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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – DISCRETE MATHEMATICS – SMTA 1302

UNIT I: LOGIC

Statements - Truth tables - Connectives - Equivalent Propositions - Tautological Implications - Normal forms -Predicate Calculus, Inference theory for Propositional Calculus and Predicate Calculus.

Propositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B,…, P,Q,…). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- "Man is Mortal", it returns truth value "TRUE"
- "12 + 9 = 3 − 2", it returns truth value "FALSE"

The following is not a Proposition −

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

Connectives

In propositional logic generally we use five connectives which are − OR (∨), AND (∧), Negation/ NOT (¬), If-then/Conditional (→), If and only if/ Biconditional (↔).

OR (∨) : The OR operation of two propositions A and B (written as A ∨ B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows −

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

AND (∧) : The AND operation of two propositions A and B (written as A ∧ B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A ∧ B
True	True	True
True	False	False
False	True	False
False	False	False

Negation (¬) : The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows –

A	¬A
True	False
False	True

If-then /Conditional (→): An implication A→ B is False if A is true and B is false. The rest of the cases are true.Here A is called Hypothesis or antecedent and q is called consequent or conclusion.

The truth table is as follows −

A	B	A→ B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (↔) : A↔B is bi-conditional logical connective which is true when p and q are both false or both are true.

The truth table is as follows −

A	**B**	A↔B
True	True	True
True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example − Prove [(A → B) ∧ A] → B is a tautology

The truth table is as follows −

A	**B**	**A → B**	**(A → B) ∧ A**	**[(A → B) ∧ A] → B**
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A → B) ∧ A] → B is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove (A ∨ B) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows −

A	B	A ∨ B	¬A	¬B	(¬A) ∧ (¬B)	(A ∨ B) ∧ [(¬A) ∧ (¬B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A ∨ B) ∧ [(¬A) ∧ (¬B)] is "False", it is a contradiction

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove (A ∨ B ∨) **∧** (¬A) a contingency

The truth table is as follows −

A	B	A ∨ B	¬A	(A ∨ B) ∧ (¬A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A ∨ B) **∧** (¬A) has both "True" and "False", it is a contingency.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions −

- The truth tables of each statement have the same truth values.

- The bi-conditional statement X ↔Y is a tautology.

Example − Prove ¬(A ∨ B) and [(¬A) ∧ (¬B)] are equivalent

Testing by 1st method (Matching truth table)

A	B	A ∨ B	¬ (A ∨ B)	¬A	¬B	[(¬A) ∧ (¬B)]
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of ¬ (A ∨ B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality)

A	B	¬ (A ∨ B)	[(¬A) ∧ (¬B)]	[¬ (A ∨ B)] ⇔ [(¬A) ∧ (¬B)]
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬ (A ∨ B)] ⇔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

EQUIVALENT LAWS

Equivalence	Name of Identity
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws
$p \wedge F \equiv F$ $p \vee T \equiv T$	Domination Laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative Laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Ditributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$	Absorption Laws
$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$	Negation Laws

Logical Equivalences involving Conditional Statements

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Logical Equivalences involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

A conditional statement has two parts − **Hypothesis** and **Conclusion**.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

Inverse, Converse, andContra-positive

Inverse –An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q". The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

Converse −The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If *q*, then *p*". The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do not do your homework".

Contra-positive –The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the inverse will be "If not *q*, then not *p*". The Contra-positive of "If you do your homework, you will not be punished" is "If you will be punished, you do your homework".

Example:

Give the converse and the Contra positve of the implication " If it is raining then I get wet".

Solution :

P : It is raining Q : I get wet

Converse : $Q \to P$: If I get wet, then it is raining.

Contrapositive : $\neg Q \to \neg P$: If I do not get wet, then it is not raining

DUALITY PRINCIPLE

Duality principle set states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said **self-dual** statement.

Examples : i) The dual of $(A \cap B) \cup C$ is $(A \cup B) \cap C$

ii) The dual of $P \wedge Q \wedge F$ is $P \vee Q \vee T$

Example : 1

Construct a truth table for $(p \to q) \to (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Example 2: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

.

Solution : The truth tables for these compound proposition is as follows.

1	2	3	4	5	6	7	8
P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$6 \leftrightarrow 7$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

We can observe that the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q.

Example 3: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution : The truth tables for these compound proposition as follows.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As the truth values of $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Example 4 : Determine whether each of the following form is a tautology or a contradiction or neither :

i) $(P \wedge Q) \rightarrow (P \vee Q)$

ii) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

iii) $(\neg P \wedge \neg Q) \rightarrow (P \rightarrow Q)$

iv) $(P \rightarrow Q) \wedge (P \wedge \neg Q)$

v) $[P \wedge (P \rightarrow \neg Q) \rightarrow Q]$

Solution:

i) The truth table for $(p \wedge q) \rightarrow (p \vee q)$

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Here all the entries in the last column are 'T'.

$\therefore (p \wedge q) \rightarrow (p \vee q)$ is a tautology.

ii) The truth table for $(p \vee q) \wedge (\neg p \wedge \neg q)$ is

1	2	3	4	5	6	
p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg P \wedge \neg q$	$3 \wedge 6$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

The entries in the last column are 'F'. Hence $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

iii) The truth table is as follows.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \to q$	$(\neg p \wedge \neg q) \to (p \to q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Here all entries in last column are 'T'.

$\therefore$ $(\neg p \wedge \neg q) \to (p \to q)$ is a tautology.

iv) The truth table is as follows.

p	q	$\neg q$	$p \wedge \neg q$	$p \to q$	$(p \to q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

All the entries in the last column are 'F'. Hence it is contradiction.

v) The truth table for $[p \wedge (p \rightarrow \neg q) \rightarrow q]$

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$	$[p \wedge (p \rightarrow \neg q) \rightarrow q]$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

The last entries are neither all 'T' nor all 'F'.

$\therefore$ $[p \wedge (p \rightarrow \neg q) \rightarrow q]$ is a neither tautology nor contradiction. It is a Contingency.

Example 5: Symbolize the following statement

Let p, q, r be the following statements:
p: I will study discrete mathematics
q: I will watch T.V.
r: I am in a good mood.
Write the following statements in terms of p, q, r and logical connectives.
(1) If I do not study and I watch T.V., then I am in good mood.
(2) If I am in good mood, then I will study or I will watch T.V.
(3) If I am not in good mood, then I will not watch T.V. or I will study.
(4) I will watch T.V. and I will not study if and only if I am in good mood.
Solution:

(1) $(\neg p \wedge q) \rightarrow r$

(2) $r \rightarrow (p \vee q)$

(3) $\neg r \rightarrow (\neg q \vee p)$

(4) $(q \wedge \neg p) \leftrightarrow r$

Example 6: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution :

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv$	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv$	$F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
	$\equiv$	$(\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
	$\equiv$	$(\neg p \wedge \neg q)$	by the identity law for **F**

Example 7: **Show that $\neg (p\leftrightarrow q) \equiv (p\vee q) \wedge \neg (p\wedge q)$ without constructing the truth table**

Solution :

$\neg (p\leftrightarrow q) \equiv (p\vee q) \wedge \neg (p\wedge q)$

$\neg (p\leftrightarrow q) \equiv \neg (p\to q) \wedge (q\to p)$

$\equiv \neg (\neg p\vee q) \wedge (\neg q\vee p)$

$\equiv \neg (\neg p\vee q) \wedge \neg q) \vee((\neg p\vee q) \wedge p)$

<u>**Elementary Product:**</u> A product of the variables and their negations in a formula is called an elementary product. If P and Q are any two atomic variables, then p, $\neg p \wedge q$, $\neg q \wedge p \wedge \neg p$ are some examples of elementary products.

<u>**Elementary Sum:**</u> A sum of the variables and their negations in a formula is called an elementary sum. If P and Q are any two atomic variables, then p, $\neg p \vee q$, $\neg q \vee p$ are some examples of elementary sums.

Normal Forms

We can convert any proposition in two normal forms −

1. Conjunctive normal form 2. Disjunctive normal form

Conjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs.

Examples

- $(P \cup Q) \cap (Q \cup R)$

- (¬P ∪ Q ∪ S ∪¬T)

Disjunctive Normal Form

A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs.

Examples

- (P ∩ Q) ∪ (Q ∩ R)
- (¬P ∩ Q ∩ S ∩¬T)

Predicate Logic deals with predicates, which are propositions containing variables.

Functionally Complete set

A set of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only this set of logical operators. ∨, ∧, and ¬form a functionally complete set of operators.

Minterms: For two variables p and q there are 4 possible formulas which consist of conjunctions of p,q or its negation given by $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$ and $\neg p \wedge \neg \neg q$

Maxterms: For two variables p and q there are 4 possible formulas which consist of disjunctions of p,q or its negation given by $p \vee q$, $p \vee \neg q$, $\neg p \vee q$ and $\neg p \vee \neg q$

Principal Disjunctive Normal Form: For a given formula an equivalent formula consisting of disjunctions of minterms only is known as principal disjunctive normal form(PDNF)

Principal Conjunctive Normal Form: For a given formula an equivalent formula consisting of conjunctions of maxterms only is known as principal conjunctive normal form(PCNF)

Obtain DNF of $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$.

Solution:

$Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge (\neg((P \vee R) \wedge Q)$ (Demorgan law)

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$ (Demorgan law)

$\Leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q)$

(Extended distributed law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law)

Obtain Pcnf and Pdnf of the formula $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

Solution:

Let S = $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

P	Q	¬ P	¬ Q	$\neg P \vee \neg Q$	$P \leftrightarrow \neg Q$	S	Minterm	Maxterm
T	T	F	F	F	F	T	$P \wedge Q$	
T	F	F	T	T	T	T	$P \wedge \neg Q$	
F	T	T	F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	T	F	F		$P \vee Q$

PCNF: $P \vee Q$ and PDNF: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Inference Theory

The theory associated with checking the logical validity of the conclusion of the given set of premises by using Equivalence and Implication rule is called **Inference theory**

Direct Method

When a conclusion is derived from a set of premises by using the accepted rules of reasoning is called **direct method**.

Indirect method

While proving some results regarding logical conclusions from the set of premises, we use negation of the conclusion as an additional premise and try to arrive at a contradiction is called **Indirect method**

Consistency and Inconsistency of Premises

A set of formular $H_1, H_2, \ldots, H_m$ is said to be **inconsistent** if their conjunction implies Contradiction.

A set of formular $H_1, H_2, \ldots, H_m$ is said to be **consistent** if their conjunction implies Tautology.

Rules of Inference

Rule P: A premise may be introduced at any point in the derivation

Rule T: A formula S may be introduced at any point in a derivation if S is tautologically implied by any one or more of the preceeding formula.

Rule CP: If S can be derived from R and set of premises , then R S can be derived from the set of premises alone.

Rules of Inference

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Rule of inference to build arguments

Example:

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny.
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. We will be home by sunset

Example 1. Show that R is logically derived from $P \rightarrow Q$, $Q \rightarrow R$, and P

Solution.	{1}	(1)	$P \rightarrow Q$	Rule P
	{2}	(2)	P	Rule P
	{1, 2}	(3)	Q	Rule (1), (2) and I11
	{4}	(4)	$Q \rightarrow R$	Rule P
	{1, 2, 4}	(5)	R	Rule (3), (4) and I11.

Example 2. Show that S V R tautologically implied by (P V Q) ∧ ($P \rightarrow R$) ∧ ($Q \rightarrow S$).

Solution .	{1}	(1)	P V Q	Rule P
	{1}	(2)	7P → Q	T, (1), E1 and E16
	{3}	(3)	Q → S	P
	{1, 3}	(4)	7P → S	T, (2), (3), and I13
	{1, 3}	(5)	7S → P	T, (4), E13 and E1
	{6}	(6)	P → R	P
	{1, 3, 6}	(7)	7S → R	T, (5), (6), and I13
	{1, 3, 6)	(8)	S V R	T, (7), E16 and E1

Example 3. Show that 7Q, P→ Q => 7P

Solution .

Solution .	{1}	(1) P → Q	Rule P
	{1}	(2) 7P → 7Q	T, and E 18
	{3}	(3) 7Q	P
	{1, 3}	(4) 7P	T, (2), (3), and I11 .

Example 4 .Prove that R ∧ (P V Q) is a valid conclusion from the premises PVQ ,
Q → R, P → M and 7M.

Solution .	{1}	(1) P → M	P
	{2}	(2) 7M	P
	{1, 2}	(3) 7P	T, (1), (2), and I12
	{4}	(4) P V Q	P
	{1, 2 , 4}	(5) Q	T, (3), (4), and I10.
	{6}	(6) Q → R	P
	{1, 2, 4, 6}	(7) R	T, (5), (6) and I11
	{1, 2, 4, 6}	(8) R ∧ (PVQ)	T, (4), (7), and I9.

Example 5 .Show that R → S can be derived from the premises
P → (Q → S), 7R V P , and Q.

Solution.	{1}	(1) 7R V P	P
	{2}	(2) R	P, assumed premise
	{1, 2}	(3) P	T, (1), (2), and I10
	{4}	(4) P → (Q → S)	P
	{1, 2, 4}	(5) Q → S	T, (3), (4), and I11
	{6}	(6) Q	P
	{1, 2, 4, 6}	(7) S	T, (5), (6), and I11
	{1, 4, 6}	(8) R → S	CP.

Example 6.Show that P → S can be derived from the premises, 7P V Q, 7Q V R, and R → S .

Solution.

{1}	(1)	7P V Q	P
{2}	(2)	P	P, assumed premise
{1, 2}	(3)	Q	T, (1), (2) and I11
{4}	(4)	7Q V R	P
{1, 2, 4}	(5)	R	T, (3), (4) and I11
{6}	(6)	R → S	P
{1, 2, 4, 6}	(7)	S	T, (5), (6) and I11
{2, 7}	(8)	P → S	CP

Predicate Logic

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

Eg.
" x is a Man"
Here **Predicate** is " is a Man" and it is denoted by M and **subject** "x" is denoted by x.
Symbolic form is M(x).

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol ∀.

∀x P(x) is read as for every value of x, P(x) is true.

Example − "Man is mortal" can be transformed into the propositional form ∀x P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.∃x P(x) is read as for some values of x, P(x) is true.

Example − "Some people are dishonest" can be transformed into the propositional form ∃x P(x) where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Nested Quantifiers

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Eg.2.

"Every apple is red".

The above statement can be restated as follows

For all x, if x is an apple then x is red

Now, we will translate it into symbolic form using universal quantifier.

Define $A(x)$: x is an apple.

$R(x)$: x is red.

∴ We write (*) into symbolic form as

$$(\forall x)\ (A(x) \rightarrow R(x))$$

Eg.3. ***"Some men are clever".***

The above statement can be restated as

"there is an x such that x is a man and x is clever".

We will translate it into symbolic form using Existential quantifier.

Let $M(x)$: x is a man

and $C(x)$: x is clever

∴ We write (B) into symbolic form as

$$(\exists x)\ (M(x) \wedge C(x))$$

Inference theory for Predicate calculus

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(y)}$	Rule US: Universal Specification
$\dfrac{P(c) \text{ for any c}}{\therefore \forall x P(x)}$	Rule UG: Universal Generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for any c}}$	Rule ES: Existential Specification
$P(c)$ for any c $\therefore \exists x P(x)$	Rule EG: Existential Generalization

Problem : 1 Show that $(\exists x)\ M(x)$ follows logically from the premises $(x)\ (H(x) \rightarrow M(x))$ and $(\exists x)\ H(x)$

Solution :		
1)	$(\exists x)\ H(x)$	rule P
2)	$H(y)$	ES
3)	$(x)\ (H(x) \rightarrow M(x))$	P
4)	$H(y) \rightarrow M(y)$	US
5)	$M(y)$	T, (2)
6)	$(\exists x)\ M(x)$	EG

Problem : 2

Symbolize the following statements:

(a) All men are mortal

(b) All the world loves a lover

(c) X is the father of mother of Y

(d) No cats has a tail

(e) Some people who trust others are rewarded

Solution:

(a) Let M(x): x is a man H(x): x is Mortal
$(\forall x)(M(x) \to H(x))$

(b) Let P(x): x is a person L(x): x is a lover R(x,y): x loves y
$(x)(P(x) \to (y)(P(y) \wedge L(y) \to R(x,y)))$

(c) Let P(x): x is a person F(x,y): x is the father of y
M(x,y): x is the mother of y $(\exists z)(P(z) \wedge F(x,z) \wedge M(z,y))$

(d) Let C(x): x is a cat T(x): x has a tail

$(\forall x)(C(x) \to \neg T(x))$

(e) Let P(x): x is a person T(x): x trust others R(x): x is rewarded

$(\exists x)(P(x) \wedge T(x) \wedge R(x))$

Problem: 3

Use the indirect method to prove that the conclusion $\exists z Q(z)$ follows from the premises $\forall x(P(x) \to Q(x))$ and $\exists y P(y)$

Solution:

1	$\neg \exists z Q(z)$	P(assumed)
2	$\forall z \neg Q(z)$	T, (1)
3	$\exists y P(y)$	P
4	$P(a)$	ES, (3)
5	$\neg Q(a)$	US, (2)
6	$P(a) \wedge \neg Q(a)$	T, (4),(5)
7	$\neg(P(a) \to Q(a))$	T, (6)
8	$\forall x(P(x) \to Q(x))$	P
9	$P(a) \to Q(a)$	US, (8)
10	$P(a) \to Q(a) \wedge \neg(P(a) \to Q(a))$	T,(7),(9) contradiction

Problem: 4

Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Solution:

1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
2) $P(a) \wedge Q(a)$	ES, 1
3) $P(a)$	Rule T, 2
4) $Q(a)$	Rule T, 2
5) $(\exists x)P(x)$	EG, 3
6) $(\exists x)Q(x)$	EG, 4
7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T, 5, 6

ASSIGNMENT PROBLEMS

1. Write the statement in symbolic form "Some real numbers are rational".
2. Symbolize the expression "x is the father of the mother of y"
3. Symbolize the expression "All the world loves a lover"
4. Write the negation of the statement "If there is a will, then there is a way".
5. Construct the truth table for $\neg(p \wedge q)$
6. Find the CNF and DNF of $\neg(p \vee q) \leftrightarrow (p \wedge q)$
7. Show that $P \rightarrow Q, Q \rightarrow \neg R, R, P \vee (J \wedge S)$ imply $J \wedge S$
8. Show that $P \rightarrow Q, P \rightarrow R, Q \rightarrow R, P$ are inconsistent.
9. Prove that $(\exists x)(P(x) \wedge Q(x) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
10. Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \rightarrow W(x,y)$ and $\neg W(a,b)$
11. Show that $\neg P \vee Q, \neg Q \vee R, R \rightarrow S \Rightarrow P \rightarrow S$
12. Show that $\neg(P \wedge \neg Q) \wedge \neg Q \vee R \wedge \neg R \Rightarrow \neg P$

13.Show that P is equivalent to $\neg\neg P, P \wedge P, P \vee P, P \wedge (P \vee Q), (P \wedge Q) \vee (P \wedge \neg Q)$

14.Indicate which one are tautologies (or) contradictions

(a) $(P \wedge Q) \Leftrightarrow P$ (b) $P \rightarrow P \vee Q$

15.If R:Ram is rich, H:Ram is happy ,Write in symbolic form

(a) Ram is poor but happy (b) Ram is poor or unhappy

(c) Ram is neither rich nor happy

16.Show that the hypothesis, "It is not sunny this afternoon and it is colder than yesterday," " We will go swimming only if it is sunny," "If we do not go swimming then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset "lead to the conclusion "we will be home by sunset".

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – DISCRETE MATHEMATICS – SMTA 1302

UNIT II SET THEORY

Basic concepts of Set theory - Laws of Set theory - Partition of set, Relations - Types of Relations: Equivalence relation, Partial ordering relation - Graphs of relation - Hasse diagram, Functions: Injective, Surjective, Bijective functions, Compositions of functions, Identity and Inverse functions.

The concept of a set is used in various disciplines and particularly in computers.

Basic Definition:

1. "A collection of well defined objects is called a set".

The capitals letters are used to denote sets and small letters are used for denote objects of the set. Any object in the set is called element or member of the set. If *x* is an element of the set *X*, then we write $x \in X$, to be read as 'x *belongs to X'* , and if *x* is not an element of *X*, the we write $x \notin X$ to be read as *' x does not belongs to X'*.

2. The number of elements in the set A is called *cardinality* of the set A, denoted by |*A*| or *n(A)* . We note that in any set the elements are distinct. The collection of sets is also a set.

$S = \{P_1, \{P_2, P_3\}, P_4, P_5\}$

Here $\{P_2, P_3\}$ itself one set and it is one element of *S* and |*S*|=4.

3. Let A and B be any two sets. If every element of A is an element of B, then A is called a *subset* of B is denote by $'A \subseteq B'$.

We can say that A *contained (included)* in B, (or) B *contains (includes)* A.

Symbolically, $A \subseteq B$ (or) $B \supseteq A$

Logically, $A \subseteq B = (x\forall)\{x \in A \rightarrow x \in B\}$

Let $A = \{1,2,3,4,5\},\ B = \{1,2,4\},\ C = \{1,5\},\ D = \{2\},\ E = \{1,4,2\}$

Then $B \subseteq A,\ C \subseteq A,\ D \subseteq A, D \subseteq B$

$C \nsubseteq B$, since $5 \in C \Rightarrow 5 \notin B,\ E \subseteq B$ and $B \subseteq E$.

Some of the important properties of set inclusion.

For any sets A, B and C

$A \subseteq A$ (Reflexive)

$(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$ (Transitive)

Note that $A \subseteq B$ does not imply $B \subseteq A$ except for the following case.

4. Two sets A and B are said to be *equal* if and only if $A \subseteq B$ and $B \subseteq A$,

$i.e., \quad A = B \Leftrightarrow (A \subseteq B\ and\ B \subseteq C)$

Example $\{1,2,4\} = \{4,1,2\}$ and $P = \{\{1,2\},4\},\ Q = \{1,2,4\}$ then $P \neq Q$

Since $\{1,2\} \in P$ and $\{1,2\} \notin Q$ eventhough $1,2 \in Q$.

The equality of sets is reflexive, symmetric, and transitive.

5. A set A is said to be a *proper subset* of a set B if $A \subseteq B$ and $A \neq B$. Symbolically it is written as $A \subset B. i.e.,\ A \subset B \Leftrightarrow (A \subseteq B\ \wedge\ A \neq B)$

$\subset$ is also called a *proper inclusion.*

6. A set is said to be *universal set* if it includes every set under our discussion. A universal set is denoted by $\cup$ or E.

In other words, if *p(x)* is a predicate.$E = \{x | p(x) \vee \neg p(x)\}$

One can observe that universal set contains all the sets.

7. A set is said to be *empty set* or *null set* if it does not contain any element, which id denoted by $\emptyset$.

In other words, if *p(x)* is a predicate.$\emptyset = \{x | p(x) \vee \neg p(x)\}$

One can observe that null set is a subset for all sets.

8. For a set A, the set of all subsets of A is called the *power set* of A. The power set of A is denoted by$\rho(A)$ or $2^{\wedge}$ $i.e.,\ \rho(A) = \{S \mid S \subseteq A\}$

Example, Let $A = \{a, b, c\}$

Then $\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\},\ A\}$

Then set $\emptyset$ and A are called *improper subsets* of A and the remaining sets are called *proper subsets* of A.

One can easily note that the number of elements of $\rho(A)$ is $2^{|A|}.i.e.,\ |\rho(A)| = 2^{|A|}$

SOME OPERATIONS ON SETS

1. Intersection of sets

Definition:

Let A and B be any two sets, the *intersection* of A and B is written as $A \cap B$ is the set of all elements which belong to both A and B.

Symbolically

$A \cap B = \{ x \mid x \in A\ and\ x \in B\}$

Example$A = \{1,2,3,4,5,6\},\ B = \{2,4,6,8\}$ then $A \cap B = \{2,4,6\}$. From the definition of intersection it follows that for any sets A,B,C and universal set E.

$A \cap A = A$ $\qquad A \cap B = B \cap A$ $\qquad A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap E = A$ $\qquad A \cap \emptyset = \emptyset$

2. Disjoint sets

Definition:

Two set A and B are called *disjoint* if and only if $A \cap B = \emptyset$, that is, A and B have no element in common.

Example $A = \{1,2,3\}$ $B = \{5,7,9\}$ $C = \{3,4\}$

$A \cap B = \emptyset,\ A \cap C = \{3\},\ B \cap C = \emptyset$

A and B are disjoint and B and C also, but A and C are not disjoint.

3. Mutually disjoint sets

Definition:

A collection of sets is called a *disjoint collection*, if for every pair of sets in the collection, are disjoint. The elements of a disjoint collection are said to be *mutually disjoint.*

Let $A = \{A_i\}_{i \in I}$ be an indexed set, A is mutually disjoint if and only if

$$A_i \cap A_j = \emptyset \text{ for all } i, j \in I, i \neq j.$$

Example

$$A_1 = \{\{1,2\},\{3\}\}, \qquad A_2 = \{\{1\},\{2,3\}\}, \qquad A_3 = \{\{1,2,3\}\}$$

Then $A = \{A_1, A_2, A_3\}$ is a disjoint collection of sets.

$$A_1 \cap A_2 = \emptyset, \qquad A_1 \cap A_3 = \emptyset \text{ and } \qquad A_2 \cap A_3 = \emptyset$$

4. Unions of sets

Definition:

The *union* of two sets A and B, written as $A \cup B$, is the set of all elements which are elements of A or the elements of B or both.

Symbolically $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Example Let $A = \{1,2,3,4,5,6\}$ $B = \{2,4,6,8\}$ then $A \cup B = \{1,2,3,4,5,,6,8\}$

From the union, it is clear that, for any sets A, B,C, and universal set E.

$A \cup A = A$ $\quad A \cup B = B \cup A$ $\quad A \cup (B \cup C) = (A \cup B) \cup C$

$A \cup E = E$ $\quad A \cup \emptyset = A$

5. Relative complement of a set

Definition:

Let A and B are any two sets. The *relative complement* of B in A, written $A - B,$ is the set of elements of A which are not elements of B.

Symbolically $A - B = \{\, x \mid x \in A \; or \; x \notin B\}$

Note that $A - B = A \cap \bar{B}$.

Example Let $A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A - B = \{1,3,5\}$

$B - A = \{8\}$

It is clear from the definition that, for any set A and B.

$A - B = \emptyset$

$A - B \neq B - A$

$A - \emptyset = A$

6. Complement of a set

Definition:

Let A be any set, and E be universal. The relative complement of A in E is called *absolute complement or complement* of A. The complement of A is denoted by $\bar{A}$ (or A^C or $\backsim A$)

Symbolically

$E - A = \bar{A} = \{\, x \mid x \in E \; and \; x \notin A\}$

Example Let $E = \{1,2,3,4, \ldots\}$ be universal set and

$A = \{2,4,6,8, \ldots\}$ be any set in E.

Then

$\bar{A} = \{1,3,5,7, \ldots\}$

From the definition, for any sets A$\bar{\bar{A}} = A$ $\quad \bar{\emptyset} = E$

$\bar{E} = \emptyset \quad A \cup \bar{A} = EA \cap \bar{A} = \emptyset$

7. Boolean sum of sets

Definition:

Let A and B are any two sets. The *symmetric difference or Boolean sum* of A and B is the set A+B defined by

$A + B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$

(or) $A + B = \{\, x \mid x \in A \; and \; x \notin B\} \cup \{\, x \mid x \in B \; and \; x \notin A\}$

Example Let

$A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A + B = \{1,3,5,8\}$ From the definition, for any sets A and B.

$A + A = \emptyset\,, \;\; A + \emptyset = A$

$A + E = \bar{A}\,, \;\; A + B = B + A$ and

$A + (B + C) = (A + B) + C$

8. The principle of duality

If we interchange the symbols $\cap$, $\cup$, E and $\emptyset$, $\subseteq$ and $\supseteq$, $\subset$ and $\supset$, in a set equation or expression. We obtain a new equation or expression is said to be *dual* of the original on *(primal).*

" If T is any theorem expressed in terms of $\cap$,$\cup$ and $-$ deducible from the given basic laws, then the dual of T is also a theorem".

Note that, the theorem T is proved in *m* steps, then dual of T also proved in *m* step.

Example The dual of $A \cap \overline{A} = \emptyset$ is given by $A \cup \overline{A} = E$.

Remark: Dual (Dual T) =T.

Identities on sets

$A \cup A = A$	Idempotent laws
$A \cap A = A$	
$A \cup B = B \cup A$	Commutative laws
$A \cap B = B \cap A$	
$(A \cup B) \cup C = A \cup (B \cup C)$	Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$A \cup (A \cap B) = A$	Absorption laws
$A \cap (A \cup B) = A$	
$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	De Morgan's laws

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

$A \cup \emptyset = A$ $A \cap \emptyset = \emptyset$

$A \cup E = E$ $A \cap E = A$

$A \cup \bar{A} = E$ $A \cap \bar{A} = \emptyset$

$\bar{\emptyset} = E$ $\bar{E} = \emptyset$ $\bar{\bar{A}} = A$

PROBLEMS

1.$S = \{a, b, p, q\}$, $Q = \{a, p, t\}$. Find $S \cup Q$ and $S \cap Q$?

Solution:

$S \cup Q = \{a, b, p, q, t\}$

$S \cap Q = \{a, p\}$

2. If $A = \{a, b, c\}$. Find $\rho(A)$?

Solution:

$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$ and

$|A| = 3$

$|\rho(A)| = 2^3 = 8$

3. Write all proper subsets of $A = \{a, b, c\}$.

Solution:

The proper subsets are

$\rho(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

4. Show that $A \subseteq B \Leftrightarrow A \cap B = A$.

Solution:

If $A \subseteq B$, then $\forall\, x \in A \Longrightarrow x \in B$

Now , let

$x \in A \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow\ x \in A \cap B$

$A = A \cap B$

If $A \cap B = A$, then

Let $x \in A$, $\quad x \in A \cap B \ \Longrightarrow x \in B$

Therefore $A \subseteq B$.

5. If $A = \{2,5,6,7\}, B = \{1,2,3,4\},\ C = \{1,3,5,7\}$. Find $A - B, A - C, C - B$ and $B - C$.

Solution:

$A - B = \{5,6,7\}$

$A - C = \{2,6\}$

$C - B = \{5,7\}$

$B - C = \{2,4\}$

6. If $A = \{2,3,4\}, B = \{1,2\},\ C = \{4,5,6\}$. Find $A + B, B + C, A + C, A + B + C$ and $(A + B) + (B + C)$.

Solution:

$A + B = \{1,3,4\}$

$B + C = \{1,2,4,5,6\}$

$A + C = \{2,3,5,6\}$

$A + B + C = \{1,3,5,6\}$

$(A + B) + (B + C) = \{2,3,5,6\}$

Note that

$A + (B + B) + C = A + (\emptyset) + C = A + C = \{2,3,5,6\}$

7. Show that $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Solution:

Let

$x \in A \Rightarrow x \in A\ (or)\ x \in B$

$\Rightarrow x \in A \cup B$

$\Rightarrow A \subseteq A \cup B$

Now let $x \in A \cap B \Rightarrow x \in A\ and\ x \in B$

$\Rightarrow x \in A$

$A \cap B \subseteq A$

Hence $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Remark: $B \subseteq A \cup B, A \cap B \subseteq B$ and $A \cap B \subseteq A \cup B$.

8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.

Solution:

$x \in A - (A \cap B) \Leftrightarrow x \in A\ and\ x \notin (A \cap B)$

$\Leftrightarrow x \in A\ and\ \{x \notin A\ or\ x \notin B\}$

$\Leftrightarrow \{x \in A \; and \; x \notin A\} (or) \{x \in A \; and \; x \notin B\}$

$\Leftrightarrow \; \emptyset \; (or) \{x \in A \; and \; x \notin B\}$

$\Leftrightarrow x \in A \; and \; x \notin B$

$A - (A \cap B) \subseteq A - B$ and $A - B \subseteq A - (A \cap B)$

Therefore $A - (A \cap B) = A - B$.

9. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution:

$x \in A \cup (B \cap C) \Leftrightarrow x \in A \; or \; x \in B \cap C$

$\Leftrightarrow x \in A \; or \; \{x \in B \; and \; x \in C\}$

$\Leftrightarrow \{x \in A \; or \; x \in B\} \; and \; \{x \in A \; or \; x \in C\}$

$\Leftrightarrow \{x \in A \cup B\} \; and \; \{x \in A \cup C\}$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

10. Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

Solution:

Let $x \in \overline{(A \cup B)} \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \; and \; x \notin B$

$\Leftrightarrow x \in \bar{A} \; and \; x \in \bar{B}$

$\Leftrightarrow x \in \bar{A} \cap \bar{B}$

Therefore $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

11. Show that $(A - B) - C = A - (B \cup C)$.

Solution:

$(A-B)-C=(A-B)\cap \bar{C}$ $\qquad$ $(P-Q=P\cap \bar{Q})$

$=(A\cap \bar{B})\cap \bar{C}$

$=A\cap (B\cap \bar{C})$ $\qquad$ (Associative)

$=A\cap (\overline{B\cup C})$ $\qquad$ (De Morgan's law)

12. Show that $A\cap (B-C)=(A\cap B)-(A\cap C)$

Solution:

Let $(A\cap B)-(A\cap C)$

$=(A\cap B)\cap (\overline{A\cap C})$

$=(A\cap B)\cap (\bar{A}\cup \bar{C})$

$=(A\cap B\cap \bar{A})\cup (A\cap B\cap \bar{C})$

$=((A\cap \bar{A})\cap B)\cup (A\cap B\cap \bar{C})$

$=(\emptyset\cap B)\cup (A\cap B\cap \bar{C})$

$=\emptyset\cup (A\cap B\cap \bar{C})$

$=A\cap (B\cap \bar{C})$

$=A\cap (B-C)$

ASSIGNMENT PROBLEMS

Part –A

1. Define a set
2. Define subset of a set. What is mean by proper subset?

(i) Find all subset of $A = \{1,2,3\}$

(ii)Find all proper subsets of A.

3. Define power set.
4. Define disjoint sets with example?
5. If $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8,10\}$. Find $A \cup B, A \cap B, a - B, B - A,$

   $A + B,$ and $B + A$?

6. Which of the following sets are empty?
7. $\{x \mid x \in R, x + 6 = 6\}$
8. $\{x \mid x\ is\ a\ real\ integer\ such\ that\ x^2 + 1 = 0\}$
9. $\{x \mid x\ is\ a\ real\ integer\ and\ x^2 - 4 = 0\}$
10. State duality principle in set theory.
11. Define cardinality of a set.
12. If a set A has *n* elements, then the number of elements of power set of A is……..
13. Find the intersection of the following sets

    (i) $\{x \mid x^2 - 1 = 0\}, \{x \mid x^2 + 2x + 1 = 0\}$
14. Write the dual of $A \cap \bar{A} = \emptyset$.
15. Let A, B and C sets, such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, can we conclude that B=C.
16. State De Morgan's Laws.
17. Whether the union of sets is commutative or not?

PART –B

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. Verify the De Morgan's laws

   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
3. Show that the intersection of sets is associative.
4. Show that $A - (B - C) = (A - B) \cup (A \cap C)$.
5. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. Let $A_i = \{1,2,3, \dots\}$ for $i = 1,2,3, \dots$ find (a) $\cup_{i=1}^{n} A_i$ (b) $\cap_{i=1}^{n} A_i$
7. Prove that $A - (A - B) \subset B$.
8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
9. Prove that $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$.
10. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that B=C.(cancelation law)
11. Show that $A - (B \cup C) = (A - B) \cap (A - C)$.
12. Show that $A + A = \emptyset$, where + is the symmetric difference of sets.
13. Show that $(R \subseteq S)$ and $(S \subset Q)$ imply $R \subset Q$.
14. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$.

CARTESIAN PRODUCT OF SETS

The *Cartesian product* of the sets A and B, is written an $A \times B$, is the set of all ordered pairs in which the first elements are in A and the second elements are in B.

$i.e.\ A \times B = \{\langle x, y\rangle | x \in A\ and\ x \in B\}$

For example

Let $A = \{1,2\}, B = \{a, b, c\},\ c = \{\alpha, \beta\}$

Now

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 1, c\rangle \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, c\rangle\}$

$A \times C = \{\langle 1, \alpha\rangle, \langle 1, \beta\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle\}$

$A \times B = \{\langle \alpha, a\rangle, \langle \alpha, b\rangle, \langle \alpha, c\rangle \langle \beta, a\rangle, \langle \beta, b\rangle, \langle \beta, c\rangle\}$

It is clear from the definition

$A \times B \neq B \times A$ and $\langle\langle a, b\rangle, c\rangle \in (A \times B) \times C$, is an ordered triple then $\langle a, b\rangle \in A \times B$ and $c \in C$.

Now , $A \times (B \times C) = \{\langle a, \langle b, c\rangle\rangle \,|\, a \in A \; and \; \langle b, c\rangle \in \langle B, C\rangle\}$

Note that $\langle a, \langle b, c\rangle\rangle$ is not an ordered triple.

This fact show that $(A \times B) \times C \neq A \times (B \times C)$

i.e. Cartesian product is not associative.

Now

$A \times A = A^2 = \{\langle x, y\rangle, \forall x, y \in A\}$ and $A^n = A^{n-1} \times A$.

Note that if A has *n* elements and B has *m* elements $A \times B$ has *nm* elements.

PROBLEMS

1.If $A = \{1,2,3\}$, $B = \{a, b\}$. Find $A \times B, B \times A$ and $A \times A$ and $A^2 \times B$

Solution :

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$B \times A = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle a, 3\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle b, 3\rangle\}$

$A^2 = A \times A = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

$A^2 \times B = \{\langle 1,1,a\rangle, \langle 1,1,b\rangle, \langle 1,2,a\rangle, \langle 1,2,b\rangle, \langle 1,3,a\rangle, \langle 1,3,b\rangle, \langle 2,1,a\rangle, \langle 2,1,b\rangle,$

$\langle 2,2,a\rangle, \langle 2,2,b\rangle, \langle 2,3,a\rangle, \langle 2,3,b\rangle, \langle 3,1,a\rangle, \langle 3,1,b\rangle, \langle 3,2,a\rangle, \langle 3,2,b\rangle, \langle 3,3,a\rangle, \langle 3,3,b\rangle\}$

2.Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (B \cap C) \Leftrightarrow x \in A \; and \; y \in B \cap C$

$\Leftrightarrow x \in A \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{x \in A \; and \; y \in B\} \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times B\} \; and \; \{\langle x, y\rangle \in A \times C\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3.Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (A \cap B) \times (C \cap D) \Leftrightarrow x \in (A \cap B) \; and \; y \in (C \cap D)$

$\Leftrightarrow \{x \in A \; and \; x \in B\} \; and \; \{y \in C \; and \; y \in D\}$

$\Leftrightarrow \{x \in A \; and \; y \in C\} \; and \; \{x \in B \; and \; y \in D\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times C\} \; and \; \{\langle x, y\rangle \in B \times D\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times C) \cap (B \times D)\}$.

ASSIGNMENT PROBLEMS

Part A

1. Define Cartesian product of sets? Given an example?
2. If $A = \{0,1\}$, find A^2.
3. If $A = \{1,2,3\}$ and $B = \{a, b\}$, find $A \times B, B \times A, A^2$.
4. True or False
   I. If $A = \{1,3,5,7,9\}$, the $\{\forall x \in A, x + 2 \; is \; a \; prime \; number\}$
   II. If $A = \{1,2,3,4,5\}$, the $\{\ni x \in A, x + 3 = 10\}$
5. If $A \times B = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$

Part B

6. If A,B and C are sets, prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. Prove that $(A \times C) - (B \times C) = (A - B) \times C$.
8. If $A = \{a, b\}$ and $B = \{1,2\}$,and $C = \{2,3\}$, find
   I. $A \times (B \cup C)$
   II. $(A \times B) \cup (A \times C)$
   III. $A \times (B \cap C)$
   IV. $(A \times B) \cap (A \times C)$
9. Show that the Cartesian product is not commutative? It is commutative only for equality of sets?

RELATIONS

Binary relation

Any set of ordered pairs defines a *binary relation.*

If *x* and *y* are binary related, under the relation R, the we write $\langle x, y\rangle \in R$ or xRy. If not the case we write $\langle x, y\rangle \notin R$.

1. Example $F = \{\langle x, y\rangle \mid x \text{ is the father of } y\}$

$L = \{\langle x, y\rangle \mid x \text{ and } y \text{ are real number and } x < y\}$

Then F, L are binary relations.

2.Example Let A and B be any two sets, then any non empty subset R of $A \times B$ is called a *binary relation.*

Now

$A = \{1,2,3\}$

$B = \{a, b\}$ then

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

Let

$R_1 = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$R_2 = \{\langle 1, b\rangle, \langle 3, a\rangle\}$

$R_3 = \{\langle 2, a\rangle\}$

Then R_1,R_2 and R_3 are binary relations A to B.

Let S be any binary relation. The *domain* of S is the set of all elements *x* such that for some $y, \langle x, y\rangle \in S$.

$D(S) = \{x \mid \langle x, y\rangle \in S, \text{for some } y\}$

Similarly, the *range* of S is the set of all elements *y* such that, for some $x, \langle x, y\rangle \in S$

$i.e.\quad R(S) = \{y \mid \langle x, y\rangle \in S, \text{for some } x\}$

Let

$S = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle, \langle 3,a\rangle\}$

$D(S) = \{1,2,3\}$

$R(S) = \{a,b\}$

If $S \subseteq X \times Y$, then clearly $D(S) \subseteq X$ and $R(S) \subseteq Y$.

In case of $X = Y$, then the relation defined on $X \times X$ is called *an universal relation* in X.

If $X = \emptyset$, then a relation on $X \times X$ is called *void relation* in X.

Since relations are sets, then we can have their union and intersection and so on.

$R \cup S = \{\langle x,y\rangle \mid xRy \text{ or } xSy\}$

$R \cap S = \{\langle x,y\rangle \mid xRy \text{ and } xSy\}$

$R - S = \{\langle x,y\rangle \mid xRy \text{ and } \langle x,y\rangle \notin S\}$

$R + S = \{\langle x,y\rangle \mid \langle x,y\rangle \text{ is either in } R \text{ or in } S \text{ but not in both}\}$

Properties of Binary relations

1. Reflexive

Let R be a binary relation defined on X.

Then R is *reflexive* if, for every $x \in X, \langle x,y\rangle \in R$.

Example:

Let

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 2,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ are defined on X.

Then R is reflexive, but S is not reflexive. Since $\langle 2,2\rangle \notin S$ and $2 \in X$.

2. Symmetric

A relation R from X to Y is *symmetric* if every $x \in X$ and $y \in Y$, whenever $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

That is, if $xRy \Rightarrow yRx$, then R is symmetric

Example:

Let

$X = \{1,2\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$ are defined on X.

Then R is symmetric, but S is not symmetric. Since $\langle 1,2\rangle \in S$ but $\langle 1,2\rangle \notin S$.

3. Transitive

A relation R is *transitive* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.

That is, if $xRy \wedge yRz$, then R is transitive.

Example:

Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,3\rangle, \langle 2,1\rangle\}$

Then R is transitive, but S is not transitive. Since $\langle 2,1\rangle \in S$ and $\langle 1,2\rangle \in S$ but $\langle 2,2\rangle \notin S$.

4.Irreflexive

A relation R in a set X is *irreflexive* if, for every $x \in X, \langle x, x\rangle \notin R$.

Example:

Let

$A = \{1,2,3\}$

$R = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 2,2\rangle, \langle 1,3\rangle\}$

Then R is irreflexive, but S is not reflexive. Since $\langle 3,3\rangle \notin S$ and $\langle 1,1\rangle \in S$.

5. Antisymmetric

A relation R in a set X is *antisymmetric* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then$x = y$.

That is, if $xRy \wedge yRx \Rightarrow x = y$, then R is antisymmetric.

Example:

Let

X be the set of all subsets of E.

R be the inclusion relation $(\subseteq)$ defined on X.

$A \subseteq B \wedge B \subseteq A \Rightarrow A = B$

Therefore R is antisymmetric in X.

6. Relation matrix

Let $X = \{x_1, x_2, \dots x_m\}$, $Y = \{y_1, y_2, \dots y_m\}$ are ordered sets, R be a relation defined from X to Y, then the *relation matrix* of R, is defined as

$M_R = (r_{ij})\ i : 1 \to m, j : 1 \to n$

Example 1:

Let $X = \{1,2,3\}$ $Y = \{a, b\}$

$R = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 2,a\rangle,\langle 3,b\rangle\}$ be a relation from X to Y. Then $M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Let

$R = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 1,3\rangle,\langle 2,2\rangle,\langle 3,1\rangle,\langle 3,2\rangle\}$ be a relation on $X = \{1,2,3\}$. Then $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

7. Composition of Binary Relations

The concept of composition of relation is different from union and intersection of two relations.

Definition:

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite $R \circ S$ is a relation from X to Z defined by

The operation $\circ$ in $R \circ S$ is called *" composition of relations".*

Example.

Let

$R = \{\langle 1,2\rangle,\langle 2,3\rangle,\langle 3,4\rangle,\langle 2,2\rangle\}$

$S = \{\langle 2,3\rangle,\langle 4,1\rangle,\langle 4,3\rangle,\langle 2,1\rangle\}$. Then

$R \circ S = \{\langle 1,3\rangle,\langle 1,1\rangle,\langle 3,1\rangle,\langle 3,3\rangle,\langle 2,3\rangle,\langle 2,1\rangle\}$

$S \circ R = \{\langle 2,4\rangle,\langle 4,2\rangle,\langle 4,4\rangle,\langle 2,2\rangle\}$

Note that

$R \circ R = R^2$

$R \circ R \circ R = R^2 \circ R = R^3$

$R^{n-1} \circ R = R^n$ etc.,

Definition:

The relation matrix for $R \circ S$ is given by $M_{R \circ S} = M_R \odot M_S$ where $\odot$ is defined as follows.

$M_R \odot M_S = \langle m_{ij} \rangle$ where $m_{ij}(\langle i,j \rangle th\ \ element)$ is 1 if and only if row *i* of M_R and column *j* of M_S have a 1 in the same relative position *k*, for some *k*.

Example:

Let

$R = \{\langle 1,2 \rangle, \langle 1,5 \rangle, \langle 2,2 \rangle, \langle 3,4 \rangle, \langle 5,1 \rangle, \langle 5,5 \rangle\}$

$S = \{\langle 1,3 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 4,2 \rangle, \langle 4,4 \rangle, \langle 5,2 \rangle, \langle 5,3 \rangle\}$. Then

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
M_{R \circ S} &= M_R \odot M_S \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
\text{and} \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$R^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \langle 5,5\rangle\}$

Definition

Let R be a relation from X to Y. The *converse* of R, is written as $\tilde{R}$, is a relation from Y to X such that$xRy \Leftrightarrow x\tilde{R}y$.

Example:

If $R = \{\langle 1,a\rangle, \langle 2,b\rangle, \langle 2,a\rangle, \langle b,3\rangle$

$\tilde{R} = \{\langle a,1\rangle, \langle b,2\rangle, \langle a,2\rangle, \langle b,3\rangle$

Also it is clear that

1. $R = S \Leftrightarrow \tilde{R} = \tilde{S}$
2. $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$
3. $\widetilde{R \cup S} = \tilde{R} \cup \tilde{S}$

Result: The relation matrix $M_{\tilde{R}}$ is the transpose of the relation M_R.

$i.e. M_{\tilde{R}} = transpose\ of\ M_R$

Example:

Let

$R = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle$

$\tilde{R} = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,3\rangle$

We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[M_R]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{\tilde{R}}$$

EQUIVALENCE RELATION

Definition:

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 1:

Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$ is an equivalence relation on X.

Example 2:

Equality of subsets on a universal set is an equivance relation.

Example 3:

Let

$X = \{1,2,3, \dots 7\}$

$R = \{\langle x, y\rangle \mid x - y \; is \; divisible \; by \; 3\}$

Now, $\forall x \in X, x - x = 0$ is divisible by 3.

Therefore $\forall x \in X, \langle x, x\rangle \in R$ (reflexive)

For any $x, y \in X$

Let $\langle x, x\rangle \in R \Rightarrow x - y$ is divisible by 3 we have $-(x - y) = y - x$ is also divisible by 3.

$\langle y, x\rangle \in R$ (symmetric)

Let $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R$

$\Rightarrow x - y$ is divisible by 3 and $y - z$ is divisible by 3.

$\Rightarrow (x - y)+ (y - z)$ is divisible by 3.

$\Rightarrow x - z$ is divisible by 3.

Therefore $\langle x, y\rangle \in R$ (Transitive)

Therefore R is an equivalence relation on X.

EQUIVALENCE CLASSES

Definition:

Let R be an equivalence relation on a set X. For any $x \in X$, the set $[x]_R \subseteq X$ given by

$[x]_R = \{y \mid xRy\ for\ y \in X\}$

is called an R-*equivalence class* generated by $x \in X$.

Therefore, an equivalence class $[x]_R$ of $x \in X$ is the set of all elements which are related to *x* by an equivalence relation R on X.

Example:

Let Z be the set of all integers and R be the relation called *"congruence modulo 4"* defined by

$R = \{\langle x, y\rangle \mid (x - y)\ is\ divisible\ by\ 4, for\ x\ and\ y \in Z\}$ (or $x \equiv y(mod\ 4)$)
Now, we determine the equivalence classes generated by R.

$[0]_R = \{\ldots - 8, -4,0,4,8 \ldots\}$

$[1]_R = \{\ldots - 7, -3,1,5,9 \ldots\}$

$[2]_R = \{\ldots - 6, -2,2,6,10 \ldots\}$

$[3]_R = \{\ldots - 5, -1,3,7,11 \ldots\}$

Note that

$[0]_R = [4]_R, [1]_R = [5]_R, \ldots etc.$

Therefore $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R, [3]_R\}$

In a similar manner, we get the equivalence classed generated by the relation *"congruence modulo m"* for any integer *m.*

Therefore, an equivalence relation R on X, will divide the set X into an *equivalence classes,* and they are called *portion* of X.

PARTIAL ORDERED RELATION

A relation R on a set X is said to be a partial ordered relation, if R satisfies reflexive, antisymmetric, and transitive.

Example:

Let $\rho(A)$ be the power set of a set A.

Define a subset relation $(\subseteq)$ on $\rho\ (A)$, then $\subseteq$ is a partial ordered relation.

Usually we denote the partial ordered relations as $' \leq '$ is said to be *partially ordered set* (or) *poset,* which is denoted by $\langle X, \leq \rangle$. We will study more about posets in the subsequent sections.

1. Closures of a relation

Let R be a relation on the set X.

2. Reflexive closure

We have the relation R is reflexive if and only if the relation.

$R = \{\langle x, y \rangle \mid \forall\ x \in X\}$ is contained in R.

i.e. R is reflexive $\Leftrightarrow I \subset R$.

Definition:

Let R be a relation on X, then the smallest reflexive relation on X, containing R, is called *reflexive closure* of R.

Therefore $R_1 = R \cup I$ is the reflexive closure of R.

3. Symmetric closure

We have, the relation R is symmetric if $\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in \tilde{R}$

$i.e. \tilde{R} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$

Definition:

Let R be a relation X, then smallest symmetric relation on X, containing R, is called the *symmetric closure* of R.

Therefore $R \cup \tilde{R}$ is the symmetric of R.

4. Transitive closure

We have, the relation R is transitive, if $\langle x, y\rangle \in R \; and \; \langle y, z\rangle \in R$ then $\langle x, z\rangle \in R$.

Definition:

A relation R^+ is said to be the *transitive closure* of the relation R on X if R^+ is the smallest transitive relation on X, containing R,

i.e R^+ is the transitive closure of R, if

I. $R \subseteq R^+$
II. R^+ is transitive on X
III. There is no transitive relation R_1 on X, such that $R \subset R_1 \subset R^+$

Remarks:

1. The transitive closure of R can be obtained by

$$R^+ = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{i=1}^{\infty} R^i$$

2. We know that $\langle x,z\rangle \in R^2$ if and only if there is an element y such that $\langle x,y\rangle \in R$ and $\langle y,z\rangle \in R$.

   Therefore $\langle a,b\rangle \in R^n$ if and only if we can find a sequence $x_1, x_2, \ldots . x_{n-1}$ in X such that $\langle a,x_1\rangle, \langle x_1,x_2\rangle, \ldots \langle x_{n-1},b\rangle$ are all in R.

   The sequence $a, x_1, x_2, \ldots . x_{n-1}, b$ is said to be a *chain* of length *n* from a to b in R. Here $x_1, x_2, \ldots . x_{n-1}$ are called interval vertices of the chain in R. Note that the interval vertices need not be distinct.

PROBLEMS

1. If $P = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$, $Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

Find (i) $P \cup Q, P \cap Q, \tilde{P}, \tilde{P} \cup Q$ (ii) domains of $P, P \cup Q, P \cap Q$ and (iii) ranges of $Q, P \cup Q, P \cap Q$.

Solution:

$P \cup Q = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$

$P \cap Q = \{\langle 2,4\rangle\}$

$\tilde{P} = \{\langle 2,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

$\tilde{P} \cup Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 2,1\rangle, \langle 4,3\rangle\}$

Domain of $P = \{1,2,3\}$

Domain of $(P \cup Q) = D(P \cup Q) = \{1,2,3,4\}$

Domain of $(P \cap Q) = D(P \cap Q) = \{2\}$

Range of $Q = R(Q) = \{2,3,4\}$

Range of $(P \cup Q) = R(P \cup Q) = \{2,3,4\}$

Range of $(P \cap Q) = R(P \cap Q) = \{4\}$

It is clear that

$D(P \cup Q) = D(P) \cup D(Q)$ and

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

In general $D(P) = R(\tilde{P})$ and $R(P) = D(\tilde{P})$.

2.Let $X = \{1,2,3,4\}$ and $R = \{\langle x, y\rangle \mid x, y \in X\ and\ (x - y)\ is\ an\ integeral$

$non\ zeromultiple\ of\ 2\}$ $S = \{\langle x, y\rangle \mid x, y \in X\ and\ (x - y)\ is\ an\ integeral$

$non\ zeromultiple\ of\ 3\}$. Find $R \cup S$ and $R \cap S$?

Solution:

Given that $R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$ and

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$ $R \cup S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,1\rangle, \langle 4,2\rangle\}$

$R \cap S = \emptyset$

Remarks:

$D(R) = \{1,2,3,4\}$

$R(R) = \{1,2,3,4\}$

$D(S) = \{1,4\}$

$R(S) = \{1,4\}$

3.Let $S = \{\langle x, x^2\rangle \mid x \in N\}$ and $T = \{\langle x, 2x\rangle \mid x \in N\}$, where $= \{0,1,2, \dots\}$. Find the range of S and T, find $S \cup T$ and $S \cap T$?

Solution:

$S = \{\langle x, x^2 \rangle \mid x \in N\}$

$= \{\langle 0,0 \rangle, \langle 1,1 \rangle, \langle 2,4 \rangle, \langle 3,9 \rangle, \langle 4,16 \rangle, \dots\dots \}$ and

$T = \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle 0,0 \rangle, \langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \langle 4,8 \rangle, \dots\dots \}$

$R(S) = \{x^2 \mid x \in N\}$

$= \{0,1,4,9,16,25 \dots\dots \}$

$R(T) = \{2x \mid x \in N\}$

$= \{0,2,4,6,8,10, \dots\dots \}$

$S \cup T = \{\langle x, x^2 \rangle \mid x \in N\} \cup \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = x^2\ (or) 2x\}$

$= \{\langle 0,0 \rangle, \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \langle 3,9 \rangle, \dots\dots \}$

$S \cap T = \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = 2x\ and\ y = x^2\}$

(Now $y = 2x$ and $y = x^2 \Rightarrow 2x = x^2\ i.e.\ x = 0\ or\ x = 2$

$x = 0\ y = 0\ and\ x = 2 \Rightarrow y = 4$)

$S \cap T = \{\langle 0,0 \rangle, \langle 2,4 \rangle \}$

4. Given an example which is neither reflexive nor irreflexive?

Solution:

Let $X = \{1,2,3,4\}$ and

$R = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle, \langle 4,1 \rangle, \langle 4,4 \rangle\}$

Then R is not reflexive, since $\langle 2,2 \rangle \notin R$, for $2 \in X$ and R is not irreflexive, since $1 \in X$, and $\langle 1,1 \rangle \in R$.

5. Test whether the following relations are transitive or not on

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle \}$

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$

$T = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$.

Solution: The relation R and T are transitive.

Since, in R, we have $\langle 1,1\rangle \in R$, then check any other pair starting with $\langle 1,z\rangle \in R$, then we must have $1R1 \wedge 1Rz \Rightarrow 1Rz$ i.e., $\langle 1,z\rangle \in R$, but there is no pair staring with 1. So, pass on to next pair $\langle 2,2\rangle$ then we check any other pair starting with 2, and so on.

In T, we have $\langle 1,1\rangle \in T$, then there are two pairs $\langle 1,2\rangle$ and $\langle 1,3\rangle$ must be the transitive of $\langle 1,1\rangle \in T$, then we must have $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in T. Then pass to $\langle 1,2\rangle \in T$ the transitive pairs are $\langle 2,1\rangle, \langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must have the pairs $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle$ in T.

Then pass to $\langle 1,3\rangle \in T$, find the transitive pairs of $\langle 1,3\rangle$ and so on, for all pairs in T. Hence T is a transitive relation.

The relation S is not transitive, since for $\langle 1,2\rangle \in S$, the transitive pairs are $\langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in S but $\langle 1,3\rangle \notin S$.

6. Let R denotes a relation on the set of pairs of positive $N \times N$ integers such that $\langle x,y\rangle R \langle u,v\rangle$ if and only if $xv = yu$. Show that R is an equivalence relations.

Solution:

Let

$P = \{\langle x,y\rangle \mid x \text{ and } y \text{ are positive integer}\}$

Now R is a relation defined on P as

$\langle x,y\rangle R\,\langle u,v\rangle \;\Leftrightarrow\; xv = yu$ for $\langle x,y\rangle, \langle u,v\rangle \in P$.

Let $\langle x,y\rangle, \langle u,v\rangle$ and $\langle m,n\rangle \in P$.

I. R is reflexive:

We have

$\langle x,y\rangle R\,\langle x,y\rangle \;\Leftrightarrow\; xy = yx$ (RHS) is true.

II. R is symmetric:

Let $\langle x,y\rangle R\,\langle u,v\rangle \;\Leftrightarrow\; xv = yu$

$\Leftrightarrow\; yu = xv$

$\Leftrightarrow\; uy = vx$

$\Leftrightarrow \langle u,v\rangle R\,\langle x,y\rangle$

III. R is transitive:

Let $\langle x,y\rangle R\,\langle u,v\rangle$ and $\langle u,v\rangle R\,\langle m,n\rangle$

$\Leftrightarrow\; (xv = yu)$ and $(un = vm)$

$\Leftrightarrow\; (xv = yu)$ and $(u = \frac{vm}{n})$

$\Leftrightarrow\; xv = y(\frac{vm}{n})$

$\Leftrightarrow\; xn = ym$

$\Leftrightarrow\; \langle u,v\rangle R\,\langle m,n\rangle$

Therefore R is reflexive, symmetric , and transitive.

Hence R is an equivalence relation.

7. Let R and S are equivalence relations on X, show that $R \cap S$ also equivalent? Whether $R \cup S$ is also an equivalent relation. If not given an example.

Solution:

Given let R and S are equivalence relations on X.

Let x, y and $z \in X$.

(i) We have $\langle x, x\rangle \in R$ and $\langle x, x\rangle \in S \Rightarrow \langle x, x\rangle \in R \cap S, \ \forall x \in X$.

Therefore $R \cap S$ is reflexive.

(ii)Let $\langle x, y\rangle \in R \cap S \Rightarrow \langle x, y\rangle \in R$ and $\langle x, y\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R$ and $\langle y, x\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R \cap S$

Therefore $R \cap S$ is symmetric.

(iii) Let $\langle x, y\rangle \in R \cap S$ and $\langle y, z\rangle \in R \cap S$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle x, y\rangle \in S)$ and $(\langle y, z\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$ and $(\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow \langle x, y\rangle \in R$ and $\langle x, z\rangle \in S$

$\Rightarrow \langle x, z\rangle \in R \cap S$

Therefore $R \cap S$ is transitive.

Hence $R \cap S$ is equivalence.

8. Prove that the relation *"congruence modulo m"* over the set of positive integers is an equivalence relation?

Show also that if $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) = (y_1 + y_2)$.

Solution:

Let N be the set of all positive integers we have *"congruence modulo m"* relation on N as $x \equiv y (mod\ m) \Leftrightarrow m | x - y$, for $x, y \in N$.

Let $x, y, z \in N$

(i) We have

$x - x = 0 = 0m$

Therefore $x \equiv x \ (mod\ m)$ for $x \in N$.

"Congruence modulo m" is reflexive.

(ii)Let

$x \equiv y (mod\ m)$

$\Rightarrow\ m |\ x - y$

$\Rightarrow x - y = km$, for some integer $k \in Z$

$\Rightarrow y - x = (-k)m$, for some integer $-k \in Z$

$\Rightarrow y \equiv x \ (mod\ m)$

"congruence modulo m" is symmetric on N.

(iii) Let

$x \equiv y (mod\ m)$ and $y \equiv z \ (mod\ m)$

$\Rightarrow x - y = k_1 m$, and $y - x = k_2 m$ for some integer $k_1, k_2 \in Z$

$\Rightarrow (x - y) + (y - z) = (k_1 + k_2)m$

$\Rightarrow x - z = (k_1 + k_2)m$ for some integer $k_1 + k_2$

$\Rightarrow x \equiv z \ (mod\ m)$

"Congruence modulo m" is transitive on N.

Hence *"congruence modulo m"* is an equivalence relation.

Let $x_1 \equiv y_1 (mod\ m)$ and $x_2 \equiv y_2 (mod\ m)$.

Then $m |\ x_1 - y_1$ and $m |\ x_2 - y_2$

i.e., $x_1 - y_1 = k_1 m$ and $x_2 - y_2 = k_2 m$

Now

$(x_1 - y_1) + (x_2 - y_2) = k_1 m + k_2 m$

$(x_1 + x_2) - (y_1 + y_2) = (k_1 + k_2)m$

$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$

$(x_1 + x_2) \equiv (y_1 + y_2)(mod\, m)$

9. Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$ be a relation defined on A. Find the transitive closure of R?

Solution:

The matrix of the relation R is given by

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
\text{and} \\
M_{R^3} &= M_{R^2} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$M_{R^4} = M_{R^3} \odot M_R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

As $|A| = 4$, we get

$$M_{R^+} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Hence

$R^+ = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$

ASSIGNMENT PROBLEMS

Part -A

1. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$ and $S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 1,3\rangle, \langle 3,3\rangle\}$ be any relations on $X = \{1,2,3\}$. Find $R \cup S, R \cap S, \tilde{R}, R(R), R(\tilde{S}), D(R \cup S), R(R \cap S)$.
2. Give an example for reflexive, symmetric, transitive and irreflexive relations.
3. Give an example of a relation which is neither reflexive nor irreflexive.
4. Give an example of a relation which is neither symmetric not antisymmetric?
5. Find the graph of the relation $R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

6. Find the relation matrix of
   $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$
7. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$ and $= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$. Find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $R \circ R \circ S$ and $S \circ S \circ S$?
8. Define equivalence relation and equivalence classes?
9. Define Poset?
10. Define reflexive closure?
11. Define transitive closure of the relation R?
12. Let $R = \{\langle 1,2\rangle, \langle 3,5\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$ be a relation $A = \{1,2,3,4,5,6\}$. Identify the root of the tree of R.
13. Determine whether the relation R is a partial ordered on the set Z, where Z is set of positive integer, and aRb if and only if a=2b.
14. The following relations are on $\{1,3,5\}$. Let R be a relation, xRy if and only if $y = x + 2$, and let S be a relation, xSy if and only if $x \leq y$. Find $R \circ S$ and $S \circ R$?
15. True or False: The relation $<$ on Z^+ is not a partial order since it is not reflexive.

Part B

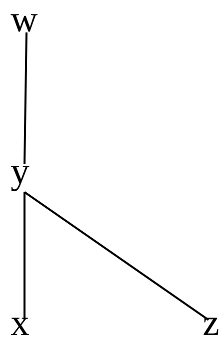
1. Show that the intersection of equivalence relations is an equivalence relation.
2. Determine whether the relations represented by the following zero-one matrices are equivalence relations.

$a)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $\quad$ $b)\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. If R and S are symmetric, show that $R \cup S$ and $R \cup S$ are symmetric.
4. Let L be set of all straight lines in the Euclidean plane and R be the relation in L defined by $xRy \Leftrightarrow x$ is perpendicular to y. Is R is Reflexive? Symmetric? Antisymmetric? Transive?
5. Consider the subsets $A = \{1,7,8\}$, $B = \{1,6,9,10\}$ and $C = \{1,9,10\}$ where $E = \{1,2,3 \dots \dots 10\}$ is an universal set. List the non empty minsets generated by A,B and C . Do they form a partition on E?
6. Let $X = \{1,2,3, \dots \dots 20\}$ and $R = \{\langle x,y \rangle \mid x - y \text{ is divisible by } 5\}$ be a relation on X. Show that R is an equivalent relation and find the partition of X induced by R.
7. If R is an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes constitute a partition on A.
8. Given the relation matrix M_R and M_S. Explain how to find $M_{R \circ S}$, $M_{S \circ R}$and M_{R^2}?
9. Let A be s set of books. Let R be a relation on A such that $\langle a,b \rangle \in R$ if ' book a' with cost more and contains fever pages then ' book b'. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?
10. Let R be a binary relation on the set of all positive integers such that $R = \{\langle a,b \rangle \mid a = b^2\}$. Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation?

HASSE DIAGRAM

A partial ordering $\leq$ on a finite set P can be represented in a plane by means of a diagram called *Hasse diagram* or a *partially ordered set set diagram* of $\langle P, \leq \rangle$. If $x \ll y$, then we place y above x, and draw a line (edge) between them. The upward direction indicates successor and downward direction indicates the predecessor. And the incomparable elements are in the same horizontal line.



y is immediate successor of x (or) x is immediate predecessor of y.

z is immediate predecessor of y, and x and y are incomparable.

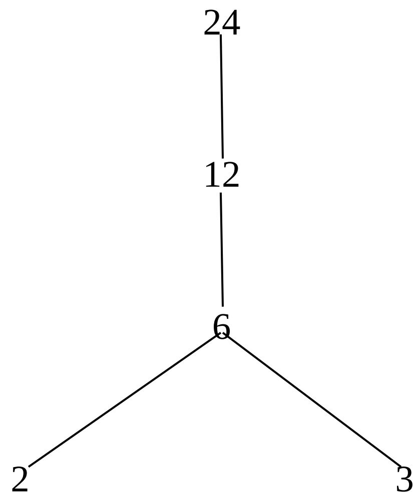
x is predecessor of w but not immediate predecessor.

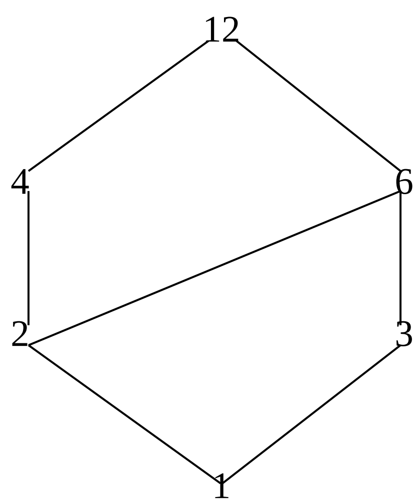
PROBLEMS

1.Let

$P_1 = \{2,3,6,12,24\}$

$P_2 = \{1,2,3,4,6,12\}$ and $\leq$ be a relation such that $x \leq y$ if and only if $x|y$.

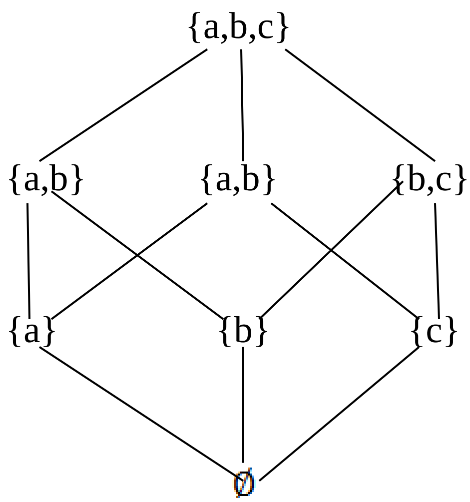

24
12
6
2
3


12
4
6
2
3
1

2.Let

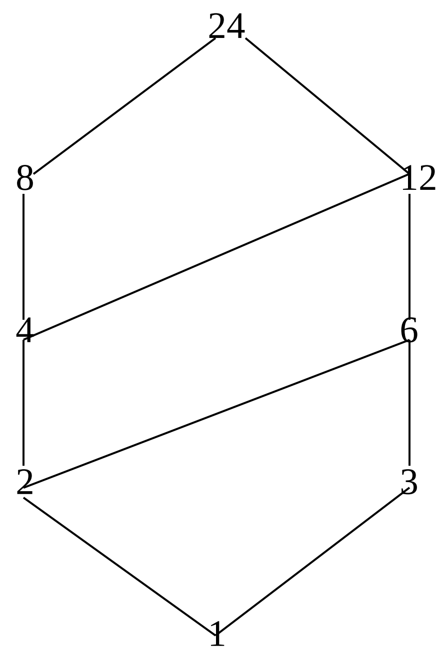
$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\},\}$ be the power set of $\{a,b,c\}$.

Consider the inclusion $(\subseteq)$ relation as the partial ordering on $\rho(A)$, then the Hasse diagram of $\langle \rho(A), \subseteq \rangle$ is



3.Let us consider the set of all divisor of 24, then it is a poset which is denoted by D_{24}

That is $D_{24} = \{1,2,3,4,6,8,12,24\}$ and let the divisor relation be partial ordering.

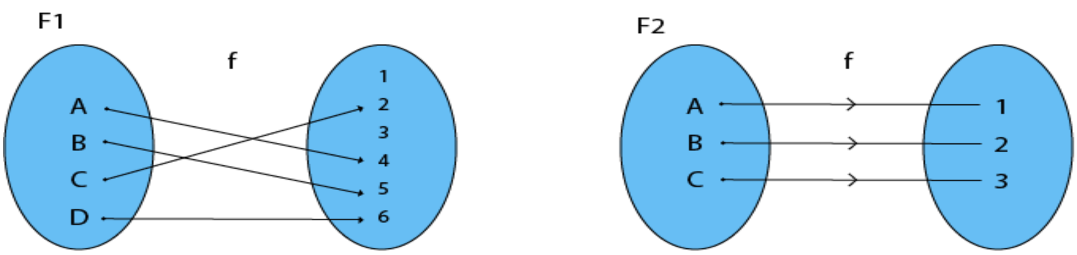

24
8
12
4
6
2
3
1

FUNCTIONS

A function in set theory world is simply a mapping of some (or all) elements from Set A to some (or all) elements in Set B. In the example above, the collection of all the possible elements in A is known as the **domain**; while the elements in A that act as inputs are specially named **arguments**. On the right, the collection of all possible outputs (also known as “range” in other branches), is referred to as the **codomain**; while the collection of actual output elements in B mapped from A is known as the **image.**

Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

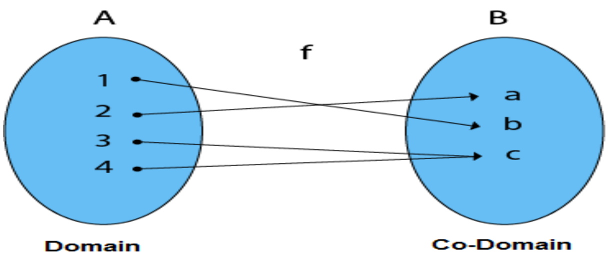


F1 and F2 show one to one Function

2. Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

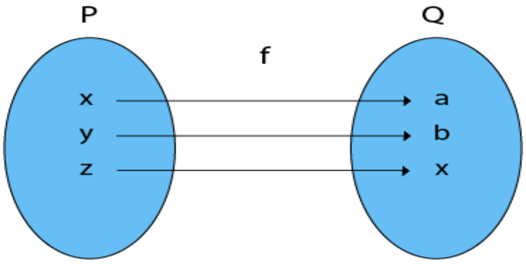
Example: Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



Example:

1. Consider P = {x, y, z}
2. Q = {a, b, c}
3. and f: P ⟶ Q such that
4. f = {(x, a), (y, b), (z, c)}

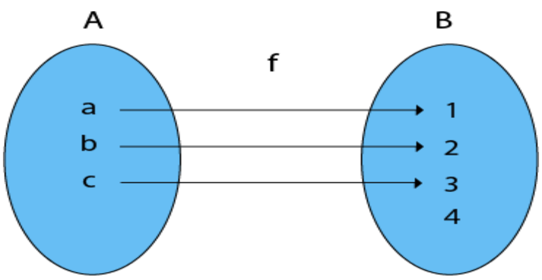
The f is a one-to-one function and also it is onto. So it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

Example:

1. Consider, A = {a, b, c}
2. B = {1, 2, 3, 4} and f: A ⟶ B such that
3. f = {(a, 1), (b, 2), (c, 3)}
4. In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1, 2, 3, 4}

Therefore, it is an into function



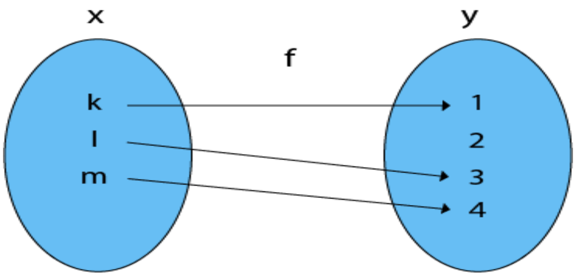
5. One-One Into Functions: Let f: X ⟶ Y. The function f is called one-one into function if different elements of X have different unique images of Y.

Example:

1. Consider, X = {k, l, m}
2. Y = {1, 2, 3, 4} and f: X ⟶ Y such that

3. $f = \{(k, 1), (l, 3), (m, 4)\}$

The function f is a one-one into function

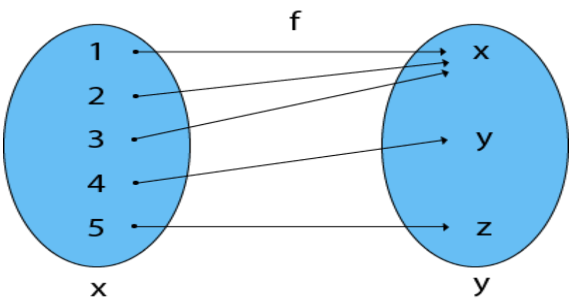


6. Many-One Functions: Let f: X → Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

Example:

1. Consider X = {1, 2, 3, 4, 5}
2. Y = {x, y, z} and f: X → Y such that
3. $f = \{(1, x), (2, x), (3, x), (4, y), (5, z)\}$

The function f is a many-one function



Example 1: Test whether the function f:R→R, f(x) = |x| + x is one-one onto function

<u>Solution:</u>

(1) Given f(x) = |x| + x

f(3) = |3|+3 = 6

f(-3) = |-3|+(-3) = 0

f(2) = |2|+2= 4

f(-2) = |-2|+(-2) = 0

f(-3) = f(-2) = 0

0 has more than one pre-image. Thus f(x) is not 1-1 function

(2) The range of f is the set of non-negative real numbers.

$\therefore$ f is not onto function

Example 2: Let S = {x, x^2/ x$\in$N} and T ={(x,2x)/x$\in$N } where N ={1,2….}. Find the range of S and T. Find S$\cup$T and S$\cap$T

Solution:

S = {x, x^2/ x$\in$N}
S = {(1,1), (2,4), (3,9), (4,16), ………}
T = {(x,2x) /x$\in$N }
S = {(1,2), (2,4), (3,6), (4,8), ………}
Range of S = {1, 4, 9, ………………………}
Range of T = {1, 4, 6, 8, …………………..}
S$\cup$T = {(1,1), (2,4), (3,9), (4,16), (1,2), (3,6), (4,8), ………}
S$\cap$T = {(2,4)}

Example 3: If f: R$\rightarrow$R, g: R$\rightarrow$R are defined by f(x) = x^2-2, g(x) = x+4, find (fog) and (gof) and check whether these functions are injective, surjective and bijective

Solution:

fog(x) = f[g(x)] = f(x+4) =$(x+4)^2$-2 = x^2+8x+14-----------------(1)
gof(x) = g[f(x)] = g(x^2-2) = x^2+2---------------------------------(2)
Given f: R$\rightarrow$R g: R$\rightarrow$R
f(x) = x^2-2

(1) f(1) = 1^1-2 = -1
f(-1) = $(-1)^2$-2 = -1
i.e., f(x_1) = f(x_2) does not imply $x_1 = x_2$
Hence f is not 1-1 function

(2) Let f: R$\rightarrow$R
Let y$\in$R. Suppose x$\in$R such that f(x) = y
x^2-2 = y
x^2 = y+2
x =$\sqrt{}$y+2
f($\sqrt{}$y+2) = ($\sqrt{}$y+2$)^2$-2=y+2-2 = y
for any y$\in$R There exist at least one element $\sqrt{}$y+2$\in$R such that f($\sqrt{}$y+2)=y

$\therefore$ f is on to function

g(x) = x+4

(1) g(x_1) = g(x_2)
x_1+4 = x_2+4
$x_1 = x_2$

g is 1-1 function

(2) g: R→R

Let y ∈R. Suppose x∈R such that f(x) = y

x = y-4 for any y∈R

There exist at least one element y-4∈R such that

g(y-4) = y

∴ g is on to function

As f is not 1-1 but onto, f is not bijective

As g is 1-1 and onto, g is bijective

Theorem 1 : A function f:A→B has an inverse if and only if it is bijective.

Proof.

Suppose g is an inverse for f (we are proving the implication ⇒). Since $g \circ f = I_A$, $g \circ f = I_A$ is injective, so is f . Since $f \circ g = i_B$, $f \circ g = i_B$ is surjective, so is f. Therefore f is injective and surjective, that is, bijective.

Conversely, suppose f is bijective. Let g:B→A, g:B→A be a pseudo-inverse to f. since ff is surjective, $f \circ g = i_{B,}$ $f \circ g = i_B$, and since f is injective, $g \circ f = i_{A,}$ $g \circ f = i_A$.

Theorem 2: Let A and B be nonempty sets, and suppose f : A → B is invertible. Then f −1 : B → A is also invertible, and $(f^{-1})^{-1} = f$.

Proof. f^{-1} is invertible if there is a function g : A → B that satisfies $g \circ f^{-1} = I_B$ and $f^{-1} \circ g = I_A$; and in that case the function g is the unique inverse of f^{-1} . Since g = f is such a function, it follows that f^{-1} is invertible and f is its inverse.

Theorem 3: If f:A→B has an inverse function then the inverse is unique.

Proof.

Suppose g_1 and g_2 are both inverses to f. Then

$g_1 = g_1 \circ I_B = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = I_A \circ g_2 = g_2, g_1 = g_1 \circ I_B = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = I_A \circ g_2 = g_2,$

proving the theorem

Theorem 4 : If f : A → B and g : B → C are one-one, then gof : A → C is also one-one.

Proof:

A function f : A → B is defined to be one-one, if the images of distinct elements of A under f are distinct, i.e. for every $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Given that f: A → B and g: B → C are one-one.

For any $x_1, x_2 \in A$
$f(x_1)=f(x_2) \Rightarrow x_1=x_2$ …(i)
$g(x_1)=g(x_2) \Rightarrow x_1=x_2$ …(ii)
To show: If $gof(x_1) = gof(x_2)$, then $x_1 = x_2$
Let $gof(x_1) = gof(x_2)$
$\Rightarrow g[f(x_1)] = g[f(x_2)]$
$\Rightarrow f(x_1) = f(x_2)$ …from (i)
$\Rightarrow x_1 = x_2$ …from (ii)
Hence, the functions gof: A → C are one-one.

Theorem 5: If f : A → B and g : B → C are onto, then gof : A → C is also onto.

Proof:
Let us consider an arbitrary element z ∈ C
'.' g is onto ∃ a pre-image y of z under the function g such that g (y) = z ………(i)
Also, f is onto, and hence, for y Î B, there exists an element x ∈ A such that f (x) = y ……(ii)
Therefore, gof (x) = g (f (x)) = g (y) from (ii)
= z from (i)
Thus, corresponding to any element z ∈ C, there exists an element x ∈ A such that gof (x) = z.
Hence, gof is onto.
Note: In general, if gof is one-one, then f is one-one. Similarly, if gof is onto, then g is onto.
The composition of functions can be considered for n number of functions.

Theorem 6: If f : X → Y, g : Y → Z and h : Z → S are functions, then ho(gof) = (hog) o f.

Proof: Let x ∈ A
LHS: ho(gof) (x)
= h(gof(x))
= h(g(f(x))), ∀ x in X
RHS: (hog) of f(x)
= hog(f(x))
= h(g(f (x))), ∀ x in X.
LHS = RHS
Hence, ho(gof) = (hog)of.

The composition of functions satisfies the associative property.

Theorem 7: Let f : X → Y and g : Y → Z be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$

Proof:

Given that f: A → B and g: B → C are bijective.
Then gof :A → C is also bijective. Therefore $(gof)^{-1}$:C → A exists.

Also f^{-1}: B→A and g^{-1}: C→B exists. Therefore $f^{-1}og^{-1}$:C → A exists.

Also we know that

$f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$
$g \circ g^{-1} = I_c$ and $g^{-1} \circ g = I_B$

Consider

$$
\begin{aligned}
(f^{-1}og^{-1}) \text{ o } (gof) &= (f^{-1}o(g^{-1}og)of) \\
&= (f^{-1}o(I_B of)) \\
&= I_A
\end{aligned}
$$

Also

$$
\begin{aligned}
(gof) \text{ o}(f^{-1}og^{-1}) &= (go(f \text{ o} f^{-1})og^{-1}) \\
&= (go(I_B og^{-1})) \\
&= I_A
\end{aligned}
$$

Hence, $(gof)^{-1} = f^{-1}og^{-1}$.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – DISCRETE MATHEMATICS – SMTA 1302

GROUP THEORY

Course Contents: Groups – Properties of groups – Semi group and Monoid (definition and examples only) – Subgroups, Cosets – Lagranges Theorem

Binary Operations

Definition

Let S be a set. A **binary operation** on S is a mapping $*: S \times S \to S$, which we will usually denote by $*(a, b) = a * b$.

Also, we've written $*$ as a **function** from $S \times S$ to S, which means two things in particular:

1. The operation $*$ is **well-defined**: given $a, b \in S$, there is exactly one $c \in S$ such that $a * b = c$. In other words, the operation is defined for *all* ordered pairs, and there is no ambiguity in the meaning of $a * b$.
2. S is **closed** under $*$: for all $a, b \in S$, $a * b$ is again in S.

Example

Here are some examples of binary operations.

- Addition and multiplication on Z are binary operations.
- Addition and multiplication on Z_n are binary operations.
- Addition and multiplication on M_n(R) are binary operations.

- The following are non -examples.
  - Define $*$ on R by $a * b = a/b$. This is not a binary operation, since it is not defined everywhere. In particular, $a * b$ is undefined whenever $b = 0$.
  - Define $*$ on R by $a * b = c$, where c is some number larger than $a + b$. This is not well-defined, since it is not clear exactly what $a * b$ should be. This sort of operation is fairly silly, and we will rarely encounter such things in the wild. It's more likely that the given set is not closed under the operation.
  - Matrix multiplication *is* a binary operation on GL_n(R). Recall from linear algebra that the determinant is multiplicative, in the sense that

$$\det(AB) = \det(A)\det(B).$$

Properties of binary operation

i) A binary operation $*$ on a set S is **commutative** if

$$a * b = b * a \text{ for all } a, b \in S.$$

Example

Let's ask whether some of our known examples of binary operations are actually commutative.

1. + and · on Z and Z_n are commutative.

2. Matrix multiplication is not commutative (on both $M_n(R)$ and $GL_n(R)$).

ii) A binary operation $*$ on a set S is **associative** if

$$(a * b) * c = a * (b * c) \text{ for all } a, b, c \in S.$$

Example

The following are examples of associative (and non- associative) binary operations.

1. + and · on Z (and Z_n) are associative.
2. Matrix multiplication is associative.
3. Subtraction on Z is a binary operation, but it is not associative. For example,

$$(3 - 5) - 1 = -2 - 1 = -3,$$

While $3 - (5 - 1) = 3 - 4 = -1$.

4. The **cross product** on R^3 is a binary operation, since it combines two vectors to produce a new vector. However, it is not associative, since

$$a \times (b \times c) = (a \times b) \times c - b \times (c \times a).$$

5. (Composition of functions) Let S be a set, and define

$$F(S) = \{\text{functions } f : S \to S\}.$$

if $f, g, h \in F(S)$, then $(f \circ g) \circ h = f \circ (g \circ h)$. To show that two functions are equal, we need to show that their values at any element $x \in S$ are equal.

For any $x \in S$, we have

$$(f \circ g) \circ h(x) = (f \circ g)(h(x)) = f(g(h(x))) \text{ and}$$

$$f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x)))$$

In other words, $(f \circ g) \circ h(x) = f \circ (g \circ h)(x)$ for all $x \in S$,
so $(f \circ g) \circ h = f \circ (g \circ h)$, and composition of functions is associative.

Groups

A **group** is a set G together with a binary operation $*: G \times G \to G$ satisfying

Closure:

For all $a, b \in G$, we have $a * b \in G$

Associativity:

For all $a, b, c \in G$, we have $a * (b * c) = (a * b) * c$.

Identity:

There exists an element $e \in G$ with the property that

$$e * a = a * e = a \text{ for all } a \in G.$$

Inverses:

For every $a \in G$, there is an element $a^{-1} \in G$ with the property that

$$a * a^{-1} = a^{-1} * a = e.$$

Example

Here are some examples of groups and not a group

1. $(\mathbb{Z}, +)$ is a group, as we have already seen.
2. $(M_n(\mathbb{R}), +)$ is a group.
3. $(\mathbb{Z}_n, +_n)$ is a group.
4. $(\mathbb{Z}, \cdot)$ is *not* a group, since inverses do not always exist. However, $(\{1, -1\}, \cdot)$ is a group. We do need to be careful here—the restriction of a binary oper- ation to a smaller set need not be a binary operation, since the set may not be closed under the operation. However, $\{1, -1\}$ is definitely closed under multiplication, so we indeed have a group.
5. $(M_n(\mathbb{R}), \cdot)$ is not a group, since inverses fail. However, $(GL_n(\mathbb{R}), \cdot)$ is a group. We already saw that it is closed, and the other axioms hold as well.
6. $(\mathbb{Z}_n, \cdot_n)$ is not a group, again because inverses fail. However, $(\mathbb{Z}_n^\times, \cdot_n)$ will be a group. Again, we just need to verify closure: if $a, b \in \mathbb{Z}_n^\times$, then a and b are both relatively prime to n. But then neither a nor b shares any prime divisors with n, so ab is also relatively prime to n. Thus $ab \in \mathbb{Z}_n^\times$.

Definition

A group $(G, *)$ is said to be **abelian** if $*$ is commutative, i.e.

$$a * b = b * a$$

for all $a, b \in G$. If a group is not commutative, we'll say that it is **nonabelian**.

Definition

The **order** of a group G, denoted by $|G|$, is the number of elements in G.

If a group G has infinitely many elements, we will write $|G| = \infty$.[3]

Definition

A group G is said to be **finite** if $|G| < \infty$.

Example

For any n, the additive group Z_n is a finite group, with

$|Z_n| = n$.

Cayley Tables

One of the things that makes finite groups easier to handle is that we can write down a table that completely describes the group. We list the elements out and multiply "row by column."

Example

Let's look at Z_3, for example. We'll write down a "multiplication table" that tells us how the group operation works for any pair of elements. As we mentioned, each entry is computed as "row times column":

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(Of course we have to remember that "times" really means "plus" in this example.) This is called a **group table** (or a **Cayley table**).

Exercise

Show that any group of order 4 is abelian. [Hint: Compute all possible Cayley tables. Up to a reordering of the elements, there are two possible tables.]

Exercise

Show that any group of order 5 is abelian. [Hint: There is only one possible Cayley table, up to relabeling.

Basic Properties of Groups

Property.1

Let G be a group. The identity element $e \in G$ is unique, *i.e., there is only one element e of G with the property that* $ae = ea = a$ *for all* $a \in G$.

Proof

For this proof, we need to use the standard mathematical trick for proving uniqueness: we assume that there is another gadget that behaves like the one in which we're interested, and we prove that the two actually have to be the same.
Suppose there is another $f \in G$ with the property that

for all $a \in G$. Then in particular, $af = fa = a$

But since e is an identity, Therefore $ef = fe = e$Since e is unique, $f.e = e.f = f$.

Property.2.

(Cancellation laws)**.** *Let G be a group, and let a, b, c ∈ G. Then:*

(a) *If $a*b = b*c$, then $b = c$.*

(b) *If $b*a = c*a$, then $b = c$.*

Proof. (a) Suppose that $ab = ac$. Multiply both sides on the left by a^{-1}:

$$a^{-1}(ab) = a^{-1}(ac).$$

By associativity, this is the same as

$$(a^{-1}a)b = (a^{-1}a)c,$$

and since $a^{-1}a = e$, we have $eb = ec$

Since e is the identity, $b = c$. The same sort of argument works for (b), except we multiply the equation on the right by a^{-1}.

The cancellation laws actually give us a very useful corollary. You may have already guessed that this result holds, but we will prove here that inverses in a group are unique.

Property 3.

Let G be a group. Every $a \in G$ has a unique *inverse, i.e. to each*

$a \in G$ there is exactly one element a^{-1} with the property that

$$aa^{-1} = a^{-1}a = e.$$

Proof. Let $a \in G$, and suppose that $b \in G$ has the property that $ab = ba = e$. Then in particular,

$$ab = e = aa^{-1},$$

and by cancellation, $b = a^{-1}$. Thus a^{-1} is unique.

Property 4.

If $a \in G$, then $(a^{-1})^{-1} = a$.

Proof. By definition, $a^{-1}(a^{-1})^{-1} = e$. But $a^{-1}a = aa^{-1} = e$ as well, so by unique-ness of inverses, $(a^{-1})^{-1} = a$.

Property 5.

For any $a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$.

Proof. We'll explicitly show that $b^{-1}a^{-1}$ is the inverse of ab by computing:

$$\begin{aligned}(ab)(b^{-1}a^{-1}) &= ((ab)b^{-1})a^{-1}\\ &= (a(bb^{-1}))a^{-1}\\ &= (ae)a^{-1}\\ &= aa^{-1}\\ &= e.\end{aligned}$$

Of course we also need to check that $(b^{-1}a^{-1})(ab) = e$, which works pretty much the same way:

$$\begin{aligned}(b^{-1}a^{-1})(ab) &= b^{-1}(a^{-1}(ab))\\ &= b^{-1}((a^{-1}a)b)\\ &= b^{-1}(eb)\\ &= b^{-1}b\\ &= e.\end{aligned}$$

Thus $(ab)^{-1} = b^{-1}a^{-1}$.

Property. 6. *The equations $ax = b$ and $xa = b$ have unique solutions in G.*

Proof. The solution to $ax = b$ is $x = a^{-1}b$, and for $xa = b$ it is $x = ba^{-1}$. These are unique since inverses are unique.

Property.7. *Let G be a group, and let a, b G. If either ab ∈ = e or ba = e, then*

$b = a^{-1}$.

Proof. This really amounts to solving the equation $ax = e$ (or $xa = e$). We know from Proposition5, that there is a unique solution, namely $x = a^{-1}e = a^{-1}$ (in either case). Therefore, if $ab = e$ or $ba = e$, then b is a solution to either $ax = e$ or $xa = e$, so $b = a^{-1}$.

Exercise

Prove that if G is a group and $a, b \in G$ with $ab = a$, then $b = e$.

The Order of an Element and Cyclic Groups

Let G be a group. We say that an element $a \in G$ has **finite order** if there exists $n \in \mathbb{Z}^+$ such that $a^n = e$. The smallest such integer is called the **order** of a, denoted by $o(a)$ (or $|a|$). If no such integer exists, we say that a has **infinite order**.

Example.1

The identity element in any group has order 1.

Example.2

2. In $\mathbb{Z}_{12}$, we see that $o(2) = 6$ and $o(3) = 4$.

We'll calculate the powers of 2 first:

$1 \cdot 2 = 2$
$2 \cdot 2 = 2 +_{12} 2 = 4$
$3 \cdot 2 = 2 +_{12} 2 +_{12} 2 = 6$
$4 \cdot 2 = 8$
$5 \cdot 2 = 10$
$6 \cdot 2 = [12]_{12} = 0$
$7 \cdot 2 = [14]_{12} = 2$
$8 \cdot 2 = [16]_{12} = 4$

and so on. What about powers of 3?

$1 \cdot 3 = 3$
$2 \cdot 3 = 3 +_{12} 3 = 6$
$3 \cdot 3 = 3 +_{12} 3 +_{12} 3 = 9$
$4 \cdot 3 = [12]_{12} = 0$
$5 \cdot 3 = [15]_{12} = 3$
$6 \cdot 3 = [18]_{12} = 6$

and so on. Notice that the lists repeat after a while. In particular, we reach 0 (i.e., the identity) after a certain point. We quantify this phenomenon by saying that these elements have **finite order**.

Proposition

Let G be a finite group. Then every element $a \in G$ has finite order.

Proof. Consider the set $\{a^n : n \geq 0\} = \{e, a, a^2, \ldots\}$. Since G is finite, this list of powers can't be infinite. (This follows from the Pigeon- hole principle, for instance. We have an infinite list of group elements that need to fit into only finitely many slots.) Therefore, two different powers of a must coincide, say $a^i = a^j$, with $j \neq i$. We can assume that $j > i$. Then

$$a^{j-i} = a^j a^{-i} = a^i a^{-i} = e,$$

so a has finite order. (In particular, $o(a) \leq j - i$.) Since $a \in G$ was arbitrary, the result follows.

Let's get on with proving some facts about order. First, we'll relate the order of an element to that of its inverse.

Proposition

Let G be a group and let $a \in G$. Then $o(a) = o(a^{-1})$.

Proof. Suppose first that a has finite order, with $o(a) = n$. Then

$$(a^{-1})^n = a^{-n} = (a^n)^{-1} = e^{-1} = e,$$

so $o(a^{-1}) \leq n = o(a)$. On the other hand, if we let $m = o(a^{-1})$, then

$$a^m = ((a^{-1})^{-1})^m = (a^{-1})^{-m} = ((a^{-1})^m)^{-1} = e,$$

so $n \leq m$. Thus $n = m$, or $o(a) = o(a^{-1})$.

Now suppose that a has infinite order. Then for all $n \in \mathbb{Z}^+$, we have $a^n \neq e$.
But then

$$(a^{-1})^n = a^{-n} = (a^n)^{-1} \neq e$$

for all $n \in \mathbb{Z}^+$, so a^{-1} must have infinite order as well.

Let's continue with our investigation of basic properties of order. The first one says that the only integers m for which $a^m = e$ are the multiples of $o(a)$.

Proposition

If $o(a) = n$ and $m \in \mathbb{Z}$, then $a^m = e$ if and only if n divides m.

Proof. If $n \mid m$, it is easy. Write $m = nd$ for some $d \in \mathbb{Z}$. Then

$$a^m = a^{nd} = (a^n)^d = e^d = e.$$

On the other hand, if $m \geq n$, we can use the Division Algorithm to write

$m = qn + r$ with $0 \le r < n$. Then

$$e = a^m = a^{qn+r} = a^{qn}a^r = (a^n)^q a^r = ea^r = a^r,$$

so $a^r = e$. But $r < n$, and n is the smallest positive power of a which yields the identity. Therefore r must be 0, and n divides m.

Note that this tells us something more general about powers of a: when we proved that elements of finite groups have finite order, we saw that $a^i = a^j$ implied that $a^{j-i} = e$. This means that $n = o(a)$ divides $j - i$. In other words, i and j must be congruent mod n.

Proposition

Let G be a group, $a \in G$ an element of finite order n. Then

$a^i = a^j$ *if and only if* $i \equiv j(\text{mod} n)$.

Along the same lines, we observed that if $a^i = a^j$ with $j > i$, then $a^{j-i} = e$, so a has to have finite order. Taking the contrapositive of this statement, we get the following result.

Cyclic Groups

Let's take a few steps back now and look at the bigger picture. That is, we want to investigate the structure of the set (a) for $a \in G$. What do you notice about it?

- **Closure:** $a^i a^j = a^{i+j} \in (a)$ for all $i, j \in$ Z.
- **Identity:** $e = a^0 \in (a)$
- **Inverses:** Since $(a^j)^{-1} = a^{-j}$, we have $(a^j)^{-1} \in (a)$ for all $j \in$ Z.

It other words, (a) is itself a group. That is, the set of all powers of a group element is a group in its own right. We will investigate these sorts of objects further in the next section, but let's make the following definition now anyway.

Definition

For $a \in G$, the set (a) is called the **cyclic subgroup** generated by a.

For now, let's look at a particular situation. Is G ever a cyclic subgroup of itself? That is, can a "generate" the whole group G? Yes, this does happen some times, and such groups are quite special.

Definition

A group G is called **cyclic** if $G = (a)$ for some $a \in G$. The element a is called a **generator** for G.

Example

1. One of our first examples of a group is actually a cyclic one: Z forms a cyclic group under addition. What is a generator for Z? Both 1 and −1 generate it, since every integer $n \in Z$ can be written as a "power" of 1 (or −1):

   These are actually the only two generators.

2. How about a finite cyclic group? For any n, Z_n is cyclic, and 1 is a generator in much the same way that 1 generates Z. There are actually plenty of other generators, and we can characterize them by using our knowledge of greatest common divisors. We'll postpone this until we've made a couple of statements regarding cyclic groups.

3. The group (Q, +) is not cyclic. (This is proven in Saracino.)

4. The dihedral group D_3 is not cyclic. The rotations all have order 3, so

$$(r_1) = (r_2) = \{i, r_1, r_2\}.$$

On the other hand, all of the reflections have order 2, so

$$(m_1) = \{i, m_1\}, \quad (m_2) = \{i, m_2\}, (m_3) = \{i, m_3\}.$$

Now let's start making some observations regarding cyclic groups. First, if $G = (a)$ is cyclic, how big is it? It turns out that our overloading of the word "order" was fairly appropriate after all, for $|G| = o(a)$.

Theorem

If $G = (a)$ is cyclic, then $|G| = o(a)$.

Proof. If a has infinite order, then $|G|$ must be infinite. On the other hand, if $o(a) = n$, then we know that $a^i = a^j$ if and only if $i \equiv j$ mod n, so the elements of G are $\{e, a, a^2, . . . , a^{n-1}\}$of which there are $n = o(a)$.

If we pair this result with Theorem, we can characterize the generators of any finite cyclic group.

Proposition

Let G be a finite cyclic group. Then for any $b \in G$, we have

$o(b) \mid |G|$.

Theorem 1.

Every cyclic group is abelian.

Proof. Let G be a cyclic group and let a be a generator for G, i.e. $G = (a)$. Then given two elements $x, y \in G$, we must have $x = a^i$ and $y = a^j$ for some $i, j \in Z$. Then

$$xy = a^i a^j = a^{i+j} = a^{j+i} = a^j a^i = yx,$$

and it follows that G is abelian.

Remark

The converse to Theorem1 is not true. That is, there are abelian groups that are not cyclic. Saracino gives the example of the non-cyclic group (Q, +). However, this is a good place to introduce a different group—the **Klein 4-group**, denoted V_4. The Klein 4-group is an abelian group of order 4. It has elements $V_4 = \{e, a, b, c\}$, with

$$a^2 = b^2 = c^2 = e \text{ and } ab = c, bc = a, ca = b.$$

Note that it is abelian by a previous exercise (Exercise 2.1).[8] However, it is not cyclic, since every element has order 2 (except for the identity, of course). If it were cyclic, there would necessarily be an element of order 4.

Subgroups

Let $(G, *)$ be a group. A **subgroup** of G is a nonempty subset

$H \subseteq G$ with the property that $(H, *)$ is a group.

Note that in order for H to be a subgroup of G, H needs to be a group with respect to the operation that it inherits from G. That is, H and G *always* carry the same binary operation. Also, we'll write

$$H \le G$$

to denote that H is a subgroup of G. Finally, if we want to emphasize that $H \le G$ but $H f= G$, we will say that H is a **proper** subgroup of G.

le

Let's look at the group Z (under addition, of course). Define 2Z = {even integers} = {2n : $n \in$ Z}. Is 2Z a subgroup of Z? We need to check that 2Z itself forms a group under addition:

- **Closure:** If $a, b \in$ 2Z, then $a = 2n$ and $b = 2m$ for some $n, m \in$ Z. Then

  $a + b = 2n + 2m = 2(n + m) \in$ 2Z, so 2Z is indeed closed under addition.

- **Associativity:** Z is already associative, so nothing changes when we pass to a subset of Z.
- **Identity:** The identity for addition on Z is 0, which is even: $0 = 2 \cdot 0 \in$ 2Z.
- **Inverses:** If $a \in$ 2Z, then $a = 2n$ for some $n \in$ Z, and $-a = -2n = 2(-n) \in$ 2Z.

Therefore, (2Z, +) is a group, hence a subgroup of Z.

Examples:.

1. Every group G has two special subgroups, namely

$$\{e\} \text{ and } G.$$

These are called the **trivial subgroups** of G.[9]

2. We saw earlier that 2Z is a subgroup of Z. There is nothing special about 2 in this example: for any $n \in Z^+$,

$$nZ = \{na : a \in Z\}$$

is a subgroup of Z. The exact same computations that we performed for 2Z will show that $nZ \leq Z$.

3. The rational numbers Q form an additive subgroup of R.

4. Here is an example from linear algebra. Consider the n-dimensional vector space R^n. Then R^n is, in particular, an abelian group under addition, and any vector subspace of R^n is a subgroup of R^n.[10] If H is a subspace of R^n, then it is closed under addition, and closure under scalar multiplication guarantees that $0 \in H$ and for $v \in H$, $-v = -1 \cdot v \in H$.

Definition

The group (a) is called the **cyclic subgroup** generated by a.

When we say that a "generates" (a), we mean that that (a) is created entirely out of the element a. In a certain sense, (a) is the *smallest* possible subgroup of G which contains a. Let's try to make this more precise. If $H \leq G$ and $a \in H$, then H must contain the elements

$$a, a^2, a^3, \ldots,$$

since H is closed. It also must contain e and a^{-1}, hence all of the elements

$$\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots,$$

i.e. all powers of a. That is, $(a) \subset H$, and we have proven the following fact:

Theorem

Let G be a group and let $a \in G$. Then (a) is the smallest subgroup of G containing a, in the sense that if $H \leq G$ and $a \in H$, then $(a) \subseteq H$.

Of course we've already encountered several examples of cyclic subgroups in our studies thus far.

Example

1. Our first example of a subgroup, $2Z \leq Z$, is a cyclic sub- group, namely (2). Similarly, nZ is cyclic for any $n \in Z$.

2. The subgroup consisting of rotations on D_n,

$$H = \{i, r_1, r_2, \ldots, r_{n-1}\} \leq D_n,$$

is cyclic since $H = (r_1)$.

3. All the proper subgroups of Z_4 and V_4 that we listed are cyclic. In addition, Z_4 is a cyclic subgroup of itself, but V_4 is not.
4. The trivial subgroup $\{e\}$ is always a cyclic subgroup, namely (e).

Theorem

Let G be a group. A nonempty subset $H \subseteq G$ is a subgroup if and only if whenever $a, b \in H$, $ab^{-1} \in H$.

Proof. Suppose that $H \leq G$, and let $a, b \in H$. Then $b^{-1} \in H$, so $ab^{-1} \in H$ since H is closed.

Conversely, suppose that $ab^{-1} \in H$ for all $a, b \in H$. Then for any $a \in H$, we can take $a = b$ and conclude that

$$e = aa^{-1} \in H,$$

so H contains the identity. Since $e \in H$, for any $a \in H$ we have

$$a^{-1} = ea^{-1} \in H,$$

so H is closed under taking inverses. Finally, we claim that H is closed under the group operation. If $a, b \in H$, then $b^{-1} \in H$, so $b^{-1}a^{-1} \in H$, and therefore

$$ab = \left((ab)^{-1}\right)^{-1} = (b^{-1}a^{-1})^{-1} \in H.$$

Thus H is closed, hence a subgroup of G.

The next criterion is quite interesting. It obviously reduces the number of things that one needs to check, but it only works for a *finite* subset of a group G.

Theorem

Let G be a group and H a nonempty ***finite*** *subset of G. Then H is a subgroup if and only if H is closed under the operation on G.*

Proof. If H is a subgroup, then it is obviously closed by hypothesis.

On the other hand, we are assuming that H is closed, so we need to verify that $e \in H$ and that for every $a \in H$, $a^{-1} \in H$ as well. Since $\{e\} \leq G$, we may assume that H is nontrivial, i.e. that H contains an element a distinct from the identity. Since H is closed, the elements

$$a, a^2, a^3, \ldots$$

are all in H, and since H is finite, this list cannot go on forever. That is, we must eventually have duplicates on this list, so

$$a^i = a^j$$

for some $1 \leq i < j \leq |H|$. Since $i < j$, $j - i \geq 0$ and we have

$$a^i = a^j = a^{j-i}a^i,$$

and using cancellation, we get
$a^{j-i} = e.$

Therefore, $e \in H$. Now observe that $j - i - 1 \geq 0$, so $a^{j-i-1} \in H$, and

$$aa^{j-i-1} = a^{j-i} = e,$$

so $a^{-1} = a^{j-i-1} \in H$. Therefore, H is a subgroup of G.

This theorem has an easy corollary, which is useful when the group is finite.

Corollary

*If G is a **finite** group, a subset H ⊆ G is a subgroup of G if and only if it is closed under the operation on G.*

Definition

The number of distinct (right) cosets of H in G is called the

index of H in G, denoted by

$$[G : H].$$

The set of all right cosets of H in G is denoted by G/H, so

$$\#(G/H) = [G : H].$$

Subgroups of Cylic Groups

Let's return now to the cyclic case. There is one very important thing that we can say about cyclic groups, namely that their subgroups are always cyclic.

Theorem

A subgroup of a cyclic group is cyclic.

Proof. Let $G = (a)$ be a cyclic group and let H be a subgroup of G. We may assume that $H f= \{e\}$, since $\{e\}$ is already known to be cyclic. Then H contains an element other than e, which must have the form a^m for some $m \in Z$ since G is cyclic. Assume that m is the *smallest* positive integer for which $a^m \in H$. We claim that $H = (a^m)$. To do this, we need to show that if $a^n \in H$, then a^n is a power of a .

Suppose that $a^n \in H$, and use the Division Algorithm to write $n = qm + r$, where $0 \leq r < m$. Then

$$a^n = a^{qm+r} = a^{qm}a^r = (a^m)^q a^r.$$

Since H is a subgroup, $(a^m)^{-q} \in H$, hence $(a^m)^{-q}a^n \in H$, and it follows that

$$a^r = (a^m)^{-q}a^n$$

is in H. But $r < m$ and we have assumed that m is the smallest positive integer such that $a^m \in H$, so we must have $r = 0$. In other words, $a^n = (a^m)^q$, so $a^n \in (a^m)$. Since a^n was an arbitrary element of H, we have shown that $H \subseteq (a^m)$. Since $a^m \in H$, we also have $(a^m) \subseteq H$, so $H = (a^m)$, and H is cyclic. This theorem has a particularly nice corollary, which tells us a lot about the structure of Z as an additive group.

Lagrange's Theorem

Theorem

Let G be a finite group and let $H \leq G$. Then $|H|$ divides $|G|$.

Proof. Let $Ha_1, \ldots, Ha_k$ denote the distinct cosets of H in G. That is, $a_1, \ldots, a_k$ all represent different cosets of H, and these are all the cosets. We know that the cosets of H partition G, so

$|G| = O(Ha_1) + \cdots + O(Ha_k)$.

(Here O means the *cardinality* of the set, or simply the number of elements in that set.) Therefore, it will be enough to show that each coset has the same number of elements as H.

We need to exhibit a bijection between H and Ha_i for each i. For each $i = 1, \ldots, k$, define a function $f_i : H \to Ha_i$ by

$$f(h) = ha_i.$$

If we can prove that f is a bijection, then we will have

$$|H| = O(Ha_i)$$

for all i. if $h_1, h_2 \in H$ with $f(h_1) = f(h_2)$, then

$$h_1 a_i = h_2 a_i,$$

which implies that $h_1 = h_2$, so f is one-to-one. To see that it is onto, take $h \in H$; then $f(h) = ha_i$.

Thus all the cosets have the same number of elements, namely $|H|$, and really says that

$$|G| = \underbrace{|H| + \cdots + |H|}_{k \text{ times}} = k|H|.$$

this implies $|H|$ *divides* $|G|$.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – DISCRETE MATHEMATICS – SMTA 1302

UNIT IV: COMBINATORICS

COURSE CONTENT: Mathematical induction – The basics of counting – The pigeonhole principle – Permutations and combinations – Recurrence relations – Solving linear recurrence relations – Generating functions – Inclusion and exclusion principle and its applications

MATHEMATICAL INDUCTION

Example 1: Show that $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let P(n) : $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)}$

1.P(1): $\frac{1}{1.2} = \frac{1}{1(1+1)}$ is true.

2.ASSUME

P(k): $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{k(k+1)}$

$= \frac{k}{k+1}$ is true. -> (1)

CLAIM : P(k+1) is true.

P(k+1) = $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} \quad \frac{1}{(k+1)(k+2)}$ using (1)

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{(k.k)+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$

$= \frac{(k+1)}{(k+2)}$

$= \frac{k+1}{(k+1)+1}$

= P(k+1) is true.

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION

$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ Is true for all n .

EXAMPLE 2 : Using mathematical induction prove that if

n>=1, then 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! - 1

SOLUTION:

Let p(n) : 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! - 1

1.P(1) : 1.1! = (1+1)! – 1 is true

2 . ASSUME p(k) : 1.1! + 2.2! +3.3! +……+k.k!

= (k+1)! – 1 is true

CLAIM : p(k+1) is true.

P(k+1) = 1.1! + 2.2! +3.3! +……+k.k! +(k+1)(k+1)!

= (k+1)!-1 +(k+1)(k+1)!

= (k+1)! [(1+k+1)] – 1

= (k+1)! (k+2) – 1

= (k+2) ! – 1

= [(k+1) + 1]! - 1

P(k+1)is true.

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

P(n) : 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! – 1 , n>=1

EXAMPLE 3 : Use mathematical induction , prove that $\sum^{n}_{m=0} 3^m = \frac{(3\wedge n+1)-1}{2}$

SOLUTION:

Let p(n): $3^0 + 3^1 + \ldots\ldots 3^n = \frac{(3\wedge n+1)-1}{2}$

1.p(0) : $3^0 = \frac{(3\wedge 0+1)-1}{2} = \frac{2}{2} = 1$ is true .

2.ASSUME

P(k):): $3^0 + 3^1 + \ldots\ldots\ 3^n = \frac{(3\wedge k+1)-1}{2}$ is true.

CLAIM : p(k+1)is true.

P (k+1) :): $3^0 + 3^1 + 3^2 + \ldots\ldots + 3^k + 3^{k+1}$

$= \frac{(3\wedge k+1)-1}{2} + 3^{k+1}$ using (1)

$= \frac{(3\wedge k+1)+2.(3\wedge k+1)-1}{2}$

$= \frac{3(3\wedge k+1)-1}{2}$

$= \frac{(3\wedge k+2)-1}{2}$

$= \frac{(3\wedge(k+1)+1)-1}{2}$

P(k+1)is true.

By the principle of mathematical induction.

P(n): $\sum^{n}_{m=0} 3^m = \frac{(3\wedge n+1)-1}{2}$ is true for n>=0

EXAMPLE 4 :Use mathematical induction , prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

SOLUTION:

Let p(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

1.p(2): that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$ = (1.707) > $\sqrt{2}$ + (1.414) is true

2.ASSUME

P(k): that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$ is true -> (1)

CLAIM : p(k+1) is true.

P(k+1) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\sqrt{k} + \frac{1}{\sqrt{k+1}}$ using (1)

$\frac{\sqrt{k}\ \sqrt{k+1}\ +1}{\sqrt{k+1}}$

$\frac{\sqrt{k(k+1)}\ +1}{\sqrt{k+1}}$

$> \frac{\sqrt{k.k}\ +1}{\sqrt{k+1}}$

$> \frac{k+1}{\sqrt{k+1}}$

$> \sqrt{k+1}$

P(k+1) $> \sqrt{k+1}$

P(K+1) is true

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION.

that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} +\dots+ \frac{1}{\sqrt{n}} > \sqrt{n+1}$

EXAMPLE 5: Using mathematical induction ,prove that $1^2 + 3^2 + 5^2 + \dots\dots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

SOLUTION :

Let p(n): $1^2 + 3^2 + 5^2 + \dots\dots (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$

1.p(1): $1^2 = \frac{1}{3} 1(2-1)(2+1) = \frac{1}{3} . 3$

=1 is true.

2.ASSUME p(k)is true.

$1^2 + 3^2 + 5^2 + \dots\dots (2k-1)^2 = \frac{1}{3} n(2k-1)(2k+1)$ -> (1) Is true.

CLAIM : p(k+1) is true.

P (k+1) $= \frac{1}{3} k (2k-1) (2k+1) +(2k+1)^2$ using (1)

$= \frac{1}{3} (2k+1) [k(2k-1) +3(2k+1)]$

$= \frac{1}{3} (2k+1) (2k2+5k+3)$

$= \frac{1}{3} (2k+1)(2k+3)(k+1)$

$= \frac{1}{3} (k+1) [2(k+1)-1][2(k+1)+1]$

P(k+1) is true .

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

$P(n) = 1^2 + 3^2 + 5^2 + \ldots\ldots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

EXAMPLE 6: Use mathematical induction to show that $n^3 - n$ is divisible by 3. For $n \in Z^+$

SOLUTION:

Let p(n): $n^n - n$ is divisible by 3.

1. p(1): $1^3 - 1$ is divisible by 3,is true.
2. ASSUME p(k): $k^3 - k$ is divisible by 3. -> (1)

CLAIM : p(k+1) is true .

P (k+1): $(k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= (k^3 - k) + 3(k^2 + k)$ ->(2)

(1) => $k^3 - k$ Is divisible by 3 and $3(k^2 + k)$ is divisible by 3 ,we have equation (2) is divisibleby 3

Therefore P(k+1) is true.

By the principle of mathematical induction , $n^3 - n$ is divisible by 3.

Strong Induction:

There is another form of mathematics induction that is often useful in proofs.In this form we use the basis step as before, but we use a different inductive step. We assume that p(j)is true for j=1…,k and show that p(k+1)must also be true based on this assumption . This is called strong Induction (and sometimes also known as the second principles of mathematical induction).

We summarize the two steps used to show that p(n)is true for all positive integers n.

Basis Step : The proposition P(1) is shown to be true

Inductive Step: It is shown that

$[P(1) \wedge P(2) \wedge \ldots\ldots \wedge P(k)] \rightarrow P(k+1)$

NOTE:

The two forms of mathematical induction are equivalent that is, each can be shown to be valid proof technique by assuming the other

EXAMPLE 1: Show that if n is an integer greater than 1, then n can be written as the product of primes.

SOLUTION:

Let P(n) be the proportion that n can be written as the product of primes

Basis Step : P(2) is true , since 2 can be written as the product of one prime

Inductive Step: Assume that P(j) is positive for all integer j with j<=k. To complete the Inductive Step, it must be shown that P(k+1) is trueunder the assumption.

There are two cases to consider namely

i) When (k+1) is prime
ii) When (k+1) is composite

Case 1 : If (k+1) is prime, we immediately see that P(k+1) is true.

Case 2: If (k+1) is composite

Then it can be written as the product of two positive integers a and b with 2<=a<b<=k+1. By the Innduction hypothesis, both a and b can be written as the product of primes, namely those primes in the factorization of a and those in the factorization of b .

WELL ORDERING PROPERTY

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non-empty set of non negative integers has a least element.

The well-ordering property can often be used directly in the proof.

PERMUTATION AND COMBINATION

PERMUTATION

A Permutation is an arrangement of set of n objects in a definite order taken some or all at a time.

Example: 1.Three letters a,b,c can be arranged

abc, acb, bac, bca, cab, cba. We have taken all the three for arrangement.

2. Using the three letters a,b,c the total no. of arrangements or permutation taking two at a time.

ab, bc, ac, ba, cb, ca.

The no. of permutation of n objects taken r at a time is denoted by P(n,r) or nP_r and is defined as

$$nP_r = \frac{n!}{(n-r)!} \text{ where } r \leq n.$$

Corollary

If r= n,

$$nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Permutation with repetition

Let $P(n; n_1, n_2, \ldots, n_r)$ denote the no. of permutation of n objects of which n_1 are alike n_2 are alike …n_r are alike then the formula is given by

$$P(n; n_1, n_2, n_3, \ldots n_r) = \frac{n!}{n_1! n_2! n_3! \ldots n_r!}$$

Circular Permutation

Arrangement of objects in a circle is called Circular Permutation. A circular Permutation of n different objects is (n-1)!

Solved Problems

1. Find the value of n if $nP_5 = 42nP_3$ where n>4

Solution

$$\frac{n!}{(n-5)!} = 42 \frac{n!}{(n-3)!}$$

$$\frac{1}{(n-5)!} = 42 \frac{1}{(n-3)(n-4)(n-5)!}$$

$(n-3)(n-4) = 42$

$n^2-7n-30 = 0$

$(n-10)(n+3) = 0$

n = 10, -3

Since n is positive, n = 10

2. How many four digit nos. can be formed by using the digits 1 to 9. If repetition of digits are not allowed.

Solution

$$9P_4 = \frac{9!}{5!} = \frac{9\times8\times7\times6\times5!}{5!}$$

= 3024.

3. Find the no. of permutations of the letters of the word ALLAHABAD.

Solution

There are 9 letters in this word. To form different words containing all these 9 letters is

$$= \frac{9!}{4!2!}$$

4. (i) A committee of 3 is to be chosen out of 5 English, 4 French, 3 Indians and the committee to contain one each. In how many ways can this be done? (ii) In how many arrangements one particular Indian can be chosen?

Solution

(i)One English member can be chosen in 5 ways
One French member can be chosen in 4 ways
One Indian member can be chosen in 3 ways
No of ways the committee can be formed = 5x4x3 = 60 ways.

(ii)Since the Indian member is fixed, we have to fill the remaining two places choosing one from English and French each. This can be done in 5x4 = 20 ways.

5. There are 5 trains from Chennai to Delhi and back to Chennai. In how many ways a person go Chennai to Delhi and return to Chennai.

Solution

5 x 4 = 20.

6. There is a letter lock with three rings, each ring with 5 letters and the password is unknown. How many different useless attempts are made to open the lock.

Solution

Total no. of attempts = 5 x 5 x 5 = 5^3

Only one will unlock , so the total no. of useless attempts is (5^3-1) = 125 -1
= 124.

7. (i) Find the no. of arrangements of the letters of the word ELEVEN,(ii) How many of them begin and end with E. (iii) How many of them have three E's together. (iv) How many begin with E and end with N.

Solution

(i) $\frac{6!}{3!} = 6 \times 5 \times 4 = 120$ ways.

(ii) First and last places are fixed, the remaining 4 places are done in 4! ways.

(iii) Treat the 3 E's as a single element.

Therefore, this single element along with L,V,N can be arranged in 4! ways.

(iv) $\frac{4!}{2!} = 4 \times 3 = 12$.

8. There are 6 different books on Physics, 3 on Chemistry, 2 on Mathematics. In how many ways can they be arranged on a shelf if the books of the same subject are always together?

Solution

Considering Physics books, Chemistry books, Mathematics books as three elements, three elements can be arranged in 3! ways. Also
Physics books can themselves be arranged in 6! Ways
Chemistry books can themselves be arranged in 3! Ways
Mathematics books can themselves be arranged in 2! Ways
No.of arrangements = 3! 6! 3! 2!

9. Find the no. of arrangements in which 6 boys and 4girls can be arranged in a line such that all the girls sit together and all the boys sit together.

Solution

The no. of arrangement with all the girls sit together and all the boys sit together is 2! 4! 6! ways.

10. Find the no. of ways in which 10 exam papers can be arranged so that 2 particular papers may not come together.

Solution

2 particular papers should not come together. The remaining 8 papers can be arranged in 8! ways.The 2 papers can be filled in 9 gaps in between these 8 papers in $9P_2$ ways.

11. In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution

The 5 lions should be arranged in the 5 places marked ‘L’.
This can be done in 5! ways.
The 4 tigers should be in the 4 places marked ‘T’.
This can be done in 4! ways.
Therefore, the lions and the tigers can be arranged in 5!*4!= 2880 ways

12. In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?

Solution

5 boys can be seated in a row in 5! ways.
Also the girls can be seated in 3! ways
The 3 girls can be filled in the 6 gaps between the boys in $6P_3$ ways.
Total no of arrangements = 5! x 3! x $6P_3$ = 1440

13. There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairy tales, novels and plays.

Solution

There are 4 books on fairy tales and they have to be put together.
They can be arranged in 4! ways.

Similarly, there are 5 novels.
They can be arranged in 5! ways.

And there are 3 plays.
They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in 4!*5!*3!= 17280 ways

13. Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 5.3. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution

First, we consider the books on fairy tales, novels and plays as single objects.
These three objects can be arranged in 3!=6 ways.
Let us fix one of these 6 arrangements.
This may give us a specific order, say, novels -> fairy tales -> plays.

Given this order, the books on the same subject can be arranged as follows.
The 4 books on fairy tales can be arranged among themselves in 4!=24 ways.
The 5 novels can be arranged in 5!=120 ways.
The 3 plays can be arranged in 3!=6 ways.

For a given order, the books can be arranged in 24*120*6=17280 ways.
Therefore, for all the 6 possible orders the books can be arranged in 6*17280= 103680 ways.

14. In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution

Let 4 girls be one unit and now there are 6 units in all.
They can be arranged in 6! ways.In each of these arrangements 4 girls can be arranged in 4! ways.
=> Total number of arrangements in which girls are always together =6!*4!=720*24= 17280.

15. How many arrangements of the letters of the word 'BENGALI' can be made

(i) If the vowels are never together.
(ii) If the vowels are to occupy only odd places.

Solution

There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants.

(i) Considering vowels a, e, i as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.
=> Total number of words =5!*3!
=120*6=720
So there are total of 720 ways in which vowels are ALWAYS TOGEGHER.

Now,
Since there are no repeated letters, the total number of ways in which the letters of the word 'BENGALI' cab be arranged:
=7!=5040

So,
Total no. of arrangements in which vowels are never together:
=ALL the arrangements possible – arrangements in which vowels are ALWAYS TOGETHER
=5040–720=4320

ii) There are 4 odd places and 3 even places. 3 vowels can occupy 4 odd places in 4P3 ways and 4 constants can be arranged in 4P4 ways.
=> Number of words $=4P_3*4P_4= 576$.

16. In how many ways 5 gentlemen and 3 ladies can be arranged along a round table so that no 2 ladies are together.

Solution:

The 5 gentlemen can be arranged in a round table in (5-1)! = 4! ways.

Since no 2 ladies are together, they can occupy the 5 gaps in between the gentlemen in $5P_3$ ways.

Therefore, total no. of arrangements = $5P_3$ x 4!

COMBINATION

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S1,S2,S3,S4. Then you can choose two pairs in the following ways:

1. {S1,S2}   2. {S1,S3}   3. {S1,S4}
4. {S2,S3}   5. {S2,S4}   6. {S3,S4}

[Observe that {S1,S2} is the same as {S2,S1}. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know that it can be done in 4P4=12 ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below:

1. {S1,S2} -> S1 on Monday and S2 on Tuesday or S2 on Monday and S1 on Tuesday
2. {S1,S3} -> S1 on Monday and S3 on Tuesday or S3 on Monday and S1 on Tuesday
3. {S1,S4} -> S1 on Monday and S4 on Tuesday or S4 on Monday and S1 on Tuesday
4. {S2,S3} -> S2 on Monday and S3 on Tuesday or S3 on Monday and S2 on Tuesday
5. {**S2**,S4} -> S2 on Monday and S4 on Tuesday or S4 on Monday and S2 on Tuesday
6. {S3,S4} -> S3 on Monday and S4 on Tuesday or S4 on Monday and S3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.
This argument holds good in general as we can see from the following theorem.

Theorem

Let n≥1 be an integer and r≤n. Let us denote the number of ways of choosing r objects out of n objects by nCr. Then
nCr=$\frac{nP_r}{r!}$.

Example: Find the number of subsets of the set {1,2,3,4,5,6,7,8,9,10,11} having 4 elements.

Solution

Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

$11C_4 = \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330$

Example: 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution

For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $12C_4$=495.

Therefore, we can draw 495 quadrilaterals.

Example: In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution

Number of ways of choosing 2 black pens from 5 black pens
$=5C_2=\frac{5P_2}{2!}=\frac{5\times4}{1\times2}=10$

Number of ways of choosing 2 white pens from 3 white pens
$=3C_2=\frac{3P_2}{2!}=\frac{3\times2}{1\times2}=3$

Number of ways of choosing 2 red pens from 4 red pens
$=4C_2=\frac{4P_2}{2!}=\frac{4\times3}{1\times2}=6$

=> By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in 10*3*6= 180ways.

Example: A question paper consists of 10 questions divided into two parts A and B. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part A and at least 2 from part B. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution

The candidate has to select six questions in all of which at least two should be from Part A and two should be from Part B. He can select questions in any of the following ways:

	Part A	**Part B**
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is:
$5C_2*5C_4$=10*5=50

If the candidate follows choice (ii), the number of ways in which he can do so is: $5C_3*5C_3=10*10=100$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is:$5C_4*5C_2=5*10=50$

Therefore, the candidate can select the question in 50+100+50= 200 ways**.**

Example: A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when:**(i)** At least 2 women are included?**(ii)** At most 2 women are included?

Solution

(i) When at least 2 women are included.
The committee may consist of

3 women, 2 men: It can be done in $4C*6C_2$ ways
Or, 4 women, 1 man: It can be done in $4C_4*6C_1$ ways
or, 2 women, 3 men: It can be done in $4C_2*6C_3$ ways

=> Total number of ways of forming the committee:
$=4C_3*6C_2+4C_4*6C_1+4C_2*6C_3= 186$ ways

(ii) When at most 2 women are included
The committee may consist of

2 women, 3 men: It can be done in $4C_2*6C_3$ ways
Or, 1 women, 4 men: It can be done in $4C_1*6C_4$ ways
Or, 5 men: It can be done in $6C_5$ ways

=> Total number of ways of forming the committee:
$=4C_2*6C_3+4C_1*6C_4+6C_5= 186$ ways

Example: The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and at least 4 bowlers?

Solution

We are to choose 11 players including 1 wicket keeper and 4 bowlers
or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players
$=2C_1*5C_4*9C_6=840$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players
$=2C_1*5C_5*9C_5=252$

=> Total number of ways of selecting the team:
=840+252= 1092

Example: There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf?

Solution

4 novels can be selected out of 5 in $5C_4$ ways.
2 biographies can be selected out of4 in $4C_2$ ways.
Number of ways of arranging novels and biographies:
$=5C_4*4C_2=30$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6!=720 ways.
By the Counting Principle, the total number of arrangements =30*720= **21600**

Example: From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels?

Solution

From 5 consonants, 3 consonants can be selected in $5C_3$ ways.
From 4 vowels, 2 vowels can be selected in $4C_2$ ways.
Now with every selection, number of ways of arranging 5 letters is $5P_5$

Total number of words $=5C_3*4C_2*5P_5=7200$.

Binomial Theorem

$(a+b)^n = nC_0a^n+ nC_1a^{n-1}b+\ldots+ nC_nb^n$

Example: Find the coefficient of the independent term of x in expansion of $(3x - (2/x^2))^{15}$.

Solution

The general term of $(3x - (2/x^2))^{15}$ is written, as $T_{r+1} = {}^{15}C_r\,(3x)^{15-r}\,(-2/x^2)^r$. It is independent of x if, 15 - r - 2r = 0 => r = 5

$\therefore\ T_6 = {}^{15}C_5(3)^{10}(-2)^5 = -\ {}^{16}C_5\ 3^{10}\ 2^5$.

Example: Find the value of the greatest term in the expansion of $\sqrt{3}(1+(1/\sqrt{3}))^{20}$.

Solution

Let T_{r+1} be the greatest term, then $T_r < T_{r+1} > T_{r+2}$

Consider : $T_{r+1} > T_r$

$=> {}^{20}C_r\ (1/\sqrt{3})^r > 20C_{r-1}(1/\sqrt{3})^{r-1}$

$=> ((20)!/(20-r)!r!)\ (1/(\sqrt{3})^r) > ((20)!/(21-r)!(r-1)!)\ (1/(\sqrt{3})^{r-1})$

$=> r < 21/(\sqrt{3}+1)$

$=> r < 7.686$ .... (i)

Similarly, considering $T_{r+1} > T_{r+2}$

$=> r > 6.69$ .... (ii)

From (i) and (ii), we get

r = 7

Hence greatest term = T_8 = 25840/9

RECURRENCE RELATIONS

Definition

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence , namely $a_0, a_{1,\ldots,} a_{n-1}$,for all integers n with n>=0,where n_0 is a non –ve integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a recurrence relation satisfies a recurrence relation , then the sequence is called a solution of the recurrence relation.

For example ,we consider the famous Fibonacci sequence

0,1,1,2,3,5,8,13,21,.....,

which can be represented by the recurrence relation.

$F_n=F_{n-1}+F_{n-2}, n>=2$

& $F_0=0, F_1=1$. Here $F_0=0$ & $F_1=1$ are called initial conditions.

It is a second order recurrence relation.

Solving Linear Homogenous Recurrence Relations with Constants Coefficients.

Step 1: Write down the characteristics equation of the given recurrence relation .Here ,the degree of character equation is 1 less than the number of terms in recurrence relations.

Step 2: By solving the characteristics equation first out the characteristics roots.

Step 3: Depends upon the nature of roots ,find out the solution a_n as follows:

Case 1: Let the roots be real and distinct say $r_1,r_2,r_3.....,r_n$ then

$$A_n= \alpha_1 r_1^n+ \alpha_2 r_2^n+ \alpha_3 r_3^n+........+ \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, ...,\alpha_n$ are arbitrary constants.

Case 2: Let the roots be real and equal say $r_1=r_2=r_3=r_n$ then

$$A_n= \alpha_1 r_1^n+ n\alpha_2 r_2^n+n^2 \alpha_3 r_3^n+........+n^2 \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, ...,\alpha_n$ are arbitrary constants.

Case 3: When the roots are complex conjugate, then

$$a_n=r^n(\alpha_1 \cos n\theta+ \alpha_2 \sin n\theta)$$

Case 4: Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation.

$S(k)=2[S(k-1)]^2-kS(k-3)$ cannot be solved.

Example If sequence $a_n=3.2^n$,n>=1, then find the recurrence relation.

Solution:

For n>=1

$a_n=3.2^n$,

now, $a_{n-1}=3.2^{n-1,}$

$=3.2^n / 2$

$a_{n-1}=a^n/2$

$a_n = 2(a_n-1)$

$a_n = 2a_n-1$, for $n\geq 1$ with $a_n=3$

Example

Find the recurrence relation for S(n) = 6(-5),$n\geq 0$

Sol :

Given $S(n) = 6(-5)^n$

$S(n-1) = 6(-5)^{n-1}$

$=6(-5)^n / -5$

$S(n-1) = S(n) / -5$

$S_n = -5.5\ (n-1)$, $n\geq 0$ with s(0) =6

Example Find the relation from $Y_k =A.2^k + B.3^k$

Sol :

Given $Y_k =A.2^k + B.3^k$ -------------→(1)

$Y_{k+1} =A.2^{k+1} + B.3^{k+1}$

$=A.2^k.2 + B3^k.3$

$Y_{k+1} =2A.2^k + 3B.3^k$ -----------------------→(2)

$Y_{k+2} =4A.2^k + 9B.3^k$ -----------------------→(3)

(3) – 5(2) + 6(1)

➔ $y_{k+2} - 5y_{k+1} + 6y_k = 4A.2^k + 9B.3^k - 10A.2^k - 15B.3^k + 6A.2^k + 6B.3^k$

$= 0$

∴ $Y_{k+1} - 5y_{k+1} + 6y_k = 0$ in the required recurrence relation.

Example

Solve the recurrence relation defind by $S_o = 100$ and S_k (1.08) S_{k-1} for $k \geq 1$

Sol ;

Given $S_0 = 100$

$S_k = (1.08) S_{k-1}$, $k \geq 1$

$S_1 = (1.08) S_0 = (1.08)100$

$S_2 = (1.08) S_1 = (1.08)(1.08)100$

$= (1.08)^2 100$

$S_3 = (1.08) S_2 = (1.08)(1.08)^2 100$

$= = (1.08)^3 100$

$S_k = (1.08) S_{k-1} = (1.08)^k 100$

Example Find an explicit formula for the Fibonacci sequence .

Sol ;

Fibonacci sequence 0,1,2,3,4........ satisify the recurrence relation

$fn = f_{n-1} + f_{n-2}$

$fn - f_{n-1} - f_{n-2} = 0$

& also satisfies the initial condition $f_0=0, f_1=1$

Now , the characteristic equation is

$r_2 - r - 1 = 0$

Solving we get $r = 1 \pm 1+4 / 2$

$= 1 \pm 5 / 2$

$fn = \alpha_1 \ (1+ 5 / 2)^n + \alpha_2 (1- 5 / 2)^n$ ----→(A)

given f_0 =0 put n=0 in (A) we get

$f0 = \alpha_1 \ (1+ 5 / 2)^0 + \alpha_2 (1- 5 / 2)^0$

(A)➔ α1 +α2 =0 ------------------------→(1)

given f_1 =1 put n=1 in (A) we get

$f_1 = \alpha_1 \ (1+ 5 / 2)^1 + \alpha_2 (1- 5 / 2)^1$

(A)➔$(1+ 5 / 2)^n + \alpha_2 (1- 5 / 2)^n \alpha_2 = 1$ -------→(2)

To solve(1) and (2)

(1) X(1+ 5 / 2) => $(1+ 5 / 2) \alpha_1 + (1+ 5 / 2) \alpha_2$ =0 -----→(3)

$(1+ 5 / 2) \alpha_1 + (1+ 5 / 2) \alpha_2$ = 1------→(2)

(-) (-) (-)

$1/2 \alpha_2 + 5/2 \alpha_2 -1/2 \alpha_2 + 5/2 \alpha_2 = -1$

2 5 $d_2 = -1$

$\alpha_2 = - 1/5$

Put $\alpha_2 = -1/5$ in eqn (1) we get α_1 1/5

Substituting these values in (A) we get

Solution $fn=1/5 (1+5/ 2)^n -1/5 (1+5/ 2)^n$

Example

Solve the recurrence equation

$a_n = 2a_{n-1} – 2a_{n-2}$, n$\geq$ 2 & a_0 =1 & a_1=2

Sol :

The recurrence relation can be written as

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is

r2 – 2r -2 =0

Roots are r= 2$\pm$ 2i / 2

=1$\pm$ i

LINEAR NON HOMOGENEOUS RECRRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation of the form

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+\ldots\ldots\ldots c_k\ a_{n-k}+F(n)\ldots\ldots\ldots\ldots$ (A)

Where $c_1, c_2, \ldots c_k$ are real numbers and F(n) is a function not identically zero depending only on n,is called a non-homogeneous recurrence relation with constant coefficient.

Here ,the recurrence relation

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+\ldots\ldots\ldots c_k\ a_{n-k}+F(n)\ldots\ldots\ldots\ldots$ (B)

Is called Associated homogeneous recurrence relation.

NOTE:

(B) is obtained from (A) by omitting F(n) for example ,the recurrence relation

$a_n = 3\ a_{n-1}+2_n$ is an example of non-homogeneous recurrence relation .Its associated

Homogeneous linear equation is

$a_n = 3\ a_{n-1}$ [By omitting F(n) = 2n]

PROCEDURE TO SOLVE NON-HOMOGENEOUS RECURRENCE RELATIONS:

The solution of non-homogeneous recurrence relations is the sum of two solutions.

1.solution of Associated homogeneous recurrence relation (By considering RHS=0).

2.Particular solution depending on the RHS of the given recurrence relation

STEP1:

a) if the RHS of the recurrence relation is

$a_0 + a_1$ n .... a_r n^r, then substitute

c_0 +c_1 n + c_2 n^2+……….. c_r $(n-1)^r$ in place of $a_n - 1$ ……….and so on ,in the LHS of the given recurrence relation

(b) if the RHS is a^n then we have

Case1:if the base a of the RHS is the characteristric root,then the solution is of the can^n .therefore substitute ca^n in place of a_n ,ca^{n-1} in place of c(n-1) a_{n-1} etc..

Case2: if the base a of RHS is not a root , then solution is of the form ca^n therefore substitute ca^n in place of a_n , ca^{n-1} in place of a_{n-1} etc..

STEP2:

At the end of step-1, we get a polynomial in 'n' with coefficient c_0,c_1……on LHS

Now, equating the LHS and compare the coefficients find the constants c_0,c_1,….

Example

Solve $a_n = 3\ a_{n-1}$ +2n with a_1 =3

Solution:

Give the non-homogeneous recurrence relation is

$a_n - 3\ a_{n-1}$ -2n=0

It's associated homogeneous equation is

$a_n - 3\ a_{n-1}$ =0 [omitting f(n) =2n]

It's characteristic equation is

r-3=0 => r=3

now, the solution of associated homogeneous equation is

$a_n\,(n) = \propto ,3^{n}$

To find particular solution

Since F(n) =2n is a polynomial of degree one,then the solution is of the from

$a_n = c_n$ +d (say) where c and d are constant

Now, the equation

$a_n = 3\ a_{n-1}$ +2n becomes

c_n +d =3(c(n-1)+d)+2n

[replace a_n by c_n +d a_{n-1} by c(n-1)+d]

- ⇨ c_n +d =3cn-3c+3d+2n
- ⇨ 2cn+2n-3c+2d=0
- ⇨ (2+2c)n+(2d-3c)=0
- ⇨ 2+2c=0 and 2d-3c=0
- ⇨ Saving we get c=-1 and d=-3/2 therefore cn+d is a solution if c=-1 and d=-3/2

a_n (p)=-n-3/2

Is a particular solution.

General solution

$a_n = a_n(\mathrm{n}) + a_n(\mathrm{p})$

$a_n = \propto 3^{n}$ -n-3/2 ………………..(A)

Given $a_1 = 3$ put n=1 in (A) we get

$a_1 = \propto\ 1(3)^{1}$ -1-3/2

$3 = 3\propto_1 - 5/2$

$3 \propto_1 = 11/2$

$\propto_1 = 11/6$

Substituting $\propto_1 = 11/6$ in (A) we get

General solution

$a_n = -n - 3/2 + (11/6)3^n$

Example:

Solve s(k)-5s(k-1)+6s(k-2)=2

With s(0)=1 ,s(1)=-1

Solution:

Given non-homogeneous equation can be written as

$a_n = 5a_{n-1} + 6a_{n-2} - 2 = 0$

The characteristic equation is

$r^2 - 5r + 6 = 0$

roots are r=2,3

the general solution is

$3_n(n) = \propto_1(2)^n + \propto_2(3)^n$

To find particular solution

As RHS of the recurrence relation is constant ,the solution is of the form C , where C is a constant

Therefore the equation

$a_n - 5a_{n-1} - 6a_{n-2} - 2 = 2$

c-5c+6c=2

$2c=2$

$c=2$

the particular solution is

$s_n(p)=1$

the general solution is

$s_n = s_n(n)+ s_n(p)$

$s_n = \propto_1(2)^n+\propto_2(3)^n+1$……….(A)

$s_n = \propto_1(2)^n+\propto_2(3)^n+1$……….(A)

Given $s_0=1$ put n=0 in (A) we get

$s_0 = \propto_1(2)^0+\propto_2(3)^0+1$

$s_0 = \propto_1+\propto_2+1$

(A) => $s_0=1 = \propto_1+\propto_2+1$

$\propto_1+\propto_2=0$……………(1)

Given a_1=-1 put n=1 in(A)

⇨ $S_1 = \propto_1(2)^1+\propto_2(3)^1+1$

⇨ (A) $-1=\propto_1(2)+\propto_2(3)+1$

⇨ $2\propto_1+3\propto_2=-2$……….(1)

$\propto_1+\propto_2=0$

$2\propto_1+3\propto_2=-2$……………….(2)

By solving (1) and (2)

$\propto_1=2, \propto_2=-2$

Substituting $\propto_1=2, \propto_2=-2$ in (A) we get

Solution is

⇨ $S_{(n)} = 2.(2)^n - 2.(3)^n + 1$

Example

Solve $\boldsymbol{a_n - 4\, a_{n-1} + 4\, a_{n-2} = 3n + 2^n}$

$\boldsymbol{a_0 = a_1 = 1}$

Solution:

The given recurrence relation is non-homogeneous

Now, its associated homogeneous equation is,

$a_n - 4\, a_{n-1} + 4\, a_{n-2} = 0$

Its characteristic equation is

$r^2 - 4r + 4 = 0$

$r = 2, 2$

solution , $a_n(n) = \propto_1 (2)^n + n \propto_2 (2)^n$

$a_n(n) = (\propto_1 + n \propto_2) 2^n$

To find particular solution

The first term in RHS of the given recurrence relation is 3n.therefore ,the solution is of the form $c_1 + c_2 n$

Replace a_n by $c_1 + c_2 n$, a_{n-1} by $c_1 + c_2(n-1)$

And a_{n-2} by $c_1 + c_2(n-2)$ we get

$(c_1 + c_2 n) - 4(c_1 + c_2(n-1)) + 4(c_1 + c_2(n-2)) = 3n$

⇨ $c_1 - 4c_1 + 4c_1 + c_2 n - 4c_2 n + 4c_2 n + 4c_2 - 8c_2 = 3n$

⇨ $c_1 + c_2 n - 4c_2 = 3n$

Generating function:

The generating function for the sequence 'S' with terms $a_0, a_1, \ldots\ldots a_n$

Of real numbers is the infinite sum.

Equating the corresponding coefficient we have

c_1-4c_2=0 and c_2=3

c_1=12 and c_2=3

Given a_0=1 using in (2)

(2) => $\propto_1$+12=1

Given a_1=1 using in (2)

(2)=> ($\propto_1$+$\propto_2$)2+12+3+1/2 .2=1

=> (2 $\propto_1$+2 $\propto_2$)+16=1……………..(14)

(3) $\propto_1$=-11

Using in (4) we have $\propto_2$=7/2

Solution a_n=(-11+7/2n)2^n+12+3n+1/2$n^2 2^n$

G(x)=G(s,x)= $a_0+a_1x+, \ldots\ldots a_nx^n+\ldots\ldots = \sum_{n=0}^{\infty} a^n x^n$

For example,

i) the generating function for the sequence 'S' with the terms 1,1,1,1…..i.s given by,

G(x)=G(s,x)= $\sum_{n=0}^{\infty} x^n$=1/1-x

ii)the generation function for the sequence 'S' with terms 1,2,3,4…..is given by

G(x)=G(s,x)= $\sum_{n=0}^{\infty}(n+1)x^n$

=1+2x+3x^2+……..

=$(1-x)^{-2}$=1/$(1-x)^2$

2.Solution of recurrence relation using generating function

Procedure:

Step1:rewrite the given recurrence relation as an equation with 0 as RHS

Step2:multiply the equation obtained in step(1) by x^n and summing if form 1 to ∞ (or 0 to ∞) or (2 to ∞).

Step3:put G(x)= $\sum_{n=0}^{\infty} a^n x^n$ and write G(x) as a function of x

Step 4:decompose G(x) into partial fraction

Step5:express G(x) as a sum of familiar series

Step6:Express a_n as the coefficient of x^n in G(x)

The following table represent some sequence and their generating functions

step1	**sequence**	**generating function**
1	1	1/1-z
2	$(-1)^n$	1/1+z
3	a^n	1/1-az
4	$(-a)^n$	1/1+az
5	n+1	$1/1-(z)^2$
6	n	$1/(1-z)^2$
7	n^2	$z(1+z)/(1-z)^3$
8	na^n	$az/(1-az)^2$

Eg:use method of generating function to solve the recurrence relation

$a_n=3a_{n-1}+1$; $n\geq1$ given that $a_0=1$

solution:

let the generating function of $\{a_n\}$ be

$$G(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$a_n=3a_{n-1}+1$$

multiplying by x^n and summing from 1 to ∞,

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^n)+\sum_{n=1}^{\infty}(x^n)$$

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^{n-1})+\sum_{n=1}^{\infty}(x^n)$$

$$G(x)-a_0=3xG(x)+x/1-x$$

$$G(x)(1-3x)=a_0+x/1-x$$
$$=1+x/1-x$$

G(x)(1-3x)=1=x+x/1-x

G(x)=1/(1-x)(1-3x)

By applying partial fraction

G(x)=-1/2/1-x+3/2/1-3x

$G(x)=-1/2(1-x)^{-1}+3/2(1-3x)^{-1}$

$G(x)[1-x-x^2]=a_0-a_1x-a_0x$

$G(x)[1-x-x^2]= a_0-a_0x+a_1x$

$G(x)=1/1-x-x^2 \quad [a_0=1, a_1=1]$

$$=\frac{1}{(1-1+\sqrt{5}\ \ x/2)(1-1-\sqrt{5}\ \ x/2)}$$

$$=\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}$$

Now,

$$1/1-x-x^2 =\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}\text{...........}(1)$$

$$1=A[1-(\frac{1+\sqrt{5}}{2})x)]+B[1-(\frac{1-\sqrt{5}}{2})x)]\text{......}(2)$$

Put x=0 in (2)

(2)=> A+B=1

Put $x= 2/1-\sqrt{5}$ in (2)

$(2)\Rightarrow \quad 1=B[1-\frac{1+\sqrt{5}}{1-\sqrt{5}}]$

$1=B[\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}]$

$1=B[\frac{-2\sqrt{5}}{1-\sqrt{5}}]$

$B=\frac{1-\sqrt{5}}{-2\sqrt{5}}$

(3) => $A=\frac{1+\sqrt{5}}{2\sqrt{5}}$

Sub A and B in (1)

$G(x)=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)[1-(\frac{1+\sqrt{5}}{2})x]^{-1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)[1-(\frac{1-\sqrt{5}}{2})x]^{-1}$

$=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)[1+(\frac{1+\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^2+\ldots\ldots$

$=\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)[1+(\frac{1-\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^2+\ldots\ldots$

a_n=coefficient of x^n in G(x)

solving we get

$a_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

Pigeon Hole Principle :

If $(n+1)$ pigeon occupies 'n' holes then atleast one hole has more than 1 pigeon.

Proof :

Assume $(n+1)$ pigeon occupies 'n' holes.

Claim : Atleast one hole has more than one pigeon.

Suppose not, *i.e.,* Atleast one hole has not more than one pigeon.

Therefore, each and every hole has exactly one pigeon.

Since, there are 'n' holes, which implies, we have totally 'n' pigeon.

which is a $\Rightarrow\Leftarrow$ to our assumption that there are $(n+1)$ pigeon.

Therefore, atleast one hole has more than 1 pigeon.

THE PIGEONHOLE PRINCIPLE

If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon. Generalized pigeonhole principle is: - If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

Example1: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution: Here n = 12 months are the Pigeonholes
And k + 1 = 3
K = 2

Example2: Show that at least two people must have their birthday in the same month if 13 people are assembled in a room.

Solution: We assigned each person the month of the year on which he was born. Since there are 12 months in a year.

So, according to the pigeonhole principle, there must be at least two people assigned to the same month.

THE PRINCIPLE OF INCLUSION –EXCLUSION

Assume two tasksT_1 and T_2 that can be done at the same time(simultaneously) now to find the number of ways to do one of the two tasks $T_1$and T_2, if we add number ways to do each task then it leads to an over count. since the ways to do both tasks are counted twice. To correctly count the number of ways to do each of the two tasks and then number of ways to do both tasks

i.e $\wedge(T_1 v T_2)=\wedge(T_1)+\wedge(T_2)-\wedge(T_1\wedge T_2)$

this technique is called the principle of Inclusion –exclusion

FORMULA:$_4$

1) $|A_1 v A_2 v A_3|=|A_1|+|A_2|+|A_3|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_2\wedge A_3|+|A_1\wedge A_2\wedge A_3|$

2) $|A_1 v A_2 v A_3 v A_4|=|A_1|+|A_2|+|A_3|+|A_4|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_1\wedge A_4|-|A_2\wedge A_3|-|A_2\wedge A_4|-|A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3|+|A_1\wedge A_2\wedge A_4|+|A_1\wedge A_3\wedge A_4|+|A_2\wedge A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3\wedge A_4|$

Example

A survey of 500 from a school produced the following information.200 play volleyball,120 play hockey,60 play both volleyball and hockey. How many are not playing either volleyball or hockey?

Solution:

Let A denote the students who volleyball

Let B denote the students who play hockey

It is given that

n=500

|A|=200

|B|=120

|A^B|=60

Bt the principle of inclusion-exclusion, the number of students playing either volleyball or hockey

|AvB|=|A|+|B|-|A^B|

|AvB|=200+120-60=260

The number of students not playing either volleyball or hockey=500-260

=240

Example

In a survey of 100 students it was found that 30 studied mathematics,54 studied statistics,25 studied operation research,1 studied all the three subjects.20 studied mathematics and statistic,3 studied mathematics and operation research And 15 studied statistics and operation research

1.how many students studied none of these subjects?

2.how many students studied only mathematics?

Solution:

1) Let A denote the students who studied mathematics

Let B denote the students who studied statistics

Let C denote the student who studied operation research

Thus |A|=30 ,|B|=54 ,|C|=25 ,|A^B|=20 ,|A^C|=3 ,|B^C|=15 ,and |A^B^C|=1

By the principle of inclusion-exclusion students who studied any one of the subject is

|AvBvC|=|A|+|B|+|C|=|A^B|-|A^C|-|B^C|+|A^B^C|

=30+54+25-20-3-15+1

=110-38=72

Students who studied none of these 3 subjects=100-72=28

2) now ,

The number of students studied only mathematics and statistics=n(A^B)-n(A^B^C)

=20-1=19

The number of students studied only mathematics and operation research=n(A^C)-n(A^B^C)

=3-1=2

Then The number of students studied only mathematics =30-19-2=9

Example

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:

Let A denote the set of positive integers not exceeding 1000 are divisible by 7

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11.

Then |A|=[1000/7]=[142.8]=142

|B|=[1000/11]=[90.9]=90

|A^B|=[1000/7*11]=[12.9]=12

The number of positive integers not exceeding 1000 that are divisible either 7 or 11 is |AvB|

By the principle of inclusion –exclusion

|AvB|=|A|+|B|-|A^B|

=142+90-12=220

There are 220 positive integers not exceeding 1000 divisible by either 7 or 11

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – DISCRETE MATHEMATICS – SMTA 1302

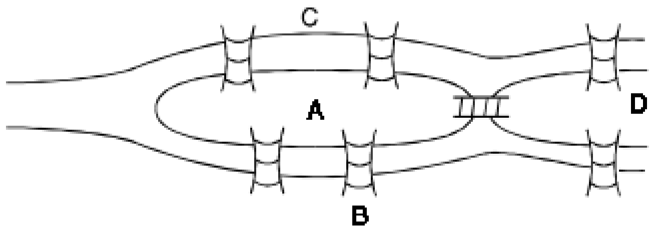
UNIT V - GRAPH THEORY

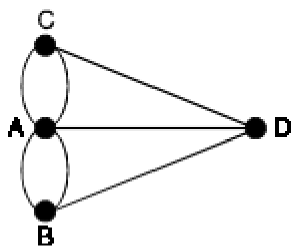
COURSE CONTENT: Introduction to graphs – Types of graphs (directed and undirected) – Basic terminology – Sub graphs – Representing graphs as incidence and adjacency matrix – Graph Isomorphism – Connectedness in Simple graphs, Paths and Cycles in graphs - Euler and Hamiltonian paths (statement only) – Tree – Binary tree (Definition and simple problems)

INTRODUCTION

The concept of graph theory is considered to have originated in 1736 with the publication of Euler's solution of the Konigsberg bridge problem. Euler (1707–1782) is regarded as the father of graph theory.

The Konigsberg Bridge Problem: The city of Konigsberg was located on the Pregel river in Prussia. The river divided the city into four separate landmasses, including the island of Kneiphopf. These four regions were linked by seven bridges as shown in the diagram. Residents of the city wondered if it were possible to leave home, cross each of the seven bridges exactly once, and return home. The Swiss mathematician Leonhard Euler thought about this problem and gave a solution.





The key to Euler's solution was in a very simple abstraction of the puzzle. Let us redraw our diagram of the city of Konigsberg by representing each of the land masses as a vertex and representing each bridge as an edge connecting the vertices corresponding to the land masses. We now have a graph that encodes the necessary information. The problem reduces to finding a "closed walk" in the graph which traverses each edge exactly once, this is called an Eulerian circuit. Euler proved such a circuit does not exist.

Graph theory is the study of points, lines and the ways in which sets of points can be connected by lines or arcs. Graphs in this context differ from the more familiar coordinate plots that portray mathematical relations and functions.

Graph theory has many colourful applications in many branches such as Physics, Chemistry, Communication Science, Computer technology, Electrical and Civil engineering, Architecture, Operations research, Genetics, Sociology, Economics etc.. It has proven useful in the design of integrated circuits (IC s) for computers and other electronic devices. These components more often called chips, contain complex, layered microcircuits that can be represented as sets of points interconnected by lines or arcs. Using graph theory, engineers develop chips with maximum component density and minimum total interconnecting conductor length. This is important for optimizing processing speed and electrical efficiency.

BASIC TERMINOLOGIES OF GRAPHS

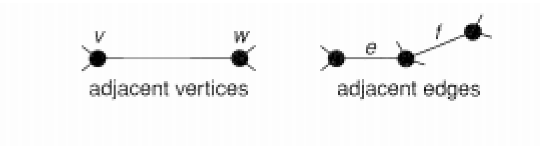
A graph is usually denoted as G = (V, E), where V is called the **vertex set** of G and E is the **edge set** of G**.** The elements of the set V are called **vertices** or **points** or **nodes** and the members of the set E are called **edges** or lines or **arcs**.

The number of vertices in a graph G is called the **order of the graph** and is denoted by $|V|$. The number of edges in a graph is called the **size of the graph** and is denoted by $|E|$. A graph is **finite** if both its vertex set and edge set are finite. Otherwise it is an **infinite graph.** We study only finite graphs, so the term **graph** means only **finite graphs**.

A graph with p vertices and q edges is called a (p, q) graph. A graph with one vertex i.e., a (1, 0) graph is called **trivial graph** and all other graphs are non trivial. A graph with zero edges i.e., a (p, 0) graph is called **empty or null or void graph**. Each graph has a diagram associated with it. These diagrams are useful for understanding problems involving such graphs.

Adjacency

Two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and *f* are adjacent if they have a vertex in common

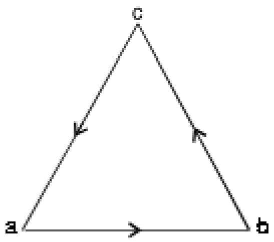


DIRECTED AND UNDIRECTED GRAPHS

Directed graph

A directed graph G consists of a set V of vertices and a set E of edges such that $e \epsilon E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called **directed graph or digraph.**

In the diagram of directed graph, each edge is represented by an arrow or directed curve from initial point to the terminal point.



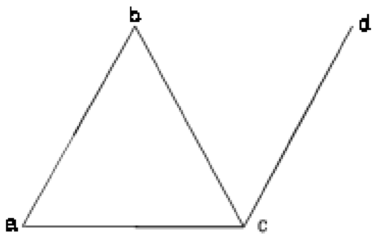
Suppose e = (a, b) is a directed edge in a digraph, then

(i) a is called the initial vertex of e and b is the terminal vertex of e
(ii) e is said to be incident from vertex to vertex b.

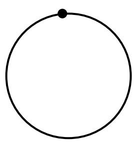
Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge e ∈ E is associated with an unordered pair of vertices. In other words, if each edge of the graph G has no direction then the graph is called un-directed graph.

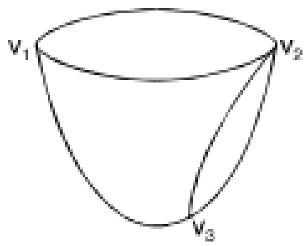
Figure given below is an example of an undirected graph. An edge joining the vertex pair a and b can be referred as either (a, b) or (b, a).



Loop : An edge of a graph that joins a vertex to itself is called loop.
Example:

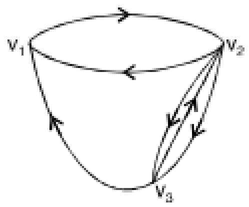


Multigraph: Two or more edges of a graph G joining the same pair of vertices are called multiple edges or parallel edges. The corresponding graph is called multigraph. In a multigraph no loops are allowed.



Un-directed multigraph

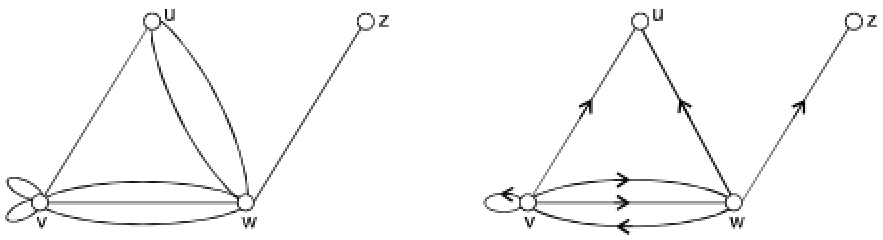
In the above figure there are two parallel edges joining nodes v_1, v_2 and v_2, v_3.



Directed multigraph

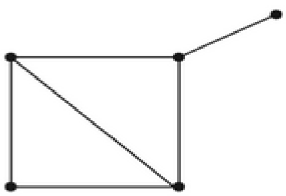
In the above figure there are two parallel edges associated with vertices v_2 and v_3

Pseudo graph: A graph, in which loops and multiple edges are allowed, is called a pseudo graph.



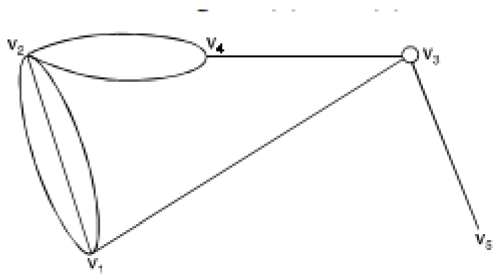
Un-directed Pseudo graph Directed Pseudo graph

Simple graph: A graph with no loops and multiple edges is called a simple graph.


(a) Simple graph

DEGREE OF A VERTEX:

For an undirected graph, the number of edges incident on a vertex v_i with self-loops counted twice is called the degree of a vertex v_i and is denoted by deg (v_i) or deg v_i or d(v_i). The degree of a vertex is also referred to as its **valency.**
For example let us consider the graph G given below. The degrees of vertices are deg (v_1) = 4, deg (v_2) = 5, deg (v_3) = 5, deg (v_4) = 3, and deg (v_5) = 1.



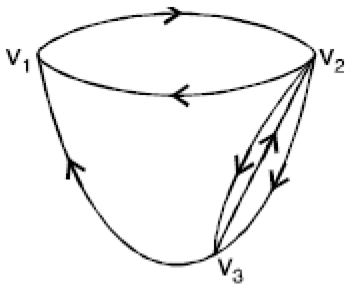
Isolated vertex: A vertex having no incident edge on it is called an isolated vertex. In other words vertex with zero degree is called an isolated vertex.

Pendent vertex or end vertex: A vertex of degree one, is called a pendent vertex or an end vertex and the corresponding edge is called the pendant edge. The vertex to which an end vertex is adjacent is called **support vertex**. In the above Figure, v_5 is a pendent vertex.

Degree Sequence: The vertex degrees of a graph arranged in non-increasing order is called degree sequence of the graph G. The degree sequence of the above graph is 5, 5, 4, 3, 1

IN DEGREE and OUT DEGREE of a Vertex

In a digraph G, the number of edges beginning at vertex v_i is called the out degree of a vertex v_i, denoted by $deg_G^+(v_i)$ or out deg (v_i).
: In a digraph G, the number of edges ending at vertex v_i is called the in degree of a vertex v_i , denoted by $deg_G^-(v_i)$ or in deg (v_i).
A vertex with zero in degree is called a **source** and a vertex with zero out degree is called a **sink**.
The sum of the in degree and out degree of a vertex is called the total degree of the vertex.

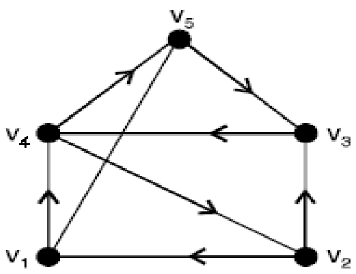


$$deg_G^-(v_1) = 2, deg_G^+(v_1) = 1, deg_G^-(2) = 2, deg_G^+(v_2) = 3, deg_G^-(v_3) = 2, deg_G^+(v_3) = 2$$

Note: For any directed graph the following property is true

$$\sum_{v\in V} deg^-(v) = \sum_{v\in V} deg^+(v) = |E|$$

Problem. *Find the in-degree and out-degree of each vertex of the following directed graph*



Solution.

in-degree $v1$ = 2, out-degree $v1$ = 1

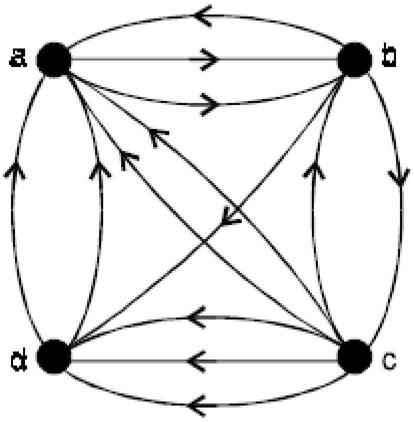
in-degree $v2$ = 2, out-degree $v2$ = 2

in-degree $v3$ = 2, out-degree $v3$ = 1

in-degree $v4$ = 2, out-degree $v4$ = 2

in-degree $v5$ = 0, out-degree $v5$ = 3

Problem. *Find the in-degree and out-degree of each vertex of the following directed graph*



Solution.

in-degree *a* = 6, out-degree *a* = 1

in-degree *b* = 1, out-degree *b* = 5

in-degree *c* = 2, out-degree *c* = 5

in-degree *d* = 2, out-degree *d* = 2.

Theorem 1: (THE HANDSHAKING THEOREM)

Statement: If G = (V, E) be an undirected graph with e edges, then $\sum_{v\epsilon V} deg_G(v) = 2e$. i.e., the sum of degrees of the vertices is an undirected graph is even.

(or)

If V = {v_1, v_2, ... v_n} is the vertex set and E is the edge set of a non directed graph G then $\sum_{i=1}^{n} deg_G(v_i) = 2|E|$

Proof :

Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees equals twice the number of edges.

Thus $\sum_{i=1}^{n} deg_G(v_i) = 2|E|$

Note : This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shaken must be even that is why the theorem is called handshaking theorem.

Corollary 1: In a non directed graph, the total number of odd degree vertices is even.

Proof :

Let G = (V, E) a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i\epsilon V} deg_G(v_i) = \sum_{v_i\epsilon U} deg_G(v_i) + \sum_{v_i\epsilon W} deg_G(v_i)$

$\Rightarrow 2e - \sum_{v_i\epsilon U} deg_G(v_i) = \sum_{v_i\epsilon W} deg_G(v_i)$

$\Rightarrow \sum_{v_i \epsilon W} deg_G(v_i)$ is also even
∴ The number of odd vertices in G is even.

Theorem 2: If G is a directed graph, then $\sum_{i=1}^{n} deg_G^{+}(v_i) = \sum_{\text{i=1}}^{\text{n}} deg_G^{-}(v_i) = |E|$
Proof : Since when the degrees are summed, each edge contributes a count of one to the degree of each of the two vertices on which the edge is incident.

Corollary 2 : In any undirected graph there is an even number of vertices of odd degree.
Proof : Let W be the set of vertices of odd degree and let U be the set of vertices of even degree. Then $\sum_{v \epsilon U} deg_G(v) + \sum_{v \epsilon W} deg_G(v) = \sum_{v \epsilon V} deg_G(v) = 2|E|$
Certainly, $\sum_{v \epsilon U} deg_G(v)$ is even. Hence $\sum_{v \epsilon W} deg_G(v)$ is even.
$\Rightarrow |W|$ is even.

Corollary 3 : If k = δ(G) is the minimum degree of all the vertices of a non directed graph G, then

$$k|V| \leq \sum_{v \epsilon V} deg_G(v) = 2|E|$$

In particular, if G is a k-regular graph, then

$$k|V| = \sum_{v \epsilon V} deg_G(v) = 2|E|$$

Problem. Show that the total number of odd degree vertices of a (p, q)-graph is always even. Solution. Let v_1, v_2 ...... v_k be the odd degree vertices in G.
Then, we have $\sum_{i=1}^{p} deg_G(v_i) = 2q$= even number
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i) + \sum_{i=k+1}^{p} deg_G(v_i) =$ even number
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i)$ = even number $- \sum_{i=k+1}^{p} deg_G(v_i)$
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i)$ = even number – even number
= even number.
⇒ This implies that number of terms in the left-hand side of the equation is even.
Therefore, k is an even number.

Problem. Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2.
Solution. Suppose the graph with 6 vertices has e number of edges. Therefore by Handshaking lemma. $\sum_{i=1}^{6} deg_G(v_i) = 2|e|$
⇒ $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$
Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence the above equation becomes, (4 + 4) + (2 + 2 + 2 + 2) = 2e
⇒ 16 = 2e ⇒ e = 8.
Hence the number of edges in a graph with 6 vertices with given condition is 8.

Problem. How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree 2?
Solution. Suppose these are n vertices in the graph with 6 edges. Also given the degree of each vertex is 2.
By handshaking lemma, $\sum_{i=1}^{n} deg_G(v_i) = 2|e|$= 2× 6 = 12
⇒ $d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 12$
⇒ $\underbrace{2 + 2 + \cdots + 2}_{n\ times} = 12$
⇒ 2n = 12
⇒ n = 6 vertices are needed.

Problem. It is possible to draw a simple graph with 4 vertices and 7 edges ? Justify.
Solution. In a simple graph with n-vertices, the maximum number of edges will be $\frac{n(n-1)}{2}$.
Hence a simple graph with 4 vertices will have at most $\frac{4\times\times 3}{2} = 6$edges.
Therefore, a simple graph with 4 vertices cannot have 7 edges.
Hence such a graph does not exist.

Problem. Show that there exists no simple graph corresponds to the following degree sequence : (i) 0, 2, 2, 3, 4 (ii) 1, 1, 2, 3 (iii) 2, 2, 3, 4, 5, 5 (iv) 2, 2, 4, 6.
Solution. (i) to (iii) : There are odd number of odd degree vertices in the graph.
Hence there exists no graph corresponds to this degree sequence.
(iv) Number of vertices in the graph is four and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exceed one less than the number of vertices.

Problem. Show that the following sequence 6, 5, 5, 4, 3, 3, 2, 2, 2 is graphical.
Solution.
We can reduce the sequence as follows :
Given sequence 6, 5, 5, 4, 3, 3, 2, 2, 2
Reducing first 6 terms by 1 counting from second term 4, 4, 3, 2, 2, 1, 2, 2.
Writing in decreasing order 4, 4, 3, 2, 2, 2, 2, 1
Reducing first 4 terms by 1 counting from second 3, 2, 1, 1, 2, 2, 1
Writing in decending order 3, 2, 2, 2, 1, 1, 1
Reducing first 3 terms by 1, counting from second 1, 1, 1, 1, 1, 1

Sequence 1, 1, 1, 1, 1, 1 is graphical.
Hence the given sequence is also graphical

Problem. Show that the sequence 6, 6, 6, 6, 4, 3, 3, 0 is not graphical.
Solution. To prove that the sequence is not graphical.
The given sequence is 6, 6, 6, 6, 4, 3, 3, 0
Resulting the sequence 5, 5, 5, 3, 2, 2, 0
Again consider the sequence 4, 4, 2, 1, 1, 0
Repeating the same 3, 1, 0, 0, 0
Since there exists no simple graph having one vertex of degree three and other vertex of degree one. The last sequence is not graphical.
Hence the given sequence is also not graphical.

Problem. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$. . Solution. By the handshaking theorem,

$\sum_{i=1}^{n} deg_G(v_i) = 2|e|$where e is the number of edges with n vertices in the graph G.

$\Rightarrow d(v_1) + d(v_2) + ... + d(v_n) = 2e$ .... (1)

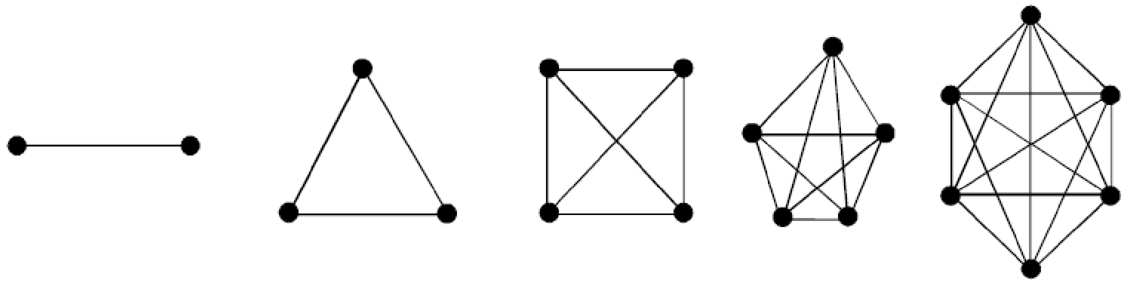
We know that the maximum degree of each vertex in the graph G can be (n – 1).
Therefore, equation (1) reduces $\underbrace{(n-1) + (n-1) + \cdots + (n-1)}_{n\ times} = 2e$

$\Rightarrow n(n - 1) = 2e$
$\Rightarrow e = \frac{n(n-1)}{2}$.
Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

SOME SPECIAL GRAPHS:

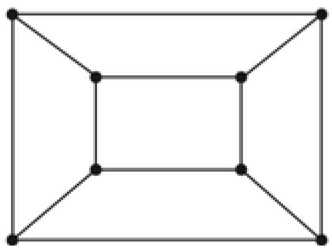
COMPLETE GRAPH

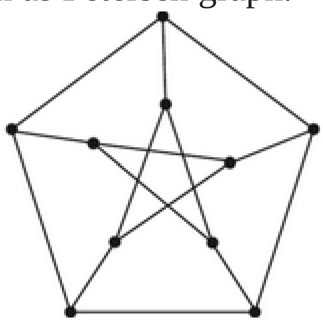
A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.
i.e., if G contains exactly one edge between each pair of distinct vertices.
A complete graph is usually denoted by **K*n*.** It should be noted that K*n* has exactly $\frac{n(n-1)}{2}$edges.
The figure given below shows complete graphs K_1 to K_6



REGULAR GRAPH

A graph in which all vertices are of **equal degree,** is called a **regular graph.**
If the degree of each vertex is *r*, then the graph is called a regular **graph of degree *r*.**
Note 1: Every null graph is regular of degree zero.
Note 2: The complete graph K*n* is a regular of degree $n - 1$.
Note 3: If G has *n* vertices and is regular of degree *r*, then G has edges.
Note 4: The figure given below shows 3 regular graphs which are also called as cubic graphs. The socond graph is also known as Petersen graph.





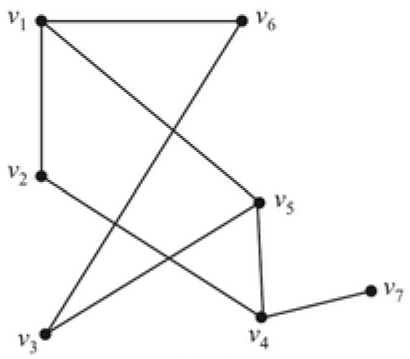
BIPARTITE GRAPH

A graph G is said to be **bipartite** if its vertex set can be partitioned into two subsets such that no two vertices in the same partition are adjacent. In other words if the simple graph G(V, E) can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 to a vertex in V_2 and no edge in G connects either two vertices in V_1 or V_2 then G is called a **bipartite graph.**
If each vertex of V_1 is connected with every vertex of V_2 by an edge, Then G is said to be a **complete bipartite graph.** If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph is denoted by $K_{m, n}$.

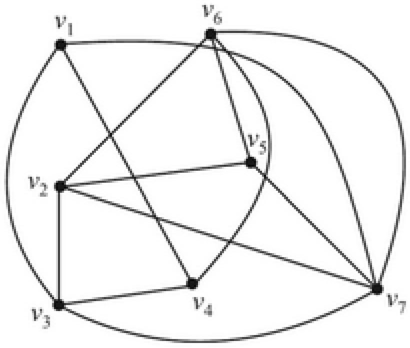
The following figure shows bipartite and complete bipartite graph

THE COMPLEMENT OF A GRAPH

Let G be a simple graph. The complement of G denoted by G^c has the same vertex set as G and two vertices in G and G^c are adjacent if and only if they are not adjacent in G.
The graph G and its complement G^c are depicted below



(a) G

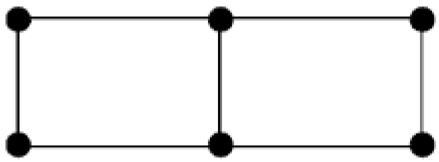


(b) G^c

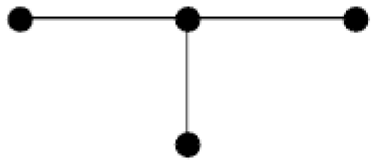
SUBGRAPH

If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G or G as a supergraph of H.
In the figure given below G_1 is a subgraph of graph G.

G :

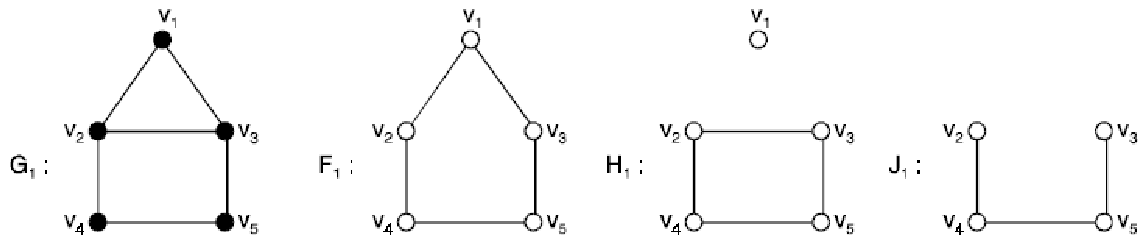


G_1 :



SPANNING SUBGRAPH

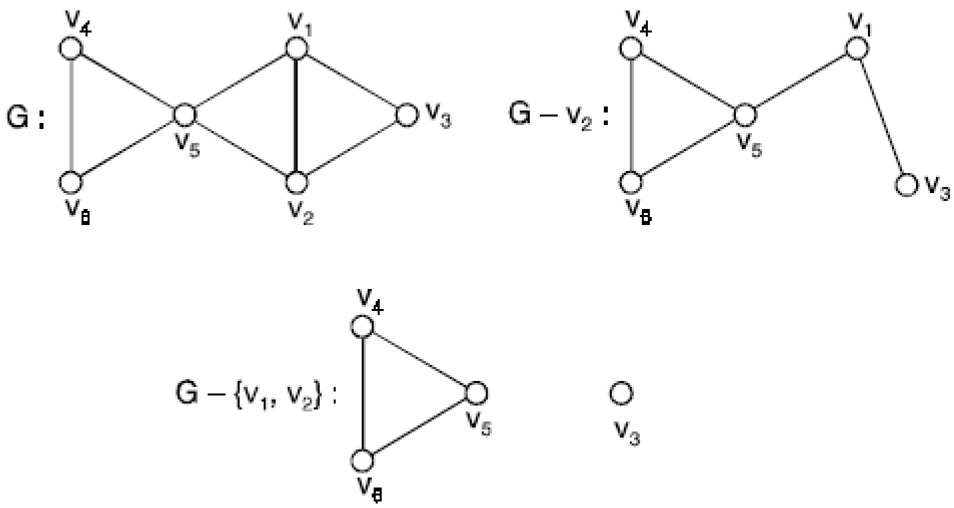
A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G$.
If $V(H) \subset V(G)$ and $E(H) \subset E(G)$ then H is called a **proper subgraph** of G.
If V(H) = V(G) then we say that H is a **spanning subgraph** of G.
A spanning subgraph need not contain all the edges in G. The graphs F1 and H1 of the figure shown below are spanning subgraphs of G1, but J1 is not a spanning subgraph of G1.



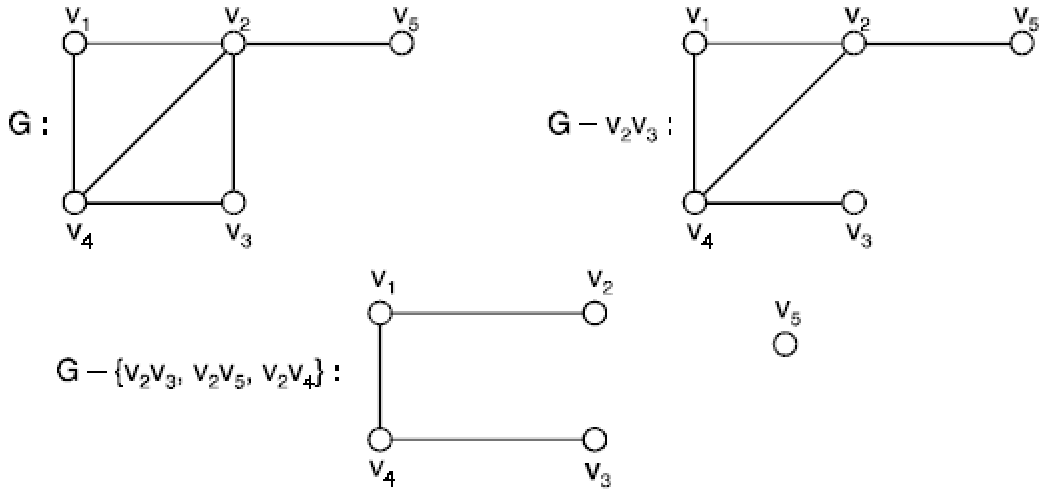
Removal of a vertex and an edge

The removal of a vertex *vi* from a graph G result in that subgraph G – *vi* of G containing of all vertices in G except *vi* and all edges not incident with *vi*. Thus G – *vi* is the maximal subgraph of G not containing *vi*. On the otherhand, the removal of an edge *xj* from G yields the spanning subgraph G – *xj*
containing all edges of G except *xj*. Thus G – *xj* is the maximal subgraph of G not containing edge *xj*.

The following figure shows deletion of vertices and deletion of edges from a graph



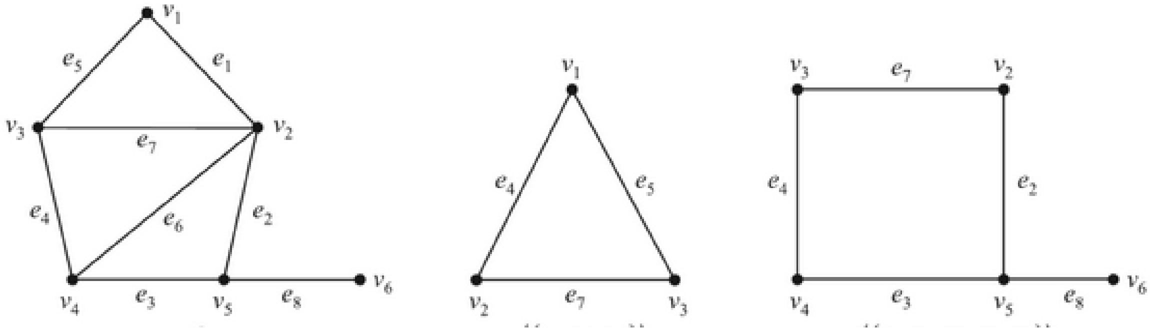
The following figure shows deletion of edges from a graph



INDUCED SUB GRAPH:

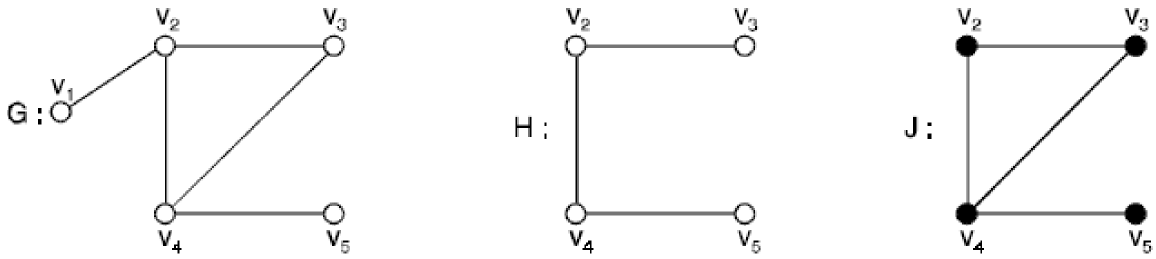
Let G be a graph with vertex set V(G), edge set E(G) and S be a non empty subset of V(G). A subgraph of G whose vertex set is S and all edges of G which have both their ends in S is known as the subgraph induced by S and is denoted by G[S] or $< S >$. Any subgraph induced by a set of vertices will be called a **vertex induced subgraph or simply an induced sub graph**. In other words a sub graph H of a graph G where V(H) ⊆ V(G) and E(H) consists of only thoe edges that are incident on the elements of V(H), is called an **induced sub graph** of G.

Let M be a non empty subset of E(G). A subgraph of G whose edge set is M and whose vertices are the ends of edges in M, is said to be a subgraph induced by M and is denoted by G[M] or $< M >$. The second figure below displays the vertex induced sub graph of graph G induced by vertex set $\{v_1, v_2, v_3\}$ and the third image in the figure shown below is the edge induced sub graph of G induced by the edge set $\{ e_2, e_3, e_4, e_7, e_8\}$

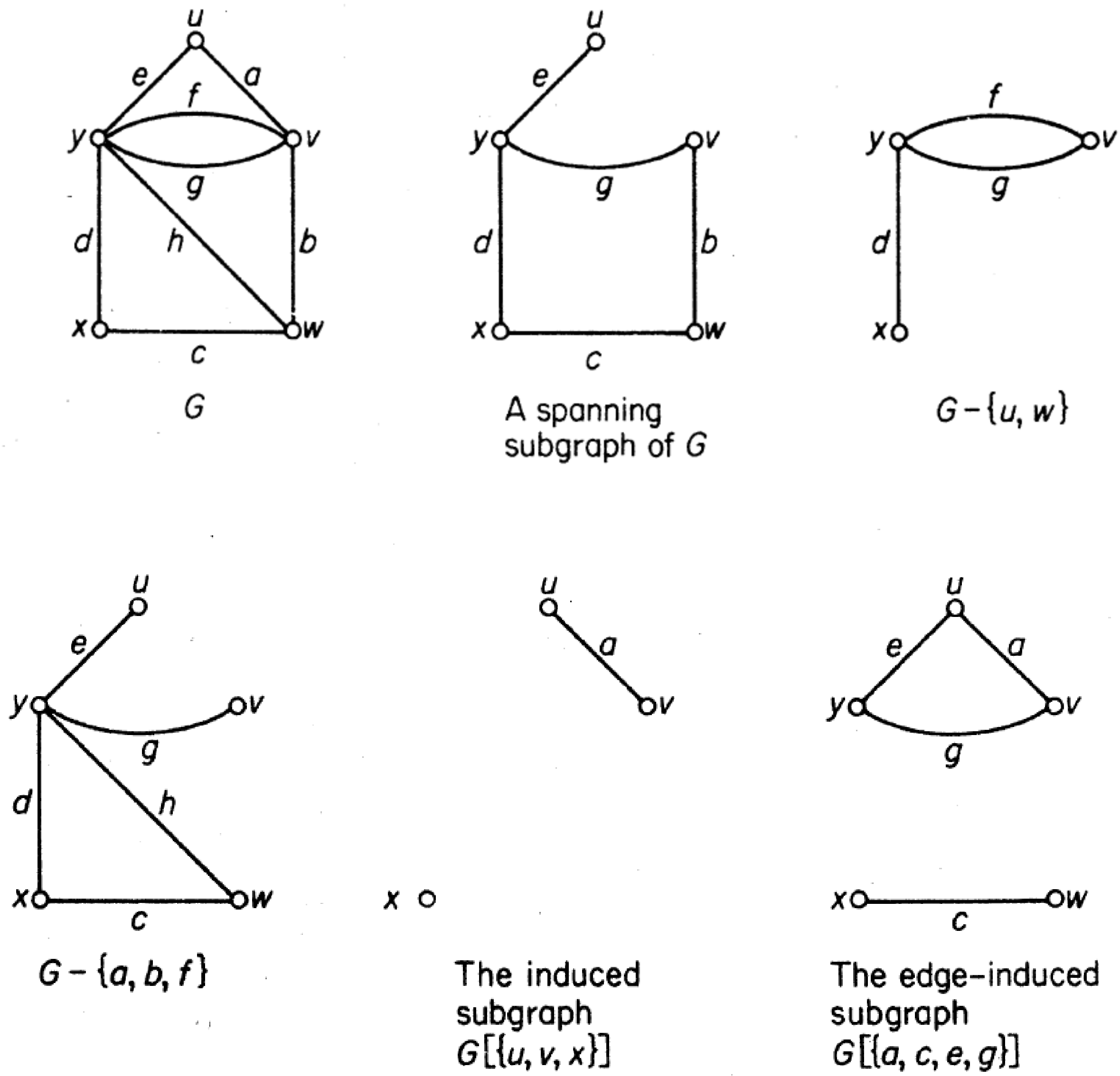


Graph G Induced sub graphs of graph G

Example H is not an induced subgraph since $v4v1$ ∈E(G), but $v4v3$ ∉E(H).



Example for spanning sub graph, vertex induced sub graph and edge induced sub graph



GRAPHS ISOMORPHISM

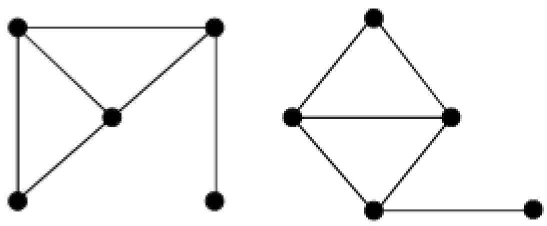
Let G1 = (V1, E1) and G2 = (V2, E2) be two graphs. A function $f : V1 \rightarrow V2$ is called a graphs isomorphism if
(*i*) *f* is one-to-one and onto.
(*ii*) for all $a, b \in V1$, $\{a, b\} \in E1$ if and only if $\{f(a), f(b)\} \in E2$ when such a function exists, G1 and G2 are called isomorphic graphs and is written as $G1 \cong G2$.
In other words, two graphs G1 and G2 are said to be isomorphic to each other if there is a one to- one correspondence between their vertices and between edges such that incidence relationship is preserved. It is written as $G1 \cong G2$ or G1 = G2.

The necessary conditions for two graphs to be isomorphic are

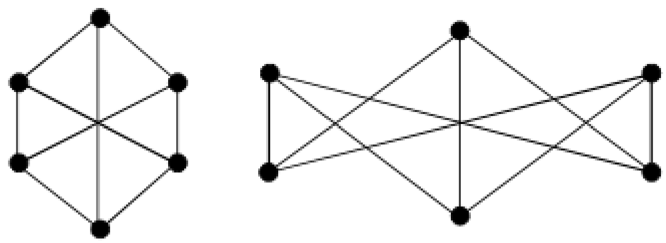
1. Both must have the **same number of vertices**
2. Both must have the **same number of edges**
3. Both must have **equal number of vertices** with the **same degree.**
4. They must have the same degree sequence and same cycle vector ($c_1, ..., c_n$), where c_i is
the number of cycles of length *i*.

The isomorphic pair of graphs are shown below

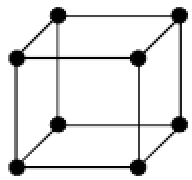
Example 1:

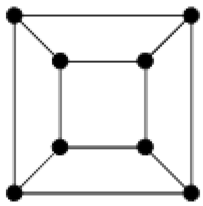


Example 2:

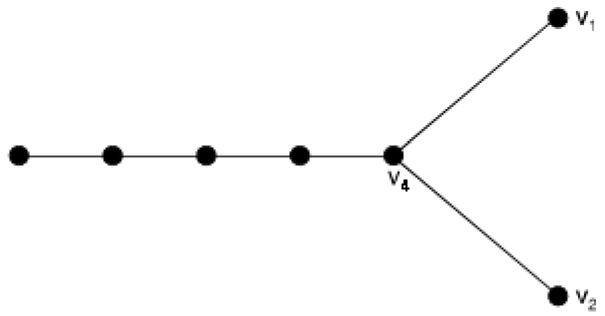


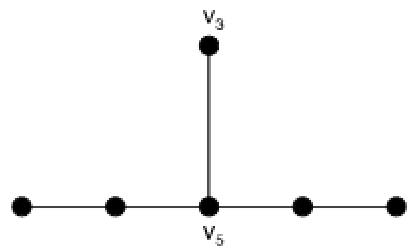
Example 3:



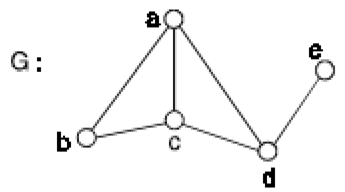


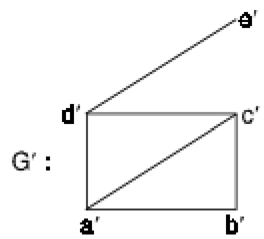
Example of two graphs that are not isomorphic





Problem. *Show that the following graphs are isomorphic*

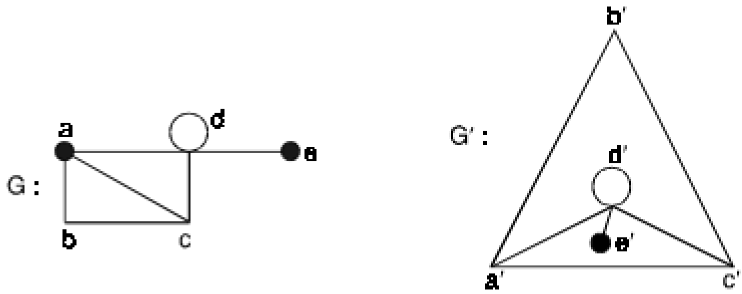




Solution. Let f : G → G′ be any function defined between two graphs degrees of the graph G and
G′ are as follows :
deg (G) deg (G′)
deg (a) = 3 deg (a') = 3
deg (b) = 2 deg (b') = 2
deg (c) = 3 deg (c') = 3
deg (d) = 3 deg (d') = 3
deg (e) = 1 deg (e') = 1
Each has 5-vertices and 6-edges.
$d(a) = d(a') = 3$
$d(b) = d(b') = 2$
$d(c) = d(c') = 3$
$d(d) = d(d') = 3$

$d(e) = d(e') = 1$
Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.
Therefore, the given two graphs are isomorphic.

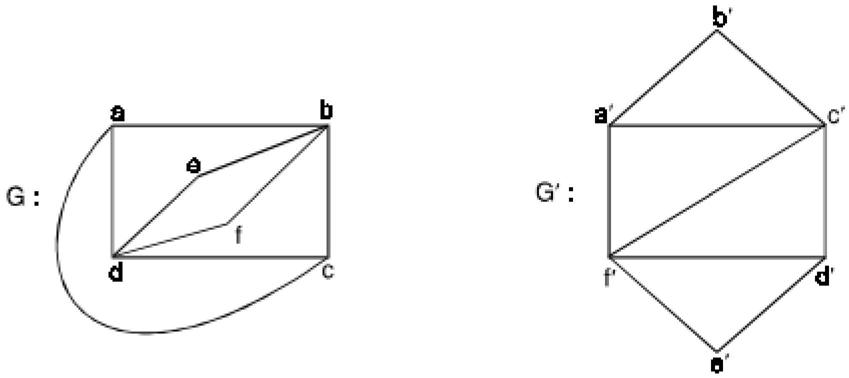
Problem. *Show that the following graphs are isomorphic.*



Solution. Let f : G →G′be any function defined between two graphs degrees of the graphs G
and G′are as follows :
deg (G) deg (G′)
deg (a) = 3 deg (a') = 3
deg (b) = 2 deg (b') = 2
deg (c) = 3 deg (c') = 3

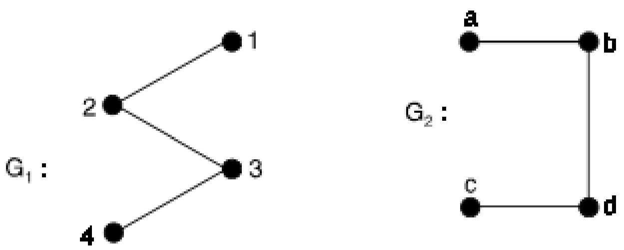
deg (d) = 5 deg (d') = 5
deg (e) = 1 deg (e') = 1
Each has 5-vertices, 6-edges and 1-circuit.
deg(a) = deg(a') = 3
deg(b) = deg(b') = 2
deg(c) = deg(c′) = 3
deg(d) = deg(d') = 5
deg(e) = deg(e') = 1
Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.
Therefore, the given two graphs G and G′are isomorphic.

Problem. *Are the 2-graphs, is given below, is isomorphic ? Give a reason.*

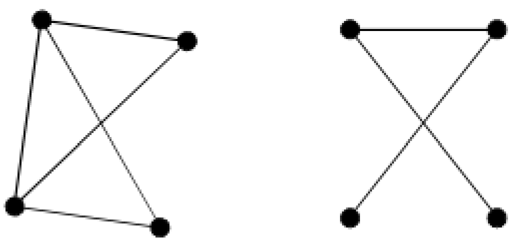


Solution. Let us enumerate the degree of the vertices
Vertices of degree 4 : $b - f'$
$d - c'$
Vertices of degree 3 : $a - a'$
$c - d'$
Vertices of degree 2 : $e - b'$
$f - e'$
Now the vertices of degree 3, in G are *a* and *c* and they are adjacent in G′, while these are *a*′□and
d′□which are not adjacent in G′.
Hence the 2-graphs are not isomorphic.

Problem. *For each pair of graphs shown, either label the graphs so as to exhibit an isomorphism or explain why the graphs are not isomorphic.*

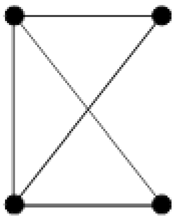


(*i*)

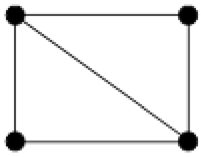


(a) (b)

(ii)

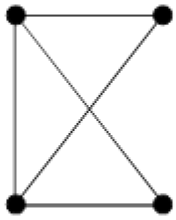


(a)

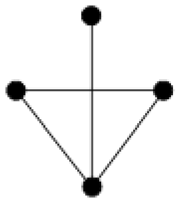


(b)

(iii)

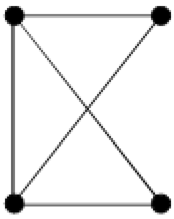


(a)

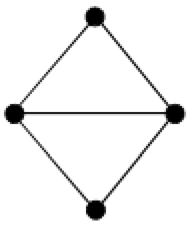


(b)

(iv)



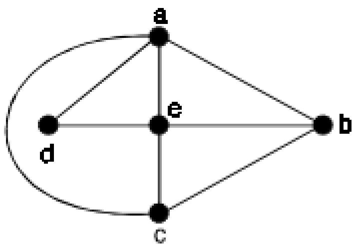
(a)

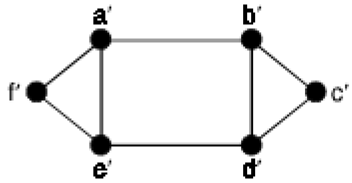


(b)

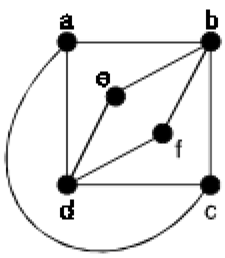
Problems. *Are the 2-graphs, is given below, is isomorphic ? Give a reason.*

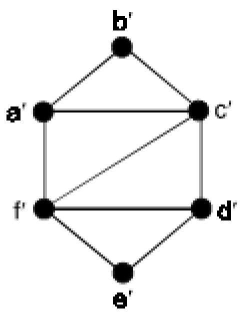
(i)



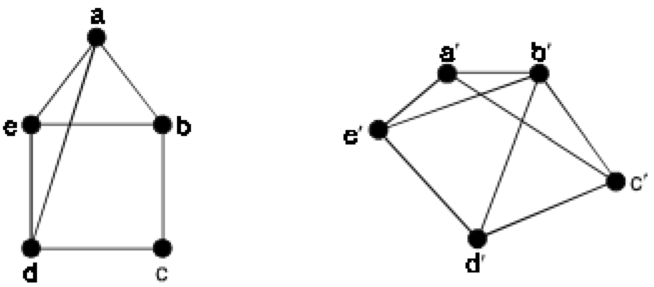


(ii)





(iii)



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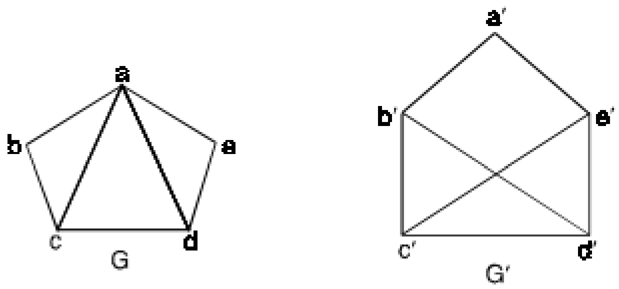
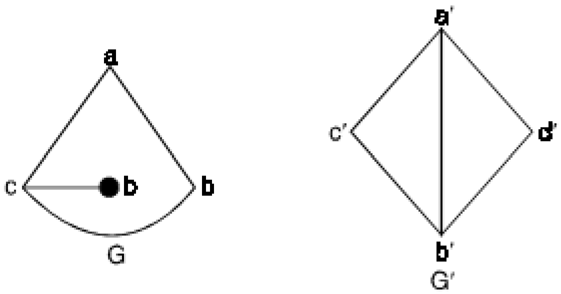
Problem. *Find whether the following pairs of graphs are isomorphic or not*

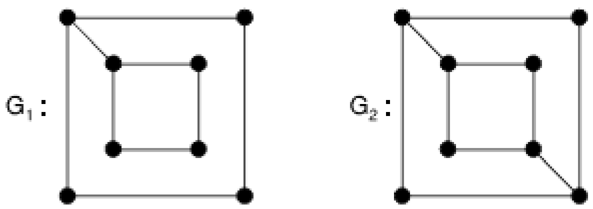
(i)

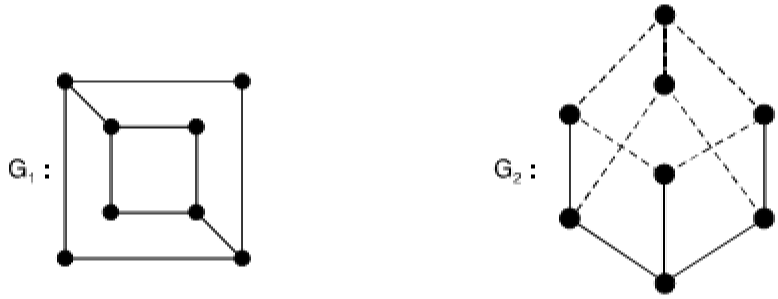


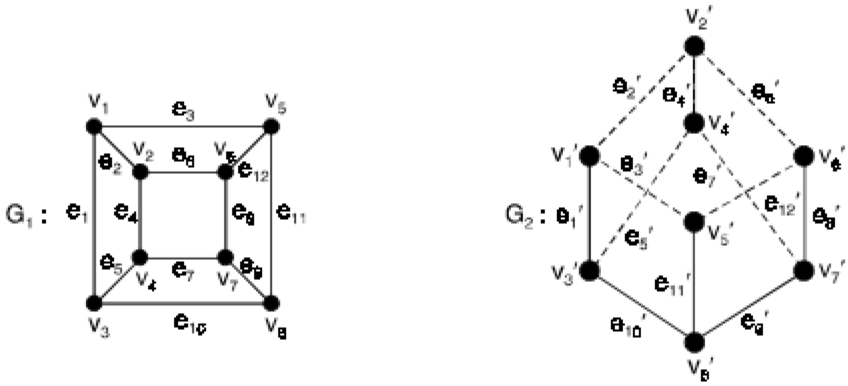
(ii)

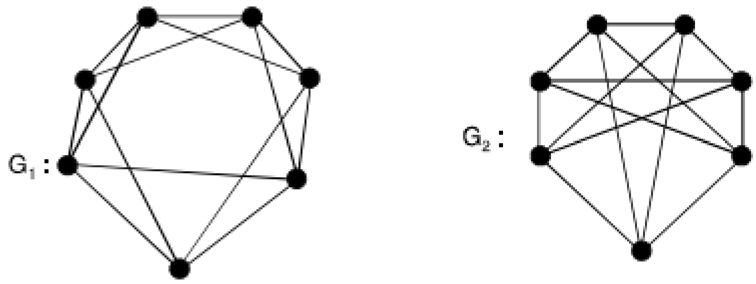


Problem. *Consider two graphs G1 and G2 as shown below, show that the graphs G1 and*
G2 are isomorphic.









REPRESENTATION OF GRAPHS

Although a diagrammatic representation of a graph is very convenient for a visual study but this
is only possible when the number of nodes and edges is reasonably small.Two types of representation are given below :

Matrix representation

The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily
applied to study the structural properties of graphs from an algebraic point of view. There are number of

matrices which one can associate witch any graph. We shall discuss adjacency matrix and the incidence
matrix.

ADJACENCY MATRIX

Representation of undirected graph

The adjacency matrix of a graph G with *n* vertices and no parallel edges is an *n* by *n* matrix A = {aij}
whose elements are given by aij =

$$\begin{cases} 1 & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0 & \text{if there is an edge between } i\text{th and } j\text{th vertices} \end{cases}$$

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices. Hence, there are as many as *n* ! different adjacency matrices for a graph with *n* vertices, since there are *n* ! different ordering of *n* vertices. However, any two such adjacency matrices are closely related in that one can be obtained from
the other by simply interchanging rows and columns.
There are a number of observations that one can make about the adjacency matrix A of a graph G. They are
(*i*) A is symmetric *i.e. aij* = *aji* for all *i* and *j*
(*ii*) The entries along the principal diagonal of A all zeros if and only if the graph has no self loops. A self loop at the vertex corresponding to *aij* = 1.
(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.
(*iv*) The (*i, j*) entry of A*m* is the number of paths of length (no. of occurrence of edges) *m* from vertex *vi* to vertex *vj*.
(*v*) If G be a graph with *n* vertices *v*1, *v*2, ... *vn* and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix and B = A + A2 + A3 + ... + A*n* – 1
Then G is a connected graph if B has no zero entries of the main diagonal.
This result can be also used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex *v*1 is represented by a 1 at the (*i, j*)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (*i, j*)th entry equals the number of edges these are associated to {*vi* – *vj*}.

All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

Representation of directed graph

The adjacency matrix of a diagonal D, with *n* vertices is the matrix A = $\{aij\}n \times n$ in which

$$aij = \begin{cases} 1 & \text{if arc } \{vi - vj\} \text{ is in D} \\ 0 & \text{otherwise} \end{cases}$$

One can make a number of observations about the adjacency matrix of a diagonal.

Observations

(*i*) A is not necessary symmetric, since there may not be an edges from *vi* to *vj* when there is an edge from *vi* to *vj*.

(*ii*) The sum of any column of *j* of A is equal to the number of arcs directed towards *vj*

(*iii*) The sum of entries in row *i* is equal to the number of arcs directed away from vertex *vi* (out degree of vertex *vi*)

(*iv*) The (*i, j*) entry of A*m* is equal to the number of path of length *m* from vertex *vi* to vertex *vj* entries of AT. A shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices.

In the adjacency matrix for a directed multigraph *aij* equals the number of edges that are associated to (*vi*, *vj*).

INCIDENCE MATRIX

Representation of undirected graph

Consider a undirected graph G = (V, E) which has *n* vertices and *m* edges all labelled. The
incidence matrix B = $\{bij\}$, is then $n \times m$ matrix,

where $b_{ij} = \begin{cases} 1 & when\ edge\ e_j\ is\ incident\ with\ vertex\ v_i \\ 0 & otherwise \end{cases}$

We can make a number of observations about the incidence matrix B of G.

(*i*) Each column of B comprises exactly two unit entries.
(*ii*) A row with all 0 entries corresponds to an isolated vertex.
(*iii*) A row with a single unit entry corresponds to a pendent vertex.
(*iv*) The number of unit entries in row *i* of B is equal to the degree of the corresponding vertex v_i.
(*v*) The permutation of any two rows (any two columns) of B corresponds to a labelling of the vertices (edges) of G.
(*vi*) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.
(*vii*) If G is connected with *n* vertices then the rank of B is $n - 1$.

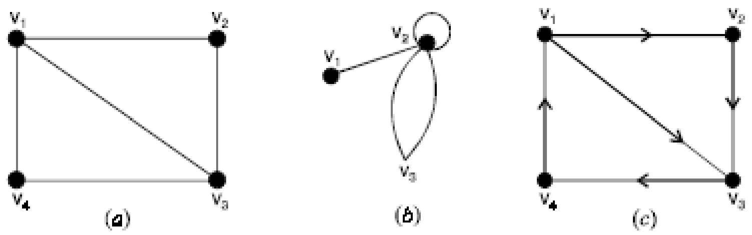
Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

Representation of directed graph

The incidence matrix B = $\{b_{ij}\}$ of digraph D with *n* vertices and *m* edges is the $n \times m$ matrix in which

$$b_{ij} = \begin{cases} 1 & if\ arc\ j\ is\ directed\ away\ from\ vertex\ v_i \\ -1 & if\ arc\ j\ is\ directed\ towards\ vertex\ v_i \\ 0 & otherwise \end{cases}$$

Problem 14. *Use adjacency matrix to represent the graphs shown in Figure below*



Solution. We order the vertices in Figure (*a*) as v_1, v_2, v_3 and v_4.
Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We order the vertices in Figure (*b*) as v_1, v_2 and v_3. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Taking the order of the vertices in Figure (*c*) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

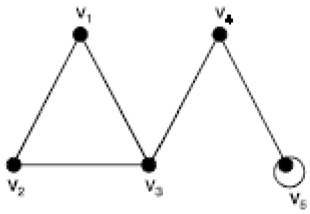
Problem 15. *Draw the undirected graph represented by adjacency matrix A given by*

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.

Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5.Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is drawn in Figure below.

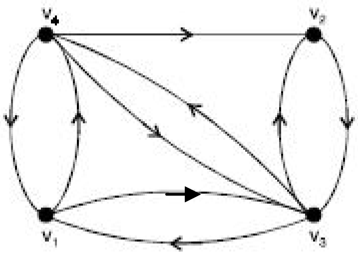


Problem 16. *Draw the digraph G corresponding to adjacency matrix*

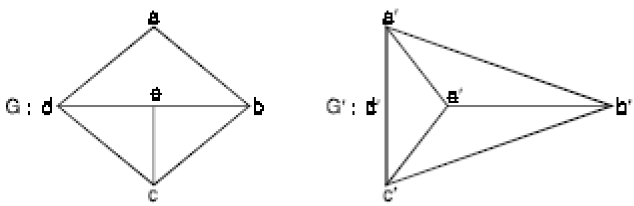
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4 . Draw an edge from v_i to v_j where $a_{ij} = 1$.

The required graph is shown in Figure below.



Problem 17. *Show that the graphs G and G ' are isomorphic*



Solution. Consider the map f : G →G′ defined as $f(a) = d'$, $f(b) = a'$, $f(c) = b'$, $f(d) = c'$ and $f(e) = e'$

The adjacency matrix of G for the ordering a, b, c, d and e is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

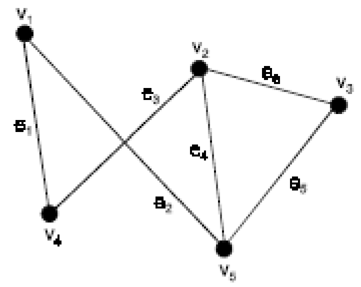
The adjacency matrix of G′ for the ordering d', a', b', c' and e' is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array}\begin{array}{c} \begin{matrix} d' & a' & b' & c' & e' \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e., $A(G) = A(G')$

Therefore G and G' are isomorphic.

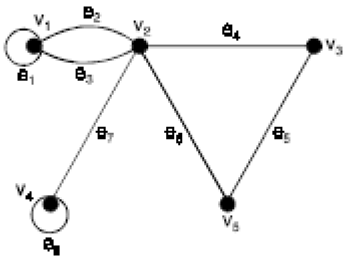
Problem 18. *Represent the graph shown in Figure below, with an incidence matrix.*



Solution. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

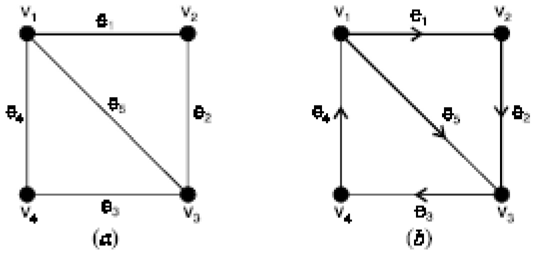
Problem 19. *Represent the Pseudo graph shown in Figure below, using an incidence matrix.*



Solution. The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problem 20. *Find the incidence matrix to represent the graph shown in Figure below :*



Solution.
The incidence matrix of Figure (*a*) is obtained by entering for row *v* and column *e* is 1 if *e* is incident on *v* and 0 otherwise. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the graph of Figure (*b*) is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Problems for practice

1. *Draw the undirected graph G corresponding to adjacency matrix*

WALKS, PATHS AND CYCLES

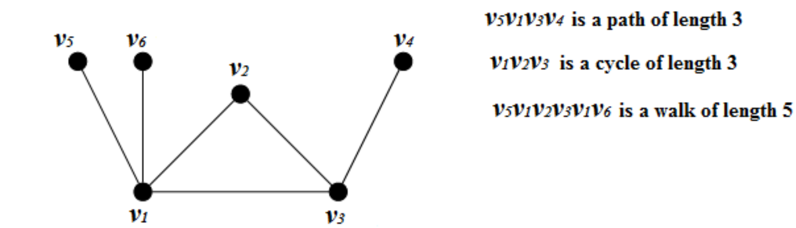
Definition

A walk in G is a sequence of vertices $v_0,\ v_1, \dots, v_k$ and a sequence of edges$(v_i,\ v_{1+1}) \in E(G)$. A walk is a path if all v_i are distinct. v_0 is the initial vertex and v_k is the terminal vertex. A zero length walk is just a single vertex v_0. If for such a path with $k \geq 2$, $(v_0,\ v_k)$is also an edge in G, then $v_0,\ v_1,\dots,\ v_k,\ v_0$ is a cycle. For multigraphs, we also consider loops and pairs of multiple edges to be cycles.

Definition

The length of a path, cycle or walk is the number of edges in it.

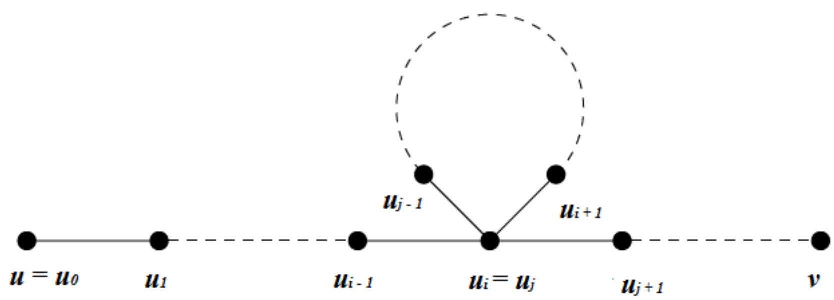
Example



Proposition: Every walk from *u* to *v* in G contains a path between *u* and *v*.

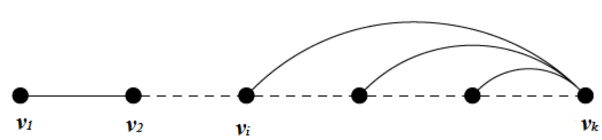
Proof.

By induction on the length *l* of the walk $u = u_0,\ u_1, \dots, v_l = v$.If $l = 1$ then our walk is also a path. Otherwise, if our walk is not a path there is $u_i = u_j$with $i < j$, then $u = u_0,\ u_1,\ u_i, u_{j+1}, v$ is also a walk from *u* to *v* which is shorter. We can use induction to conclude the proof.



Proposition: Every G with minimum degree $\delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$.

Proof. Let $v_1,\ v_2, \dots, v_k$ be a longest path in G. Then all neighbors of v_k belong to $v_1, v_2, \dots, v_{k-1}$so $k - 1 \geq \delta$ and $k \geq \delta + 1$, and our path has at least δ edges. Let $i\ (1 \leq i \leq k)$be the minimum index such that $(v_i,\ v_k) \in E(G)$. Then the neighbors of v_k are among$v_1,\ v_2, \dots, v_{k-1}$, so $k - i \geq \delta$. Then $v_i,\ v_{i+1}, \dots, v_k$ is a cycle of length at least $\delta + 1$.

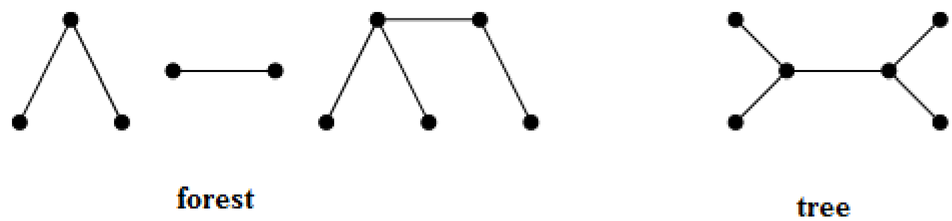


TREES

Definition:

A graph having no cycle is acyclic. A tree is a connected acyclic graph. A leaf (or pendant vertex) is a vertex of degree 1. A forest is an acyclic graph. A tree is a connected forest. A subforest is a subgraph of a forest. A connected subgraph of a tree is a subtree. A spanning tree of a connected graph is a subtree that includes all the vertices of that graph. The edges of a spanning tree are called branches.

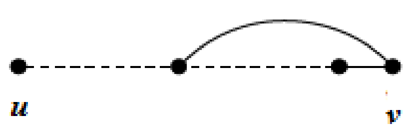
Example:



Lemma: Every finite tree with at least two vertices has at least two leaves. Deleting a leaf from an n-vertex tree produces a tree with $n - 1$ vertices.

Proof.

Every connected graph with at least two vertices has an edge. In an acyclic graph, the end points of a maximum path have only one neighbor on the path and therefore have degree 1.Hence the endpoints of a maximum path provide the two desired leaves.



Suppose v is a leaf of a tree G, and let $G' = G - v$. If $u, w \in V(G')$, then no u, w-path P in G can pass through the vertex v of degree 1, so P is also present in G'. Hence G' is connected. Since deleting a vertex cannot create a cycle, G' is also acyclic. We conclude that G' is a tree with $n - 1$vertices.

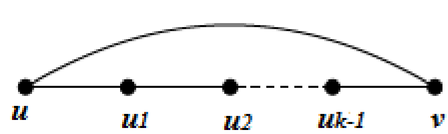
Theorem: For an n-vertex simple graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices).
(a) G is connected and has no cycles.
(b) G is connected and has $n - 1$ edges.
(c) G has $n - 1$ edges and no cycles.
(d) For every pair $u, v \in V(G)$, there is exactly one $u, v -$ path in G.
To prove this theorem we will need a small lemma.

Definition: An edge of a graph is a cut-edge if its deletion disconnects the graph.

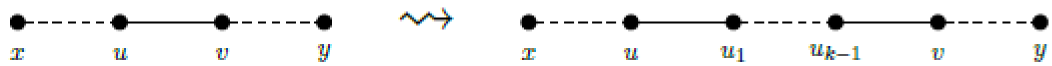
Lemma: An edge contained in a cycle is not a cut-edge.

Proof of the lemma:

Let (u, v) belong to a cycle.



Then any path x ... y in G which uses the edge (u, v) can be extended to a walk in $G - (u, v)$ asfollows:



Proof of Theorem:

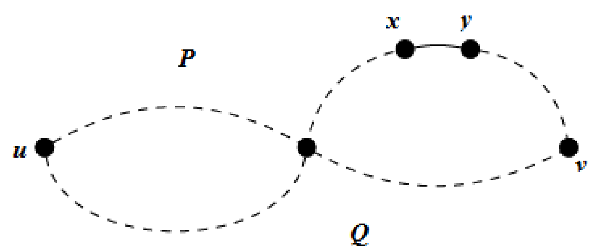
We first demonstrate the equivalence of (a), (b), (c) by proving that any two of {connected, acyclic, $n - 1$ edges} implies the third.

(a) ⇒ (b), (c): We use induction on n. For n = 1, an acyclic 1-vertex graph has no edge. For the induction step, suppose n > 1, and suppose the implication holds for graphs with fewer than n vertices. Given G, the Lemma provides a leaf v and states that $G' = G - v$ is acyclic and connected. Applying the induction hypothesis to G'yields $e(G') = n - 2$, and hence$e(G) = n - 1$.

(b) ⇒ (a), (c): Delete edges from cycles of G one by one until the resulting graph G' is acyclic. By Lemma, G is connected. By the paragraph above, G' has $n - 1$ edges. Since this equals$|E(G)|$, no edges were deleted, and G itself is acyclic.

(c) ⇒ (a), (b): Suppose G has k components with orders $n_1, . . . n_k$. Since G has no cycles, each component satisfies property (a), and by the first paragraph the ith component has $n_i - 1$ edges. Summing this over all components yields $e(G) = \sum(n_i - 1) = n - k$. We are given $e(G) = n - 1$, so $k = 1$, and G is connected.

(a) ⇒ (d): Since G is connected, G has at least one u, v −path for each pair $u, v \in V(G)$. SupposeG has distinct u, v −paths P and Q. Let $e = (x, y)$ be an edge in P but not in Q. The concatenation of P with the reverse of Q is a closed walk in which e appears exactly once. Hence, $(P \cup Q) - e$ is an x, y −walk not containing e. Thus we have a cycle with e and contradicts the hypothesis that G is acyclic. Hence G has exactly one u, v −path.



(d) ⇒ (a): If there is a u; v-path for every u; v ε V (G), then G is connected. If G has a cycle C, then G has two paths between any pair of vertices on C.

Definition:

Given a connected graph G, a spanning tree T is a subgraph of G which is a tree and contains every vertex of G.

Corollary:

(a) Every connected graph on n vertices has at least $n - 1$ edges and contains a spanning tree;

(b) Every edge of a tree is a cut-edge;

(c) Adding an edge to a tree creates exactly one cycle.

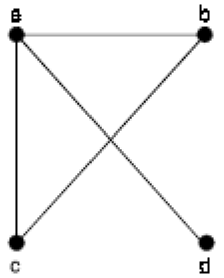
Proof.

(a) Delete edges from cycles of G one by one until the resulting graph G_0 is acyclic. By Lemma, G is connected. The resulting graph is acylic so it is a tree. Therefore G had at least $n - 1$edges and contains a spanning tree.

(b) Note that deleting an edge from a tree T on n vertices leaves $n - 2$ edges, so the graph is disconnected by (a).

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

2. *Use an adjacency matrix to represent the graph shown in Figure below*

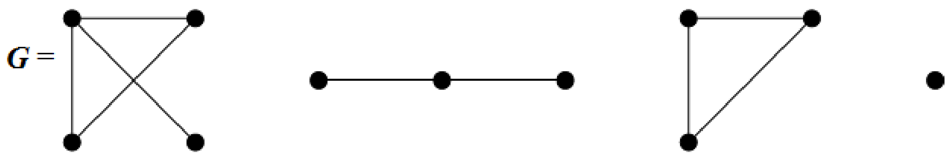


3. *Draw a graph with the adjacency matrix*

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Definition:

A (connected) component of G is a connected subgraph that is maximal by inclusion. We say G is connected if and only if it has one connected component. The graph G which is given below has 4 connected components.



Proposition: A graph with n vertices and m edges has at least $n - m$ connected components.

Proof.

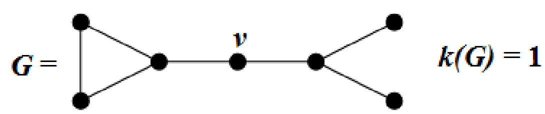
Start with the empty graph (which has n components), and add edges one-by-one. Note that adding an edge can decrease the number of components by at most 1.

Definition: (Vertex connectivity)

A vertex cut in a connected graph $G = (V, E)$ is a set $S \subseteq V$ such that $G \setminus S = G[V \setminus S]$ has more than one connected component. A cut vertex is a vertex v such that $\{v\}$is a cut.

Definition:

G is called k-connected if $|V(G)| > k$ and if $G \setminus X$ is connected for every set $X \subseteq V$with $|X| < k|$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (non-empty) graph is 0-connected and the 1-connected graphs are precisely the non-trivial connected graphs. The greatest integer k such that G is $k-$connected is the connectivity $k(G)$of G. For example, if $G = K_n$, then $k(G) = n - 1$. In the below example, deleting *v* disconnects G, so *v* is a cut vertex.



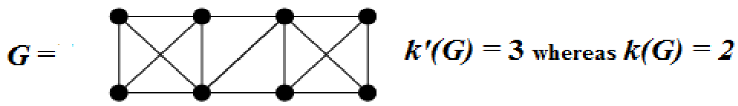
Proposition: For every graph G, $k(G) \leq \delta(G)$.

Proof.

Let $v \in V(G)$ be a vertex of minimum degree $d(v) = \delta(G)$. Then deleting *N(v)* disconnects *v* from the rest of G.

Definition: (Edge connectivity)

A disconnecting set of edges is a set $S \subseteq E(G)$ such that $G \setminus F$has more than one component. Given $S, T \subseteq V(G)$ the notation [S, T] specifies the set of edges having one end point in S and the other in T. An edge cut is an edge set of the form [S, $\bar{S}$], where S is a non-empty proper subset of V (G). A graph is k-edge-connected if every disconnecting set has at least k edges. The edge-connectivity of G, written $k^{'}(G)$, is the minimum size of a disconnecting set. One edge disconnecting G is called a bridge. For example, if $G = K_n$, then $k^{'}(G) = n - 1$.



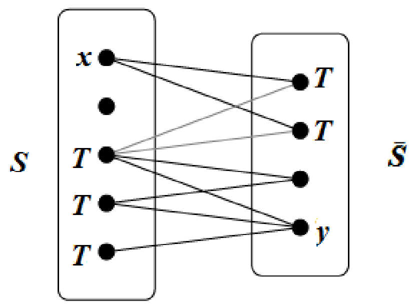
Remark: An edge cut is a disconnecting set but not the other way around. However, everyminimal disconnecting set is a cut.

Theorem:$k(G) \leq k^{'}(G) \leq \delta(G)$.

Proof.

The edges incident to a vertex *v* of minimum degree, form a disconnecting set, hence $k^{'}(G) \leq \delta(G)$. It remains to show that $k(G) \leq k^{'}(G)$. Suppose$|G| > 1$ and [S, S] is a minimum edge cut, having size $k^{'}(G)$.

If every vertex of S is adjacent to every vertex of $\bar{S}$ and $|G| = |V(G)| = n$, then $k^{'}(G) = |S||\bar{S}| = |S|(|G| - |S|)$. This expression is minimized at $|S| = 1$. By definition,$k(G) \leq |G| - 1$, so the inequality holds.



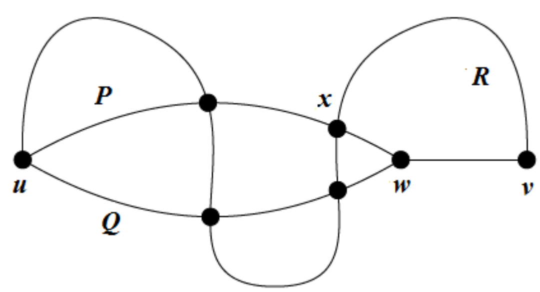
Hence we may assume there exists $x \in S$, $y \in \bar{S}$ with *x* not adjacent to *y*. Let T be the vertex set consisting of all neighbors of *x* in S and all vertices of $S \backslash x$ that have neighbours in S (illustrated below). Deleting T destroys all the edges in the cut [S, S] (but does not delete *x* or *y*), so T is a separating set. Now, by the definition of T we can injectively associate at least one edge of [S,$\bar{S}$] to each vertex in T, so .$k(G) \leq |T| \leq |[S, \bar{S}]| = k^{'}(G)$.

Definition: Two paths are internally disjoint if neither contains a non-endpoint vertex of the other. We denote the length of the shortest path from u to v (the distance from u to v) by d(u, v).

Theorem: (Whitney 1932).A graph G having at least three vertices is 2-connected if and only if each pair $u, v \in V(G)$ is connected by a pair of internally disjoint $u, v -$ paths in G.

Proof.

When G has internally disjoint u, v −paths, deletion of one vertex cannot separate u from v. Since this is given for every u, v, the condition is sufficient. For the converse, suppose that G is2-connected. We prove by induction on $d(u, v)$ that G has two internally disjoint u, v paths. When $d(u, v) = 1$, the graph $G \setminus (u, v)$ is connected, since $k^{'}(G) \geq k(G) = 2$. A $u, v -$ path in $G \setminus (u, v)$ is internally disjoint in G from the u, v −path consisting of the edge (u, v) itself.



For the induction step, we consider $d(u,v) = k > 1$ and assume that G has internally disjoint x,y −paths whenever $1 \leq d(x,y) \leq k$. Let w be the vertex before v on a shortest u,v −path. We have $d(u,w) = k - 1$, and hence by the induction hypothesis G has internally disjoint u,w − paths P andQ. Since $G \setminus w$ is connected, $G \setminus w$ contains a u,v −path R. If this path avoids P or Q, we are finished, but R may share internal vertices with both P and Q. Let x be the last vertex of R belonging to $P \cup Q$. Without loss of generality, we may assume, $x \in P$. We combine the u,x −subpath of P with the x,v −subpath of R to obtain a u,v −path internally disjoint from $Q \cup \{(w,v)\}$.

Corollary: G is 2-connected and$|V(G)| \geq 3$ if and only if every two vertices in G lie on a common cycle.

EULERIAN AND HAMILTONIAN PATHS

Definition: A trail is a walk with no repeated edges.

Definition: An Eulerian trail in a graph $G = (V,E)$ is a walk in G passing through every edge exactly once. If this walk is closed (starts and ends at the same vertex) it is called an Eulerian tour.

Theorem: A connected graph has an Eulerian tour if and only if each vertex has even degree.
In order to prove this theorem we use the following lemma.

Lemma: Every maximal trail in a graph where all the vertices have even degree is a closed trail.

Proof.

Let T be a maximal trail. If T is not closed, then T has an odd number of edges incident to the final vertex v. However, as v has even degree, there is an edge incident to v that is not in T. This edge can be used to extend T to a longer trail, contradicting the maximality of T.

Proof of Theorem

To see that the condition is necessary, suppose G has an Eulerian tour C. If a vertex v was visited k times in the tour C, then each visit used 2 edges incident to v (one in comingedge and one outgoing edge). Thus, $d(v) = 2k$, which is even.

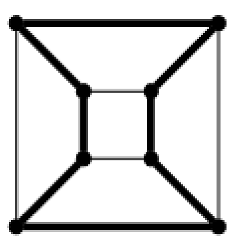
To see that the condition is sufficient, let G be a connected graph with even degrees. Let $T = e_1 e_2 \ldots e_l$ (where $e_i = (v_{i-1}, v_i)$)be a longest trail in G. Then, by Lemma, T is closed, that is, $v_0 = v_l$. If T does not include all the edges of G then, since G is connected, there is an edge outside of T such that $e = (u, v_i)$ for some vertex v_i in T. But then $T' = e e_{i+1} \ldots e_l e_1 e_2 \ldots e_i$ is a trail in G which is longer than T, contradicting the fact that T is a longest trail in G. Thus, we conclude that T includes all the edges of G and so it is an Eulerian tour.

HAMILTON PATHS AND CYCLES

Definition: A Hamilton path/cycle in a graph G is a path/cycle visiting every vertex of G exactly once. A graph G is called Hamiltonian if it contains a Hamilton cycle.

Hamilton cycles were introduced by Kirkman in 1985, and were named after Sir William Hamilton, who produced a puzzle whose goal was to find a Hamilton cycle in a specific graph.

Example: Hamilton cycle in the skeleton of the 3-dimensional cube.

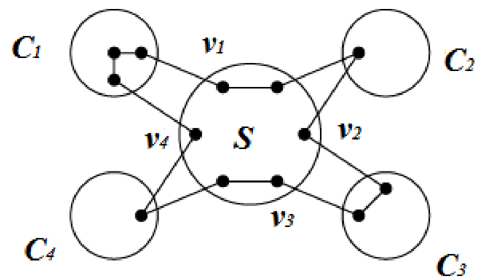


Proposition 5.3.

Theorem: If G is Hamiltonian then for any set $S \subseteq V(G)$ the graph $G \setminus S$has at most $|S|$connected components.

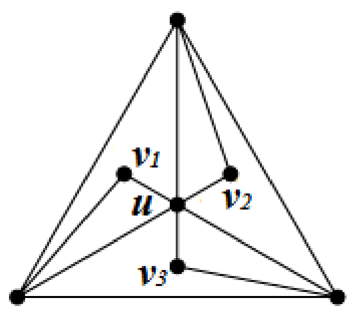
Proof.

Let $C_1, C_2, \ldots, C_k$ be the components of $G \setminus S$. Imagine that we are moving along a Hamilton cycle in some order, vertex-by-vertex (in the picture below, we are moving clockwise, starting from some vertex in C_1, say). We must visit each component of $G \setminus S$at least once, when we leaveC_i forthe first time, let v_i be the subsequent vertex visited (which must be in S). Each v_i must be distinct because a cycle cannot intersect itself. Hence, S must have at least as many vertices as the number of connected components of $G \setminus S$.



Example:

The condition in Proposition is not sufficient to ensure that a graph is Hamiltonian. The graph G above satisfies the condition of Proposition, but is not Hamiltonian. Indeed, one would need to include all the edges incident to the vertices v_1, v_2 and v_3 in a Hamiltoncycle of G, however, in that case the vertex u would have degree at least 3 in that Hamilton cycle, which is impossible. We also give some sufficient conditions for Hamiltonicity.



Theorem: (Dirac 1952). If G is a simple graph with $n \geq 3$ vertices and if $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Proof.

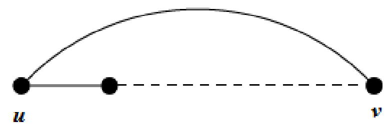
The condition that $n \geq 3$ must be included since K_2 is not Hamiltonian but satisfies $\delta(G) = \frac{|K_2|}{2}$. If there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree, so we may restrict our attention to maximal non-Hamiltonian graphs G with minimum degree at least $\frac{n}{2}$. By "maximal" we mean that for every pair (u, v) of non-adjacent vertices of G, the graph obtained from G by adding the edge $e = (u, v)$ is Hamiltonian.

The maximality of G implies that G has a Hamilton path, say from $u = v_1$, to $v = v_n$, because every Hamilton cycle in $G \cup \{e\}$ must contain the new edge e. We use most of this path $v_1, v_2, \ldots, v_n$with a small switch, to obtain a Hamilton cycle in G. If some neighbor of u immediately follows a neighbor of v on the path, say (u; vi+1) ∈ E(G) and (v; vi) ∈ E(G), then G has the Hamilton cycle$(u, v_{i+1}, v_{i+2}, \ldots, v_{n-1}, v, v_i, v_{i-1}, \ldots, v_2)$ shown below.

To prove that such a cycle exists, we show that there is a common index in the sets S and T defined by $S = \{i: (u, v_{i+1} \in E(G)\}$ and T= $\{i: (v, v_i \in E(G)\}$. Summing the sizes of these sets, yields$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n$. Neither S nor T contains the index n. This implies that $|S \cup T| < n$, and hence $|S \cap T| \geq 1$, as required. This is a contradiction.

It can be observed that this argument uses only that $d(u) + d(v) \geq n$. Therefore, we can weaken the requirement of minimum degree $\frac{n}{2}$ to require only that $d(u) + d(v) \geq n$ whenever *u* is not adjacent to *v*.

(c) Let $u, v \in T$. There is a unique path in T between u and v, so adding an edge (u, v) closes this path to a unique cycle.



Theorem: A connected graph has at least one spanning tree.

Proof.

Consider the connected graph G with n vertices and m edges. If $m = n - 1$, then G is a tree. Since G is connected, $m \geq n - 1$. We still have to consider the case $m \geq n$, where there is a circuit in G. We remove an edge e from that circuit. G − e is now connected. We repeat until there are $n - 1$ edges. Then, we are left with a tree.

Theorem: If a tree is not trivial, then there are at least two pendant vertices.

Proof.

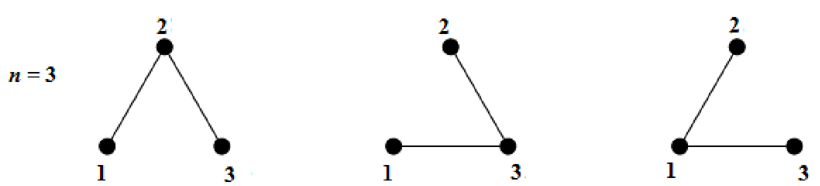
If a tree has n ≥2 vertices, then the sum of the degrees is 2(n − 1). If every vertex has a degree ≥ 2, then the sum will be $\geq 2n$. On the other hand, if all but one vertex have degree ≥ 2, then the sum would be $\geq 1 + 2(n - 1) = 2n - 1$. This is because a cut vertex of a tree is not a pendant vertex. A forest with k components is sometimes called a k-tree. (So a 1-tree is a tree.)

Theorem (Cayley's Formula). There are n^{n-2}trees with vertex set *n*.

Question: What is the number of spanning trees in a labeled complete graph on n vertices?

By Cayley's formula, it is n^{n-2}.

Example:



Theorem: If G is a tree, then the number of edges in G= $n - 1$.

Proof.

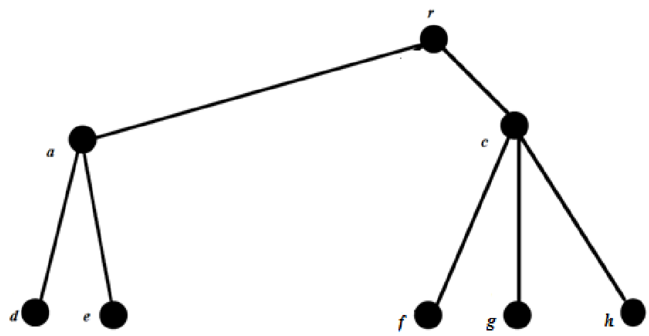
Let us denote the number of edges in G by *m*. By induction on *n*, when n = 1, G is isomorphic to K_1 and so the number of edges in G is $m = 0 = n - 1$. Suppose the theorem is true for all trees on fewer than *v* vertices and let G be a tree on $n \geq 2$ vertices. Let $(u, v) \in E(G)$, then $G - (u, v)$ contains no $u, v -$ path, since (u, v) is the unique $u, v -$ path in G. Thus $G - (u, v)$ is disconnected so $\omega(G - uv) = 2$.The components G_1 and G_2 of $G - (u, v)$, being acyclic are trees. Moreover, each has fewer than n vertices. Therefore by induction hypothesis, $E(G_i) = V(G_i) - 1$, for $i = 1, 2$. Thus$E(G) = E(G_i) + E(G_i) + 1 = V(G_1) + V(G_2) + 1 = V(G) - 1 = n - 1$.

CONNECTIVITY

Definition:

A graph G is connected if, for all pairs $u, v \in V(G')$, there is a path in G from *u* to *v*.

Note that it suffices for there to be a walk from u to v, by Proposition



Definition: The left (right) subtree of a vertex v in a binary tree is the binary subtree spanning the left (right)-child of v and all of its descendants.

Theorem: The complete binary tree of height h has $2^{h+1} - 1$ vertices.

Corollary: Every binary tree of height h has at most $2^{h+1} - 1$vertices.

Expression Trees

An expression tree is a special type of a binary tree that represents an algebraic expression in such a way that stores its structure and shows how the order of operations applies. This is a very important type of a tree in computer science. We're interested in a few different operators. We break these operators down into two categories:

- Binary Operators - operators that take two inputs
    - +
    - - (here, subtraction)
    - *
    - / (both integer and floating-point division)
    - % (modulus)
    - ^ or ** (exponentiation)

- Unary Operators - operators that take one input
    - - (here, negation)

Note that we don't mention parentheses. The expression tree's structure removes the need to talk about parentheses, as the structure encodes precedence.

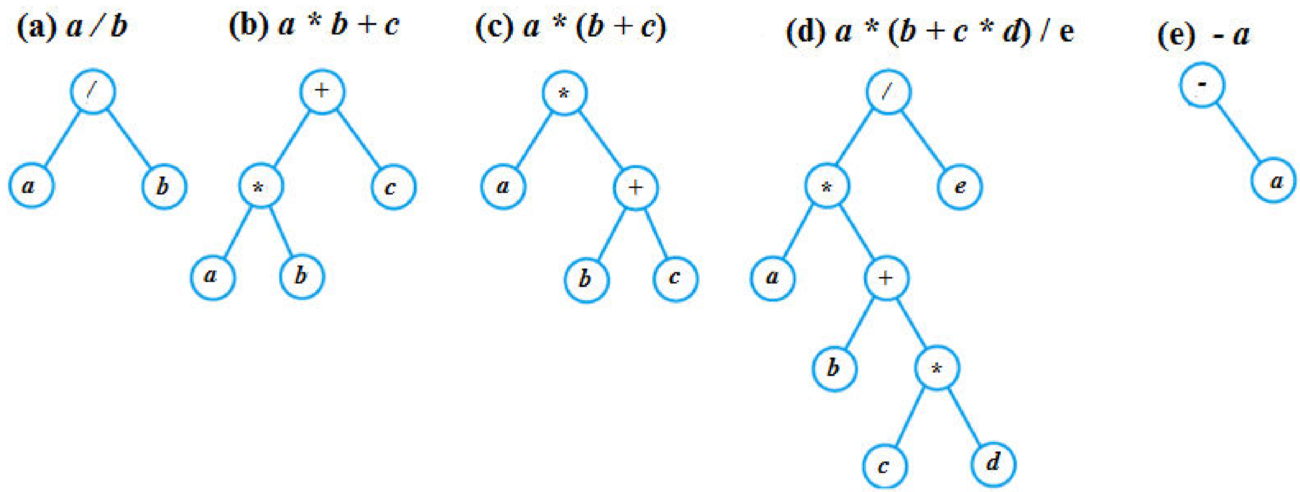
When we have a single expression based on a binary operator, we draw the expression tree as follows:

- The operator is the root of the tree.
- The operands are the children. Because some operations are *not* commutative, order does matter. The operand before the operator is the left child and the operand after the operator is the right child. Thus, we get a tree with a root and two children. For example see figure (a).

When we have a single expression based on a unary operator, we draw the expression tree as follows:

- The operator is the root of the tree.
- The operand is the child.

Thus, we get a tree with a root and one child. (It's really more of a linear structure than a tree, but it does fit the definition of a tree. We'll find that these kinds of trees are interesting when we join them together as part of more complicated expressions.) The Expression tree for –a is in figure (e). Note that we could treat negation as multiplication by -1 and eliminate the need for unary trees if we'd like to have all nodes in our tree having exactly 2 children (or no child). When we wish to work with more complicated expressions, we invoke the recursive nature of binary trees. When an operand is an expression rather than a single variable or constant, we simply put the expression tree for that expression in lieu of theoperand. Figures (b), (c) and (d) are examples of such expression trees.


(a) a / b
(b) a * b + c
(c) a * (b + c)
(d) a * (b + c * d) / e
(e) - a