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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I -ENGINEERING MATHEMATICS-III– SMTA1301

UNIT- I

Complex Variables

Introduction to Complex Numbers

A general form of a complex number is $z = x + iy$ when x and y are real and $i = \sqrt{-1}$. Here x is called the real part and y is the imaginary part of z.

A conjugate of a complex number z is $\overline{z} = x - iy$. Then

$$z + \overline{z} = 2x \Rightarrow x = \frac{1}{2}[z + \overline{z}]$$

$$z - \overline{z} = 2iy \Rightarrow y = \frac{1}{2i}[z - \overline{z}]$$

$$z\,\overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

The complex number $z = x + iy$ can be represented by a point (x, y) in a complex plane. The modulus (absolute value) of z is given by

$$|z| = \sqrt{x^2 + y^2}$$

The distance between the points z_1 and z_2 is $|z_1 - z_2|$.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the distance

$$\begin{aligned} z_1 z_2 &= |z_1 - z_2| \\ &= |(x_1 + iy_1) - (x_2 + iy_2)\} \\ &= |(x_1 - x_2) + i(y_1 - y_2)| \end{aligned}$$

Polar form of a complex number : Let the polar coordinates of the point (x, y) be (r, θ), then

$$z = x + iy = r[\cos\theta + i\sin\theta] = r\,e^{i\theta}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

Squaring and adding, we get

$$x^2 + y^2 = r^2$$

$$\therefore \; r = \sqrt{x^2+y^2}$$

Dividing the above results, we get

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The number r is called the modulus value of z and θ is called the amplitude (argument) of the complex number z.

Euler's Formula

We know $\quad e^{in\theta} = \cos n\theta + i \sin n\theta$

Demoivre's theorem for positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

Note : $e^{-in\theta} = \cos(n\theta) - i \sin(n\theta)$

Functions of a Complex Variable

Let $z = x + iy$ and $\omega = u + iv$. If z and ω are two complex variables and if for each value of z in a complex plane there corresponds one or more values of ω, then ω is called to be a function of z.

We can write $\quad \omega = f(z) = u + iv = u(x, y) + i\, v(x, y)$.

Here u and v are real functions of the real variables x and y.

For example $\quad f(z) = z^2$

$$= (x^2 - y^2) + i\,(2xy)$$

Singled Valued Function

A function $f(z)$ is called a single valued function of z if for each value of z in the domain R, there is only one value of ω.

For example, $\quad f(z) = z^3, \dfrac{1}{z}$

$$f(z) = \frac{1}{z^2+1}$$

If there is more than one value of ω corresponding to a given value of z, then $f(z)$ is called multiple-valued function.

For example, $\quad f(z) = z^{1/4}, \sqrt{z}$

We can represent $z = x + iy$ and $\omega = u + iv$ on separate complex planes called z-plane and ω-plane respectively. The relation $\omega = f(z)$ gives the correspondence between the points (x, y) of the z-plane and the points (u, v) of the ω-plane.

Limits :

$$\text{Let } z = x + iy$$

$$\text{Let } z_0 = x_0 + iy_0$$

$$\underset{z \to z_0}{\text{Lt}} \omega = \underset{z \to z_0}{\text{Lt}} f(z) = \omega_0$$

$$\underset{z \to z_0}{\text{Lt}} f(z) = \underset{z \to z_0}{\text{Lt}} (u + iv) \qquad [\because f(z) = u + iv]$$

$$= \underset{\substack{x \to x_0 \\ y \to y_0}}{\text{Lt}} (u + iv) = u_0 + iv_0$$

In symbols, we write

$$\underset{z \to z_0}{\text{Lt}} f(z) = l$$

Continuity of f (z) :

A function $f(z)$ is said to be continuous at $z = z_0$ if

$$\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0).$$

If $f(z)$ is continuous in any region R of the z-plane, if it is continuous at every point of that region.

Derivatives of f (z)

Let $\omega = f(z)$ be a single-valued function of the variable z. The derivative of $f(z)$ is defined as

$$\frac{d\omega}{dz} = f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] \text{ if limits exists.}$$

Partial derivative of u :

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}\right]$$

$$\frac{\partial u}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}\right]$$

Analytic Functions

A single valued function $f(z)$ which possesses a unique derivative with respect to z at all points of a region R is called an analytic function. It is also called a ***Regular function*** or ***Holomorphic*** function.

Singular Point : A point at which an analytic function $f(z)$ ceases to possess a derivative is called a singular point of the function or singularity of $f(z)$.

The necessary and sufficient conditions for the derivative of the function $f(z)$.

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ (C-R equations).

Derivation of Cauchy-Reimann Equations
(Necessary condition for a function $f(z)$ to be analytic)

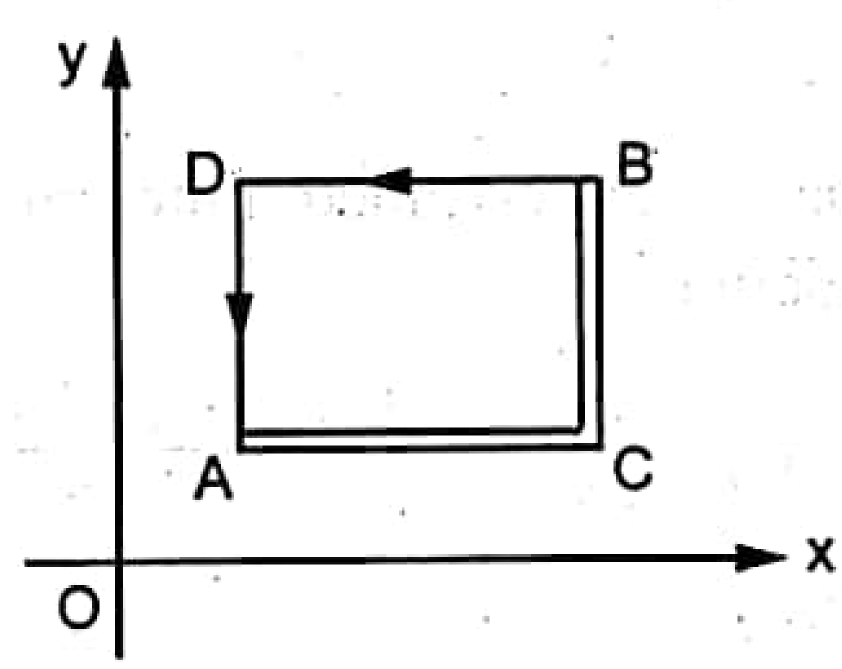
Let us assume that $f(z) = u + iv$ is analytic in a region R of the z-plane.

i.e., $f(z)$ has a derivative everywhere in the region R.

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right]$$

We know $z = x + iy$, $\Delta z = \Delta x + i\,\Delta y$.

Δz approaches to zero along any path in R.



We can write $f'(z)$ as below :

$$f'(z) = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{\text{Lt}} \frac{[\,u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)\,] - [\,u(x,y) + i\,v(x,y)\,]}{\Delta x + i\,\Delta y} \quad \ldots (1)$$

Now we choose the path BCA. Let $\Delta y \to 0$, first and then $\Delta x \to 0$.

$$[\Delta z = \Delta x]$$

$\therefore$ The above equation (1) becomes

$$f'(z) = \underset{\Delta x \to 0}{\text{Lt}} \frac{[u(x+\Delta x, y) + i\,v(x+\Delta x, y)] - [u(x, y) + i\,v(x, y)]}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \underset{\Delta x \to 0}{\text{Lt}} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \ldots (2)$$

Secondly we choose the path BDA. Let $\Delta x \to 0$ first and then $\Delta y \to 0$. ($\Delta z = i\,\Delta y$).

Therefore, the equation (1) becomes,

$$f'(z) = \underset{\Delta y \to 0}{\text{Lt}} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x, y) + i\,v(x, y)]}{i\,\Delta y}$$

$$= \frac{1}{i} \underset{\Delta y \to 0}{\text{Lt}} \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y} + \underset{\Delta y \to 0}{\text{Lt}} \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}$$

$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots (3)$$

From (2) and (3), we have

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots (4)$$

The above equation (4) are called Cauchy-Riemann's equations.

The CR-equations can also be written as

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

Note :

(i) To check the given function is analytic or not, we can use the CR equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(ii) To find the derivative of $f(z)$, we can use

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(iii) To find $f(z)$ or $f'(z)$ in terms of z, we can substitute $x = z$ and $y = 0$ on both sides.

(iv) Recall the following formulae :

$$\begin{aligned}
\sin(ix) &= i\sinh x \\
\cos(ix) &= \cosh x \\
\sin(0) &= 0, \quad \cos(0) = 1 \\
\sinh(0) &= 0, \quad \cosh(0) = 1 \\
\frac{d}{dx}(\sin x) &= \cos x, \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\sinh x) &= +\cosh x \\
\frac{d}{dx}(\cosh x) &= +\sinh x \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\cos(x+y) &= \cos x\cos y - \sin x\sin y
\end{aligned}$$

Example 1 ***Prove that $f(z) = z^2$ is an analytic function.***

Solution : Given :

$$\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 + i^2y^2 + 2ixy \\
&= x^2 - y^2 + i2xy
\end{aligned}$$

$\therefore u = x^2 - y^2$	$v = 2xy$
$\frac{\partial u}{\partial x} = 2x$	$\frac{\partial v}{\partial x} = 2y$
$\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial y} = 2x$

Here $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C.R. equations are satisfied.

$\therefore f(z)$ is analytic function.

Example 2 ***Test the analyticity of $f(z) = e^z$.***

Solution : Given :

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x [\cos y + i \sin y] \\ &= e^x \cos y + i e^x \sin y \end{aligned}$$

Here

$u = e^x \cos y$	$v = e^x \sin y$
$\dfrac{\partial u}{\partial x} = e^x \cos y$	$\dfrac{\partial v}{\partial x} = e^x \sin y$
$\dfrac{\partial u}{\partial y} = -e^x \sin y$	$\dfrac{\partial v}{\partial y} = e^x \cos y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z) = e^z$ is analytic function.

Example 3 ***Test whether the function $f(z) = \cos z$ is analytic or not.***

Solution : Given :

$$\begin{aligned} f(z) &= \cos z \\ &= \cos(x + iy) \\ &= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\ &= \cos(x)\cosh y - \sin(x)\, i \sinh y \\ &= \cos x \cosh y + i(-\sin x \sinh y) \end{aligned}$$

Here

$u = \cos x \cosh y$	$v = -\sin x \sinh y$
$\dfrac{\partial u}{\partial x} = -\sin x \cosh y$	$\dfrac{\partial v}{\partial x} = -\cos x \sinh y$
$\dfrac{\partial u}{\partial y} = \cos x \sinh y$	$\dfrac{\partial v}{\partial y} = -\sin x \cosh y$

Here $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\therefore$ $f(z) = \cos z$ is analytic function.

Example 4 ***Discuss the analyticity of $f(z) = \log z$.***

Solution : We know $\log z = \dfrac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\left(\dfrac{y}{x}\right)$

$$u = \frac{1}{2}\log(x^2+y^2). \qquad v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \qquad = -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The partial derivatives are continuous except at $x = 0$, $y = 0$. CR equations are satisfied.

Its derivative is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{x-iy}{x^2+y^2} = \frac{(x-iy)}{(x-iy)(x+iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z}$$

Hence $f(z) = \log z$ is analytic everywhere except at $z = 0$, (at the origin).

Example 5 ***Prove that $f(z) = \sin z$ is analytic function and hence find the derivative.***

Solution : Given : $f(z) = \sin z = \sin(x+iy)$

$$= \sin(x+iy)$$

$$= \sin(x)\cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$\dfrac{\partial u}{\partial x} = \cos x \cosh y$	$\dfrac{\partial v}{\partial x} = -\sin x \sinh y$
$\dfrac{\partial u}{\partial y} = \sin x \sinh y$	$\dfrac{\partial v}{\partial y} = \cos x \cosh y$

Here CR equations are satisfied.

Consider $\quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$f'(z) = \cos x \cosh y + i(-\sin x \sinh y)$$

To find $f'(z)$ in terms of z, let us substitute $x = z$ and $y = 0$ on both sides,

$$f'(z) = \cos z \cdot 1 + i(-\sin z \cdot 0)$$

$$f'(z) = \cos z$$

Note : Here after we can use this method to find $f(z)$ or $f'(z)$ by substituting $x = z$ and $y = 0$.

$\boxed{\textit{Example 6}}$ ***Prove that $f(z) = z^3$ is analytic function.***

Solution : Given : $f(z) = z^3$

$$= (x + iy)^3$$

$$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$u = x^3 - 3xy^2$	$v = 3x^2y - y^3$
$\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$	$\dfrac{\partial v}{\partial x} = 6xy$
$\dfrac{\partial u}{\partial y} = -6xy$	$\dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore$ CR equations are satisfied.

Hence $f(z)$ is an analytic function.

Example 7 ***Show that $f(z) = |z|^2$ is differentiable only at the origin.***

Solution : Given : $f(z) = |z|^2$

$$= x^2 + y^2 \qquad [\because |z|^2 = z\bar{z} = x^2 + y^2]$$

$$\therefore u = x^2 + y^2, \qquad v = 0$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

Here CR equations are satisfied only when $x = 0$ and $y = 0$.

Note that CR equations are not satisfied for other values. Thus $f(z) = |z|^2$ is differentiable only at the origin.

Example 8 ***Prove that sin (x − iy) is not analytic.***

Solution : $f(z) = \sin(x - iy)$

$$= \sin x \cosh y - i \cos x \sinh y$$

$$u = \sin x \cosh y \qquad v = -\cos x \sinh y$$

$$u_x = \cos x \cosh y \qquad v_x = \sin x \sinh y$$

$$u_y = \sin x \sinh y \qquad v_y = -\cos x \cosh y$$

CR equations are not satisfied.

$\therefore f(z)$ is not analytic.

Example 9 ***Prove that $f(z) = e^{2z}$ is analytic and find its derivative.***

Solution : Given : $f(z) = e^{2z}$

$$= e^{2(x+iy)} = e^{2x} e^{i2y}$$

$$= e^{2x} [\cos 2y + i \sin 2y]$$

$$= e^{2x} \cos 2y + i e^{2x} \sin 2y$$

$$u = 2e^{2x} \cos 2y \qquad v = e^{2x} \sin 2y$$

$$\frac{\partial u}{\partial x} = 2e^{2x} \cos 2y \qquad \frac{\partial v}{\partial x} = 2e^{2x} \sin 2y$$

$$\frac{\partial u}{\partial y} = -2e^{2x} \sin 2y \qquad \frac{\partial v}{\partial y} = 2e^{2x} \cos 2y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

CR equations are satisfied.

$$\text{Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (2\,e^{2x}\cos 2\,y) + i\,(2\,e^{2x}\sin 2\,y)$$

Put $x = z$ and $y = 0$ on both sides,

$$f'(z) = 2\,e^{2z}$$

Example 10 ***Prove that $\frac{d}{dz}$ [sin z] = cos z by using complex variables.***

Solution : Given : $f(z) = \sin(z)$

$$= \sin(x + iy)$$

$$= \sin(x)\cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$ $\qquad$ $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\qquad$ $\frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y,$ $\qquad$ $\frac{\partial v}{\partial y} = \cos x \cosh y$

$\therefore$ CR equations are satisfied.

$$\text{Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cos x \cosh y + i(-\sin x \sinh y)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = \cos z$$

$$\therefore \frac{df(z)}{dz} = \frac{d}{dz}(\sin z) = \cos z$$

Example 11 ***Prove that e^x [cos y + i sin y] is an analytic function. (or) Prove that e^z is an analytic function and hence find its derivative.***

Solution : Given : $f(z) = e^z = e^{x+iy}$

$$= e^x\,e^{iy}$$

$$= e^x[\cos y + i \sin y]$$

Here

$$\begin{array}{rcl|rcl} u &=& e^x \cos y & v &=& e^x \sin y \\ \dfrac{\partial u}{\partial x} &=& e^x \cos y & \dfrac{\partial v}{\partial x} &=& e^x \sin y \\ \dfrac{\partial u}{\partial y} &=& -e^x \sin y & \dfrac{\partial v}{\partial y} &=& e^x \cos y \end{array}$$

$\therefore$ CR equations are satisfied.

$$\text{We know } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i\, e^x \sin y \qquad \dots (1)$$

$$= e^x [\cos y + i \sin y]$$

$$= e^x e^{iy} = e^{x+iy}$$

$$f'(z) = e^z$$

Note : Here we can also use $x = z$, and $y = 0$ in (1) to get $f'(z) = e^z$.

Example 12 ***Test whether f (z) = cosh z is analytic or not.***

Solution : Given : $f(z) = \cosh z$

$$= \cos(iz) = \cos[i(x+iy)]$$

$$= \cos(ix - y)$$

$$= \cos(ix)\cos(y) + \sin(ix)\sin(y)$$

$$= \cosh x \cos y + i \sinh x \sin(y)$$

$$\begin{array}{rcl|rcl} u &=& \cosh x \cos y & v &=& \sinh x \sin y \\ \dfrac{\partial u}{\partial x} &=& \sinh x \cos y & \dfrac{\partial v}{\partial x} &=& \cosh x \sin y \\ \dfrac{\partial u}{\partial y} &=& -\cosh x \sin y & \dfrac{\partial v}{\partial y} &=& \sinh x \cos y \end{array}$$

CR equations are satisfied.

Example 13 ***Prove that f (z) = sinh z is analytic and find its derivative.***

Solution : $f(z) = \sinh z$ $\qquad \sin(ix) = i \sinh x$

$$= \frac{1}{i}\sin[iz] \qquad \sinh x = \frac{1}{i}\sin ix$$

$$= -i \sin [\, i(x+iy)\,]$$
$$= -i\,[\, \sin (ix - y)\,]$$
$$= -i\,[\, \sin (ix) \cos y - \cos (ix) \sin (y)\,]$$
$$= -i\,[\, i \sinh x \cos y - \cosh x \sin y\,]$$
$$f(z) = \sinh x \cos y + i \cosh x \sin y$$

$u = \sinh x \cos y$	$v = \cosh x \sin y$
$\dfrac{\partial u}{\partial x} = \cosh x \cos y$	$\dfrac{\partial v}{\partial x} = \sinh x \sin y$
$\dfrac{\partial u}{\partial y} = -\sinh x \sin y$	$\dfrac{\partial v}{\partial y} = \cosh x \cos y$

CR equations are satisfied.

For derivative of $f(z)$, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cosh x \cos y + i \sinh x \sin y$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = \cosh z$$

Milne-Thomson Method to find $f(z)$

This method can be used to find an analytic function $f(z)$ when u or v is given.

Let us assume that the real part of $f(z)$ is given. Then we can find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right), \text{ using CR equations.}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Put $x = z$ and $y = 0$ on both sides, we get

$$f'(z) = \frac{\partial}{\partial x} u(z, 0) - i \frac{\partial u(z, 0)}{\partial y} \qquad \ldots (1)$$

which is a function of z.

Integrating (1), we get $f(z)$ in terms of z.

Note : If the imaginary part of $f(z)$ is given, we can find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

For this consider

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}, \text{ using CR equations.}$$

Put $x = z$ and $y = 0$ on both sides, we get

$$f'(z) = \frac{\partial v(z, 0)}{\partial y} + i \frac{\partial v(z, 0)}{\partial x}$$

Integrating (2), we get $f(z)$ in terms of z. This method is called Milne-Thomson method.

Method of find $f(z)$ when u is given

$\boxed{\textit{Example 1}}$ ***Find an analytic function $f(z)$ whose real part is given by $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.***

Solution : Given :

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial u}{\partial y} = 0 - 6xy + 0 - 6y + 0$$

$$= -6xy - 6y$$

Consider $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

Here u is given and using CR equations

$$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= [3x^2 - 3y^2 + 6x] + i[6xy + 6y]$$

Put $x = z$ and $y = 0$ on both sides

$$f'(z) = 3z^2 + 6z$$

Integrating, we get

$$f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + C$$

$$f(z) = z^3 + 3z^2 + C, \qquad \text{C is a complex constant.}$$

Example 2 ***Find an analytic function $f(z)$ whose real part is given as $u = y + e^x \cos y$.***

Solution : Given : $u = y + e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = 1 - e^x \sin y$$

Consider $$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial u}{\partial x}\right) + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= e^x \cos y + i(-1 + e^x \sin y)$$

Put $x = z$ and $y = 0$ on both sides,

$$f'(z) = e^z - i$$

Integrating, we get $$f(z) = e^z - iz + C$$

Example 3 ***Find an analytic function whose real part is given by $u = \frac{x}{x^2 + y^2}$.***

Solution : Given : $u = \frac{x}{x^2 + y^2}$

$$u_x = \frac{(x^2 + y^2)\, 1 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{0 - x\,(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

Let $f(z) = u + iv$

$$
\begin{aligned}
f'(z) &= u_x + i\,v_x \\
&= u_x - i\,u_y \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2} - i\,\frac{2\,xy}{(x^2+y^2)^2}
\end{aligned}
$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\frac{z^2}{z^4} = -\frac{1}{z^2}$$

Integrating, we get $\quad f(z) = \frac{1}{z} + C$

Example 4 ***Find $f(z)$ which is analytic, given***

$$u = \frac{1}{2}\log(x^2+y^2).$$

Solution : Given : $\quad u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{2\,x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{2\,y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

Consider

$$
\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \\
&= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) \\
&= \frac{x}{x^2+y^2} + i\left(-\frac{y}{x^2+y^2}\right)
\end{aligned}
$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = \frac{z}{z^2} + i(0) = \frac{1}{z}$$

Integrating, we get $\quad f(z) = \log z + C$

Example 5 ***If $u = \frac{y}{x^2+y^2}$ find an analytic function $f(z)$.***

Solution : Given : $\quad u = \frac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{0 - y\,(2\,x)}{(x^2+y^2)^2} = \frac{-2\,xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\,1 - y\,(2\,y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Consider
$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= \left[\frac{-2\,xy}{(x^2+y^2)^2}\right] + i\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

Put $x = z,\ y = 0$, we get

$$f'(z) = i\left[\frac{-z^2}{z^4}\right] = i\left(-\frac{1}{z^2}\right)$$

Integrating, we get

$$f(z) = i\cdot\frac{1}{z} + C$$

$$= \frac{i}{z} + C \text{ where } C \text{ is complex constant}$$

$\boxed{\textit{Example 6}}$ ***Find an analytic function $f(z) = u + iv$ if u is given by $u = \cos x \cosh y$.***

Solution : Given :

$$u = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y$$

Consider
$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= -\sin x \cosh y + i\,(-\cos x \sinh y)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\sin z + 0$$

Integrating, we get
$$f(z) = \cos z + C$$

Example 7 ***Find an analytic function $f(z)$ whose real part is given by $u = e^{2x}[x \cos 2y - y \sin 2y]$.***

Solution : Given : $u = e^{2x} x \cos 2y - e^{2x} y \sin 2y$

$$\frac{\partial u}{\partial x} = [e^{2x} + 2x\, e^{2x}] \cos 2y - 2\, e^{2x} y \sin 2y$$

$$= e^{2x}[\cos 2y + 2x \cos 2y - 2y \sin 2y]$$

$$\frac{\partial u}{\partial y} = -2\, e^{2x} x \sin 2y - e^{2x}[\sin 2y + 2y \cos 2y]$$

$$= -e^{2x}[2x \sin 2y + \sin 2y + 2y \cos 2y]$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\, \dfrac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= e^{2x}[\cos 2y + 2x \cos 2y - 2y \sin 2y] + i\,[e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y)]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = e^{2z}[1 + 2z] + 0$$

Integrating, we get

$$f(z) = \int (2z + 1)\, e^{2z}\, dz + C$$

For using Bernouli's formula

Put	$u = 2z + 1$		$v = e^{2z}$	
	$u' = 2$		$v_1 = \dfrac{e^{2z}}{2}$	
	$u'' = 0$		$v_2 = \dfrac{e^{2z}}{4}$	

$$\int uv\, dx = uv_1 - u'\, v_2 + u''\, v_3 - \ldots\ldots\ldots$$

$$f(z) = (2z + 1)\, \frac{e^{2z}}{2} - 2\, \frac{e^{2z}}{4} + C$$

$$= z\ e^{2z} + \frac{1}{2}\ e^{2z} - \frac{1}{2}\ e^{2z} + C$$

$$f(z) = z\ e^{2z} + C$$

$\boxed{\textit{Example 8}}$ ***Find the analytic function $f(z) = u + iv$ if***

$$u = e^{-x}\,[\,(x^2 - y^2)\cos y + 2\,xy\sin y\,].$$

Solution : $u_x = e^{-x}\,[\,2\,x\cos y + 2\,y\sin y\,] -$

$$e^{-x}\,[\,(x^2 - y^2)\cos y + 2\,xy\sin y\,].$$

$$u_y = e^{-x}\,[\,-2\,y\cos y - y^2\sin y + 2x\,(y\cos y + \sin y)\,]$$

At $x = z,\ y = 0,$

$$u_x = e^{-z}\,[\,2\,z\,] - e^{-z}\,[\,(z^2)\,] = e^{-z}\,[\,2\,z - z^2\,]$$

$$u_y = e^{-z}\,[\,0\,]$$

$$\therefore\ F'(z) = u_x + i\,v_x$$

$$= u_x + i\,(-u_y)$$

$$F'(z) = e^{-z}\,[\,2\,z - z^2\,]$$

$$F(z) = \int (2\,z - z^2)\,e^{-z}\ dz + C$$

Using Bernouli's formula, we get

$u = 2\,z - z^2$	$v = e^{-z}$
$u' = 2 - 2\,z$	$v_1 = -e^{-z}$
$u'' = -2$	$v_2 = e^{-z}$
$u''' = 0$	$v_3 = -e^{-z}$

$$\therefore \int uv\ dx = u\ v_1 - u'\ v_2 + u''\ v_3 - \ldots\ldots\ldots$$

$$\therefore\ F(z) = -(2\,z - z^2)\,e^{-z} - (2 - 2\,z)\,e^{-z} + 2\,(e^{-z}) + C$$

$$= e^{-z}\,[\,-2\,z + z^2 - 2 + 2\,z + 2\,] + C$$

$$F(z) = z^2\,e^{-z} + C$$

Example 9 ***An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the complex potential function.***

Solution :

$$\text{Let } F(z) = \phi + i\psi$$

$$\text{Given } \phi = 3x^2y - y^3$$

$$\therefore \frac{\partial \phi}{\partial x} = 6xy, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2$$

$$\text{Consider } F'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \phi}{\partial x} + i\left(-\frac{\partial \phi}{\partial y}\right)$$

$$= 6xy - i(3x^2 - 3y^2)$$

Put $x = z$, $y = 0$, we get

$$F'(z) = -i\,3z^2$$

Integrating, we get $\quad F(z) = -iz^3 + C$

Note : If we take $F(z) = \phi + i\psi$ and it is analytic then the CR equations are

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Example 10 ***Find an analytic function $f(z) = u + iv$, whose real part is given by $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.***

Solution : Let $f(z) = u + iv$, and $u_x = v_y$, $u_y = -v_x$

$$\text{Given : } u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x = \frac{(\cosh 2y - \cos 2x)\,2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} \quad [\because \cos^2 2x + \sin^2 2x = 1]$$

$$u_y = \frac{0 - 2\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\text{Consider } f'(z) = u_x + i\,v_x$$

$$= u_x - i\,u_y$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Put $x = z, y = 0$, we get

$$f'(z) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{(1 - \cos 2z)} = -\frac{1}{\sin^2 2z}$$

$$f'(z) = -\operatorname{cosec}^2 2z$$

Integrating, we get

$$f(z) = \cot z + C$$

Note : In the same way we can find $f(z)$, where

$$u = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} \text{ is given.}$$

Method of Finding F(z) = *u* + *iv* when *v* is given

Example 1 ***Find an analytic function $f(z)$ where $v = 2xy$.***

Solution : Given : $v = 2xy$

$$\frac{\partial v}{\partial x} = 2y \text{ and } \frac{\partial v}{\partial y} = 2x$$

We know $f'(z) = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}$ [Here u is not given]

$$= \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x} \qquad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = 2z$$

Integrating, we get

$$f(z) = z^2 + C$$

$\boxed{\textit{Example 2}}$ ***Find an analytic function f (z) whose imaginary part is given by $v = e^x \sin y$.***

Solution : Given :

$$v = e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \text{ and } \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \text{ Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i\, e^x \sin y$$

$$= e^x [\cos y + i \sin y]$$

Put $x = z$ and $y = 0$ on both sides,

$$f'(z) = e^z$$

Integrating, we get $\quad f(z) = e^z + C$

$\boxed{\textit{Example 3}}$ ***If $v = -\sin x \sinh y$, find a function foz which is regular.***

Solution : Given :

$$v = -\sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$$

$$\text{Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= (-\sin x \cosh y) + i(-\cos x \sinh y)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\sin z$$

Integrating, we get $\quad f(z) = \cos z + C$

Example 4 ***Find an analytic function $f(z)$ whose imaginary part is $v = x^3 - 3xy^2 + 2x + 1$.***

Solution : Given :

$$
\begin{aligned}
v &= x^3 - 3xy^2 + 2x + 1 \\
v_x &= 3x^2 - 3y^2 + 2 \\
v_y &= -6xy \\
v_x(z, 0) &= 3z^2 + 2 \\
v_y(z, 0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Consider } F'(z) &= u_x + i v_x \\
&= v_y + i v_x
\end{aligned}
$$

Putting $x = z, y = 0$, we get

$$
\begin{aligned}
F'(z) &= v_y(z, 0) + i v_x(z, 0) \\
&= 0 + i(3z^2 + 2)
\end{aligned}
$$

$$
\begin{aligned}
\text{Integrating, we get } F(z) &= i \int (3z^2 + 2)\, dz + C \\
&= i[z^3 + 2z] + C
\end{aligned}
$$

Example 5 ***If $u = \dfrac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$, then find the corresponding analytic function $f(z)$.***

[***Ans :*** $f(z) = \sec z + C$]

$\boxed{\textit{Example 6}}$ ***Find a regular $f(z)$ whose imaginary part is given $v = e^{-x}[x \cos y + y \sin y]$.***

Solution : Given :

$$
\begin{aligned}
v &= e^{-x}[x \cos y + y \sin y] \\
v_x &= e^{-x}[\cos y] - e^{-x}[x \cos y + y \sin y] \\
&= e^{-x}[\cos y - x \cos y - y \sin y] \\
v_y &= e^{-x}[-x \sin y + y \cdot \cos y + \sin y]
\end{aligned}
$$

$$
\begin{aligned}
\text{Consider } F'(z) &= u_x + i\, v_x \\
&= v_y + i\, v_x
\end{aligned}
$$

At $x = z,\ y = 0$, we get

$$
\begin{aligned}
F'(z) &= v_y(z, 0) + i\, v_x(z, 0) \\
&= 0 + i\, e^{-z}[1 - z]
\end{aligned}
$$

Integrating, we get

$$
\begin{aligned}
F(z) &= i \int (1 - z)\, e^{-z}\, dz + C \\
&= i[-(1 - z)\, e^{-z} - (-1)\, e^{-z}] + C \\
&= i[-1\, e^{-z} + z\, e^{-z} + e^{-z}] + C \\
F(z) &= i[z\, e^{-z}] + C
\end{aligned}
$$

$\boxed{\textit{Example 7}}$ ***Find the regular function $f(z)$ whose imaginary part is given by $v = e^{-x}[x \sin y - y \cos y]$.***

Solution : Given :

$$
\begin{aligned}
v &= e^{-x}[x \sin y - y \cos y] \\
\frac{\partial v}{\partial x} &= e^{-x}[1 \cdot \sin y] - e^{-x}[x \sin y - y \cos y] \\
&= e^{-x}[\sin y - x \sin y + y \cos y] \\
\frac{\partial v}{\partial y} &= e^{-x}[x \cos y - \cos y + y \sin y]
\end{aligned}
$$

$$
\begin{aligned}
\text{Consider } f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\
&= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}
\end{aligned}
$$

$$= e^{-x}[x \cos y - \cos y - y \sin y] + i\, e^{-x}[\sin y - x \sin y + y \cos y]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = e^{-z}[z-1] + i\, e^{-z}[0]$$
$$= (z-1)\, e^{-z}$$

Integrating, we get

$$f(z) = \int (z-1)\, e^{-z}\, dz$$
$$= -(z-1)\, e^{-z} - e^{-z} \cdot 1 + C$$
$$= -z\, e^{-z} + e^{-z} - e^{-z} + C$$
$$f(z) = -z\, e^{-z} + C$$

u	$=$	$z-1,$	v	$=$	e^{-z}
u'	$=$	$1,$	v_1	$=$	$-e^{-z}$
u''	$=$	$0,$	v_2	$=$	e^{-z}

Example 8 ***Find the analytic function whose imaginary part is $e^{x^2-y^2} \sin(2xy)$.***

Solution :

Given : $v = e^{x^2-y^2} \sin(2xy)$

$$\frac{\partial v}{\partial x} = e^{x^2-y^2}(2x) \sin(2xy) + e^{x^2-y^2} \cos 2xy\,(2y)$$

$$\frac{\partial v}{\partial y} = e^{x^2-y^2}(-2y) \sin 2xy + e^{x^2-y^2} \cos(2xy)(2x)$$

We know $f(z) = u + iv$

$$f'(z) = u_x + i\, v_x$$
$$= v_y + i\, v_x \qquad [\because u_x = v_y]$$
$$= 2\, e^{x^2-y^2}[-y \sin 2xy + x \cos 2xy]$$
$$+ i\, 2\, e^{x^2-y^2}[x \sin 2xy + y \cos 2xy]$$

Put $x = z$ and $y = 0$,

$$f'(z) = 2\, e^{z^2}[0 + z] + i\, 2\, e^{z^2}[0]$$
$$f'(z) = 2z\, e^{z^2}$$

Integrating $\quad f(z) = \int 2z\, e^{z^2}\, dz + C$

Put $z^2 = t$, $\therefore\ 2z\, dz = dt$

$$\therefore f(z) = \int e^t\, dt + C$$

$$\therefore f(z) = e^t + C$$

$$f(z) = e^{z^2} + C$$

$\boxed{\textit{Example 9}}$ ***Construct the analytic function whose imaginary part is $e^{-x}[x\cos y + y\sin y]$ and which equals 1 at the origin.***

Solution : Given :. $\quad v = e^{-x}[x\cos y + y\sin y]$

$$v_x = e^{-x}[1\cdot\cos y + 0] - e^{-x}[x\cos y + y\sin y]$$

$$v_y = e^{-x}[-x\sin y + 1\cdot\sin y + y\cos y]$$

Consider $F'(z) = u_x + i\, v_x$

$$= v_y + i\, v_x$$

$$= e^{-x}[-x\sin y + \sin y + y\cos y] + i\, e^{-x}[\cos y - x\cos y - y\sin y]$$

Put $x = z$ and $y = 0$, we get

$$F'(z) = e^{-z}[0] + i\, e^{-z}[1 - z]$$

Integrating, we get $\quad F(z) = i\int (1-z)\, e^{-z}\, dz + C$

Using integration by parts, we get

$$u = 1 - z, \qquad dv = e^{-z}\, dz$$

$$du = -dz, \qquad v = -e^{-z}$$

$$F(z) = i\left[-(1-z)\, e^{-z} - \int -e^{-z}(-dz)\right] + C$$

$$= i[-(1-z)\, e^{-z} + e^{-z}] + C$$

$$F(z) = i z\, e^{-z} + C$$

Given $F(0) = 1 \Rightarrow C = 1$

$$\therefore f(z) = i z\, e^{-z} + 1$$

Example 10 *If $v = e^x\,[x \sin y + y \cos y]$ is an imaginary part of an analytic function $f(z)$, find $f(z)$ in terms of z.*

Solution : Given :

$$v = e^x (x \sin y + y \cos y)$$

$$v_x = e^x (x \sin y + y \cos y) + e^x (\sin y)$$

$$= e^x (x \sin y + y \cos y + \sin y)$$

$$v_y = e^x (x \cos y + \cos y - y \sin y)$$

Consider

$$f'(z) = u_x + i\,v_x$$

$$= v_y + i\,v_x$$

$$= e^x (x \cos y + \cos y - y \sin y) + i\,e^x (x \sin y + y \cos y + \sin y)$$

Put $x = z,\ y = 0$ on both sides,

$$f'(z) = e^z (z + 1)$$

Integrating, we get

$$f(z) = \int (z + 1)\,e^z\,dz$$

$$= (z + 1)\,e^z - e^z + C$$

$$f(z) = z\,e^z + C$$

Method of finding $f(z)$ when $u - v$ is given

Let $f(z) = u + iv$ and is an analytic function.

$$f(z) = u + iv \quad \ldots (1)$$

$$i\,f(z) = iu - v \quad \ldots (2)$$

Adding (i) and (ii), we get

$$(1 + i)\,f(z) = (u - v) + i\,(u + v) \quad \ldots (3)$$

Let $U = u - v$, $V = u + v$ and $F(z) = (1 + i)\,f(z)$.

Then (iii) becomes,

$$F(z) = U + iV \quad \ldots (4)$$

If $u - v$ is given in the problem, then

(a) Substitute $u - v = U$. (Now U is known)

(b) Find $F(z)$ by usual method.

(c) Equate $F(z) = (1+i) f(z)$

$$\therefore f(z) = \frac{1}{1+i} F(z)$$

This is a procedure to find $f(z)$ if $u - v$ is given.

Note : If $u + v$ is given in the problem, we can use the similar method as above.

$$\text{Let} \quad f(z) = u + iv \qquad \dots (1)$$

$$i f(z) = iu - v \qquad \dots (2)$$

Adding (1) and (2),

$$(1+i) f(z) = (u - v) + i(u + v)$$

$$\text{i.e.,} \quad F(z) = U + iV$$

Here $u + v$ is given. Then

(1) Substitute $u + v = V$ [V is known]

(2) Find $F(z)$ as usual method.

(3) Equate $F(z) = (1+i) f(z)$

$$\therefore f(z) = \frac{1}{1+i} F(z)$$

Note : If $F(z) = U + iV$ is analytic, then CR equations are

$$U_x = V_y$$

$$U_y = -V_x$$

Example 11 ***If $u - v = e^x [\cos y - \sin y]$, find the corresponding analytic function $f(z) = u + iv$.***

Solution : Consider $f(z) = u + iv \qquad \dots$ (i)

$$i f(z) = iu - v \qquad \dots (ii)$$

Adding (i) and (ii),

$$(1+i) f(z) = (u - v) + i(u + v)$$

$$\text{i.e.,} \quad F(z) = U + iV$$

Here $U = u - v = e^x [\cos y - \sin y]$ is given

$$U_x = e^x [\cos y - \sin y]$$

$$U_y = e^x [-\sin y - \cos y]$$

Consider $\quad F'(z) = U_x + i V_x$

$= U_x + i(-U_y)$

$= e^x [\cos y - \sin y] + i e^x [\sin y + \cos y]$

Put $x = z,\ y = 0$, we get

$$F'(z) = e^z + i\ e^z$$

$$= (1+i)\, e^z$$

Integrating, we get

$$F(z) = (1+i)\, e^z + C$$

i.e., $\quad (1+i) f(z) = (1+i)\, e^z + C$

$$f(z) = e^z + C_1$$

Example 12 ***Find an analytic function $f(z)$ if given $u + v = x^2 - y^2 + 2xy$.***

Solution : Consider $\quad f(z) = u + iv \quad \ldots$ (i)

$i f(z) = iu - v \quad \ldots$ (ii)

Adding $\quad (1+i) f(z) = (u - v) + i(u + v) \quad \ldots$ (iii)

(iii) can be written as $\quad F(z) = U + iV$

where $u - v = U,\ u + v = V,\ (1+i) f(z) = F(z)$.

Given $\quad V = u + v = x^2 - y^2 + 2xy$

$V_x = 2x + 2y$

$V_y = -2y + 2x$

Consider $\quad F'(z) = U_x + i\ V_x$

$= V_y + i\ V_x$

$= (-2y + 2x) + i(2x + 2y)$

Put $x = z,\ y = 0$ on both sides,

$$F'(z) = 2z + i\,2z$$

$$= 2(1+i)\, z$$

Integrating $\quad F(z) = (1+i)\, z^2 + c$

i.e., $\quad (1+i) f(z) = (1+i)\, z^2 + c$

$$\therefore\ f(z) = z^2 + \frac{c}{1+i}$$

$$f(z) = z^2 + c_1$$

Example 13 ***Find an analytic function***

$f(z) = u + iv$ if $u - v = (x-y)(x^2 + 4xy + y^2)$.

Solution : Consider

$$f(z) = u + iv$$

$$i\,f(z) = iu - v$$

$$(1+i)\,f(z) = (u-v) + i\,(u+v)$$

$$F(z) = U + iV$$

Here let

$$\begin{aligned} U &= (x-y)(x^2+4xy+y^2) \\ &= x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3 \\ &= x^3 + 3x^2y - 3xy^2 - y^3 \\ U_x &= 3x^2 + 6xy - 3y^2 \\ U_y &= 3x^2 - 6xy - 3y^2 \\ F'(z) &= U_x + iV_x \\ &= U_x - iU_y \\ &= (3x^2 + 6xy - 3y^2) - i\,(3x^2 - 6xy - 3y^2) \end{aligned}$$

Put $x = z,\ y = 0$ on both sides,

$$\begin{aligned} F'(z) &= 3z^2 - i\,3z^2 \\ &= 3(1-i)z^2 \end{aligned}$$

Integrating

$$F(z) = (1-i)z^3 + c$$

i.e.,

$$(1+i)\,f(z) = (1-i)z^3 + c$$

$$\therefore\ f(z) = \left(\frac{1-i}{1+i}\right) z^3 + \frac{c}{(1+i)}$$

Now

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1-i-i-1}{1+1}$$

$$= \frac{-2i}{2} = -i$$

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = c_1$$

$$\therefore\ f(z) = -iz^3 + c_1$$

Harmonic Function

A function $f(x, y)$ is called Harmonic if it satisfies Laplace equation

$$f_{xx} + f_{yy} = 0$$

i.e., The solution of Laplace equation is called Harmonic function.

Example 1 ***A function $f = x^2 - y^2$ is harmonic.***

Solution : Given :

$$\begin{aligned} f &= x^2 - y^2 \\ f_x &= 2x \\ f_{xx} &= 2 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} f &= x^2 - y^2 \\ f_y &= -2y \\ f_{yy} &= -2 \end{aligned}$$

$$\therefore\ f_{xx} + f_{yy} = 2 + (-2) = 0$$

Example 2 ***A function $f = \frac{1}{2} \log (x^2 + y^2)$ is harmonic.***

Solution : Given :

$$\begin{aligned} f &= \frac{1}{2} \log (x^2 + y^2) \\ f_x &= \frac{1}{2} \frac{1}{x^2 + y^2} (2x) \\ &= \frac{x}{x^2 + y^2} \\ f_y &= \frac{y}{x^2 + y^2} \\ f_{xx} &= \frac{(x^2 + y^2) \cdot 1 - x\,(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ f_{yy} &= \frac{(x^2 + y^2)\, 1 - y\,(2y)}{(x^2 + y^2)^2} \\ &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \end{aligned}$$

$\therefore\ f_{xx} + f_{yy} = 0 \Rightarrow f$ is harmonic function.

Example 3 ***Prove that $f = e^x \sin y$ satisfies Laplace equation.***

Solution : Given : $\quad f = e^x \sin y \qquad\qquad f_y = e^x \cos y$

$$f_{xx} = e^x \sin y \qquad f_{yy} = -e^x \sin y$$

$$f_{xx} + f_{yy} = e^x \sin y - e^x \sin y = 0$$

$\therefore$ f is harmonic function which satisfies Laplace equation.

$\boxed{\textit{Example 4}}$ ***Prove that the real part of an analytic function satisfies Laplace equation (Harmonic function).***

Solution : Proof : Given : $f(z) = u + iv$ is analytic.

$\therefore$ It satisfies CR equations.

$$U_x = V_y \qquad \ldots \text{(i)}$$

$$U_y = -V_x \qquad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to x,

$$u_{xx} = v_{xy}$$

Differentiating (ii) partially with respect to y,

$$u_{yy} = -v_{yx}$$

Adding the above two equations, we get

$$u_{xx} + u_{yy} = 0$$

$\Rightarrow$ The real part u satisfies Laplace equation.

i.e., u is a harmonic function.

Note : If $f(z)$ is analytic function, then u is a harmonic function.

$\boxed{\textit{Example 5}}$ ***Prove that an imaginary part of an analytic function satisfies Laplace equation (harmonic function).***

Solution : Given : $f(z) = u + iv$ is an analytic function.

$$\therefore\ u_x = v_y \qquad \ldots \text{(i)}$$

$$u_y = -v_x \qquad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to y, we get

$$u_{yx} = v_{yy}$$

Differentiating (ii) partially with respect to x, we get

$$u_{xy} = -v_{xx}$$

$$-u_{xy} = v_{xx}$$

Adding the above two equation, we get

$$v_{xx} + v_{yy} = 0$$

v satisfies Laplace equation.

$\Rightarrow$ v is a harmonic function.

Note : If $f(z)$ is analytic then v is harmonic. The real and imaginary parts of an analytic functions are harmonic.

$\boxed{\text{Example 6}}$ ***Prove that*** $u = \frac{1}{2} \log (x^2 + y^2)$ ***is a real part of an analytic function*** $f(z)$.

Solution : Given :

$$u = \frac{1}{2} \log (x^2 + y^2)$$

$$u_x = \frac{x}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2)\, 1 - x\, (2\, x)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{y}{x^2 + y^2}$$

$$u_{yy} = \frac{(x^2 + y^2)\, 1 - y\, (2\, y)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = 0$$

u is a harmonic function.

u is a real part of an analytic function $f(z)$.

$\boxed{\text{Example 7}}$ ***Prove that*** $e^x \sin y$ ***is an imaginary part of an analytic function*** $f(z)$.

Solution : We know that the real and imaginary parts of an analytic functions are harmonic.

Given :

$$v = e^x \sin y$$

$$\therefore v_x = e^x \sin y, \qquad v_y = e^x \cos y$$

$$v_{xx} = e^x \sin y, \qquad v_{yy} = -e^x \sin y$$

$$\therefore v_{xx} + v_{yy} = 0$$

v is harmonic function.

$\therefore$ v is an imaginary part of an analytic function.

$\boxed{\text{Example 8}}$ ***Check the function $x^2 + y^2$ is a real part of an analytic function $f(z)$ or not.***

Solution : Let

$$u = x^2 + y^2$$

$$u_x = 2x, \qquad u_y = 2y$$

$$u_{xx} = 2 \qquad u_{yy} = 2$$

$$u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$$

u is not harmonic.

$\therefore$ u is not a real part of analytic function.

$\boxed{\text{Example 9}}$ ***Prove that an analytic function with constant real part is constant.***

Solution : Given : $f(z) = u + iv$ is an analytic function.

Also given u = constant (c_1)

$$u_x = 0$$

$$u_y = 0$$

Since $f(z)$ is analytic, then it satisfies

$$u_x = v_y \text{ and } u_y = -v_x$$

$$v_y = 0, \quad v_x = 0 \qquad [\because u_x = u_y = 0]$$

$\Rightarrow$ V is constant (c_2).

$$\therefore f(z) = u + iv$$

$$= c_1 + i c_2$$

$$= \text{constant}$$

$\Rightarrow$ If u is constant then $f(z)$ is constant.

$\boxed{\textit{Example 10}}$ ***Prove that an analytic function with constant imaginary part is constant.***

Solution : Proof : Given : $\quad v = \text{constant } (c_1)$

$$\therefore v_x = 0, \quad v_y = 0$$

Since $f(z) = u + iv$ is analytic, it satisfies

$$u_x = v_y \text{ and } u_y = -v_x$$

$$\Rightarrow u_x = 0, \quad u_y = 0 \qquad [\because v_x = v_y = 0]$$

$\Rightarrow$ u is constant (c_2)

$$\therefore f(z) = u + iv$$
$$= c_2 + i\,c_1 = \text{constant}$$

$\therefore$ If v is constant then $f(z)$ is constant.

$\boxed{\textit{Example 11}}$ ***Prove that an analytic function with constant modulus is constant.***

Solution : Proof : Consider $f(z) = u + iv = u(x, y) + i\,v(x, y)$

$$|f(z)| = \sqrt{u^2 + v^2}$$

Given that $\sqrt{u^2 + v^2} = \text{constant } (c)$

Squaring $u^2 + v^2 = c^2 \qquad \ldots \text{(i)}$

Differentiating (i) partially with respect to x,

$$2\,u\,u_x + 2\,v\,v_x = 0$$

$$u\,u_x + v\,v_x = 0 \qquad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to y,

$$2\,u\,u_y + 2\,v\,v_y = 0$$

$$u\,u_y + v\,v_y = 0$$

$$u\,(-v_x) + v\,u_x = 0 \qquad [\because \text{CR equation}]$$

$$v\,u_x + (-u)\,v_x = 0 \qquad \ldots \text{(iii)}$$

For solving u_x and v_x from (ii) and (iii),

$$\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -u^2 - v^2 = -(u^2 + v^2)$$

$$= -c^2, \text{ using (i)}$$

$$\neq 0$$

$$\therefore u_x = 0 \text{ and } v_x = 0$$

Since $f(z)$ is analytic, it satisfies

$$u_x = v_y \text{ and } u_y = -v_x$$

$$\therefore v_y = 0 \text{ and } u_y = 0 \qquad [\because u_x = 0,\ v_x = 0]$$

$\Rightarrow u_x = 0,\ u_y = 0,\ v_x = 0,\ v_y = 0.$

$\Rightarrow u = \text{constant } (c_1)$ and $v = \text{constant } (c_2)$

$$\therefore f(z) = c_1 + i c_2$$

$$= \text{constant}$$

$\boxed{\textit{Example 12}}$ ***Prove that*** $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$

if $f(z)$ is a regular function.

Solution : Proof : We know that $f(z) = u + iv$

$$\text{Then } |f(z)|^2 = u^2 + v^2$$

$$\text{Also } f'(z) = u_x + i v_x$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

Given $f(z) = u + iv$ is analytic, therefore

$$u_x = v_y,\ u_y = -v_x \text{ and}$$

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

Now consider $\qquad |f(z)|^2 = u^2 + v^2 \qquad \ldots (1)$

Differentiating (1) partially with respect to x,

$$\frac{\partial}{\partial x} |f(z)|^2 = 2 u u_x + 2 v v_x$$

$$\frac{\partial^2}{\partial x^2} |f(z)|^2 = 2 [u u_{xx} + u_x u_x + v v_{xx} + v_x v_x]$$

$$= 2 [u u_{xx} + u_x^2 + v v_{xx} + v_x^2] \qquad \ldots (2)$$

Similarly differentiating (1) partially with respect to y twice

$$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2[u\,u_{yy} + u_y^2 + v\,v_{yy} + v_y^2] \quad \ldots (3)$$

Adding (2) and (3), using Laplace equation,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2[u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

$$\because u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$$

Using CR equations on RHS, we get

$$= 2[u_x^2 + v_x^2 + v_x^2 + u_x^2]$$

$$= 4[u_x^2 + v_x^2]$$

$$= 4|f(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Example 13 ***If $f(z)$ is a holomorphic function of z, show that***

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Solution : Let $f(z) = u + iv = u(x,y) + i\,v(x,y)$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$= (u^2+v^2)^{1/2}$$

$$\frac{\partial}{\partial x}[|f(z)|] = \frac{1}{2}(u^2+v^2)^{-1/2}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]$$

$$= \frac{1}{(u^2+v^2)^{1/2}}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]$$

$$\left\{\frac{\partial}{\partial x}[|f(z)|]\right\}^2 = \frac{1}{u^2+v^2}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]^2$$

$$= \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right]$$

Similarly

$$\left\{\frac{\partial}{\partial y}[|f(z)|]\right\}^2 = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2\,uv\,\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$$

Adding, we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 =$$

$$\frac{1}{u^2+v^2}\left\{u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right.$$

$$\left. + 2\,uv\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\right\}$$

Using CR equations, we get

$$= \frac{1}{u^2+v^2}\left[u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \right.$$

$$\left. v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\,uv\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right]\right]$$

$$= \frac{1}{u^2+v^2}\left[(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$= |f'(z)|^2$$

Orthogonal System

Orthogonal curves : Two curves are said to be orthogonal to each other, if they intersect at right angles at each of their points of intersection.

If m_1 and m_2 are slopes of the two curves, then $m_1 m_2 = -1$.

Let $f(z) = u + iv$ is an analytic function, then the family of curves $u = c_1$ and $v = c_2$ are orthogonal. The real and imaginary parts of an analytic function forms an orthogonal system.

Example 1 ***If $f(z) = u + iv$ is analytic, prove that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal.***

Solution : Given : $u(x, y) = c_1$ ... (1)

Differentiating partially with respect to x,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

Let m_1 is the slope of the curve $u = c_1$.

$$\therefore m_1 = -\frac{u_x}{u_y} \quad \text{... (2)}$$

Also $v(x, y) = c_2$ ... (3)

Differentiating partially with respect to x,

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

Since $f(z)$ is analytic, it satisfies CR equations.

$$\therefore \frac{dy}{dx} = \frac{-\left(-\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}$$

Let m_2 is the slope of the curve $v = c_2$.

$$\therefore m_2 = \frac{u_y}{u_x}$$

Then $m_1 m_2 = -1$.

The family of curves are orthogonal.

$\boxed{\textit{Example 2}}$ ***Consider the analytic function*** $f(z) = z^2 = (x^2 - y^2) + i\,(2\,xy)$.

Solution : Let $x^2 - y^2 = c_1$; $2\,xy = c_2$.

Differentiating with respect to x on both sides,

$$2\,x - 2\,y\,\frac{dy}{dx} = 0\,;2\left[\,x\,\frac{dy}{dx} + y \cdot 1\right] = 0$$

$$x - y\,\frac{dy}{dx} = 0\,; \qquad x\,\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{x}{y}\,; \qquad \frac{dy}{dx} = -\frac{y}{x}$$

$$m_1 = \frac{x}{y}\,; \qquad m_2 = -\frac{y}{x}$$

Then $\quad m_1\,m_2 = \left(\frac{x}{y}\right)\left(\frac{-y}{x}\right) = 1$

$\boxed{\textit{Example 3}}$ ***Consider an analytic function*** $f(z) = e^z$.

$$e^z = e^x \cos y + i\,e^x \sin y$$

Solution : $\quad$ Let $u = c_1$,

i.e., $\quad e^x \cos y = c_1$

$$e^x \cos y + e^x\,(-\sin y)\,\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = \cot y$$

$$m_1 = \cot y$$

Let $v = c_2$

$$e^x \sin y = c_2$$

$$e^x \sin y + e^x \cos y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\tan y$$

$$m_2 = -\tan y$$

$$\therefore\ m_1\,m_2 = (\cot y)\,(-\tan y) = -1$$

$\boxed{\textit{Example 4}}$ ***If $f(z) = \sin z$ is an analytic function, prove that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.***

Solution : Given :

$$
\begin{aligned}
f(z) &= \sin z = \sin(x + iy) \\
&= \sin x \cos(iy) + \cos(x)\sin(iy) \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned}
$$

Consider

$$
\begin{aligned}
u(x, y) &= c_1 \\
\sin x \cosh y &= c_1 \qquad \ldots (1)
\end{aligned}
$$

Differentiating (1) partially with respect to x, we get

$$\sin x \sinh y \frac{dy}{dx} + \cos x \cosh y = 0$$

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{\cos x \cosh y}{\sin x \sinh y} \\
m_1 &= -\cot x \coth y
\end{aligned}
$$

Again consider

$$
\begin{aligned}
v(x, y) &= c_2 \\
\cos x \sinh y &= c_2 \qquad \ldots (2)
\end{aligned}
$$

Differentiating partially with respect to x, we get

$$-\sin x \sinh y + \cos x \cosh y \frac{dy}{dx} = 0$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x \sinh y}{\cos x \cosh y} \\
m_2 &= \tan x \tanh y \\
\therefore m_1 m_2 &= -1 \\
u(x, y) &= c_1 \text{ and } v(x, y) = c_2 \text{ are orthogonal.}
\end{aligned}
$$

Note : For any analytic function $F(z) = u + iv$, the family of curves $u = c_1$, $v = c_2$ forms an orthogonal system.

HARMONIC CONJUGATES

We know that the real and imaginary parts of an analytic function $f(z) = u + iv$ are Harmonic Functions (satisfies Laplace equation). Here u and v are called Harmonic conjugates. *i.e.*, u is harmonic conjugate to v and v is harmonic conjugate to u.

Result (i) : If $f(z) = u + iv$ is analytic then u and v are harmonic functions.

For example, $f(z) = x^2 - y^2 + i\,2\,xy = z^2$ is analytic and $u = x^2 - y^2$, $v = 2\,xy$ are harmonic.

Result (ii) : If u and v are harmonic, then $f(z) = u + iv$ need not be harmonic. For example, $u = x^2 - y^2$, $v = e^x \sin y$ are harmonic but $u + iv = f(z)$ is not analytic.

Result (iii) : Since u is a function of x and y,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Similarly we can write $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Method of Finding Harmonic Conjugates

Given $f(z) = u + iv$ is analytic function, $u(x, y)$ is the real part of $f(z)$ and harmonic.

$$\therefore\ u_x = V_y,\quad u_y = -v_x,\quad u_{xx} + u_{yy} = 0.$$

Since v is a Harmonic conjugate and a function of x and y, we write,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using CR equations, we have

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Integrating, we get $$v = \int -\frac{\partial u}{\partial y}\, dx + \int \frac{\partial u}{\partial x}\, dy + \text{constant}$$

$$\text{Let } M = -\frac{\partial u}{\partial y},\quad N = \frac{\partial u}{\partial x} \qquad \ldots \text{(i)}$$

$$V = \int M\, dx + \int N\, dy + C \qquad \ldots (1)$$

(i) Integrate M with respect to x by treating y as a constant.

(ii) Integrate N with respect to y by deleting the terms containing x.

In the same way we can find u if v is given.

$$u = \int M\, dx - \int N\, dy$$

(i) Integrate M with respect to x by treating y as a constant.

(ii) Integrate the second integral N with respect to y by deleting the terms which contains x.

This method is explained clearly by the following examples.

Example 1 ***If $u = x^2 - y^2$ is a real part of an analytic function $f(z)$, find its harmonic conjugate v.***

Solution : Given : $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y$$

$$\text{Consider } dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy$$

$$= -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy = 2y\, dx + 2x\, dy$$

Integrating $\displaystyle\int dv = 2y\int dx + 0$

$$v = 2yx + c$$

[$\because$ By deleting the term containing x in the second integral]

$$v = 2xy + c, \text{ where } c \text{ is a constant.}$$

Example 2 ***Prove that $u = e^x \cos y$ is a harmonic function and find its harmonic conjugate.***

Solution : Given : $u = e^x \cos y$

$$u_x = e^x \cos y, \qquad u_y = -e^x \sin y$$

$$u_{xx} = e^x \cos y, \qquad u_{yy} = -e^x \cos y$$

$$\therefore\ u_{xx} + u_{yy} = 0$$

$\Rightarrow$ u is a harmonic function.

To find its harmonic conjugate, consider

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$

Integrating on both sides, we get

$$v = \sin y\int e^x\, dx + 0 \quad \text{[by deleting the term containing } x\text{]}$$

$$v = e^x \sin y + c$$

Example 3 ***If $u = \frac{1}{2}\log(x^2 + y^2)$ is a real part of an analytic function $f(z)$, find v.***

Solution : Given : $u = \frac{1}{2}\log(x^2 + y^2)$

$$u_x = \frac{x}{x^2+y^2}, \quad u_y = \frac{y}{x^2+y^2}$$

Consider $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$= \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$= \frac{x\,dy - y\,dx}{x^2\left[1 + \left(\frac{y}{x}\right)^2\right]}$$

$$= \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} \qquad \left[\because \int \frac{1}{x^2 + a^2} dx = \tan^{-1}\left(\frac{x}{a}\right)\right]$$

Integrating, we get $v = \tan^{-1}\left(\frac{y}{x}\right) + c$

$\boxed{\textit{Example 4}}$ ***Show that the function $u = x^4 - 6x^2y^2 + y^4$ is harmonic and find its harmonic conjugate.***

Solution : Given :

$$u = x^4 - 6x^2y^2 + y^4$$

$$u_x = 4x^3 - 12xy^2$$

$$u_{xx} = 12x^2 - 12y^2$$

$$u_y = -12x^2y + 4y^3$$

$$u_{yy} = -12x^2 + 12y^2$$

$$\therefore u_{xx} + u_{yy} = 0$$

u is harmonic function.

Consider $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= (12x^2y - 4y^3)\,dx + (4x^3 - 12xy^2)\,dy$$

Integrating, $$v = 12\,y \int x^2\,dx - 4\,y^3 \int dx + 0$$

$$= 12\,y\,\frac{x^3}{3} - 4\,y^3\,x + c$$

$$= 4\,x^3y - 4\,xy^3 + c$$

$\boxed{\text{Example 5}}$ ***If $u = 3\,x^2y + 2\,x^2 - y^3 - 2\,y^2$ is the real part of an analytic function $f(z)$, find v.***

Solution : Given : $$u = 3\,x^2y + 2\,x^2 - y^3 - 2\,y^2$$

$$u_x = 6\,xy + 4\,x, \quad u_y = 3\,x^2 - 3\,y^2 - 4\,y$$

Consider $$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

$$= (-3\,x^2 + \cancel{3\,y^2} + 4\,y)\,dx + (6\,xy + 4\,x)\,dy$$

Integrating, we get $$v = -3\,\frac{x^3}{3} + 3\,y^2\,x + 4\,yx + c$$

$$v = -x^3 + 3\,xy^2 + 4\,xy + c$$

$\boxed{\text{Example 6}}$ ***If $v = 2\,xy$ is the imaginary part of an analytic function $f(z)$, find its conjugate.***

Solution : Given :. $v = 2\,xy\,; \quad \therefore\ v_x = 2\,y\,; \quad v_y = 2\,x$

Consider $$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= \frac{\partial v}{\partial y}\,dx + \left(-\frac{\partial v}{\partial x}\right)dy$$

(y – constant) (delete x terms)

$$du = 2\,x\,dx - 2\,y\,dy$$

Integrating on both sides, we get

$$u = (x^2 - y^2) + c$$

$\boxed{\text{Example 7}}$ ***If $e^x \sin y$ is an imaginary part of a regular function $f(z)$, find u.***

Solution : Given : $$v = e^x \sin y$$

$$v_x = e^x \sin y, \quad v_y = e^x \cos y$$

$$\text{Consider} \quad du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= \frac{\partial v}{\partial y}\,dx + \left(-\frac{\partial v}{\partial x}\right)dy$$

$$du = e^x \cos y\,dx - e^x \sin y\,dy$$

$$\text{Integrating} \qquad \int du = \cos y \int e^x\,dx - 0$$

$$u = \cos y\, e^x + c$$

$$u = e^x \cos y + c$$

$\boxed{\textit{Example 8}}$ ***If $f(z) = u + iv$ is an analytic function, and $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, find u.***

Solution : Given :
$$v = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$v_x = 2x + \frac{(x^2+y^2)\,1 - x\,(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = -2y + \frac{0 - x\,(2y)}{(x^2+y^2)^2}$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2}$$

Consider,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= \frac{\partial v}{\partial y}\,dx + \left(-\frac{\partial v}{\partial x}\right)dy$$

$$= \left[-2y - \frac{2xy}{(x^2+y^2)^2}\right]dx - \left[-2x - \frac{y^2-x^2}{(x^2+y^2)^2}\right]dy$$

Integrating, we get

$$u = -2xy + \frac{y}{x^2+y^2} + 0 + c$$

$$u = -2xy + \frac{y}{x^2+y^2} + c$$

Example 9 **If $\omega = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, determine ϕ.**

[*Ans* : $\phi = -2xy + \dfrac{y}{x^2+y^2}$]

Example 10 ***In a two dimensional flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$. Find the velocity potential ϕ.***

Solution : Let $f(z) = \phi + i\psi$

Given : $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\psi_x = -\frac{y}{x^2+y^2}$$

$$\psi_y = \frac{x}{x^2+y^2}$$

Consider $d\phi = \dfrac{\partial\phi}{\partial x}\,dx + \dfrac{\partial\phi}{\partial y}\,dy$

$$= \frac{\partial\psi}{\partial y}\,dx + \left(-\frac{\partial\psi}{\partial x}\right)dy$$

$$= \frac{x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy$$

Integrating, we get $\phi = \dfrac{1}{2}\displaystyle\int \frac{2x}{x^2+y^2}\,dx + 0 + c$

$$\phi = \frac{1}{2}\log(x^2+y^2) + c$$

Cauchy-Riemann Equations in Polar Form

Consider a function $f(z) = u + iv$ and $z = r\, e^{i\theta}$.

$$f(z) = f(r\, e^{i\theta}) = u(r, \theta) + i\, v(r, \theta) \quad \ldots (1)$$

Differentiating (1) partially, with respect to r, we get

$$f'(z)\, e^{i\theta} = \frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} \quad \ldots (2)$$

Differentiating (1) partially with respect to θ, we get

$$f'(z)\, r\, e^{i\theta}\, i = \frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}$$

$$f'(z)\, e^{i\theta} = \frac{1}{i\,r}\left[\frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{-i}{r}\left[\frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[-i\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\,\frac{\partial u}{\partial \theta}\right] \quad \ldots (3)$$

Equating (2) and (3) of RHS, we get

$$\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\,\frac{\partial u}{\partial \theta}\right]$$

Equating real and imaginary parts, we get

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\,\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\,\frac{\partial u}{\partial \theta} \end{aligned}} \quad \ldots (4)$$

The above equation given by (4) is called CR equations in polar form.

Note : Consider the equation (2),

$$f'(z)\, e^{i\theta} = \frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right] \quad \ldots (5)$$

This equation can be used to find the derivative of $f(z)$.

This equation can be used to find the derivative of $f(z)$.

$\boxed{\text{Example 11}}$ ***Prove that the function $f(z) = z^n$ is analytic and hence find its derivative.***

Solution : Let $z = r\, e^{i\theta}$

$$z^n = (r\, e^{i\theta})^n = r^n \cdot e^{in\theta} = r^n\,[\cos n\theta + i \sin n\theta]$$

Here $u = r^n \cos n\theta$, $v = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = n\, r^{n-1} \cos n\theta \quad \Bigg| \quad \frac{\partial v}{\partial r} = n\, r^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -n\, r^n \sin n\theta \quad \Bigg| \quad \frac{\partial v}{\partial \theta} = n\, r^n \cos n\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

CR equations in polar form satisfied.

$\therefore f(z) = z^n$ is a regular function of z.

For derivative of $f(z)$, consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right]$$

$$= e^{-i\theta}\,[n\, r^{n-1} \cos n\theta + i\, n\, r^{n-1} \sin n\theta]$$

$$= e^{-i\theta} \cdot n\, r^{n-1}\,[\cos n\theta + i \sin n\theta]$$

$$= e^{-i\theta}\, n\, r^{n-1}\, e^{in\theta}$$

$$= n\,(r\, e^{i\theta})^{n-1}$$

$$f'(z) = n\, z^{n-1}$$

$$\therefore \frac{d}{dz}[z^n] = n\, z^{n-1}$$

$\boxed{\textit{Example 12}}$ ***Prove that $f(z) = \log z$ is a regular function of z and find its derivative.***

Solution : Given :

$$f(z) = \log z, \quad \text{let } z = r\,e^{i\theta}$$

$$= \log(r\,e^{i\theta})$$

$$= \log r + \log(e^{i\theta})$$

$$= \log r + i\theta \log e$$

$$= \log r + i\theta \qquad [\because \log_e e = 1]$$

Here $\quad u = \log r, \qquad v = \theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0, \qquad \frac{\partial v}{\partial \theta} = 1$$

Here CR equation in polar form satisfied.

$f(z) = \log z$ is analytic function of z.

For derivative of $f(z)$, consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= e^{-i\theta}\left[\frac{1}{r}+0\right] = (r\,e^{i\theta})^{-1}$$

$$= z^{-1}$$

$$f'(z) = \frac{1}{z}$$

i.e., $\quad \dfrac{d}{dz}[\log z] = \dfrac{1}{z}$

Laplacian Operator

In cartesian coordinates, the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In polar coordinates, the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Note : If $f(z) = u(r, \theta) + i\, v(r, \theta)$ is an analytic function, then $u(r, \theta)$ and $v(r, \theta)$ are Harmonic functions.

Harmonic Functions

We know that CR equations in polar form is

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

Consider $\quad r\, u_r = v_\theta \qquad \ldots (1)$

Differentiating (1) partially with respect to θ, we get

$$r\, u_{\theta r} = v_{\theta\theta}$$

$$\therefore \frac{1}{r} v_{\theta\theta} = u_{\theta r} \qquad \ldots (2)$$

Now consider $\quad r\, v_r = -u_\theta \qquad \ldots (3)$

Differentiating (3), partially with respect to r, we get

$$r\, v_{rr} + v_r = -u_{r\theta} \qquad \ldots (4)$$

Since $u_{\theta r} = u_{r\theta}$, (4) becomes,

$$r\, v_{rr} + v_r = -\frac{1}{r} v_{\theta\theta}$$

$$r^2 v_{rr} + r\, v_r + v_{\theta\theta} = 0$$

$$\Rightarrow \quad v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$$

The above equation is Laplace equation, and we can say v is a harmonic function.

Similarly, we can prove

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$\Rightarrow$ u is also Harmonic function

CONFORMAL MAPPING

Mapping (Transformation)

A curve C in the z-plane is mapped into the respective curve C_1 in the ω-plane by the given function $\omega = f(z)$ which defines a mapping (transformation) of the z-plane into the ω-plane.

Some standard transformations :

(i) Translation by $\omega = z + c$

(ii) Magnification and rotation by $\omega = cz$

(iii) Inversion and reflection by $\omega = \frac{1}{z}$

(iv) Bilinear transformation $\omega = \frac{az + b}{cz + d}$.

Here a, b, c, d are complex constants.

Conformal Mapping (Conformal Transformation)

Let two curves C_1 and C_2 in the z-plane intersect at the point P and the corresponding curves C_3 and C_4 in the ω-plane intersect at the point Q. If the angle of intersection of the curves at P and Q are the same in magnitude and sense, then the transformation is conformal or mapping is conformal.

Note : The transtormation by the function (analytic) $\omega = f(z)$ is conformal if $f'(z) \neq 0$.

Critical point : A point at which the derivative of $f(z)$ equals to zero (the mapping is not conformal). *i.e.*, A point at which $f'(z) = 0$ is called a critical point of the transformation $\omega = f(z)$.

For example, consider $\omega = z^2$, then $\frac{d\omega}{dz} = 2z$.

$$\frac{d\omega}{dz} = 2z = 0$$

$$z = 0$$

$z = 0$ is a critical point of the transformation $\omega = z^2$.

Example : Consider $\omega = z + \frac{1}{z} = \frac{z^2 + 1}{z}$

$$\frac{d\omega}{dz} = \frac{z(2z) - (z^2 + 1)}{z^2}$$

$$= \frac{z^2 - 1}{z^2}$$

The critical points are $\frac{d\omega}{dz} = 0$

$$z^2 - 1 = 0$$

$$z^2 = 1$$

$$z = \pm 1$$

Fixed Points (Invariant Points)

Fixed points of a mapping $\omega = f(z)$ are points that are mapped on to themselves (image is same as z).

Fixed points are obtained by $f(z) = z$.

Example 1 *Find the invariant points of* $\omega = \dfrac{1}{z - 2i}$.

Solution :

$$\frac{1}{z-2i} = z$$

$$1 = z^2 - 2iz$$

$$z^2 = 2iz - 1 = 0$$

$$\therefore z = i, i$$

Example 2 *Find the points at which the transformation* $\omega = \sin z$ *is not conformal.*

Solution :

$$f'(z) = 0 \Rightarrow \cos z = 0$$

$$z = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots\ldots\ldots$$

Example 3 *Find the invariant points of the transformation* $\omega = \dfrac{1+iz}{1-iz}$.

Solution :

$$\frac{1+iz}{1-iz} = z$$

$$\therefore iz^2 + (i-1)z + 1 = 0$$

$$\therefore z = \frac{1}{2}\left[1 + i \pm \sqrt{6i}\right]$$

Example 4 *Consider* $\omega = f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{1}{2}(z^2 + 1)$.

Solution : The invariants points are obtained from

$$f(z) = z$$

$$\frac{1}{2}(z^2 + 1) = z$$

$$z^2 + 1 = 2z^2$$

$$z^2 = 1$$

$$z = \pm 1$$

Isogonal Transformation (Isogonal Mapping)

If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves at Q of ω-plane only in magnitude then the transformation is called Isogonal.

Example 1 ***Discuss the transformation*** $\omega = f(z) = z^2$.

Solution : Given :

$$
\begin{aligned}
f(z) &= z^2 \\
u + iv &= (x + iy)^2 \\
&= (x^2 - y^2) + i\,2\,xy \\
u &= x^2 - y^2, \quad v = 2\,xy
\end{aligned}
$$

Case (i) : Let $\quad u = \text{constant } C_1$

$$\therefore \; x^2 - y^2 = C_1 \text{ which is a rectangular hyperbola.}$$

Similarly if $v = C_2$, then

$$
\begin{aligned}
2\,xy &= C_2 \\
xy &= \frac{C_2}{2} \text{ which also represents rectangular hyperbola.}
\end{aligned}
$$

$\therefore$ A pair of lines $u = C_1$, $v = C_2$ parallel to the axes in the ω-plane, mapping into the pair of orthogonal rectangular hyperbolas in the z-plane.

Case (ii) : Let $x = c$, a constant.

$u = c^2 - y^2$	$v = 2\,cy$
$y^2 = c^2 - u$	$y = \dfrac{v}{2\,c}$
	$y^2 = \dfrac{v^2}{4\,c^2}$

Eliminating y from the above equations,

$$
\begin{aligned}
c^2 - u &= \frac{v^2}{4\,c^2} \\
v^2 &= 4\,c^2\,(c^2 - u)
\end{aligned}
$$

which represents a parabola.

Let $y = \text{constant } (k)$.

Then

$$x^2 - k^2 = u, \qquad 2xk = v$$

$$x^2 = u + k^2, \qquad x = \frac{v}{2k}$$

$$x^2 = \frac{v^2}{2k}$$

Eliminating x from the above equations, we get

$$u + k^2 = \frac{v^2}{2k^2}$$

$$v^2 = 2k^2(u + k^2) \text{ which is also parabola.}$$

Here the pair of lines $x = c$ and $y = k$ parallel to the axes in the z-plane map into orthogonal parabolas in the ω-plane. The critical point of mapping $\omega = z^2$ is $z = 0$. (not conformal at $z = 0$).

Example 2 ***Discuss the transformation*** $\omega = z + \frac{1}{z}$.

Solution : Let $z = r(\cos\theta + i\sin\theta)$ in polar form.

Given :

$$\omega = z + \frac{1}{z}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r[\cos\theta + i \text{ isn } \theta]}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\therefore \cos\theta = \frac{u}{\left(r + \frac{1}{r}\right)}, \quad \sin\theta = \frac{v}{\left(r - \frac{1}{r}\right)}$$

We know $\cos^2\theta + \sin^2\theta = 1$.

$$\therefore \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r + \frac{1}{r}\right)^2} = 1 \qquad \text{... (1)}$$

For r = constant (c), the equation (1) represents an ellipse.

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Again consider $\omega = u + iv = r(\cos\theta + i\sin\theta)$.

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$r + \frac{1}{r} = \frac{u}{\cos\theta}, \qquad r - \frac{1}{r} = \frac{v}{\sin\theta}$$

$$\left(\frac{r^2+1}{r}\right) = \frac{u}{\cos\theta}, \qquad \left(\frac{r^2-1}{r}\right) = \frac{v}{\sin\theta}$$

$$\left(\frac{r^2+1}{r}\right)^2 = \frac{u^2}{\cos^2\theta}, \qquad \left(\frac{r^2-1}{r}\right)^2 = \frac{v^2}{\sin^2\theta}$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(\frac{r^4+1+2r^2}{r^2}\right) - \left(\frac{r^4+1-2r^2}{r^2}\right)^2$$

$$= \frac{r^4+1+2r^2-r^4-1+2r^2}{r^2}$$

$$= 4$$

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad \text{... (2)}$$

For θ = constant of the z-plane transforms into a family of hyperbolas.

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$$

Example 3 ***Discuss the transformation*** $\omega = z + \dfrac{k^2}{z}$.

Solution : (Solve the problem as above.)

Example 4 ***Discuss the transformation*** $\omega = \cosh z$.

Solution : Given : $\omega = f(z) = \cosh(z)$

$$u + iv = \cosh x \cos y + i \sinh x \sin y$$

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y \qquad \text{... (1)}$$

$$\therefore \cosh x = \frac{u}{\cos y}, \qquad \sinh x = \frac{v}{\sin y}$$

We know that $\cosh^2 x - \sinh^2 x = 1$ (eliminating y).

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad \dots (2)$$

i.e., The lines parallel to x-axis (y = constant) in the z-plane mapping into hyperbola.

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$$

We know that $\cos^2 y + \sin^2 y = 1$. For eliminating y from the given equation (1),

$$\cos y = \frac{u}{\cosh x}, \quad \sin y = \frac{v}{\sinh x}$$

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1 \qquad \dots (3)$$

i.e., The lines parallel to Y-axis (x = constant) in the z-plane mapping into ellipse in the ω-plane.

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \qquad \dots (4)$$

Example 5 ***Discuss the transformation*** $\omega = \frac{1}{z}$.

Solution : Given : $\omega = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{u}{v} = -\frac{x}{y} \Rightarrow y = -\frac{v}{u}x$$

Substituting the value of y in u,

$$u = \frac{x}{x^2 + \frac{v^2}{u^2}\cdot x^2} = \frac{u^2 x}{(u^2+v^2)\,x^2} = \frac{u^2}{(u^2+v^2)\,x}$$

$$\therefore x = \frac{u}{u^2+v^2}$$

$$\therefore\ y = -\frac{v}{u}\cdot x = \frac{-v}{u}\left(\frac{u}{u^2+v^2}\right) = -\left(\frac{v}{u^2+v^2}\right)$$

$$\therefore\ x = \frac{u}{u^2+v^2} \text{ and } y = -\frac{v}{u^2+v^2} \qquad \ldots (1)$$

Now consider $\omega = \frac{1}{z}$. $\quad \therefore\ z = \frac{1}{\omega}$

$$x + iy = \frac{1}{(u+iv)}\frac{(u-iv)}{(u-iv)}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\therefore\ x = \frac{u}{u^2+v^2} \text{ and } y = -\frac{v}{u^2+v^2} \qquad \ldots (2)$$

Consider the equation,

$$a\,(x^2+y^2) + b\,x + c\,y + d = 0 \qquad \ldots (3)$$

For $a = 0$, this represents a straight line and for $a \neq 0$, this represents a circle.

For the transformation $\omega = \frac{1}{z}$, we can substitute the value of x and y in (3).

$$a\left[\frac{1}{u^2+v^2}\right] + b\left[\frac{u}{u^2+v^2}\right] + c\left[\frac{-u}{u^2+v^2}\right] + d = 0$$

$$a + bu - cv + d\,(u^2+v^2) = 0$$

i.e.,

$$d\,(u^2+v^2) + bu - cv + a = 0 \qquad \ldots (4)$$

If $d \neq 0$, this (4) represents a circle in the ω-plane.

If $d = 0$, it represents a straight line.

The transformation $\omega = \frac{1}{z}$ transforms circles into circles. It is called circular transformation.

Example 6 ***Find the mapping of the circle $|\, z \,| = c$ by the transformation $\omega = 2\,z$.***

Solution : Given : $\quad \omega = 2\,z = 2\,(x+iy) = 2\,x + i\,2\,y$

$$u + iv = 2\,x + i\,2\,y$$

$$\therefore u = 2x, \quad v = 2y$$

Consider $|z| = c$. $\quad \therefore \sqrt{x^2+y^2} = c$

$$x^2+y^2 = c^2 \text{ (circle)}$$

$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = c^2$$

$$\frac{u^2}{4} + \frac{v^2}{4} = c^2$$

$$u^2+v^2 = 4c^2$$

$$u^2+v^2 = (2c)^2$$

This is an equation of the circle centre at the origin and radius $2c$.

$\boxed{\textit{Example 7}}$ ***Find the mapping of the circle*** $|z| = k$ ***by the transformation*** $f(z) = z+2+3i$.

Solution : Given : $\quad \omega = z+2+3i$

$$u+iv = x+iy+2+3i$$

$$u+iv = (x+2)+i(y+3)$$

$$u = x+2, \qquad v = y+3$$

$$\therefore x = u-2, \qquad y = v-3$$

Consider, $\quad |z| = k \Rightarrow x^2+y^2 = k^2$

$$(u-2)^2+(v-3)^2 = k^2$$

which is an equation of a circle with centre (2, 3) and radius k.

$\boxed{\textit{Example 8}}$ ***Find the image of the circle*** $|z-1| = 1$ ***in the complex plane under the mapping*** $\omega = \dfrac{1}{z}$.

Solution : $\quad \omega = \dfrac{1}{z}$

$$u+iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \text{ and } v = \frac{-y}{x^2+y^2}$$

The equation of the circle is $|z-1|=1$.

$$\begin{aligned} \text{i.e.,} \qquad |x+iy-1| &= 1 \\ |(x-1)+iy| &= 1 \\ (x-1)^2+(y)^2 &= (1)^2 \\ x^2+1-2x+y^2 &= 1 \\ x^2+y^2 &= 2x \\ \frac{x}{x^2+y^2} &= \frac{1}{2} \end{aligned}$$

$$\text{i.e.,} \qquad u = \frac{1}{2} \qquad \because \frac{x}{x^2+y^2} = u$$

$$\begin{aligned} 2u &= 1 \\ 2u-1 &= 0 \end{aligned}$$

which is a straight line.

Example 9 ***Find the image of $|z-2i|=2$ under the mapping $\omega=\frac{1}{z}$.***

Solution : Given : $\omega = \frac{1}{z}$

$$u+iv = \frac{1}{x+iy} \qquad \therefore u = \frac{x}{x^2+y^2}$$

$$v = \frac{-y}{x^2+y^2}$$

Also given $|z-2i| = 2$

$$\begin{aligned} |x+iy-2i| &= 2 \\ |x+i(y-2)| &= 2 \\ x^2+(y-2)^2 &= 4 \\ x^2+y^2+4-4y &= 4 \\ x^2+y^2-4y &= 0 \\ x^2+y^2 &= 4y \\ 4 &= \frac{x^2+y^2}{y} \end{aligned}$$

$$\frac{1}{4} = \frac{y}{x^2+y^2}$$

$$\frac{1}{4} = -v \qquad \left[\because v = \frac{-y}{x^2+y^2}\right]$$

$$4v + 1 = 0 \text{ which is a straight line.}$$

Example 10 ***Discuss the transformation $\omega = \sin z$.***

Solution : Given :

$$\omega = f(z) = \sin(z)$$

$$u + iv = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\therefore u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

$$\sin x = \frac{u}{\cosh y}, \qquad \cos x = \frac{v}{\sinh y} \qquad \dots (1)$$

We know $\sin^2 x + \cos^2 x = 1.$

$$\therefore \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

For $y = \text{constant } (c_1)$, say $\cosh^2(y) = a^2$, $\sinh^2(y) = b^2$,

then
$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \text{ (Ellipse)}$$

Similarly from (1),

$$\cosh y = \frac{u}{\sin x}, \quad \sinh y = \frac{v}{\cos x}$$

We know that $\cosh^2 y - \sinh^2 y = 1.$

$$\therefore \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

For $x = \text{constant } (c_2)$, say

$$\sin^2 x = A^2$$

$$\cos^2 x = B^2$$

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \text{ (Hyberbola)}$$

Example 11 ***Discuss the transformation $\omega = \cos z$.***

Solution : Consider

$$\omega = \cos(z)$$

$$u + iv = \cos(x + iy)$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$u = \cos x \cosh y, \qquad v = -\sin x \sinh y$$

$$\cos x = \frac{u}{\cosh y}, \qquad \sin x = -\frac{v}{\sinh y}$$

For eliminating x, consider $\cos^2 x + \sin^2 x = 1$.

$$\therefore \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

For $y = c$, $\cosh^2 y = a^2$ (say), $\sinh^2 (y) = b^2$.

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad \text{(Ellipse)}$$

For eliminating y, consider $\cosh^2 y - \sinh^2 y = 1$.

$$\therefore \cosh y = \frac{u}{\cos x}, \quad \sinh y = \frac{-v}{\sin x}$$

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x}$$

$$\therefore \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

For x = constant, say $\cos^2 x = A^2$, $\sin^2 x = B^2$.

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \quad \text{(Hyperbola)}$$

Example 12 ***Discuss the transformation $\omega = \sinh z$.***

Solution : Given : $\omega = \sinh z = \sinh (x + iy)$

$$= \frac{1}{i} \sin (ix - y)$$

$$u + iv = \sinh x \cos y + i \cosh x \sin y$$

$$u = \sinh x \cos y \qquad v = \cosh x \sin y \ldots \text{(i)}$$

$$\sinh x = \frac{u}{\cos y}, \qquad \cosh x = \frac{v}{\sin y}$$

We know $\cosh^2 x - \sinh^2 x = 1$ (for eliminating y)

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

For $y = c$,

$$\frac{v^2}{\sin^2 c} - \frac{u^2}{\cos^2 c} = 1$$

$$\frac{v^2}{a^2} - \frac{u^2}{b^2} = 1 \text{ for } a = \sin c\,;\ b = \cos c.$$

$$\boxed{\frac{v^2}{a^2} - \frac{u^2}{b^2} = 1} \text{ which is a confocal hyperbola.}$$

From (i) $\cos y = \dfrac{u}{\sinh x}$, $\sin y = \dfrac{v}{\cosh x}$

We know $\cos^2 y + \sin^2 y = 1$

$$\frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1$$

For x = constant, say $\sinh x = A$, $\cosh x = B$.

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \text{ which is an ellipse.}$$

Bilinear Transformation

The transformation of the form

$$\omega = \frac{az+b}{cz+d} \quad \dots (1)$$

where a, b, c, d are complex constants is known as Bilinear transformation if $ad - bc \neq 0$. It is also called Mobius transformation or Linear fractional transformation.

The condition $ad - bc \neq 0$ means that the transformation is conformal.

Note :

$$\omega = \frac{az+b}{cz+d} \quad \dots (1)$$

$$\frac{d\omega}{dz} = \frac{(cz+d)\,a - (az+b)\,c}{(cz+d)^2}$$

$$= \frac{acz + ad - acz - bc}{(cz+d)^2}$$

$$= \frac{ad - bc}{(cz+d)^2}$$

The Bilinear transformation (1) is conformal if $\frac{d\omega}{dz} \neq 0$.

i.e., $ad - bc \neq 0$.

Note : If $ad - bc = 0$ then $\frac{d\omega}{dz} = 0$.

i.e., Every point of the z-plane is a critical point.

The inverse mapping of (1) is also bilinear transformation.

i.e., $$z = \frac{-d\omega + b}{c\omega - a}$$

The invariant points of a bilinear transformation,

$$z = \frac{az + b}{cz + d} \qquad [\because \omega = z;\ f(z) = z]$$

$$cz^2 + dz = az + b$$

$$cz^2 + (d - a)z - b = 0$$

The roots of this equation is invariant point or fixed point of the transformation.

Note :

(i) A bilinear transformation maps circles into circles.

(ii) A bilinear transformation preserves cross-ratio of four points.

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_1 - \omega_4)(\omega_3 - \omega_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

(OR)

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_4 - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

Example 13 ***Find the Mobius transformation that maps the points $z = 1, i, -1$ into the points $\omega = 2, i, -2$.***

Solution : Let $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$$\omega_1 = 2,\ \omega_2 = i,\ \omega_3 = -2$$

We know $$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_4 - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)} \qquad \ldots (1)$$

Put $z_4 = z$, $\omega_4 = \omega$, in (1),

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega-\omega_1)(\omega_2-\omega_3)} = \frac{(z_1-z_2)(z_3-z)}{(z-z_1)(z_2-z_3)}$$

$$\frac{(2-i)(-2-\omega)}{(\omega-2)(i+2)} = \frac{(1-i)(-1-z)}{(z-1)(i+1)}$$

$$\frac{(\omega+2)}{(\omega-2)}\frac{(2-i)}{(2+i)} = \frac{(z+1)}{(z-1)}\frac{(1-i)}{(1+i)}$$

$$\frac{(\omega+2)}{(\omega-2)} = \frac{(z+1)}{(z-1)}\frac{(1-i)(2+i)}{(1+i)(2-i)}$$

$$= \frac{(z+1)}{(z-1)}\frac{(2+i-2i+1)}{(2-i+2i+1)}$$

$$= \frac{(z+1)}{(z-1)}\frac{(3-i)}{(3+i)}$$

$$\frac{(\omega+2)}{(\omega-2)} = \frac{3z-iz+3-i}{3z+iz-3-i}$$

Using componendo and dividendo

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{a-c} = \frac{c+d}{c-d}\text{, we get}$$

$$\frac{(\omega+2)+(\omega-2)}{(\omega+2)-(\omega-2)} = \frac{(3z-iz+3-i)+(3z+iz-3-i)}{(3z-iz+3-i)-(3z+iz-3-i)}$$

$$\frac{2\omega}{4} = \frac{6z-2i}{-2iz+6}$$

$$\frac{\omega}{2} = \frac{2(3z-i)}{2(-iz+3)}$$

$$\omega = \frac{2[3z-i]}{[-iz+3]}$$

$$i.e.,\ \omega = \frac{-6z+2i}{iz-3}$$

Example 14 ***Find the invariant points of the transformation***

$$\omega = -\frac{2z+4i}{iz+1}.$$

Solution : $-\dfrac{2z+4i}{iz+1} = z$ $\qquad [\because \omega = z]$

$$
\begin{aligned}
2z + 4i &= -z(iz+1) \\
2z + 4i &= -iz^2 - z \\
iz^2 + 3z + 4i &= 0 \\
\therefore\ z &= \frac{-3 \pm \sqrt{9 - 4(i)(4i)}}{2i} \\
&= \frac{-3 \pm 5}{2i} = \frac{1}{i}, \ \frac{-4}{i} = -i,\ 4i
\end{aligned}
$$

Example 15 ***Find the bilinear transformation which maps the points $z = 1, i, -1$ into points $\omega = 0, 1, \infty$***

Solution : We know that

$$
\frac{(\omega_1 - \omega_2)(\omega_3 - \omega)}{(\omega - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)} \qquad \ldots \text{(i)}
$$

Here $\omega_3 = \infty$ is given. Equation (1) can be written as

$$
\frac{(\omega_1 - \omega_2)\,\omega_3\left(1 - \dfrac{\omega}{\omega_3}\right)}{(\omega - \omega_1)\,\omega_3\left(\dfrac{\omega_2}{\omega_3} - 1\right)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)}
$$

$$
\begin{aligned}
\frac{(\omega_1 - \omega_2)}{(\omega - \omega_1)(-1)} &= \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)} \\
\frac{-(0 - 1)}{(\omega - 0)} &= \frac{(1 - i)(-1 - z)}{(z - 1)(i + 1)} \\
+\frac{1}{\omega} &= +\frac{(z+1)}{(z-1)}\frac{(1-i)}{(1+i)} \\
\frac{1}{\omega} &= \frac{(z+1)}{(z-1)}\frac{(1-i)}{(1+i)} \\
\omega &= \frac{(z-1)}{(z+1)}\frac{(1+i)}{(1-i)} \\
\omega &= \frac{z + iz - 1 - i}{z - iz + 1 - i} \\
\omega &= \frac{(1+i)z - (1+i)}{(1-i)z + (1-i)} \text{ which is of the form } \frac{az+b}{cz+d}.
\end{aligned}
$$

Example 16 ***Find the linear fractional transformation which maps the points z = –1, 0, 1 into ω = 0, i, 3i.***

Solution : We know that

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega-\omega_1)(\omega_2-\omega_3)} = \frac{(z_1-z_2)(z_3-z)}{(z-z_1)(z_2-z_3)}$$

$$\frac{(0-i)(3i-\omega)}{(\omega-0)(i-3i)} = \frac{(-1-0)(1-z)}{(z+1)(0-1)}$$

$$\frac{(-i)(3\,i-\omega)}{\omega\,(-2\,i)} = \frac{(-1)(1-z)}{(-1)(z+1)}$$

$$\frac{(3\,i-\omega)}{2\,\omega} = \frac{(1-z)}{(z+1)}$$

$$(z+1)(3\,i-\omega) = 2\,\omega\,(1-z)$$

$$3\,iz-z\omega+3\,i-\omega = 2\,\omega-2\,z\omega$$

$$3\,i\,(z+1) = 2\,\omega-2\,z\omega+z\omega+\omega$$

$$= 3\,\omega-z\omega = \omega\,(3-z)$$

$$\therefore \omega = \frac{3\,i\,(z+1)}{(3-z)}$$

$$\omega = -3\,i\left(\frac{z+1}{z-3}\right)$$

Example 17 ***Find the Mobius transformation which maps from (∞, i, 0) into (0, i, ∞).***

Solution : Substituting in the above formula,

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega-\omega_1)(\omega_2-\omega_3)} = \frac{(z_1-z_2)(z_3-z)}{(z-z_1)(z_2-z_3)}$$

Taking z_1 and ω_3 outside and substitute, we get

$$\frac{(0-i)(1-0)}{(\omega-0)(-1)} = \frac{(1-0)(0-z)}{(0-1)(i-0)}$$

$$\frac{(-i)}{-\omega} = \frac{(1)(-z)}{(-i)}$$

$$\frac{i}{\omega} = \frac{z}{i}$$

$$\boxed{\omega = -\frac{1}{z}}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II -ENGINEERING MATHEMATICS-III– SMTA1301

UNIT - II

Complex Integration

Introduction :

Consider a continuous function f (z) of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0)\,,\, P_1(z_1)\,,\, \ldots\ldots ,P_i(z_i),\ldots\ldots. ,P_n(z_n) = B.$$

Let $\delta z_i = z_i - 1$ and ζ_i be any point on the arc $P_{i-1} P_i$. The limit of the sum

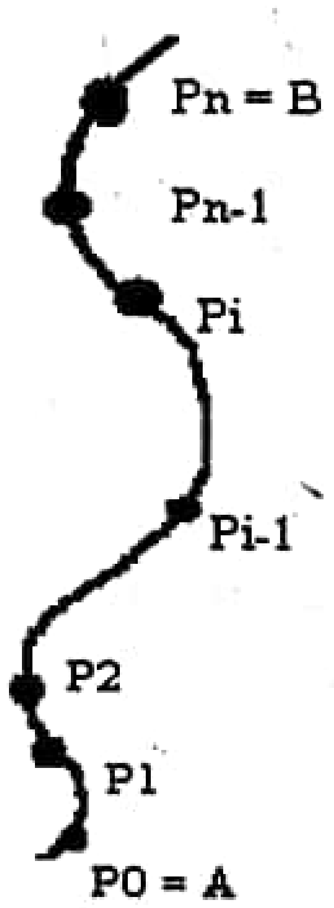
$\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the **line integral of f(z)** taken along the path C , i.e.

$$\int_C f(z)dz\,.$$

Writing $f(z) = u(x,y) + iv(x,y)$ and nothing that $dz = dx + i\,dy$,

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.



Note :

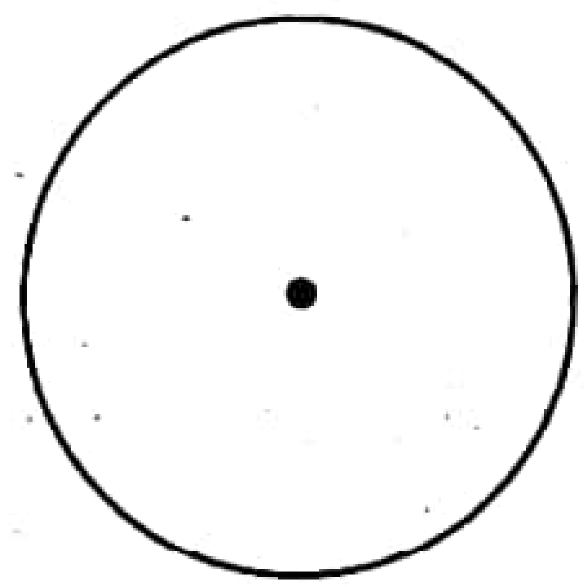
If $\omega = f(z) = u\,(x, y) + i\,v\,(x, y)$

then $\int_C f(z)dz = \int_C (u + iv)\; d\,(x + iy)$

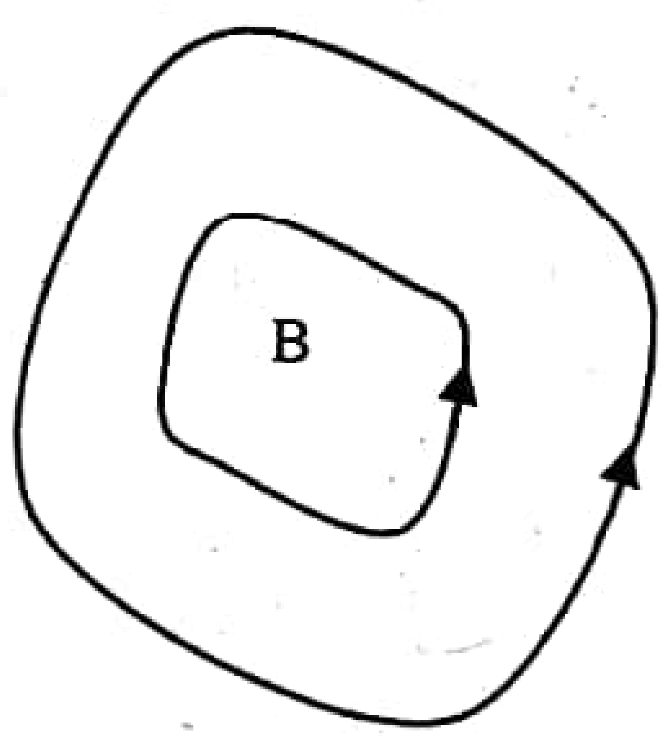
$$= \int_C (u + iv)\;(dx + idy)$$

$$= \int_C (udx + vdy) + i\int_C (vdx + udy)$$

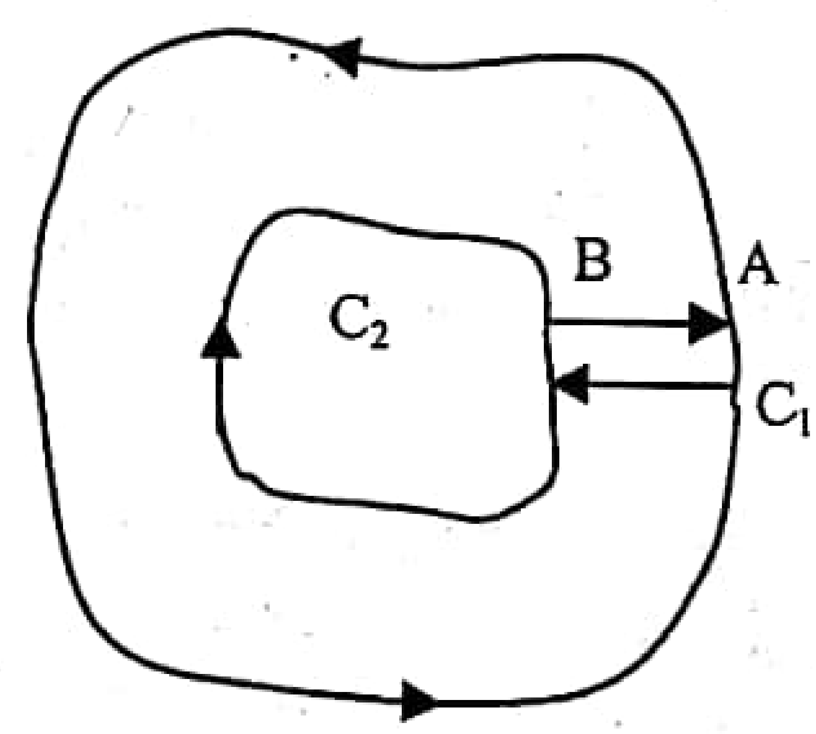
Simply connected Region: A simply connected region is one in which any closed curve lying entirely within it can be contracted to a point without passing out of the region.



Simply Connected Region



Multi-connected region

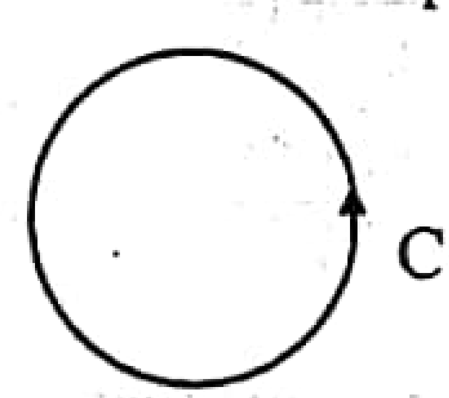


Simply connected region

CAUCHY'S THEOREM

Theorem :

If f(z) is an analytic function and f '(z) is continuous at each point within and an a closed curve C , then $\int_C f(z)\,dz = 0$.



Proof:

Consider f(z) = u(x,y) +iv(x,y) and z = x+iy , dz =dx + idy

$$\therefore \int_C f(z)\,dz = \int_C (udx - vdy) + i \int_C (vdx + udy) \qquad \ldots\ldots(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. We know Green's theorem is.

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

using this in (1)

$$\int_C f(z)\,dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\,dy + \iint \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \quad \ldots(2)$$

Now f(z) being analytic, u and v necessarily satisfy the Cauchy-Riemann equations

$$\text{i.e. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad \ldots\ldots\ldots(3)$$

Substituting (3) in (2) we have

$$\int_c f(z)\,dz = \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0$$

Hence $\int_c f(z)\,dz = 0$

Extension of Cauchy's Theorem.

If f(z) is analytic in the region D between two simple closed curves C and C_1, then, $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz.$

To prove this, we need t o introduce the cross-cut AB. Then $\int f(z)dz = 0$ where the path is as indicated by arrows in Fig.(1) .i.e. along AB-along C_1 in clockwise sense & along BA – along C in anti – clockwise sense

$$\text{i.e. } \int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{BA} f(z)\,dz + \int_C f(z)\,dz = 0.$$

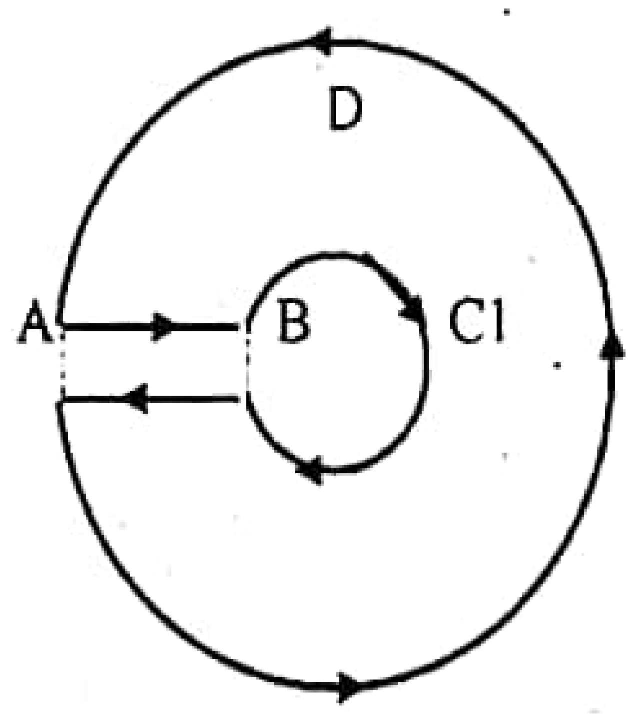


Fig.(1)

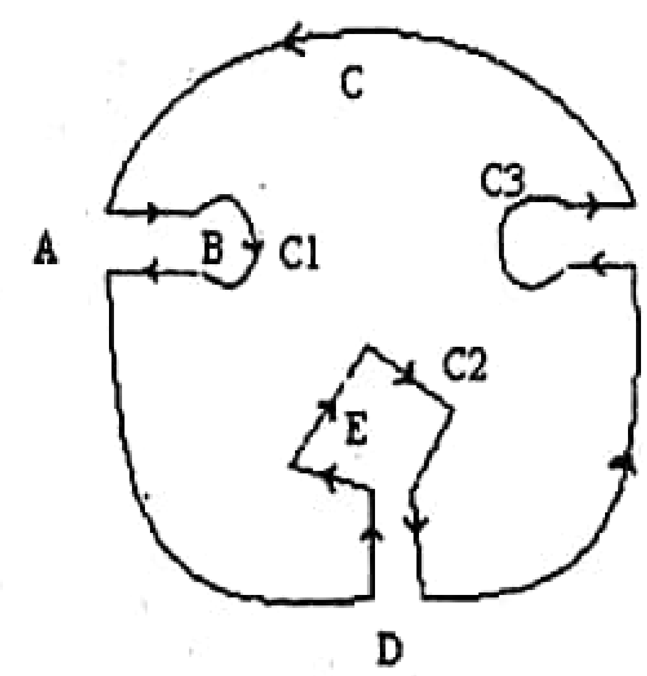


Fig.(2)

But, since the integrals along AB and along BA cancel , it follows that

$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz = 0.$$

Reversing the direction of the integral around C_1 and transposing , we get

$\displaystyle\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$ each integration being taken in the anti-clockwise sense.

If $C_1, C_2, C_3, \ldots\ldots.$be any number of closed curves within C (Fig-2) then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \ldots..$$

CAUCHY'S INTEGRAL FORMULA

Theorem :

If f(z) is analytic within and on a closed curve and if a is any point within C, then $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{z-a}$.

Proof :

Consider the function f(z) / (z-a) which is analytic at all points within C except at z = a. With the point a as center and radius r , draw a small circle C_1 lying entirely within C.

Now f(z) / (z-a) being analytic in the region enclosed by C and C1, we have by Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\, d\theta = i \int_{C_1} f(a+re^{i\theta}) d\theta. \qquad \text{......(1)}$$

In the limiting form, as the circle C_1 shrinks to the point a , i.e. as $r \to 0$, the integral (1) will approach to

$$i \int_{C_1} . f(a) d\theta = i f(a) \int_0^{2\pi} d\theta = 2\pi i\, f(a). \text{thus} \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

i.e. $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

which is the desired Cauchy's integral formula.

$$\Rightarrow \int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

Cauchy's integral formula for derivative of an analytic function:-

We know Cauchy's integral formula is

$$F(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

Differentiating both sides of (2) w.r.t.a,

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{\partial}{\partial a} \left[\frac{f(z)}{z-a} \right] dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz \qquad \text{......(3)}$$

similarly, $$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz \qquad \text{......(4)}$$

and in general, $$f'(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz. \qquad \text{......(5)}$$

thus it follows from the results (2) to (5) that if a function f(z) is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C. Incidently we have established a remarkable fact that **an analytic function possesses derivatives of all orders a| these are themselves all analytic.**

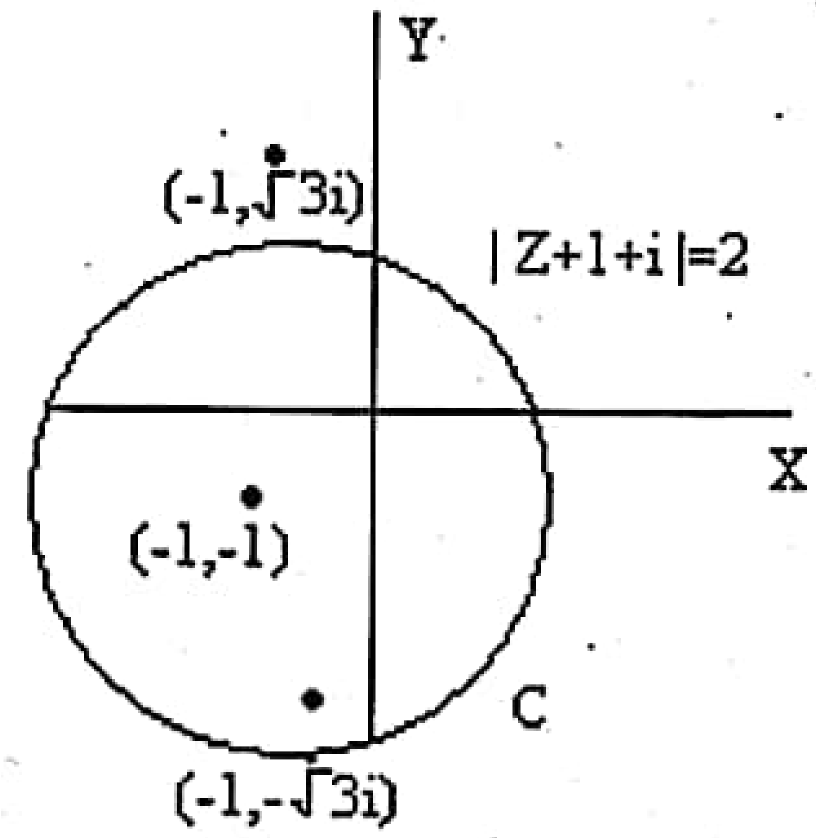
Example 1: Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dx$ where C is the circle

(i) $|z| = 1$, (ii) $|z| = \frac{1}{2}$.

(i) Here $f(z) = z^2 - z + 1$ and a =1.

Since f(z) is analytic within and on circle

$C : |z| = 1$ and a = 1 lies on C.



$\therefore$ By Cauchy's Integral Formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1$ i.e. $\int_C \frac{z^2 - z + 1}{z-1} dz = 2\pi i$.

(ii) In this case , a = 1 lies outside the circle $C : |z| = \frac{1}{2}$. So $\frac{(z^2 - z + 1)}{(z - 1)}$ analytic everywhere within C.

$\therefore$ By Cauchy's Theorem $\int_C \frac{z^2 - z + 1}{z - 1} dz = 0$.

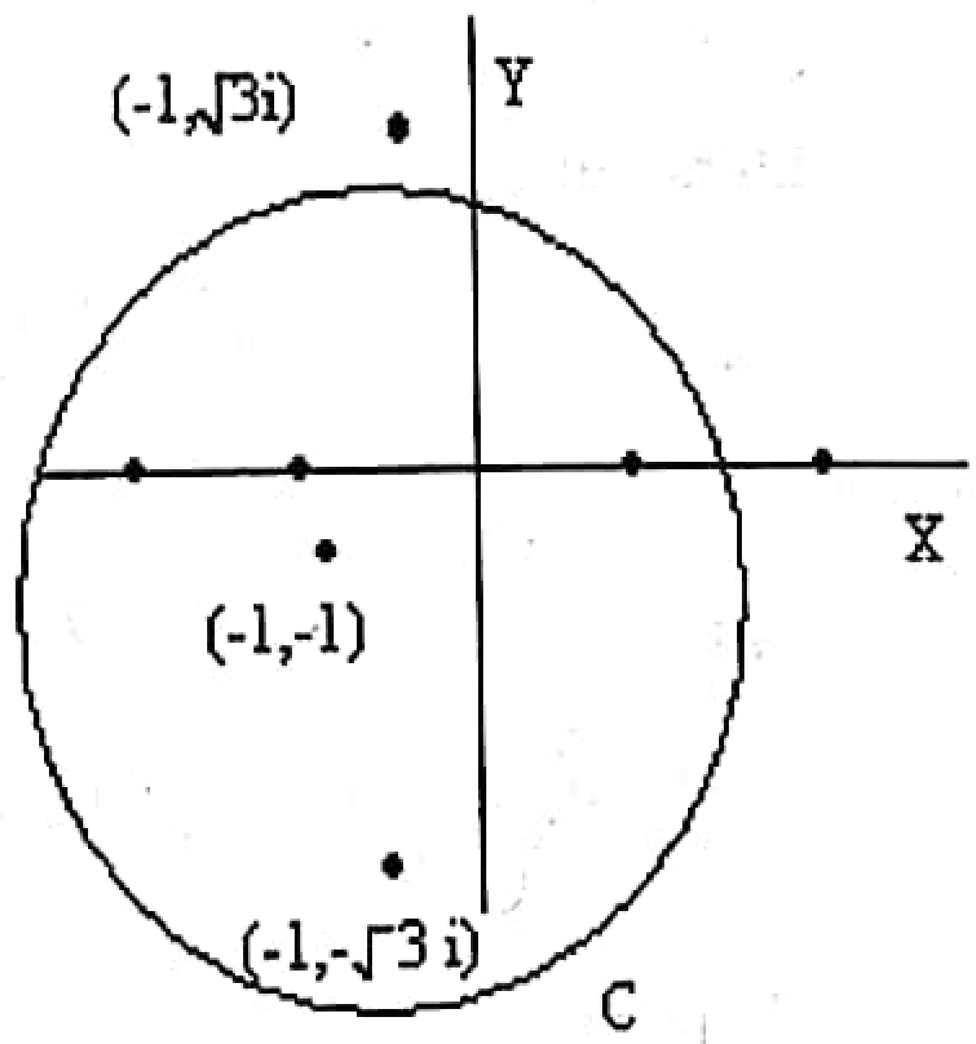
Example 2:

Using Cauchy's integral formula, Evaluate $\int_C \frac{z+1}{z^2 + 2z + 4} dz$ where c is t| circle $|z + 1 + i| = 2$

Solution:

$|z + 1 + i| = |z - (-1 - i)|$ is the circle with centre at z = -1-I and radius 2 units

The function $\dfrac{z+1}{z^2+2z+4}$ will cease to be analytic where $z^2+2z+4=0$



$$z = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm i2\sqrt{3}}{2}$$

$$z = -1 \pm i\sqrt{3}$$

$$z = -1 + i\sqrt{3}, -1 - i\sqrt{3}$$

$$\therefore \frac{(z+1)}{z^2+2z+4} = \frac{z+1}{(z+1-i\sqrt{3})(z+1+i\sqrt{3})}$$

The above function is analytic at all points except at the points $-1+i\sqrt{3}$ lies outside c and $-1-i\sqrt{3}$ lies inside c.

$\therefore$ we consider the function $f(z) = \dfrac{z+1}{z+1-i\sqrt{3}}$

by cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

here $a = -1 - i\sqrt{3}$, (lies inside c) $\therefore \displaystyle\int \frac{\frac{z+1}{z+1-i\sqrt{3}}}{z-(-1-i\sqrt{3})}\,dz = 2\pi i\, f(a)$

$=2\pi i f(-1-i\sqrt{3})$

$$= 2\pi i\left(\frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}}\right)$$

substitution in f(z)

$$= 2\pi i\left(\frac{-i\sqrt{3}}{-2i\sqrt{3}}\right) = \pi i$$

Example 3:

Using cauchy's integral formula evaluate $\int_c \frac{z+4}{z^2+2z+5}dz$ where c is circle $|z+1-i|=2$.

Solution :

$|z+1-i| = |z-(-1+i)|$ is the circle with center at (-1+i) and radius 2 units. The function $\frac{z+4}{z^2+2z+5}$ will cease to be regular where $z^2+2z+5=0$

$$\text{i.e., } z^2+2z+5=0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$z = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

$$\therefore z = -1+2i, \quad -1-2i$$

$$\frac{z+4}{(z^2+2z+5)} = \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}$$

The above function is analytic at all points except at z=-1+2i which lies inside c and z = -1 --2i which lies outside c.

$\therefore$ We consider the function

$$f(z) = \frac{\dfrac{z+4}{[z-(-1-2i)]}}{[z-(-1+2i)]} = \frac{f(z)}{z-a}$$

$\therefore$ By cauchy integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Takng a = -1+2i (lies inside c)

$$\int_c \frac{\left(\dfrac{z+4}{z+1+2i}\right)}{[z-(-1+2i)]} dz = 2\pi i\, f(-1+2i)$$

$$= 2\pi i\left(\frac{-1+2i+4}{-1+2i+1+2i}\right)$$

$$= 2\pi i\left(\frac{2i+3}{4i}\right) = \frac{\pi}{2}(2i+3)$$

Example 4:

Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is $|z|=3$ using cauchy integral formula.

Solution:

$|z|=3$ is a circle with center at the origin and radius 3 units

consider

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

put z = 1 A = -1

put $z = 2 \quad B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = - \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz + \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

Since z=1, and z=2 lies inside c and $f(z) = \sin \pi z^2 + \cos \pi z^2$

By cauchy integral formula

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$
$$= -2\pi i\, (\sin \pi + \cos \pi) + 2\pi i\, (\sin 2\pi + \cos 2\pi)$$
$$= 2\pi i\, (1+1)$$
$$= 4\pi i$$

Example 5:

Using cauchy integral formula evaluate $\int_c \frac{dz}{(z^2+1)(z^2-4)}$

where c is $\int_c \frac{dz}{(z^2+1)(z^2-4)}$ where c is $|z| = \frac{3}{2}$

Solution :

$|z| = \frac{3}{2}$ is the circle with center at the origin and radius 3/2 units.

$$\frac{1}{(z^2+1)(z^2-4)} = \frac{1}{(z+i)(z-i)(z+2)(z-2)}$$

The above function is analytic at all points excepts at $z = i$, $-i$ which lies inside c and $z = \pm 2$ which lies outside c

$\therefore$ we consider the function

$$f(z) = \frac{\frac{1}{z^2-4}}{(z+i)(z-i)}$$

Now

$$\frac{1}{(z+1)(z-i)}=\frac{A}{(z+i)}+\frac{B}{(z-i)}$$

$$1 \quad = A(z-i)+B(z+i)$$

$$\text{Put} \quad z=i,\ B=\frac{1}{2i}=-\frac{i}{2}$$

$$\text{Put} \quad z=-i,\ B=-\frac{1}{2i}=\frac{i}{2}$$

$$\therefore \frac{1}{(z+i)(z-i)}=\frac{\frac{i}{2}}{(z+i)}-\frac{\frac{i}{2}}{(z-i)}$$

$$\therefore \int_c \left[\frac{\frac{i}{2}}{(z+i)}-\frac{\frac{i}{2}}{(z-i)}\right]\frac{1}{z^2-4}\,dz=\frac{i}{2}\int_c \frac{\left(\frac{1}{z^2-4}\right)}{z+i}\,dz-\frac{i}{2}\int_c \frac{\left(\frac{1}{z^2-4}\right)}{z-i}\,dz$$

taking a = i , -i (which lie inside c)

By cauchy integral formula

$$\int_c \frac{f(z)}{z-a}\,dz=2\pi i\, f(a)$$

$$=\left(\frac{i}{2}\right)2\pi i\, f(-i)-\left(\frac{i}{2}\right)2\pi i\, f(i)$$

$$=\left(\frac{i}{2}\right)2\pi i\left[\frac{1}{-5}-\frac{1}{-5}\right]$$

$$=-\pi\left[-\frac{1}{5}+\frac{1}{5}\right]$$

$$=0$$

Example 6: Evaluate $\int_c \frac{z^2 dz}{(z-1)^2 (z^2+1)}$ where c is $|z-2| = 2$. Using cauchy integrai formula.

Solution :

$|z-2| = 2$ is a circle with center at 2 and radius 2 units consider.

$$\frac{z^2}{(z-1)^2(z^2-1)} = \frac{z^2}{(z-1)^3(z+1)}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-3)^3} + \frac{D}{(z+1)}$$

$$z^2 = A(z-1)^2(z+1) + B(z-1)(z+1) + C(z+1) + D(z-1)^3$$

put $z = 1$,

$$c = \frac{1}{2}$$

put $z = -1$

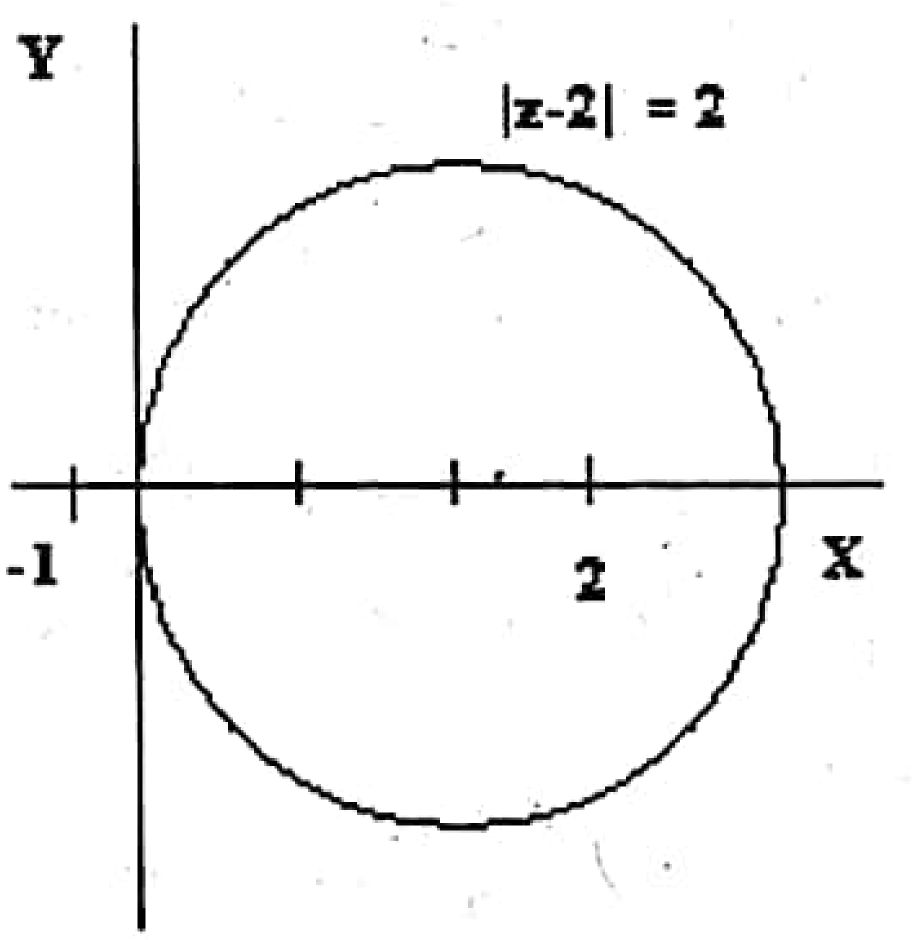
$$-8D = 1,$$

$$D = -\frac{1}{8}$$

Coefficient of z^3, $A + D = 0$

$$A = -D$$

$$A = \frac{1}{8}$$



equating constant coefficient

$A - B + C - D = 0$

$$B = \frac{1}{8} + \frac{1}{2} + \frac{1}{8}$$

$$= \frac{1+4+1}{8} = \frac{6}{8}$$

$$B = \frac{3}{4}$$

$$\int_c \frac{z^2}{(z-1)^2(z^2-1)}\,dz = \frac{1}{8}\int_c \frac{1}{(z-1)}\,dz + \frac{3}{4}\int_c \frac{1}{(z-1)^2} + \frac{1}{2}\int_c \frac{dz}{(z-1)^3} - \frac{1}{8}\int_c \frac{dz}{(z+1)}$$

Since the point z=1 lies inside c and z = -1 lies outside c. By cauchy integral formula & its derivatives we have

$$= \frac{1}{8}2\pi i\, f(1) + \frac{3}{4}(2\pi i)\, f'(1) + \frac{1}{2}\frac{(2\pi i)\, f''(1)}{2!} + 0$$

$$= \frac{1}{8}2\pi i + \frac{3}{4}2\pi i + \frac{1}{2}\frac{(2\pi i)}{2!}$$

$$= \frac{\pi}{4}i + \frac{3}{2}\pi i + \frac{\pi}{2}i = \frac{\pi i + 6\pi i + 2\pi i}{4} = \frac{9\pi i}{4}$$

$[\because\ f(z)=1 \quad f(1)=1 \quad f'(1)=1 \quad f''(1)=1]$

Example 7: Evaluate using Cauchy's integral formula :

$\int_C \frac{e^{2z}}{(z-1)(z-2)}$, where C is the circle $|z| = 3$

Solution: $f(z) = e^{2z}$ is analytic within the circle $C : |z| = 3$ and the two singular points $a = 1$ and $a = 2$ lie inside C.

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz = \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$$

$$= 2\pi i e^4 - 2\pi i e^2 = 2\pi I\,(e^4 - e^2)$$

[By Cauchy's integral formula]

Example 8:

Evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$ where c is the circle $|z| = 3$.

Solution :

Here $|z| = 3$ is a circle with center at the origin and radius 3 units.

Also $f(z) = \cos \pi z^2$

and consider $\dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$$1 = A(z-2) + B(z-1)$$

$$\text{put } z = 1, \qquad A = -1$$
$$\text{put } z = 2, \qquad B = 1$$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\int_C \frac{\cos \pi z^2}{(z-1)} dz + \int_C \frac{\cos \pi z^2}{(z-2)} dz$$

Since z =1 and z = 2 lies inside c. By cauchy integral formula we have

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i\, [-\cos \pi + \cos 4\pi]$$

$$= 2\pi i\, [-(-1)+1] \;=\; 4\pi i$$

Example 8:

Evaluate $\displaystyle\int_C \frac{(z+1)\, dz}{(z^2+2z+4)^2}$ where c is $|z+1+i| = 2$ using cauchy integral formula.

Solution :

$|z+1+i| = 2$ is a circle with centre $(-1, -i)$ and radius 2 units.

$$\frac{z+1}{(z^2+2z+4)^2} = \frac{z+1}{\left[z-\left(-1-\sqrt{3}i\right)\right]^2\left[z-\left(-1+\sqrt{3}i\right)\right]^2}$$

The above function is analytic at all points except at $z = -1-\sqrt{3}$ I which lies inside c and $z = -1+\sqrt{3}$ I which lies outside c.

$\therefore$ Consider the function

$$f(z) = \frac{\dfrac{z+1}{\left[z-(-1+\sqrt{3}i\right]^2}}{\left[z-\left(-1-\sqrt{3}i\right)\right]^2}$$

$\therefore$ By cauchy integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^2} dz = 2\pi i\, f'(a)$$

taking $\quad a = -1-\sqrt{3}\,i$

$\therefore \qquad = 2\pi i\, f'(-1-\sqrt{3}i)$

But $\qquad f(z) = \dfrac{z+1}{\left[z-\left(-1+\sqrt{3}i\right)\right]^2}$

$$= \frac{z+1}{(z-\alpha)^2}\ \alpha = -1+\sqrt{3}i$$

$$f'(z) = \frac{(z-\alpha)^2 - 2(z+1)(z-\alpha)}{(z-\alpha)^4} = \frac{-(z+\alpha+2)}{(z-\alpha)^3}$$

$$f'(a) = f'\left(-1-\sqrt{3}i\right)$$

$$= \frac{-\left[-1-\sqrt{3}i-1+\sqrt{3}i+2\right]}{\left(-1-\sqrt{3}i+1-\sqrt{3}i\right)^3} = \frac{0}{-\left(2\sqrt{3}i\right)^3} = 0$$

$$\therefore \int_c \frac{(z+1)\,dz}{(z^2+2z+4)^2} = 2\pi i\ f'(-1-\sqrt{3}i)$$

$$= 0 \qquad\qquad \because f'(-1-\sqrt{3}i) = 0$$

Example 10 :

Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where c is $|z|=2$ using cauchy integral formula.

Solution :

$|z|=2$ is a circle with centre at the origin and radius 2 units

Here $f(z) = e^{2z}$

Clearly $z = -1$ lies inside c

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = \int_c \frac{e^{2z}}{[z-(-1)]^4} dz$$

since $z = -1$ lies inside c

By cauchy integral formula for derivatives

$$f'''(a) = \frac{3!}{2\pi i} \int_c \frac{f(z)}{(z-a)^4} dz$$

$$\therefore \int_c \frac{e^{2z}}{[z-(-1)]^4} dz = \frac{2\pi i\, f'''(a)}{3!}$$

$$= 2\pi i\, f'''(-1) \qquad \ldots(1)$$

since $f(z) = e^{2z}$

$$f'(z) = 2e^{2z}$$
$$f''(z) = 4e^{2z}$$
$$f'''(z) = 8e^{2z}$$
$$f'''(-1) = 8e^{-2} \qquad \ldots(2)$$

Therefore (2) in (1) we get

$$\int_c \frac{e^{2z}}{[z-(-1)]^4} dz = \frac{2\pi i \times 8e^{-2}}{6}$$

$$= \frac{8}{3}\pi i e^{-2}$$

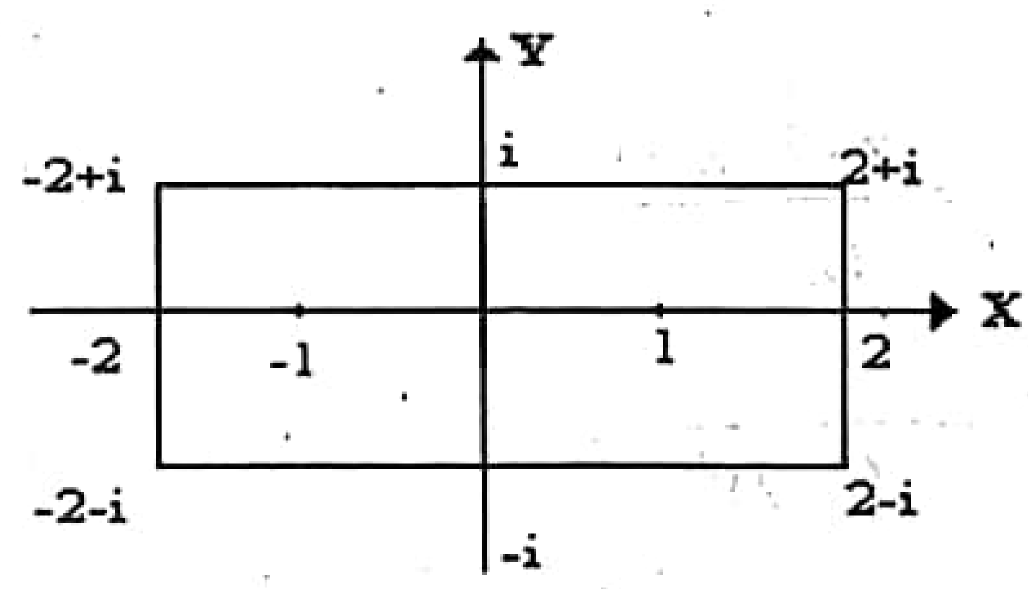
Example 11:

$\int_C \frac{\cos \pi z}{z^2 - 1} dz$ around a rectangle with vertices $2 \pm i, -2 \pm i$.

Solution :

$f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points a=1 and a = -1 lie inside this rectangle.

$$\therefore \int_C \frac{\cos \pi z}{z^2 - 1} dz = \frac{1}{2} \int_C \left(\frac{1}{z-1} - \frac{1}{z+1} \right) \cos \pi z \, d$$



$$= \frac{1}{2} \int_C \frac{\cos \pi z}{z-1} dz - \int_C \frac{\cos \pi z}{z+1} dz$$

$=$

$$\frac{1}{2}\{2\pi i \cos \pi (1)\} - \frac{1}{2}\{2\pi i \cos \pi(-1)\} = 0.$$

[By Cauchy's integral formula]

Example 12:

Evaluate $\int_c \frac{(z-1)}{(z+1)^2 (z-2)} dz$ where c is circle $|z - i| = 2$

Solution :

$|z - i| = 2$ is a circle with centre at i and radius 2 units.

Consider $\frac{z-1}{(z+1)^2(z-2)}$

The above function is analytic at all except at z= - 1 which lies inside 'c'.

$$\therefore \text{ we consider } f(z) = \frac{z-1}{z-2}$$

$$\therefore \int_c \frac{\left(\frac{z-1}{z-2}\right)}{(z-(-1))^2} dz = 2\pi i\, f'(-1) \qquad \text{....(1)}$$

($\because$ using cauchy integral formula taking a = -1)

since $f(z) = \frac{z-1}{z-2}$

$$f'(z) = \frac{(z-2)-(z-1)}{(z-2)^2}$$

$$= \frac{z-2-z+1}{(z-2)^2}$$

$$= -\frac{1}{(z-2)^2}$$

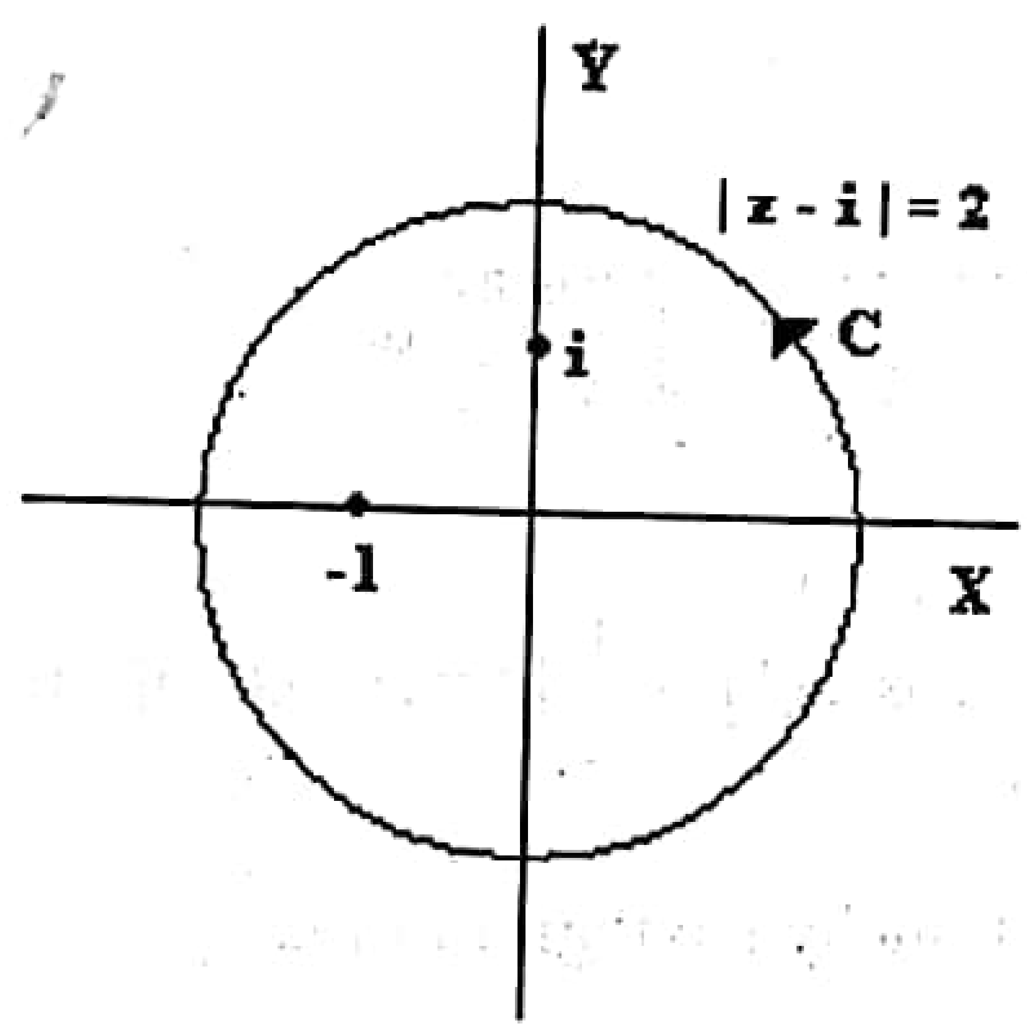
$$\therefore f'(-1) = -\frac{1}{9} \qquad \text{......(2)}$$

Substitute (2) in (1) we get

$$\int_C \frac{z-1}{(z+1)^2 (z-2)} dz = 2\pi i\, f'(-1)$$

$$= 2\pi i\, [-\frac{1}{9}]$$

$$= -\frac{2\pi i}{9}$$



Example 13:

Evaluate

(i) $$\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\, dz$$, where C is the circle $|z| = 1$.

(ii) $$\int_C \frac{e^{2z}}{(z+1)^4}\, dz$$, where C is the circle $|z| = 2$.

Solution:

(i) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (0.5 approx.) lies within C.

$\therefore$ By cauchy's integral formula $f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\, dz$,

We get $$\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\, dz = \pi i \left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i\,(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i$$

(ii) $f(z) = e^{2z}$ is analytic within the circle C : $|z| = 2$. Also $z = -1$ lies inside C.

$\therefore$ By cauchy's integral formula : $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{4'}}$

We get $$\int_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}\left[8e^{2z}\right]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

Example 14:

Evaluate $$\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\, dz,$$ where c is $|z| = 1$

Solution :

Here $f(z) = \sin^6 z$ $\quad |z| = 1$ is the circle with center at the origin and radius 1 units clearly $z = \frac{\pi}{6}$ lies inside $|z| = 1$

$\therefore$ By cauchy integral formula for derivatives

$$\int_C \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$

$$\therefore \int_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''(\pi/6) \qquad .....(1)$$

But $\quad f(z) = \sin^6 z$

$f'(z) = 6 \sin^5 z \cos z$

$f''(z) = 6 [\sin^5 z(-\sin z) + \cos z(5 \sin^4 z)]$

$= 6 [-\sin^6 z + 5 \cos z \sin^4 z]$

$$\therefore f''\left(\frac{\pi}{6}\right) = 6\left[-\sin^6\left(\frac{\pi}{6}\right) + 5 \cos\left(\frac{\pi}{6}\right) x \sin^4\left(\frac{\pi}{6}\right)\right]$$

$$= 6\left[-\frac{1}{64} + \frac{5}{16} x \frac{3}{4}\right] \qquad ....(2)$$

$$= \frac{21}{16}$$

Substitute (2) in (1) we have

$$\int_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!}\left(\frac{21}{16}\right) = \frac{21\pi i}{16}$$

Taylor's series:

If f(z) is analytic inside a circle C with centre at a , then for z inside C , $f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots\dots + \frac{f^n(a)}{n!}(z-a)^n + \dots\dots$(1)

Note: If a = 0 in Taylor's series we get Maclaurin's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{f^n(0)}{n!}$$

Note: Complex analytic functions can always be represented by power series of the form (1)

Complex analytic functions can always be represented by power series of the form (1)

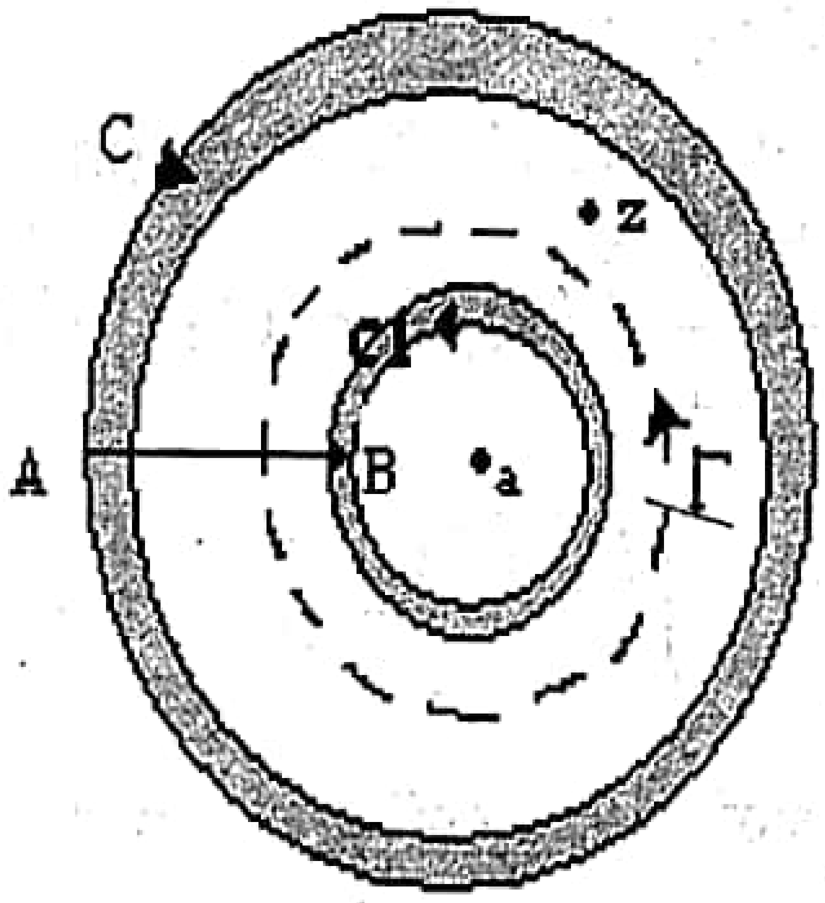
Laurent's Series:

If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a , then for all z in R

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \dots$

f(z) =

$$\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$



Γ being any curve in R', encircling C_1

Where $a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{-n+1}}\,dt$$

Note: $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called integral part and $\sum_{n=0}^{\infty} b_n (z-a)^{-n}$ is called principle part of the Laurents series.

Note:

(i) To obtain Taylor's series or Laurent's series simply expand f(z) by Binomial theorem.

(ii) Laurent's series of a given analytic function f(z) in its annulus of convergence is unique.

(iii) If $|z|<1$, then (We Know)

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots\dots\dots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \dots\dots\dots$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots\dots\dots$$

$$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots\dots\dots$$

Example 1:

Find the Laurents series Expansion of $\frac{1}{z^2-z-2}$ in the region $1<|z|<2$

Solution: $f(z) = \frac{1}{z^2-z-2} = \frac{1}{(z+1)(z-2)}$

$$\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$$

$1=A(z-2)+B(z+1)$

put $z = 2$ $\qquad B = \frac{1}{3}$

put z = -1 $\qquad A = -\frac{1}{3}$

$$f(z) = \frac{1}{z^2 - z - 2} = \frac{1}{(z+1)(z-2)}$$

$$\frac{1}{(z+1)(z-2)} = \frac{-1}{3(z+1)} + \frac{1}{3(z-2)}$$

$$= \frac{1}{3z(1+1/z)} - \frac{1}{6(1-z/2)}$$

$$= -\frac{1}{3z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$f(z) = -\frac{1}{3z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 - \ldots\right) - \frac{1}{6}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2 + \ldots\right)$$

In the first series the expansion in valid $\left|\frac{1}{z}\right| < 1$, i.e. $1 < |z|$

In the second series the expansion in valid $\left|\frac{z}{2}\right| < 1$, $|z| < 2$

∴ The series is valid when $1 < |z| < 2$.

Example 2: Obtain the expansion of the function $\frac{z-1}{z^2}$ in Taylors series of powers of (z - 1) and state the region of validity.

Solution: $f(z) = \frac{z-1}{z^2}$

$$= \frac{1}{z} - \frac{1}{z^2}$$

The Taylors series at z = 1 is

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^n(1) \qquad \ldots(1)$$

Now $f(z) = \frac{1}{z} - \frac{1}{z^2}$

$f(1) = 0$ ...(2)

$f'(z) = -\frac{1}{z^2} + \frac{(-1)(-2)}{z^3}$

$f''(z) = \frac{(-1)(-2)}{z^3} + \frac{(-1)(-2)(-3)}{z^4}$

....

....

$f^n(z) = \frac{(-1)^n n!}{z^{n+1}} + \frac{(-1)^{n+1}(n+1)!}{z^{n+2}}$

$\therefore f^n(1) = (-1)^n n! + (-1)^{n+1}(n+1)!$

$=(-1)^n n![1-(n+1)]$

$=(-1)^n n!(-n)$

$f^n(1)=(-1)^{n+1} n.n!$ ...(3)

Substitute (2) & (3) in (1) we have

$$f(z) = \sum_{n=1}^{\infty} n(-1)^{n+1}(z-1)^n$$

f(z) is analytic at $z = 0$. Also $|z - 1| < 1$ is the region of converges.

Hence the region of validity $|z-1|<1$

Example 3: Obtain the Taylors series of expansion of log $(1 + z)$ when $|z|<1$.

Solution: Let $f(z) = \log(1 + z)$

$f(0) = \log(1) = 0$ ...(1)

$f'(z) = \frac{1}{1+z}$

$$f''(z) = -\frac{1}{(1+z)^2}$$

$$f'''(z) = \frac{(-1)(-2)}{(1+z)^3} = \frac{2!(-1)^2}{(1+z)^3}$$

....

....

$$f^n(z) = \frac{(-1)(-2)\ldots-(n-1)}{(1+z)^n} = \frac{(n-1)!(-1)^{n-1}}{(1+z)^n}$$

$$\therefore f^n(0) = (n-1)!\,(-1)^{n-1} \qquad \ldots(2)$$

The Taylors series at $z = 0$ is

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^n(0) \qquad \ldots(3)$$

substitute (1) & (2) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}(n-1)!(-1)^{n-1}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

Example 4: Expand cos z in a Taylors series about $z = \frac{\pi}{4}$

Solution:

$$f(z) = \cos z \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(z) = -\sin z \quad f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f''(z) = -\cos z \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(z)=\sin z \quad f'''\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

The Taylors series about $z = a$ is

$$f(z)=f(a)+\sum_{n=1}^{\infty}\frac{(z-a)^n}{n!}f^n(a)$$

$$=f(a)+\frac{\left(z-\frac{\pi}{4}\right)}{1!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\ldots$$

$$\cos z=\frac{1}{\sqrt{2}}\left[1-\frac{\left(z-\frac{\pi}{4}\right)}{1!}-\frac{\left(z-\frac{\pi}{4}\right)}{2!}+\ldots\right]$$

Example 5: Find Taylors expansion of

(i) $f(z)=\dfrac{1}{(z+1)^2}$ about the point $z = - i$.

(ii) $f(z)=\dfrac{2z^3+1}{z^2+z}$ about the point $z = i$

(i) To expand f(z) about $z = - i$ i.e. in power of z+i, put z+i=t. Then

$$f(z)=\frac{1}{(t-i+1)^2}=(1-i)^{-2}[1+t/(1-i)]^{-2}$$

$$=\frac{i}{2}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}-\frac{4t^3}{(1-i)^3}+\ldots\right]$$

(Expanding by Binomial theorem)

$$=\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(ii) $f(z) = \dfrac{2z^3+1}{z(z+1)} = 2z-2+\dfrac{2z+1}{z(z+1)} = (2i-2)+2(z-i)+\dfrac{1}{z}+\dfrac{1}{z+1}$ ...(1)

(By partial fractions)

To expand 1/z and 1/(z+1) about z - i = t, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1+\frac{t}{i}\right)^{-1}$$ (Expanding by Binomial theorem)

$$= \frac{1}{i}\left[1-\frac{t}{i}+\frac{t^2}{i^2}-\frac{t^3}{i^3}+\frac{t^4}{i^4}-....\infty\right]$$

$$= \frac{1}{i}+\frac{t}{1}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\frac{t^4}{i^5}-....\infty$$

$$= -i+(z-i)+\sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{i^{n+1}} \qquad ...(2)$$

and $\dfrac{1}{z+1} = \dfrac{1}{t+i+1} = \dfrac{1}{1+i}\left(1+\dfrac{t}{1+i}\right)^{-1}$ (Expanding by Binomial theorem)

$$= \frac{1}{1+i}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\frac{t^4}{(1+i)^4}-...\infty\right]$$

$$= \frac{1-i}{2}-\frac{t}{2i}+\left[\frac{t^2}{(1+i)^3}-\frac{t^3}{(1+i)^4}+\frac{t^4}{(1+i)^5}-...\infty\right]$$

$$= \frac{1}{2}-\frac{i}{2}-\frac{z-i}{2i}+\sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{(1+i)^{n+1}} \qquad ...(3)$$

Substituting from (2) and (3) in (1) we get

$$f(z) = \left(2i-2-i+\frac{1}{2}-\frac{i}{2}\right)+\left(2+1-\frac{1}{2i}\right)(z-i)+\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

$$= \left(\frac{i}{2}-\frac{3}{2}\right)+\left(3+\frac{i}{2}\right)(z-i)+\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

Example 6: Find the Laurents series expansion of $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ above $z = 1$

Solution: $f(z) = \dfrac{e^{2z}}{(z-1)^3}$

Here we have to expand f(z) in Laurents series as powers of (z - 1)

Put $z - 1 = u$ i.e., $z = u + 1$

$$\therefore f(z) = \frac{e^{2u+2}}{u^3} = \frac{e^2}{u^3}\left[1 + \frac{(2u)}{1!} + \frac{(2u)^2}{2!} + \ldots\right]$$

$$= e^2\left[\frac{1}{u^3} + \frac{2u}{u^3} + \frac{(2u)^2}{2u^3} + \frac{(2u)^3}{3!\,u^3} + \ldots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2}{3}(z-1) + \ldots\infty\right]$$

The series is valid when $|z - 1| > 0$

Example 7: Find the Laurents series of $f(z) = \dfrac{1}{(z-1)(z-2)}$ in $|z| > 2$

Solution: $f(z) = \dfrac{1}{(z-1)(z-2)}$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)} \quad \text{(using partial fraction)}$$

In the region $|z| > 2$ the Laurents series is

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z)=-\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+\ldots\right)+\frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^{2}+\ldots\right)$$

Example 8: Find the Laurents expansion of $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in

(i) $|z|>3$ (ii) $2<|z|<3$

Solution:

$$f(z)=\frac{z^2}{(z+2)(z+3)}=A+\frac{B}{(z+2)}+\frac{C}{(z+3)}$$

$$z^2-1=A(z+2)(z+3)+B(z+3)+C(z+2)$$

put $z=-3$ $\quad -C=8 \quad \therefore C=8$

put $z=-2$ $\quad B=3$

Equating the coefficient of z^2, $\quad A=1$

$$\therefore f(z)=1+\frac{3}{(z+2)}-\frac{8}{(z+3)}$$

(i) $|z|>3$

$$\therefore f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z)=1+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^{2}-\ldots\right]-\frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^{2}-\ldots\right]$$

In the above expansion the first series is valid when $\left|\frac{2}{z}\right| < 1$ i.e. $2 < |z|$

In the second series valid for $\left|\frac{3}{z}\right| < 1$ i.e. $3 < |z|$

$\therefore$ The whole expansion is valid when $|z| > 3$

(ii) $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right) - \ldots\right] - \frac{3}{8}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \ldots\right]$$

In the above expansion the first series is valid when $\left|\frac{2}{z}\right| < 1$ i.e. $2 < |z|$

In the second expansion is valid when $\left|\frac{z}{3}\right| < 1$ i.e. $|z| < 3$

$\therefore$ The whole expansion is valid $2 < |z| < 3$

Example 9: Find the Laurents Expansion of the function $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in the annulus $1 < |z+1| < 3$

Solution: put $z + 1 = u$

$z = u - 1$

$$f(z) = \frac{7(u-1)-2}{(u-1)u(u-3)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$=-\frac{3}{u}+\frac{1}{u-1}+\frac{2}{u-3}$$

(using partial fraction), $1<|u|<3$

$$=-\frac{3}{u}+\frac{1}{u\left(1-\frac{1}{u}\right)}-\frac{2}{3\left(1-\frac{u}{3}\right)}$$

$$=-\frac{3}{u}+\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$=-\frac{3}{u}+\frac{1}{u}+\left[1+\frac{1}{u}+\frac{1}{u^2}+\ldots\right]-\frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\ldots\ldots\right]$$

$$=\left[\frac{-2}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\ldots\ldots\ldots\right]-\frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\ldots\ldots\right]$$

$$=\left[\frac{-2}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\ldots\right]-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\left(\frac{(z+1)}{3}\right)^2+\ldots\ldots\right]$$

clearly this series is valid in the region $1<|z+1|<3$

Example 10:

Find the Laurents series expansion of the function $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3<|z+2|<5$

Solution :

$$\text{Put } z+2=u$$
$$z=u-2$$

$$\therefore\ f(z)=\frac{(u-2)^2-6(u-2)-1}{(u-2-1)(u-2-3)u}$$

$$=\frac{u^2-4u+4-6u+12-1}{(u-3)(u-5)u}=\frac{u^2-10u+15}{u(u-3)(u-5)}$$

and the region is $3 < |u| < 5$

$$= \frac{1}{u} + \frac{1}{u-3} - \frac{1}{u-5}$$ (using partial fractions)

$$= \frac{1}{u} + \frac{1}{u\left(1-\frac{3}{u}\right)} + \frac{1}{5\left(1-\frac{u}{5}\right)}$$

$$= \frac{1}{u} + \frac{1}{u}\left(1-\frac{3}{u}\right)^{-1} + \frac{1}{5}\left(1-\frac{u}{5}\right)^{-1}$$

$$= \frac{1}{u} + \frac{1}{u}\left[1+\frac{3}{u}+\frac{3^2}{u^2}+\ldots\right] + \frac{1}{5}\left[1+\frac{u}{5}+\frac{u^2}{5^2}+\ldots\right]$$

$$= \left(\frac{2}{u}+\frac{3}{u^2}+\frac{3^2}{u^3}+\ldots\right) + \frac{1}{5}\left(1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\ldots\right)$$

Clearly this series valid when $\left|\frac{3}{u}\right| < 1$ i.e., $3 < |u|$ and $\left|\frac{u}{5}\right| < 1$ i.e. $|u| < 5$

i.e. $3 < |u| < 5$

i.e. in the annulus $3 < |z+2| < 5$

Example 11: Find the Laurents series about the point $z = 0$ for the function $\frac{z^2-1}{z^2+5z+6}$ in the region $2 < |z| < 3$.

Solution: $f(z) = \frac{z^2-1}{z^2+5z+6}$

$$=\frac{z^2-1}{(z+3)(z+2)}$$

$$=A+\frac{B}{(z+3)}+\frac{C}{(z+2)}$$ (since the Nr. Degree is equal to Dr. degree

$$=1+\frac{3}{(z+2)}-\frac{8}{(z+3)}$$ (using partial fraction)

Now in the region $2<|z|<3$ the Laurents series is

$$=1+\frac{3}{\left(z+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$==1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$==1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\ldots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\ldots\right]$$

clearly the above expansion is valid for $2<|z|<3$

Example 12: Find the Laurents series expansion for the function

$f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$ about $z=-2$.

Solution: Given $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$

Putting $z+2=u$

$$f(z)=(u-3)\sin\left(\frac{1}{u}\right)$$

$$=(u-5)\left[\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}-\ldots\infty\right]$$

$$f(z) = 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \ldots$$

Example 13: Expand f(z) = 1/(z – 1)(z – 2) in the region:

(a) $|z| < 1$,
(b) $1 < |z| < 2$,
(c) $|z| > 2$,
(d) $0 < |z-1| < 1$.

Solution:

(a) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(i)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \qquad \ldots(ii)$$

For | z | < 1, both | z/2 | and | z | are less than 1. Hence (ii) gives an expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + \left(1 + z + z^2 + z^3 + \ldots\right)$$

$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots$ which is a Taylor's series.

(b) For $1 < |z| < 2$, we write (i) as

$$f(z) = -\frac{1}{2}\frac{1}{(1 - z/2)} - \frac{1}{z(1 - z^{-1})} \qquad \ldots(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (iii) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) - \frac{1}{z}\left(1 + z^{-1} + z^{-2} + z^{-3} + \ldots\right)$$

$$= \ldots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \ldots$$

which is a Laurent's series.

(c) For $|z| > 2$, we write (i) as

$$f(z) = \frac{1}{z(1-2z^{-1})} - \frac{1}{z(1-z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + \ldots) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= \ldots + 8z^{-4} + 4z^{-3} + 2z^{-2} - 1 - z - z^{2} - \ldots$$

(d) For $0 < |z-1| < 1$, we write (i) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots]$$

(1) Zeroes of an analytic function

Def . A zero of an analytic function f(z) is that value of z for which f(z) = 0.

(2) Singularities of an analytic function

Def . A singular point of a function is the point at which the function ceases to be analytic.

(i) **Isolated Singularity .** It z =a is a singularity of f(z) such that f(z) is analytic at each point in its neighbourhood (i.e., there exists a circle with centre a which has no other singularity) , then z = a is called an **isolated singularity.**

In such a case, f(z) can be expanded in a Laurent's series around z = a, giving

$$f(z) = a_0 + a_1 (z - 1) + a_2 (z - a)^2 + ... + b_1 (z - 1)^{-1} + b_2 (z - a)^{-2} + ... \quad ...(1)$$

For example , f(z) = cot (π / z) is not analytic where tan (π / z)=0 i.e., at the points π / z = 4π or z = 1 / n (n = 1,2,3,....)

Thus z = 1 , ½ , 1/3 , ... are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large , z=0 is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus z = 0 is the *non – isolated singularity* of f(z).

(ii) **Removable Singularity .** If all the negative powers of (z –a) in (1) are zero , then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ .Here the singularity can be removed by defining f(z) at z =a in such a way that it becomes analytic at z = a. Such a singularity is called a *removable singularity.*

Thus if $\underset{x\to a}{\mathrm{Lt}}\, f(z)$ exists finitely , then z = a is a removable singularity

(iii) **Poles .** If all the negative powers of (z – a) in (i) after the n^{th} are missing, then the singularity at z = a is called a **pole of order n**

A pole of first order is called a simple pole.

(iv) **Essential singularity .** If the number of negative powers of (z – a) in (1) is infinite , then z = a is called an essential singularity . In this case,

$\underset{x\to a}{\mathrm{Lt}}\, f(z)$ does not exist.

Example 1 :

Find the nature of singularities of the function

(i) $\dfrac{z - \sin z}{z^2}$

Solution :

Here z = 0 is a singularity.

Also $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\ldots\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \ldots\ldots$$

Since there are no negative powers of z in the expansion , z=0 is a *removable singularity.*

(ii) $(z+1)\sin\dfrac{1}{z-2}$

Solution :

$$(z+1)\sin\frac{1}{z-2}=(t+2+1)\ \sin\frac{1}{t} \qquad \text{where } t=z-2$$

$$=(t+3)\left\{\frac{1}{t}-\frac{1}{3!\,t^3}+\frac{1}{5!\,t^5}-......\right\}$$

$$=\left(1-\frac{1}{3!t^2}+\frac{1}{5!\,t^4}-.....\right)+\left(\frac{3}{t}-\frac{1}{2t^3}+\frac{3}{5!\,t^5}-...\right)$$

$$=1+\frac{3}{t}-\frac{1}{6t^2}-\frac{1}{2t^3}+\frac{1}{120t^4}-......$$

$$=1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+.....$$

Since there are infinite number of terms in the negative powers of (z – 2) is an *essential singularity.*

(iii) $\dfrac{1}{\cos\ z-\sin z}$

Solution: Poles of f(z) = $\dfrac{1}{\cos\ z\ -\sin z}$ are given by equating the denominator to zero , i.e., by cos z – sin z = 0 or tan z = 1 or z = π / 4 is a simple pole of f(z).

Example :

What type of singularity have the following functions :

(i) $\dfrac{1}{1-e^z}$

Solution : Poles of f(z) = $\dfrac{1}{(1-e^z)}$ are found by equating to zero \1 – $e^z = 0$ or $e^z = 1$ =

$e^{2n\pi I}$

$\therefore \quad z = 2n\pi i\,(n = 0, \pm 1, \pm 2, \ldots\ldots.)$

Clearly f(z) has a simple pole at $z = 2\pi i$.

(ii) $\dfrac{e^{2z}}{(z-1)^4}$

Solution :

$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}.e^{2t} \qquad \text{where } t = z-1$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \ldots\ldots\right\}$$

$$= e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \ldots\ldots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \ldots\right\}$$

since there are finite (4) number of terms containing negative powers of (z-1),

$\therefore$ z=1 is a pole of 4 th order.

(iii) ze^{1/z^2}

Solution : $f(z) = ze^{1/z^2} = z\left\{1 + \dfrac{1}{1!z^2} + \dfrac{1}{2!z^4} + \dfrac{1}{3!z^6} + \ldots\right\}$

$$= z + z^{-1} + \frac{z^{-3}}{2} + \frac{z^{-5}}{6} + \ldots\infty$$

since there are infinite number of terms in the negative powers of z , therefore z = 0 is an essential singularity of f(z).

RESIDUES

The co-efficient of $(z-a)^{-1}$ in the expansion of f(z) around an isolated singularity is called the **residue** of f(z) at that point. Thus in the Laurent's series expansion of f(z) around z = a i.e., $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots, + a_{-1}(z-a)^{-2} + \ldots\ldots$, the residue of f(z) at z = a is a_{-1}.

$$\therefore \qquad \text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$$\text{i.e.,} \qquad \int_C f(z)\,dz = 2\pi i \text{ Res } f(a) \qquad \ldots\ldots\ldots\ldots(1)$$

CALCULATION OF RESIDUES

(1) If f(z) has a simple pole at z = a , then

$$\textbf{Res f(a)} = \underset{z\to a}{\text{Lt}}\left[(z-a)f(z)\right]$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2\ldots\ldots\ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughput by z – a , we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots\ldots + c_{-1}$$

Taking limits as z → a , we get

$$\underset{z\to a}{\text{Lt}}\left[(z-a)f(z)\right] = c_{-1} = \text{Res } f(a)$$

(2) Another formula for Res f(a):

Let $f(z) = \phi(z)/\Psi(z)$, where $\Psi(z) = (z-a)F(z)$, $F(a) \neq 0$

$$\text{Then } \underset{z\to a}{\text{Lt}}\left[(z-a)\varphi(z)/\psi(z)\right] = \underset{z\to a}{\text{Lt}} \frac{(z-a)\left[\varphi(a) + (z-a)\varphi'(a) + \ldots\ldots\right]}{\psi(a) + (z-a)\psi'(a) + \ldots\ldots}$$

$$= \underset{z\to a}{\text{Lt}} \frac{\varphi(a) + (z-a)\varphi'(a) + \ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots}, \quad \text{since } \psi(a) = 0$$

Thus $\quad$ **Res f(a)** $= \dfrac{\varphi(a)}{\psi'(a)}$

(3) If f(z) has a pole of order n at z = a , then

$$\textbf{Res f(a)} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$$

Example :

Find the poles and residues of $f(z) = \dfrac{z}{z^2 - 3z + 2}$

Solution :

$$f(z) = \frac{z}{(z-2)(z-1)}$$

To find poles of f(z) put Dr = 0

(ie) (z -2) (z – 1) = 0

∴ $\quad z = 2, 1$ are two simple poles of f(z)

Residue of f(z) at z = 2

$$= \underset{z\to 2}{\text{Lt}}\left[(z-2)\frac{z}{(z-2)(z-1)}\right]$$

$$= \frac{2}{2-1} = 2$$

Residue of f(z) at z = 1

$$= \underset{z\to 1}{\text{Lt}}\left[(z-1)\frac{z}{(z-2)(z-1)}\right]$$

$$= \frac{1}{1-2} = -1$$

Example :

Find the poles and residues of $f(z) = \cot z$.

Solution : $f(z) = \cot z$

$= \cos z / \sin z$

This is of the form

$$f(z) = \phi(z) / \psi(z)$$

poles , $\sin z = 0$

$$z = n\pi \qquad z = 0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots\ldots$$

$\therefore \varphi(a) \neq 0$ and $\psi(a) = 0$

$\therefore$ Residue at $z = a$ is $\dfrac{\varphi(a)}{\Psi'(a)}$

$$z = a = 0, \pm\pi, \pm 2\pi, \ldots$$

$$\text{Residue of } f(z) = \frac{\cos z}{\frac{d}{dz}(\sin z)}$$

$$= \frac{\cos z}{\cos z}$$

$$= 1$$

Example : Find the poles and residues of $f(z) = \dfrac{ze^z}{(z-a)^3}$.

Solution: $f(z) = \dfrac{ze^z}{(z-a)^3}$

$\therefore z = a$ is a pole of order 3.

$$= \frac{1}{(m-1)} \underset{z \to a}{\text{Lt}} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Here m = 3

$$=\frac{1}{2!}\underset{z\to a}{\text{Lt}}\frac{d^2}{dz^2}(z-a)^3\frac{ze^z}{(z-a)^3}$$

$$=\frac{1}{2!}\underset{z\to a}{\text{Lt}}\frac{d^2}{dz^2}\left(ze^z\right)$$

$$=\frac{1}{2}\underset{z\to a}{\text{Lt}}\frac{d}{dz}\left(ze^z+e^z\right)$$

$$=\frac{1}{2}\underset{z\to a}{\text{Lt}}(e^z+ze^z+e^z)$$

$$=\frac{1}{2}(2e^a+ae^a)$$

$$=\frac{1}{2}e^a(2+a)$$

Example: Evaluate the residue at the poles for the function

$$f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$$

Solution: $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ is pole of order 2

$z=2i$ is a simple pole

Residue of f(z) at z = - 1 (pole of order 2)

$$\therefore \underset{z\to -1}{\text{Lt}}\frac{d}{dz}(z+1)^2\frac{z^2-2z}{(z+1)^2(z+4)}$$

$$= \underset{z\to -1}{\text{Lt}} \frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$= \underset{z\to -1}{\text{Lt}} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$$

$$= \frac{(5)(-4)-(3)(-2)}{25}$$

$$= \frac{14}{25}$$

Residue of f(z) at z = 2i (simple pole)

$$= \underset{z\to 2i}{\text{Lt}} (z-2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \underset{z\to 2i}{\text{Lt}} \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \frac{(2i)^2-2(2i)}{(2i+1)^2(4i)}$$

$$= \frac{-4-4i}{(-4+1+4i)4i}$$

$$= \frac{-4(1+i)}{4i(-3+4i)}$$

$$= \frac{-(1+i)}{-3i-4} = \frac{(1+i)}{3i+4} \times \frac{(-3i+4)}{(-3i+4)}$$

$$= \frac{7+i}{25}$$

Residue of f(z) at z = -2i (simple pole)

$$= \underset{z \to -2i}{Lt} (z+2i) \frac{z^2 - 2z}{(z+1)^2 (z+2i)(z-2i)}$$

$$= \underset{z \to -2i}{Lt} \frac{z^2 - 2z}{(z+1)^2 (z-2i)}$$

$$= \frac{(-2i)^2 - 2(-2i)}{(-2i+1)^2 (-4i)}$$

$$= \frac{-4 - +4i}{(-4i)(-4-4i+1)}$$

$$= \frac{-4+4i}{(-4i)(-3-4i)}$$

$$= \frac{1-i}{(i)(3+4i)}$$

$$= \frac{(1-i)}{(3i-4)} \times \frac{(-3i-4)}{(-3i-4)}$$

$$= \frac{(1-i)(3i+4)}{(3i-4)(3i+4)}$$

$$\frac{(1-i)(3i+4)}{(3i-4)(3i+4)}$$

Example: Find poles and residues of $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+1)}$

Solution: $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+1)}$

Poles of f(z) is

$z = -1$ is pole of order 2

$z = i$ is a simple pole

$z = - i$ is a simple pole

Residue at $z = - 1$ (pole of order 2)

$$= \underset{z\to -1}{Lt} \frac{d}{dz}(z+1)^2 \frac{z^2 - 2z}{(z+1)^2(z^2+1)}$$

$$= \underset{z\to -1}{Lt} \frac{d}{dz}\left(\frac{z^2 - 2z}{z^2+1}\right)$$

$$= \underset{z\to -1}{Lt} \frac{(z^2+1)(2z-2)-(z^2 2z)(2z)}{(z^2+1)^2}$$

$$= \frac{(2)(-4)-(3)(-2)}{4}$$

$$= \frac{-8+6}{4}$$

$$= -\frac{2}{4} = -\frac{1}{2}$$

Residue of f(z) at $z = i$ (simple pole)

$$= \underset{z\to -1}{Lt} (z-i)\frac{z^2 - 2z}{(z+1)^2(z^2+1)(z-i)}$$

$$= \underset{z\to i}{Lt} \frac{z^2 - 2z}{(z+1)^2(z+i)}$$

$$= \frac{(i)^2 - 2(i)}{(i+1)^2(2i)}$$

$$= \frac{-(1+2i)}{-4} = \frac{1+2i}{4}$$

Residue of f(z) at z = I (simple pole)

$$= \underset{z\to -i}{\mathrm{Lt}} (z+i) \frac{z^2-2z}{(z+1)^2(z+i)(z-i)}$$

$$= \underset{z\to -i}{\mathrm{Lt}} (z+i) \frac{z^2-2z}{(z+1)^2(z+i)(z-i)}$$

$$= \frac{(-i)^2-2(-i)}{(-i+1)^2(-2i)}$$

$$= \frac{-1+2i}{(-2i)(-2i)} = \frac{-1+2i}{-4} = \frac{1-2i}{4}$$

RESIDUE THEOREM

If f(z) is analytic in a closed curve C except at a finite number of singular points within C, then

$$\int_C f(z)dz = 2\pi i \text{ x (sum of the residues at the singular points within C)}$$

Let us surround each of the singular points a_1, a_2, a_3,.....,a_n by a small circ such that it encloses no other singular point. Then these circles C_1, C_2,... ,C_n together with C, form a multiply connected region in which f(z) is analytic.

∴ Applying Cauchy's theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ...+ \int_{C_n} f(z)dz \qquad \text{by (1)}$$

$$= 2\pi i[\mathrm{Res}\, f(a_1) + \mathrm{Res}\, f(a_2) + ... + \mathrm{Res}\, f(a_n)]$$

which is the desired result.

Example: Use residue theorem to evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz$

Where c is $|z-2|=2$.

Solution: $f(z)=\frac{3z^2+2}{(z-1)(z^2+9)}$

Poles are

$z=1$ is a simple pole

$z=\pm 3i$ are two simple poles.

Here c is the circle $|z-2|=2$.

$\therefore$ $z=1$ is only pole lies inside c.

$\therefore$ By cauchy residue theorem

$\int_C f(z)dz = 2\pi i$ (sum of the residue of f(z) ...(1)

at the poles which lies inside c)

$\therefore$ Residue of f(z) at $z=1$ (simple pole)

$$= \underset{z\to 1}{\mathrm{Lt}}(z-1)\frac{3z^2+2}{(z-1)(z^2+9)}$$

$$=\frac{5}{10}=\frac{1}{2}$$

$$\int_C f(z)dz = 2\pi(1/2)$$

$$=\pi i$$

Example: Determine poles and residues of $f(z) = \dfrac{z}{(1-z)^2(z+2)}$ and hence evaluate

$\int_C f(z)dz$ where c is the curve $|z| = 5/2$

Solution: $f(z) = \dfrac{z}{(1-z)^2(z+2)}$

$\therefore$ poles are z = 1 is poles of order 2 and z = -2 is a simple pole.

Here c is the circle $|z| = 5/2$.

$\therefore$ z = 1 and z = -2 are lying inside c.

$\therefore$ Residue of f(z) at z = 1 (pole of order 2)

$$= \underset{z\to 1}{\mathrm{Lt}} \frac{d}{dz}(z-1)^2 \frac{z}{(1-z)^2(z+2)}$$

$$= \underset{z\to 1}{\mathrm{Lt}} \frac{d}{dz}\left(\frac{z}{z+2}\right)$$

$$= \underset{z\to 1}{\mathrm{Lt}} \frac{(z+2)(1)-(z)(1)}{(z+2)^2}$$

$$\frac{3-1}{9} = \frac{2}{9}$$

Residue of f(z) at z = - 2 (simple pole)

$$= \underset{z\to -2}{\mathrm{Lt}} (z+2)\frac{z}{(1-z)^2(z+2)}$$

$$= \frac{(-2)}{(-2-1)^2} = -\frac{2}{9}$$

$\therefore$ By Cauchy residue theorem

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residues of f(z) at the poles which lies inside c)}$$

$$= 2\pi i\left(\frac{2}{9}\right)-\left(\frac{2}{9}\right)$$

$$= 0$$

Example: Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where $|z| = 3$.

Solution: $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are

$z = 1$ simple pole

$z = 2$ simple pole

Here the circle is $|z| = 3$

$\therefore$ Both $z = 1$ & $z = 2$ lies inside c

$\therefore$ Residue of f(z) at $z = 1$

$$= \lim_{z\to 1} (z-1)\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin \pi + \cos \pi}{-1} = \frac{-1}{-1} = 1$$

Residue of f(z) at $z = 2$.

$$= \lim_{z\to 2} (z-2)\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1} = \frac{1}{1} = 1$$

$\therefore$ By residue theorem

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residues at the interior poles)}$$

$$= 2\pi i\,(1+i)$$

$$= 4\pi i$$

Example :

Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z| = 3/2$

Solution :

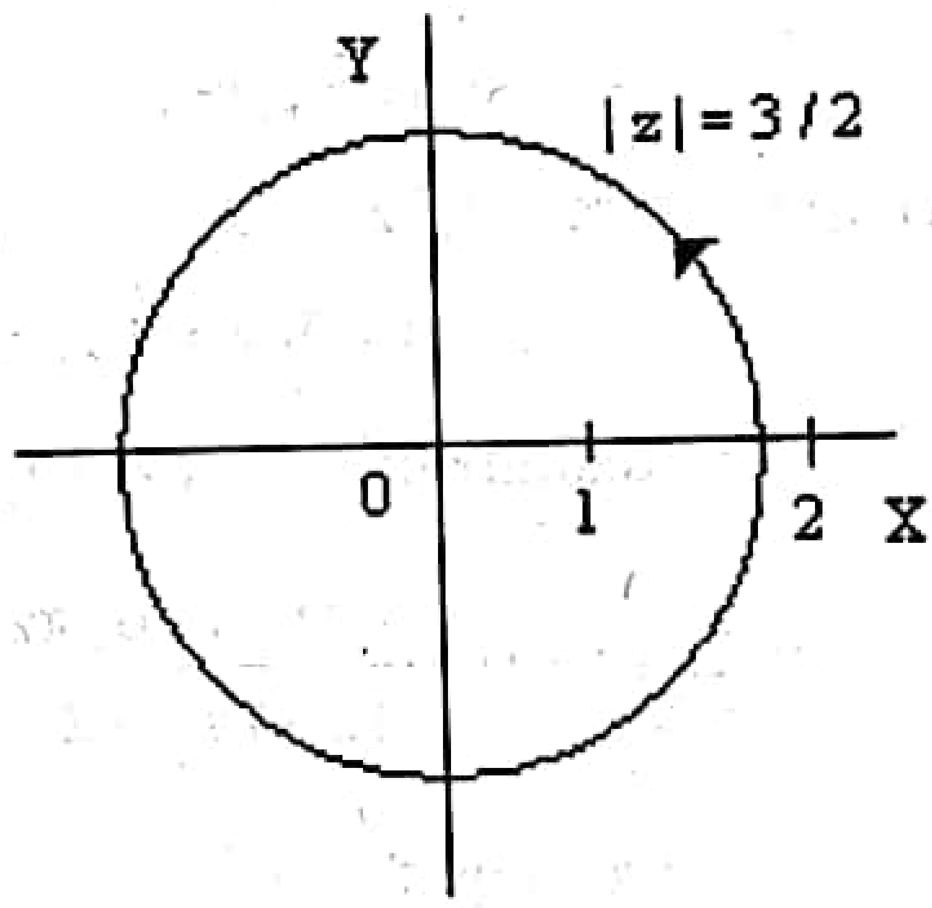
$$f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

$\therefore$ The poles are

$z = 0$ simple pole

$z = 1$ simple pole

$z = 2$ simple pole



Here the circle is $|z| = 3/2$

$\therefore$ $z = 0$ & $z = 1$ lie inside

c and $z = 2$ lies outside c.

$\therefore$ Residue of $f(z)$ at $z = 0$ simple pole

$$= \underset{z \to 0}{\text{Lt}}\; z\,\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4}{2} = 2$$

Residue of f(z) at z =1 is

$$\underset{z\to 1}{Lt}\ (z-1)\frac{4-3z}{z(z-2)}$$

$$=\frac{1}{1(-1)}=-1$$

$\therefore$ By cauchy integral theorem

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of the residues at the interior poles})$$

$$= 2\pi i \quad (2-1)$$

$$= 2\pi i$$

Example :

Evaluate $\int_C \frac{dz}{(x^2+4)^2}$ around the closed contour $|z-i| = 2.$

Solution :

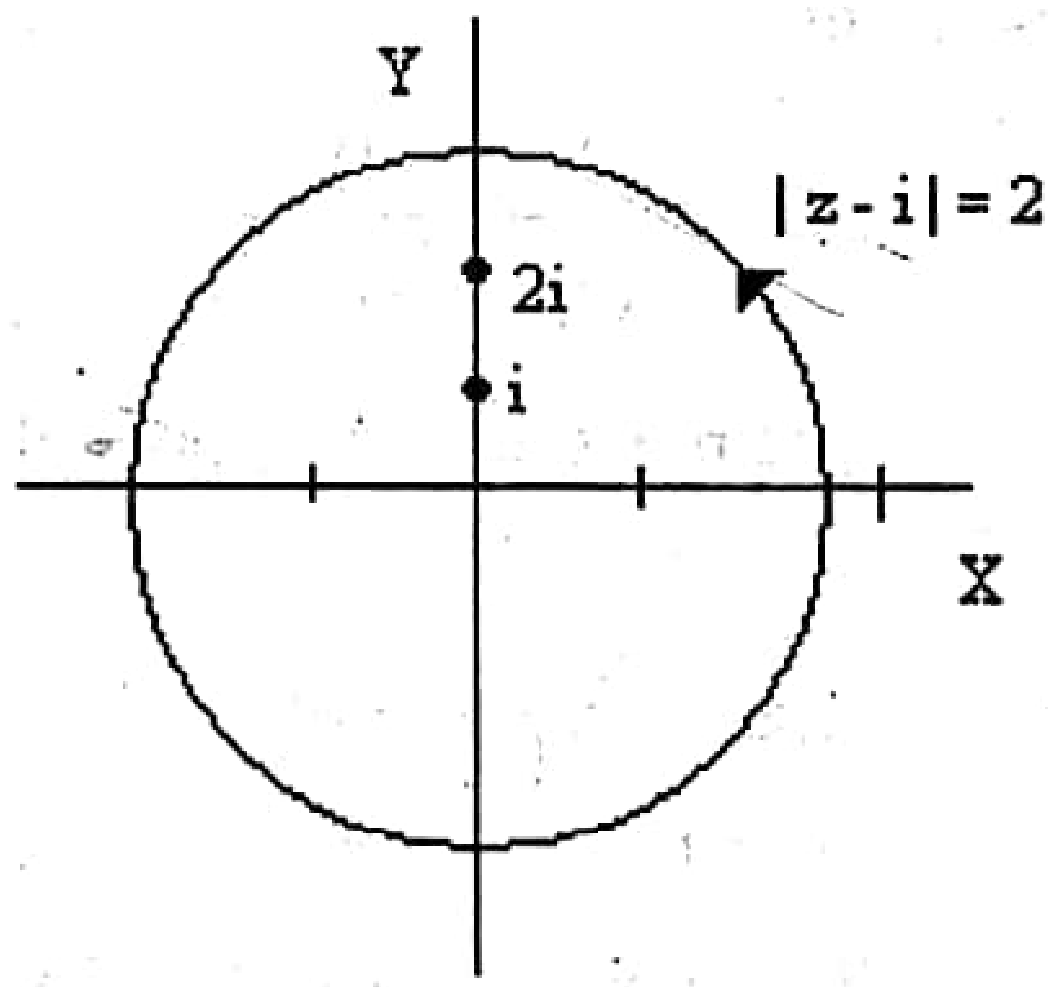
$$f(z) = \frac{1}{(z^2+4)^2}$$

The poles are $z = \pm 2i$ are pole of order 2.

Here the circle is $|z-i|=2$

$\therefore$ $z = 2i$ is the only pole lies inside c.

$\therefore$ Residue of f(z) at $z = 2i$ (pole of order 2)



$$= \underset{z\to 2i}{Lt}\ \frac{d}{dz}(z-2i)^2.\frac{1}{(z-2i)^2(z+2i)^2}$$

$$= \underset{z\to 2i}{Lt} \frac{(z+2i)^2(0)-1(2)(z+2i)}{(z+2i)^4}$$

$$= \underset{z\to 2i}{Lt} \frac{-(2z+4i)}{(z+2i)^4} = \frac{-(4i+4i)}{(4i)^4} = -\frac{i}{32}$$

$\therefore$ By cauchy's integral theorem

$$\int_C f(z)\ dz = 2\pi i \quad (\text{ sum of the residues at the interior poles})$$

$$= 2\pi i\left(\frac{-i}{32}\right) = \frac{\pi}{16}$$

Example : Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}$ where c is the circle $|z-i| = 2$.

Solution :

$$f(z) = \frac{(z-1)}{(z+1)^2(z-2)}$$

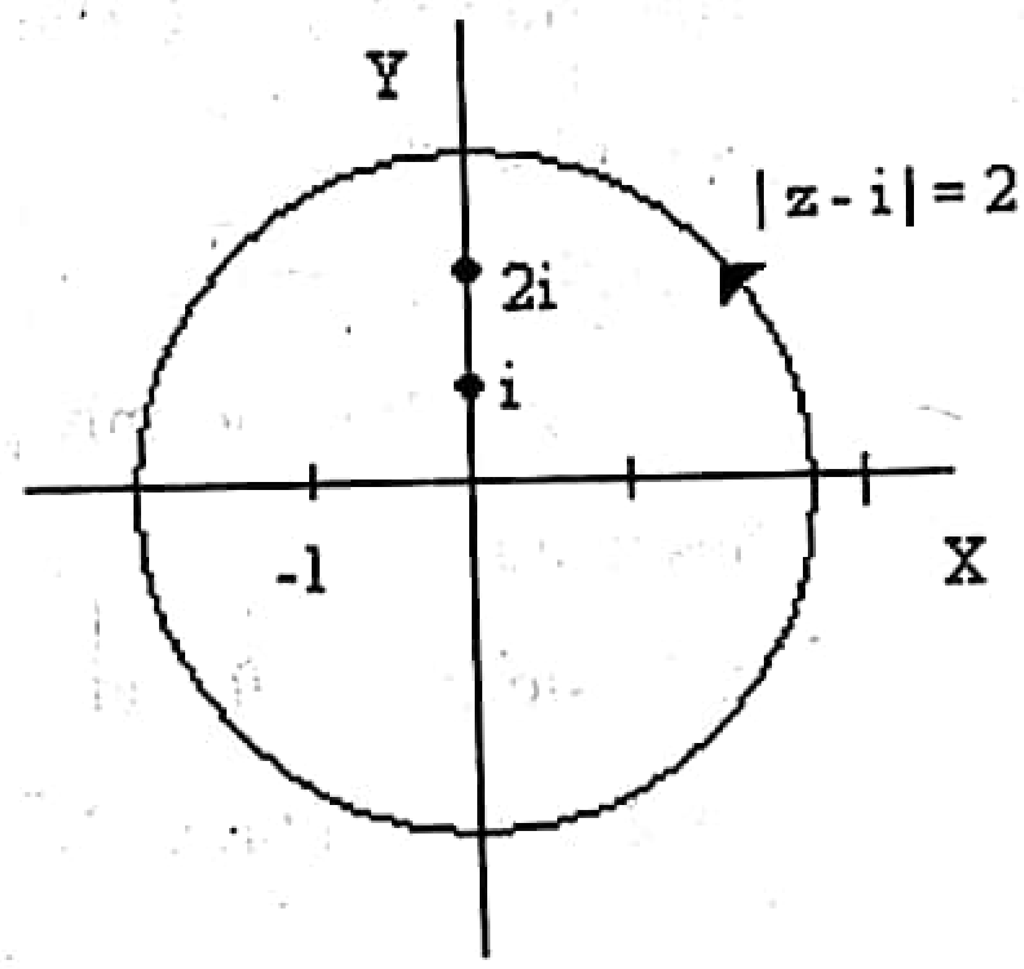
The poles are $z = -1$ pole of order 2

& $z = 2$ simple pole

Here the circle is $|z-i| = 2$

Therefore $z = -1$ is the only pole

Lies inside c.



Therefore Residue of f(z) at

z = -1 (pole of order 2)

$$= \underset{z\to -1}{Lt} \frac{d}{dz}(z+1)^2 \cdot \frac{(z-1)}{(z+1)^2(z-2)}$$

$$= \underset{z\to -1}{Lt} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \underset{z\to -1}{Lt} \frac{(z-2)(1)-(z-1)\,(1)}{(z-2)^2}$$

$$= \frac{(-3)-(-2)}{9} = -\frac{1}{9}$$

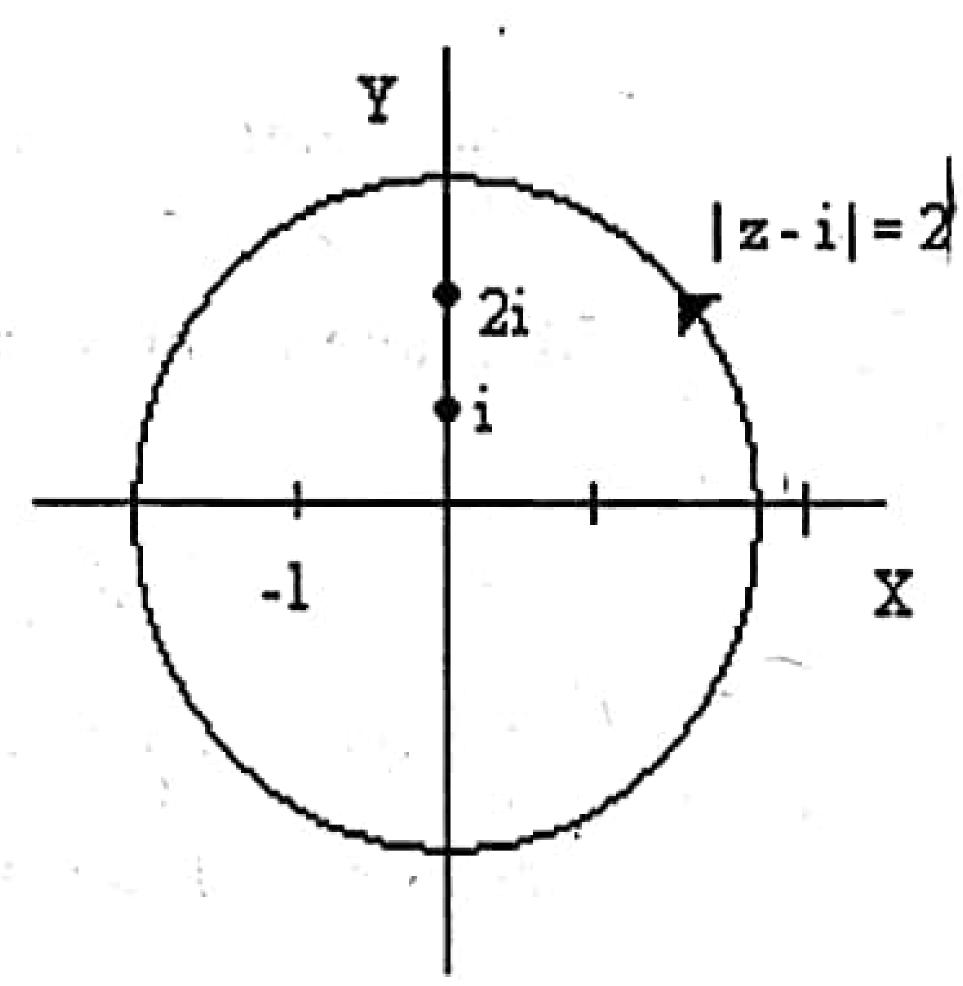
$\therefore$ By Residue theorem

$$\int_C f(z)\, dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Example :

Find the poles and residues of

$$f(z) = \frac{z-3}{(z+1)^2(z-2)}$$



The poles are z = -1 pole of order 2

& z = 2 simple pole

Here the circle is $|z-i| = 2$

Here z = -1 is the only pole lies

Inside c.

Therefore Residue at z = -1

(pole of order 2)

$\boxed{\textit{Example 4}}$ ***If $f(z) = \sin z$ is an analytic function, prove that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.***

Solution : Given :
$$\begin{aligned} f(z) &= \sin z = \sin(x + iy) \\ &= \sin x \cos(iy) + \cos(x)\sin(iy) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

$$\text{Consider } u(x, y) = c_1$$

$$\sin x \cosh y = c_1 \qquad \ldots (1)$$

Differentiating (1) partially with respect to x, we get

$$\sin x \sinh y \frac{dy}{dx} + \cos x \cosh y = 0$$

$$\frac{dy}{dx} = -\frac{\cos x \cosh y}{\sin x \sinh y}$$

$$m_1 = -\cot x \coth y$$

$$\text{Again consider } v(x, y) = c_2$$

$$\cos x \sinh y = c_2. \qquad \ldots (2)$$

Differentiating partially with respect to x, we get

$$-\sin x \sinh y + \cos x \cosh y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\sin x \sinh y}{\cos x \cosh y}$$

$$m_2 = \tan x \tanh y$$

$$\therefore m_1 m_2 = -1$$

$$u(x, y) = c_1 \text{ and } v(x, y) = c_2 \text{ are orthogonal.}$$

Note : For any analytic function $F(z) = u + iv$, the family of curves $u = c_1$, $v = c_2$ forms an orthogonal system.

Example :

Evaluate $\int_C \frac{z-3}{z^2+2z+5} dz$, where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

Solution:

The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by $z^2+2z+5=0$

i.e., by $z = \frac{-2\pm\sqrt{(4-20)}}{2} = -1\pm 2i$

(i) Both the poles $z=-1+2i$ and $z=-1-2i$ lie outside the circle $|z|=1$.

Therefore, f(z) is analytic everywhere within C.

Hence by Cauchy's theorem, $\int_C \frac{z-3}{z^2+2z+5} dz = 0$

(ii) Here only one pole $z=-1+2i$ lies inside the circle C: $|z+1-i|=2$.

Therefore, f(z) is analytic within C except at this pole.

$\therefore$ Res $f(-1+2i) = \underset{z\to -1+2i}{Lt} [\{z-(-1+2i)\}f(z)] = \underset{z\to -1+2i}{Lt} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$

$= \underset{z\to -1+2i}{Lt} \frac{(z-3)}{z+1+2i} = \frac{-4+2i}{4i} = i+1/2$

Hence by residue theorem $\int_C f(z)\,dz = 2\pi i \text{ Res } f(-1+2i) = 2\pi i\,(i+1/2) = \pi(i-2)$

(iii) Here only one pole $z=-1-2i$ lies inside the circle C: $|z+1+i|=2$.

Therefore, f(z) is analytic within C except at this pole.

$\therefore$ Res $f(-1-2i) = \underset{z\to -1-2i}{Lt} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$

$$= \underset{z\to -1-2i}{Lt} \frac{(z-3)}{z+1-2i} = \frac{-4-2i}{-4i} = 1/2 - i$$

Hence by residue theorem

$$\int_C f(z)\,dz$$

$$= 2\pi i \text{ Res } f(-1-2i)$$

$$= 2\pi i\,(1/2 - i)$$

$$= \pi\,(2+i)$$

Example :

Evaluate $\int_C \tan z\,dz$ where C is the circle $|z| = 2$.

Solution:

The poles of $f(z) = \sin z / \cos z$ are given by $\cos z = 0$ i.e., $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$ off these many poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$$\therefore \quad \text{Res } f(\pi/2) = \underset{z\to \pi/2}{Lt} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z\to \pi/2}{Lt} \left(\frac{\sin z}{-\sin z}\right) = -1$$

$$\text{Similarly Res } f(-\pi/2) = \underset{z\to -\pi/2}{Lt} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1$$

Hence by residue theorem ,

$$\int_C f(z)\,dz = 2\pi i \;\{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i\,(-1-1) = -4\pi i$$

Example :

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$, where C is the circle $|z|=3$.

Solution :

$$f(z) = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$$ is analytic within the circle $|z|=3$

excepting the poles $z=1$ and $z=2$.

Since $z=1$ is a pole of order 2.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\left\{(z-1)^2 f(z)\right\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2)f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$

Hence by Residue Theorem,

$$\int_C f(z)\, dz = 2\pi i \left[\text{Res } f(1) + \text{Res } f(2)\right] = 2\pi i(2\pi + 1 + 1) = 4\pi(\pi + 1) i$$

CONTOUR INTEGRATION

VALUATION OF REAL DEFINITE INTEGRALS

any important definite integrals can be evaluated by applying the Residue theorem to operly chosen integrals.

a) Integration around the circle: An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form

$\int_C f(z)dz$, where f(z) is a rational function of z and C is a unit circle $|z| = 1$.

Hence the integral is equal to $2\pi i$ times the sum of the residues at those poles of f(z) which are within C.

Procedure: Integrals of the form $\int_0^{2\pi} \varphi(\cos\theta, \sin\theta)d\theta$ where φ is a rational function of $\cos\theta$ and $\sin\theta$.

Working rule: put $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

since $z = e^{i\theta}$

$$dz = i e^{i\theta}\, d\theta$$

$$d\theta = \frac{dz}{ie^{i\theta}}$$

$$= \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \varphi(\cos\theta, \sin\theta) d\theta = \int_C \varphi\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\frac{dz}{iz}\right]$$ where c is the unit circle

$|z| = 1$

$$= \int f(z) dz$$

$\therefore$ By cauchy residue theorem

$= 2\pi i$ (sum of the residues of f(z) at e poles which lies inside c)

Example: Using method of contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Solution: put $z = e^{i\theta}$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{dz/iz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$ where c is the unit circle $|z| = 1$

$$= \int_C \frac{dz/iz}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)}$$

$$= \int_C \frac{dz/iz}{\frac{4z+z^2+1}{2z}}$$

$$= \frac{2}{i} \int_C \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i} \int_C f(z)dz$$

By cauchy residue theorem

$$= \frac{2}{i} 2\pi i \text{ (sum of the residue of f(z) at the poles lies inside c)}$$

$$= 4\pi \text{ (sum of the residues of f(z) at the poles inside c)}$$

The poles of f(z) are given by the roots of $z^2 + 4z + 1 = 0$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

i.e., $z = -2 + \sqrt{3}$ & $z = 2 - \sqrt{3}$

i.e., $\alpha = -2 + \sqrt{3}$, $\beta = -2 - \sqrt{3}$

But $z = \alpha$ lies inside c

Residue of f(z) at $z = \alpha$.(simple pole).

Residue at the simple pole is given by $\underset{z \to \alpha}{Lt} (z-\alpha) f(z)$

Hence $\underset{z \to \alpha}{Lt} (z-\alpha) \frac{1}{(z-\alpha)(z-\beta)}$

$$= \frac{1}{\alpha - \beta}$$

$$= \frac{1}{(-2+\sqrt{3}-(-2-\sqrt{3}))}$$

$$= \frac{1}{2\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{dz}{2+\cos\theta} = 4\pi\left(\frac{1}{2\sqrt{3}}\right)$$

$$= \frac{2\pi}{\sqrt{3}}$$

Example: Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \int_0^{2\pi} \frac{R.P.e^{i2\theta}}{5+4\cos\theta} d\theta$$

$$= R.P. \int_0^{2\pi} \frac{(e^{i\theta})^2}{5+4\cos\theta} d\theta$$

put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$= R.P. \int_C \frac{z^2 dz / iz}{5+4\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$=\frac{1}{i}\,\text{R.P.}\int_C \frac{z^2\,dz}{5z+2z^2+2}$$

$$=\frac{1}{i}\,\text{R.P.}\int_C f(z)dz$$

$$\text{where } f(z)=\frac{z^2}{2z^2+5z+2}$$

$$=\frac{1}{i}\,\text{R.P.}\,2\pi i\text{ (sum of the residue of f(z) at its interior poles)}$$

$$=\text{R.P. }2\pi\text{ (sum of the residue of f(z) at its interior poles)}$$

For poles of f(z) put Dr.=0

i.e., $2z^2 + 5z + 2 = 0$

$2z(z+2) + 1(z+2) = 0$

$(2z+1)(z+2) = 0$

$z = -2,\ -1/2$

But only $z = -1/2$ lies inside c,

Hence Residue of f(z) at $z = -1/2$ is

$$= \underset{z\to-\frac{1}{2}}{\text{Lt}}\left(z+\frac{1}{2}\right)f(z)$$

$$= \underset{z\to-\frac{1}{2}}{\text{Lt}}\left(z+\frac{1}{2}\right)\frac{z^2}{(z+2)(2z+1)}$$

$$= \underset{z\to -\frac{1}{2}}{\mathrm{Lt}}\left(z+\frac{1}{2}\right)\frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1/4}{(2)(3/2)} = \frac{1}{12}$$

$$\therefore \text{R.P.} \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta = 2\pi\left(\frac{1}{12}\right)$$

$$\text{R.P.} \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta} d\theta = \left(\frac{\pi}{6}\right)$$

$$\text{i.e., } \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \left(\frac{\pi}{6}\right)$$

Example: By integrating around a unit circle, evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$.

Putting $z = e^{i\theta}$, $d\theta = dz / iz$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

and $\cos 3\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right) = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

Hence the given integral $I = \int_C \frac{\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)}{5-2\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z^2-5z+2)}dz = -\frac{1}{2i}\int_C \frac{(z^6+1)dz}{z^3(2z-1)(z-2)}$$

$$= -\frac{2}{2i}\int f(z)dz \text{ where C is the unit circle } |z| = 1.$$

Now f(z) has a pole of order 3 at z = 0 and simple poles at z = ½ and z = 2. Of these only z = 0 and z = ½ lie within the unit circle.

$$\therefore \operatorname{Res} f(1/2) = \operatorname*{Lt}_{z\to 1/2} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \operatorname*{Lt}_{z\to 1/2}\left(\frac{z^6+1}{2z^3(z-2)}\right) = -\frac{65}{24}$$

$$\operatorname{Res} f(0) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right)_{z=0}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right]_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z=0$$

$$= \left[\frac{d}{dz}\left(\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^4}\right)\right]_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2 5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2x16} = \frac{84}{32} = \frac{21}{8}$$

$$\text{Hence } I = -\frac{1}{2i}[2\pi i(\operatorname{Res} f(1/2) + \operatorname{Res} f(0))] = -\pi\left(-\frac{65}{24}+\frac{21}{8}\right) = -\pi\left(-\frac{1}{12}\right) = \frac{\pi}{12}.$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III -ENGINEERING MATHEMATICS-III– SMTA1301

UNIT-III
Z -TRANSFORMATIONS AND DIFFERENCE EQUATIONS

Definition:

Let $\{f(n)\}$be a sequence defined for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$then the Z-transform of $f(n)$is defined as $Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} = F(z)$ which is known as two sided or Bilateral Z-transform of $f(n)$.

If $f(n) = 0$ for $n < 0$, then the Z-transform reduces to one sided or Unilateral Z-transform and is defined as $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = F(z)$.

Z-transform for discrete values of t:

If the function $f(t)$ is defined at discrete values of t, where $t = nT,\ n = 0,1,2,3,\ldots\infty$,T being the sampling period, then $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n} = F(z)$.

Z-transform of standard functions:

1. $Z\{a^n\} = \frac{z}{z-a}$ if $|z| > |a|$

   Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}$$

Note:

(i) When a = 1, $Z\{1\} = \frac{z}{z-1}$

(ii) When a = -1, $Z\{(-1)^n\} = \frac{z}{z+1}$

2. $Z(k) = k\frac{z}{z-1}$

Proof: By definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$= \sum_{n=0}^{\infty} k z^{-n} = k\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$Z\{k\} = k\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right] = k\left[1 - \frac{1}{z}\right]^{-1} = k\left[\frac{z-1}{z}\right]^{-1} = k\frac{z}{z-1}$$

3. $Z\{n\} = \frac{z}{(z-1)^2}$

Proof: By definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{n\} = \sum_{n=0}^{\infty} n\, z^{-n} = \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right] = \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} = \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z^2}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\therefore Z\{n\} = \frac{z}{(z-1)^2}$$

4. $Z\left\{\frac{1}{n}\right\} = log\left(\frac{z}{z-1}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$Z\left\{\frac{1}{n}\right\} = \sum_{n=0}^{\infty}\frac{1}{n}\mathrm{z}^{-\mathrm{n}} = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{\mathrm{z}}\right)^{\mathrm{n}} = \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 \ldots$$

$$= -log\left(1 - \frac{1}{z}\right) = -log\left(\frac{z-1}{z}\right) \left[\because \log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty\right]\right.$$

$$\therefore\ Z\left\{\frac{1}{n}\right\} = log\left(\frac{z}{z-1}\right)$$

5. $Z\left\{\frac{1}{n+1}\right\} = zlog\left(\frac{z}{z-1}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$Z\left\{\frac{1}{n+1}\right\}=\sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{z}\right)^{n}=1+\frac{1}{2}\left(\frac{1}{z}\right)+\frac{1}{3}\left(\frac{1}{z}\right)^{2}+\frac{1}{4}\left(\frac{1}{z}\right)^{3}+\cdots$$

$$=z\left[\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^{2}+\frac{1}{3}\left(\frac{1}{z}\right)^{3}+\cdots\right]$$

$$=-z\log\left(1-\frac{1}{z}\right)=-z\log\left(\frac{z-1}{z}\right)\left[\because \log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\infty\right]\right.$$

$$\therefore\ Z\left\{\frac{1}{n+1}\right\}=z\log\left(\frac{z}{z-1}\right)$$

6. $Z\left\{\frac{1}{n!}\right\}=e^{\frac{1}{z}}$

Proof: By the definition $Z\{f(n)\}=\sum_{n=0}^{\infty}f(n)\,z^{-n}$

$$Z\left\{\frac{1}{n!}\right\}=\sum_{n=0}^{\infty}\frac{1}{n!}z^{-n}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^{n}=1+\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^{2}+\frac{1}{3!}\left(\frac{1}{z}\right)^{3}+\cdots$$

$$\therefore\ Z\left\{\frac{1}{n!}\right\}=e^{\frac{1}{z}}$$

7. $Z\{na^n\}=\frac{az}{(z-a)^2}$

Proof: By the definition $Z\{f(n)\}=\sum_{n=0}^{\infty}f(n)\,z^{-n}$

$$Z\{na^n\}=\sum_{n=0}^{\infty}n\,a^n\,z^{-n}=\sum_{n=0}^{\infty}n\left(\frac{a}{z}\right)^{n}=\frac{a}{z}+2\left(\frac{a}{z}\right)^{2}+3\left(\frac{a}{z}\right)^{3}+\cdots$$

$$=\frac{a}{z}\left[1+2\left(\frac{a}{z}\right)+3\left(\frac{a}{z}\right)^{2}+\cdots\right]$$

$$=\frac{a}{z}\left(1-\frac{a}{z}\right)^{-2}=\frac{a}{z}\left(\frac{z-a}{z}\right)^{-2}=\frac{az}{(z-a)^2}.$$

8. $Z\{\cos n\theta\}=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$ and $Z\{\sin n\theta\}=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$

Proof: Let $a=e^{i\theta}$

We know that $Z\{a^n\}=\frac{z}{z-a}\ \Rightarrow Z\left\{\left(e^{i\theta}\right)^{n}\right\}=\frac{z}{z-e^{i\theta}}=\frac{z}{z-(\cos\theta+i\sin\theta)}$

$$Z\{(cos\theta + isin\theta)^n\} = \frac{z}{(z - cos\theta) - isin\theta}$$

$$Z\{cosn\theta + isinn\theta\} = \frac{z[(z - cos\theta) + isin\theta]}{\left((z - cos\theta)\right)^2 + sin^2\theta} = \frac{z(z - cos\theta) + izsin\theta}{z^2 - 2zcos\theta + cos^2\theta + sin^2\theta}$$

$$Z\{cosn\theta\} + iZ\{sinn\theta\} = \frac{z(z - cos\theta) + izsin\theta}{z^2 - 2zcos\theta + 1} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1} + i\frac{zsin\theta}{z^2 - 2zcos\theta + 1}$$

Equate the real and imaginary parts on both sides, we get

$$Z\{cosn\theta\} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}$$

$$Z\{sinn\theta\} = \frac{zsin\theta}{z^2 - 2zcos\theta + 1}$$

Note: When $\theta = \frac{\pi}{2}$

$$Z\left\{cosn\frac{\pi}{2}\right\} = \frac{z^2}{z^2+1} \text{ and } Z\left\{sinn\frac{\pi}{2}\right\} = \frac{z}{z^2+1}$$

9. $Z\{coshn\theta\} = \frac{z(z - cosh\theta)}{z^2 - 2zcosh\theta + 1}$

Proof: $Z\{coshn\theta\} = Z\left\{\frac{e^{n\theta} + e^{-n\theta}}{2}\right\} = \frac{1}{2}Z\left\{\left(e^{\theta}\right)^n + \left(e^{-\theta}\right)^n\right\}$

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} + \frac{z}{z - e^{-\theta}}\right] = \frac{z}{2}\left[\frac{z - e^{-\theta} + z - e^{\theta}}{(z - e^{\theta})(z - e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - \left(e^{\theta} + e^{-\theta}\right)}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right] = \frac{z(z - cosh\theta)}{z^2 - 2zcosh\theta + 1}$$

10. $Z\{sinhn\theta\} = \frac{zsinh\theta}{z^2 - 2zcosh\theta + 1}$

Proof: $Z\{sinhn\theta\} = Z\left\{\frac{e^{n\theta} - e^{-n\theta}}{2}\right\}$

$$= \frac{1}{2}Z\left\{\left(e^{\theta}\right)^n - \left(e^{-\theta}\right)^n\right\}$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} - \frac{z}{z - e^{-\theta}}\right] = \frac{z}{2}\left[\frac{z - e^{-\theta} - z + e^{\theta}}{(z - e^{\theta})(z - e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{\left(e^{\theta} + e^{-\theta}\right)}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right] = \frac{zsinh\theta}{z^2 - 2zcosh\theta + 1}$$

11. **Z-Transform of unit step function**:

Unit step function is denoted by $u(n)$ and is defined by $u(n) = \begin{cases} 1, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$

By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$\therefore\ Z\{u(n)\} = \sum_{n=0}^{\infty} u(n)\, \mathrm{z}^{-\mathrm{n}} = \sum_{n=0}^{\infty} 1.\, \mathrm{z}^{-\mathrm{n}} = \sum_{n=0}^{\infty} \left(\frac{1}{\mathrm{z}}\right)^{\mathrm{n}}$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots \infty = \left[1 - \frac{1}{z}\right]^{-1} = \left[\frac{z-1}{z}\right]^{-1} = \frac{z}{z-1}$$

$$\therefore\ Z\{u(n)\} = \frac{z}{z-1}$$

Note: $Z\{u(n-k)\} = \mathrm{z}^{-\mathrm{k}} Z\{u(n)\} = \mathrm{z}^{-\mathrm{k}} \frac{z}{z-1}$

12. **Z-Transform of unit impulse function**:

Unit impulse function is denoted by $\delta(n)$ and is defined by $\delta(n) = \begin{cases} 1, for\ n = 0 \\ 0, for\ n \neq 0 \end{cases}$

By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$\therefore\ Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)\, \mathrm{z}^{-\mathrm{n}} = 1$$

Note: $Z\{\delta(n-k)\} = \mathrm{z}^{-\mathrm{k}} Z\{\delta(n)\} = \frac{1}{\mathrm{z}^{\mathrm{k}}}$

Properties of Z-transform:

1. Linearity property

If $Z\{f(n)\} = F(z)\ and Z\{g(n)\} = G(z)$ then $Z[af(n) \pm bg(n)] = aZ\{f(n)\} \pm bZ\{g(n)\}$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$Z\{[af(n) \pm bg(n)]\} = \sum_{n=0}^{\infty} [af(n) \pm bg(n)]\, \mathrm{z}^{-\mathrm{n}} = a \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}} \pm b \sum_{n=0}^{\infty} g(n)\, \mathrm{z}^{-\mathrm{n}}$$

$$= aZ\{f(n)\} \pm bZ\{g(n)\} = aF(z) \pm bG(z)$$

2. Damping Rule

If $Z\{f(n)\} = F(z)$,then $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n)\, \mathrm{z}^{-\mathrm{n}} = \sum_{n=0}^{\infty} f(n)\, \left(\frac{\mathrm{z}}{\mathrm{a}}\right)^{-\mathrm{n}} = F\left(\frac{z}{a}\right)$$

Note: $Z\{a^{-n} f(n)\} = F(az)$

3. Differentiation in Z-domain

If $Z\{f(n)\} = F(z)$,then $Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$

Proof: By the definition $\boldsymbol{F}(\boldsymbol{z}) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

Differentiate w.r.t. z on both sides we get,

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} f(n)(-\mathrm{n})\, \mathrm{z}^{-\mathrm{n}-1} = \mathrm{z}^{-1} \sum_{n=0}^{\infty} -nf(n)\, \mathrm{z}^{-\mathrm{n}}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} nf(n)\, \mathrm{z}^{-\mathrm{n}}$$

$$-z\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} nf(n)\, \mathrm{z}^{-\mathrm{n}}$$

$$\therefore\ Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$$

4. Time shifting property

If $Z\{f(n)\} = F(z)$, then

(i) $Z\{f(n-k)\} = z^{-k} F(z)$

(ii) $Z\{f(n+k)\} = z^k\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, \mathrm{z}^{-\mathrm{n}}$

$$Z\{f(n-k)\} = \sum_{n=0}^{\infty} f(n-k)\, \mathrm{z}^{-\mathrm{n}}$$

Put $n - k = m \Rightarrow n = m + k$

$$Z\{f(n-k)\} = \sum_{m=-k}^{\infty} f(m)\,\mathrm{z}^{-(\mathrm{m+k})} = z^{-k}\sum_{m=-k}^{\infty} f(m)\,\mathrm{z}^{-\mathrm{m}} = z^{-k}\sum_{m=0}^{\infty} f(m)\,\mathrm{z}^{-\mathrm{m}}$$

$\therefore\; Z\{f(n-k)\} = z^{-k}F(z)$

Now

$$Z\{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)\,\mathrm{z}^{-\mathrm{n}}$$

Put $n+k=m \Rightarrow n = m-k$

$$Z\{f(n+k)\} = \sum_{m=k}^{\infty} f(m)\,\mathrm{z}^{-(\mathrm{m-k})} = z^{k}\sum_{m=k}^{\infty} f(m)\,\mathrm{z}^{-\mathrm{m}}$$

$$= z^{k}\left[\sum_{m=k}^{\infty} f(m)\,\mathrm{z}^{-\mathrm{m}} + \sum_{m=0}^{k-1} f(m)\,\mathrm{z}^{-\mathrm{m}} - \sum_{m=0}^{k-1} f(m)\,\mathrm{z}^{-\mathrm{m}}\right]$$

$$= z^{k}\left[\sum_{m=0}^{\infty} f(m)\,\mathrm{z}^{-\mathrm{m}} - \sum_{m=0}^{k-1} f(m)\,\mathrm{z}^{-\mathrm{m}}\right]$$

$$\therefore\; Z\{f(n+k)\} = z^{k}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

Note:

$$Z\{f(t+kT)\} = Z\{f_{n+k}\} = z^{k}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

Problems:

1. Find the Z-transform of $\frac{(n+1)(n+2)}{2}$

Solution: $Z\left\{\frac{(n+1)(n+2)}{2}\right\} = Z\left\{\frac{n^2+3n+2}{2}\right\} = \frac{1}{2}[Z\{n^2\} + 3Z\{n\} + 2Z\{1\}]$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}\right]$$

2. Find the Z-transform of $\frac{1}{n(n+1)}$

Solution: Let $f(n) = \frac{1}{n(n+1)}$

By partial fraction $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)}$

$\Rightarrow 1 = A(n+1) + Bn$

When n = -1⇒B = -1 and n = 0 ⇒ A = 1

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1} \Rightarrow \mathrm{Z}\left\{\frac{1}{n(n+1)}\right\} = \mathrm{Z}\left\{\frac{1}{\mathrm{n}}\right\} - \mathrm{Z}\left\{\frac{1}{\mathrm{n}+1}\right\}$$

$$= \log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right) - \mathrm{z}\log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right) = (1-\mathrm{z})\log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right)$$

3. Find the Z-transform of $\frac{2n+3}{(n+1)(n+2)}$

Solution: Let $f(n) = \frac{2n+3}{(n+1)(n+2)}$

By partial fraction $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$

⇒2n+3= $A(n+2) + B(n+1)$

When n = -2⇒B = 1 and n = -1 ⇒ A = 1

$$\therefore \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2} \Rightarrow \mathrm{Z}\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = \mathrm{Z}\left\{\frac{1}{\mathrm{n}+1}\right\} + \mathrm{Z}\left\{\frac{1}{\mathrm{n}+2}\right\}$$

$$= \mathrm{z}\log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right) + \mathrm{z}^2\log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right) - \mathrm{z} = (\mathrm{z}^2+\mathrm{z})\log\left(\frac{\mathrm{z}}{\mathrm{z}-1}\right) - \mathrm{z}$$

4. Find the Z-transform of $ab^n + 2n$.

   Solution: $Z\{ab^n + 2n\} = aZ\{b^n\} + 2Z\{n\} = a\frac{z}{z-b} + 2\frac{z}{(z-1)^2}$.

5. Find the Z-transform of $f(n) = \begin{cases} 1, for\ n = k \\ 0, otherwise \end{cases}$

   Solution: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\,\mathrm{z}^{-\mathrm{n}}$

   $$Z\{f(n)\} = 1.z^{-k} = \frac{1}{z^k}$$

6. Find the Z- transform of $f(n-5)$

   Solution: We know that $Z\{f(n-k)\} = z^{-k}F(z)$

   $\therefore Z\{f(n-5)\} = z^{-5}Z\{f(n)\} = z^{-5}\frac{z}{z-1}$ (since f(n) is a unit step function)

7. Find the Z- transform of $2^n\delta(n-3)$

Solution: We know that $Z\{\delta(n-k)\} = z^{-k} = \frac{1}{z^k}$

$\therefore Z\{2^n \delta(n-3)\} = [z^{-3} Z\{\delta(n)\}]_{z\to\frac{z}{2}}$

$= \left[\frac{1}{z^3}\right]_{z\to\frac{z}{2}} = \frac{1}{\left(\frac{z}{2}\right)^3} = \frac{8}{z^3}$ (since $\delta(n)$ is a unit impulse function)

8. Find the Z-transform of $\frac{1}{(n+2)!}$

Solution: Let $f(n+2) = \frac{1}{(n+2)!} \Rightarrow f(n) = \frac{1}{n!}$

By shifting theorem

$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$

$\therefore Z\{f(n+2)\} = z^2[F(z) - f(0) - f(1)z^{-1}]$

$F(z) = Z\{f(n)\} = Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

$\therefore Z\left\{\frac{1}{(n+2)!}\right\} = z^2\left[e^{\frac{1}{z}} - 1 - z^{-1}\right]$

9. Find the Z-transform of $r^n cosn\theta \ and \ r^n sinn\theta$

Solution: We know that

$$Z\{cosn\theta\} = \frac{z(z-cos\theta)}{z^2 - 2zcos\theta + 1}$$

By damping rule $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$

$$Z\{r^n cosn\theta\} = \left\{\frac{z(z-cos\theta)}{z^2 - 2zcos\theta + 1}\right\}_{z\to\frac{z}{r}} = \frac{\frac{z}{r}\left(\frac{z}{r} - cos\theta\right)}{\left(\frac{z}{r}\right)^2 - 2\frac{z}{r}cos\theta + 1}$$

$$\therefore Z\{r^n cosn\theta\} = \frac{z(z - rcos\theta)}{z^2 - 2zrcos\theta + r^2}$$

Also $Z\{sinn\theta\} = \frac{zsin\theta}{z^2-2zcos\theta+1}$

$$Z\{r^n sinn\theta\} = \left\{\frac{zsin\theta}{z^2 - 2zcos\theta + 1}\right\}_{z\to\frac{z}{r}} = \frac{\frac{z}{r}sin\theta}{\left(\frac{z}{r}\right)^2 - 2\frac{z}{r}cos\theta + 1}$$

$$\therefore Z\{r^n sinn\theta\} = \frac{zrsin\theta}{z^2 - 2zrcos\theta + r^2}$$

10. Find the Z-transform of $n\ conn\theta$

Solution: We know that

$$Z\{cosn\theta\} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}$$

By the property of Z-transform $Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$

$$Z\{n\ cosn\theta\} = -z\frac{d}{dz}\left\{\frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}\right\}$$

$$= -z\left[\frac{(z^2 - 2zcos\theta + 1)(2z - cos\theta) - (z^2 - zcos\theta)(2z - 2cos\theta)}{(z^2 - 2zcos\theta + 1)^2}\right]$$

$$\therefore Z\{n\ cosn\theta\} = \frac{z(z^2cos\theta - 2z + cos\theta)}{(z^2 - 2zcos\theta + 1)^2}$$

11. Find the Z-transform of $sin^2\left(\frac{n\pi}{4}\right)$

Solution:

$$sin^2\left(\frac{n\pi}{4}\right) = \frac{1-cos2\left(\frac{n\pi}{4}\right)}{2} = \frac{1-cos\left(\frac{n\pi}{2}\right)}{2}$$

$$Z\left\{sin^2\left(\frac{n\pi}{4}\right)\right\} = Z\left\{\frac{1 - cos\left(\frac{n\pi}{2}\right)}{2}\right\} = \frac{1}{2}\left[Z\{1\} - Z\left\{cos\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

Theorems on Z-transform

1. First Shifting Theorem

If $Z\{f(t)\} = F(z)\ then\ Z\{e^{-at}f(t)\} = F(ze^{aT})$

Proof: By the definition of Z-transform $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{e^{-at}f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n} = \sum_{n=0}^{\infty} f(nT)\, (ze^{aT})^{-n}$$

$$\therefore Z\{e^{-at}f(t)\} = F(ze^{aT})$$

Note: $Z\{e^{at}f(t)\} = F\left(\frac{z}{e^{aT}}\right)$

2. **Second Shifting theorem**

If $Z\{f(t)\} = F(z)$ *then* $Z\{f(t+T\} = z\{F(z) - f(0)\}$

Proof: By the definition of Z-transform $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{f(t+T)\} = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n} = \sum_{n=0}^{\infty} f((n+1)T)\, z^{-n}$$

Put n+1 = m $\quad Z\{f(t+T)\} = \sum_{m=1}^{\infty} f(mT) z^{-(m-1)} = z\sum_{m=1}^{\infty} f(mT) z^{-m}$

$$= z\left[\sum_{m=1}^{\infty} f(mT) z^{-m} - f(0)\right] = z\{F(z) - f(0)\}$$

3. **Initial Value theorem**

If $Z\{f(n)\} = F(z)$, *then* $f(0) = \lim_{z\to\infty} F(z)$

Proof: By the definition $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots = f(0) + \frac{f(1)}{z} + \frac{f(1)}{z^2} + \cdots$$

Taking limit as $z \to \infty$ on both sides

$$\lim_{z\to\infty} F(z) = f(0) + 0 + \cdots$$

$$\Rightarrow f(0) = \lim_{z\to\infty} F(z)$$

4. **Final Value Theorem**

If $Z\{f(n)\} = F(z)$, *then* $\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)\, F(z)$

Proof: By the definition $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{f(n+1)-f(n)\}=\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}$$

$$\Rightarrow z\{F(z)-f(0)\}-F(z)=\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}$$

$$(z-1)F(z)-zf(0)=\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}$$

Taking limit $z\to 1$ on both sides

$$\lim_{z\to 1}[(z-1)F(z)-zf(0)]=\lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}$$

$$\lim_{z\to 1}(z-1)\,F(z)-f(0)=f(1)-f(0)+f(2)-f(1)+f(3)-f(2)+\cdots.f(\infty)$$

$$\lim_{z\to 1}(z-1)\,F(z)=f(\infty)=\lim_{n\to\infty}f(n)$$

$$\therefore \lim_{n\to\infty}f(n)=\lim_{z\to 1}(z-1)\,F(z)$$

Convolution of sequences

The convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$ is defined as
$f(n)*g(n)=\sum_{k=0}^{n}f(k)\,g(n-k)$

5. **Convolution Theorem**

If $Z\{f(n)\}=F(z)$ and $Z\{g(n)\}=G(z), then$ $Z\{f(n)*g(n)\}=F(z)G(z)$

Proof: By the definition $\boldsymbol{F}(\boldsymbol{z})=Z\{f(n)\}=\sum_{n=0}^{\infty}f(n)\,z^{-n}$

$$Z\{f(n)*g(n)\}=\sum_{n=0}^{\infty}[f(n)*g(n)]\,z^{-n}$$

$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\,g(n-k)\,z^{-n}$ (By the definition of convolution)

By changing the order of summation

$$Z\{f(n)*g(n)\}=\sum_{k=0}^{n}f(k)\sum_{n=0}^{\infty}g(n-k)z^{-n}$$

$$=\sum_{k=0}^{\infty}f(k)\,Z\{g(n-k)\}=\sum_{k=0}^{\infty}f(k)\,z^{-k}G(z)=F(z)G(z)$$

Problems:

1. Find the Z-transform of

(i) $f(t) = e^{-at}$

(ii) $f(t) = e^{at}$

(iii) $f(t) = cosat$

(iv) $f(t) = sinat$

Solution:

(1) By first shifting property $Z\{e^{-at}f(t)\} = F(ze^{aT})$

$$\therefore Z\{e^{-at}(1)\} = Z\{1\}_{z\to ze^{aT}} = \left[\frac{z}{z-1}\right]_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}}{ze^{aT}-1}$$

(2) $Z\{e^{at}(1)\} = Z\{1\}_{z\to \frac{z}{e^{aT}}} = \left[\frac{z}{z-1}\right]_{z\to \frac{z}{e^{aT}}}$

$$= \frac{\frac{z}{e^{aT}}}{\frac{z}{e^{aT}}-1} = \frac{z}{z-e^{aT}}$$

(3) By the definition $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{cosat\} = \sum_{n=0}^{\infty} cosanT\, z^{-n} = \sum_{n=0}^{\infty} cosn(aT)\, z^{-n}$$

$$= \frac{z(z-cosaT)}{z^2-2zcosaT+1}$$

(4) $Z\{sinat\} = \sum_{n=0}^{\infty} sinanT\, z^{-n} = \sum_{n=0}^{\infty} sinn(aT)\, z^{-n}$

$$= \frac{zsinaT}{z^2-2zcosaT+1}$$

2. Find the Z-transform of sin(t+T)

Solution: Let $f(t+T) = \sin(t+T)$ implies $f(t) = sint$

By second shifting theorem $Z\{f(t+T)\} = z[F(z) - f(0)]$

$f(0) = sin0 = 0 \; and \; F(z) = Z\{f(t)\} = Z\{sint\} = \dfrac{zsinT}{z^2 - 2zcosT + 1}$

$\therefore Z\{\sin(t+T)\} = z\left[\dfrac{zsinT}{z^2 - 2zcosT + 1} - 0\right] = \dfrac{z^2 sinT}{z^2 - 2zcosT + 1}$

3. Find the Z-transform of $(t+T)e^{-(t+T)}$

Solution:

Let $f(t+T) = \sin(t+T)$ implies $f(t) = te^{-t}$

By second shifting theorem $Z\{f(t+T)\} = z[F(z) - f(0)]$

$f(0) = 0e^0 = 0 \; and \; F(z) = Z\{f(t)\} = Z\{te^{-t}\} = Z\{t\}_{z \to ze^T}$

$= \left[\dfrac{Tz}{(z-1)^2}\right]_{z \to ze^T} = \dfrac{Tze^T}{(ze^T - 1)^2}$

$\therefore Z\left\{(t+T)e^{-(t+T)}\right\} = z\left[\dfrac{Tze^T}{(ze^T-1)^2} - 0\right] = \dfrac{Tz^2e^T}{(ze^T-1)^2}$

4. Find the initial value of $F(z) = \frac{z}{(z-1)(z-2)}$

Solution:

By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$= \lim\limits_{z\to\infty} \dfrac{z}{(z-1)(z-2)}$

$= \lim\limits_{z\to\infty} \dfrac{z}{z^2\left(1 - \frac{1}{z}\right)\left(1 - \frac{2}{z}\right)} = 0$

5. If $F(z) = \frac{1+z^{-1}}{1-0.25z^{-2}}, find \; f(0) and \; f(\infty)$

Solution: By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$f(0) = \lim\limits_{z\to\infty} \dfrac{1+z^{-1}}{1-0.25z^{-2}} = 1$

By final value theorem, $f(\infty) = \lim_{z\to 1}(z-1)\,F(z) = \lim_{z\to 1}(z-1)\frac{z(z+1)}{z^2-0.25} = 0$

6. If $F(z) = \frac{2z^2+3z+14}{(z-1)^4}, find \; f(2) and \; f(3)$

Solution: By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$$= \lim_{z\to\infty} \frac{2z^2+3z+14}{(z-1)^4} = \lim_{z\to\infty} \frac{z^2\left(2+\frac{3}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}$$

$\therefore f(0) = 0$

$$f(1) = \lim_{z\to\infty}[z(F(z) - f(0))]$$

$$= \lim_{z\to\infty} z\frac{2z^2+3z+14}{(z-1)^4} = \lim_{z\to\infty} z^3 \frac{\left[2+\frac{3}{z}+\frac{14}{z^2}\right]}{z^4\left(1-\frac{1}{z}\right)^4}$$

$\therefore f(1) = 0$

$$f(2) = \lim_{z\to\infty}[z^2(F(z) - f(0) - f(1)z^{-1})]$$

$$= \lim_{z\to\infty} z^2\frac{2z^2+3z+14}{(z-1)^4} = \lim_{z\to\infty} z^4 \frac{\left[2+\frac{3}{z}+\frac{14}{z^2}\right]}{z^4\left(1-\frac{1}{z}\right)^4}$$

$\therefore f(2) = 2$

$$f(3) = \lim_{z\to\infty}[z^3(F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2})]$$

$$= \lim_{z\to\infty}\left[z^3\left(\frac{2z^2+3z+14}{(z-1)^4} - \frac{2}{z^2}\right)\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+14z^2-2z^4+8z^3-12z^2+8z-2}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{11z^3+2z^2+8z-2}{z^2(z-1)^4}\right] = \lim_{z\to\infty} z^6\left[\frac{11+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$\therefore f(3) = 11$

7. Verify initial and final value theorem for $f(t) = e^{-at}\cos bt$.

Solution: Given $f(t) = e^{-at}\cos bt$.

$$F(z) = Z\{f(t)\} = Z\{e^{-at}\cos bt\} = [Z\{\cos bnT\}]_{z\to ze^{aT}}$$

$$= \left[\frac{z(z-\cos bT)}{z^2 - 2z cosbT + 1}\right]_{z\to ze^{aT}}$$

$$F(z) = \frac{ze^{aT}(ze^{aT} - \cos bT)}{(ze^{aT})^2 - 2ze^{aT} cosbT + 1}$$

By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

Consider L.H.S $f(0) = \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-at} \cos bt = 1 \ldots\ldots\ldots (1)$

Consider R.H.S $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{ze^{aT}(ze^{aT} - \cos bT)}{(ze^{aT})^2 - 2ze^{aT} cosbT+1}$

$$= \lim_{z\to\infty} \frac{z^2 e^{2aT}(1 - \frac{\cos bt}{ze^{aT}})}{z^2 e^{2aT}\left[1 - \frac{2\cos bt}{ze^{aT}} + \frac{1}{z^2 e^{2aT}}\right]} = 1 \ldots\ldots\ldots (2)$$

From (1) and (2) L.H.S= R.H.S . Hence Initial Value theorem verified.

By final value theorem, $f(\infty) = \lim_{z\to 1}(z-1)\, F(z)$

Consider L.H.S $f(\infty) = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-at} \cos bt = 0 \ldots\ldots\ldots (1)$

Consider R.H.S $\lim_{z\to 1}(z-1)\, F(z) = \lim_{z\to 1}(z-1)\, F(z)$

$$= \lim_{z\to 1}(z-1) \frac{ze^{aT}(ze^{aT} - \cos bT)}{(ze^{aT})^2 - 2ze^{aT} cosbT + 1} = 0 \ldots\ldots\ldots (2)$$

From (1) and (2) L.H.S= R.H.S . Hence Final Value theorem verified.

8. Find the Z-transform of the convolution of $f(n) = a^n U(n)\ and\ g(n) = b^n U(n)$

Solution: We know that $Z\{f(n) * g(n)\} = Z\{f(n)\}Z\{g(n)\}$

$\therefore Z\{a^n U(n) * b^n U(n)\} = Z\{a^n U(n)\}Z\{b^n U(n)\}$

$$= \frac{z}{z-a}\frac{z}{z-b} = \frac{z^2}{(z-a)(z-b)}$$

9. Find the Z-transform of $5^n * cosn\theta\ \ and\ sinn\frac{\pi}{2} * cosn\frac{\pi}{2}$

Solution: We know that $Z\{f(n) * g(n)\} = Z\{f(n)\}Z\{g(n)\}$

$\therefore Z\{5^n * cosn\theta\} = Z\{5^n\}Z\{cosn\theta\}$

$$= \frac{z}{z-5} \frac{z(z-cos\theta)}{z^2 - 2zcos\theta + 1} = \frac{z^2(z-cos\theta)}{(z-5)(z^2-2zcos\theta+1)}$$

We know that $Z\left\{cosn\frac{\pi}{2}\right\} = \frac{z^2}{z^2+1}$ $and\ Z\left\{sinn\frac{\pi}{2}\right\} = \frac{z}{z^2+1}$

$$\therefore Z\left\{sinn\frac{\pi}{2} * cosn\frac{\pi}{2}\right\} = Z\left\{sinn\frac{\pi}{2}\right\} Z\left\{cosn\frac{\pi}{2}\right\}$$

$$= \frac{z}{z^2+1} \frac{z^2}{z^2+1} = \frac{z^3}{(z^2+1)^2}$$

The Inverse Z-Transform

If $Z[f(n)] = F(Z)$ then $Z^{-1}F(Z) = f(n)$ is called inverse Z-transform of $F(Z)$

Example:

1. $Z[a^n] = \frac{z}{z-a} z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Methods of finding inverse Z-transforms:

1. Method of partial fraction
2. Method of residues
3. Convolution method

Partial Fraction Method:

1. Find the inverse Z-transform of $\frac{10z}{z^2-3z+2}$

   Solution:

$$F(Z) = \frac{10z}{z^2-3z+2} = \frac{10z}{(z-1)(z-2)}$$

$$\frac{F(z)}{10z} = \frac{10z}{z^2-3z+2} = \frac{10z}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put z =1 A =1

Put z = 2 B = 1

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\frac{F(z)}{10z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$F(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-10z}{z-1}\right] + Z^{-1}\left[\frac{10z}{z-2}\right]$$

$$= -10\, Z^{-1}\left[\frac{z}{z-1}\right] + 10Z^{-1}\left[\frac{z}{z-2}\right]$$

$$Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1)^n + 10(2)^n, \quad n \geq 0$$

2. Find the inverse Z-transform of $\frac{z-4}{(z-1)(z-2)^2}$

Solution:

$$F(Z) = \frac{z-4}{(z-1)(z-2)^2}$$

$$\frac{z-4}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z-4 = A(z-2)^2 + B(z-1) + c(z-1)(z-2)$$

Put z =1 A = -3

Put z = 2 B = -2

Equating coefficient of z^2 A + C = 0, C = 3

$$\frac{z-4}{(z-1)(z-2)^2} = \frac{-3}{z-1} - \frac{2}{(z-2)^2} + \frac{3}{z-2}$$

$$Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right] = -3Z^{-1}\left[\frac{1}{z-1}\right] - Z^{-1}\left[\frac{2}{(z-2)^2}\right] + 3\, Z^{-1}\left[\frac{1}{z-2}\right]$$

$$Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right] = -3(1)^{n-1} - (n-1)(2)^{n-1} + 3(2)^{n-1}, \quad n \geq 1$$

3. Find the inverse Z-transform of $\frac{z^2+3z}{(z-1)^2(z^2+1)}$

Solution:

$$F(Z) = \frac{z^2 + 3z}{(z-1)^2(z^2+1)}$$

$$\frac{F(z)}{z} = \frac{z^2 + 3z}{(z-1)^2(z^2+1)}$$

$$\frac{z^2 + 3z}{(z-1)^2(z^2+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{Cz+D}{z^2+1}$$

$$z^2 + 3 = A(z-1)(z^2+1) + B(z^2+1) + (Cz+D)(z-1)^2 \quad \ldots\ldots\ldots.(1)$$

Solving (1) we get A= -1 , B = 2, C = 1, D = 0

$$\frac{F(z)}{z} = \frac{-1}{z-1} + \frac{2}{(z-2)^2} + \frac{z}{z^2+1}$$

$$Z^{-1}[F(z)] = -Z^{-1}\left[\frac{z}{z-1}\right] + 2Z^{-1}\left[\frac{z}{(z-2)^2}\right] + Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$Z^{-1}\left[\frac{z^2+3z}{(z-1)^2(z^2+1)}\right] = -(1)^n - 2n + cos\frac{n\pi}{2}$$

Method of Residues

To find inverse Z- transform using residue theorem

If $Z[f(n)] = F(Z)$, then f(n) which gives the inverse Z-transform of $F(Z)$ is obtained from the following result $f(n) = \frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz$

Where C is the closed contour which encloses all the poles of the integrand.

By Residue theorem,

$$\int_C z^{n-1}F(z)\,dz = 2\pi i\left[sum\ of\ residues of\ z^{n-1}F(z)\ at its\ poles\right]$$

Substituting (2) in (1)

1. Find the inverse Z – transform of $\frac{z}{(z-1)(z-2)}$

Solution: Let $F(z) = \frac{1}{(z-1)(z-2)} f(n) = Z^{-1}F(Z)$

$$z^{n-1}F(z) = \frac{z^n}{(z-1)(z-2)}$$

The poles are z = 1, z = 2 (simple poles)

f(n)= sum of the residues $z^{n-1}F(z)$ at its

poles $\operatorname{Res}\left\{z^{n-1}F(z)\right\}_{z=1} = \lim_{z\to 1}(z-1)\dfrac{z^n}{(z-1)(z-2)} = -(1)^n$

$$\operatorname{Res}\left\{z^{n-1}F(z)\right\}_{z=2} = \lim_{z\to 2}(z-2)\frac{z^n}{(z-1)(z-2)} = 2^n$$

f(n)= sum of the residues $z^{n-1}F(z)$ at its poles

$= 2^n - (1)^n, \qquad n \geq 0$

2. Find $Z^{-1}\left\{\frac{2z^2+4z}{(z-2)^3}\right\}$ by Residue theorem.

Solution: Let $F(z) = \frac{2z^2+4z}{(z-2)^3}$

$$f(n) = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\,dz = \frac{1}{2\pi i}\int_C z^{n-1}\frac{2z^2+4z}{(z-2)^3}\,dz$$

$f(n) = \frac{1}{2\pi i}\int_C z^n\frac{(2z+4)}{(z-2)^3}dz$……………..(1)

Let $\varphi(z) = z^n\frac{(2z+4)}{(z-2)^3}$

Equate the denominator to zero, we get

z = 2 is a pole of order 3

$$\underset{z=2}{Res}\,\varphi(z) = \frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}\left[(z-2)^3 z^n\frac{(2z+4)}{(z-2)^3}\right]$$

$$= \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}[z^n(2z+4)] = \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}[2z^{n+1}+4z^n)]$$

$$= \frac{1}{2}\lim_{z\to 2}\frac{d}{dz}[2(n+1)z^n + 4nz^{n-1})] = \frac{1}{2}\lim_{z\to 2}[2n(n+1)z^{n-1} + 4n(n-1)z^{n-2}]$$

$$= n(n+1)(2)^{n-1} + 2n(n-1)(2)^{n-2} = n(2)^{n-1}[n+1+n-1] = n^2 2^n$$

By Residue theorem

$f(n) = \frac{1}{2\pi i}2\pi i[Res_{z=2}\,\varphi(z)]$ (by (1)

$\therefore\ f(n) = n^2 2^n$

3. Find $Z^{-1}\left\{\frac{z}{z^2+2z+2}\right\}$ by

Residue theorem**.**

Solution: Let

$$f(n) = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\,dz = \frac{1}{2\pi i}\int_C z^{n-1}\frac{z}{z^2+2z+2}\,dz$$

$$F(z) = \frac{z}{z^2+2z+2}$$

$$f(n) = \frac{1}{2\pi i}\int_C z^n \frac{1}{z^2+2z+2}\,dz$$

……………..(1) Let

$$\varphi(z) = z^n\frac{1}{z^2+2z+2}$$

Equate the denominator to zero, we get

$$z^2+2z+2 = 0 \Rightarrow z = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i = -1-i, -1+i$$

$$\underset{z=-1-i}{Res}\,\varphi(z) = \lim_{z\to-1-i}[z-(-1-i)]z^n\frac{1}{[z-(-1-i)][z-(-1+i)]}$$

$$= \frac{(-1-i)^n}{-2i} = \frac{1}{-2i}\left[\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)^n\right]$$

$$\underset{z=-1+i}{Res}\,\varphi(z) = \lim_{z\to-1+i}[z-(-1+i)]z^n\frac{1}{[z-(-1-i)][z-(-1+i)]}$$

$$= \frac{(-1+i)^n}{2i} = \frac{1}{2i}\left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n\right]$$

By Residue theorem

$$f(n) = \frac{1}{2\pi i}2\pi i[Res_{z=-1-i}\,\varphi(z) + Res_{z=-1+i}(\varphi(z))]$$

$$f(n) = \frac{1}{2i}\left\{\left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n\right] - \left[\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)^n\right]\right\}$$

$$= \frac{\left(\sqrt{2}\right)^n}{2i}\left[\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4} - \cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right] = \left(\sqrt{2}\right)^n\sin\frac{3n\pi}{4}$$

$$\therefore\ f(n) = \left(\sqrt{2}\right)^n\sin\frac{3n\pi}{4}$$

Convolution Method:

1. Find the inverse Z-transform of $\frac{z^2}{(z-a)^2}$ using convolution theorem.

Solution:

By convolution theorem

$$Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\}Z^{-1}\{G(z)\}$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-a}\right\}$$

$$= Z^{-1}\left\{\frac{z}{z-a}\right\} Z^{-1}\left\{\frac{z}{z-a}\right\}$$

$$= a^n * a^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} a^k\, a^{n-k} = \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} 1 = (n+1)a^n$$

$$\therefore Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = (n+1)a^n$$

2. Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z+1)}$ using convolution theorem.

Solution:

By convolution theorem $Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\}Z^{-1}\{G(z)\}$

$$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z+1)}\right\} = Z^{-1}\left\{\frac{8z^2}{8\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right\} = Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z+\frac{1}{4}\right)}\right\}$$

$$= Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)}\right\} Z^{-1}\left\{\frac{z}{\left(z+\frac{1}{4}\right)}\right\}$$

$$= \left(\frac{1}{2}\right)^n * \left(-\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(-\frac{1}{4}\right)^{n-k} = \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[-\frac{\frac{1}{2}}{\frac{1}{4}}\right]^k = \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} [-2]^k$$

$$= \left(-\frac{1}{4}\right)^n [1 + (-2)^1 + (-2)^2 + \cdots + (-2)^n]$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right] \quad \left[Since\ a + ar + ar^2 + ar^3 + \cdots. + ar^n = \frac{1-r^{n+1}}{1-r}, when\ r < 1\right.$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{3}\right] = \frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$$

$$\therefore Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z+1)}\right\} = \frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$$

3. Find the inverse Z-transform of $\frac{z^2}{(z-1)(z-3)}$ using convolution theorem.

Solution:

By convolution theorem $Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\}Z^{-1}\{G(z)\}$

$$Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = Z^{-1}\left\{\frac{z}{(z-1)} \cdot \frac{z}{(z-3)}\right\}$$

$$= Z^{-1}\left\{\frac{z}{(z-1)}\right\}Z^{-1}\left\{\frac{z}{(z-3)}\right\}$$

$$= (1)^n * (3)^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} (1)^k (3)^{n-k} = (3)^n \sum_{k=0}^{n} \left[\frac{1}{3}\right]^k$$

$$= (3)^n \left[1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n\right]$$

$$= (3)^n \left[\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\left(\frac{1}{3}\right)}\right] \quad \left[Since\ a + ar + ar^2 + ar^3 + \cdots . + ar^n = \frac{1-r^{n+1}}{1-r}, when\ r < 1\right.$$

$$= \frac{3}{2}(3)^n\left[1 - \left(\frac{1}{3}\right)^{n+1}\right] = \frac{1}{2}[(3)^{n+1} - 1]$$

$$\therefore Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = \frac{1}{2}[(3)^{n+1} - 1]$$

4. Find the inverse Z-transform of $\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$ using convolution theorem.

Solution:

By convolution theorem $Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\}Z^{-1}\{G(z)\}$

$$Z^{-1}\left\{\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right\} = Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z-\frac{1}{4}\right)}\right\}$$

$$= Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)}\right\}Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{4}\right)}\right\} = \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^{n} \sum_{k=0}^{n} \left[\frac{\frac{1}{2}}{\frac{1}{4}}\right]^{k} = \left(\frac{1}{4}\right)^{n} \sum_{k=0}^{n} [2]^{k}$$

$$= \left(\frac{1}{4}\right)^{n} [1 + (2)^{1} + (2)^{2} + \cdots + (2)^{n}]$$

$$= \left(\frac{1}{4}\right)^{n} \left[\frac{(2)^{n+1}-1}{2-1}\right] \quad \left[Since\ a + ar + ar^{2} + ar^{3} + \cdots . + ar^{n} = \frac{r^{n+1}-1}{r-1}, when\ r > 1\right.$$

$$= \left(\frac{1}{4}\right)^{n} [(2)^{n+1} - 1] = 2\left(\frac{1}{2}\right)^{n} - \left(\frac{1}{4}\right)^{n}$$

$$\therefore Z^{-1}\left\{\frac{z^{2}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right\} = 2\left(\frac{1}{2}\right)^{n} - \left(\frac{1}{4}\right)^{n}$$

Difference Equations:

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Example:

1. $y_{n+2} - y_{n+1} + y_n = 5$
2. $a_0 u_{n+1} + a_1 u_n = g(n)$

Formation of difference equation

1. Form a difference equation given $y_n = cosn\frac{\pi}{2}$

Solution:

Given $y_n = cosn\frac{\pi}{2}$…………..(1)

$$y_{n+1} = \cos(n+1)\frac{\pi}{2} = cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -sin\frac{n\pi}{2}$$

$y_{n+1} = -sin\frac{n\pi}{2}$………………..(2)

$$y_{n+2} = -sin\frac{(n+1)\pi}{2} = -sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -cos\frac{n\pi}{2}$$

$y_{n+2} = -cos\frac{n\pi}{2}$……………….(3)

$y_{n+2} = -y_n$ (By (1)

$\therefore y_{n+2} + y_n = 0$

2. Form a difference equation given $u_n = \frac{1}{2}n(n+1)$

Solution:

Given $u_n = \frac{1}{2}n(n+1) = \frac{n^2}{2} + \frac{n}{2}$.........(1)

$u_{n+1} = \frac{(n+1)^2}{2} + \frac{(n+1)}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1$.........(2)

$u_{n+2} = \frac{(n+1)^2}{2} + \frac{3(n+1)}{2} + 1 = \frac{n^2}{2} + \frac{5n}{2} + 3$..............(3)

(3) - (2) gives $u_{n+2} - u_{n+1} = n + 2$............(4)

(2) - (1) gives $u_{n+1} - u_n = n + 1$.............(5)

(4) – (5) gives $u_{n+2} - 2u_{n+1} + u_n = 1$ which is the required difference equation.

3. Form a difference equation given $y_n = a(2)^n + b(-3)^n$

Solution:

Given $y_n = a(2)^n + b(-3)^n$.............(1)

$y_{n+1} = a(2)^{n+1} + b(-3)^{n+1} = 2a(2)^n - 3b(-3)^n$..........(2)

$y_{n+2} = 2a(2)^{n+1} - 3b(-3)^{n+1} = 4a(2)^n + 9b(-3)^n$.........(3)

Eliminating 2^n and $(-3)^n$ from equations (1) (2) and (3)

$$\begin{vmatrix} y_n & a & b \\ y_{n+1} & 2a & -3b \\ y_{n+2} & 4a & 9b \end{vmatrix} = 0$$

$$\Rightarrow y_n[18ab + 12ab] - y_{n+1}[9ab - 4ab] + y_{n+2}[-3ab - 2ab] = 0$$

$\Rightarrow -5aby_{n+2} - 5aby_{n+1} + 30aby_n = 0$

$\div\, by - 5ab \text{ ,we get}$

$y_{n+2} + y_{n+1} - 6y_n = 0$

4. Form a difference equation by eliminating $2^n, 3^n$ *and* 4^n from the sequence

$$u_n = \frac{1}{2}(2)^n - 3^n + \frac{1}{2}(4)^n$$

Solution:

Given $u_n = \frac{1}{2}(2)^n - 3^n + \frac{1}{2}(4)^n$ ....(1)

$$u_{n+1} = \frac{1}{2}(2)^{n+1} - 3^{n+1} + \frac{1}{2}(4)^{n+1} = (2)^n - 3.3^n + 2(4)^n \text{ ....(2)}$$

$$u_{n+2} = (2)^{n+1} - 3.3^{n+1} + 2(4)^{n+1} = 2(2)^n - 9.3^n + 8(4)^n \text{ ....(3)}$$

$$u_{n+3} = 2(2)^{n+1} - 9.3^{n+1} + 8(4)^{n+1} = 4(2)^n - 27.3^n + 32(4)^n \text{ ....(4)}$$

$$\begin{vmatrix} u_n & \frac{1}{2} & -1 & \frac{1}{2} \\ u_{n+1} & 1 & -3 & 2 \\ u_{n+2} & 2 & -9 & 8 \\ u_{n+3} & 4 & -27 & 32 \end{vmatrix} = 0$$

$$\Rightarrow u_n \begin{vmatrix} \frac{1}{2} & -3 & 2 \\ 2 & -9 & 8 \\ 4 & -27 & 32 \end{vmatrix} - u_{n+1} \begin{vmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 2 & -9 & 8 \\ 4 & -27 & 32 \end{vmatrix} + u_{n+2} \begin{vmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & -3 & 2 \\ 4 & -27 & 32 \end{vmatrix} - u_{n+3} \begin{vmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & -3 & 2 \\ 2 & -9 & 8 \end{vmatrix} = 0$$

Expanding by usual determinant method, we get

$u_{n+3} - 9u_{n+2} + 26u_{n+1} - 24u_n = 0$ which is the required difference equation.

Solution of Difference Equation Using Z-Transform:

The following results are used to solve difference equation.

(i) $Z\{y(n+1)\} = zY(z) - zy(0)$

(ii) $Z\{y(n+2)\} = z^2Y(z) - z^2$

(iii) $Z\{y(n+3)\} = z^3Y(z) - z^3$

And so on…

1.Solve

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n \; given \; y_0 = y_1 = 0$ using Z-transform.

Solution:

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Taking Z-transform on both sides,

$$Z\{y_{n+2}\} + 6Z\{y_{n+1}\} + 9Z\{y_n\} = Z\{2^n\}$$

$$z^2Y(z) - z^2y(0) - zy(1) + 6[zY(z) - zy(0)] + 9Y(z) = \frac{z}{z-2}$$

$$z^2Y(z) + 6zY(z) + 9Y(z) = \frac{z}{z-2}$$

$$Y(z)[z^2 + 6z + 9] = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

By partial fraction $\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$……………..(1)

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put z =-3 $\Rightarrow C = -\frac{1}{5}$ Put z =2 $\Rightarrow A = \frac{1}{25}$

Equate the co-efficient of z^2 on both sides, we get

A + B = 0 $\Rightarrow B = -A = -\frac{1}{25}$

Equation (1) becomes $\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{\frac{1}{25}}{z-2} - \frac{\frac{1}{25}}{z+3} - \frac{\frac{1}{5}}{(z+3)^2}$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$\Rightarrow y(n) = \frac{1}{25}Z^{-1}\left\{\frac{z}{z-2}\right\} - \frac{1}{25}Z^{-1}\left\{\frac{z}{z+3}\right\} - \frac{1}{5}Z^{-1}\left\{\frac{z}{(z+3)^2}\right\}$$

$$\therefore y(n) = \frac{1}{25}\,(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n$$

2.Solve

$y_{n+2} + y_n = 2 \; given \; y_0 = y_1 = 0$ using Z-transform.

Solution:

Given $y_{n+2} + y_n = 2$

Taking Z-transform on both sides,

$$Z\{\,y_{n+2}\} + Z\{\,y_n\} = 2Z\{\,1\}$$

$$z^2Y(z) - z^2y(0) - zy(1) + Y(z) = 2\frac{z}{z-1}$$

$$z^2Y(z) + Y(z) = 2\frac{z}{z-1}$$

$$Y(z)[z^2+1] = 2\frac{z}{z-1}$$

$$Y(z) = 2\frac{z}{(z-1)(z^2+1)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{2}{(z-1)(z^2+1)}$$

By partial fraction $\frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{(z^2+1)}$……………..(1)

$$2 = A(z^2+1) + (Bz+C)(z-1)$$

Put z =1 $\Rightarrow A = 1$

Equate the co-efficient of z^2 on both sides, we get

A + B = 0 ⇒ $B = -A = -1$

Equate the co-efficient of z on both sides, we get

-B + C = 0 ⇒ C = B =- 1

Equation (1) becomes $\frac{Y(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{1}{z-1} + \frac{-z-1}{(z^2+1)}$

$$Y(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$\Rightarrow y(n) = Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z^2}{z^2+1}\right\} - Z^{-1}\left\{\frac{z}{z^2+1}\right\}$$

$$\therefore y(n) = 1 - cos\frac{n\pi}{2} - sin\frac{n\pi}{2}$$

3.Solve

$y(n+3) - 6y(n+2) + 12y(n+1) - 8y(n) = 0 \; given \; y(0) = -1;$

$y(1) = 0; and \; y(2) = 1$ using Z-transform.

Solution:

Given $y(n+3) - 6y(n+2) + 12y(n+1) - 8y(n) = 0$

Taking Z-transform on both sides,

$$Z\{y(n+3)\} - 6Z\{y(n+2)\} + 12Z\{y(n+1)\} - 8Z\{y(n)\} = 0$$

$$z^3Y(z) - z^3y(0) - z^2y(1) - zy(2) - 6[z^2Y(z) - z^2y(0) - zy(1)] + 12[zY(z) - zy(0)] - 8Y(z) = 0$$

$$z^3Y(z) + z^3 - z - 6z^2Y(z) - 6z^2 + 12zY(z) + 12z - 8Y(z) = 0$$

$$Y(z)[z^3 - 6z^2 + 12z - 8] = -z^3 + 6z^2 - 11z$$

$$Y(z) = \frac{-z[z^2 - 6z + 11]}{[z^3 - 6z^2 + 12z - 8]}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{-[z^2 - 6z + 11]}{(z-2)^3}$$

By partial fraction $\frac{-[z^2-6z+11]}{(z-2)^3} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3}$..................(1)

$$-[z^2 - 6z + 11] = A(z-2)^2 + B(z-2) + C$$

Put z = 2 $\Rightarrow C = -3$

Equate the co-efficient of z^2 on both sides, we get

A = -1

Equate the co-efficient of z on both sides, we get

-4A + B = 6 ⇒ B = 2

Equation (1) becomes $\frac{Y(z)}{z} = \frac{-[z^2-6z+11]}{(z-2)^3} = \frac{-1}{z-2} + \frac{2}{(z-2)^2} - \frac{3}{(z-2)^3}$

$$Y(z) = \frac{-z}{z-2} + \frac{2z}{(z-2)^2} - \frac{3z}{(z-2)^3}$$

$$\Rightarrow y(n) = -Z^{-1}\left\{\frac{z}{z-2}\right\} + Z^{-1}\left\{\frac{2z}{(z-2)^2}\right\} - 3Z^{-1}\left\{\frac{z}{(z-2)^3}\right\}$$

$$\therefore y(n) = -(2)^n + n(2)^n - \frac{3}{8}n(n-1)(2)^n$$

4.Solve $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2, given\ that\ \ y(0) = 3 and\ y(1) = -2$ using Z-transform.

Solution: Given $y(n) + 3y(n-1) - 4y(n-2) = 0$.................(1)

Changing 'n' into 'n+2' we get,

$y(n+2) + 3y(n+1) - 4y(n) = 0$......................(2)

Taking Z-transform on both sides,

$$Z\{y(n+2)\} + 3Z\{y(n+1)\} - 4Z\{\mathrm{y(n)}\} = 0$$

$$z^2Y(z) - z^2y(0) - zy(1) + 3[zY(z) - zy(0)] - 4Y(z) = 0$$

$$z^2Y(z) - 3z^2 + 2z + 3zY(z) - 9z - 4Y(z) = 0$$

$$Y(z)[z^2 + 3z - 4] = 3z^2 + 7z$$

$$Y(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4} = \frac{3z^2 + 7z}{(z+4)(z-1)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{3z+7}{(z-1)(z+4)}$$

By partial fraction $\frac{3z+7}{(z-1)(z+4)} = \frac{A}{z-1} + \frac{B}{z+4}$.................(3)

$$3z + 7 = A(z+4) + B(z-1)$$

Put z =-4 $\Rightarrow B = 1$ Put z =1 $\Rightarrow A = 2$

Equation (3) becomes $\frac{Y(z)}{z} = \frac{3z+7}{(z-1)(z+4)} = \frac{2}{z-1} + \frac{1}{z+4}$

$$Y(z) = 2\frac{z}{z-1} + \frac{z}{z+4}$$

$$\Rightarrow y(n) = 2Z^{-1}\left\{\frac{z}{z-1}\right\} + Z^{-1}\left\{\frac{z}{z+4}\right\}$$

$\therefore y(n) = 2\,(1)^n + (-4)^n$

5.Solve

$x(n+2) + 4x(n+1) + 4x(n) = n, given\ x(0) = 0\ and\ x(1) = 1$ using

Z-transform.

Solution:

Given $x(n+2) + 4x(n+1) + 4x(n) = n$

Taking Z-transform on both sides,

$Z\{x(n+2)\} + 4Z\{x(n+1)\} + 4Z\{\mathrm{x(n)}\} = Z\{n\}$

$$z^2X(z) - z^2x(0) - zx(1) + 4[zX(z) - zx(0)] + 4Y(z) = \frac{z}{(z-1)^2}$$

$$z^2X(z) - z + 4zX(z) + 4X(z) = \frac{z}{(z-1)^2}$$

$$X(z)[z^2 + 4z + 4] = \frac{z}{(z-1)^2} + z$$

$$X(z) = \frac{z}{(z-1)^2(z+2)^2} + \frac{z}{(z+2)^2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{(z-1)^2(z+2)^2} + \frac{1}{(z+2)^2} = \frac{z^2 - 2z + 2}{(z-1)^2(z+2)^2}$$

By partial fraction $\frac{z^2-2z+2}{(z-1)^2(z+2)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2} + \frac{D}{(z+2)^2}$.................(1)

$$z^2 - 2z + 2 = A(z-1)(z+2)^2 + B(z+2)^2 + C(z-1)^2(z+2) + D(z-1)^2$$

Put z = 1 ⇒ $B = \frac{1}{9}$ Put z = -2 ⇒ $D = \frac{10}{9}$

Equate the co-efficient of z^3 on both sides, we get

A +C = 0

Equate the co-efficient of z^2 on both sides, we get

3A +B + D =1 ⇒ 3A = 1 – B - D = 1- $\frac{1}{9} - \frac{10}{9} = -\frac{2}{9}$

$$\therefore A = -\frac{2}{27}$$

$$C = -A \Rightarrow C = \frac{2}{27}$$

Equation (1) becomes $\frac{X(z)}{z} = \frac{z^2-2z+2}{(z-1)^2(z+2)^2} = \frac{-\frac{2}{27}}{z-1} + \frac{\frac{1}{9}}{(z-1)^2} + \frac{\frac{2}{27}}{z+2} + \frac{\frac{10}{9}}{(z+2)^2}$

$$X(z) = -\frac{2}{27}\frac{z}{z-1} + \frac{1}{9}\frac{z}{(z-1)^2} + \frac{2}{27}\frac{z}{z+2} + \frac{10}{9}\frac{z}{(z+2)^2}$$

$$\Rightarrow x(n) = -\frac{2}{27}Z^{-1}\left\{\frac{z}{z-1}\right\} + \frac{1}{9}Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} + \frac{2}{27}Z^{-1}\left\{\frac{z}{z+2}\right\} + \frac{10}{9}Z^{-1}\left\{\frac{z}{(z+2)^2}\right\}$$

$$\therefore x(n) = -\frac{2}{27}\,(1)^n + \frac{1}{9}n + \frac{2}{27}(-2)^n - \frac{5}{9}n(-2)^n$$

*************ALL THE BEST*****************

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV -ENGINEERING MATHEMATICS-III– SMTA1301

UNIT IV

PARTIAL DIFFERENTIAL EQUATIONS

1.1 INTRODUCTION

Partial differential equations are found in problems involving wave phenomena, heat conduction in homogeneous solids and potential theory. As an equation containing ordinary differential coefficients is called an ordinary differential equation, an equation containing partial differential coefficients is called a partial differential equation.

1.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equation can be formed by

(i) Eliminating arbitrary constants

(ii) Eliminating arbitrary functions.

Note:

1. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the process of elimination results in a partial differential equation of the first order.

2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to a partial differential equation of second or higher orders.
3. If the partial differential equation is formed by eliminating arbitrary functions, the order of the equation will be, in general, equal to the number of arbitrary functions eliminated.

1.3 ELIMINATION OF ARBITRARY CONSTANTS

Consider the functional relation among

$$x, y, z, \text{ i.e. } f(x, y, z, a, b) = 0 \quad (1)$$

where a and b are arbitrary constants to be eliminated.

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0, \text{ i.e. } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \quad (2)$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0, \text{ i.e. } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \quad (3)$$

Equations (2) and (3) will contain a and b.

If we eliminate a and b from equations (1), (2) and (3), we get partial differential equation (involving p and q) of the first order.

1.4 ELIMINATION OF ARBITRARY FUNCTIONS

Let us consider the relation

$$f(u, v) = 0 \quad (1)$$

where u and v are functions of x, y, z and f is an arbitrary function to be eliminated. Differentiating (1) partially with respect to x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \quad (2)$$

[since u and v are functions of x, y, z and z is in turn, a function of x, y]

Differentiating (2) partially with respect to y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad (3)$$

Instead of eliminating f, let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3).

EXAMPLE 1

Form the partial differential equation by eliminating a and b from

$$z = \left(x^2 + a^2\right)\left(y^2 + b^2\right)$$

SOLUTION

$$z = \left(x^2 + a^2\right)\left(y^2 + b^2\right) \qquad -(1)$$

Differentiating (1) partially w.r.t x and y we get

$$p = \frac{\partial z}{\partial x} = \left(y^2 + b^2\right)2x \qquad -(2)$$

$$q = \frac{\partial z}{\partial y} = \left(x^2 + a^2\right)2y \qquad -(3)$$

Multiplying Eqn. (2) and Eqn (3) $\Rightarrow$

$$pq = \left(x^2 + a^2\right)\left(y^2 + b^2\right)4xy$$

$$pq = 4xyz \text{ [using (1)]}$$

EXAMPLE 2

Find the differential equation of all spheres of fixed radius having their centres in the xy plane.

SOLUTION

A point lying in the xy plane is of the form (a,b,0). Let the fixed radius be c.

Equation of the sphere is

$$(x-a)^2 + (y-b)^2 + z^2 = c^2 (a, b \text{ arbitrary constants}) \tag{1}$$

Differentiating partially w.r.t. x we get

$$2(x-a) + 2z\,\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad x - a = -zp \tag{2}$$

Differentiating partially w.r.t. y we have

$$2(y-b) + 2z\,\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad y - b = -zq \tag{3}$$

Substituting in (1)

$$(-zp)^2 + (-zq)^2 + z^2 = c^2$$

$$\Rightarrow \qquad z^2(p^2 + q^2 + 1) = c^2$$

$$\text{i.e.,} \qquad z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = c^2.$$

Note: Here z is given as an implicit function of x and y.

EXAMPLE 3

Obtain the partial differential equation by eliminating arbitrary constants a and b from

$(x-a)^2 + (y-b)^2 + z^2 = 1$.

SOLUTION

$(x-a)^2+(y-b)^2+z^2=1 \quad -(1)$

Differentiating (1) partially w.r.t x and y we get

$2(x-a)+2zp=0$

$\Rightarrow x-a=-zp \quad -(2)$

$2(y-b)+2zq=0$

$\Rightarrow y-b=-zq \quad -(3)$

Substituting (2) & (3) in (1) we get

$z^2p^2+z^2q^2+z^2=1$

i.e., $z^2\left(p^2+q^2+1\right)=1$

EXAMPLE 4

Form the partial differential equation from the following relation by eliminating the arbitrary constants, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

SOLUTION

Differentiating partially w.r.t. x and y we get

$$\frac{2x}{a^2}+\frac{1}{c^2}\,2zp=0 \quad \Rightarrow \quad \frac{x}{a^2}+\frac{zp}{c^2}=0 \tag{1}$$

$$\frac{2y}{b^2}+\frac{1}{c^2}\,2zq=0 \quad \Rightarrow \quad \frac{y}{b^2}+\frac{zq}{c^2}=0 \tag{2}$$

Differentiating (1) partially w.r.t. x

$$\frac{1}{a^2}+\frac{1}{c^2}\left(zr+p^2\right)=0 \tag{3}$$

Eliminating $\frac{1}{a^2}$ and $\frac{1}{c^2}$ from (1) and (3) we get

$$\begin{vmatrix} x & zp \\ 1 & zr+p^2 \end{vmatrix}=0 \quad \Rightarrow \quad xzr+xp^2-zp=0$$

$$\text{i.e.,} \qquad xz\,\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0.$$

EXAMPLE 5

Form the partial differential equation by eliminating the arbitrary function 'f' from

(i) $z = e^{ay} f(x + by)$; and

(ii) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

SOLUTION

(i) $$z = e^{ay}.f(x + by)$$

i.e. $$e^{-ay}z = f(x + by) \qquad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$e^{-ay}p = f'(u) \cdot 1 \qquad (2)$$

$$e^{-ay}q - ae^{-ay}z = f'(u)b \qquad (3)$$

where $u = x + by$

Eliminating $f'(u)$ from (2) and (3), we get

$$\frac{q - az}{p} = b$$

i.e. $$q = az + bp$$

(ii) $$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

i.e. $$z - y^2 = 2f\left(\frac{1}{x} + \log y\right) \qquad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$p = 2f'(u).\left(\frac{-1}{x^2}\right) \qquad (2)$$

and $$q - 2y = 2f'(u).\left(\frac{1}{y}\right) \qquad (3)$$

where $u = \frac{1}{x} + \log y$
Dividing (2) by (3), we have

$$\frac{p}{q - 2y} = \frac{-y}{x^2}$$

i.e. $$px^2 + qy = 2y^2$$

which is the required partial differential equation.

EXAMPLE 6

Form the differential equation by eliminating f and g from $z = xf(ax + by) + g(ax + by)$.

SOLUTION

$$z = x \cdot f(u) + g(u) \tag{1}$$

where $u = ax + by$.
Differentiating partially with respect to x and y,

$$p = xf'(u) \cdot a + f(u) + g'(u) \cdot a \tag{2}$$

$$q = xf'(u) \cdot b + g'(u) \cdot b \tag{3}$$

$$r = x \cdot f''(u)a^2 + f'(u) \cdot 2a + g''(u) \cdot a^2 \tag{4}$$

$$s = xf''(u)ab + f'(u)b + g''(u)ab \tag{5}$$

$$t = xf''(u)b^2 + g''(u) \cdot b^2 \tag{6}$$

$[(4) \times b - (5) \times 2a]$ gives

$$br - 2as = -a^2b[xf''(u) + g''(u)] \tag{7}$$

$$= -a^2b \times \frac{1}{b^2}t, \text{ from (6)}$$

i.e.
$$b^2\frac{\partial^2 z}{\partial x^2} - 2ab\frac{\partial^2 z}{\partial x \partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0$$

EXAMPLE 6

Form the differential equation by eliminating the arbitrary functions f and g from

$$z = f(x+iy) + (x+iy)g(x-iy), \quad \text{where} \quad i = \sqrt{-1} \quad \text{and} \quad x+iy \neq z$$

$$z = f(u) + (x+iy)g(v) \tag{1}$$

where $u = x + iy$ and $v = x - iy$.

SOLUTION

Differentiating partially with respect to x and y,

$$p = f'(u) \cdot 1 + (x+iy)g'(v) \cdot 1 + g(v) \tag{2}$$

$$q = f'(u) \cdot i + (x+iy)g'(v)(-i) + g(v) \cdot i \tag{3}$$

$$r = f''(u) \cdot 1 + (x+iy)g''(v) \cdot 1 + 2g'(v) \cdot 1 \tag{4}$$

$$s = f''(u) \cdot i + (x+iy)g''(v)(-i) \tag{5}$$

$$t = f''(u)(-1) + (x+iy)g''(v) \cdot (-1) + 2g'(v) \tag{6}$$

Adding (4) and (6), we get

$$r + t = 4g'(v) \tag{7}$$

From (2) and (3), we get

$$p + iq = 2(x+iy)g'(v) \tag{8}$$

Eliminating $g'(v)$ from (7) and (8), we get

$$r + t = 2\frac{(p + iq)}{x + iy}$$

i.e $$(x + iy)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 2\left(\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right)$$

EXAMPLE 7

Form the differential equation by eliminating f and ϕ from $z = xf(y/x) + y\phi(x)$.

$$z = xf(u) + y\phi(x) \tag{1}$$

where $u = \dfrac{y}{x}$.

SOLUTION

Differentiating partially with respect to x and y, we get

$$p = xf'(u) \cdot \left(-\frac{y}{x^2}\right) + f(u) + y\phi'(x)$$

i.e. $$p = -\frac{y}{x} \cdot f'(u) + f(u) + y\phi'(x) \tag{2}$$

$$q = x \cdot f'(u) \cdot \frac{1}{x} + \phi(x)$$

i.e. $$q = f'(u) + \phi(x) \tag{3}$$

$$r = -\frac{y}{x} \cdot f''(u)\left(-\frac{y}{x^2}\right) + y\phi''(x)$$

i.e. $$r = \frac{y^2}{x^3}f''(u) + y\phi''(x) \tag{4}$$

$$s = -\frac{y}{x^2}f''(u) + \phi'(x) \tag{5}$$

$$t = \frac{1}{x}f''(u) \tag{6}$$

Eliminating $f''(u)$ from (5) and (6), we get

$$s + \frac{y}{x}t = \phi'(x) \tag{7}$$

From (2) and (3), we get

$$px + qy = \{xf(u) + y\phi(x)\} + xy\phi'(x)$$

i.e.

$$px + qy = z + xy\phi'(x) \tag{8}$$

Eliminating $\phi'(x)$ from (7) and (8), we get

$$xys + y^2t = px + qy - z$$

i.e.

$$xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

1.5 SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

A solution of a P.D.E. which contains as many arbitrary constants as the number of independent variables is called *the complete solution* or *complete integral* of the equation.

A solution of a P.D.E. which contains as many arbitrary functions as the order of the equation is called the *general solution* or *general integral* of the equation.

1.6 PROCEDURE TO FIND GENERAL SOLUTION

Let $$F(x, y, z, p, q) = 0 \tag{1}$$

be a first order P.D.E. Let its complete solution be

$$\phi(x, y, z, a, b) = 0 \tag{2}$$

where a and b are arbitrary constants.

Let $b = f(a)$ [or $a = g(b)$], where 'f' is an arbitrary function.

Then (2) becomes

$$\phi[x, y, z, a, f(a)] = 0 \tag{3}$$

Differentiating (2) partially with respect to a, we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \tag{4}$$

Theoretically, it is possible to eliminate 'a' between (3) and (4).

This eliminant, which contains the arbitrary function 'f', is the general solution of (1).

A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary functions in the general solution is called *a particular solution* or *particular integral* of the P.D.E.

1.7 PROCEDURE TO FIND SINGULAR SOLUTION

Let $$F(x, y, z, p, q) = 0 \tag{1}$$

be a first order P.D.E.

Let its complete solution be

$$\phi(x, y, z, a, b) = 0 \tag{2}$$

Differentiating (2) partially with respect to a and then b, we have

$$\frac{\partial \phi}{\partial a} = 0 \tag{3}$$

and $$\frac{\partial \phi}{\partial b} = 0 \tag{4}$$

The eliminant of a and b from equations (2), (3) and (4), if it exists, is the singular solution of the P.D.E. (1).

1.8 COMPLETE SOLUTIONS OF FIRST ORDER NON-LINEAR P.D.E

A P.D.E., the partial derivatives occurring in which are of the first degree, is said to be *linear*; otherwise it is said to be *non-linear*.

Type I

Equations of the form $f(p, q) = 0$, i.e. the P.D.E.s that contain p and q only explicitly.

For equations of this type, it is known that a solution will be of the following form,

$$z = ax + by + c \tag{1}$$

But this solution contains three arbitrary constants, whereas the number of independent variables is two. Hence if we can reduce the number of arbitrary constants in (1) by one, it becomes the complete solution of the equation $f(p, q) = 0$. Now from (1), $p = a$ and $q = b$. If (1) is to be a solution of $f(p, q) = 0$, the values of p and q obtained from (1) should satisfy the given equation.

i.e. $$f(a, b) = 0$$

Solving this, we can get $b = \phi(a)$, where ϕ is a known function. Using this value of b in (1), the complete solutiion of the given P.D.E. is

$$z = ax + \phi(a)y + c \tag{2}$$

The general solution can be obtained from (2) by the method given earlier.

To find the singular solution, we have to eliminate a and c from

$$z = ax + \phi(a)y + c, \quad x + \phi'(a)y = 0 \quad \text{and} \quad 1 = 0$$

of which the last equation is absurd. Hence there is no singular solution for equations of type I.

Type II

Clairaut's type, i.e. the P.D.E.s of the form

$$z = px + qy + f(p, q) \tag{1}$$

For equations of this type also, it is known that a solution will be of the form

$$z = ax + by + c \tag{2}$$

If we can reduce the number of arbitrary constants in (2) by one, it becomes the complete solution of (1).

From (2) we get $p = a$ and $q = b$.

As before,
$$z = ax + by + f(a, b) \tag{3}$$

From (2) and (3), we get $c = f(a, b)$

Thus the complete solution of (1) is given by (3).

The singular and general integral are obtained in the usual manner

Type III

Equations not containing x and y explicitly, i.e. equations of the form

$$f(z, p, q) = 0 \tag{1}$$

For equations of this type, it is known that a solution will be of the form

$$z = \phi(x + ay) \tag{2}$$

where 'a' is an arbitrary constant and ϕ is a specific function to be found out.

Putting $x + ay = u$, (2) becomes $z = \phi(u)$ or $z(u)$

$\therefore$
$$p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

and
$$q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

If (2) is to be a solution of (1), the values of p and q obtained should satisfy (1).

$$\text{i.e. } f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \tag{3}$$

From (3), we can get

$$\frac{dz}{du} = \psi(z, a) \tag{4}$$

Now (4) is an ordinary differential equation, which can be solved by the variable separable method.

The solution of (4), which will be of the form $g(z, a) = u + b$ or $g(z, a) = x + ay + b$, is the complete solution of (1).

The general and singular solutions of (1) can be found out by the usual methods.

Type IV

Equations of the form

$$f(x, p) = g(y, q) \tag{1}$$

that is equations which do not contain z explicitly and in which terms containing p and x can be separated from those containing q and y.

To find the complete solution of (1), we assume that $f(x, p) = g(y, q) = a$, where 'a' is an arbitrary constant.

Solving $f(x, p) = a$, we can get $p = \phi(x, a)$ and solving $g(y, q) = a$, we can get $q = \psi(y, a)$.

Now

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \text{ or } p dx + q dy$$

i.e.

$$dz = \phi(x, a)dx + \psi(y, a)dy$$

Integrating with respect to the concerned variables, we get

$$z = \int \phi(x, a)dx + \int \psi(y, a)dy + b \tag{2}$$

The complete solution of (1) is given by (2), which contains two arbitrary constants a and b.

The general and singular solutions of (1) are found out by the usual methods.

Type V

Equations of the form $f(x^m p, y^n q) = 0$ or $f(x^m p, y^n q, z) = 0$,
where m and n are constants, each not equal to 1.
We make the transformations $x^{1-m} = X$ and $y^{1-n} = Y$.

Then $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = (1-m)x^{-m} P$, where $P \equiv \dfrac{\partial z}{\partial X}$ and
$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = (1-n)y^{-n} Q$, where $Q \equiv \dfrac{\partial z}{\partial Y}$.

Therefore the equation $f(x^m p, y^n q) = 0$ reduces to $f\{(1-m)P, (1-n)Q\} = 0$, which is a type I equation.

The equation $f(x^m p, y^n q, z) = 0$ reduces to $f\{(1-m)P, (1-n)Q, z\} = 0$, which is a type III equation.

Type VI

Equations of the form $f(px, qy) = 0$ or $f(px, qy, z) = 0$

These equations correspond to m = 1 and n = 1 of the type A equations.

The required transformations are

$$\log x = X \text{ and } \log y = Y$$

In this case, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{1}{x}$ or $px = P$ and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{1}{y}$ or $qy = Q$, where $P \equiv \dfrac{\partial z}{\partial X}$ and $Q \equiv \dfrac{\partial z}{\partial Y}$.

Therefore the equation $f(px, qy) = 0$ reduces to $f(P, Q) = 0$, which is a type I equation.

The equation $f(px, qy, z) = 0$ reduces to $f(P, Q, z) = 0$, which is a type III equation.

EXAMPLE 1

Solve the equation $pq + p + q = 0$.

SOLUTION

This equation contains only p and q explicitly.

$\therefore$ Let a solution of the equation be

$$z = ax + by + c \tag{1}$$

From (1), we get $p = a$ and $q = b$.

Since (1) is a solution of the given equation,

$$ab + a + b = 0$$

$$\therefore \qquad b = -\frac{a}{a+1} \tag{2}$$

Using (2) in (1), the required complete solution of the equation

$$z = ax - \frac{a}{a+1}y + c \tag{3}$$

To find the general solution, we put $c = f(a)$ in (3), where 'f' is an arbitrary function.

$$\text{i.e.} \qquad z = ax - \frac{a}{a+1}y + f(a) \tag{4}$$

Differentiating (4) partially with respect to a, we get

$$x - \frac{1}{(a+1)^2}y + f'(a) = 0 \tag{5}$$

Eliminating a between (4) and (5), we get the required general solution.

To find the singular solution, we have to differentiate (3) partially with respect to a and c.

When we differentiate (3) partially with respect to c, we get $0 = 1$, which is absurd.

Hence, no singular solution exists for the given equation.

EXAMPLE 2

Solve the equation $Z = px + qy + c\sqrt{1 + p^2 + q^2}$.

SOLUTION

The given equation

$$z = px + qy + c\sqrt{1 + p^2 + q^2} \tag{1}$$

is a Clairaut's type equation.

$\therefore$ Its complete solution is

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \tag{2}$$

where a and b are arbitrary constants and c is a given constant.

The general solution of (1) is found out from (2) as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to a and then b.

$$0 = x + \frac{ca}{\sqrt{1 + a^2 + b^2}} \tag{3}$$

and

$$0 = y + \frac{cb}{\sqrt{1 + a^2 + b^2}} \tag{4}$$

From (3) and (4), we get $\frac{a}{b} = \frac{x}{y}$ or $\frac{a}{x} = \frac{b}{y} = k$, say

$\therefore$ $$a = kx \text{ and } b = ky$$

Using these values in (3), we have

$$\frac{kc}{\sqrt{1 + k^2(x^2 + y^2)}} = -1$$

since k is negative.

i.e. $$1 + k^2(x^2 + y^2) = k^2c^2$$

or $$k^2(c^2 - x^2 - y^2) = 1$$

i.e. $$k = -\frac{1}{\sqrt{c^2 - x^2 - y^2}}$$

$\therefore$ $$a = -\frac{x}{\sqrt{c^2 - x^2 - y^2}}, \quad b = -\frac{y}{\sqrt{c^2 - x^2 - y^2}}$$

and $$\sqrt{1 + a^2 + b^2} = \frac{c}{\sqrt{c^2 - x^2 - y^2}}$$

Using these values in (2), the singular solution of (1) is got as

$$z = -\frac{x^2}{\sqrt{c^2 - x^2 - y^2}} - \frac{y^2}{\sqrt{c^2 - x^2 - y^2}} + \frac{c^2}{\sqrt{c^2 - x^2 - y^2}}$$

i.e. $$z = \sqrt{c^2 - x^2 - y^2} \quad \text{or}$$

$$x^2 + y^2 + z^2 = c^2$$

EXAMPLE 3

Solve the equation $z^2(p^2 + q^2 + 1) = c^2$, where c is a constant.

SOLUTION

The given equation

$$z^2(p^2 + q^2 + 1) = c^2 \tag{1}$$

does not contain x and y explicitly.

Therefore (1) has a solution of the form

$$z = z(x + ay) \tag{2}$$

where $z(u) = z(x + ay)$ is a function of $(x + ay)$, where a is an arbitrary constant.

From (2), we have $p = \frac{dz}{du}$ and $q = \frac{dz}{du} \cdot a$

Since (2) is a solution of (1), we get

$$z^2 \left\{ \left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 + 1 \right\} = c^2$$

i.e.

$$(1 + a^2) \left(\frac{dz}{du} \right)^2 = \frac{c^2}{z^2} - 1$$

i.e.

$$\sqrt{1 + a^2} \frac{dz}{du} = \frac{\sqrt{c^2 - z^2}}{z}$$

i.e.

$$\sqrt{1 + a^2} \frac{z dz}{\sqrt{c^2 - z^2}} = du \tag{3}$$

Integrating (3), the complete solution of (1) is

$$-\frac{1}{2} \sqrt{1 + a^2} \int \frac{-2z dz}{\sqrt{c^2 - z^2}} = u + b$$

i.e.

$$-\sqrt{1 + a^2} \sqrt{c^2 - z^2} = x + ay + b \text{ or}$$

$$(1 + a^2)(c^2 - z^2) = (x + ay + b)^2 \tag{4}$$

The general and singular solutions of (1) are found out from (4) as usual.

EXAMPLE 4

Solve the equation

$$p^2(1 + x^2)y = qx^2$$

SOLUTION

The given equation, which does not contain z, can be rewritten as

$$p^2\frac{(1+x^2)}{x^2}=\frac{q}{y}=a,\ \text{say} \tag{1}$$

$$p=\frac{\sqrt{a}\cdot x}{\sqrt{1+x^2}} \quad \text{and} \quad q=ay$$

$$dz=p\,dx+q\,dy$$

$$=\sqrt{a}\cdot\frac{x}{\sqrt{1+x^2}}dx+ay\,dy \tag{2}$$

Integrating (2), we get the complete solution of the given equation as

$$z=\sqrt{a(1+x^2)}+\frac{ay^2}{2}+b \tag{3}$$

where a and b are arbitrary constants.

From (3), we get the general solution as usual. Singular solution does not exist.

EXAMPLE 5

Solve the equation $\dfrac{x^2}{p}+\dfrac{y^2}{q}=z$.

SOLUTION

The given equation does not belong to any of the standard types.
It can be rewritten as

$$\frac{1}{px^{-2}}+\frac{1}{qy^{-2}}=z \tag{1}$$

As equation (1) contains px^{-2} and qy^{-2}, we make the substitutions $X = x^3$ and $Y = y^3$. [Refer to type A equations]

Then $P = \dfrac{\partial z}{\partial X} = p \cdot \dfrac{1}{3x^2}$ or $px^{-2} = 3P$ and similarly $qy^{-2} = 3Q$.

Then (1) becomes

$$\frac{1}{P} + \frac{1}{Q} = 3Z \tag{2}$$

As (2) does not contain X and Y explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

Since (3) is a solution of (2), we get

$$\frac{dz}{du}(1 + a) = 3az\left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du}\left(3az\frac{dz}{du} - a - 1\right) = 0$$

$$\text{As} \quad \frac{dz}{du} \neq 0, \qquad 3az\frac{dz}{du} = a + 1 \tag{4}$$

Solving (4), $\int 3az \, dz = (a + 1)u + b$

i.e. $\quad \frac{3}{2}az^2 = (a + 1)(X + aY) + b$

which is the complete solution of equation (2).

$\therefore$ The complete solution of equation (1) is

$$\frac{3}{2}az^2 = (a + 1)(x^3 + ay^3) + b$$

where a and b are arbitrary constants.

The general and singular solutions are found out as usual.

EXAMPLE 6

Solve the equation

$$yp = 2xy + \log q$$

SOLUTION

The given equation, which does not contain z, can be rewritten as

$$p - 2x = \frac{1}{y}\log q = a, \text{ say} \tag{1}$$

$\therefore$ $$p = 2x + a \quad \text{and} \quad q = e^{ay}$$

Now $$dz = p\,dx + q\,dy$$

i.e. $$dz = (2x + a)dx + e^{ay}dy \tag{2}$$

Integrating (2), we get

$$z = x^2 + ax + \frac{1}{a}e^{ay} + b \tag{3}$$

where a and b are arbitrary constants.
Equation (3) is the complete solution of the given equation.
General solution is found out as usual.
Singular solution does not exist.

1.8 GENERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations, for which the general solution can be obtained directly, can be divided into the following three categories

1. Equations that can be solved by direct (partial) integration.

2. **Lagrange's linear equation of the first order,**

3. Linear partial differential equations of higher order with constant coefficients,

1.9 LAGRANGE'S LINEAR EQUATION

A linear partial differential equation of the first order, which is of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z, is called *Lagrange's linear equation*.

General solution of Lagrange's linear equation

The general solution of the equation $Pp + Qq = R$ is $f(u, v) = 0$, where 'f' is an arbitrary function and $u(x, y, z) = a$ and $v(x, y, z) = b$ are independent solutions of the simultaneous differential equations $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$.

Working rule to solve $Pp + Qq = R$

(i) To solve $Pp + Qq = R$, we form the corresponding subsidiary simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(ii) Solving these equations, we get two independent solutions $u = a$ and $v = b$.

(iii) Then the required general solution is $f(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$.

1.10 SOLUTION OF THE SIMULTANEOUS EQUATIONS $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Method of grouping

By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, eventhough P; Q; R are, in general, functions of x, y, z. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings.

Method of Multipliers

If we can find a set of three quantities l, m, n, which may be constants or functions of the variables x, y, z, such that $lP + mQ + nR = 0$, then a solution of the simultaneous equations is found out as follows.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

Since $lP + mQ + nR = 0$, $l dx + m dy + n dz = 0$. If $l dx + m dy + n dz$ is an exact differential of some function $u(x, y, z)$, then we get $du = 0$. Integrating this, we get $u = a$, which is a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Similarly, if we can find another set of independent multipliers l', m', n', we can get another independent solution $v = b$.

EXAMPLE 1

Solve the equations $\dfrac{\partial^2 z}{\partial x^2} = xy$

SOLUTION

$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrating both sides of (1) partially with respect to x (i.e. treating y as a constant),

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + \phi(y) \tag{2}$$

Integrating (2) partially with respect to x,

$$z = \frac{x^3}{6}y + f(y) + x \cdot \phi(y) \tag{3}$$

where $f(y)$ and $\phi(y)$ are arbitrary functions. Equation (3) is the required general solution of (1).

EXAMPLE 2

By changing the independent variables by the transformations $u = x - y$ and $v = x + y$, show that the equation $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$ can be transformed as $\dfrac{\partial^2 z}{\partial v^2} = 0$ and hence solve it.

SOLUTION

$u = x - y$ and $v = x + y$

$x = \dfrac{u + v}{2}$ and $y = \dfrac{v - u}{2}$

If we express x and y in z in terms of u and v, z becomes a function of u and v.

$$\begin{aligned} z_x &= \frac{\partial z}{\partial x} = z_u \cdot u_x + z_v \cdot v_x, \text{ where } z_u = \frac{\partial z}{\partial u} \text{ and } u_x = \frac{\partial u}{\partial x}, \text{ etc.} \\ &= z_u + z_v \end{aligned}$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = -z_u + z_v$$
$$z_{xx} = (z_{uu} + z_{uv}) + (z_{vu} + z_{vv}) = z_{uu} + 2z_{uv} + z_{vv}$$
$$z_{xy} = (-z_{uu} + z_{uv}) + (-z_{vu} + z_{vv}) = -z_{uu} + z_{vv}$$
$$z_{yy} = z_{uu} - z_{uv} + (-z_{vu} + z_{vv}) = z_{uu} - 2z_{uv} + z_{vv}$$

Using these values in the given equation $z_{xx}+2z_{xy}+z_{yy} = 0$, it becomes $4z_{vv} = 0$.

i.e.
$$\frac{\partial^2 z}{\partial v^2} = 0 \tag{1}$$

Integrating (1) partially with respect to v,

$$\frac{\partial z}{\partial v} = g(u) \tag{2}$$

Integrating (2) partially with respect to v,

$$z = v \cdot g(u) + f(u) \tag{3}$$

$\therefore$ The solution of the given equation is

$$z = f(x-y) + (x+y)g(x-y)$$

EXAMPLE 3

Solve the equation $x^2p + y^2q + z^2 = 0$.

SOLUTION

The given equation

$$x^2p + y^2q = -z^2 \tag{1}$$

is a Lagrange's linear equation with $P = x^2$, $Q = y^2$ and $R = -z^2$

The subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Taking the first two ratios, we get an ordinary differential equation in x and y, namely, $\frac{dx}{x^2} = \frac{dy}{y^2}$.

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} - a$

i.e. $$\frac{1}{x} - \frac{1}{y} = a \qquad (1)$$

Taking the last two ratios, we get the equation $\frac{dy}{y^2} = \frac{-dz}{z^2}$

$$\frac{dy}{y^2} = \frac{dz}{dz^2}$$

Integrating, we get $\frac{-1}{y} = \frac{1}{z} - b$

Solving,

$$\frac{1}{y} + \frac{1}{z} = b \qquad (2)$$

$\therefore$ The general solution of the given equation is $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$, where '$f$' is an arbitrary function.

EXAMPLE 4

Solve the equation $(x - 2z)p + (2z - y)q = y - x$.

SOLUTION

This is Lagrange's linear
equation with $P = x - 2z$, $Q = 2z - y$ and $R = y - x$.

The subsidiary equations are

$$\frac{dx}{x - 2z} = \frac{dy}{2z - y} = \frac{dz}{y - x} \qquad (1)$$

Using the multipliers 1, 1, 1, each ratio in (1) $= \frac{dx + dy + dz}{0}$

$$\therefore \quad dx + dy + dz = 0$$

Integrating, we get, $$x + y + z = a \qquad (2)$$

Using the multipliers $y, x, 2z$, each ratio in (1) $= \dfrac{y\,dx + x\,dy + 2z\,dz}{0}$

$$\therefore \qquad d(xy) + 2z\,dz = 0$$

Integrating, we get $$xy + z^2 = b \qquad (3)$$

Therefore the general solution of the given equation is $f(x+y+z,\ xy+z^2) = 0$

EXAMPLE 5

Solve the equation $(x^2 - y^2 - z^2)p + 2xyq = 2zx$.

SOLUTION

This is Lagrange's linear

equation with $P = x^2 - y^2 - z^2$, $\quad Q = 2xy$, $\quad R = 2zx$.

The subsidiary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \qquad (1)$$

Taking the last two ratios, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log a$

i.e. $$\frac{y}{z} = a \qquad (2)$$

Using the multipliers x, y, z, each of the ratios in (1) $= \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$ (3)

Taking the last ratio in (1) and the ratio in (3),

$$\frac{dz}{2zx} = \frac{\frac{1}{2}d(x^2+y^2+z^2)}{x(x^2+y^2+z^2)}$$

i.e. $$\frac{dz}{z} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

Integrating, we get $\log b + \log z = \log(x^2+y^2+z^2)$

i.e. $$\frac{x^2+y^2+z^2}{z} = b \qquad (4)$$

Therefore the general solution of the given equation is $f\left(\dfrac{y}{z},\ \dfrac{x^2+y^2+z^2}{z}\right) = 0$.

EXAMPLE 6

Solve the equation $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

SOLUTION

This is a Lagrange's linear equation with $P = x^2(y-z)$, $Q = y^2(z-x)$, $R = z^2(x-y)$.

The subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad (1)$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, each of the ratios in (1) $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$$\therefore \quad \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get $-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -a$

or
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \quad (2)$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get $\log x + \log y + \log z = \log b$

or
$$xyz = b \quad (3)$$

Therefore the general solution of the given equation is $f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \quad xyz\right) = 0$.

1.13 LINEAR P.D.E.S OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

Linear partial differential equations of higher order with constant coefficients may be divided into two categories as given below.

(i) Equations in which the partial derivatives occurring are all of the same order (of course, with degree 1 each) and the coefficients are constants. Such equations are called *homogeneous linear P.D.E.s* with constant coefficients.

(ii) Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called *non-homogeneous linear P.D.E.s* with constant coefficients.

Any equation of the form

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots\ldots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \ldots(1)$$

where $a_1, a_2, \ldots\ldots, a_n$ are all constants, is called a homogeneous linear partial differential equation of the nth order with constant coefficients.

It is called homogeneous because all the terms contain derivatives of the same order *i.e.,* nth order.

Writing D for $\frac{\partial}{\partial x}$ and D′ for $\frac{\partial}{\partial y}$ (1) becomes

$$\left[D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n\right] z = F(x, y)$$

or

$$\phi(D, D')z = F(x, y)$$

where $\phi(D, D') = D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n$

As in the case of ordinary linear differential equations with constant coefficients the complete solution consists of two parts.

(I) The Complementary Function (C.F.): It is the complete solution of the equation $\phi(D, D') z = 0$. It must contain n arbitrary functions where n is the order of the differential equation.

(II) The Particular Integral (P.I): It is a particular solution (free from arbitrary constants) of $\phi(D, D')z = F(x, y)$.

Then complete solution C.S. is Z = C.F. + P.I.

WORKING RULE FOR FINDING THE C.F.

Consider the equation $\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

which in the symbolic form is

$$[D^2 + a_1 DD' + a_1 D'^2]Z = 0 \qquad ...(1)$$

Its symbolic form equated to zero is

$$D^2 + a_1, D D' + a_2 D'^2 = 0 \qquad ...(2)$$

It is called **Auxiliary Equation (A.E.)**

(2) can be expressed as quadratic in $\frac{D}{D'}$ *i.e.*, $\left(\frac{D}{D'}\right)^2 + a_1\left(\frac{D}{D'}\right) + a_2 = 0$

Let its roots be m_1 and m_2

Case I. When the roots of the auxiliary equation are distinct then (1) can be put in the form

$$(D - m_1 D')(D - m_2 D')z = 0 \qquad ...(3)$$

Now the solution of $(D - m_2 D')z = 0$ is the solution of (3) $\therefore$ to solve it we have $\frac{\partial z}{\partial x} - m_2 \frac{\partial z}{\partial y} = 0$ or $p - m_2 q = 0$, which is Lagrange's form $\therefore$ its auxiliary equation are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

Taking 1st and 2nd terms of the ratio $m_1 dx = -dy$. Integrate $y + m_2 x = a$

Also $\qquad dz = 0 \quad \Rightarrow \quad z = b.$

$\therefore$ Solution of $\quad (D - m_2 D')z = 0$ is $z = f_2(y + m_2 x)$ $\qquad ...(4)$

Similarly from (3) the other factor $(D - m_1 D')z = 0$ will give the solution $z = f_1(y + m_1 x)$

Hence the complete solution of (1) is $z = f_1(y + m_1 x) + f_2(y + m_2 x)$

Case II. When the roots of the auxiliary equation are equal *i.e.*, $m_1 = m_2 = m$ (say) then (1) can be written as

$$(D - mD')(D - mD')\,z = 0 \qquad \text{...[4(}a\text{)]}$$

Let $\qquad (D - mD')\,z = u \qquad$ ...(5)

Then [4(a)] becomes $(D - mD')\,u = 0$

Its solution as proved in 1st case is $\quad u = f(y + mx)$

Substituting the value of 'u' in (5), we get $(D - m\,D')\,z = f\,(y + mx)$

or
$$\frac{\partial z}{\partial x} - m\,\frac{\partial z}{\partial y} = f\,(y + mx)$$

or $\qquad p - mq = f(y + mx)$, which is Lagrange's form

$\therefore\;$ Its auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{f\,(y + mx)}$

Taking 1st and 2nd terms of the ratio $-m\,dx = dy$. Integral

$$y + mx = a \qquad \text{...(6)}$$

Taking 1st and 3rd terms and substituting $y + mx = a$

$$\frac{dx}{1} = \frac{dz}{f\,(a)}$$

Integrating, we get $\qquad z = f(a)\,x + b$

$\therefore \qquad \boxed{z - f(a)x = b} \qquad$ ...(7)

Combining (6) and (7) complete solution of (1) is $z - f\,(a)\,x = f\,(y + mx)$

or
$$\boxed{z = f\,(y + mx) + xf\,(y + mx)}$$

Note 1. Auxiliary equation of (l) *i.e.*, $D^2 + a_1 DD' + a_2 a'^2 = 0$, which is quadratic in $\frac{D}{D'}$

i.e., $\left(\frac{D}{D'}\right)^2 + a_1\left(\frac{D}{D'}\right) + a_2 = 0$ can also be represented by $m_2 + a_1 m + a_2 = 0$, where m is obtained by replacing D by m and D′ by 1 so for convinence A.E. of (1) can be written as $m_2 + a_1 m + a_2 = 0$.

Note 2. Generalised form of the results obtained in case I and case II

Case I. If the distinct roots of an A.E. are $m_1, m_2, m_3, \ldots\ldots$, then

$\text{C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$ and so on.

Case II. If the roots of the A.E. are $m_1, m_2, m_3, \ldots\ldots$ and only two roots are equal *i.e.*, only $m_1 = m_2 = m$ and all other are distinct.

Then $\text{C.F.} = f_1(y + mx) + xf_2(y + mx) + f_3(y + m_3 x)$ and so on.

Case III. If the roots of the auxiliary equation are $m_1, m_2, m_3, m_4, \ldots\ldots$ and three roots are equal *i.e.*, $m_1 = m_2 = m_3 = m$ the $\text{C.F.} = f_1(y + mx) + xf_2(y + mx) + x_2 f_3(y + mx) + f_4(y + m_4 x) + \ldots\ldots$

We can continue this process of equal roots to any number of times.

TABLE FOR FINDING COMPLEMENTARY FUNCTIONS

Step I. Write the equation in symbolic form *i.e.*, $(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots + a_n D'^n) z = 0$

Step II. Write the auxiliary equation (putting $D = m$, $D' = 1$)

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0$$

Step III. Solve it for m. We will get exactly n values of m.

Step IV. Write C.F. as follows.

Roots of A.E.	C.F.
(1) $m_1, m_2, m_3, \ldots\ldots$(all distinct)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \ldots..$
(2) $m_1, m_1, m_3, \ldots\ldots$(two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \ldots..$
(3) $m_1, m_1, m_1, m_4, \ldots\ldots$(three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + f_4(y + m_4 x) + \ldots..$
(4) $m_1, m_1, \ldots\ldots$ r times, $m_{r+1} \ldots\ldots$ (r equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) \ldots..\ x^{r-1} f_r(y + m_1 x) + f_{r+1}(y + m_{r+1} x) \ldots..$

EXAMPLE 1

Solve the following equation

$$\frac{\partial^2 z}{\partial x^2} + 4\,\frac{\partial^2 z}{\partial x\,\partial y} - 5\,\frac{\partial^2 z}{\partial y^2} = 0$$

SOLUTION

Symbolic form of the given equation is $(D^2 + 4\,DD' - 5\,D'^2)\,z = 0$

A.E. is $\quad m^2 + 4m - 5 = 0$

$(m-1)(m+5) = 0 \quad \therefore \quad m = 1, -5$

Required solution is $z = f_1\,(y + x) + f_2\,(y - 5x)$

EXAMPLE 2

Solve the following equation

$(D^3 - 6D^2D' + 12\,DD'^2 - 8D'^3)z = 0.$

SOLUTION

$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)\,z = 0$

A.E. is $m^3 - 6m^2 + 12m - 8 = 0$

$(m-2)^3 = 0 \quad \therefore \quad m = 2, 2, 2$

Reqd. solution is $z = f_1\,(y + 2x) + x f_2\,(y + 2x) + x^2 f_3\,(y + 2x)$

RULES FOR FINDING PARTICULAR INTEGRALS

The complete solution of the homogeneous equation is

$$\frac{\partial^n z}{\partial x^n} + a_1\,\frac{\partial^n z}{\partial x^{n-1}\,\partial y} + a_2\,\frac{\partial^n z}{\partial x^{n-2}\,\partial y^2} \,.......\, a_n\,\frac{\partial^n z}{\partial y^n} = F(x, y)$$

i.e., $\quad (D^n + a_1\,D^{n-1}\,D' + a_2\,D^{n-2}\,D'^2 + ... + a_n\,D'^n)\,z = F(x, y)$

is known if C.F. and P.I. are obtained.

To Find Particular Integral

(*i*) **When $F(x, y) = e^{ax+by}$**

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by} = \frac{1}{\phi(a, b)} e^{ax+by}$$

[*i.e.*, put $D = a$ and $D' = b$] provided $\phi(a, b) \neq 0$

If $f(a, b) = 0$; it is called case of failure.

(*ii*) **When $F(x, y) = \sin(ax + by)$**

$$\text{P.I.} = \frac{1}{\phi\left\{D^2, DD', D'^2\right\}} \sin(ax + by) = \frac{1}{\phi\left[-a^2, -ab, -b^2\right]} \sin(ax + by)$$

[*i.e.*, put $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$] provided $\phi(-a^2, -ab, -b^2) \neq 0$

If $\phi(-a^2, -ab, -b^2) = 0$ then it is called a case of failure.

(*iii*) A similar rule holds for $\cos(ax + by)$

(*iv*) When $F(x, y) = x^m y^n$, where m, n are positive integers. $\left\{\begin{array}{l}\text{In each case expand by} \\ \text{Binomial theorem}\end{array}\right.$

$$\text{P.I.} = \frac{1}{\phi(D, D')} x^m y^n = \left[\phi(D, D')\right]^{-1} x^m y^n$$

If $m < n$ expand $[\phi(D, D')]^{-1}$ in powers of $\frac{D}{D'}$

and if $m > n$ $[\phi(D, D')]^{-1}$ in powers of $\frac{D'}{D}$

Also we have $$\frac{1}{D} F(x, y) = \int_{y\ \text{constant}} F(x, y)\, dx$$

and $$\frac{1}{D'} F(x, y) = \int_{x\ \text{constant}} F(x, y)\, dy$$

TABLE FOR FINDING P.I.

We know from Symbolic form $\text{P.I.} = \dfrac{1}{\phi(\text{D}, \text{D}')} \text{F}(x, y)$

S.No.	Function	P.I.
1.	When $\text{F}(x, y) = e^{ax+by}$	$\text{P.I.} = \dfrac{1}{\phi(a, b)} e^{ax+by}$ [Put $\text{D} = a$, $\text{D}' = b$] provided $\phi(a, b) \neq 0$
2.	When $\text{F}(x, y) = \sin(ax + by)$	$\text{P.I.} = \dfrac{1}{\phi(\text{D}^2, \text{DD}', \text{D}^2)} \sin(ax + by)$ $= \dfrac{\sin(ax+by)}{\phi(-a^2, -ab, -b^2)}$ Put $\text{D}^2 = -a^2$, $\text{DD}' = -ab$, $\text{D}'^2 = -b^2$ where $\phi(-a^2, -ab, -b^2) \neq 0$
3.	When $\text{F}(x, y) = \cos(ax + by)$	$\text{P.I.} = \dfrac{\cos(ax+by)}{\phi(-a^2, -ab, -b^2)}$ provided $\phi(-a^2, -ab, -b^2) \neq 0$
4.	When $\text{F}(x, y) = x^m y^n$	$\text{P.I.} = \dfrac{1}{\phi(\text{D}, \text{D}')} x^m y^n$ $= [\phi(\text{D}, \text{D}')^{-1} x^m y^n$ If $m < n$ expand $[\phi(\text{D}, \text{D}')^{-1}$ in power of $\dfrac{\text{D}}{\text{D}'}$ If $m > n$ expand $[\phi(\text{D}, \text{D}')^{-1}$ in power of $\dfrac{\text{D}'}{\text{D}}$
5.	When $\text{F}(x, y) = e^{ax+by}\,\text{V}$, where V is a function of x and y	$\text{P.I.} = \dfrac{1}{\phi(\text{D}, \text{D}')} e^{ax+by} = e^{ax+by} \dfrac{1}{\phi(\text{D}+a)(\text{D}'+b)} \text{V}$
6.	When $\text{F}(x, y)$ is any function of (x, y)	Resolve $\dfrac{1}{\phi(\text{D}, \text{D}')}$ into partial fractions and apply $\dfrac{1}{\text{D} - m\text{D}'} \text{F}(x, y) = \int \text{F}(x, c - mx)\, dx$, where $c = y + mx$

EXAMPLE 3

Solve the equation

$$(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$$

SOLUTION

The auxiliary equation is $m^3 - 3m + 2 = 0$

i.e. $\quad (m-1)(m^2+m-2) = 0$

i.e. $\quad (m-1)^2(m+2) = 0$

$\therefore \quad m = 1, 1, -2$

$\therefore \quad \text{C.F.} = xf_1(y+x) + f_2(y+x) + f_3(y-2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3 - 3DD'^2 + 2D'^3}(e^{2x-y} + e^{x+y}) \\
&= \frac{1}{(D+2D')(D-D')^2}e^{2x-y} + \frac{1}{(D-D')^2(D+2D')}e^{x+y} \\
&= \frac{1}{9} \cdot \frac{1}{D+2D'}e^{2x-y} + \frac{1}{9} \cdot \frac{1}{(D-D')^2}e^{x+y} \\
&= \frac{1}{9}\left[xe^{2x-y} + \frac{x^2}{2}e^{x+y}\right]
\end{aligned}$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y+x) + f_2(y+x) + f_3(y-2x) + \frac{x}{9}e^{2x-y} + \frac{x^2}{18}e^{x+y}$$

EXAMPLE 4

Solve the equation $\quad (D^2 - 3DD' + 2D'^2)z = 2\cosh(3x+4y)$

SOLUTION

The auxiliary equation is $$m^2 - 3m + 2 = 0$$

i.e. $$(m-1)(m-2) = 0$$

$\therefore$ $$m = 1, 2$$

$\therefore$ The C.F. of the given P.D.E. $= f_1(y+x) + f_2(y+2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 3DD' + 2D'^2} 2\cosh(3x+4y) \\
&= \frac{1}{D^2 - 3DD' + 2D'^2}\left[e^{3x+4y} + e^{-(3x+4y)}\right] \\
&= \frac{1}{3^2 - 3.3.4 + 2.4^2} e^{3x+4y} + \frac{1}{(-3)^2 - 3(-3)(-4) + 2(-4)^2} e^{-(3x+4y)} \\
&= \frac{1}{5}\left[e^{3x+4y} + e^{-(3x+4y)}\right] \\
&= \frac{2}{5}\cosh(3x+4y)
\end{aligned}$$

$\therefore$ The general solution of the given equation is $z = f_1(y+x) + f_2(y+2x) + \frac{2}{5}\cosh(3x+4y)$.

EXAMPLE 5

Solve the equation
$$(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3$$

SOLUTION

The auxiliary equation is

$$m^3 - 7m - 6 = 0, \quad \text{i.e.} \quad (m+1)(m^2 - m - 6) = 0$$

i.e.
$$(m+1)(m+2)(m-3) = 0$$

$\therefore \qquad m = -1, -2, 3$

$\therefore \qquad \text{C.F.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3) \\
&= \frac{1}{D^3}\left\{1 - \frac{\left(7DD'^2 + 6D'^3\right)}{D^3}\right\}^{-1}(x^2 + xy^2 + y^3) \\
&= \frac{1}{D^3}\left[1 + \frac{D'^2}{D^3}(7D + 6D') + \cdots\right](x^2 + xy^2 + y^3)
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{D^3} + \frac{1}{D^6}(7DD'^2 + 6D'^3)\right](x^2 + xy^2 + y^3) \\
&= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}\{7D \cdot (2x + 6y) + 36\} \\
&= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}(50) \\
&= \frac{x^5}{3.4.5} + y^2 \cdot \frac{x^4}{2.3.4} + y^3 \cdot \frac{x^3}{1.2.3} + 50 \cdot \frac{x^3}{1.2.3} \\
&= \frac{1}{60}x^5 + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3
\end{aligned}$$

$\therefore$ The general solution is

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x^5}{60} + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

EXAMPLE 6

Solve the equation

$$(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y}$$

SOLUTION

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\therefore \qquad m = 1, 1$

$\therefore \qquad \text{C.F.} = xf_1(y+x) + f_2(y+x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-D')^2}e^{x+y}(x^2y^2) \\
&= e^{x+y}\frac{1}{[(D+1)-(D'+1)]^2}x^2y^2 \\
&= e^{x+y}\frac{1}{(D-D')^2}x^2y^2 \\
&= e^{x+y}\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left(1+\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left\{x^2y^2+\frac{2}{D}(2x^2y)+\frac{3}{D^2}(2x^2)\right\}
\end{aligned}$$

$$\begin{aligned}
&= e^{x+y}\left[y^2\cdot\frac{1}{D^2}(x^2)+4y\cdot\frac{1}{D^3}(x^2)+6\cdot\frac{1}{D^4}(x^2)\right] \\
&= \left(\frac{1}{12}x^4y^2+\frac{1}{15}x^5y+\frac{1}{60}x^6\right)e^{x+y}
\end{aligned}$$

$\therefore$ General solution is

$$z = xf_1(y+x) + f_2(y+x) + \left(\frac{1}{12}y^2+\frac{1}{15}xy+\frac{1}{60}x^2\right)x^4e^{x+y}$$

EXAMPLE 7

Solve the equation

$$(D^2 - 5\,DD' + 6\,D'^2)\,z = y \sin x$$

SOLUTION

The auxiliary equation is $m^2 - 5m + 6 = 0$

i.e. $$(m-2)(m-3) = 0$$

$\therefore$ $$m = 2, 3$$

$\therefore$ $$\text{C.F.} = \phi_1(y+2x) + \phi_2(y+3x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-2D')(D-3D')} y \sin x \\
&= \frac{1}{D-2D'} \left[\int (a-3x) \sin x \, dx \right]_{a \to y+3x} \\
&= \frac{1}{D-2D'} [(a-3x)(-\cos x) + 3(-\sin x)]_{a \to y+3x} \\
&= \frac{1}{D-2D'} [-y \cos x - 3 \sin x] \\
&= -\left\{ \int [(a-2x)\cos x + 3 \sin x] dx \right\}_{a \to y+2x} \\
&= -\left[(a-2x)\sin x + 2(-\cos x) - 3\cos x \right]_{a \to y+2x} \\
&= 5 \cos x - y \sin x
\end{aligned}$$

$\therefore$ General solution is

$$z = \phi_1(y+2x) + \phi_2(y+3x) + 5\cos x - y \sin x$$

*********************** ALL THE BEST**********************************

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V -ENGINEERING MATHEMATICS-III– SMTA1301

UNIT V

THEORY OF SAMPLING AND TESTING OF HYPOTHESIS

Population:
The group of individuals, under study is called is called population.

Sample:
A finite subset of statistical individuals in a population is called Sample.

Sample size:
The number of individuals in a sample is called the Sample size.

Parameters and Statistics:
The statistical constants of the population are referred as Parameters and the statistical constants of the Sample are referred as Statistics.

Standard Error :
The standard deviation of sampling distribution of a statistic is known as its standard error and is denoted by (S.E)

Test of Significance :
It enable us to decide on the basis of the sample results if the deviation between the observed sample statistic and the hypothetical parameter value is significant or the deviation between two sample statistics is significant.

Null Hypothesis:
A definite statement about the population parameter which is usually a hypothesis of no-difference and is denoted by $H_{o.}$

Alternative Hypothesis:
Any hypothesis which is complementary to the null hypothesis is called an Alternative Hypothesis and is denoted by $H_{1.}$

Errors in Sampling:
Type I and Type II errors.
Type I error: Rejection of H_0 when it is true.
Type II error: Acceptance of H_0 when it is false.

Two types of errors occur in practice when we decide to accept or reject a lot after examining a sample from it. They are Type 1 error occurs while rejecting H_o when it is true. Type 2 error occurs while accepting H_o when it is wrong.

Critical region:

A region corresponding to a statistic t in the sample space S which lead to the rejection of H_o is called Critical region or Rejection region. Those regions which lead to the acceptance of H_o are called Acceptance Region.

Level of Significance :

The probability α that a random value of the statistic "t" belongs to the critical region is known as the level of significance. In otherwords the level of significance is the size of the type I error. The levels of significance usually employed in testing of hypothesis are 5% and 1%.

One tail and two tailed test:

A test of any statistical hyposthesis where the alternate hypothesis is one tailed(right tailed/ left tailed) is called one tailed test.

For the null hypothesis H_0 if $\mu = \mu_0$ then.

$H_1 = \mu > \mu_0$ (Right tail)

$H_1 = \mu < \mu_0$ (Left tail)

$H_1 = \mu \# \mu_0$ (Two tail test)

Types of samples :

Small sample and Large sample

Small sample (n≤<30) : "Students t test, F test , Chi Square test

Large sample (n>30) : Z test.

95 % confidence limit for the population mean μ in a small test.

Let x be the sample mean and n be the sample size. Let s be the sample S.D.

Then $\bar{x} \pm t_{0.05}$ (s/√n-1)

Application of t – distribution

When the size of the sample is less than 30, „t" test is used in (a) single mean and (b) difference of two means.

Distinguish between parameters and statistics.

Statistical constant of the population are usually referred to as parameters. Statistical measures computed from sample observations alone are usually referred to as statistic.

In practice, parameter values are not known and their estimates based

Write short notes on critical value.

The critical or rejection region is the region which corresponds to a predetermined level of significanceα. Whenever the sample statistic falls in the critical region we reject the null hypothesis as it will be considered to be probably false. The value that separates the rejection region from the acceptance region is called the critical value.

Define level of significance explain.

The probability α that a random value of the statistic„t‟ belongs to the critical region is known as the level of significance. In other words level of significance is the size of type I error. The levels of significance usually employed in testing of hypothesis are 5% and 1%.

Outline the assumptions made when the't' test us applied for difference of means.

(i) Degree of freedom is $n_1 + n_2 - 2$.

(ii) The two population variances are believed to be equal.

(iii) $S = \sqrt{\frac{(n_1 s_1^2 + n_2 s_2^2)}{(n_1 + n_2 - 2)}}$ is the standard error.

<u>Type I Student t test for single mean</u>

$$|t| = \frac{\bar{x} - \mu}{s / \sqrt{n-1}}$$

Where $\bar{x}$ the sample mean, μ is is the population mean, s is the SD and n is the number of observations.

Problems :

1. The mean weakly sales of soap bars in departmental stores were 146.3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a SD of 17.2. Was the advertising campaign successful?

Solution:

Calculated t value = 1.97 and Tabulated Value = 1.72(at 5% level of significance with 21 degrees of freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. A sample of 26 bulbs gives a mean life of 990 hours with SD of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not upto the standard.

Solution:
Calculated t value = 2.5
Tabulated Value = 1.708(at 5% level of significance with 25 degrees of freedom)
Calculated value > Tabulated value, Reject Ho (Null hypothesis)

3. The average breaking strength of steel rod is specified to be 18.5 thousand pounds. To test this sample of 14 rods was tested. The mean and SD obtained were 17.85 and 1.955 respectively. Is the result of the experiment significant?

Solution:
Calculated t value = 1.199
Tabulated Value = 2.16(at 5% level of significance with 13 degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

4. Find the confidence limits of the mean of the population for a random sample of size 16 from a normal population with mean 53 and SD √10 with t value at 5% for 15 Degrees of freedom is 2.13.

Solution
(54.68, 51.31)

<u>Type II Student t test when SD not given</u>

$|t| = (\bar{x} - \mu) / (s/\sqrt{n})$

Where $\bar{x} = \Sigma(x)/n$ and $s^2 = 1/(n-1)\,\Sigma(x-\bar{x})^2$

PROBLEMS

Students t test where SD of the sample is not given directly)

1. A random sample of 10 boys had the following IQ"s 70,120,110,101,88,83,95,98,107,100. Do these data support the assumption of a population mean IQ of 100? Find the reasonable range in which most of the mean IQ values of samples of 10 boys lie?

Solution:
Calculated t value = 0.62
Tabulated Value = 2.26(at 5% level of significance with 9 degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)
95% confidence limits: (86.99, 107.4)

2. The heights of 10 males of a given locality are found to be 70,67,62,68,61,68,70,64,64,66 inches. Is it reasonable to believe that the average height is greater than 64 inches Test at 5%.

Solution:
Calculated t value = 2
Tabulated Value = 1.833(at 5% level of significance with 9 degrees of freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

3. Certain pesticide is packed into bags by a machine. A random sample of 10 bags is drawn and their contents are found to be as follows: 50,49,52,44,45,48,46,45,49,45. Test if the average packing to be taken 50 grams

Solution:

Calculated t value = 3.19

Tabulated Value = 2.262 (at 5% level of significance with 9 degrees of Freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

Type III Student t test for difference of means of two samples

To test the significant difference between two mean $\bar{x}_1$ and $\bar{x}_2$ of sample sizes n_1 and n_2 use the statistic.

$$|t| = (\bar{x}_1 - \bar{x}_2) / s\sqrt{((1/n_1) + (1/n_2))}$$

Where $s^2 = (n_1 s_1^2 + n_2 s_2^2)/ (n_1 + n_2 - 2)$

s_1 and s_2 being the sample standard deviations degree of freedom being $n_1 + n_2 - 2$.

PROBLEMS

1. Samples of two types of electric light bulbs were tested for length of life and following data were obtained.

Type I	Type II
Sample size $n_1 = 8$	$n_2 = 7$
Sample means $x_1 = 1234$ hrs	$x_2 = 1036$ hrs
Sample S.D. $s_1 = 36$ hrs	$s_2 = 40$ hrs

Is the difference in the means sufficient to warrant that type I is superior to type II regarding length of life.

Solution:

Calculated t value = 9.39

Tabulated Value = 1.77 (at 5% level of significance with 13 degrees of freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. Below are given the gain in weights (in N) of pigs fed on two diets A and B.

Diet A	25	32	30	34	24	14	32	24	30	31	35	25		
Diet B	44	34	22	10	47	31	40	32	35	18	21	35	29	22

Test if the two diets differ significantly as regards their effect on increase in weight.

Solution:

Calculated t value = 0.609

Tabulated Value = 2.06 (at 5% level of significance with 25 degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

3. The nicotine content in milligrams of two samples of tobacco were found to be as follows:

Sample A	24	27	26	21	25	
Sample B	27	30	28	31	22	36

Can it be said that two samples come from normal populations having the same mean.

Solution:

Calculated t value = 1.92
Tabulated Value = 2.262 (at 5% level of significance with 9 degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

4. The means of two random samples of sizes 9 and 7 are given as 196.42 and 198.82. The sum of the squares of the deviations from mean is 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same normal population?

Solution:

Calculated t value = 2.63
Tabulated Value = 2.15 (at 5% level of significance with 14 degrees of freedom)
Calculated value > Tabulated value, Reject Ho (Null hypothesis)

F - TEST

To test if the two samples have come from same population we use F test (OR) To test if there is any significant difference between two estimates of population variance.

F= GREATER VARIANCE/SMALLER VARIANCE
(OR)
$F = S_1^2 / S_2^2$
Where
$S_1^2 = \Sigma (x- \bar{x})^2 / n_1-1$
$S_2^2 = \Sigma (y- \bar{y})^2 / n_2-1$
Where n_1 is the first sample size and n_2 is the second sample size

1. Applications of F-test.

To test whether if there is any significant difference between two estimates of population variance. To test if the two samples have come from the same population we use f test.

2. Uses f test in sampling

To test whether there is any significant difference between two estimates of population variance. To test if the two samples have come from the same population.

If the sample variance S^2 is not given we can obtain the population variance by using the relation

$S_1^2 = n_1 s_1^2/(n_1-1)$ and $S_2^2 = n_2 s_2^2/(n_2-1)$

If we have to test whether the samples come from the same normal population then the problem has to be solved by both the t test and the f tests.

(i) To test the equality of the variances by F test
(ii) To test the equality of means by t test

Problems

1. In one sample of 8 observations the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observation it was 102. 6. Test whether this difference is significant at 5 % level.

Solution:

Calculated F value = 1.057
Tabulated Value = 3.29 (at 5% level of significance with (7,9) degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

2. Two random samples gave the following results.

Sample	Size	Sample mean	Sum of squares of deviations from the mean
1	10	15	90
2	12	14	108

Test whether the samples come from the same normal population.

Solution:
Calculated F value = 1.018
Tabulated Value = 2.9 (at 5% level of significance with (9,11) degrees of freedom)
By t test Calculated t value = 0.74
Tabulated Value = 2.086 (at 5% level of significance).
In both the tests of sampling
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

3. The time taken by workers in performing a job by method I and method II is given below.

Method I	20	16	26	27	23	22	
Method II	27	33	42	35	32	34	38

Do the data show that the variances of time distribution from population from which these samples are drawn do not differ significantly?
Solution:
Calculated F value = 1.37
Tabulated Value = 4.95 (at 5% level of significance with (6,5) degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)
4. The nicotine content in milligrams of two samples of tobacco were found to be as follows:

Sample A	24	27	26	21	25	
Sample B	27	30	28	31	22	36

Can it be said that two samples come from normal populations having the same variances.
Solution:
Calculated F value =4.07
Tabulated Value = 6.26 (at 5% level of significance with (5,4) degrees of freedom)
Calculated value < Tabulated value, Accept Ho (Null hypothesis)

CHI-SQUARE TEST

CHI-SQUARE TEST FORMULAE

$$\psi^2 = \Sigma \frac{(O - E)^2}{E}$$

Where O is the observed frequency and E is the Expected frequency

1. Define Chi square test of goodness of fit.

Under the test of goodness of fit we try to find out how far observed values of a given phenomenon are significantly different from the expected values. The Chi square statistic can be used to judge the difference between the observed and expected frequencies.

2. Give the main use of Chi-square test.

To test the significance of discrepancy between experimental values and the theoretical values, obtained under some theory or hypothesis.

3. Write the condition for the application of ψ^2 test.

ψ^2 test can be applied only for small samples.

4. How is the number of degrees of freedom of chi-square distribution fixed for testing the goodness of fit of a poisson distribution for the given data.

Degree of freedom = n – 1 where n is the no. of observations.

CHI-SQUARE TEST FOR INDEPENDENCE OF ATTRIBUTES

An attribute means a quality or characteristic. Eg. Drinking, smoking, blindness, honesty

2 X 2 CONTINGENCY TABLE

Consider any two attributes A and B. A and B are divided into two classes.

OBSERVED FREQUENCIES

A	a	b
B	c	d

EXPECTED FREQUENCIES

E(a)= (a+c)(a+b)/N	E(b)=(b+d)(a+b)/N	a+b
E(c) = (a+c)(c+d)/N	E(d)=(b+d)(c+d)/N	c+d
a+c	b+d	N(Total frequencies)

PROBLEMS

1. A die is thrown 264 times with the following results. Show that the die is biased

No appeared on the die	1	2	3	4	5	6
Frequency	40	32	28	58	54	60

Solution:

Calculated ψ^2 value =17.6362

Tabulated Value = 11.07 (at 5% level of significance with 5 degrees of freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. 200 digits were chosen at random from a set of tables. The frequencies of the digits were

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	18	19	23	21	16	25	22	20	21	15

Use the ψ^2 test to assess the correctness of the hypothesis that the digits were distributed in the equal number in the tables from which these were chosen.

Solution:

Calculated ψ^2 value = 4.3

Tabulated Value = 16.919 (at 5% level of significance with 9 degrees of freedom)

Calculated value< Tabulated value, Accept Ho (Null hypothesis)

3. Two groups of 100 people each were taken for testing the use of a vaccine 15 persons contracted the disease out of the inoculated persons while 25 contracted the disease in the other group. Test the efficiency of the vaccine using chi square test.

Solution:

Calculated ψ^2 value = 3.125

Tabulated Value = 3.184 (at 5% level of significance with 1 degrees of freedom)

Calculated value< Tabulated value, Accept Ho (Null hypothesis)

4. In a certain sample of 2000 families 1400 families are consumers of tea. Out of 1800 Hindu families, 1236 families consume tea. Use Chi square test and state whether there is any significant difference between consumption of tea among Hindu and Non – Hindu families.

Solution:

Calculated ψ^2 value = 15.238

Tabulated Value = 3.841 (at 5% level of significance with 1 degrees of freedom)

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

5. Given the following contingency table for hair colour and eye colour. Find the value of Chi-Square and is there any good association between the two

Hair colour / Eye colour	Fair	Brown	Black
Grey	20	10	20
Brown	25	15	20
Black	15	5	20

Solution:

Calculated ψ^2 value = 3.6458

Tabulated Value = 9.488 (at 5% level of significance with 4 degrees of freedom)

Calculated value < Tabulated value, Accept Ho (Null hypothesis)

LARGE SAMPLES

TEST OF SIGNIFICANCE OF LARGE SAMPLES

If the size of the sample n>30 then that sample is called large sample.

Type 1. Test of significance for single proportion

Let p be the sample proportion and P be the population proportion, we use the statistic Z= (p-P) / $\sqrt{(PQ/n)}$

Limits for population proportion P are given by p±3$\sqrt{(PQ/n)}$
Where q = 1-p

1. A manufacture claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 were faulty. tEst his claim at 5% level of significance.
Solution:
Calculated Z value = 2.59
Tabulated Value = 1.96 (at 5% level of significance) Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers.
Solution:
Calculated Z value = 2.04
Tabulated Value = 1.645 (at 5% level of significance) Calculated value > Tabulated value, Reject Ho (Null hypothesis)

3. A die is thrown 9000 times and of these 3220 yielded 3 or 4. Is this consistent with the hypothesis that the die was unbiased?
Solution:
Calculated Z value = 4.94 since z>3
Calculated value > Tabulated value, Reject Ho (Null hypothesis)

4 A random sample of 500 apples were taken from the large consignment and 65 were found to be bad. Find the percentage of bad apples in the consignment.
Solution:
(0.175, 0.085) Hence percentage of bad apples in the consignment lies between 17.5% and 8.5%

Type II Test of significance for difference of proportions

Let n_1 and n_2 are the two sample sizes and sample proportions are p_1 and p_2

$Z = \frac{(p_1 - p_2)}{\sqrt{pq(1/n_1 + 1/n_2)}}$ where $p= (n_1p_1+n_2p_2)/n_1+n_2$ and $q=1-p$

Proplems

1. Before an increase in excise duty on tea, 800 persons out of a sample of 1000 persons were found to be tea drinkers. After an increase in duty 800 people were tea drinkers in the sample of 1200 people. Using standard error of proportions state whether there is a significant decrease in the consumption of tea after the increase in the excise duty.

Solution:

Calculated Z value = 6.972

Tabulated value at 5% (one tail) = 1.645

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations.

Solution:

Calculated Z value = 2.55

Tabulated value at 5% = 1.96

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

<u>Type III Test of significance for single Mean</u>

$z= \bar{x} - \mu / (\sigma/\sqrt{n})$ where $\bar{x}$ is the same mean

μ is the population mean, s is the population S.D.

n is the sample size.

The values of $\bar{x} \pm 1.96\ (\sigma/\sqrt{n})$ are called 95% confidence limits for the mean of the population corresponding to the given sample.

The values of $\bar{x} \pm 2.58\ (\sigma/\sqrt{n})$ are called 99% confidence limits for the mean of the population corresponding to the given sample.

PROBLEMS

1. A sample of 900 members has a mean of 3.4 cms and SD 2.61 cms. Is the sample from a large population of mean is 3.25 cm and SD 2.61 cms. If the population is normal and its mean is unknown find the 95% confidence limits of true mean.

Solution:

Calculated Z value = 1.724

Tabulated value at 5% = 1.96

Calculated value < Tabulated value, Accept Ho (Null hypothesis)

Limits (3.57, 3.2295)

2. An insurance agent has claimed that the average age of policy holders who issue through him is less than the average for all agents which is 30.5 years. A random sample of 100 policy holders who had issued through him gave the following age distribution.

Age	**16-20**	**21-25**	**26-30**	**31-35**	**36-40**
No of persons	**12**	**22**	**20**	**30**	**16**

Test the significant difference at 5% level of significance.

Solution:

Calculated Z value = 2.68

Tabulated value at 5% = 1.645

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

3 Write down the test statistic for single mean for large samples.

$\mu = \overline{X} - \mu / (\sigma/\sqrt{n})$ where $\overline{X}$ is the same mean

μ is the population mean, s is the population S.D.

n is the sample size.

4. The mean score of a random sample of 60 students is 145 with a SD of 40. Fine the 95 % confidence limit for the population mean.

Solution $\overline{z} = X \pm 1.96\ (\sigma/\sqrt{n})$

$= 145 \pm (1.96)\ (40/\sqrt{60})$

$= 145 \pm 10.12$

$= 155.12$ or 134.88

$\therefore$ The confidence limits are 155.12 and 134.88.

<u>Type IV Test of significance for Difference of means</u>

$$Z= (\bar{x}_1 - \bar{x}_2) / \sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}$$

PROBLEMS

1. The means of 2 large samples of 1000 and 2000 members are 67.5 inches and 68 inches respectively. Can the samples be regarded as drawn from the same population of SD 2.5 inches.

Solution:

Calculated Z value = 5.16

Tabulated value at 5% = 1.96

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

2. The mean yield of wheat from a district A was 210 pounds with SD 10 pounds per acre from a sample of 100 plots. In another district the mean yield was 220 pounds with sD 12 pounds from a sample of 150 plots. Assuming that the SD of yield in the entire state was 11 pounds test whether there is any significant difference between the mean yield of crops in the two districts.

Solution:

Calculated Z value = 7.041

Tabulated value at 5% = 1.96

Calculated value > Tabulated value, Reject Ho (Null hypothesis)

PRACTICE PROBLEMS

1. Ten cartoons are taken at random from an automatic filling machine. The mean net weight of 10 cartoons is 11.802 and SD is 0.15. Does the sample mean differ significantly from the weight of 12?

Solution:

Calculated t value = 4

Tabulated Value = 2.26(at 5% level of significance with 9 degrees of freedom)

Calculated value > Tabulated value, Reject Ho(Null hypothesis)

2. A random sample of size 20 from a normal population gives a sample mean of 42 and sample SD 6. Test if the population mean is 44?

Solution:

Calculated t value = 1.45

Tabulated Value = 2.09(at 5% level of significance with 19 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis)

3. A machine which produces mica insulating washers for using electric devices is said to turn out washers having a thickness of 10 mm. A sample of 10 washers has an average of 9.52 mm with SD of 0.6 mm. calculate student‟s t test.

Solution:

Calculated t value = 2.528

Tabulated Value = 2.26(at 5% level of significance with 9 degrees of freedom)

Calculated value > Tabulated value, Reject Ho(Null hypothesis)

4. The mean lifetime of 25 fans produced by a company is computed to be 1570 hours with SD 120 hrs. The company claims that the average life of fans produced by them is 1600 hours. Is the claim acceptable.

Solution:

Calculated t value = 1.22

Tabulated Value = 2.06(at 5% level of significance with 24 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis)

5. From a population of students 10 are selected. Their weekly packet money observed as 20,22,21,15,25,19,18,20,21,22. Test if the sample supports that on an average student get Rs.25 as packet money.

Solution:

Calculated t value = 1.89

Tabulated Value = 2.26 (at 5% level of significance with 24 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis).

6. Ten individuals are chosen from random and their heights are found to be in inches 63,63,64,65,66,69,69,70,70,71. Discuss the solution that the mean height of the universe is 65?

Solution :

Calculated t value = 2.02

Tabulated Value = 2.26 (at 5% level of significance with 9 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis).

7. An IQ test was given to 5 persons before and after they were trained. Results are given below.

IQ before training	110	120	123	132	125
IQ after training	120	118	125	136	121

Test if there is any change in the IQ after the training program.

Solution :

Calculated t value = 0.816

Tabulated Value = 2.78 (at 5% level of significance with 4 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis).

8. Memory capacity of 10 girls were tested before and after training. State if the training was effective or not

Before	12	14	11	8	7	10	3	0	5	6
After	15	16	10	7	5	12	10	2	3	8

Solution :

Calculated t value = 1.3646

Tabulated Value = 2.26 (at 5% level of significance with 9 degrees of freedom)

Calculated value < Tabulated value, Accept Ho(Null hypothesis).

9. 1.Two random samples gave the following results. Test whether the samples come from the same normal population.

Sample	Size	Sample Mean	Sum of squares of deviations from the mean
1	10	15	90
2	12	14	108

Solution:

Calculated F =1.018,Tabulated F for (9,11) d.f at 5% level=2.90. Since Calculated F < Tabulated F, the null hypothesis H_0 is accepted. Calculated t =0.74,Tabulated t for 20 d.f at 5% level=2.086. Since Calculated t < Tabulated t, the null hypothesis H_0 is accepted.

10. The fatality rate of typhoid patients is believed to be 17.26%. In a certain year 640 patients suffering from typhoid were treated in a metropolitan hospital and only 63 patients died. Can you consider the hospital efficient?

Ans:z=4.96, H_0 rejected.

11. A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leave without making a purchase. A random sample of 50 shoppers showed that 35 of them left without making a purchase. Are these sample results consistent with the claim of the salesman?

Ans:z=1.443, H_0 accepted

12. In a large city A, 20% of a random sample of 900 school boys had a slight physical defect. In another large city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant?

Ans:z=0.92, H_0 accepted.

13. Before and increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty 800 out of a sample of 1200 persons. Find whether there is a significant decrease in the consumption of tea after the increase in duty.

Ans: z=6.82, H_0 is rejected.

14. A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160cm. Can it be reasonably regarded that, in the population, the mean height is 165cm, and the SD is 10cm?

Ans:z=5, H_0 rejected.

15. A simple sample of heights of 6400 English men has a mean of 170cm and SD of 6.4cm, while a sample of heights of 1600 Americans has a mean of 172cm and a SD of 6.3cm. Do the data indicate that Americans, on the average taller than Englishmen?
Ans: z=11.32, H_0 rejected.

16. The average marks scored by 32 boys is 72 with SD of 8, while that for 36 girls is 70 with SD of 6. Test at 1% level whether boys perform better than girls.
Ans:z-1.15, H_0 accepted.

17. A random sample of 600 men chosen from a certain city contained 400 smokers. In another sample of 900 men chosen from another city, there were 450 smokers. Do the data indicate that (i)the cities are significantly different with respect to smoking habit among men? and (ii)the first city contains more smokers than the second?
Ans:z=6.49,(i)yes (ii)yes

18. In a college, 60 junior students are found to have a mean height of 171.5cm and 50 senior students are found to have a mean height of 173.8 cm. Can we conclude, based on these data, that the juniors are shorter than the seniors at 1% level assuming that the SD of students of that college is 6.2cm?
Ans:No, z=1.937

19. Tests made on the breaking strength of 10 pieces of a metal gave the following results: 578,572,570,568,572,570, 570,572,596 and 584kg. Test if the mean breaking strength of the wire can be assumed as 577 kg?
Ans:yes,t=0.65

20. A mechinist is expected to make engine parts with axle diameter of 1.75cm. A random sample of 10 parts shows a mean diameter of 1.85cm, with SD of 0.1cm. On the basis of this sample, would you say that the work of the machinist is inferior?
Ans: yes, t=3

21. A certain injection administered to each of the 12 patients resulted in the following increases of blood pressure: 5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4. Can it be

concluded that the injection will be in general, accompanied by an increase in BP?
Ans: yes, t=2.89

22. The mean life time of a sample of 25 bulbs is found as 1550h, with SD of 120h. The company manufacturing the bulbs claims that the average life of their bulbs is 1600h. Is the claim acceptable?
Ans: yes, t=2.04

23. Two independent samples of sizes 8 and 7 contained the following values: Sample 1: 19, 17, 15, 21, 16, 18, 16, 14 and Sample 2: 15, 14, 15, 19, 15, 18, 16. Is the difference between the sample means significant?
Ans:No,t=0.93

24. The average production of 16 workers in a factory was 107 with SD of 9, while 12 workers in another comparable factory had an average production of 111 with SD of 10. Can we say that the production rate of workers in the latter factory is more than that in the former factory?
Ans: No, t=1.067

25. The following table gives the number of fatal road accidents that occurred during the 7 days of the week. Find whether the accidents are uniformly distributed over the week.
Ans:χ^2=4.17, accidents occur uniformly

Day	**Sun**	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**
Number	8	14	16	12	11	14	9

26. 1000 families were selected at random in a city to test the belief that high income families usually send their children to public schools and the low income families often said their children to government schools. From the following results test whether income and type of schooling are independent.
Ans:χ^2=22.5, reject H_0

Income	**School**	
	Public	**Govt.**
Low	370	430
High	130	70

27. Three samples are taken comprising 120 doctors, 150 advocates and 130 university teachers. Each person chosen is asked to select one of the three categories that best represents his feeling toward a certain national policy. The three categories are in favour of the policy(F), against the policy(A), and indifferent toward the policy(I). The results of the interviews are given below. On the basis of this data can it be concluded that the views Doctors, Advocates, and University teachers are homogeneous in so far as National policy under discussion is concerned.

Ans:χ^2=27.237, reject H_0

Occupation	Reaction		
	F	**A**	**I**
Doctors	80	30	10
Advocates	70	40	40
University teachers	50	50	30

28. A marketing agency gives you the following information about age groups of the sample informants and their liking for a particular model of scooter which a company plans to introduce. On the basis of the data can it be concluded that the model appeal id independent of the age group of the informants?

Ans:χ^2=42.788, reject H_0

	Age group of informants		
	Below 20	**20 – 39**	**40 – 59**
Liked	125	420	60
Disliked	75	220	100

29. A certain drug is claimed to be effective in curing cold. In an experiment on 500 persons with cold, half of them were given the drug and half of them were given the sugar pills. The patient"s reaction to the treatment are recorded and given below. On the basis of this data, can it be concluded that the drug and sugar pills differ significantly in curing cold?

Ans:χ^2=3.52, do not differ significantly

	Helped	**Harmed**	**No effect**
Drug	150	30	70
Sugar Pills	130	40	80

********************* ALL THE BEST *********************