,\ u_3 = 67.50$

Solution:

	U1	U2	U3
	U4	U5	U6
	U7	U8	U9

Take the coordinate system with origin at the center of the square.

Since the boundary conditions are symmetrical about the x, y axes and x=y, we have $u_1 = u_3 = u_7 = u_9$.

$$u_2 = u_4 = u_6 = u_8$$

We need to find u_1, u_2, u_5 *only*.

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \ldots\ldots\ldots (1)$$

At i=-1, j=-1, we have, $u_2 + u_4 - 4u_1 = 8 \Rightarrow u_2 - 2u_1 = 4$ ………………(2)

At i=0, j=1, we have, $u_1 + u_3 + u_5 - 4u_2 = 0 \Rightarrow 2u_1 + u_5 - 4u_2 = 0 \ldots\ldots\ldots (3)$

At i=0, j=0, we have, $u_2 + u_4 + u_6 + u_8 - 4u_5 = 0 \Rightarrow u_2 - u_5 = 0$……………………(4)

From (2), $u_1 = \frac{1}{2}(u_2 - 4)$

From (4), $u_5 = u_2$

Using in (3), we get $u_2 = -2$.

Therefore, $u_2 = -2 = u_5$, $u_1 = -3$

Solve $\Delta^2 u = 8x^2y^2$ for square mesh given u=0 on the 4 boundaries dividing the square into 16 sub-squares of length 1 unit.

Solution:

	U1	U2	U3
	U4	U5	U6
	U7	U8	U9

Take the coordinate system with origin at the center of the square.

Since the boundary conditions are symmetrical about the x, y axes and x=y, we have $u_1 = u_3 = u_7 = u_9$.

$$u_2 = u_4 = u_6 = u_8$$

We need to find u_1, u_2, u_5 *only*.

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \dots\dots\dots (1)$$

At i=-1, j=-1, we have, $u_2 + u_4 - 4u_1 = 8 \Longrightarrow u_2 - 2u_1 = 4$ ....(2)

At i=0, j=1, we have, $u_1 + u_3 + u_5 - 4u_2 = 0 \Rightarrow 2u_1 + u_5 - 4u_2 = 0 \dots\dots\dots (3)$

At i=0, j=0, we have, $u_2 + u_4 + u_6 + u_8 - 4u_5 = 0 \Rightarrow u_2 - u_5 = 0$.......................(4)

From (2), $u_1 = \frac{1}{2}(u_2 - 4)$

From (4), $u_5 = u_2$

Using in (3), we get $u_2 = -2$.

Therefore, $u_2 = -2 = u_5$, $u_1 = -3$

Solve $u_t = u_{xx}$ subject to u(0,t)=0, u(1,t)=0 and u(x,0)=sin(πx), $o < x < 1$ by Bender-Schmidt method.

Solution: Since h and K are not given we will select them properly and use Bender-Schmidt method.

K=$\frac{a}{2}h^2$=$\frac{1}{2}h^2$, since a=1.

Since range of x is (0, 1), take h=0.2.

Hence k==$\frac{1}{2}0.2^2$=0.02.

The formulae is $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

We form the table.

J I	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

Solve numerically, $4u_{xx} = u_{tt}$ with the boundary conditions u(0,t)=0, u(4, t)=0 and the initial conditions $u_t(x,0) = 0$ and

U(x, 0) =x (4-x), taking h=1. (For 4 time steps).

Solution: Since a^2=4, h=1, k=$\frac{h}{a}$=1/2.

Taking k=1/2 , we use the formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

From u(0,t)=0⟹ U along x=0 $are\ all\ zero$.

From u(4,t) =0⇒ u along x=4 are all zero.

U(x,0)=x(4-x)⇒ u(0,0)=0, u(1,0)=3, u(2,0)=4, u(3,0)=3.

Now, we fill up the row t=0 using the above values,

$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0}+u_{i-1,0}}{2}$,

$u_{1,1} = \frac{u_{2,0}+u_{0,0}}{2}$=2, $u_{2,1} = \frac{u_{3,0}+u_{1,0}}{2} = 3$, $u_{3,1} = \frac{u_{4,0}+u_{2,0}}{2} = 2$, $u_{4,1} = 0$

T x	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	-2	-3	-2	0
2	0	-3	-4	-3	0
2.5	0	-2	-3	-2	0
3	0	0	0	0	0
3.5	0	2	3	2	0
4	0	3	4	3	0

Solve , $25u_{xx} = u_{tt}$ for u at the pivotal points, given u (0, t)= u(5,t)=0, $u_t(x,0) = 0$ and u(x,0)= 2x for 0<x<2.5,=10-2x, for 2.5<x<5 for one half period of vibration.

Solution:

a^2=25, a=5

Period of vibration=2l/a=$\frac{2x5}{5}$=2 seconds.

Half period=1 second. We want values up to t=1 second. Taking h=1, k=$\frac{h}{a}$=1/5.

Step-size in t-direction=1/5.

The explicit scheme is, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

As explained in previous problem, we have u(0,0)=0,u(1, 0)=2, u(2,0)=4, u(3,0)=4, u(4,0)=2, u(5,0)=0.

$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0}+u_{i-1,0}}{2}$,

$u_{1,1} = \frac{u_{2,0}+u_{0,0}}{2}=2, u_{2,1} = \frac{u_{3,0}+u_{1,0}}{2} = 3, u_{3,1} = \frac{u_{4,0}+u_{2,0}}{2} = 3, u_{4,1} = 2.$

T x	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	-1	-1	-1	-1	0
4/5	0	-2	-3	-3	-2	0
1	0	-2	-4	-4	-2	0

Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 unit satisfying the boundary conditions:

U(0, y)=0 for 0≤y$\leq 4, u(4,y) = 12 + y \; for \; 0 \leq y \leq 4, u(x,0) = 3x \; for \; 0 \leq x \leq 4, u(x,4) = x^2$for 0≤ $x \leq 4$.

Solution:

We divide the square mesh into 16 sub-squares of side 1 unit and calculate the numerical values of u on the boundary
using given analytical expressions.

0	1	4	9	16
	U1	U2	U3 15	
0	U4	U5	U6 14	
0	U7	U8	U9 13	
0	3	6	9 12	

Let the internal grid points be $u_1, u_2, u_3, \ldots\ldots, u_9$.

Rough values: $u_5 = \frac{1}{4}$(4+6+14+0) =6 (SFPF)

$u_1 = \frac{1}{4}$(0+6+4+0)=2.5 (DFPF)

$u_3 = \frac{1}{4}$(16+6+14+4)=10 (DFPF)

$u_7 = \frac{1}{4}$(0+6+6+0)=3 (DFPF)

$u_9 = \frac{1}{4}$(6+6+14+12)=9.5 (DFPF)

We use SFPF to get the other values of u.

$u_2 = \frac{1}{4}(2.5+6+4+10)=5.625$ (SFPF)

$u_4 = \frac{1}{4}(0+6+2.5+3)=3.125$ (SFPF)

$u_6 = \frac{1}{4}(10+6+14+9.5)=9.875$ (SFPF)

$u_8 = \frac{1}{4}(6+6+3+9.5)=6.125$ (SFPF)

Now we proceed for iteration using always SFPF.

U1 2.4375 2.3672	U2 5.6094 5.5888	U3 9.8711 9.8652
U4 2.8594 2.8698	U5 6.1172 6.1209	U6 9.8721 9.8731
U7 2.9948 3.0057	U8 6.153 6.1582	U9 9.5063 9.5078

Repeating one more iteration, we conclude, correct to 2 decimals,

$u_1 = 2.37$, $u_2 = 5.59$, $u_3 = 9.87$, $u_4 = 2.88$, $u_5 = 6.13$, $u_6 = 9.88$, $u_7 = 3.01$, $u_8 = 6.16$, $u_9 = 9.51$