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SCHOOL OF SCIENCE AND HUMANITIES

Department of Mathematics

UNIT – I – LOGIC – SMTA1208

LOGIC

<u>Propositional Logic – Definition</u>

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- "Man is Mortal", it returns truth value "TRUE"
- "12 + 9 = 3 − 2", it returns truth value "FALSE" The following is not a Proposition
- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

<u>Connectives</u>

In propositional logic generally we use five connectives which are − OR (∨), AND (∧), Negation/ NOT (¬), Implication / if-then (→), If and only if (↔).

<u>OR (∨)</u> : The OR operation of two propositions A and B (written as A ∨ B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows −

A	B	A ∨ B	
True	True	True	
True	False	True	
False	True	True	
False	False False	False False False	False False

<u>AND (∧)</u> : The AND operation of two propositions A and B (written as A ∧ B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A ∧ B
True	True	False

True	False	False
False	True	False
False	False	True

<u>Negation (¬)</u> : The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows –

A	**¬A**
True	False
False	True

<u>Implication / if-then (→):</u> An implication A→ B is False if A is true and B is false. The rest of the cases are true.

The truth table is as follows −

A	**B**	**A→ B**
True	True	True
True	False	False
False	True	True
False	False	True

<u>If and only if (</u>↔) : A↔B is bi-conditional logical connective which is true when p and q are both false or both are true.

The truth table is as follows −

A	**B**	**A↔B**
True	True	True

True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example − Prove [(A → B) ∧ A] → B is a tautology

The truth table is as follows −

A	**B**	**A → B**	**(A → B) ∧ A**	**[(A → B) ∧ A] → B**
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A → B) ∧ A] → B is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove (A ∨ B) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows −

A	**B**	**A ∨ B**	**¬A**	**¬B**	**(¬A) ∧ (¬B)**	**(A ∨ B) ∧ [(¬A) ∧ (¬B)]**
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A ∨ B) ∧ [(¬A) ∧ (¬B)] is "False", it is a contradiction

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove (A ∨ B ∨) ∧ (¬A) a contingency

The truth table is as follows −

A	**B**	**A ∨ B**	**¬A**	**(A ∨ B) ∧ (¬A)**
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A ∨ B) ∧ (¬A) has both "True" and "False", it is a contingency.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions −

- The truth tables of each statement have the same truth values.
- The bi-conditional statement X ↔Y is a tautology.

Example − Prove ¬(A ∨ B) and [(¬A) ∧ (¬B)] are equivalent

Testing by 1st method (Matching truth table)

A	**B**	**A ∨ B**	**¬ (A ∨ B)**	**¬A**	**¬B**	**[(¬A) ∧ (¬B)]**
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of ¬ (A ∨ B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality)

A	**B**	**¬ (A ∨ B)**	**[(¬A) ∧ (¬B)]**	**[¬ (A ∨ B)] ⇔ [(¬A) ∧ (¬B)]**
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬ (A ∨ B)] ⇔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

EQUIVALENT LAWS

Equivalence	Name of Identity
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws
$p \wedge F \equiv F$ $p \vee T \equiv T$	Domination Laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative Laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Ditributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$	Absorption Laws
$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$	Negation Laws

Logical Equivalences involving Conditional Statements

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Logical Equivalences involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

A conditional statement has two parts − **Hypothesis** and **Conclusion**.

Example of Conditional Statement − “If you do your homework, you will not be punished.” Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

Inverse, Converse, andContra-positive

Inverse –An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse will be “If not p, then not q”. The inverse of “If you do your homework, you will not be punished” is “If you do not do your homework, you will be punished.”

Converse −The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse will be “If *q*, then *p*”. The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do not do your homework”.

Contra-positive –The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is “If p, then q”, the inverse will be “If not *q*, then not *p*”. The Contra-positive of "If you do your homework, you will not be punished” is "If you will be punished, you do your homework”.

Example:

Give the converse and the Contra positve of the implication " If it is raining then I get wet".

Solution :

P : It is raining Q : I get wet

Converse : $Q \to P$: If I get wet, then it is raining.

Contrapositive : $\neg Q \to \neg P$: If I do not get wet, then it is not raining

DUALITY PRINCIPLE

Duality principle set states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said **self-dual** statement.

Examples : i) The dual of (A ∩ B) ∪ C is (A ∪ B) ∩ C
ii) The dual of P ∧ Q ∧ F is P ∨ Q ∨ T

Example : 1

Construct a truth table for $(p \to q) \to (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Example 2: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

.

Solution : The truth tables for these compound proposition is as follows.

1	2	3	4	5	6	7	8
P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$6 \leftrightarrow 7$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

We can observe that the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q.

Example 3: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution : The truth tables for these compound proposition as follows.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As the truth values of $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Example 4 : Determine whether each of the following form is a tautology or a contradiction or neither :

i) $(P \wedge Q) \rightarrow (P \vee Q)$

ii) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

iii) $(\neg P \wedge \neg Q) \rightarrow (P \rightarrow Q)$

iv) $(P \rightarrow Q) \wedge (P \wedge \neg Q)$

v) $[P \wedge (P \rightarrow \neg Q) \rightarrow Q]$

Solution:

i) The truth table for $(p \wedge q) \rightarrow (p \vee q)$

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Here all the entries in the last column are 'T'.

$\therefore (p \wedge q) \rightarrow (p \vee q)$ is a tautology.

ii) The truth table for $(p \vee q) \wedge (\neg p \wedge \neg q)$ is

1	2	3	4	5	6	
p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg P \wedge \neg q$	$3 \wedge 6$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

The entries in the last column are 'F'. Hence $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

iii) The truth table is as follows.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \to q$	$(\neg p \wedge \neg q) \to (p \to q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Here all entries in last column are 'T'.

$\therefore$ $(\neg p \wedge \neg q) \to (p \to q)$ is a tautology.

iv) The truth table is as follows.

p	q	$\neg q$	$p \wedge \neg q$	$p \to q$	$(p \to q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

All the entries in the last column are 'F'. Hence it is contradiction.

v) The truth table for $\left[p \wedge (p \rightarrow \neg q) \rightarrow q\right]$

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$	$\left[p \wedge (p \rightarrow \neg q) \rightarrow q\right]$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

The last entries are neither all 'T' nor all 'F'.

$\therefore$ $\left[p \wedge (p \rightarrow \neg q) \rightarrow q\right]$ is a neither tautology nor contradiction. It is a

Contingency.

Example 5: Symbolize the following statement

Let p, q, r be the following statements:
p: I will study discrete mathematics
q: I will watch T.V.
r: I am in a good mood.
Write the following statements in terms of p, q, r and logical connectives.
(1) If I do not study and I watch T.V., then I am in good mood.
(2) If I am in good mood, then I will study or I will watch T.V.
(3) If I am not in good mood, then I will not watch T.V. or I will study.
(4) I will watch T.V. and I will not study if and only if I am in good mood.
Solution:

(1) $(\neg p \wedge q) \rightarrow r$

(2) $r \rightarrow (p \vee q)$

(3) $\neg r \rightarrow (\neg q \vee p)$

(4) $(q \wedge \neg p) \leftrightarrow r$

Elementary Product: A product of the variables and their negations in a formula is called an elementary product. If P and Q are any two atomic variables, then p, ¬ p □ q ,
¬ q □p □ ¬ p are some examples of elementary products.

Elementary Sum: A sum of the variables and their negations in a formula is called an elementary sum. If P and Q are any two atomic variables, then p, ¬ p □ q, ¬ q □ p are some examples of elementary sums.

Normal Forms

We can convert any proposition in two normal forms −

1. Conjunctive normal form 2.Disjunctive normal form

Conjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs.

Examples

- $(P \cup Q) \cap (Q \cup R)$
- $(\neg P \cup Q \cup S \cup \neg T)$

Disjunctive Normal Form

A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs.

Examples

- $(P \cap Q) \cup (Q \cap R)$
- $(\neg P \cap Q \cap S \cap \neg T)$

Predicate Logic deals with predicates, which are propositions containing variables.

Functionally Complete set

A set of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only this set of logical operators. □, □, and ¬form a functionally complete set of operators.

Minterms: For two variables p and q there are 4 possible formulas which consist of conjunctions of p,q or its negation given by p □ q, p □ ¬q, ¬ p □ q and ¬ p □¬ ¬ q

Maxterms: For two variables p and q there are 4 possible formulas which consist of disjunctions of p,q or its negation given by p □ q, p □ ¬ q, ¬ p □q and ¬ p □ ¬ q

Principal Disjunctive Normal Form: For a given formula an equivalent formula consisting of disjunctions of minterms only is known as principal disjunctive normal form(PDNF)

<u>Principal Conjunctive Normal Form</u>: For a given formula an equivalent formula consisting of conjunctions of maxterms only is known as principal conjunctive normal form(PCNF)

Obtain DNF of $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$.

Solution:

$Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge (\neg((P \vee R) \wedge Q)$ (D e morgan law)

$\Leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$ (D e morgan law)

$\Leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q)$

(E xtended distributed law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (N egation law)

$\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (N egation law)

Obtain Pcnf and Pdnf of the formula $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

Solution:

Let S = $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

P	Q	$\neg$ P	$\neg$ Q	$\neg P \vee \neg$ Q	$P \leftrightarrow \neg Q$	S	Minterm	Maxterm
T	T	F	F	F	F	T	$P \wedge Q$	
T	F	F	T	T	T	T	$P \wedge \neg Q$	
F	T	T	F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	T	F	F		$P \vee Q$

PCNF: $P \vee Q$ and PDNF: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Inference Theory

The theory associated with checking the logical validity of the conclusion of the given set of premises by using Equivalence and Implication rule is called **Inference theory**

Direct Method

When a conclusion is derived from a set of premises by using the accepted rules of reasoning is called **direct method**.

Indirect method

While proving some results regarding logical conclusions from the set of premises, we use negation of the conclusion as an additional premise and try to arrive at a contradiction is called **Indirect method**

<u>Consistency and Inconsistency of Premises</u>

A set of formular $H_1, H_2, \ldots, H_m$ is said to be **inconsistent** if their conjunction implies Contradiction.

A set of formular $H_1, H_2, \ldots, H_m$ is said to be **consistent** if their conjunction implies Tautology.

<u>Rules of Inference</u>

Rule P: A premise may be introduced at any point in the derivation

Rules of Inference

TABLE 1 Rules of Inference.		
Rule of Inference	***Tautology***	***Name***
p $p \to q$ $\therefore q$	$[p \land (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \land (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$[(p) \land (q)] \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$[(p \lor q) \land (\neg p \lor r)] \to (q \lor r)$	Resolution

<u>Rule of inference to build arguments</u>

Example:

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny.
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. We will be home by sunset

p	It is sunny this afternoon	1.	$\neg p \wedge q$
q	It is colder than yesterday	2.	$r \rightarrow p$
r	We go swimming	3.	$\neg r \rightarrow s$
s	We will take a canoe trip	4.	$s \rightarrow t$
t	We will be home by sunset (the conclusion)	5.	t

propositions   hypotheses

Example 1.Show that R is logically derived from $P \rightarrow Q$, $Q \rightarrow R$, and P

Solution.	{1}	(1)	$P \rightarrow Q$	Rule P
	{2}	(2)	P	Rule P
	{1, 2}	(3)	Q	Rule (1), (2) and I11
	{4}	(4)	$Q \rightarrow R$	Rule P
	{1, 2, 4}	(5)	R	Rule (3), (4) and I11.

Example 2.Show that $S \vee R$ tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Solution .	{1}	(1)	$P \vee Q$	Rule P
	{1}	(2)	$7P \rightarrow Q$	T, (1), E1 and E16
	{3}	(3)	$Q \rightarrow S$	P
	{1, 3}	(4)	$7P \rightarrow S$	T, (2), (3), and I13
	{1, 3}	(5)	$7S \rightarrow P$	T, (4), E13 and E1
	{6}	(6)	$P \rightarrow R$	P
	{1, 3, 6}	(7)	$7S \rightarrow R$	T, (5), (6), and I13
	{1, 3, 6)	(8)	$S \vee R$	T, (7), E16 and E1

Example 3. Show that 7Q, P→ Q => 7P

Solution .	{1}	(1) P → Q	Rule P
	{1}	(2) 7P → 7Q	T, and E 18
	{3}	(3) 7Q	P
	{1, 3}	(4) 7P	T, (2), (3), and I11 .

Example 4 .Prove that R ^ (P V Q) is a valid conclusion from the premises PVQ ,
Q → R, P → M and 7M.

Solution .	{1}	(1) P → M	P
	{2}	(2) 7M	P
	{1, 2}	(3) 7P	T, (1), (2), and I12
	{4}	(4) P V Q	P
	{1, 2 , 4}	(5) Q	T, (3), (4), and I10.
	{6}	(6) Q → R	P
	{1, 2, 4, 6}	(7) R	T, (5), (6) and I11
	{1, 2, 4, 6}	(8) R ^ (PVQ)	T, (4), (7), and I9.

Example 5 .Show that R → S can be derived from the premises
P → (Q → S), 7R V P , and Q.

Solution.	{1}	(1) 7R V P	P
	{2}	(2) R	P, assumed premise
	{1, 2}	(3) P	T, (1), (2), and I10
	{4}	(4) P → (Q → S)	P
	{1, 2, 4}	(5) Q → S	T, (3), (4), and I11
	{6}	(6) Q	P
	{1, 2, 4, 6}	(7) S	T, (5), (6), and I11
	{1, 4, 6}	(8) R → S	CP.

Example 6.Show that $P \rightarrow S$ can be derived from the premises, 7P V Q, 7Q V R, and $R \rightarrow S$.

Solution.

{1}	(1)	7P V Q	P
{2}	(2)	P	P, assumed premise
{1, 2}	(3)	Q	T, (1), (2) and I11
{4}	(4)	7Q V R	P
{1, 2, 4}	(5)	R	T, (3), (4) and I11
{6}	(6)	$R \rightarrow S$	P
{1, 2, 4, 6}	(7)	S	T, (5), (6) and I11
{2, 7}	(8)	$P \rightarrow S$	CP

Predicate Logic

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

Eg.
" x is a Man"
Here **Predicate** is " is a Man" and it is denoted by M and **subject** "x" is denoted by x.
Symbolic form is M(x).

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol ∀.

∀x P(x) is read as for every value of x, P(x) is true.

Example − "Man is mortal" can be transformed into the propositional form ∀x P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.∃x P(x) is read as for some values of x, P(x) is true.

Example − "Some people are dishonest" can be transformed into the propositional form ∃x P(x) where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Nested Quantifiers

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Eg.2.

"***Every apple is red***".

The above statement can be restated as follows

For all x, if x is an apple then x is red

Now, we will translate it into symbolic form using universal quantifier.

Define A (x) : x is an apple.

R (x) : x is red.

∴ We write (*) into symbolic form as

$$(\forall x)\ (A(x) \rightarrow R(x))$$

Eg.3. ***"Some men are clever".***

The above statement can be restated as

"there is an x such that x is a man and x is clever".

We will translate it into symbolic form using Existential quantifier.

Let $M(x)$: x is a man

and $C(x)$: x is clever

$\therefore$ We write (B) into symbolic form as

$$(\exists x)\,(M(x) \wedge C(x))$$

Inference theory for Predicate calculus

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(y)}$	Rule US: Universal Specification
$\dfrac{P(c) \text{ for any c}}{\therefore \forall x P(x)}$	Rule UG: Universal Generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for any c}}$	Rule ES: Existential Specification
$P(c)$ for any c $\therefore \exists x P(x)$	Rule EG: Existential Generalization

Problem : **Show that $(\exists x)\ M(x)$ follows logically from the premises $(x)\ (H(x) \to M(x))$ and $(\exists x)\ H(x)$**

Solution :	1)	$(\exists x)\ H(x)$	rule P
	2)	$H(y)$	ES
	3)	$(x)\ (H(x) \to M(x))$	P
	4)	$H(y) \to M(y)$	US
	5)	$M(y)$	T, (2)
	6)	$(\exists x)\ M(x)$	EG

Symbolize the following statements:

(a) **All men are mortal**

(b) **All the world loves a lover**

(c) X is the father of mother of Y (d)No cats has a tail

(e) Some people who trust others are rewarded

Solution:

(a) Let M(x): x is a man H(x): x is Mortal

$(\forall x)(M(x) \rightarrow H(x))$

(b) Let P(x): x is a person L(x): x is a lover R(x,y): x loves y

$(x)(P(x) \rightarrow (y)(P(y) \wedge L(y) \rightarrow R(x,y)))$

(c) Let P(x): x is a person F(x,y): x is the father of y

M(x,y): x is the mother of y $(\exists z)(P(z) \wedge F(x,z) \wedge M(z,y))$

(d) Let C(x): x is a cat T(x): x has a tail

$(\forall x)(C(x) \rightarrow \neg T(x))$

(e) Let P(x): x is a person T(x): x trust others R(x): x is rewarded

$(\exists x)(P(x) \wedge T(x) \wedge R(x))$

Use the indirect method to prove that the conclusion $\exists z Q(z)$ follows from the premises $\forall x(P(x) \rightarrow Q(x))$ and $\exists y P(y)$

Solution:

1	$\neg \exists z Q(z)$	P(assumed)
2	$\forall z \neg Q(z)$	T, (1)
3	$\exists y P(y)$	P
4	$P(a)$	ES, (3)
5	$\neg Q(a)$	US, (2)
6	$P(a) \wedge \neg Q(a)$	T, (4),(5)
7	$\neg(P(a) \rightarrow Q(a))$	T, (6)
8	$\forall x(P(x) \rightarrow Q(x))$	P
9	$P(a) \rightarrow Q(a)$	US, (8)
10	$P(a) \rightarrow Q(a) \wedge \neg(P(a) \rightarrow Q(a))$	T,(7),(9) contradiction

Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$

Solution:

1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
2) $P(a) \wedge Q(a)$	ES, 1
3) $P(a)$	Rule T, 2
4) $Q(a)$	Rule T, 2
5) $(\exists x) P(x)$	EG, 3
6) $(\exists x) Q(x)$	EG, 4
7) $(\exists x) P(x) \wedge (\exists x) Q(x)$	Rule T, 5, 6

ASSIGNMENT PROBLEMS

1. Write the statement in symbolic form "Some real numbers are rational".
2. Symbolize the expression "x is the father of the mother of y"
3. Symbolize the expression "All the world loves a lover"
4. Write the negation of the statement "If there is a will, then there is a way".
5. Construct the truth table for $\neg(p \wedge q)$
6. Find the CNF and DNF of $\neg(p \vee q) \leftrightarrow (p \wedge q)$
7. Show that $P \to Q, Q \to \neg R, R, P \vee (J \wedge S)$ imply $J \wedge S$
8. Show that $P \to Q, P \to R, Q \to R, P$ are inconsistent.
9. Prove that $(\exists x)(P(x) \wedge Q(x) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
10. Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \to W(x,y)$ and $\neg W(a,b)$
11. Show that $\neg P \vee Q, \neg Q \vee R, R \to S \Rightarrow P \to S$
12. Show that $\neg(P \wedge \neg Q) \wedge \neg Q \vee R \wedge \neg R \Rightarrow \neg P$

13.Show that P is equivalent to $\neg\neg P, P \wedge P, P \vee P, P \wedge (P \vee Q), (P \wedge Q) \vee (P \wedge \neg Q)$

14.Indicate which one are tautologies (or) contradictions

(a) $(P \wedge Q) \Leftrightarrow P$ (b) $P \rightarrow P \vee Q$

15.If R:Ram is rich, H:Ram is happy ,Write in symbolic form

(a) Ram is poor but happy (b) Ram is poor or unhappy

(c) Ram is neither rich nor happy

16.Show that the hypothesis, "It is not sunny this afternoon and it is colder than yesterday," " We will go swimming only if it is sunny," "If we do not go swimming then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset "lead to the conclusion "we will be home by sunset".

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SCHOOL OF SCIENCE AND HUMANITIES

Department of Mathematics

UNIT – II – SET THEORY – SMTA1208

SET THEORY

Basic concepts of Set theory -Laws of Set theory -Partition of set, Relations -Types of Relations: Equivalence relation, Partial ordering relation-Graphs of relation-Hasse diagram, Functions: Injective, Surjective, Bijective functions, Compositions of functions, Identity and Inverse functions.

The concept of a set is used in various disciplines and particularly in computers.

Basic Definition:

1. "A collection of well-defined objects is called a **set**".

The capital letters are used to denote sets and small letters are used for denote objects of the set. Any object in the set is called element or member of the set. If *x* is an element of the set *X*, then we write $x \in X$, to be read as 'x *belongs to X'*, and

If *x* is not an element of *X*, the we write $x \notin X$ to be read as *'x does not belong to X'.*

2. The number of elements in the set A is called ***cardinality*** of the set A, denoted by |*A*| or *n(A)*. We note that in any set the elements are distinct. The collection of sets is also a set.

$$S = \{P_1, \{P_2, P_3\}, P_4, P_5\}$$

Here $\{P_2, P_3\}$ itself one set and it is one element of *S* and |*S*|=4.

3. Let A and B be any two sets. If every element of A is an element of B, then A is called a ***subset*** of B is denote by $'A \subseteq B'$.

We can say that A ***contained(included)*** in B, (or) B ***contains(includes)***A.

Symbolically, $A \subseteq B$(or)$B \supseteq A$

Logically.

$$A \subseteq B = (x\forall)\{x \in A \to x \in B\}$$

Let $A = \{1,2,3,4,5\},\ B = \{1,2,4\},\ C = \{1,5\},\ D = \{2\},\ E = \{1,4,2\}$

Then $B \subseteq A,\ C \subseteq A,\ D \subseteq A, D \subseteq B$

$C \nsubseteq B$, since $5 \in C \Rightarrow 5 \notin B,\ E \subseteq B$ and $B \subseteq E$.

Some of the important properties of set inclusion.

For any sets A, B and C $A \subseteq A$

(Reflexive)

$(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$ (Transitive)

Note that $A \subseteq B$ does not imply $B \subseteq A$ except for the following case.

4. Two sets A and B are said to be ***equal*** if and only if $A \subseteq B$ and $B \subseteq A$,

$i.e., \quad A = B \Leftrightarrow (A \subseteq B \ and\ B \subseteq C)$

Example $\{1,2,4\} = \{4,1,2\}$ and $\quad P = \{\{1,2\},4\},\ Q = \{1,2,4\}$ then $P \neq Q$

Since $\{1,2\} \in P$ and $\{1,2\} \notin Q$ even though $1,2 \in Q$.

The equality of sets is reflexive, symmetric, and transitive.

5. A set A is said to be a ***proper subset*** of a set B if $\quad A \subseteq B$ and $A \neq B$.
Symbolically it is written as $\quad A \subset B. i.e.,\ A \subset B \Leftrightarrow (A \subseteq B\ \wedge\ A \neq B)$

$\subset$ is also called a *proper inclusion.*

6. A set is said to be ***universal set*** if it includes every set under our discussion. A universal set is denoted by $\cup$ or E.

In other words, if *p(x)* is a predicate. $\quad E = \{x | p(x) \vee \neg p(x)\}$

One can observe that universal set contains all the sets.

7. A set is said to be ***empty set* or *null set*** if it does not contain any element, which is denoted by . $\emptyset$

In other words, if *p(x)* is a predicate. $\emptyset = \{x | p(x) \lor \neg p(x)\}$

One can observe that null set is a subset for all sets.

8. For a set A, the set of all subsets of A is called the ***power set*** of A. The power set of A is denoted by $\rho(A)$ or $2^\wedge$ $i.e., \ \rho(A) = \{S \mid S \subseteq A\}$

Example, Let $A = \{a, b, c\}$

Then $\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \ A\}$

Then set $\emptyset$ and A are called ***improper subsets*** of A. A and the remaining sets are called ***proper subsets*** of A.

One can easily note that the number of elements of $\rho(A)$ is

$2^{|A|}. i.e., \ |\rho(A)| = 2^{|A|}$

SOMEOPERATIONS ONSETS

1. Intersection of sets Definition:

Let A and B be any two sets, the *intersection* of A and B is written as $A \cap B$ is the set of all elements which belong to both A and B.

Symbolically

$A \cap B = \{ x \mid x \in A \, and \, x \in B\}$

Example $A = \{1,2,3,4,5,6\}, \ B = \{2,4,6,8\}$ then $A \cap B = \{2,4,6\}$. From the definition of intersection, it follows that for any sets A, B, C and universal set E.

$A \cap A = A$ $\quad$ $A \cap B = B \cap A$ $\quad$ $A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap E = A$ $\quad$ $A \cap \emptyset = \emptyset$

2. Disjoint sets

Definition:

Two sets A and B are called *disjoint* if and only if $A \cap B = \emptyset$, that is, A and B have no element in common.

Example$A = \{1,2,3\}$ $B = \{5,7,9\}$ $C = \{3,4\}$

$A \cap B = \emptyset,\ A \cap C = \{3\},\ B \cap C = \emptyset$

A and B are disjoint and B and C also, but A and C are not disjoint.

3. Mutually disjoint sets

Definition:

A collection of sets is called a *disjoint collection*, if for every pair of sets in the collection are disjoint. The elements of a disjoint collection are said to be *mutually disjoint.*

Let $A = \{A_i\}_{i \in I}$ be an indexed set, Ais mutually disjoint if and only if
for all $i, j \in I,\ i \neq j$ $A_i \cap A_j = \emptyset$

Example

$A_1 = \{\{1,2\},\{3\}\}, \qquad A_2 = \{\{1\},\{2,3\}\}, \qquad A_3 = \{\{1,2,3\}\}$

Then$A = \{A_1, A_2, A_3\}$ is a disjoint collection of sets.

$A_1 \cap A_2 = \emptyset, \qquad A_1 \cap A_3 = \emptyset$and $\qquad A_2 \cap A_3 = \emptyset$

4. Unions of sets

Definition:

The *union* of two sets A and B, written as $A \cup B$, is the set of all elements which are elements of A or the elements of B or both.

Symbolically$A \cup B = \{x \mid x \in A \ or \ x \in B\}$

Example Let $A = \{1,2,3,4,5,6\}$$B = \{2,4,6,8\}$ then $A \cup B = \{1,2,3,4,5,,6,8\}$

From the union, it is clear that, for any sets A, B, C, and universal set E.

$A \cup A = A \qquad A \cup B = B \cup A \qquad A \cup (B \cup C) = (A \cup B) \cup C$

$A \cup E = E \quad A \cup \emptyset = A$

5. Relative complement of a set

Definition:

Let A and B are any two sets. The *relative complement* of B in A, written $A - B$,is the set of elements of A which are not elements of B.

Symbolically$A - B = \{ x \mid x \in A \, or \, x \notin B\}$

Note that $A - B = A \cap \bar{B}$.

Example Let $A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$then

$A - B = \{1,3,5\}$

$B - A = \{8\}$

It is clear from the definition that, for any set A and B.

$A - B = \emptyset$

$A - B \neq B - A$

$A - \emptyset = A$

6. Complement of a set

Definition:

Let A be any set, and E be universal. The relative complement of A in E is called *absolute complement or complement* of A. The complement of A is denoted by $\bar{A}$Symbolically (or A^C or $\backsim A$)

$E - A = \bar{A} = \{ x \mid x \in E \, and \, x \notin A\}$

Example Let $E = \{1,2,3,4, \ldots\}$ be universal set and $A = \{2,4,6,8, \ldots\}$ be any set in E.

Then

$\bar{A} = \{1,3,5,7, \ldots\}$

From the definition, for any sets A $\quad \bar{\bar{A}} = A \quad \bar{\emptyset} = E$

$\bar{E} = \emptyset \quad A \cup \bar{A} = E A \cap \bar{A} = \emptyset$

7. Boolean sum of set

Definition:

Let A and B are any two sets. The *symmetric difference or Boolean sum* of A and B is the set A+B defined by

$$A + B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$$

(or) $A + B = \{ x \mid x \in A \text{ and } x \notin B\} \cup \{ x \mid x \in B \text{ and } x \notin A\}$

Example Let

$A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A + B = \{1,3,5,8\}$ From the definition, for any sets A and B.

$A + A = \emptyset, \quad A + \emptyset = A$

and $A + E = \bar{A}, \quad A + B = B + A$

$A + (B + C) = (A + B) + C$

8. The principle of duality

If we interchange the symbols $\cap$, $\cup$, E and $\emptyset$, $\subseteq$ and $\supseteq$, $\subset$ and $\supset$, in a set equation or expression. We obtain a new equation or expression is said to be *dual* of the original on *(primal).*

"If T is any theorem expressed in terms of $\cap$, $\cup$ and $-$ deducible from the given basic laws, then the dual of T is also a theorem".

Note that, the theorem T is proved in *m* steps, then dual of T also proved in *m* step.

Example The dual of $A \cap \overline{A} = \emptyset$ is given by $A \cup \overline{A} = E$.

Remark: Dual (Dual T) = T.

Identities on sets

$A \cup A = A$ Idempotent laws

$A \cap A = A$

$A \cup B = B \cup A$ Commutative laws

$A \cap B = B \cap A$

$(A \cup B) \cup C = A \cup (B \cup C)$ Associative laws

$(A \cap B) \cap C = A \cap (B \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (A \cap B) = A$ Absorption laws

$A \cap (A \cup B) = A$

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$ De Morgan's laws

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

$A \cup \emptyset = A$ $\quad$ $A \cap \emptyset = \emptyset$

$A \cup E = E$ $\quad$ $A \cap E = A$

$A \cup \bar{A} = E$ $\quad$ $A \cap \bar{A} = \emptyset$

$\bar{\emptyset} = E$ $\quad$ $\bar{E} = \emptyset$ $\quad$ $\bar{\bar{A}} = A$

PROBLEMS

1.$S = \{a, b, p, q\}$,$Q = \{a, p, t\}$.Find $S \cup Q$and$S \cap Q$?

Solution:

$S \cup Q = \{a, b, p, q, t\}$

$S \cap Q = \{a, p\}$

2. If$A = \{a, b, c\}$. Find$\rho(A)$?

Solution:
$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

$|A| = 3$

$|\rho(A)| = 2^3 = 8$

3. Write all proper subsets of $A = \{a, b, c\}$.

Solution:

The proper subsets are

$\rho(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

4. Show that$A \subseteq B \Leftrightarrow A \cap B = A$.

Solution:

If$A \subseteq B$, then$\forall\ x \in A \Longrightarrow x \in B$ Now, let

$x \in A \Leftrightarrow x \in A$ and$x \in B$

$\Leftrightarrow\ x \in A \cap B$

$A = A \cap B$

If $A \cap B = A$, then

Let $x \in A, x \in A \cap B \Rightarrow x \in B$

Therefore $A \subseteq B$.

5. If $A = \{2,5,6,7\}, B = \{1,2,3,4\},\ C = \{1,3,5,7\}$. Find $A - B, A - C, C - B$ and $B - C$.

Solution:

$A - B = \{5,6,7\}$

$A - C = \{2,6\}$

$C - B = \{5,7\}$

$B - C = \{2,4\}$

6. If $A = \{2,3,4\}, B = \{1,2\},\ C = \{4,5,6\}$. Find $A + B, B + C, A + C, A + B + C$ $(A + B) + (B + C)$.

Solution:

$A + B = \{1,3,4\}$
$B + C = \{1,2,4,5,6\}$

$A + C = \{2,3,5,6\}$

$A + B + C = \{1,3,5,6\}$

$(A + B) + (B + C) = \{2,3,5,6\}$

Note that

$A + (B + B) + C = A + (\emptyset) + C = A + C = \{2,3,5,6\}$

7. Show that $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Solution:

Let

$x \in A \implies x \in A\ (or)\ x \in B$

$\implies x \in A \cup B$

$\implies A \subseteq A \cup B$

Now let $x \in A \cap B \implies x \in A\ and\ x \in B$

$\implies x \in A$

$A \cap B \subseteq A$

Hence $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Remark: $B \subseteq A \cup B, A \cap B \subseteq B$ and $A \cap B \subseteq A \cup B$.

8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.

Solution:

$x \in A-(A \cap B) \Leftrightarrow x \in A \; and \; x \notin (A \cap B)$

$\Leftrightarrow x \in A \; and \; \{x \notin A \; or \; x \notin B\}$

$\Leftrightarrow \{x \in A \; and \; x \notin A\}(or)\, \{x \in A \; and \; x \notin B\}$

$\Leftrightarrow \; \emptyset \; (or)\{x \in A \; and \; x \notin B\}$

$\Leftrightarrow x \in A \; and \; x \notin B$

$A-(A \cap B) \subseteq A-B$ and $A-B \subseteq A-(A \cap B)$

Therefore $A-(A \cap B)=A-B$.

9. Show that $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$

Solution:

$x \in A \cup (B \cap C) \Leftrightarrow x \in A \; or \; x \in B \cap C$

$\Leftrightarrow x \in A \; or \; \{x \in B \; and \; x \in C\}$

$\Leftrightarrow \{x \in A \; or \; x \in B\} \; and \; \{x \in A \; or \; x \in C\}$

$\Leftrightarrow \{x \in A \cup B\} \; and \; \{x \in A \cup C\}$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$.

10. Show that $\overline{(A \cup B)}=\bar{A} \cap \bar{B}$.

Solution:

Let $x \in \overline{(A \cup B)} \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \; and \; x \notin B$

$\Leftrightarrow x \in \bar{A} \; and \; x \in \bar{B}$

$\Leftrightarrow x \in \bar{A} \cap \bar{B}$

Therefore $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

11. Show that $(A - B) - C = A - (B \cup C)$.

Solution:

$(A - B) - C = (A - B) \cap \bar{C}$ $\qquad (P - Q = P \cap \bar{Q})$

$= (A \cap \bar{B}) \cap \bar{C}$

$= A \cap (B \cap \bar{C})$ (Associative)

$= A \cap (\overline{B \cup C})$ (De Morgan's law)

12. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

Solution:

Let $(A \cap B) - (A \cap C)$

$= (A \cap B) \cap (\overline{A \cap C})$

$= (A \cap B) \cap (\bar{A} \cup \bar{C})$

$= ((A \cap \bar{A}) \cap B) \cup (A \cap B \cap \bar{C})$

$= (A \cap B \cap \bar{A}) \cup (A \cap B \cap \bar{C})$

$= (\emptyset \cap B) \cup (A \cap B \cap \bar{C})$

$= \emptyset \cup (A \cap B \cap \bar{C})$

$= A \cap (B \cap \bar{C})$

$= A \cap (B - C)$

ASSIGNMENT PROBLEMS

Part –A

1. Define a set
2. Define subset of a set. What is meant by proper subset?

(i) Find all subsets of $A = \{1,2,3\}$

(ii) Find all proper subsets of A.

3. Define power set.
4. Define disjoint sets with example?
5. If $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8,10\}$. Find $A \cup B, A \cap B, a - B, B - A,$ and $A + B,$ $B + A?$
6. Which of the following sets are empty? 7. $\{x \mid x \in R, x + 6 = 6\}$

8. $\{x \mid x \textit{ is a real integer such that } x^2 + 1 = 0\}$

9. $\{x \mid x \textit{ is a real integer and } x^2 - 4 = 0\}$

10. State duality principle in set theory.

11. Define cardinality of a set.

12. If a set A has *n* elements, then the number of elements of power set of A is……..

13. Find the intersection of the following sets

(i) $\{x \mid x^2 - 1 = 0\}, \{x \mid x^2 + 2x + 1 = 0\}$ 14. Write the dual of $A \cap \bar{A} = \emptyset$.

15. Let A, B and C sets, such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, can we conclude that B = C.

16. State De Morgan's Laws.

17. Whether the union of sets is commutative or not?

PART–B

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. Verify the De Morgan's laws

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$,(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

3. Show that the intersection of sets is associative.
4. Show that $A - (B - C) = (A - B) \cup (A \cap C)$.
5. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. Let $A_i = \{1,2,3,\ldots\}$ for $i = 1,2,3,\ldots$ find (a) $\bigcup_{i=1}^{n} A_i$ (b) $\bigcap_{i=1}^{n} A_i$
7. Prove that $A - (A - B) \subset B$.
8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
9. Prove that $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$.
10. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that B=C. (cancelation law)
11. Show that $A - (B \cup C) = (A - B) \cap (A - C)$.
12. Show that $A + A = \emptyset$, where + is the symmetric difference of sets.
13. Show that $(R \subseteq S)$ and $(S \subset Q)$ imply $R \subset Q$.
14. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$.

CARTESIAN PRODUCT OFSETS

The *Cartesian product* of the sets A and B, is written as $A \times B$, is the set of all ordered pairs in which the first elements are in A and the second elements are in B.

$i.e.\ A \times B = \{\langle x, y\rangle | x \in A\ and\ x \in B\}$

For example

Let $A = \{1,2\}, B = \{a, b, c\},\ c = \{\alpha, \beta\}$ Now

$A \times B = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 1,c\rangle \langle 2,a\rangle, \langle 2,b\rangle, \langle 3,c\rangle\}$

$A \times C = \{\langle 1,\alpha\rangle, \langle 1,\beta\rangle, \langle 2,\alpha\rangle, \langle 2,\beta\rangle\}$

$A \times B = \{\langle \alpha,a\rangle, \langle \alpha,b\rangle, \langle \alpha,c\rangle \langle \beta,a\rangle, \langle \beta,b\rangle, \langle \beta,c\rangle\}$

It is clear from the definition

$A \times B \neq B \times A$ and $\langle\langle a,b\rangle, c\rangle \in (A \times B) \times C,$ is an ordered triple

$\langle a,b\rangle \in A \times B$ and $c \in C$ then and .

Now, $A \times (B \times C) = \{\langle a, \langle b,c\rangle\rangle \,|\, a \in A \text{ and } \langle b,c\rangle \in \langle B,C\rangle\}$

Note that $\langle a, \langle b,c\rangle\rangle$ is not an ordered triple.

This fact shows that $(A \times B) \times C \neq A \times (B \times C)$

i.e. Cartesian product is not associative.

Now

$A \times A = A^2 = \{\langle x,y\rangle, \forall x,y \in A\}$ and $A^n = A^{n-1} \times A.$

Note that if A has *n* elements and B has *m* elements $A \times B$ has *nm* elements.

PROBLEMS

1. If $A = \{1,2,3\},\ B = \{a,b\}$. Find $A \times B, B \times A$ and $A \times A$ and $A^2 \times B$

Solution:

$A \times B = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle, \langle 3,a\rangle, \langle 3,b\rangle\}$

$B \times A = \{\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle b,3\rangle\}$

$A^2 = A \times A = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

$A^2 \times B = \{\langle 1,1,a\rangle, \langle 1,1,b\rangle, \langle 1,2,a\rangle, \langle 1,2,b\rangle, \langle 1,3,a\rangle, \langle 1,3,b\rangle, \langle 2,1,a\rangle, \langle 2,1,b\rangle,$

$\langle 2,2,a\rangle, \langle 2,2,b\rangle, \langle 2,3,a\rangle, \langle 2,3,b\rangle, \langle 3,1,a\rangle, \langle 3,1,b\rangle, \langle 3,2,a\rangle, \langle 3,2,b\rangle, \langle 3,3,a\rangle, \langle 3,3,b\rangle\}$

2.Show that$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution: For any$\langle x, y\rangle$,

$\langle x, y\rangle \times (B \cap C) \Leftrightarrow x \in A \; and \; y \in B \cap C$

$\Leftrightarrow x \in A \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{x \in A \; and \; y \in B\} \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times B\} \; and \; \{\langle x, y\rangle \in A \times C\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3.Show that$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Solution: For any$\langle x, y\rangle$,

$\langle x, y\rangle \times (A \cap B) \times (C \cap D) \Leftrightarrow x \in (A \cap B) \; and \; y \in (C \cap D)$

$\Leftrightarrow \; \{x \in A \; and \; x \in B\} \; and \; \{y \in C \; and \; y \in D\}$

$\Leftrightarrow \{x \in A \; and \; y \in C\} \; and \; \{x \in B \; and \; y \in D\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times C\} \; and \; \{\langle x, y\rangle \in B \times D\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times C) \cap (B \times D)\}$.

ASSIGNMENT PROBLEMS

Part A

1. Define Cartesian product of sets? Give an example?
2. If$A = \{0,1\}$,find A^2.
3. If$A = \{1,2,3\}$and $B = \{a, b\}$, find $A \times B, B \times, A \;\; A^2$
4. True or False
   I. If$A = \{1,3,5,7,9\}$,the $\{\forall x \in A, x + 2 \; is \; a \; prime \; number\}$
   II. If$A = \{1,2,3,4,5\}$,the $\{\exists \; x \in A, x + 3 = 10\}$
5. If$A \times B = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$

Part B

6. If A, B and C are sets, prove that$A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. Prove that $(A \times C) - (B \times C) = (A - B) \times C$.
8. If$A = \{a, b\}$ and $B = \{1,2\}$,and$C = \{2,3\}$, find I.
   $A \times (B \cup C)$
   II. $(A \times B) \cup (A \times C)$
   III. $A \times (B \cap C)$
   IV. $(A \times B) \cap (A \times C)$
9. Show that the Cartesian product is not commutative? It is commutative only for equality of sets?

RELATIONS

Binary relation

Any set of ordered pairs defines a *binary relation.*

If x and y are binary related, under the relation R, then we write $\langle x, y\rangle \in R$ or xRy. If not the case we write $\langle x, y\rangle \notin R$.

1. Example $F = \{\langle x, y\rangle \mid x\ is\ the\ father\ of\ y\}$

$L = \{\langle x, y\rangle \mid x\ and\ y\ are\ real\ number\ and\ x < y\}$

Then F, L are binary relations.

2.Example Let A and B be any two sets, then any nonempty subset R of $A \times B$ is called a *binary relation.*

Now

$A = \{1,2,3\}$

$B = \{a, b\}$ then

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

Let

$R_1 = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$R_2 = \{\langle 1, b\rangle, \langle 3, a\rangle\}$

$R_3 = \{\langle 2, a\rangle\}$

Then R_1, R_2 and R_3 are binary relations A to B.

Let S be any binary relation. The *domain* of S is the set of all elements x such that for some $y, \langle x, y\rangle \in S$.

$D(S) = \{x \mid \langle x, y\rangle \in S, for\ some\ y\}$

Similarly, the *range* of S is the set of all elements y such that, for some

$x, \langle x, y\rangle \in S$

$i.e. \quad R(S) = \{y \mid \langle x,y\rangle \in S, for\ some\ x\}$

Let

$S = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle, \langle 3,a\rangle\}$

$D(S) = \{1,2,3\}$

$R(S) = \{a,b\}$

If $S \subseteq X \times Y$, then clearly $D(S) \subseteq X$ and $R(S) \subseteq Y$.

In case of $X = Y$, then the relation defined on $X \times X$ is called *a universal relation* in X.

If $X = \emptyset$, then a relation on $X \times X$ is called *void relation* in X.

Since relations are sets, then we can have their union and intersection and so on.

$R \cup S = \{\langle x,y\rangle \mid xRy\ or\ xSy\}$

$R \cap S = \{\langle x,y\rangle \mid xRy\ and\ xSy\}$

$R - S = \{\langle x,y\rangle \mid xRy\ and\ \langle x,y\rangle \notin S\}$

$R + S = \{\langle x,y\rangle \mid \langle x,y\rangle\ is\ either\ in\ R\ or\ in\ S\ but\ not\ in\ both\}$

Properties of Binary relations

1. Reflexive

Let R be a binary relation defined on X.

Then R is *reflexive* if, for every $x \in X, \langle x,y\rangle \in R$.

Example:

Let
$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 2,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ are defined on X.

Then R is reflexive, but S is not reflexive. Since $\langle 2,2\rangle \notin Sand2 \in X.$

2. Symmetric

A relation R from X to Y is *symmetric* if every $x \in X$and$y \in Y$,whenever $\langle x,y\rangle \in R$,then $\langle y,x\rangle \in R$.

That is, if $xRy \Rightarrow yRx$, then R is symmetric

Example:

Let

$X = \{1,2\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$and

$S = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$are defined on X.

Then R is symmetric, but S is not symmetric. Since$\langle 1,2\rangle \in S$but$\langle 1,2\rangle \notin S.$

3. Transitive

A relation R is *transitive* if, whenever$\langle x,y\rangle \in R$and$\langle y,z\rangle \in R$, then$\langle x,z\rangle \in R$.That is, if $xRy \wedge yRz$, then R is transitive.

Example:

Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$and

$S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,3\rangle, \langle 2,1\rangle\}$

Then R is transitive, but S is not transitive. Since $\langle 2,1\rangle \in S$and$\langle 1,2\rangle \in S$but

$\langle 2,2\rangle \notin S.$

4.Irreflexive

A relation R in a set X is *irreflexive* if, for every $x \in X, \langle x, x \rangle \notin R$

Example:

Let

$A = \{1,2,3\}$

$R = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 2,2\rangle, \langle 1,3\rangle\}$

Then R is irreflexive, but S is not reflexive. Since $\langle 3,3\rangle \notin S$ and $\langle 1,1\rangle \in S$.

5. Antisymmetric

A relation R in a set X is *antisymmetric* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $x = y$.

That is, if $xRy \wedge yRx \Rightarrow x = y$, then R is antisymmetric.

Example:

Let

X be the set of all subsets of E.

R be the inclusion relation $(\subseteq)$ defined on X.

$A \subseteq B \wedge B \subseteq A \Rightarrow A = B$

Therefore R is antisymmetric in X.

6. Relation matrix

Let $X = \{x_1, x_2, \dots x_m\}, Y = \{y_1, y_2, \dots y_m\}$ are ordered sets, R be a relation Defined from X to Y, then the *relation matrix* of R, is defined as

$$M_R = (r_{ij})\ i : 1 \to m, j : 1 \to n$$

Example 1:

Let $X = \{1,2,3\}$ $Y = \{a, b\}$

$R = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 3,b\rangle$ be a relation from X to Y. Then $M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$ be a relation on $X = \{1,2,3\}$.

Then $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

7. Composition of Binary Relations

The concept of composition of relation is different from union and intersection of two relations.

Definition:

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite $R \circ S$ is a relation from X to Z defined by

The operation $\circ$ in $R \circ S$ is called *"composition of relations"*.

Example.

Let

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$

$S = \{\langle 2,3\rangle, \langle 4,1\rangle, \langle 4,3\rangle, \langle 2,1\rangle\}$. Then

$R \circ S = \{\langle 1,3\rangle, \langle 1,1\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$

$S \circ R = \{\langle 2,4\rangle, \langle 4,2\rangle, \langle 4,4\rangle, \langle 2,2\rangle\}$

Note that

$R \circ R = R^2$

$R \circ R \circ R = R^2 \circ R = R^3$

$R^{n-1} \circ R = R^n$ etc.,

Definition:

The relation matrix for $R \circ S$ is given by $M_{R \circ S} = M_R \odot M_S$ where $\odot$ is defined as follows.

$M_R \odot M_S = \langle m_{ij} \rangle$ where $m_{ij}(\langle i,j\rangle th\ \ element)$ is1if and only if row *I* of M_R and column *j* of M_S have a1in the same relative position *k*, for some *k*.

Example:

Let

$R = \{\langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 3,4\rangle, \langle 5,1\rangle, \langle 5,5\rangle\}$

$S = \{\langle 1,3\rangle, \langle 2,5\rangle, \langle 3,1\rangle, \langle 4,2\rangle, \langle 4,4\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$Then

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
M_{R\circ S} &= M_R \odot M_S \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
\text{and} & \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$R^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \langle 5,5\rangle\}$

Definition

Let R be a relation from X to Y. The *converse* of R, is written as $\tilde{R}$, is a relation from Y to X such that

$$xRy \Leftrightarrow x\tilde{R}y$$

Example:

If$R = \{\langle 1,a\rangle, \langle 2,b\rangle, \langle 2,a\rangle, \langle b,3\rangle$

$\tilde{R} = \{\langle a,1\rangle, \langle b,2\rangle, \langle a,2\rangle, \langle b,3\rangle$

Also it is clear that 1.

2. $R = S \Leftrightarrow \tilde{R} = \tilde{S}$

3. $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$

$\widetilde{R \cup S} = \tilde{R} \cup \tilde{S}$

Result: The relation matrix $M_{\tilde{R}}$ is the transpose of the relation M_R.

$i.e. M_{\tilde{R}} = transpose\ of\ M_R$

Example:

Let

$R = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle$

$\tilde{R} = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,3\rangle$

We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[M_R]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{\tilde{R}}$$

EQUIVALENCE RELATION

Definition:

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 1:

Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$ Is an equivalence relation on X.

Example 2:

Equality of subsets on a universal set is an equivalence relation.

Example 3:

Let

$X = \{1,2,3, \dots 7\}$

$R = \{\langle x, y\rangle \mid x - y \text{ is divisible by } 3\}$

Now, $\forall\, x \in X, x - x = 0$ is divisible by 3. Therefore,

$\forall\, x \in X, \langle x, x\rangle \in R$ (reflexive)

For any $x, y \in X$

Let $\langle x, x\rangle \in R \Rightarrow x - y$ is divisible by 3 we have $-(x - y) = y - x$ is also divisible by 3.

$\langle y, x\rangle \in R$ (symmetric)

Let $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R$

$\Rightarrow x - y$ is divisible by 3 and $y - z$ is divisible by 3.

$\Rightarrow (x - y) + (y - z)$ is divisible by 3.

$\Rightarrow x - z$ is divisible by 3.

Therefore $\langle x, y \rangle \in R$ (Transitive)

Therefore, R is an equivalence relation on X.

EQUIVALENCE CLASSES

Definition:

Let R be an equivalence relation on a set X. For any $x \in X$, the set $[x]_R \subseteq X$ given by

$$[x]_R = \{y \mid xRy \; for \; y \in X\}$$

is called an R-*equivalence class* generated by $x \in X$.

Therefore, an equivalence class $[x]_R$ of $x \in X$ is the set of all elements which are related to *x* by an equivalence relation R on X.

Example:

Let Z be the set of all integers and R be the relation called *"congruence modulo 4"* defined by

$$R = \{\langle x, y \rangle \mid (x - y) \; is \; divisible \; by \; 4, for \; x \; and \; y \in Z\}$$

(or $x \equiv y (mod\ 4)$)

Now, we determine the equivalence classes generated by R.

$$[0]_R = \{\ldots -8, -4, 0, 4, 8 \ldots\}$$

$$[1]_R = \{\ldots -7, -3, 1, 5, 9 \ldots\}$$

$$[2]_R = \{\ldots -6, -2, 2, 6, 10 \ldots\}$$

$$[3]_R = \{\ldots -5, -1, 3, 7, 11 \ldots\}$$

Note that

$[0]_R = [4]_R, [1]_R = [5]_R, \dots etc.$

Therefore $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R, [3]_R\}$

In a similar manner, we get the equivalence classed generated by the relation *"congruence modulo m"* for any integer *m.*

Therefore, an equivalence relation R on X, will divide the set X into an *Equivalence classes,* and they are called *portion* of X.

PARTIAL ORDERED RELATION

A relation R on a set X is said to be a partial ordered relation, if R satisfies reflexive, antisymmetric, and transitive.

Example:

Let $\rho(A)$ be the power set of a set A.

Define a subset relation $(\subseteq)$ on $\rho\ (A)$, then $\subseteq$ is a partial ordered relation.

Usually, we denote the partial ordered relations as $' \leq '$is said to be *partially ordered set* (or) *poset,* which is denoted by $\langle X, \leq \rangle$. We will study more about posets in the subsequent sections.

1. Closures of a relation

Let R be a relation on the set X.

2. Reflexive closure

We have the relation R is reflexive if and only if the relation.

$R = \{\langle x, y \rangle \mid \forall\ x \in X\}$ is contained in R.

i.e., R is reflexive $\Leftrightarrow I \subset R.$

Definition:

Let R be a relation on X, then the smallest reflexive relation on X, containing R, is called *reflexive closure* of R.

Therefore $R_1 = R \cup I$ is the reflexive closure of R.

3. Symmetric closure

We have, the relation R is symmetric if $\langle x, y \rangle \in R \Leftrightarrow \langle y, x \rangle \in \tilde{R}$

$i.e.\, \tilde{R} = \{\langle y, x \rangle \mid \langle x, y \rangle \in R\}$

Definition:

Let R be a relation X, then smallest symmetric relation on X, containing R, is called the *symmetric closure* of R.

Therefore $R \cup \tilde{R}$ is the symmetric of R.

4. Transitive closure

We have, the relation R is transitive, if $\langle x, y \rangle \in R \text{ and } \langle y, z \rangle \in R$ then

$\langle x, z \rangle \in R.$

Definition:

A relation R^+ is said to be the *transitive closure* of the relation R on X if R^+ is the smallest transitive relation on X, containing R,

i.e., R^+ is the transitive closure of R, if

I $R \subseteq R^+$

II R^+ Is transitive on X

III There is no transitive relation R_1 on X, such $R \subset R_1 \subset R^+$

Remarks:

1. The transitive closure of R can be obtained by

$$R^+ = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{i=1}^{\propto} R^i$$

2. We know that $\langle x,z\rangle \in R^2$ if and only if there is an element *y* such that and . $\langle x,y\rangle \in R$ $\langle y,z\rangle \in R$

Therefore, $\langle a,b\rangle \in R^n$ if and only if we can find a sequence $x_1, x_2, \ldots . x_{n-1}$ in X such that $\langle a, x_1\rangle, \langle x_1, x_2\rangle, \ldots \langle x_{n-1}, b\rangle$ are all in R.

The sequence $a, x_1, x_2, \ldots . x_{n-1}, b$ is said to be a *chain* of length *n* from a to b in R. Here $x_1, x_2, \ldots . x_{n-1}$ are called interval vertices of the chain in R. Note that the interval vertices need not be distinct.

PROBLEMS

1.If $P = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$, $Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

Find(i) $P \cup Q, P \cap Q, \tilde{P}, \tilde{P} \cup Q$ (ii) domains of $P, P \cup Q, P \cap Q$and(iii)ranges of . $Q, P \cup Q, P \cap Q$

Solution:

$P \cup Q = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$

$P \cap Q = \{\langle 2,4\rangle\}$

$\tilde{P} = \{\langle 2,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

$\tilde{P} \cup Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 2,1\rangle, \langle 4,3\rangle\}$

Domain of $P = \{1,2,3\}$

Domain of $(P \cup Q) = D(P \cup Q) = \{1,2,3,4\}$

Domain of $(P \cap Q) = D(P \cap Q) = \{2\}$

Range of $Q = R(Q) = \{2,3,4\}$

Range of $(P \cup Q) = R(P \cup Q) = \{2,3,4\}$

Range of $(P \cap Q) = R(P \cap Q) = \{4\}$

It is clear that

$D(P \cup Q) = D(P) \cup D(Q)$ and

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

In general, $D(P) = R(\tilde{P})$ and $R(P) = D(\tilde{P})$.

2. Let $X = \{1,2,3,4\}$ and $R = \{\langle x, y\rangle \mid x, y \in X \text{ and } (x - y) \text{ is an integeral non zero multiple of } 2\}$ $S = \{\langle x, y\rangle \mid x, y \in X \text{ and } (x - y) \text{ is an integeral non zero multiple of } 3\}$. Find $R \cup S$ and $R \cap S$?

Solution:

Given that $R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$ and

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$ $R \cup S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,1\rangle, \langle 4,2\rangle\}$

$R \cap S = \emptyset$

Remarks:

$D(R) = \{1,2,3,4\}$

$R(R) = \{1,2,3,4\}$

$D(S) = \{1,4\}$

$R(S) = \{1,4\}$

3. Let $S = \{\langle x, x^2\rangle \mid x \in N\}$ and $T = \{\langle x, 2x\rangle \mid x \in N\}$, where $= \{0,1,2, \dots.\}$. Find the range of S and T, find $S \cup T$ and $S \cap T$?

Solution

$S = \{\langle x, x^2 \rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \langle 3,9\rangle, \langle 4,16\rangle, \dots\dots\}$

and

$T = \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 4,8\rangle, \dots\dots\ \}$

$R(S) = \{x^2 \mid x \in N\}$

$= \{0,1,4,9,16,25 \dots\dots\ \}$

$R(T) = \{2x \mid x \in N\}$

$= \{0,2,4,6,8,10, \dots\dots\ \}$

$S \cup T = \{\langle x, x^2 \rangle \mid x \in N\} \cup \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = x^2\ (or) 2x\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 3,9\rangle, \dots\dots\ \}$

$S \cap T = \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = 2x\ and\ y = x^2\}$

(Now $y = 2x$ and $y = x^2 \Rightarrow 2x = x^2\ i.e. x = 0\ or\ x = 2$

$x = 0\ y = 0\ and\ x = 2 \Rightarrow y = 4$

$S \cap T = \{\langle 0,0\rangle, \langle 2,4\rangle\}$

4. Given an example which is neither reflexive nor irreflexive?

Solution:

Let $X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}$

Then R is not reflexive, since $\langle 2,2 \rangle \notin R$, for $2 \in X$ and R is not irreflexive, since $1 \in X$, and $\langle 1,1 \rangle \in R$.

5. Test whether the following relations are transitive or not on

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle\}$

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$

$T = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$.

Solution: The relation R and T are transitive.

Since, in R, we have $\langle 1,1\rangle \in R$, then check any other pair starting with $\langle 1,z\rangle \in R$, then we must have $1R1 \wedge 1Rz \Rightarrow 1Rz$ i.e., $\langle 1,z\rangle \in R$, but there is no pair staring with 1. So, pass onto next pair $\langle 2,2\rangle$ then we check any other pair starting with 2, and so on.

In T, we have $\langle 1,1\rangle \in T$, then there are two pairs $\langle 1,3\rangle$ and $\langle 1,2\rangle$ must be the transitive of, the new must have $\langle 1,1\rangle \in T$ and $\langle 1,2\rangle$ in T $\langle 1,3\rangle$ Then pass to the transitive $\langle 1,2\rangle \in T$ pairs are $\langle 2,1\rangle, \langle 2,2\rangle$ and then we must have the pairs $\langle 2,3\rangle$ $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle$ In T.

Then pass to $\langle 1,3\rangle \in T$, find the transitive pairs of $\langle 1,3\rangle$ and soon, for all pairs in T. Hence T is a transitive relation.

The relation S is not transitive, since for $\langle 1,2\rangle \in S$, the transitive pairs are $\langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in S but $\langle 1,3\rangle \notin S$.

6. Let R denotes a relation on the set of pairs of positive $N \times N$ integers such that $\langle x,y\rangle R \langle u,v\rangle$ if and only if $xv = yu$. Show that R is an equivalence relation.

Solution:

Let

$P = \{\langle x,y\rangle \mid x \text{ and } y \text{ are positive integer}\}$

Now R is a relation defined on P as

$\langle x, y\rangle R \langle u, v\rangle \Leftrightarrow xv = yu$ for $\langle x, y\rangle, \langle u, v\rangle \in P$.

Let $\langle x, y\rangle, \langle u, v\rangle$ and $\langle m, n\rangle \in P$.

I. R is reflexive:

We have

$\langle x, y\rangle R \langle x, y\rangle \Leftrightarrow xy = yx$ (RHS) is true.

II. R is symmetric:

Let $\langle x, y\rangle R \langle u, v\rangle \Leftrightarrow xv = yu$

$\Leftrightarrow yu = xv$

$\Leftrightarrow uy = vx$

$\Leftrightarrow \langle u, v\rangle R \langle x, y\rangle$

III. R is transitive:

Let $\langle x, y\rangle R \langle u, v\rangle$ and $\langle u, v\rangle R \langle m, n\rangle$

$\Leftrightarrow (xv = yu)$ and $(un = vm)$

$\Leftrightarrow (xv = yu)$ and $(u = \frac{vm}{n})$

$\Leftrightarrow xv = y(\frac{vm}{n})$

$\Leftrightarrow xn = ym$

$\Leftrightarrow \langle u, v\rangle R \langle m, n\rangle$

Therefore, R is reflexive, symmetric, and transitive. Hence R is an equivalence relation.

7. Let R and S are equivalence relations on X, show that $R \cap S$ also equivalent? Whether $R \cup S$ is also an equivalence relation. If not give an example.

Solution:

Given let R and S are equivalence relations on X.

Let x, y and $z \in X$.

(i) We have $\langle x, x\rangle \in R$ and $\langle x, x\rangle \in S \Rightarrow \langle x, x\rangle \in R \cap S, \ \forall x \in X.$

Therefore $R \cap S$ is reflexive.

(ii) Let $\langle x,y\rangle \in R\cap S \Rightarrow \langle x,y\rangle \in R$ and $\langle x,y\rangle \in S$

and $\langle y,x\rangle \in S \quad \Rightarrow \langle y,x\rangle \in R$

$\Rightarrow \langle y,x\rangle \in R\cap S$

Therefore $R \cap S$ is symmetric.

(iii) Let $\langle x,y\rangle \in R\cap S$ and $\langle y,z\rangle \in R\cap S$

$\Rightarrow (\langle x,y\rangle \in R$ and $\langle x,y\rangle \in S)$ and $(\langle y,z\rangle \in R$ and $\langle y,z\rangle \in S)$

$\Rightarrow (\langle x,y\rangle \in R$ and $\langle y,z\rangle \in S)$ and $(\langle x,y\rangle \in R$ and $\langle y,z\rangle \in S)$ and

$\Rightarrow \langle x,y\rangle \in R \;\; \langle x,z\rangle \in S$

$\Rightarrow \langle x,z\rangle \in R\cap S$

Therefore $R \cap S$ is transitive.

Hence $R \cap S$ is equivalence.

8. Prove that the relation "*congruence modulo m*" over the set of positive integers is an equivalence relation?

Show also that if $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) = (y_1 + y_2)$.

Solution:

Let N be the set of all positive integers we have *"congruence modulo m"* relation on N as $x \equiv y (mod\, m) \Leftrightarrow m|\, x - y$, for $x, y \in N$.

Let $x, y, z \in N$

(i)We have

$$x - x = 0 = 0m$$

Therefore$x \equiv x \ (mod\ m)$for$x \in N$.

"Congruence modulo m" is reflexive. (ii)Let

$$x \equiv y(mod\ m)$$

$$\Rightarrow\ m |\ x - y$$

$\Rightarrow x - y = km,$ for some integer $k \in Z$

$\Rightarrow y - x = (-k)m$,for some integer$-k \in Z$

$$\Rightarrow y \equiv x\ (mod\ m)$$

"congruence modulo m" is symmetric on N.

(iii) Let

$x \equiv y(mod\ m)$and$y \equiv z\ (mod\ m)$

$\Rightarrow x - y = k_1 m$, and $y - x = k_2 m$ for some integer $k_1, k_2 \in Z$

$$\Rightarrow (x - y) + (y - z) = (k_1 + k_2)m$$

$\Rightarrow x - z = (k_1 + k_2)m$ for some integer $k_1 + k_2$

$$\Rightarrow x \equiv z\ (mod\ m)$$

"Congruence modulo m" is transitive on N.

Hence *"congruence modulo m"* is an equivalence relation. Let

$x_1 \equiv y_1 (mod\ m)$ and $x_2 \equiv y_2 (mod\ m)$.

Then$m|\ x_1 - y_1$and$m|\ x_2 - y_2$

i.e.,$x_1 - y_1 = k_1 m$and$x_2 - y_2 = k_2 m$

Now

$$(x_1 - y_1) + (x_2 - y_2) = k_1 m + k_2 m$$

$(x_1 + x_2) - (y_1 + y_2) = (k_1 + k_2)m$

$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$

$(x_1 + x_2) \equiv (y_1 + y_2)(mod\, m)$

9. Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle$ be a relation defined on A. Find the transitive closure of R?

Solution:

The matrix of the relation R is given by

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
\text{and} \\
M_{R^3} &= M_{R^2} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{R^4} &= M_{R^3} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

As $|A| = 4$, we get

$$
\begin{aligned}
M_{R^+} &= M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Hence

$$R^+ = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$$

ASSIGNMENT PROBLEMS

Part -A

1. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$ and $S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 1,3\rangle, \langle 3,3\rangle\}$ be any relations on $X = \{1,2,3\}$. Find $R \cup S, R \cap S, \widetilde{R}, R(R), R(\tilde{S}), D(R \cup S), R(R \cap S)$.
2. Give an example for reflexive, symmetric, transitive and irreflexive relations.
3. Give an example of a relation which is neither reflexive nor irreflexive.
4. Give an example of a relation which is neither symmetric nor antisymmetric?
5. Find the graph of the relation

   $R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

6. Find the relation matrix of

   $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$
7. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$ and $= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ Find $R \circ S$ $S \circ R$ $R \circ R$ $S \circ S$ $R \circ R \circ S$ and $S \circ S \circ S$?
8. Define equivalence relation and equivalence classes?
9. Define Poset?
10. Define reflexive closure?
11. Define transitive closure of the relation R?
12. Let $R = \{\langle 1,2\rangle, \langle 3,5\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$ be a relation $A = \{1,2,3,4,5,6\}$. Identify the root of the tree of R.
13. Determine whether the relation R is a partial ordered on the set Z, where Z is set of positive integers, and aRb if and only if a=2b.
14. The following relations are on $\{1,3,5\}$. Let R be a relation, xRy if and only if $y = x + 2$,

and let S be a relation, xSy if and only if $x \leq y$. Find $R \circ S$ and $S \circ R$?

15. True or False: The relation $<$ on Z^+ is not a partial order since it is not reflexive.

Part B

1. Show that the intersection of equivalence relations is an equivalence relation.
2. Determine whether the relations represented by the following zero-one matrices are equivalence relations.

$a)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b)\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

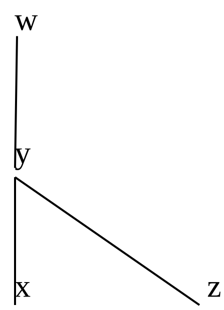
3. If R and S are symmetric, show that $R \cup S$ and $R \cup S$ are symmetric.
4. Let L be set of all straight lines in the Euclidean plane and R be the relation in L defined by $xRy \Leftrightarrow x$ is perpendicular to y. Is R is Reflexive? Symmetric? Antisymmetric? Transitive?
5. Consider the subsets $A = \{1,7,8\}$, $B = \{1,6,9,10\}$ and $C = \{1,9,10\}$ where $E = \{1,2,3 \dots \dots 10\}$ is a universal set. List the non-empty min sets generated by A, B and C. Do they form a partition on E?
6. Let $X = \{1,2,3, \dots .\,.20\}$ and $R = \{\langle x, y\rangle \mid x - y \text{ is divisible by } 5\}$ be a relation on X. Show that R is an equivalent relation and find the partition of X induced by R.
7. If R is an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes constitute a partition on A.
8. Given the relation matrix M_R and M_S. Explain how to find $M_{R \circ S}$, $M_{S \circ R}$ and M_{R^2}?
9. Let A be a set of books. Let R be a relation on A such that $\langle a, b\rangle \in R$ if 'book a' with

cost more and contains fever pages then ‘book b’. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?

10. Let R be a binary relation on the set of all positive integers such that $R = \{\langle a, b\rangle \mid a = b^2\}$. Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation?

HASSE DIAGRAM

A partial ordering $\leq$on a finite set P can be represented in a plane by means of a diagram called *Hasse diagram* or a *partially ordered set set diagram* of $\langle P, \leq\rangle$. If$x \ll y$, then we place y above x, and draw a line (edge) between them. The upward direction indicates success or and downward direction indicates the predecessor. And the incomparable elements are in the same horizontal line.



y is immediate successor of x(or)x is immediate predecessor of y. $z$is

immediate predecessor of y, and x and y are incomparable.

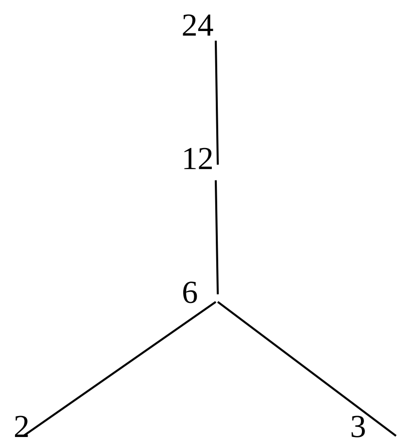
x is predecessor of w but not immediate predecessor.

PROBLEMS

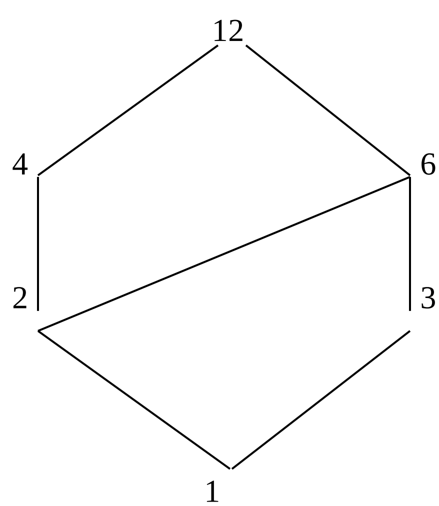
1.Let

$P_1 = \{2,3,6,12,24\}$

$P_2 = \{1,2,3,4,6,12\}$ and$\leq$ be a relation such that $x \leq y$ if and only if $x|y$.



24
12
6
2
3

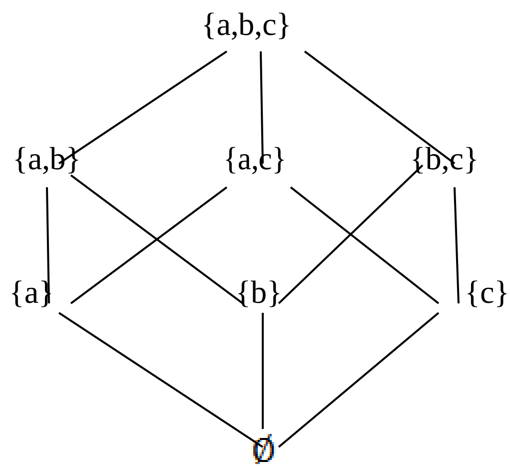


12
4
6
2
3
1

2.Let

$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\},\}$ be the power set of $\{a, b, c\}$.

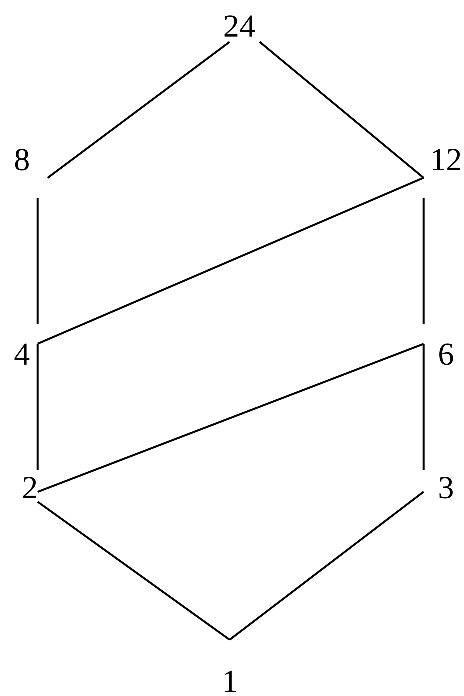
Consider the inclusion $(\subseteq)$ relation as the partial ordering on $\rho(A)$,then the Hasse diagram of $\langle \rho(A), \subseteq \rangle$ is



3.Let us consider the set of all divisor of 24, then it is a poset which is denoted by

D_{24}

That is $D_{24} = \{1,2,3,4,6,8,12,24\}$ and let the divisor relation be partial ordering.

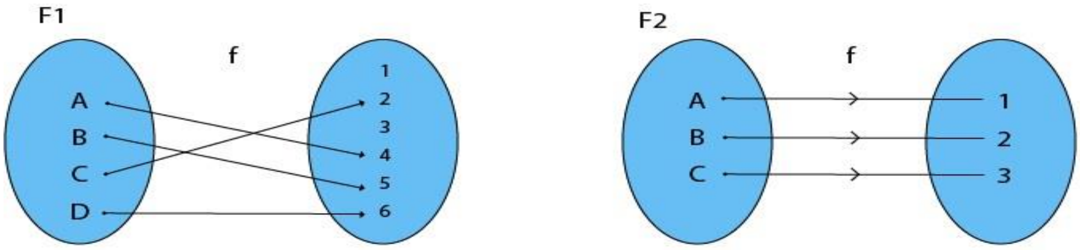

24
8
12
4
6
2
3
1

FUNCTIONS

A function in set theory world is simply a mapping of some (or all) elements from Set A to some (or all) elements in Set B. In the example above, the collection of all the possible elements in A is known as the **domain**; while the elements in A that act as inputs are specially named **arguments**. On the right, the collection of all possible outputs (also known as “range” in other branches), is referred to as the **codomain**; while the collection of actual output elements in B mapped from A is known as the **image.**

Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

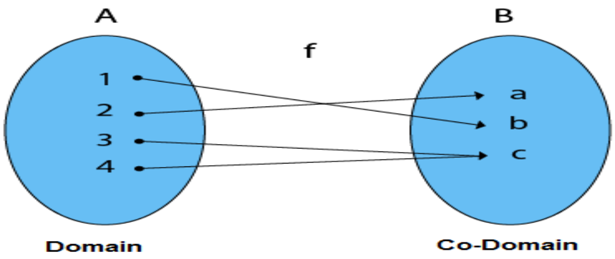


F1 and F2 show one to one Function

2. Surjective (Onto)Functions: A function in which every element of Co-Domain Set has one pre-image.

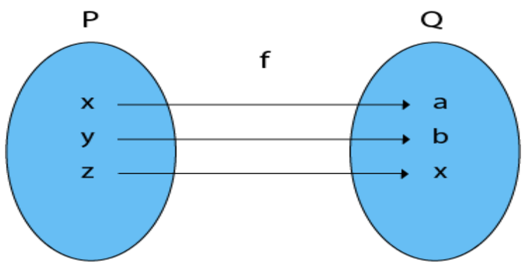
Example: Consider, A = {1,2,3,4}, B= {a, b, c} and f= {(1,b),(2,a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one-to - one) and surjective(onto) is called bijective (One-to-One Onto) Function.



Example:

1. Consider P= {x, y, z}
2. Q = {a, b, c}
3. and f: P→Q such that
4. f = {(x, a), (y, b), (z, c)}

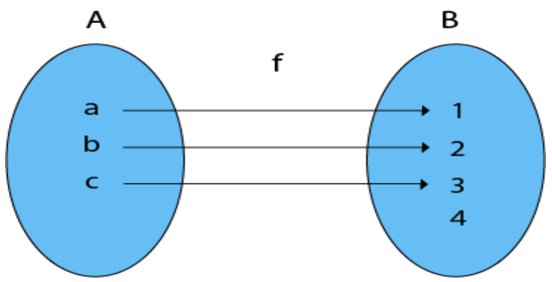
The f is a one-to-one function and also it is onto. So, it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

Example:

1. Consider, A = {a, b, c}
2. B= {1, 2,3,4} and f: A →B such that 3. f={(a,1), (b,2), (c,3)}
4. In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1,2,3,4}

Therefore, it is an into function

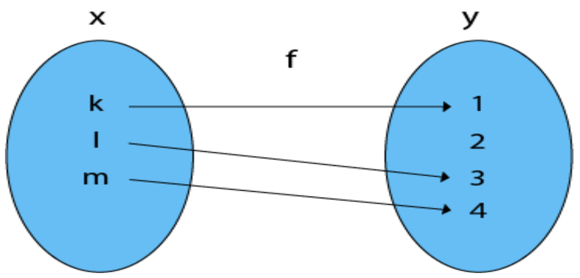


5. One – One Into Functions: Let f: X→Y. The function f is called one-one into function if different elements of X have different unique images of Y.

Example:

1. Consider, X = {k, l, m}
2. Y = {1, 2,3,4}and f: X →Y such that
3. f={(k,1), (l, 3), (m,4)}

The function f is a one-one into function

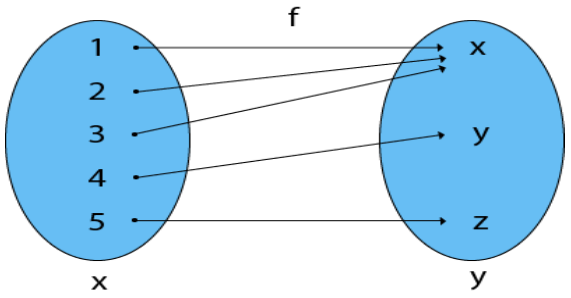


6. Many – One Functions: Let f: X→Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

Example:

1. Consider X = {1, 2, 3, 4,5}
2. Y = {x, y, z}and f: X →Y such that3.
   F= {(1, x), (2, x), (3, x), (4, y), (5, z)}

The function f is a many-one function



Example1: Test whether the function f: R→R, f(x) = |x| +x is one-one onto function
Solution:

(1) Given f(x) = |x| +x
f(3) = |3|+3=6
f(-3) = |-3|+(-3)= 0f(2) =
|2|+2= 4
f(-2) = |-2|+(-2)= 0
f(-3) = f(-2) =0
0 has more than one pre-image. Thus f(x) is not 1-1function

(2) The range of f is the set of non-negative real numbers.
∴ f is not onto function

Example2: Let S = {x, x^2/x ∈ N} and T={(x,2x)/x ∈ N} where N = {1,2….}. Find the range of S and T. Find S∪T and S∩T Solution:

S = {x, x^2/x ∈ N}
S = {(1,1), (2,4), (3,9), (4,16), ………}

$T = \{(x,2x) / x \in N\}$
$S = \{(1,2), (2,4), (3,6), (4,8), \ldots\ldots\ldots\}$
Range of S= $\{1, 4, 9, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$
Range of T = $\{1, 4, 6, 8, \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\}$
$S \cup T = \{(1,1), (2,4), (3,9), (4,16), (1,2), (3,6), (4,8), \ldots\ldots\ldots\}$
$S \cap T = \{(2,4)\}$

Example3: If f: $R \rightarrow R$, g: $R \rightarrow R$ are defined by $f(x) = x^2-2$, $g(x)= x+4$, find (fog) and (gof) and check whether these functions are injective, surjective and bijective

Solution:

$fog(x) = f[g(x)] = f(x+4) = (x+4)^2-2=x^2+8x+14$-----------------(1)

$g \text{ o } f(x) =g[f(x)] =g(x^2-2) = x^2+2$---------------------------------(2)

Given f: $R \rightarrow R$ g: $R \rightarrow R$ $f(x) =x^2-$
2

(1) $f(1) =1^1-2=-1$

$f(-1) = (-1)^2-2=-1$

i.e., $f(x1) = f(x2)$ does not imply $x1= x2$

Hence f is not1-1function

(2) Let f: $R \rightarrow R$

Let $y \in R$. Suppose $x \in R$ such that $f(x) = y$

$x^2-2=y$

$x^2=y+2$

$x=\sqrt{y+2}$

$f(\sqrt{y+2}) = (\sqrt{y+2})^2-2=y+2-2=y$

for any $y \in R$ There exist atleast one element $\sqrt{y+2} \in R$ such that $f(\sqrt{y+2}) = y$

$\therefore$ f is onto function $g(x) = x+4$

(1) $g(x1) =g(x2)$

$x1+4=x2+4$

$x1=x2$

gis1-1function

(2) g: $R \rightarrow R$

Let $y \in R$. Suppose $x \in R$ such that $f(x) =y$ $x= y-$
4 for any $y \in R$

There exist atleast one element $y-4 \in R$ such that $g(y-4)$
$=y$

$\therefore$ g is onto function

As f is not1-1but onto, f is not bijective

As g is1-1and onto, g is bijective

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SCHOOL OF SCIENCE AND HUMANITIES

Department of Mathematics

UNIT – II – COMBINATORICS AND RECURRENCE RELATIONS – SMTA1208

COMBINATORICS AND RECURRENCE RELATIONS

Generating functions - Recurrence relations – Counting: Permutations and Combinations – Principle of Inclusion and Exclusion - The pigeonhole principle – Simple Applications

Strong Induction

There is another form of mathematics induction that is often useful in proofs.In this form we use the basis step as before, but we use a different inductive step. We assume that p(j)is true for j=1…,k and show that p(k+1)must also be true based on this assumption . This is called strong Induction (and sometimes also known as the second principles of mathematical induction).

We summarize the two steps used to show that p(n)is true for all positive integers n.

Basis Step : The proposition P(1) is shown to be true

Inductive Step: It is shown that

$[P(1) \wedge P(2) \wedge \ldots\ldots\ldots \wedge P(k)] \rightarrow P(k+1)$

NOTE:

The two forms of mathematical induction are equivalent that is, each can be shown to be valid proof technique by assuming the other

EXAMPLE 1: Show that if n is an integer greater than 1, then n can be written as the product of primes.

SOLUTION:

Let P(n) be the proportion that n can be written as the product of primes

Basis Step : P(2) is true , since 2 can be written as the product of one prime

Inductive Step: Assume that P(j) is positive for all integer j with j<=k. To complete the Inductive Step, it must be shown that P(k+1) is trueunder the assumption.

There are two cases to consider namely

i) When (k+1) is prime
ii) When (k+1) is composite

Case 1 : If (k+1) is prime, we immediately see that P(k+1) is true.

Case 2: If (k+1) is composite

Then it can be written as the product of two positive integers a and b with 2<=a<b<=k+1. By the Innduction hypothesis, both a and b can be written as the product of primes, namely those primes in the factorization of a and those in the factorization of b .

WELL ORDERING PROPERTY

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non-empty set of non negative integers has a least element.

The well-ordering property can often be used directly in the proof.

Theorem :

For every non negative integer n, 5n = 0

Proof:

Basis Step: 5 – 0 = 0

Inductive Step: Suppose that 5j = 0 for all non negative integers j with o<=j<=k. Write k+1 = i+j where I and j are natural numbers less than k+1. By the induction hypothesis

5(k+1) = 5(i+j) = 5i + 5j = 0+ 0 =0

Example 1:

Among any group of 367 people, there must be atleast 2 with same birthday, because there are only 366 possible birthdays.

Example 2:

In any group of 27 English words, there must be at least two, that begins with the same letter, since there are only 26 letters in English alphabet

Example 3:

Show that among 100 people , at least 9 of them were born in the same month

Solution :

Here, No of Pigeon = m = No of People = 100

No of Holes = n = No of Month = 12

Then by generalized pigeon hole principle

{[100-1]/12}+1 = 9, were born in the same month

Combinations:

Each of the difference groups of sections which can be made by taking some or all of a number of things at a time is called a combinations.

The number of combinations of 'n' things taken 'r' as a time means the number as groups of 'r' things which can be formed from the 'n' things.

It denoted by nCr.

The value of nCr :

Each combination consists /r/ difference things which can be arranged among themselves in r! Ways. Hence the number of arrangement for all the combination is nCr x r! . This is equal to the permulations of 'n' difference things taken 'r' as a time.

$$nCr \times r! = n P r$$

$$nCr = n P r / r! \text{ --------------------} \rightarrow (A)$$

$$= n (n-1) , (n-2)........(n-r+1) / 1,2,3,.......r$$

Cor 1 : $nPr = n! / (n-r)! \text{ --------} \rightarrow (B)$

Substituting (B) in (A) we get

$nCr = n! / (n-r)!r!$

Cor 2: To prove that nCr = nCn-r

Proof :

$$nCr = n! / r!(n-r)! \text{ ---------------------} \rightarrow (1)$$

$$nCn-r = n! / (n-r)! [n-(n-r)]!$$

$$= n! / (n-r) ! r! \text{ -------------------------} \rightarrow (2)$$

From 1 and 2 we get

$$nCr=nCn-r$$

Example :

$$30C_{28} = 30 C_{30-28}$$

$$=30 C_2$$

$$30 \times 29 / 1x2$$

Example

In how many can 5 persons be selected from amongs 10 persons ?

Sol :

The selection can be done in $10C_5$ ways.

=10x9x8x7x6 / 1x2x3x4x5

= 9 x 28 ways.

Example

How many ways are there to from a commitiee , if the consists of 3 educanalis and 4 socialist if there are 9 educanalists and 11 socialists.

Sol : The 3 educanalist can be choosen from a educanalist in 9C3 ways. The socialist can be choosen from 11 socialist in 11C4 ways.

$\therefore$ By products rule , the number of ways to select the commitiee is

$=9C_3.11C_4$

= 9! / 3! 6! . 11! / 4! 7!

= 84 x 330

27720 ways.

Example

1. A team of 11 players is so be chosen from 15 members. In how ways can this be done if

i. One particular player is always included.

ii. Two such player have always to be included.

Sol : Let one player be fixed the remaining players are 14 . Out of these 14 players we have to select 10 players in $14C_{10}$ ways.

$14C_4$ ways. [.`. $nCr = nC_{n-r}$]

➔ 14x13x12x11 / 1x2x3x4

➔1001 ways.

2. Let 2 players be fixed. The remaining players are 13. Out of these players we have to select a players in $13C_9$ ways.

$13C_4$ ways [.`. $nC_r = nC_{n-r}$]

➔13x12x11x10 / 1x2x3x4 ways

➔715 ways.

Example

Find the value of 'r' if $20C_r = 20_{Cr-2}$

Sol : Given $20\ C_r = 20C_{20-(r-2)}$ ➔r=20-(r+2) ---------------→(1)

r = 9

Example

From a commitiee consisting of 6 men and 7 women in how many ways can be select a committee of

(1) 3men and 4 women.
(2) 4 members which has atleast one women.
(3) 4 persons of both sexes.
(4) 4 person in which Mr. And Mrs kannan is not included.

Sol :

(a) 3 men can be selected from 6 men is $6C_3$ ways. 4 women can be selected from 7 women in $7C_4$ ways.

.`. By product rule the committee of 3 men and 4 women can be selected in

$$6C_3 \times 7C_4 \text{ ways} = \frac{6x5x4x}{1x2x3} \; X \; \frac{7x6x5x4}{1x2x3x4}$$

$$=700 \text{ ways.}$$

(b) For the committee of atleast one women we have the following possibilities

1. 1 women and 3 men
2. 2 women and 2 men
3. 3 women and 1 men
4. 4 women and 0 men

There fore the selection can be done in

$= 7C_1 \text{ x } 6C_3 + 7C_2 \text{ x } 6C_2 + 7C_3 \text{ x} 6C_1 + 7C_4 \text{ x } 6C_6$ ways

= 7x20+21x15+35x6+35x1

=140x315x210x35

=700 ways.

(d) For the committee of bath sexes we have the following possibilities .

1. 1 men and 3 women
2. 2 men and 2 women
3. 3 men and 1 women

Which can be done in

$=6C_1x7C_3+6C_2x7C_2+6C_3x7C_1$

$=6x35+15x21+20x7$

$=210+315+140$

=665 ways.

Sol : (1) 4 balls of any colour can be chosen from 11 balls (6+5) in $11C_4$ way

=330 ways.

(2) The 2 white balls can be chosen in $6C_2$ ways. The 2 red balls can be chosen in $5C_2$ ways. Number of ways selecting 4 balls 2 must be red

$=6C_2 + 5C_2$

$$= \frac{6x5}{1 \times 2} + \frac{5x4}{1 \times 2}$$

=15 + 10

=25 ways.

Number of ways selecting 4 balls and all Of same colour is $= 6C_4 + 5C_1$

=15+5

=20ways.

Definition

A Linear homogeneous recurrence relation of degree K with constant coefficients is a recurrence relation of the form

The recurrence relation in the definition is linesr since the right hand side is the sum of multiplies of the previous terms of sequence.

The recurrence relation is homogeneous , since no terms occur that are not multiplies of the aj"s.

The coefficients of the terms of the sequence are all constants ,rather than function that depends on “n”.

The degree is k because an is exrressed in terms of the previous k terms of the sequence

Ex: The recurrence relation

$H_n = 2H_{n-1}+1$

Is not homogenous

Ex: The recurrence relation

$B_n = nB_{n-1}$

Does not have constant coefficient

Ex The relation $T(k)=2[T(k-1)]^2KT(K-3)$

Is a third order recurrence relation &

T(0),T(1),T(2) are the initial conditions.

Ex: The recurrence relation for the function

f : N->Z defined by

f(x)=2x,∀ x € N is given by

f(n+1)=f(n)+2,n>=0 with f(0)=0

f(1)=f(0)+2=0+2=2

f(2)=f(1)+2=2+2=4 and so on.

It is a first order recurrence relation.

RECURRENCE RELATIONS

Definition

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence , namely $a_0, a_1, \ldots, a_{n-1}$,for all integers n with n>=0,where n_0 is a non –ve integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a recurrence relation satisfies a recurrence relation , then the sequence is called a solution of the recurrence relation.

For example ,we consider the famous Fibonacci sequence

0,1,1,2,3,5,8,13,21,.....,

which can be represented by the recurrence relation.

$F_n=F_{n-1}+F_{n-2}, n>=2$

& $F_0=0, F_1=1$. Here $F_0=0$ & $F_1=1$ are called initial conditions.

It is a second order recurrence relation.

Solving Linear Homogenous Recurrence Relations with Constants Coefficients.

Step 1: Write down the characteristics equation of the given recurrence relation .Here ,the degree of character equation is 1 less than the number of terms in recurrence relations.

Step 2: By solving the characteristics equation first out the characteristics roots.

Step 3: Depends upon the nature of roots ,find out the solution a_n as follows:

Case 1: Let the roots be real and distinct say $r_1,r_2,r_3.....,r_n$ then

$A_n= \alpha_1 r_1^n+ \alpha_2 r_2^n+ \alpha_3 r_3^n+........+ \alpha_n r_n^n,$

Where $\alpha_1, \alpha_2, ...,\alpha_n$ are arbitrary constants.

Case 2: Let the roots be real and equal say $r_1=r_2=r_3=r_n$ then

$A_n= \alpha_1 r_1^n+ n\alpha_2 r_2^n+n^2 \alpha_3 r_3^n+........+n^2 \alpha_n r_n^n,$

Where $\alpha_{1,}\ \alpha_{2,}\ ....,\alpha_n$ are arbitrary constants.

Case 3: When the roots are complex conjugate, then

$$a_n=r^n(\alpha_1\cos n\theta+\ \alpha_2\sin n\theta)$$

Case 4: Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation.

$S(k)=2[S(k-1)]^2-kS(k-3)$ cannot be solved.

Example If sequence $a_n=3.2^n$,n>=1, then find the recurrence relation.

Solution:

For n>=1

$a_n=3.2^n$,

now, $a_{n-1}=3.2^{n-1,}$

$=3.2^n / 2$

$a_{n-1}=a^n/2$

$a_n\ = 2(a_n-1)$

$a_n = 2a_n-1$, for $n\geq 1$ with $a_n=3$

Example

Find the recurrence relation for $S(n) = 6(-5),n\geq 0$

Sol :

Given $S(n) = 6(-5)^n$

$S(n-1) = 6(-5)^{n-1}$

$=6(-5)^n / -5$

$S(n-1) = S(n) / -5$

$S_n = -5.5 (n-1)$, $n\geq 0$ with $s(0) =6$

Example Find the relation from $Y_k =A.2^k + B.3^k$

Sol :

Given $Y_k =A.2^k + B.3^k$ --------------$\rightarrow$(1)

$Y_{k+1} =A.2^{k+1} + B.3^{k+1}$

$=A.2^k.2 + B3^k.3$

$Y_{k+1} =2A.2^k + 3B.3^k$ ------------------------$\rightarrow$(2)

$Y_{k+2} =4A.2^k + 9B.3^k$ ------------------------$\rightarrow$(3)

(3) – 5(2) + 6(1)

$\rightarrow y_{k+2} -5y_{k+1} + 6y_k =4A.2^k + 9B.3^k -10A.2^k - 15B.3^k + 6A.2^k + 6B.3^k$

$=0$

$\therefore Y_{k+1}-5y_{k+1} + 6y_k = 0$ in the required recurrence relation.

Example

Solve the recurrence relation defind by $S_o = 100$ and S_k (1.08) S_{k-1} for $k\geq 1$

Sol ;

Given $S_0 = 100$

$S_k = (1.08) S_{k-1}$, $k\geq 1$

$S_1 = (1.08) S_0 = (1.08)100$

$S_2 = (1.08) S_1 = (1.08)(1.08)100$

$=(1.08)^2\ 100$

$S_3 = (1.08)\ S_2 = (1.08)(1.08)^2 100$

$= =(1.08)^3\ 100$

$S_k =(1.08)S_{k-1} = (1.08)^k 100$

Example Find an explicit formula for the Fibonacci sequence .

Sol ;

Fibonacci sequence 0,1,2,3,4........ satisify the recurrence relation

$fn= f_{n-1} + f_{n-2}$

$fn- f_{n-1} - f_{n-2} = 0$

& also satisfies the initial condition $f_0=0,f_1=1$

Now , the characteristic equation is

$r_2-r-1 =0$

Solving we get $r=1\underline{+}\ 1+4 / 2$

$= 1\underline{+}\ 5 / 2$

$fn = \alpha_1\ (1+ 5 / 2)^n + \alpha_2\ (1- 5 / 2)^n$ ----→(A)

given f_0 =0 put n=0 in (A) we get

$f0 = \alpha_1\ (1+ 5 / 2)^0 + \alpha_2\ (1- 5 / 2)^0$

(A)➔ $\alpha 1 +\alpha 2 =0$ ------------------------→(1)

given f_1 =1 put n=1 in (A) we get

$f_1 = \alpha_1\ (1+ 5 / 2)^1 + \alpha_2\ (1- 5 / 2)^1$

(A)➔$(1+ 5 / 2)^n + \alpha_2\ (1- 5 / 2)^n\ \alpha_2 = 1$ -------→(2)

To solve(1) and (2)

(1) X(1+ 5 / 2) => (1+ 5 / 2) α_1 + (1+ 5 / 2) α_2 =0 -----→(3)

(1+ 5 / 2) α_1 + (1+ 5 / 2) α_2 = 1------→(2)

(-) (-) (-)

1/2 α_2 + 5/2 α_2 -1/2 α_2 + 5/2 α_2 = -1

2 5 d_2 = -1

α_2 = - 1/5

Put α_2 = -1/5 in eqn (1) we get α_1 1/5

Substituting these values in (A) we get

Solution fn=1/5 $(1+5/ 2)^n$ -1/5 $(1+5/ 2)^n$

Example

Solve the recurrence equation

$a_n = 2a_{n-1} – 2a_{n-2}$, $n \geq 2$ & a_0 =1 & a_1=2

Sol :

The recurrence relation can be written as

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is

r2 – 2r -2 =0

Roots are r= $2 \pm 2i / 2$

$=1 \pm i$

LINEAR NON HOMOGENEOUS RECRRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation of the form

$a_n = c_1\ a_{n-1}$+$c_2\ a_{n-2}$+………..$c_k\ a_{n-k}$+F(n)……………. (A)

Where c_1, c_2 ,…. c_k are real numbers and F(n) is a function not identically zero depending only on n,is called a non-homogeneous recurrence relation with constant coefficient.

Here ,the recurrence relation

$a_n = c_1\ a_{n-1}$+$c_2\ a_{n-2}$+………..$c_k\ a_{n-k}$+F(n)……………. (B)

Is called Associated homogeneous recurrence relation.

NOTE:

(B) is obtained from (A) by omitting F(n) for example ,the recurrence relation

$a_n = 3\ a_{n-1}$+2_n is an example of non-homogeneous recurrence relation .Its associated

Homogeneous linear equation is

$a_n = 3\ a_{n-1}$ [By omitting F(n) = 2n]

PROCEDURE TO SOLVE NON-HOMOGENEOUS RECURRENCE RELATIONS:

The solution of non-homogeneous recurrence relations is the sum of two solutions.

1.solution of Associated homogeneous recurrence relation (By considering RHS=0).

2.Particular solution depending on the RHS of the given recurrence relation

STEP1:

a) if the RHS of the recurrence relation is

$a_0 + a_1$ n …. a_r n^r, then substitute

c_0 +$c_1\ n + c_2$ n^2+………..c_r $(n-1)^r$ in place of $a_n - 1$ ………..and so on ,in the LHS of the given recurrence relation

(b) if the RHS is a ^{n} then we have

Case1:if the base a of the RHS is the characteristric root,then the solution is of the can^n .therefore substitute ca^n in place of a_n ,ca^{n-1} in place of c(n-1) a_{n-1} etc..

Case2: if the base a of RHS is not a root , then solution is of the form ca^n therefore substitute ca^n in place of a_n , ca^{n-1} in place of a_{n-1} etc..

STEP2:

At the end of step-1, we get a polynomial in 'n' with coefficient c_0,c_1……on LHS

Now, equating the LHS and compare the coefficients find the constants c_0,c_1,….

Example

Solve $a_n = 3\ a_{n-1}$ +2n with a_1=3

Solution:

Give the non-homogeneous recurrence relation is

$a_n - 3\ a_{n-1}$ -2n=0

It's associated homogeneous equation is

$a_n - 3\ a_{n-1}$ =0 [omitting f(n) =2n]

It's characteristic equation is

r-3=0 => r=3

now, the solution of associated homogeneous equation is

$a_n\ (n) = \propto ,3^n$

To find particular solution

Since F(n) =2n is a polynomial of degree one,then the solution is of the from

$a_n = c_n$ +d (say) where c and d are constant

Now, the equation

$a_n = 3\ a_{n-1}$ +2n becomes

c_n +d =3(c(n-1)+d)+2n

[replace a_n by c_n +d a_{n-1} by c(n-1)+d]

- ⇨ c_n +d =3cn-3c+3d+2n
- ⇨ 2cn+2n-3c+2d=0
- ⇨ (2+2c)n+(2d-3c)=0
- ⇨ 2+2c=0 and 2d-3c=0
- ⇨ Saving we get c=-1 and d=-3/2 therefore cn+d is a solution if c=-1 and d=-3/2

a_n (p)=-n-3/2

Is a particular solution.

General solution

a_n=a_n(n)+ a_n(p)

$a_n =\propto 3^n$ -n-3/2 ………………..(A)

Given $a_1 = 3$ put n=1 in (A) we get

$a_1 =\propto 1(3)^1$ -1-3/2

$3=3\propto_1-5/2$

$3 \propto_1=11/2$

$\propto_1=11/6$

Substituting $\propto_1=11/6$ in (A) we get

General solution

$a_n=-n-3/2+(11/6)3^n$

Example:

Solve s(k)-5s(k-1)+6s(k-2)=2

With s(0)=1 ,s(1)=-1

Solution:

Given non-homogeneous equation can be written as

$a_n = 5a_{n-1} + 6a_{n-2} - 2 = 0$

The characteristic equation is

$r^2 - 5r + 6 = 0$

roots are r=2,3

the general solution is

$3_n(n) = \propto_1 (2)^n + \propto_2 (3)^n$

To find particular solution

As RHS of the recurrence relation is constant ,the solution is of the form C , where C is a constant

Therefore the equation

$a_n - 5a_{n-1} - 6a_{n-2} - 2 = 2$

c-5c+6c=2

2c=2

c=2

the particular solution is

$s_n(p) = 1$

the general solution is

$s_n = s_n(n) + s_n(p)$

$s_n = \propto_1 (2)^n + \propto_2 (3)^n + 1$..........(A)

$s_n = \propto_1 (2)^n + \propto_2 (3)^n + 1$……….(A)

Given s_0=1 put n=0 in (A) we get

$s_0 = \propto_1 (2)^0 + \propto_2 (3)^0 + 1$

$s_0 = \propto_1 + \propto_2 + 1$

(A) => $s_0 = 1 = \propto_1 + \propto_2 + 1$

$\propto_1 + \propto_2 = 0$……………(1)

Given a_1=-1 put n=1 in(A)

- ⇨ $S_1 = \propto_1 (2)^1 + \propto_2 (3)^1 + 1$
- ⇨ (A) $-1 = \propto_1 (2) + \propto_2 (3) + 1$
- ⇨ $2 \propto_1 + 3 \propto_2 = -2$……….(1)

$\propto_1 + \propto_2 = 0$

$2 \propto_1 + 3 \propto_2 = -2$……………….(2)

By solving (1) and (2)

$\propto_1 = 2, \propto_2 = -2$

Substituting $\propto_1 = 2, \propto_2 = -2$ in (A) we get

Solution is

⇨ $S_{(n)} = 2.(2)^n - 2.(3)^n + 1$

Example

Solve $a_n - 4\, a_{n-1} + 4\, a_{n-2} = 3n + 2^n$

$a_0 = a_1 = 1$

Solution:

The given recurrence relation is non-homogeneous

Now, its associated homogeneous equation is,

$a_n - 4\,a_{n-1}+4\,a_{n-2}=0$

Its characteristic equation is

$r^2-4r+4=0$

$r=2,2$

solution , $a_n(n) = \propto_1 (2)^n + n \propto_2 (2)^n$

$a_n(n) = (\propto_1 + n \propto_2)2^n$

To find particular solution

The first term in RHS of the given recurrence relation is 3n.therefore ,the solution is of the form c_1+c_2n

Replace a_n by c_1+c_2n , a_{n-1} by $c_1+c_2(n-1)$

And a_{n-2} by $c_1+c_2(n-2)$ we get

$(c_1+c_2n)-4(c_1+c_2(n-1))+4(c_1+c_2(n-2))=3n$

⇨ $c_1-4c_1+4c_1+c_2n-4c_2n+4c_2n+4c_2-8c_2=3n$

⇨ $c_1+c_2n-4c_2=3n$

Equating the corresponding coefficient we have

$c_1-4c_2=0$ and $c_2=3$

$c_1=12$ and $c_2=3$

Given $a_0=1$ using in (2)

(2) => $\propto_1+12=1$

Given $a_1=1$ using in (2)

(2)=> $(\propto_1+\propto_2)2+12+3+1/2\,.2=1$

=> $(2\propto_1+2\propto_2)+16=1$……………..(14)

(3) $\propto_1=-11$

Using in (4) we have $\propto_2=7/2$

Solution $a_n=(-11+7/2n)2^n+12+3n+1/2n^2 2^n$

Example:

HOW MANY INTEGERS BETWEEN 1 to 100 that are

i) not divisible by 7,11,or 13

ii) divisible by 3 but not by 7

Solution:

i) let A,B and C denote respectively the number of integer between 1 to 100 that are divisible by 7,11 and 13 respectively

now,

$|A| = [100/7] = 14$

$|B| = [100/11] = 9$

$|C| = [100/13] = 7$

$|A \wedge B| = [100/7] = 1$

$|A \wedge C| = [100/7*13] = 1$

$|B \wedge C| = [100/11*13] = 0$

$|A \wedge B \wedge C| = [100/7*11*13] = 0$

That are divisible by 7, 11 or 13 is $|A \vee B \vee C|$

By principle of inclusion and exclusion

$|A \vee B \vee C| = |A|+|B|+|C|-|A \wedge B|-|A \wedge C|-|B \wedge C|+|A \wedge B \wedge C|$

$= 14+9+7-(1+1+0)+0$

$= 30-2 = 28$

Now,

The number of integer not divisible by any of 7,11,and 13=total-$|A \vee B \vee C|$

$= 100-28 = 72$

ii) let A and B denote the no. between 1 to 100 that are divisible by 3 and 7 respectively

|A| =[100/3]=33

|B|=[100/7]=14

| A^B |=[100/3*7]=14

The number of integer divisible by 3 but not by 7

=|A|-| A^B |

=33-4=29

Example:

There are 2500 student in a college of these 1700 have taken a course in C, 1000 have taken a course pascal and 550 have taken course in networking .further 750 have taken course in both C and pascal ,400 have taken courses in both C and Networking and 275 have taken courses in both pascal and networking. If 200 of these student have taken course in C pascal and Networking.

i)how many these 2500 students have taken a courses in any of these three courses C ,pascal and networking?

ii)How many of these 2500 students have not taken a courses in any of these three courses C,pascal and networking?

Solution:

Let A,B and C denotes student have taken a course in C,pascal and networking respectively

Given

|A|=1700

|B|=1000

|C|=550

| A^B | =750

| A^C|=40

| B^C =275

| A^B^C |=200

Number of student who have taken any one of these course=| A^B^C |

By principle of inclusion and exclusion

|AvBvC| =|A|+|B|+|C|-|A^B|-|A^C|-|B^C|+|A^B^C|

=(1700+1000+550)-(750+400+275)+200

=3450-1425=2025

The number between 1-100 that are divisible by 7 but not divisible by 2,3,5,7= $\Bigg\}$ =|D|- | A^B^C ^CD|

=142-4=138

Example:

A survey of 500 television watches produced the following information.285 watch hockey games.195 watch football games 115 watch basketball games .70 watch football and hockey games.50 watch hockey and basketball games and 30 watch football and hockey games.how many people watch exactly one of the three games?

Solution:

H=> let television watches who watch hockey

F=> let television watches who watch football

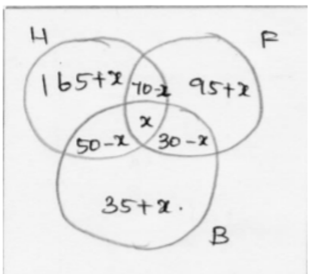
B=> let television watches who watch basketball

Given

n(H)=285,n(F)=195,n(B)=115,n(H^F)=70,n(H^B),n(F^B)=30

let x be the number television watches who watch all three games

now, we have



Given 50 members does not watch any of the three games.

Hence (165+x)+(95+x)+(35+x)+(70+x)+(50+x)+(30+x)+x=500

=445+x=500

X=55

Number of students who watches exactly one game is=165+x+95+x+35+x

=295+3*55

=460

Generating function:

The generating function for the sequence 'S' with terms $a_0,a_1,\ldots\ldots a_n$

Of real numbers is the infinite sum.

$G(x)=G(s,x)= a_0+a_1x+,\ldots\ldots a_nx^n+\ldots\ldots= \sum_{n=0}^{\infty} a^n x^n$

For example,

i) the generating function for the sequence 'S' with the terms 1,1,1,1.....i.s given by,

$G(x)=G(s,x)= \sum_{n=0}^{\infty} x^n=1/1-x$

ii)the generation function for the sequence 'S' with terms 1,2,3,4.....is given by

$G(x)=G(s,x)= \sum_{n=0}^{\infty}(n+1)x^n$

$=1+2x+3x^2+\ldots\ldots..$

$=(1-x)^{-2}=1/(1-x)^2$

2.Solution of recurrence relation using generating function

Procedure:

Step1:rewrite the given recurrence relation as an equation with 0 as RHS

Step2:multiply the equation obtained in step(1) by x^n and summing if form 1 to ∞ (or 0 to ∞) or (2 to ∞).

Step3:put G(x)= $\sum_{n=0}^{\infty} a^n x^n$ and write G(x) as a function of x

Step 4:decompose G(x) into partial fraction

Step5:express G(x) as a sum of familiar series

Step6:Express a_n as the coefficient of x^n in G(x)

The following table represent some sequence and their generating functions

step1	**sequence**	**generating function**
1	1	1/1-z
2	$(-1)^n$	1/1+z
3	a^n	1/1-az
4	$(-a)^n$	1/1+az
5	n+1	$1/1-(z)^2$
6	n	$1/(1-z)^2$
7	n^2	$z(1+z)/(1-z)^3$
8	na^n	$az/(1-az)^2$

Eg:use method of generating function to solve the recurrence relation

$a_n=3a_{n-1}+1$; $n\geq 1$ given that $a_0=1$

solution:

let the generating function of $\{a_n\}$ be

$G(x)=\sum_{n=0}^{\infty} a_n x^n$

$a_n=3a_{n-1}+1$

multiplying by x^n and summing from 1 to ∞,

$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^n)+\sum_{n=1}^{\infty}(x^n)$

$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^{n-1})+\sum_{n=1}^{\infty}(x^n)$

$G(x)-a_0=3xG(x)+x/1-x$

$G(x)(1-3x)=a_0+x/1-x$

$=1+x/1-x$

$G(x)(1-3x)=1=x+x/1-x$

$G(x)=1/(1-x)(1-3x)$

By applying partial fraction

$G(x)=-1/2/1-x+3/2/1-3x$

$G(x)=-1/2(1-x)^{-1}+3/2(1-3x)^{-1}$

$G(x)[1-x-x^2]=a_0-a_1x-a_0x$

$G(x)[1-x-x^2]= a_0-a_0x+a_1x$

$G(x)=1/1-x-x^2 \quad [a_0=1, a_1=1]$

$$=\frac{1}{(1-1+\sqrt{5}\ \ x/2)(1-1-\sqrt{5}\ \ x/2)}$$

$$=\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}$$

Now,

$$1/1-x-x^2 =\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}\ldots\ldots\ldots\ldots(1)$$

$$1=A[1-(\frac{1+\sqrt{5}}{2})x)]+ B[1-(\frac{1-\sqrt{5}}{2})x)]\ldots\ldots(2)$$

Put x=0 in (2)

(2)=> A+B=1

Put $x= 2/1-\sqrt{5}$ in (2)

(2)=> $1=B[1-\frac{1+\sqrt{5}}{1-\sqrt{5}}]$

$$1=B[\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}\ \ ]$$

$$1=B[\frac{-2\sqrt{5}}{1-\sqrt{5}}]$$

$B=\frac{1-\sqrt{5}}{-2\sqrt{5}}$

(3) => $A=\frac{1+\sqrt{5}}{2\sqrt{5}}$

Sub A and B in (1)

$G(x)=\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})[1-(\frac{1+\sqrt{5}}{2})x]^{-1}-\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})[1-(\frac{1-\sqrt{5}}{2})x]^{-1}$

$=\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})[1+(\frac{1+\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^{2}+\ldots\ldots.$

$=\frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})[1+(\frac{1-\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^{2}+\ldots\ldots.$

a_n=coefficient of x^n in G(x)

solving we get

$a_n=\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1}-\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}$

THE PRINCIPLE OF INCLUSION –EXCLUSION

Assume two tasksT_1 and T_2 that can be done at the same time(simultaneously) now to find the number of ways to do one of the two tasks $T_1$and T_2, if we add number ways to do each task then it leads to an over count. since the ways to do both tasks are counted twice. To correctly count the number of ways to do each of the two tasks and then number of ways to do both tasks

i.e $\wedge(T_1 v T_2)=\wedge(T_1)+\wedge(T_2)-\wedge(T_1\wedge T_2)$

this technique is called the principle of Inclusion –exclusion

FORMULA:4

1) $|A_1vA_2vA_3|=|A_1|+|A_2|+|A_3|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_2\wedge A_3|+|A_1\wedge A_2\wedge A_3|$

2) $|A_1vA_2vA_3vA_4|=|A_1|+|A_2|+|A_3|+|A_4|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_1\wedge A_4|-|A_2\wedge A_3|-|A_2\wedge A_4|-|A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3|+|A_1\wedge A_2\wedge A_4|+|A_1\wedge A_3\wedge A_4|+|A_2\wedge A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3\wedge A_4|$

Example

A survey of 500 from a school produced the following information.200 play volleyball,120 play hockey,60 play both volleyball and hockey. How many are not playing either volleyball or hockey?

Solution:

Let A denote the students who volleyball

Let B denote the students who play hockey

It is given that

n=500

|A|=200

|B|=120

|A^B|=60

Bt the principle of inclusion-exclusion, the number of students playing either volleyball or hockey

|AvB|=|A|+|B|-|A^B|

|AvB|=200+120-60=260

The number of students not playing either volleyball or hockey=500-260

=240

Example

In a survey of 100 students it was found that 30 studied mathematics,54 studied statistics,25 studied operation research,1 studied all the three subjects.20 studied mathematics and statistic,3 studied mathematics and operation research And 15 studied statistics and operation research

1.how many students studied none of these subjects?

2.how many students studied only mathematics?

Solution:

1) Let A denote the students who studied mathematics

Let B denote the students who studied statistics

Let C denote the student who studied operation research

Thus $|A|=30$, $|B|=54$, $|C|=25$, $|A \wedge B|=20$, $|A \wedge C|=3$, $|B \wedge C|=15$,and $|A \wedge B \wedge C|=1$

By the principle of inclusion-exclusion students who studied any one of the subject is

$$|A \vee B \vee C|=|A|+|B|+|C|=|A \wedge B|-|A \wedge C|-|B \wedge C|+|A \wedge B \wedge C|$$

$$=30+54+25-20-3-15+1$$

$$=110-38=72$$

Students who studied none of these 3 subjects=100-72=28

2) now ,

The number of students studied only mathematics and statistics=$n(A \wedge B)-n(A \wedge B \wedge C)$

$$=20-1=19$$

The number of students studied only mathematics and operation research=$n(A \wedge C)-n(A \wedge B \wedge C)$

$$=3-1=2$$

Then The number of students studied only mathematics =30-19-2=9

Example

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:

Let A denote the set of positive integers not exceeding 1000 are divisible by 7

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11 .

Then $|A|=[1000/7]=[142.8]=142$

$$|B|=[1000/11]=[90.9]=90$$

$$|A \wedge B|=[1000/7*11]=[12.9]=12$$

The number of positive integers not exceeding 1000 that are divisible either 7 or 11 is |AvB|

By the principle of inclusion –exclusion

|AvB|=|A|+|B|-|A^B|

=142+90-12=220

There are 220 positive integers not exceeding 1000 divisible by either 7 or 11

Example:

A survey among 100 students shows that of the three ice cream flavours vanilla,chocolate,and strawberry ,50 students like vanilla,43 like chocalate ,28 lik strawberry,13 like vanilla, and chocolate,11like chocalets and strawberry,12 like strawberry and vanilla and 5 like all of them.

Find the number of students surveyed who like each of the following flavours

1.chocalate but not strawberry

2.chocalate and strawberry ,but not vanilla

3.vanilla or chocolate, but not strawberry

Solution:

Let A denote the set of students who like vanilla

Let B denote the set of students who like chocalate

Let C denote the set of students who like strawberry

Since 5 students like all flavours

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SCHOOL OF SCIENCE AND HUMANITIES

Department of Mathematics

UNIT – II – NUMERICAL METHODS FOR SOLVING EQUATIONS – SMTA1208

NUMERICAL METHODS FOR SOLVING EQUATIONS

Numerical Solution of algebraic and transcendental equations: Regula Falsi method, Newton Raphson method - Numerical Solution of simultaneous linear algebraic equations: Gauss Jordan method, Gauss Jacobi method, Gauss Seidel method.

INTRODUCTION

Solution of Algebraic and Transcendental Equations

A polynomial equation of the form

$$f(x) = p_n(x) = a_0 x^{n-1} + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

is called an Algebraic equation. For example,

$x^4 - 4x^2 + 5 = 0$, $4x^2 - 5x + 7 = 0$; $2x^3 - 5x^2 + 7x + 5 = 0$ are algebraic equations.

An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example,

$$\tan x - e^x = 0; \quad \sin x - xe^{2x} = 0; \qquad x e^x = \cos x$$

are transcendental equations.

Finding the roots or zeros of an equation of the form $f(x) = 0$ is an important problem in science and engineering. We assume that $f(x)$ is continuous in the required interval. A root of an equation $f(x) = 0$ is the value of x, say $x = \alpha$ for which $f(\alpha) = 0$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x – axis (see Fig. 1)

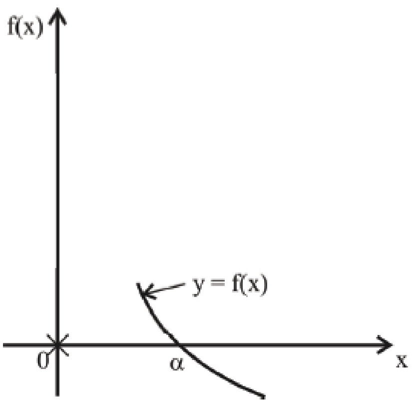

Fig. 1 Geometrical Interpretation of a root of $f(x) = 0$

A number α is a simple root of $f(x) = 0$; if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, we can write $f(x)$ as, $f(x) = (x - \alpha)\, g(x)$, $g(\alpha) \neq 0$.

A number α is a multiple root of multiplicity m of $f(x) = 0$, and $$f^m(\alpha) = 0.$$

Then, $f(x)$ can be writhen as,

$$f(x) = (x - \alpha)^m g(x),\ g(\alpha) \neq 0$$

A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of $f(x)$.

The methods of finding the roots of $f(x) = 0$ are classified as,

1. Direct Methods
2. Numerical Methods.

Direct methods give the exact values of all the roots in a finite number of steps. Numerical methods are based on the idea of successive approximations. In these methods, we start with one or two initial approximations to the root and obtain a sequence of approximations x_0, x_1, … x_k which in the limit as $k \to \infty$ converge to the exact root $x = a$. There are no direct methods for solving higher degree algebraic equations or transcendental equations. Such equations can be solved by Numerical methods. In these methods, we first find an interval in which the root lies. If a and b are two numbers such that $f(a)$ and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between a and b. We take a or b or any valve in between a or b as first approximation x_1. This is further improved by numerical methods. Here we discuss few important Numerical methods to find a root of $f(x) = 0$.

REGULA FALSI METHOD

This is another method to find the roots of $f(x) = 0$. This method is also known as Regular False Method. In this method, we choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs. Hence a root lies in between these points. The equation of the chord joining the two points.

$(a, f(a))$ and $(b, f(b))$ is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad \text{.....(5)}$$

We replace the part of the curve between the points $[a, f(a)]$ and $[b, f(b)]$ by means of the chord joining these points and we take the point of intersection of the chord with the x axis as an approximation to the root (see Fig.3). The point of intersection is obtained by putting y = 0 in (5), as

$$x = x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \qquad \text{.....(6)}$$

x_1 is the first approximation to the root of $f(x) = 0$.

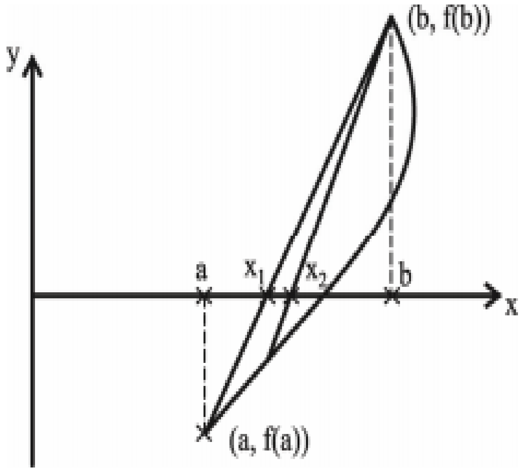

Fig. 3 Method of False Position

If $f(x_1)$ and $f(a)$ are of opposite signs, then the root lies between a and x_1 and we replace b by x_1 in (6) and obtain the next approximation x_2. Otherwise, we replace a by x_1 and generate the next approximation. The procedure is repeated till the root is obtained to the desired accuracy. This method is also called linear interpolation method or chord method.

1. Find the root of the equation $2x - \log x = 7$ which lies between 3.5 and 4 by Regula–False method. (JNTU 2006)

Solution

Given $f(x) = 2x - \log x_{10} = 7$ .....(1)

Take $x_0 = 3.5, \quad x_1 = 4$

Using Regula Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x)} . f(x_0)$$

$$x_2 = 3.5 - \frac{4 - 3.5}{(0.3979 + 0.5441)} (-0.5441)$$

$$x_2 = 3.7888$$

Now taking $x_0 = 3.7888$ and $x_1 = 4$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0)$$

$$x_3 = 3.7888 - \frac{4 - 3.7888}{0.3988} (-0.0009)$$

$$x_3 = 3.7893$$

The required root is = 3.789

2. Find a real root of $xe^x = 3$ using Regula-Falsi method.

Solution

Given $f(x) = x\,e^x - 3 = 0$

$f(1) = e - 3 = -0.2817 < 0$

$f(2) = 2e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2

Now taking $x_0 = 1$, $x_1 = 2$

Using Regula – Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$\therefore \qquad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817}$$

$$x_2 = 1.329$$

Now $f(x_2) = f(1.329) = 1.329\,e^{1.329} - 3 = 2.0199 > 0$

$f(1) = -0.2817 < 0$

∴ The root lies between 1 and 1.329 taking $x_0 = 1$ and $x_2 = 1.329$

∴ Taking $x_0 = 1$ and $x_2 = 1.329$

∴ $$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{1(2.0199) + (1.329)(0.2817)}{(2.0199) + (0.2817)}$$

$$= \frac{2.3942}{2.3016} = 1.04$$

Now $f(x^3) = 1.04\,e^{1.04} - 3 = -0.05 < 0$

The root lies between x^2 and x^3

i.e., 1.04 and 1.329 [∵ $f(x_2) > 0$ and $f(x_3) < 0$]

∴ $$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.04)(-0.05) - (1.329)(2.0199)}{(-0.05) - (2.0199)}$$

$x_4 = 1.08$ is the approximate root

3. Find a real root of $e^x \sin x = 1$ using Regula – Falsi method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

Consider $x_0 = 2$

$f(x_0) = f(2) = e^2 \sin 2 - 1 = -0.7421 < 0$

$f(x_1) = f(3) = e^3 \sin 3 - 1 = 0.511 > 0$

$\therefore$ The root lies between 2 and 3

Using Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{2(0.511)+3(0.7421)}{0.511+0.7421}$$

$$x_2 = 2.93557$$

$f(x_2) = e^{2.93557}\sin(2.93557) - 1$

$f(x_2) = -0.35538 < 0$

$\therefore$ Root lies between x_2 and x_1

i.e., lies between 2.93557 and 3

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(2.93557)(0.511) - 3(-35538)}{0.511 + 0.35538}$$

$x_3 = 2.96199$

$f(x_3) = e^{2.90199}\sin(2.96199) -1 = -0.000819 < 0$

$\therefore$ root lies between x_3 and x_1

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{2.96199(0.511)+3(0.000819)}{0.511+0.000819} = 2.9625898$$

$f(x^4) = e^{2.9625898}\sin(2.9625898) - 1$

$f(x^4) = -0.0001898 < 0$

$\therefore$ The root lies between x_4 and x_1

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{2.9625898(0.511)+3(0.0001898)}{0.511+(0.0001898)}$$

$$x_5 = 2.9626$$

we have

$$x_4 = 2.9625$$

$$x_5 = 2.9626$$

$$\therefore \quad x_5 = x_4 = 2.962$$

$\therefore$ The root lies between 2 and 3 is 2.962

4. Find a real root of $x\,e^x = 2$ using Regula – Falsi method (J

Solution

$f(x) = x\,e^x - 2 = 0$

$f(0) = -2 < 0,\quad f(1) = \text{i.e.}, -2 = (2.7183)-2$

$f(1) = 0.7183 > 0$

$\therefore$ The root lies between 0 and 1

Considering $x_0 = 0, x_1 = 1$

$f(0) = f(x_0) = -2;\quad f(1) = f(x_1) = 0.7183$

By Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{0(0.7183) - 1(-2)}{0.7183 - (-2)} = \frac{2}{2.7183}$$

$$x_2 = 0.73575$$

Now $f(x^2) = f(0.73575) = 0.73575\,e^{0.73575} - 2$

$f(x_2) = -0.46445 < 0$

and $f(x_1) = 0.7183 > 0$

$\therefore$ The root x_3 lies between x_1 and x_2

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(0.73575)(0.7183)}{0.7183 + 0.46445}$$

$$x_3 = \frac{0.52848 + 0.46445}{1.18275}$$

$$x_3 = \frac{0.992939}{1.18275}$$

$$x_3 = 0.83951 \quad f(x^3) = \frac{(0.83951)}{(0.83951)e^{-2}}$$

$$f(x_3) = (0.83951)\, e^{0.83951} - 2$$

$$f(x_3) = -0.056339 < 0$$

$\therefore$ One root lies between x_1 and x_3

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} = \frac{(0.83951)(0.7183) - 1(-0.056339)}{0.7183 + 0.056339}$$

$$x_4 = \frac{0.65935}{0.774639} = 0.851171$$

$f(x_4) = 0.851171\ e0.851171 - 2 = -0.006227 < 0$

Now x_5 lies between x_1 and x_4

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$x_5 = \frac{(0.851171)(0.7183) + (.006227)}{0.7183 + 0.006227}$$

$$x_5 = \frac{0.617623}{0.724527} = 0.85245$$

Now $f(x_5) = 0.85245\ e^{0.85245}\ e^{0.85245} - 2 = -0.0006756 < 0$

$\therefore$ One root lies between x_1 and x_5, (i.e., x_6 lies between x_1 and x_5)

Using Regula – Falsi method

$$x_6 = \frac{(0.85245)(0.7183) + 0.0006756}{0.7183 + 0.0006756}$$

$$x_6 = 0.85260$$

Now $f(x_6) = -0.00006736 < 0$

$\therefore$ One root x_7 lies between x_1 and x_6

By Regula – Falsi method

$$x_7 = \frac{x_6 f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$x_7 = \frac{(0.85260)(0.7183) + 0.0006736}{0.7183 + 0.0006736}$$

$$x_7 = 0.85260$$

From $x^6 = 0.85260$ and $x_7 = 0.85260$

$\therefore$ A real root of the given equation is 0.85260

NEWTON RAPHSON METHOD

This is another important method. Let x_0 be approximation for the root of $f(x) = 0$. Let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_1) = f(x_0 + h)$ by Taylor series, we get

$$f(x_1) = f(x_1 + h) = f(x_0) + h\ f'(x_0) + \frac{h^2}{2!}\ f''(x_0) + \ldots\ldots = 0 \qquad \ldots..(1)$$

For small valves of h, neglecting the terms with h^2, h^3 ..... etc,. We get

$$\therefore \qquad f(x_0) + h\ f'(x_0) = 0 \qquad \ldots..(2)$$

and $$h = -\frac{f(x_0)}{f^1(x_0)}$$

$\therefore$ $$x_1 = x_0 + h$$
$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Proceeding like this, successive approximation $x_2, x_3, \ldots x_{n+1}$ are given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \ldots..(3)$$

For n = 0, 1, 2, ……

Note:

(i) The approximation x_{n+1} given by (3) converges, provided that the initial approximation x_0 is chosen sufficiently close to root of $f(x) = 0$.

(ii) Convergence of Newton-Raphson method: Newton-Raphson method is similar to iteration method

$$\phi(x) = x - \frac{f(x)}{f'(x)} \qquad \ldots..(1)$$

differentiating (1) w.r.t to 'x' and using condition for convergence of iteration method i.e.

$$|\phi'(x)| < 1,$$

We get

$$\left|1 - \frac{f'(x).f'(x) - f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

Simplifying we get condition for convergence of Newton-Raphson method is

$$|f(x).f''(x)| < [f(x)]^2$$

Example 1

Using Newton-Raphson method (a) Find square root of a number (b) Find a reciprocal of a number.

Solution

(a) Let n be the number and $x = \sqrt{n}\ x^2 = n$

If $f(x) = x^2 - n = 0$ .....(1)

Then the solution to $f(x) = x^2 - n = 0$ is $x = \sqrt{n}$

$f^1(x) = 2x$

by Newton Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \left(\frac{x_i^2 - n}{2x_i}\right)$$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{x}{x_i}\right)$$

using the above formula the square root of any number 'n' can be found to required accuracy.

(b) To find the reciprocal of a number 'n'

$f(x) = \frac{1}{x} - n = 0$ .....(1)

$\therefore$ solution of (1) is $x = \frac{1}{n}$

$f^1(x) = -\frac{1}{x^2}$

Now by Newton-Raphson method,

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)}\right)$$

$$x_{i+1} = x_i - \left(\frac{\frac{1}{x_i} - N}{-\frac{1}{x_1^2}}\right)$$

$$x_{i+1} = x_i\ (2 - x_i\, n)$$

Using the above formula, the reciprocal of a number can be found to required accuracy.

Example 2

Find the reciprocal of 18 using Newton–Raphson method.

Solution

The Newton-Raphson method

$x_{i+1} = x_i\ (2 - x_i\ n)$ .....(1)

considering the initial approximate value of x as $x_0 = 0.055$ and given $n = 18$

$\therefore\ x_1 = 0.055\ [2 - (0.055)\ (18)]$

$\therefore\ x_1 = 0.0555$

$x_2 = 0.0555\ [2 - 0.0555 \times 18]$

$x_2 = (0.0555)\ (1.001)$

$x_2 = 0.0555$

Hence $x_1 = x_2 = 0.0555$

$\therefore$ The reciprocal of 18 is 0.0555.

Example 3

Find a real root for $x \tan x + 1 = 0$ using Newton–Raphson method

Solution

Given $f(x) = x \tan x + 1 = 0$

$f^1(x) = x \sec2 x + \tan x$

$f(2) = 2 \tan 2 + 1 = -3.370079 < 0$

$f(3) = 2 \tan 3 + 1 = -0.572370 > 0$

$\therefore$ The root lies between 2 and 3

Take $x_0 = \frac{2+3}{2} = 2.5$ (average of 2 and 3), By Newton-Raphson method

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)}\right)$$

$$x_1 = x_0 - \left(\frac{f(x_0)}{f^1(x_0)}\right)$$

$$x_1 = 2.5 - \frac{(-0.86755)}{3.14808}$$

$$x_1 = 2.77558$$

$$x_2 = x_1 - \frac{f(x_i)}{f^1(x_i)} ;$$

$f(x_1) = -0.06383, \qquad f^1(x_1) = 2.80004$

$$x_2 = 2.77558 - \frac{(-0.06383)}{2.80004}$$

$$x_2 = 2.798$$

$f(x_2) = -0.001080, \qquad f^1(x_2) = 2.7983$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)} = 2.798 - \frac{[-0.001080]}{2.7983}$$

$$x_3 = 2.798.$$

$\therefore \qquad x_2 = x_3$

$\therefore$ The real root of $x \tan x + 1 = 0$ is 2.798

Example 4

Find a root of $e^x \sin x = 1$ using Newton–Raphson method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

$f^1(x) = e^x \sin x + e^x \cos x$

Take $x_1 = 0$, $x_2 = 1$

$f(0) = f(x_1) = e^0 \sin 0 - 1 = -1 < 0$

$f(1) = f(x_2) = e^1 \sin(1) - 1 = 1.287 > 0$

The root of the equation lies between 0 and 1. Using Newton Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

Now consider x_0 = average of 0 and 1

$$x_0 = \frac{1+0}{2} = 0.5$$

$$x_0 = 0.5$$

$$f(x_0) = e^{0.5} \sin(0.5) - 1$$

$$f^1(x_0) = e^{0.5} \sin(0.5) + e^{0.5} \cos(0.5) = 2.2373$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.5 - \frac{(-0.20956)}{2.2373}$$

$$x_1 = 0.5936$$

$$f(x_1) = e^{0.5936} \sin(0.5936) - 1 = 0.0128$$

$$f^1(x_1) = e^{0.5936} \sin(0.5936) + e^{0.5936} \cos(0.5936) = 2.5136$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.5936 - \frac{(0.0128)}{2.5136}$$

$$\therefore \quad x_2 = 0.58854$$

similarly $\quad x_3 = x_2 - \dfrac{f(x_1)}{f^1(x_1)}$

$$f(x_2) = e^{0.58854} \sin(0.58854) - 1 = 0.0000181$$

$$f^1(x_2) = e^{0.58854} \sin(0.58854) + e^{0.58854} \cos(0.58854)$$

$$f(x_2) = 2.4983$$

$$\therefore \quad x_3 = 0.58854 - \frac{0.0000181}{2.4983}$$

$$x_3 = 0.5885$$

$$\therefore \quad x_2 - x_3 = 0.5885$$

0.5885 is the root of the equation $e^x \sin x - 1 = 0$

GAUSS ELIMINATION METHOD

This is the elementary elimination method and it reduces the system of equations to an equivalent upper – triangular system, which can be solved by back substitution.

We consider the system of n linear equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

There are two steps in the solution viz., the elimination of unknowns and back substitution.

Example 1

Solve the following system of equations using Gaussian elimination.

$$x_1 + 3x_2 - 5x_3 = 2$$

$$3x_1 + 11x_2 - 9x_3 = 4$$

$-x_1 + x_2 + 6x_3 = 5$

Solution

An augmented matrix is given by

$$\left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right].$$

We use the boxed element to eliminate any non-zeros below it.

This involves the following row operations

$$\left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right] \begin{array}{l} R2 - 3 \times R1 \\ R3 + R1 \end{array} \Rightarrow \left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 0 & 2 & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right].$$

And the next step is to use the 2 to eliminate the non-zero below it. This requires the final row operation

$$\left[\begin{array}{rrr|r} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right] \; R3 - 2 \times R2 \Rightarrow \left[\begin{array}{rrr|r} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 0 & -11 & 11 \end{array}\right].$$

This is the augmented form for an upper triangular system, writing the system in extended form we

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 2 \\ 2x_2 + 6x_3 &= -2 \\ -11x_3 &= 11 \end{aligned}$$

This gives $x_3 = -1$; $x_2 = 2$; $x_1 = -9$.

Example 2

Solve the system of equations

$2x + 4y + 6z = 22$

$3x + 8y + 5x = 27$

$-x + y + 2z = 2$

Solution

$$\begin{bmatrix} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_1{}' = 1/2R_1}$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_2}' = \boldsymbol{R_2} - \boldsymbol{3R}_1$; $\boldsymbol{R_3}' = \boldsymbol{R_3} + \boldsymbol{R}_1$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$$

$\boldsymbol{R_2}' = \mathbf{1/2}\boldsymbol{R}_2$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \boldsymbol{2R}_2$; $\boldsymbol{R_3}' = \boldsymbol{R_3} - \boldsymbol{3R_2}$

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$\boldsymbol{R_3}' = \mathbf{1/11}\boldsymbol{R}_1$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \boldsymbol{7R}_3$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \boldsymbol{7R}_3$; $\boldsymbol{R_2}' = \boldsymbol{R_2} + \boldsymbol{2R}_3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, the solution to the system is x = 3, y = 1, z = 2.

ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

As a numerical technique, Gaussian elimination is rather unusual because it is direct. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error. Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

The Jacobi Method The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal. If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal. A matrix A is diagonally dominated if, in each row, the absolute value of the entry on the diagonal is greater than the sum of the absolute values of the other entries. More compactly, A is diagonally dominated if

$$|A_{ii}| > \sum_{j, j \neq i} |A_{ij}| \quad \text{for all } i$$

To begin the Jacobi method, solve the first equation for the second equation for and so on, as follows

$$x_1 = 1/a_{11}[b_1 - a_{12}x_2 - \dots - a_{1n}x_n]$$

$$x_2 = 1/a_{22}[b_2 - a_{21}x_1 - \dots - a_{2n}x_n]$$

$$\vdots$$

$$x_n = 1/a_{nn}[b_n - a_{n1}x_1 - a_{n2}x_2 - \dots]$$

Then make an initial approximation of the solution, Initial approximation and substitute these values of into the right-hand side of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often converges to the actual solution.

GAUSS JACOBI METHOD

Example

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Solution

To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in Table

n	0	1	2	3	4	5	6	7
x_1	0.000	-0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Because the last two columns in the above table are identical, you can conclude that to three significant digits the solution is $x_1= 0.186$, $x_2 = 0.331$, $x_3 = -0.423$.

GAUSS SEIDEL METHOD

Intuitively, the Gauss-Seidel method seems more natural than the Jacobi method. If the solution is converging and updated information is available for some of the variables, surely it makes sense to use that information! From a programming point of view, the Gauss-Seidel method is definitely more convenient, since the old value of a variable can be overwritten as soon as a new value becomes available. With the Jacobi method, the values of all variables from the previous iteration need to be retained throughout the current iteration, which means that twice as much as storage is needed. On the other hand, the Jacobi method is perfectly suited to parallel computation, whereas the Gauss-Seidel method is not. Because the Jacobi method updates or 'displaces' all of the variables at the same time (at the end of each iteration) it is often called the method of simultaneous displacements. The Gauss-Seidel method updates the variables one by one (during each iteration) so its corresponding name is the method of successive displacements.

Example

Solve the following system of equations by Gauss – Seidel method

$28x +4y –z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

Solution

Since the diagonal element in given system are not dominant, we rearrange the equation as follows

$28x +4y – z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

Hence

$x =1/28[32 – 4y +z]$

$y = 1/17[35-2x -4z]$

$z = 1/10[24 – x – 3y]$

Setting y = 0 and z = 0, we get,

First iteration

$x^{(1)} = 1/28 [32- 4(0) +(0)] = 1.1429$

$y^{(1)} = 1/17 [35 – 2(1.1429) -4(0)] = 1.9244$

$z^{(1)} = 1/10 [24 – 1.1429 – 3(1.9244)] = 1.8084$

Second Iteration

$x^{(2)} = 1/28 [32- 4(1.9244) +(1.8084)] = 0.9325$

$y^{(2)} = 1/17 [35 – 2(0.9325) -4(1.8084)] = 1.5236$

$z^{(2)} = 1/10 [24 – 0.9325 – 3(1.5236)] = 1.8497$

Third Iteration

$x^{(3)} = 1/28 [32- 4(1.5236) +(1.8497)] = 0.9913$

$y^{(3)} = 1/17 [35 – 2(0.9913) -4(1.8497)] = 1.5070$

$z^{(3)} = 1/10 [24 – 0.9913 – 3(1.5070)] = 1.8488$

Fourth Iteration

$x^{(4)} = 1/28 [32- 4(1.5070) +(1.8488)] = 0.9936$

$y^{(4)} = 1/17 [35 – 2(0.9936) -4(1.8488)] = 1.5069$

$z^{(4)} = 1/10 [24 – 0.9936 – 3(1.5069)] = 1.8486$

Fifth Iteration

$x^{(5)} = 1/28 [32- 4(1.5069) +(1.8486)] = 0.9936$

$y^{(5)} = 1/17 [35 – 2(0.9936) -4(1.8486)] = 1.5069$

$z^{(5)} = 1/10 [24 – 0.9936 – 3(1.5069)] =1.8486$

Since the values of x, y, z are the same in the 4^{th} and 5^{th} Iteration, we stop the procedure here.

Hence x = 0.9936, y = 1.5069, z = 1.8486.

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Department of Mathematics

UNIT – II – NUMERICAL INTERPOLATION, DIFFERENTIATION AND INTEGRATION – SMTA1208

NUMERICAL INTERPOLATION, DIFFERENTIATION AND INTEGRATION

Interpolation: Newton's forward and backward difference interpolation formula (equal interval) -Lagrange's interpolation formula (unequal interval). Numerical Differentiation - Newton's forward and backward difference interpolation formula (equal interval). Numerical Integration: Trapezoidal rule, Simpson's 1/3rd and 3/8 th rule.

Interpolation

The process of computing intermediate values of (x_0, x_n) for a function $y(x)$ from a given set of values of a function

Gregory-Newton's forward interpolation formula

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + --- (a)$$

$$where\ u = \frac{1}{h}(x - x_0)$$

Gregory-Newton's backward interpolation formula

$$y(x) = y_n + \frac{\nabla y_n}{1}v + \frac{\nabla^2 y_n}{2}v(v+1) + \frac{\nabla^3 y_n}{6}v(v+1)(v+2) + \frac{\nabla^4 y_n}{24}v(v+1)(v+2)(v+3) + --- (b)$$

$$where\ v = \frac{1}{h}(x - x_n)$$

Remark:

(i) The process of finding the values of $y(x_i)$ outside the interval (x_0, x_n) is called *extrapolation*

(ii) The *interpolating polynomial* is a function $p_n(x)$ through the data points $y_i = f(x_i) = P_n(x_i)$ i = 0,12,..n

(iii) Gregory-Newton's forward interpolation formula (a) can be applicable if the interval difference h is constant and used to interpolate the value of $y(x_i)$ nearer to beginning value x_0 of the data set

(iv) If $y = f(x)$is the exact curve and $y = p_n(x)$ is the interpolating polynomial then the *Error in polynomial interpolation* is $y(x) - p_n(x)$ given by

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)--(x-x_n): x_0 < x < x_n, x_0 < c < x_n --- (c)$$

(v) *Error in Newton's forward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}u(u-1)(u-2)--(u-n): x_0 < x < x_n, x_0 < c < x_n ----(d)$$

(vi) *Error in Newton's backward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}v(v+1)(v+2)--(v+n): x_0 < x < x_n, x_0 < c < x_n ----(e)$$

Problem1: Estimate θ at $x = 43$ & $x = 84$ from the following table .also find $y(x)$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation

Difference Table:

x	$\theta = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	$184 = y_0$	$y_1 - y_0 = 20 = \Delta y_0$				
50	$204 = y_1$	$y_2 - y_1 = 22 = \Delta y_1$	$2 = \Delta^2 y_0$	$0 = \Delta^3 y_0$		
60	$226 = y_2$	$y_3 - y_2 = 24 = \Delta y_2$	$2 = \Delta^2 y_1$	$0 = \Delta^3 y_1$	$0 = \Delta^4 y_0$	$0 = \nabla^5 y_n$
70	$250 = y_3$	$y_4 - y_3 = 26 = \Delta y_3$	$2 = \Delta^2 y_2$	$0 = \nabla^3 y_n$	$0 = \nabla^4 y_n$	
80	$276 = y_4$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$2 = \nabla^2 y_n$			
90	$304 = y_n$					

Case (i): to find the value of θ at $x = 43$

Since $x = 43$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(43-40) = \frac{3}{10} = 0.3 \Rightarrow u-1 = -0.7, u-2 = -1.7, u-3 = -2.7 ---(2)$$

Substituting (2) in (1), we get $y(x=43) = 184 + \frac{20}{1}(\frac{3}{10}) + \frac{2}{2}(\frac{3}{10})(\frac{-7}{10}) + 0 = \frac{18979}{10} = 189.79$

Case (ii): to find the value of θ at $x = 84$

Since $x = 84$ is nearer to x_n we apply Newton's backward Interpolation

$$y(x) = y_n + \frac{\nabla y_n}{1} v + \frac{\nabla^2 y_n}{2} v(v+1) + \frac{\nabla^3 y_n}{6} v(v+1)(v+2) + \frac{\nabla^4 y_n}{24} v(v+1)(v+2)(v+3) + ---(3)$$

$$where\ v = \frac{1}{h}(x - x_n) = \frac{1}{10}(84-90) = \frac{-6}{10} \Rightarrow v+1 = \frac{4}{10}, v+2 = \frac{14}{10}, v+3 = \frac{24}{10} --- (4)$$

Substituting (4) in (3), we get $y(x=84) = 304 + \frac{28}{1}(\frac{-6}{10}) + \frac{2}{2}(\frac{-6}{10})(\frac{4}{10}) + 0 = \frac{7174}{25} = 286.96$

To find polynomial $y(x)$, from (1) we get

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(x-40) \Rightarrow u-1 = \frac{1}{10}(x-50), u-2 = \frac{1}{10}(x-60), u-3 = \frac{1}{10}(x-60) --- (2)^1$$

Substituting (4) in (3), we get

$$y(x) = 184 + \frac{20}{1}\frac{1}{10}(x-40) + \frac{2}{2}\frac{1}{10}(x-40)\frac{1}{10}(x-50) + 0 = 184 + 2x - 80 + \frac{1}{100}(x^2 - 90x + 2000)$$

$$\Rightarrow y(x) = \frac{1}{100}(x^2 + 110x + 12400) ----------- (5)$$

To Estimate θ at $x = 43$ & $x = 84$, put $x = 43$ & $x = 84$ in (5), we get

$$y(43) = \frac{1}{100}(18979) = 189.79\ and\ \ y(84) = \frac{1}{100}(28696) = 286.96$$

Problem2: Estimate the number of students whose weight is between 60 lbs and 70 lbs from the following data

Weight(lbs)	0-40	40-60	60-80	80-100	100-120
No. Students	250	120	100	70	50

Solution: let x-Weight less than 40 lbs, y-Number of Students, $\Rightarrow x_0 = 40, x_1 = 60, x_2 = 80, x_3 = 100, x_n = 120$, Here all the intervals are equal with h=x_1-x_0=20 we apply Newton interpolation

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$250 = y_0$	$y_1 - y_0 = 120 = \Delta y_0$			
60	$370 = y_1$	$y_2 - y_1 = 100 = \Delta y_1$	$-20 = \Delta^2 y_0$	$-10 = \Delta^3 y_0$	
80	$470 = y_2$	$y_3 - y_2 = 70 = \Delta y_2$	$-30 = \Delta^2 y_1$	$10 = \nabla^2 y_n$	$20 = \Delta^4 y_0 = \nabla^4 y_n$

100 $540 = y_3$ $y_n - y_{n-1} = 50 = \nabla y_n$ $-20 = \nabla^2 y_n$

120 $590 = y_n$

Case (i): to find the number of students y whose weight less than 60 lbs ($x = 60$)

From the difference table the number of students y whose weight less than 60 lbs ($x = 60$) = 370

Case (ii): to find the number of students y whose weight less than 70 lbs ($x = 70$)

Since $x = 70$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + ----(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{20}(70-40) = \frac{3}{2} \Rightarrow u-1 = \frac{3}{2}, u-2 = \frac{2}{2}, u-2 = \frac{-1}{2}, u-3 = \frac{-3}{2} -----(2)$$

Substituting (2) in (1), we get

$$y(x=70) = 250 + \frac{120}{1}(\frac{3}{2}) + \frac{-20}{2}(\frac{3}{2})(\frac{1}{2}) + \frac{-10}{6}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2}) + \frac{20}{24}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2}) = 423.59$$

The number of students y whose weight less than 70 lbs ($x = 70$) =424

Number of students whose weight is between 60 lbs and 70 lbs $=$

$\begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 70 lbs} \end{Bmatrix} - \begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 60 lbs} \end{Bmatrix}$ = 424-370 = 54

Lagrange's interpolation formula for Unequal intervals

$$y(x) = \frac{(x-x_1)(x-x_2)--(x-x_n)}{(x_0-x_1)(x_0-x_2)--(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)--(x-x_n)}{(x_1-x_0)(x_1-x_2)--(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)--(x-x_n)}{(x_2-x_0)(x_2-x_1)--(x_2-x_n)}y_2 + ----+\frac{(x-x_0)(x-x_1)--(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)--(x_n-x_{n-1})}y_n$$

Problem 3: Determine the value of $y(1)$ from the following data using Lagrange's Interpolation

x	-1	0	2	3
y	-8	3	1	12

Solution: given

x	$x_0 = -1$	$x_1 = 0$	$x_2 = 3$	$x_n = 3$
y	$y_0 = -8$	$y_1 = 3$	$y_2 = 1$	$y_n = 12$

Since the intervals ere not uniform we cannot apply Newton's interpolation.

Hence by Lagrange's interpolation for unequal intervals

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}y_n$$

$$y(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12) ----(1)$$

To compute $y(1)$ put $x = 1$ in (1), we get

$$y(x=1)=\frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(1+1)(1-2)(1\ \ -3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(1+1)(1-0)(1-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(1+1)(1-0)(1-2)}{(3+1)(3-0)(3-2)}(12)$$
$$\Rightarrow y(x=1)=2$$

To find polynomial $y(x)$, from (1) we get

$$y(x)=\frac{2}{3}(x^3-5x^2+6x)+\frac{1}{2}(x^3-4x^2+x+6)$$
$$-\frac{1}{6}(x^3-2x^2-3x)+\frac{1}{1}(x^3-x^2-2x)----(1)$$
$$y(x)=x^3(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1)+x^2(\frac{-10}{3}+\frac{-4}{2}+\frac{2}{6}-1)+x\ \ (\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2)+(\frac{6}{2})$$
$$\Rightarrow y(x)=2x^3-6x^2+3x\ \ +3-----(2)$$

To compute $y(1)$ put $x=1$ in (2), we get $y(x=1)=2-6+3+3=2$

Inverse interpolation

For a given set of values of x and y, the process of finding $x(dependent)$ given $y(independent)$ is called **Inverse interpolation**

$$x(y)=\frac{(y\ -y_1)(y\ -y_2)--(y\ -y_n)}{(y_0-y_1)(y_0-y_2)--(y_0-y_n)}x_0+\frac{(y\ -y_0)(y\ -y_2)--(y\ -y_n)}{(y_1-y_0)(y_1-y_2)--(y_1-y_n)}x_1$$
$$+\frac{(y\ -y_0)(y\ -y_1)--(y\ -y_n)}{(y_2-y_0)(y_2-y_1)--(y_2-y_n)}x_2+---+\frac{(y\ -y_0)(y\ -y_1)--(y\ -y_{n-1})}{(y_n-y_0)(y_n-y_1)--(y_n-y_{n-1})}x_n$$

Problem 4: Estimate the value of x given $y = 100$ from the following data, $y(3) = 6$ $y(5) = 24$, $y(7) = 58$, $y(9) = 108$, $y(11) = 174$

Solution: given

x	$x_0 = 3$	$x_1 = 5$	$x_2 = 7$	$x_3 = 9$	$x_n = 11$
y	$y_0 = 6$	$y_1 = 24$	$y_2 = 58$	$y_3 = 108$	$y_n = 174$

By applying Lagrange's inverse interpolation

$$x(y) = \frac{(y - y_1)(y - y_2)(y - y_3)(y - y_n)}{(y_0 - x_1)(y_0 - y_2)(y_0 - y_3)(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)(y - y_3)(y - y_n)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)(y_1 - y_n)} x_1$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_3)(y - y_n)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)(y_2 - y_n)} x_2 + \frac{(y - y_0)(y - y_1)(y - y_2)(y - y_n)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)(y_3 - y_n)} x_3$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_2)(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)(y_n - y_2)(y_n - y_{n-1})} x_n$$

$$\Rightarrow x(100) = \frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3) + \frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$

$$+ \frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7) + \frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$

$$+ \frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$

$$\Rightarrow x(100) = 0.35344 - 1.51547 + 2.88703 + 7.06759 - 0.13686 = 8.65573$$

<u>Numerical Differentiation</u>

The process of computing the derivatives of y at a given value of x using a set of given values of x and y is called Numerical differentiation.

<u>Newton's forward formula for Derivatives</u>

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{\Delta^2 y_0}{2}(2u-1) + \frac{\Delta^3 y_0}{6}(3u^2 - 6u + 2) + \frac{\Delta^4 y_0}{24}(4u^3 - 18u^2 + 22u - 6) + --- \right\}$$

$$y''(x) = \frac{d^2 y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + \frac{\Delta^3 y_0}{1}(u-1) + \frac{\Delta^4 y_0}{24}(12u^2 - 36u + 22) + --- \right\} where\ u = \frac{1}{h}(x - x_0)$$

<u>Newton's backward formula for Derivatives</u>

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2}(2v+1) + \frac{\nabla^3 y_n}{6}(3v^2 + 6v + 2) + \frac{\nabla^4 y_n}{24}(4v^3 + 18v^2 + 22v + 6) + --- \right\}$$

$$y''(x) = \frac{d^2 y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + \frac{\nabla^3 y_n}{1}(v+1) + \frac{\nabla^4 y_n}{24}(12v^2 + 36v + 22) + --- \right\} where\ v = \frac{1}{h}(x - x_n)$$

Problem 5: Find the rate of growth of population in the year 1941&1961 from the following table

year	1931	1941	1951	1961	1971
Population	40.62	60.80	79.95	103.56	132.65

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation

Difference Table: let x-year,y-Population

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	$40.62 = y_0$	$y_1 - y_0 = 20.18 = \Delta y_0$			
1941	$60.80 = y_1$	$y_2 - y_1 = 19.15 = \Delta y_1$	$-1.03 = \Delta^2 y_0$	$5.49 = \Delta^3 y_0$	
1951	$79.95 = y_2$	$y_3 - y_2 = 23.61 = \Delta y_2$	$4.46 = \Delta^2 y_1$	$1.02 = \nabla^2 y_n$	$-4.47 = \Delta^4 y_0 = \nabla^4 y_n$
1196	$103.56 = y_3$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$5.48 = \nabla^2 y_n$		
1971	$132.65 = y_n$				

Case (i): to find rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$

Since $x = 1941$ is nearer to x_0 we apply Newton's forwarded formula for derivative

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{\Delta^2 y_0}{2}(2u-1) + \frac{\Delta^3 y_0}{6}(3u^2-6u+2) + \frac{\Delta^4 y_0}{24}(4u^3-18u^2+22u-6) + --- \right\}$$

$$where\, u = \frac{1}{h}(x - x_0) = \frac{1}{10}(1941-1931) = 1$$

$$\Rightarrow y'(x=1941) = \frac{dy}{dx} = \frac{1}{10}\left\{20.18 + \frac{-1.03}{2}(2-1) + \frac{5.49}{6}(3-6+2) + \frac{-4.47}{24}(4-18+22-6) + --- \right\}$$

The rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$= $y'(1941) = 2.36425$

Case (ii): to find rate of growth of population $(\frac{dy}{dx})$in the year $(x = 1961)$

Since $x = 1961$ is nearer to x_n we apply Newton's backward formula for derivative

$$y'(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{\nabla^2 y_n}{2}(2v+1)+\frac{\nabla^3 y_n}{6}(3v^2+6v+2)+\frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$v=\frac{1}{h}(x-x_n)=\frac{1}{10}(1961-1971)=-1$$

$$\Rightarrow y'(x=1961)=\frac{dy}{dx}=\frac{1}{10}\left\{29.09+\frac{5.48}{2}(-2+1)+\frac{1.02}{6}(3-6+2)+\frac{-4.47}{24}(-4+18-22+6)+---\right\}$$

The rate of growth of population ($\frac{dy}{dx}$) in the year ($x = 1961$)= $y'(1961)=2.65525$

Problem 6 A rod is rotating in a plane, estimate the angular velocity and angular acceleration of the rod at time 6 secs from the following table

Time-t(sec)	0	0.2	0.4	0.6	0.8	1.0
Angle-θ(radians)	0	0.12	0.49	1.12	2.02	3.20

Solution: Here all the intervals are equal with h=x_1-x_0=0.2 we apply Newton interpolation

Difference Table: let x- time (sec),y-Angle (radians)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$0=y_0$	$y_1-y_0=0.12=\Delta y_0$			
	$0.12=y_1$	$y_2-y_1=0.37=\Delta y_1$	$0.25=\Delta^2 y_0$	$0.01=\Delta^3 y_0$	
	$0.49=y_2$	$y_3-y_2=0.63=\Delta y_2$	$0.26=\Delta^2 y_1$	$0.01=\Delta^3 y_1$	$0=\Delta^4 y_0$
	$1.12=y_3$	$y_4-y_3=0.90=\Delta y_3$	$0.27=\Delta^2 y_2$	$0.01=\nabla^2 y_n$	$0=\nabla^4 y_n$
	$2.02=y_4$	$y_n-y_{n-1}=1.18=\nabla y_n$	$0.28=\nabla^2 y_n$		
	$3.20=y_n$				

Case (i): to find Angular velocity ($\frac{dy}{dx}$) in time ($x = 0.6\ sec$)

Since $x = 0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y^{'}(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{\nabla^2 y_n}{2}(2v+1)+\frac{\nabla^3 y_n}{6}(3v^2+6v+2)+\frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$v=\frac{1}{h}(x-x_n)=\frac{1}{0.2}(0.6-1.0)=-2$$

$$y^{'}(x=0.6)=\frac{dy}{dx}=\frac{1}{0.2}\left\{1.18+\frac{0.28}{2}(-4+1)+\frac{0.01}{6}(12-12+2)+\frac{0}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$\Rightarrow The\, angular\, velocity\; y^{'}(x=0.6)=3.81665\, radian/\sec$$

Case (ii): to find Angular acceleration $(\frac{d^2y}{dx^2})$ in time $(x = 0.6\ sec)$

Since $x = 0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y^{''}(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_n+\frac{\nabla^3 y_n}{1}(v+1)+\frac{\nabla^4 y_n}{24}(12v^2+36v\ \ +22)+---\right\}$$

$$where\ v=\frac{1}{h}(x-x_n)=\frac{1}{0.2}(0.6-1.0)=-2$$

$$\Rightarrow y^{''}(x=0.6)=\frac{1}{0.2^2}\left\{0.28+\frac{0.01}{1}(-2+1)+0\right\}$$

$$y^{''}(0.6)=6.75\ radian/\sec^2$$

Numerical Integration

The process of evaluating an integral w.r.t x whose integrand is f(x) between the limits a and b using a given set of x and y values is called Numerical Integration.

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+y_3+y_4+--)\ where\ h=\frac{1}{n}(x_n-x_0), n-number\ of\ \text{int}ervals$$

Simpson's 1/3 rd rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+y_6+-)+4(y_1+y_3+y_5+--)\ \}$$

$$where\ h=\frac{1}{n}(x_n-x_0), n-number\ of\ \text{int}ervals$$

Simpson's 3/8 th rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0+y_n)+2(y_3+y_6+y_9+-)+3(y_1+y_2+y_4+y_5+--)\ \}$$

$$where\ h=\frac{1}{n}(x_n-x_0), n-number\ of\ \text{int}ervals$$

Remarks:

1) Geometrical interpretation of $\int_{x_0}^{x_n} y(x)dx$ is approximated by the sum of area of the trapezium
2) Simpson's $1/3$ rule is applicable when number of intervals are multiples of 2 and Simpson's $3/8$ rule is applicable when number of intervals are multiples of 3
3) The error in trapezoidal rule is $\frac{b-a}{12}h^2M$ where $M = max\{y_0'', y_1'', \dots\}$ which is of order h^2

4) The error in Simpson's $1/3$ rule rule is $\frac{b-a}{180}h^4M$ where $M = max\{y_0'''', y_2'''', \ldots\}$which is of order h^4

Problem7: Evaluate $\int_1^6 \frac{1}{1+x^2}dx$ using (i) Trapezoidal rule (ii) Simpson's $1/3$ rule (iii) Simpson's $3/8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^6 \frac{1}{1+x^2}dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x) = \frac{1}{1+x^2}, x_0 = 0, x_0 + nh = 6 ----(1)$

Choose the number of interval (n)=6 so that we can apply all rules

x	$x_0 = 0$	$x_1 = x_0 + h = 1$	$x_2 = x_1 + h = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$	$x_n = 6$
$y(x) = \frac{1}{1+x^2}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

case(i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + --)$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\left\{(1+\frac{1}{37}) + 2(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}) \right\} = 1.410799$$

Case (ii) Simpson's $1/3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + y_6 + -) + 4(y_1 + y_3 + y_5 + --) \}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{1}{3}\left\{(1+\frac{1}{37}) + 2(\frac{1}{5}+\frac{1}{17}) + 4(\frac{1}{2}+\frac{1}{10}+\frac{1}{26}) \right\} = 1.36617$$

Case(iii) Simpson's $3/8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\left\{(y_0+y_n)+2(y_3+y_6+y_9+-)+3(y_1+y_2+y_4+y_5+--)\ \right\}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3}{8}\left\{(1+\frac{1}{37})+2(\frac{1}{10})+3(\frac{1}{2}+\frac{1}{5}+\frac{1}{17}+\frac{1}{26})\ \right\}=1.35708$$

Comparison

Exact value $\int_0^6 \frac{1}{1+x^2}dx = \left[\tan^{-1}(x)\right]_{x=0}^{x=6} = \tan^{-1}(6)-\tan^{-1}(0) = 1.40565$

Hence trapezoidal rule gives better approximation than Simpson's rule.

Problem 8: By dividing the range into 10 equal part Determine the value of $\int_0^{\pi} \sin x\, dx$ using (i) Trapezoidal rule (ii) Simpson's $1/3$ rule (iii) Simpson's $3/8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^{\pi} \sin x\, dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x)=\sin x, x_0=0, x_0+nh=\pi \, and \; n=10----(1)$

$$given\, number\, of\, \text{int}\,ervals(n)=10, (1)\Rightarrow h=\frac{1}{n}(x_n-x_0)=\frac{1}{10}(\pi-0)=\frac{\pi}{10}$$

x	$x_0=0$	$x_1=x_0+h=\frac{\pi}{10}$	$x_2=x_1+h=\frac{2\pi}{10}$	$x_3=\frac{3\pi}{10}$	$x_4=\frac{4\pi}{10}$	$x_5=\frac{5\pi}{10}$	$x_6=\frac{6\pi}{10}$
$y(x)=\sin(x)$	$\sin(0)=0$	$\sin(\frac{\pi}{10})=0.30901$	$\sin(\frac{2\pi}{10})=0.58779$	$\sin(\frac{3\pi}{10})=0.80901$	$\sin(\frac{4\pi}{10})=0.95106$	$\sin(\frac{5\pi}{10})=1.0$	$\sin(\frac{6\pi}{10})=0.95106$

x	$x_7=\frac{7\pi}{10}$	$x_8=\frac{8\pi}{10}$	$x_9=\frac{9\pi}{10}$	$x_n=\pi$
$y(x)=\sin(x)$	$\sin(\frac{7\pi}{10})=0.80902$	$\sin(\frac{8\pi}{10})=0.58779$	$\sin(\frac{9\pi}{10})=0.30902$	$\sin(\frac{10\pi}{10})=0$

Case (i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + --)$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\{(0+0) + 2(0.30901 + 0.58779 + 0.80901 + 0.95106 + 1.0 + 0.95106 + 0.80901 + 0.58779 + 0.30901)\ \}$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = 1.98352$$

Case (ii) Simpson's $1/3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + y_6 + -) + 4(y_1 + y_3 + y_5 + --)\ \}$$

$$\Rightarrow \int_0^6 \sin(x)dx = \frac{\pi}{30}\{(0+0) + 2(0.58779 + 0.95106 + 0.95106 + 0.58779) + 4(0.30901 + 0.80901 + 1.0 + 0.80901 + 0.30901\}$$

$$\Rightarrow \int_0^6 \sin(x)dx = 2.00010$$

Case (iii) Simpson's $3/8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0 + y_n) + 2(y_3 + y_6 + y_9 + -) + 3(y_1 + y_2 + y_4 + y_5 + --)\ \}$$

This rule cannot be applied sin*ce n is not a multipole of* 3

Comparison

Exact value $\int_0^{\pi} \sin(x)dx = [-\cos(x)]_{x=0}^{x=\pi} = -[\cos(\pi) - \cos(0)] = 2.0$

Hence, Simpson's $1/3$ rule gives better approximation than trapezoidal rule