SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COURSE NAME: THEORY OF EQUATIONS AND MATRICES

COURSE CODE: SMTA1204

UNIT – I– THEORY OF EQUATIONS

I. Introduction

POLYNOMIAL EQUATIONS

An expression of the form

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ where n is a positive integer and $a_0, a_1 \ldots\ldots a_n$ are constants is called a polynomial in x of the nth degree, if $a_0 \neq 0$.

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ is called an algebraic equation are polynomial equation of nth degree, if $a_0 \neq 0$

Fundamental Theorem of Algebra:

Every nth degree equation has exactly n roots or imaginary.

Relation Between Roots And Coefficient Of Equation:

(i) If α, β, γ are the roots of $x^3 + p\,x^2 + p\,x + p = 0$ the sum of the roots

$s_1 = \alpha + \beta + \gamma = -p_1.$

Sum of the products of two roots taken at a time $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = p_2.$

Product of all the roots, $s_3 = \alpha\beta\gamma = -p_3.$

(ii) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$ then

Sum of the roots $s_1 = \alpha + \beta + \gamma + \delta = -p_1.$

$s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta + \beta\delta + \gamma\delta = p_2.$

Sum of the products of roots taken three at a time

$s_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -p_3.$

Product of the roots, $s_4 = \alpha\beta\gamma\delta = p_4.$

2. For the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$

i) $\Sigma\alpha^2 = p_1^2 - 2p_2$

ii) $\Sigma\alpha^3 = -p_1^3 + 3p_1p_2 - 3p_3$

iii) $\Sigma\alpha^4 = p_1^4 - 4p_1^2p_2 + 2p_2^2 + 4p_1p_3 - 4p_4$

iv) $\Sigma\alpha^2\beta = 3p_3 - p_1p_2$

v) $\Sigma\alpha^2\beta\gamma = p_1p_3 - 4p_4$

Note: For the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$ $\Sigma\alpha^2\beta^2 = p_2^2 - 2p_1p_3$

3. To remove the second term from a n^{th} degree equation, the roots must be diminished by h= $\frac{-a_1}{na_0}$ and the resultant equation will not contain the term with x^{n-1}.

4. In any equation with rational coefficients, irrational roots occur in conjugate pairs.

5. In any equation with real coefficients, complex roots occur in conjugate pairs.

6. i) If $f(x) = x^n + p_1x^{n-1} + \ldots\ldots + p_{n-1}x + p_n$ and f(a) and f(b) are of opposite sign, then at leastone real root of f(x) =0 lies between a and b.

7. (a) For a cubic equation, when the roots are

(i) In A.P., then they are taken as $a-3d, a+d, a+3d$.

(ii) In G.P., then they are taken as $\frac{a}{d^3}, \frac{a}{d}, ad, ad^3$

(iii) In H.P., then they are taken as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$.

8. If an equation is unaltered by changing x into 1/x, then the equation is called reciprocal equation.

(i) If an equation is unaltered by changing x into $\frac{1}{x}$, then it is a reciprocal equation.

(ii) A reciprocal equation $f(x) = p_0x^n + p_1x^{n-1} + \ldots\ldots\ldots + p_n = 0$ is said to be a reciprocal equation of first class $p_i = p_{n-i}$ for all i.

(iii) A reciprocal equation $f(x) = p_0x^n + p_1x^{n-1} + \ldots\ldots + p_n = 0$ is said to be a reciprocal equation of second class $p_i = p_{n-i}$ for all i.

(iv) For an odd degree reciprocal equation of class one, -1 is a root and for an odd degree reciprocal equation of class two, 1 is a root.

(v) For an even degree reciprocal equation of class two, 1 and -1 are roots.

IRRATIONAL ROOTS AND IMAGINARY ROOTS

SOLVED PROBLEM

1. Form polynomial equations of the lowest degree, with roots as given below.

(i) $1, -1, 3$

Equations having roots α, β, γ is $(x-\alpha)(x-\beta)(x-\gamma)=0$

Sol: Required equation is

$$(x-1)(x+1)(x-3)=0$$

$$\Rightarrow (x^2-1)(x-3)=0$$

$$\Rightarrow x^3-3x^2-x+3=0$$

(ii) $1 \pm 2i, 4, 2$

In an equation imaginary roots occur in conjugate pairs.

Sol: Equation having roots $\alpha, \beta, \gamma, \delta$ is $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$

Required equation is

$$\left[x-(1+2i)\right]\left[x-(1-2i)\right](x-4)(x-2)=0$$

$$\left[x-(1+2i)\right]\left[x-(1-2i)\right]$$

$$=\left[(x-1)-2i\right]\left[(x-1)+2i\right]$$

$$=(x-1)^2-4i^2$$

$$=(x-1)^2+4$$

$$=x^2-2x+1+4$$

$$= x^2 - 2x + 5$$

$$(x-4)(x-2) = x^2 - 4x - 2x + 8$$

$$= x^2 - 6x + 8$$

Required equation is

$$(x^2 - 2x + 5)(x^2 - 6x + 8) = 0$$

$$\Rightarrow x^4 - 2x^3 + 5x^2 - 6x^3 + 12x^2 - 30x + 8x^2 - 16x + 40 = 0$$

$$\Rightarrow x^4 - 8x^3 + 25x^2 - 46x + 40 = 0$$

PROBLEMS ON RELATION BETWEEN ROOTS AND COEFFICIENTS

1. **If** α, β, γ **are the roots of** $4x^3 - 6x^2 + 7x + 3 = 0$, **then find the value of** $\alpha\beta + \beta\gamma + \gamma\alpha$.

Sol: α, β, γ are the roots of $4x^3 - 6x^2 + 7x + 3 = 0$

$$\alpha + \beta + \gamma = -\frac{a_1}{a_0} = \frac{6}{4}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_2}{a_0} = \frac{7}{4}$$

$$\alpha\beta\gamma = -\frac{a_3}{a_0} = -\frac{3}{4}$$

$$\therefore \alpha\beta + \beta\gamma + \gamma\alpha = \frac{7}{4}$$

2. **If** $1, 1, \alpha$ **are the roots of** $x^3 - 6x^2 + 9x - 4 = 0$, **then find** α

Sol: $1, 1, \alpha$ are roots of $x^3 - 6x^2 - 9x - 4 = 0$

Sum $= 1 + 1 + \alpha = 6$

$\alpha = 6 - 2 = 4$

3. **If** $-1, 2$ **and** α **are the roots of** $2x^3 + x^2 - 7x - 6 = 0$, **then find** α

Sol: $-1, 2, \alpha$ are roots of $2x^3 + x^2 - 7x - 6 = 0$

Sum $= -1 + 2 + \alpha = -\frac{1}{2}$

$$\alpha = -\frac{1}{2} - 1 = -\frac{3}{2}$$

5. If $1, -2$ and 3 are roots of $x^3 - 2x^2 + ax + 6 = 0$, then find a.

Sol: $1, -2$ and 3 are roots of

$x^3 - 2x^2 + ax + 6 = 0$

$\Rightarrow 1(-2) + (-2)3 + 3.1 = a$

i.e., $a = -2 - 6 + 3 = -5$

6). If the product of the roots of $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find a.

Sol: α, β, γ are the roots of $4x^3 + 16x^2 - 9x - a = 0$

$\alpha\beta\gamma = \frac{a}{4} = 9 \Rightarrow a = 36$

7. Find the values of S_1, S_2, S_3 and S_4 for each of the following equations.

(i) $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

Sol: Given equation is

$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

We know that $s_1 = -\frac{a_1}{a_0} = \frac{16}{1} = 16$

$s_2 = \frac{a_2}{a_0} = \frac{86}{1} = 86$

$s_3 = -\frac{a_3}{a_0} = \frac{176}{1} = 176$

$s_4 = \frac{a_4}{a_0} = \frac{105}{1} = 105$

8. **Solve $x^3 - 3x^2 - 16x + 48 = 0$, given that the sum of two roots is zero.**

Sol: Let α, β, γ are the roots of

$x^3 - 3x^2 - 16x + 48 = 0$

$\alpha + \beta + \gamma = 3$

Given $\alpha + \beta = 0$ ($\because$ Sum of two roots is zero)

$\therefore \gamma = 3$

i.e. $x - 3$ is a factor of

$x^3 - 3x^2 - 16x + 48 = 0$

$$\begin{array}{c|cccc} 3 & 1 & -3 & -16 & 48 \\ & - & 3 & 0 & -48 \\ \hline & 1 & 0 & -16 & \underline{0} \end{array}$$

$x^2 - 16 = 0 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

The roots are -4,4 and 3

9. **Find the condition that $x^3 - px^2 + qx - r = 0$ may have the sum of its roots zero.**

Sol: Let α, β, γ be the roots of $x^3 - px^2 + qx - r = 0$

$\alpha + \beta + \gamma = p$ (1)

$\alpha\beta + \beta\alpha + \gamma\alpha = q$ (2)

$\alpha\beta\gamma$ (3)

Given $\alpha + \beta = 0$

($\because$ Sum of two roots is zero)

From (1), $\gamma = p$

$\because \gamma$ is a root of $x^3 - px^2 + qx - r = 0$

$\gamma^3 - p\gamma^2 + q\gamma - r = 0$

But $\gamma = p$

$\Rightarrow p^3 - p\left(p^2\right) + q\left(p\right) - r = 0$

$\Rightarrow p^3 - p^3 + qp - r = 0$

$\therefore qp = r$ is the required condition.

10. Given that the roots of $x^3 + 3px^2 + 3px + r = 0$ are in

(i) A.P., show that $2p^2 - 3qp + r = 0$

(ii) G.P., show that $p^3r = q^3$

(iii) H.P., show that $2q^3 = r(3pq - r)$

Sol: Given equation is $x^3 + 3px^2 + 3px + r = 0$

(i) The roots are in A.P.

Suppose $a - d, a, a + d = -3p$

$3a = -3p \Rightarrow a = -p$ (1)

$\because$ 'a' $x^3 + 3px^2 + 3qx + r = 0$

$\Rightarrow a^3 + 3pa^2 + 3qa + r = 0$

But $a = -p$

$\Rightarrow -p^3 + 3p(-p)^2 + 3q(-p) + r = 0$

$\Rightarrow 2p^3 - 3pq + r = 0$ is the required condition

(ii) The roots are in G.P.

Suppose the roots be $\frac{a}{R}, a, aR$

Given $\left(\frac{a}{R}\right)(a)(aR) = -r$

$\Rightarrow a^3 = -r$

$\Rightarrow a = (-r)^{1/3}$

$\because$ 'a' is a root of $x^3 + 3px^2 + 3qx + r = 0$

$$\Rightarrow \left(-r^{1/3}\right)^3 + 3p\left(-r^{1/3}\right)^2 + 3q\left(-r^{1/3}\right) + r = 0$$

$$\Rightarrow -r + 3pr^{2/3} - 3qr^{1/3} + r = 0$$

$$pr^{2/3} = qr^{1/3}$$

$$\Rightarrow pr^{1/3} = q$$

$\Rightarrow p^3 r = q$ is the required condition

SYMMETRIC FUNCTION OF THE ROOTS

Symmetric function of the roots

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ be the roots of the equation.

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots. + p_n = 0.$$

We have learned that

$$S_1 = \Sigma\, \alpha_1 = -\, p_1$$

$$S_2 = \Sigma\, \alpha_1\alpha_2 = p_2$$

$S_3 = \Sigma\, \alpha_1\, \alpha_2\, \alpha_3 = -\, p_3$

…………………….

…………………….

Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

PROBLEM

If α, β, γ **are roots of the equations** $x^3 - 10x^2 + 6x - 8 = 0$ **find** $\alpha^2 + \beta^2 + \gamma^2$

Given α, β, γ are the roots of the equation $x^3 - 10x^2 + 6x - 8 = 0$

$\alpha + \beta + \gamma = 10,\ \alpha\beta + \beta\gamma + \gamma\alpha = 6,\ \alpha\beta\gamma = 8$

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 100 - 12$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 88$

Example 1. If α, β, γ are the roots of the equations $x^3 + px^2 + qx + r = 0$, Express the value of $\Sigma\, \alpha^2\, \beta$ in terms of the coefficients.

Solution.

We have $\alpha + \beta + \gamma = -\, p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -\, r.$

$\Sigma\, \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)\,(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$= q\,(-\, p) - 3\,(-\, r)$

$= 3r - pq.$

Example 2. If α, β, γ, δ be the roots of the bi quadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, Find (1) $\Sigma\, \alpha^2$, (2) $\Sigma\, \alpha^2 \beta\gamma$, (3) $\Sigma\, \alpha^2 \beta^2$, (4) $\Sigma\, \alpha^3 \beta$ and (5) $\Sigma\, \alpha^4$.

Solution.

The relations between the roots and the coefficients are

$$\alpha + \beta + \gamma + \delta = -\, p.$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\, r$$

$$\alpha\beta\gamma\delta = s.$$

$$\Sigma\, \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2\, (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (\Sigma\, \alpha)^2 - 2\, \Sigma\, \alpha\beta$$

$$= p^2 - 2q.$$

$$\Sigma\, \alpha^2\, \beta\gamma = (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)\, (\alpha + \beta + \gamma + \delta) - 4\, \alpha\beta\gamma\delta$$

$$= (\Sigma\, \alpha\beta\gamma)\, (\Sigma\, \alpha) - 4\, \alpha\beta\gamma\delta$$

$$= pr - 4s.$$

$$\Sigma\, \alpha^2\, \beta^2 = \alpha^2\, \beta^2 + \alpha^2\, \gamma^2 + \alpha^2\, \delta^2 + \beta^2\, \gamma^2 + \beta^2\, \delta^2 + \gamma^2\, \delta^2$$

$$= (\Sigma\, \alpha\beta)^2 - 2\, \Sigma\, \alpha^2\, \beta\gamma - 6\, \alpha\beta\gamma\delta$$

$$= q^2 - 2\, (pr - 4s) - 6s$$

$$= q^2 - 2pr + 2s.$$

$$\Sigma\, \alpha^3\, \beta = (\Sigma\, \alpha^2)\, (\Sigma\, \alpha\beta) - \Sigma\, \alpha^2\, \beta\gamma$$

$$= (p^2 - 2q)\, q - (pr - 4s)$$

$$= p^2 q - 2q^2 - pr + 4s.$$

$$\Sigma\, \alpha^4 = (\Sigma\, \alpha^2)^2 - 2\, \Sigma\, \alpha^2\, \beta^2$$

$$= (p^2 - 2q)^2 - 2\, (q^2 - 2pr + 2s)$$

$$= p^4 - 4p^2 q + 2q^2 + 4pr - 4s.$$

Example 4. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, from the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$.

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

In the required equation

S_1 = Sum of the roots $= \beta + \gamma - 2\alpha + \gamma + \alpha - 2\beta + \alpha + \beta - 2\gamma$

$= 0.$

S_2 = Sum of the products of the roots taken two at a time

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta) + (\beta + \gamma - 2\alpha)(\alpha + \beta - 2\gamma) + (\alpha + \beta - 2\gamma)(\gamma + \alpha - 2\beta)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)$ + 2 similar terms

$= (-p - 3\alpha)(-p - 3\beta) + (-p - 3\alpha)(-p - 3\gamma) + (-p - 3\gamma)(-p - 3\beta)$

$= (p + 3\alpha)(p + 3\beta) + (p + 3\alpha)(p + 3\gamma) + (p + 3\gamma)(p + 3\beta)$

$= 3p^2 + 6p(\alpha + \beta + \gamma) + 9(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 3p^2 + 6p(-p) + 9q$

$= 9q - 3p^2.$

S_3 = Products of the roots

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)(\alpha + \beta + \gamma - 3\gamma)$

$= (-p - 3\alpha)(-p - 3\beta)(-p - 3\gamma)$

$= -\{p^3 + 3p^2(\alpha + \beta + \gamma) + 9p(\alpha\beta + \beta\gamma + \gamma\alpha) + 27\alpha\beta\gamma\}$

$= -\{p^3 + 3p^2(-p) + 9pq - 27r\}$

$= 2p^2 - 9pq + 27r$

Hence the required equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$

$$\text{i.e., } x^3 + (9q - 3p^2)\, x - (2p^3 - 9pq + 27r) = 0.$$

Example 5. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ prove that

(1) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = r - pq$

(2) $\alpha^3 + \beta^3 + \gamma^3 = -p^3 + 3pq - 3r$.

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r$.

(1). $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = [-(p + \alpha)(p + \beta)(p + \gamma)]$

Since $\alpha + \beta + \gamma = -p$ $\quad\therefore \alpha + \beta = -p - \gamma$

$$= -[p^3 + p^2(\alpha + \beta + \gamma) + p(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma]$$

$$= -[p^3 + p^2 \times -p + pq - r] = -[p^3 - p^3 + pq - r] = r - pq.$$

(2). $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)]$

$\sum \alpha^3 = \sum \alpha\,[\sum \alpha^2 - \sum \alpha\beta] + 3\alpha\beta\gamma;$

But $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

Therefore $\sum \alpha^3 = \sum \alpha\,[(\sum \alpha)^2 - 3\sum \alpha\beta] + 3\alpha\beta\gamma; = -p[p2 - 3q] - 3r = -p3 + 3pq - 3r$.

Exercises

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ find the value of

   (1) $(\beta + \gamma - \alpha)^3 + (\gamma + \alpha - \beta)^3 + (\alpha + \beta - \gamma)^3$.

   (2) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$.

2. If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$,

   Evaluate (1) $\Sigma\, \alpha^2 \beta\gamma$, (2) $\Sigma\, (\beta + \alpha + \delta)^2$ and (3) $\Sigma \frac{1}{\alpha^2}$.

   Answer : 1. (1).$24r - p^3$, (2).$\frac{2rp - q^2}{r}$, 2. (1).$pr - 4s$, (2).$3p^2 - 2q$,(3).$\frac{r^2 - 2qr}{s}$

UNIT – II – RECIPROCAL EQUATIONS

SUM OF THE ROOTS OF THE EQUATION

Sum of the powers of the roots of an equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of an equation f (x) = 0.The sum of the r^{th} powers of the roots

i.e., $\alpha_1^r + \alpha_1^r + \ldots\ldots + \alpha_n^r$

is usually denoted by S_r. We can easily see that S_r constitutes a symmetric function of the roots and hence we can calculate the value of S_r by the methods described in the previous article. When r is greater than 4, the calculation of S_r by the previous method becomes tedious and in those cases, the following two methods can be used profitably.

We have $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$.

Taking logarithms on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots\ldots + \frac{1}{x-\alpha_n}$$

$$\frac{xf'(x)}{f(x)} = \frac{x}{x-\alpha_1} + \frac{x}{x-\alpha_2} + \ldots\ldots + \frac{x}{x-\alpha_n}$$

$$= \frac{1}{1-\frac{\alpha_1}{x}} + \frac{1}{1-\frac{\alpha_2}{x}} + \ldots\ldots + \frac{1}{1-\frac{\alpha_n}{x}}$$

$$= \left(1 - \frac{\alpha_1}{x}\right)^{-1} + \left(1 - \frac{\alpha_2}{x}\right)^{-1} + \ldots\ldots + \left(1 - \frac{\alpha_n}{x}\right)^{-1}$$

$$= 1 + \frac{\alpha_1}{x} + \frac{\alpha_1^2}{x^2} + \ldots + \frac{\alpha_1^n}{x^n} + \ldots$$

$$+ \ 1 + \frac{\alpha_2}{x} + \frac{\alpha_2^2}{x^2} + \ldots + \frac{\alpha_2^n}{x^n} + \ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ 1 + \frac{\alpha^n}{x} + \frac{\alpha_n^2}{x^2} + \ldots + \frac{\alpha_n^n}{x^n} + \ldots$$

$$= n + (\Sigma\, \alpha_1)\frac{1}{x} + (\Sigma\, \alpha_1^2)\frac{1}{x^2} + \ldots\ldots + (\Sigma\, \alpha_1^r)\frac{1}{x^r} + \ldots$$

$$= n + S_1 \cdot \frac{1}{x} + S_2 \cdot \frac{1}{x^2} + \ldots + S_r \cdot \frac{1}{x^r} + \ldots$$

$\therefore$ S_r = Coefficient of $\frac{1}{x^r}$ in the expansion of $\frac{xf'(x)}{f(x)}$.

Example. Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$.

Solution.

The equation can be written in the form

$$f(x) = x^5 - x^2 - x - 1 = 0$$

S_r = Coefficient of $\frac{1}{x^3}$ in the expansion of $\frac{x(5x^4 - 2x - 1)}{x^5 - x^2 - x - 1}$

= Coefficient of $\frac{1}{x^3}$ in $\frac{5 - \frac{2}{x^3} - \frac{1}{x^4}}{1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}}$

= " " $(5 - \frac{2}{x^3} - \frac{1}{x^4})(1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5})^{-1}$

= " " $(5 - \frac{2}{x^3} - \frac{1}{x^4})\left\{1 + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \left(\frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5}\right)^2 + \right.$

...

= " " $(5 - \frac{2}{x^3} - \frac{1}{x^4})(1 + \frac{1}{x^3} + \ldots)$

= 3.

Newton's Theorem on the sum of the powers of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of an equation

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n = 0$$

and let be $S_r = \alpha_1^r + \alpha_2^r + \ldots\ldots + \alpha_n^r$ so that $S_0 = n$.

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n).$$

Taking logarithms on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots\ldots + \frac{1}{x - \alpha_n}$$

i.e., $f'(x) = \frac{f(x)}{x - \alpha_1} + \frac{f(x)}{x - \alpha_2} + \ldots\ldots + \frac{f(x)}{x - \alpha_n}$

By actual division, we obtain

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + p_1)\, x^{n-2} + (\alpha_1^2 + p_1\alpha_1 + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_1^{n-1} + p_1\alpha_1^{n-2} + \ldots + p_{n-1})$$

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_2 + p_1)\, x^{n-2} + (\alpha_2^2 + p_1\alpha_2 + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_2^{n-1} + p_1\alpha_2^{n-2} + \ldots + p_{n-1})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{f(x)}{x-\alpha_n} = x^{n-1} + (\alpha_n + p_1)\, x^{n-2} + (\alpha_n^2 + p_1\alpha_n + p_2)\, x^{n-3}$$

$$+ \ldots\ldots (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \ldots + p_{n-1}).$$

Adding all these functions, we get

$$f'(x) = nx^{n-1} + (S_1 + np_1)x^{n-2} + (S_2 + p_1S_1 + np_2)x^{n-3}$$

$$+ \ldots\ldots (S_{n-1} + p_1S_{n-2} + \ldots\ np_{n-1}).$$

But $f'(x)$ is also equal to

$$nx^{n-1} + (n-1)\, p_1x^{n-2} + (n-2)\, p_2x^{n-3} + \ldots + 2p_{n-2} + p_{n-1}.$$

Equating the coefficients in two values of $f'(x)$, we obtain the following relations :

$$S_1 + p_1 = 0$$

$$S_2 + p_1S_1 + 2p_2 = 0$$

$$S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0$$

$$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_{n-1} + p_1 S_{n-2} + p_2 S_{n-3} + \ldots + p_{n-2} S_1 + (n-1)\, p_{n-1} = 0$$

From these (n – 1) relations we can calculate in succession the values of S_1 , S_2 , S_3 , … S_{n-1} in terms of the coefficients p_1 , p_2 , p_3 ,…. p_{n-1} . We can extend our results to the sums of all positive powers of the roots, viz., S_n , S_{n+1} , …. S_r where $r > n$.

We have $x^{r-n} f(x) = x^r + p_1 x^{r-1} + p_2 x^{r-2} + \ldots + p_n x^{r-n}$.

Replacing in this identity, x by the roots α_1 , α_2 , α_3 ,……. , α_n, in succession and adding, we have

$$S_r + p_1 S_{r-1} + p_2 S_{r-2} + \ldots + p_n S_{r-n} = 0$$

Now giving r the values n, n +1, n +2,…… successively and observing that $S_0 = n$, we obtain from the last equation

$$S_n + p_1 S_{n-1} + p_2 S_{n-2} + \ldots + n p_n = 0$$

$$S_{n+1} + p_1 S_n + p_2 S_{n-1} + \ldots + p_n S_1 = 0$$

$$S_{n+2} + p_1 S_{n+1} + p_2 S_n + \ldots + p_n S_2 = 0$$

and so on.

Thus we get

$$S_r + p_1 S_{r-1} + p_2 S_{r-2} + \ldots + r p_r = 0 , \text{ if } r < n$$

And $S_r + p_1 S_{r-1} + p_2 S_{r-2} + \ldots\ p_n S_{r-n} = 0$, if $r \geq n$.

Cor. To find the sum of the negative integral powers of the roots of f (x) = 0, put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the transformed equation.

Example 1. Show that the sum of the eleventh powers of the roots of $x^7 + 5x^4 + 1 = 0$ is zero.

Solution.

Since 11 is greater than 7, the degree of the equation, we have to use the latter equation in Newton's theorem.

If we assume the equation as

$x^7 + p_1x^6 + p_2x^5 + p_3x^4 + p_4x^3 + p_5x^2 + p_6x + p_7 = 0$,

we have $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 0$, $p_3 = 5$, $p_7 = 1$.

$\therefore\ S_{11} + p_1S_{10} + p_2S_9 + p_3S_8 + p_4S_7 + p_5S_6 + p_6S_5 + p_7S_4 = 0$

i.e., $S_{11} + 5S_8 + S_4 = 0$ ……(1)

Again

$S_8 + p_1S_7 + p_2S_6 + p_3S_5 + p_4S_4 + p_5S_3 + p_6S_2 + p_7S_1 = 0$

i.e., $S_8 + 5S_5 + S_1 = 0$ ……(2)

Using the first equation in the Newton's theorem

$S_5 + p_1S_4 + p_2S_3 + p_3S_2 + p_4S_1 + 5p_5 = 0$

i.e., $S_5 + 5S_2 = 0$ ……(3)

Again

$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0$

i.e., $S_4 + 5S_1 = 0$ ……(4)

Again

$S_2 + p_1S_1 + 2p_2 = 0$

i.e., $S_2 = 0$ ……..(5)

Also $S_1 = 0$ ……..(6)

From (4) , (5) and (6) , we get $S_4 = 0$

From (3) , (5) , we get $S_5 = 0$

From (2), we get $S_8 = 0$

Substituting the values of S_4 , S_8 in (1) , we get $S_{11} = 0$.

Example 2. If a + b + c + d = 0, show that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}$$

Solution.

Since a + b + c + d = 0, we can consider that a, b, c, d are the roots of the equation

$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$ where $p_1 = 0$.

From Newton's theorem on the sums of powers of the roots, we get

$$S_5 + p_1S_4 + p_2S_3 + p_3S_2 + p_4S_1 = 0 \qquad \ldots\ldots(1)$$

$$S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0 \qquad \ldots\ldots(2)$$

$$S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0 \qquad \ldots\ldots(3)$$

$$S_2 + p_1S_1 + 2p_2 = 0 \qquad \ldots\ldots(4)$$

$$S_1 + p_1 = 0 \qquad \ldots\ldots(5)$$

From (5), we get $S_1 = 0$

From (4), we get $S_2 = -\,2p_2$

From (3), we get $S_3 = -\,3p_3$

From (1), we get $S_5 - 3p_2p_3 - 2p_3p_2 = 0$

i.e., $S_5 = 5p_2p_3$.

$$\therefore \frac{s_5}{5} = \frac{s_2}{2} \cdot \frac{s_3}{3}$$

i.e., $\dfrac{a^5+b^5+c^5+d^5}{5} = \dfrac{a^2+b^2+c^2+d^2}{2} \cdot \dfrac{a^3+b^3+c^3+d^3}{3}$.

Example 3. Find $\frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5}$ where α , β , γ are the roots of the equation

$x^3 + 2x^2 - 3x - 1 = 0.$

Solution.

Put $x = \frac{1}{y}$ in the equation, then the equation becomes

$$\frac{1}{y^3}+\frac{2}{y^2}-\frac{3}{y}-1=0$$

i.e., $y^3 + 3y^2 - 2y - 1 = 0$

The roots of the equation are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$$\therefore \frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5} = S_5 \text{ for the equation } y^3 + 3y^2 - 2y - 1 = 0.$$

From Newton's theorem on the sum of the powers of the roots of the equations, we get

$$S_5 + 3S_4 - 2S_3 - S_2 = 0$$

$$S_4 + 3S_3 - 2S_2 - S_1 = 0$$

$$S_3 + 3S_2 - 2S_1 - S_0 = 0$$

$$S_2 + 3S_1 - 4 = 0$$

$$S_1 + 3 = 0.$$

$\therefore$ $S_1 = -3$, $S_2 = 13$, $S_3 = -42$, $S_4 = 149$, $S_5 = -518$.

$$\therefore \frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5} = -518.$$

Transformation in general.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equations f (x) = 0, it is required to find an equation whose roots are

$\phi(\alpha_1), \phi(\alpha_2), \ldots, \phi(\alpha_n)$.

The relation between a root x of f (x) = 0 and a root y of the required equation is $y = \phi(x)$.

Now if x be eliminated between f (x) = 0 and $y = \phi(x)$, an equation in y is obtained which is the required equation.

Example 1. If α , β , γ are the roots of the equation $x^3 + px^2 + qx + r \equiv 0$, from the equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\gamma\alpha}$, $\gamma - \frac{1}{\alpha\beta}$.

Solution.

We have $\alpha - \frac{1}{\beta\gamma}$

$= \alpha - \frac{\alpha}{\alpha\beta\gamma}$

$= \alpha - \frac{\alpha}{-r}$ since $\alpha\beta\gamma = -r$

$= \alpha + \frac{\alpha}{r}$.

$\therefore$ $y = x + \frac{x}{r}$.

$\therefore$ The required equation is obtained by eliminating x between the equations

$$y = x + \frac{x}{r}. \qquad \text{……… (1)}$$

$$x^3 + px^2 + qx + r = 0 \qquad \text{………(2)}$$

From (1) , we get $x = \frac{yr}{1+r}$

Substituting this value of x in the equation (2), we get

$$r^3y^3 + pr(1+r)\,y^2 + q(1+r)^2y + (1+r)^3 = 0.$$

Example 2. If a , b , c be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $bc - a^2$, $ca - b^2$, $ab - c^2$.

Solution.

We have $bc - a^2 = \frac{abc}{a} - a^2$

$= -\frac{r}{a} - a^2$ since $abc = -r$.

Hence the required equation is obtained by eliminating x between the equations

$$y = -\frac{r}{x} - x^2 \qquad \text{........ (1)}$$

and $x^3 + px^2 + qx + r = 0$ ....(2)

From (1) , we get $x^3 + xy + r = 0$ ....(3)

Subtracting (3) from (2) , we get

$$px^2 + qx - xy = 0$$

i.e., $x(px + q - y) = 0$

i.e., $x = 0$ or $px + q - y = 0$.

x cannot be equal to zero.

$\therefore \quad px + q - y = 0$.

TRANSFORMATIONS OF EQUATIONS

1. To form an equation whose roots are k-times the roots of a given equation.

2. To form an equation whose roots are the reciprocals of the roots of a given equation.

3. To form an equation whose roots are less by ‘h’ then the roots of a given equation. (i.e., Diminishing the roots by h)

Remark:
Increasing the roots by h is equivalent to decreasing the roots by –h.

4. To form an equation in which certain specified terms of the given equation are absent.

SOLVED PROBLEMS

1. Form an equation whose roots are three times those of the equation

$$x^3 - x^2 + x + 1 = 0.$$

Solution:

To obtain the required equation, we have to multiply the coefficients of x^3, x^2, x, and 1 by 1, 3, 3^2, and 3^3 respectively.

Thus $x^3 - 3x^2 + 9x + 27 = 0$ is the desired equation.

2. Form an equation whose roots are the negatives of the roots of the equation

Solution:

By multiplying the coefficients successively by 1, -1, 1, -1 we obtain the required equation as $x^3 + 6x^2 + 8x + 9 = 0$.

3. Form an equation whose roots are the reciprocals of the roots of

$$x^4 - 5x^3 + 7x^2 - 4x + 5 = 0.$$

Solution:

We obtain the required equation, by replacing the coefficients in the reverse order, as $5x^4 - 4x^3 + 7x^2 - 5x + 1 = 0$

RECIPROCAL EQUATIONS

An equation remains unaltered if x is changed into 1/x, then the equation is called a reciprocal equation.

Reciprocal roots:

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n = 0.$$

We have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n \equiv (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n).$$

Put $x = \frac{1}{y}$, we have

$$\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \ldots + p_n$$

$$= \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\ldots\left(\frac{1}{y} - \alpha_n\right)$$

Multiplying throughout by y^n, we have

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0$$

$$= (\alpha_1\ \alpha_2 \ldots\ldots \alpha_n)\left(\frac{1}{\alpha_1} - y\right)\left(\frac{1}{\alpha_2} - y\right)\ldots\left(\frac{1}{\alpha_n} - y\right)$$

Hence the equation

$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0$ has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$

Illustrative examples:

1. Solve the equation $60x^4 - 736x^3 + a433x^2 - 736x + 60 = 0$

Solution:

The given equation is a standard reciprocal equation. Dividing throughout by x^2, we obtain,

$$60x^2 - 736x + 1433 - \frac{736}{x} + \frac{60}{x^2} = 0$$

$$60\left(x^2 + \frac{1}{x^2}\right) - 736\left(x + \frac{1}{x}\right) + 1433 = 0$$

Putting $y = x + \frac{1}{x}$ and simplifying, we obtain

$$60y^2 - 736y + 1313 = 0$$

On solving, we get $y = \frac{101}{10} \text{ or } \frac{13}{6}$

When $y = \frac{101}{10}, x + \frac{1}{x} = \frac{101}{10} \Rightarrow 10x^2 - 101x + 10 = 0$

i.e., $x = 10, \frac{1}{10}$

Similarly when $y = \frac{13}{6}$, we get $x = \frac{3}{2}, \frac{2}{3}$

Thus the roots of the given equation are $10, \frac{1}{10}, \frac{3}{2}, \frac{2}{3}$

Example 1. Find the roots of the equation $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$.

Solution.

This is a reciprocal equation of odd degree with like signs.

$\therefore$ (x+1) is a factor of $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1$

The equation can be written as

$$x^5 + x^4 + 3x^4 + 3x^3 + 3x^2 + 3x + x + 1 = 0$$

i.e., $x^4(x+1) + 3x^3(x+1) + 3x(x+1) + 1(x+1) = 0$

i.e., $(x+1)(x^4 + 3x^3 + 3x + 1) = 0$.

$\therefore$ x +1 = 0 or $x^4 + 3x^3 + 3x + 1 = 0$.

Dividing by x^2, we get $x^2 + 3x + \frac{3}{x} + \frac{1}{x^2} = 0$

$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$.

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$

$\therefore$ $z^2 - 2 + 3z = 0$

$\therefore$ $z = \frac{-3 \pm \sqrt{17}}{2}$.

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

i.e., $2x^2 + (-3 + \sqrt{17})\,x + 2 = 0$

or $2x^2 + (-3 - \sqrt{17})\,x + 2 = 0$.

From these equations x can be found.

Example 2. Solve the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.

Solution.

This is a reciprocal equation of odd degree with unlike signs.

Hence x – 1 is a factor of the left- hand side.

The equation can be written as follows:

$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^2 + 5x^2 - 5x + 6x - 6 = 0$

i.e., $6x^4(x - 1) + 5x^3(x - 1) - 38x^2(x - 1) + 5x(x - 1) + 6(x - 1) = 0$

i.e., $(x - 1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore$ $x - 1 = 0$ or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

We have to solve the equation $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

Dividing by x^2, $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

i.e., $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$

Put $x + \frac{1}{x} = z$. $\therefore\ x^2 + \frac{1}{x^2} = z^2 - 2.$

The equation becomes

$$6(z^2 - 2) + 5z - 38 = 0$$

i.e., $6z^2 + 5z - 50 = 0$

i.e., $(2z - 5)\,(3z + 10) = 0.$

$\therefore\ \ x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = -\frac{10}{3}$

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 + 10x + 3 = 0$

i.e., $(2x - 1)\,(x - 2) = 0$ or $(3x + 1)\,(x + 3) = 0$

i.e., $x = \frac{1}{2}$ or 2 or $-\frac{1}{3}$ or $-3.$

$\therefore$ The roots of the equation are $1, \frac{1}{2}, 2, -\frac{1}{3}$ and -3.

Example 3. Solve the equation $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0.$

Solution.

There is no mid-term and this is a reciprocal equation of even degree with unlike signs. We can easily see that $x^2 - 1$ is a factor of the expression on left-hand side of the equation.

The equation can be written as

$$6(x^6 - 1) - 35x(x^4 - 1) + 56x^2(x^2 - 1) = 0$$

i.e., $6(x^2 - 1)(x^4 + x^2 + 1) - 35x(x^2 - 1) + (x^2 + 1)\,) + 56x^2(x^2 - 1) = 0$

i.e., $(x^2-1)(6x^4-35x^3+62x^2-35x+6)=0$

i.e., $x = 1$ or -1 or $6x^4-35x^3+62x^2-35x+6=0$.

Dividing by x^2, we get $6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$.

$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$.

Put $x+\frac{1}{x}=z$. $\therefore$ $x^2+\frac{1}{x^2}=z^2-2$.

$\therefore \quad 6(z^2-2)-35z+62=0$

i.e., $\quad 6z^2-35z-50=0$

i.e., $\quad (3z-10)(2z-5)=0$

$z=\frac{10}{3}$ or $\frac{5}{2}$.

$\therefore \quad x+\frac{1}{x}=\frac{10}{3}$ or $x+\frac{1}{x}=\frac{5}{2}$

i.e., $3x^2-10x+3=0$ or $2x^2-5x+2=0$

i.e., $(x-3)(3x-1)=0$ or $(x-2)(2x-1)=0$

i.e., $x=3$ or $\frac{1}{3}$ or 2 or $\frac{1}{2}$

$\therefore$ The roots of the equation are $1, -1, 3, \frac{1}{3}, 2$ and $\frac{1}{2}$.

Exercises

Solve the following equations:-

1. $x^4-10x^3+26x^2-10x+1=0$.
2. $x^4+3x^3-3x-1=0$.
3. $2x^6-9x^5+10x^4-3x^3+10x^2-9x+2=0$.
4. $2x^5+x^4+x+1=12x^2(x+1)$.
5. $x^5-5x^3+5x^2-1=0$.

UNIT – III – TRANSFORMATION OF EQUATIONS

Solved Problems

☐1.- **Find the polynomial equation whose roots are the translates of those of**

$x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$ by 3

Sol: Given equation is

$$f(x) = x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$$

Required equation is $f(x+3) = 0$

$$(x+3)^5 - 4(x+3)^4 + 3(x+3)^2$$

$$-4(x+3) + 6 = 0$$

3	1	-4	0	3	-4	6
	0	3	-3	-9	-18	-66
	1	-1	-3	-6	-22	-60
	0	3	6	9	9	A_5
	1	2	3	3	-13	
	0	3	15	54	A_4	
	1	5	18	57		
	0	3	24	A_3		
	1	8	42			
	0	3	A_3			
	1	11				
	A_0	A_1				

Required equation is

$$x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$$

2. Find the polynomial equation whose roots are the translates of the roots of the equation

$x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2

Sol:

Given $f(x) = x^4 - x^3 - 10x^2 + 4x + 24 = 0$

Required equation is $f(x-2) = 0$

$(x-2)^4 + (x-2)^3 - 10(x-2)^2$

-2	1	-1	-10	4	24	
	0	-2	6	8	-24	
	1	-3	-4	12	0	A_4
	0	-2	10	-12		
	1	-5	6	0	A_3	
	0	-2	14			
	1	-7	20			
	0	-2	A_2			
	1	-9				
	A_0	A_1				

Required equation is

$x^4 - 9x^3 + 20x^2 = 0$

3. Find the polynomial equation whose roots are the translates of the equation $3x^5 + 5x^3 + 7 = 0$ by 4

sol:

Given $f(x) = 3x^5 - 5x^3 + 7 = 0$

Required equation is $f(x-4) = 0$

$3(x-4)^5 - 5(x-4)^3 + 7 = 0$

-4	3	0	-5	0	0	7
	0	-12	48	-172	688	-2752
	3	-12	43	-172	688	-2745
	0	-12	96	-556	2912	A_5
	3	-24	139	-728	3600	
	0	-12	144	-1132	A_4	
	3	-36	283	-1860		
	0	-12	192	A_3		
	3	-48	475			
	0	-12	A_2			
	3	60				
	A_0	A_1				

4. Find the equation whose roots are less by 2, than the roots of the equation

$x^5 - 3x^4 - 2x^3 + 15x^2 + 20x + 15 = 0$.

Solution:

To find the desired equation, divide the given equation successively by $x - 2$.

2	1	-3	-2	+15	+20	+15
		2	-2	-8	+14	68
	1	-1	-4	+7	+34	83
		2	+2	-4	+6	
	1	+1	-2	+3	+40	
		+2	+6	+8		
	1	+3	+4	+11		
		2	+10			
	1	+5	+14			
		+2				
	1	+7				
	1					

5. Solve the equation $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ by removing its second term.

Solution:

To remove the second term, we have to diminish the roots of the given equation by $h = \frac{-a_1}{na_0} = \frac{8}{4.1} = 2$.

Dividing the given equation successively by $x - 2$, we obtain the new equation as

$x^4 - 25x^2 + 144 = 0$

On solving, we get $x = -4, 4, -3, 3$.

Thus the roots of the original equation are $-2, 6, -1$ and 5.

PROBLEMS BASED ON REMOVAL OF SECOND TERMS

1. Transform each of the following equations into ones in which of the coefficients of thesecond highest power of x is zero and also find their transformed equations.

(i) $x^3 - 6x^2 + 10x - 3 = 0$

Given equation is $x^3 - 6x^2 + 10x - 3 = 0$.To remove the second term diminish the roots,

$$\text{By} - \frac{a_1}{na_0} = \frac{6}{3} = 2$$

2	1	-6	10	-3
	0	2	-8	$+4$
	1	-4	2	$+1$
	0	2	-4	A_3
	1	-2	-2	
	0	$+2$	A_2	
	1	0		
	A_0	A_1		

Required equation is $x^3 - 2x + 1 = 0$

(ii) $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$

- **Sol:** Given equation is $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$

  Diminishing the roots by $-\dfrac{a_1}{na_0} = \dfrac{-4}{4} = -1$

-1	1	4	2	-4	-2
	0	-1	-3	1	3
	1	3	-1	-3	1
	0	-1	-2	3	A_4
	1	2	-3	0	
	0	-1	-1	A_3	
	1	1	-4		
	0	-1	A_2		
	1	0			
	A_0	A_1			

Required equation is $x^4 - 4x^2 + 1 = 0$

(iii) $x^3 + 6x^2 + 4x + 4 = 0$

Sol: Given equation is $x^3 + 6x^2 + 4x + 4 = 0$

To remove the second term diminish the roots by

-2	1	6	4	4
	0	-2	-8	8
	1	4	-4	12
	0	-2	-4	A_3
	1	2	-8	
	0	-2	A_2	
	1	0		
	A_0	A_1		

Required equation is $x^3 - 8x + 12 = 0$

PROBLEMS ON SQUARES OF THE ROOTS

Find the polynomial equation whose roots are the squares of the roots of

$x^3 - x^2 + 8x - 6 = 0$

Sol: Let $f(x) \equiv x^3 - x^2 + 8x - 6$

The required equation is $f(\sqrt{x}) = 0$

i.e., $\left(\sqrt{x}\right)^3 - \left(\sqrt{x}\right)^2 + 8\sqrt{x} - 6 = 0$

$\Rightarrow x\sqrt{x} + x + 8\sqrt{x} - 6 = 0$

$\Rightarrow \sqrt{x}(x+8) = x + 6$

Squaring on both sides

$\Rightarrow x\left(x^2 + 16x + 64\right) = x^2 + 12x + 36$

$\Rightarrow x^3 + 16x^2 + 64x - x^2 - 12x - 36 = 0$

$\therefore x^3 + 15x^2 + 52x - 36 = 0$

Descartes' Rule of signs.

An equation f (x) = 0 cannot have more positive roots than there are changes of sign in f (x)

Let f(x) be a polynomial whose signs of the terms are

+ + - - - + - + + + - + -.

In this there are seven changes of sign including changes from + to – and from – to +. We shall show that if this polynomial be multiplied by a binomial (corresponding to a positive root) whose signs of the terms are + – , the resulting polynomial will have atleast one more change of sign than the original. Writing down only the signs of the terms in the multiplication, we have

$$
\begin{array}{r}
+\,+\,-\,-\,-\,+\,-\,+\,+\,+\,-\,+\,- \\
+\,- \\
\hline
-\,-\,+\,+\,+\,-\,+\,-\,-\,-\,+\,-\,+ \\
+\,+\,-\,-\,-\,+\,-\,+\,+\,+\,-\,+\,- \\
\hline
+\,\pm\,-\,\pm\,\pm\,+\,-\,+\,\pm\,\pm\,-\,+\,-\,+
\end{array}
$$

Here in the last line the ambiguous sign $\pm$ is placed wherever there are two different signs to be added. Here we see in the product

(1) An ambiguity replaces each continuation of sign in the original polynomial.

(2) The sign before and after an ambiguity or a set of ambiguities are unlike and

(3) A change of sign is introduced in the end.

Descartes' rule of signs for negative roots.

Let $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$.

By subtracting $-x$ instead of x in the equations, we get

$$f(-x) = (-x - \alpha_1)(-x - \alpha_2) \ldots\ldots (-x - \alpha_n).$$

The roots of $f(-x) = 0$ are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

Hence to find the maximum number of negative roots of $f(x) = 0$, it is enough to find the maximum number of positive roots of $f(-x) = 0$.

So we can enunciate Descartes' rule for negative roots as follows.

Example. Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$.

Solution.

The series of signs of the terms are $+ - - +$.

Here there are two changes of sign.

Hence there cannot be more than two positive roots.

Changing x into $-x$, the equation becomes

$$-x^5 - 6x^2 + 4x + 5 = 0$$

i.e., $x^5 + 6x^2 - 4x - 5 = 0$.

The series of the signs of the terms are

$+ + - -$.

Here there is only one change of sign.

$\therefore$ There cannot be more than one negative root.

So the equation has got at the most three real roots. The total number of roots of the equation is 5. Hence there are at least two imaginary roots of the equation. We can also determine the limits between which the real roots lie.

$x =$	$-\infty$	-2	-1	0	1	2	∞
$x^5 - 6x^2 - 4x + 5 =$	$-$	$-$	$+$	$+$	$-$	$+$	$+$

The positive roots lie between 0 and 1, and 1 and 2, the negative root between -2 and -1.

Example:

Consider the equation $f(x) = x^4 + 3x - 1 = 0$

This a polynomial equations of degree 4, and hence must have four roots.

The signs of the coefficients of $f(x)$ are $+ + -$

Therefore, the number of changes in signs = 1

By Descarte's rule of signs, number of real positive roots ≤ 1.

Now $f(-x) = x^4 - 3x - 1 = 0$

The signs of the coefficients of $f(-x)$ are $+ - -$

Therefore, the number of changes in signs = 1.

Hence the number of real negative roots of $f(x) = 0$ is ≤ 1.

Therefore, the maximum number of real roots is 2.

If the equation has two real roots, then the other two roots must be complex roots. Since complex roots occur in conjugate pairs, the possibility of one real root and three complex roots is not admissible.

Also $f(0) < 0$, and $f(1) > 0$, so $f(x) = 0$ has a real roots between 0 and 1.

Therefore, the given equation must have two real roots and two complex roots.

Problem.

Discuss the nature of roots of the equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0$.

Solution.

With $f(x) = x^9 + 5x^8 - x^3 + 7x + 2$, there are two changes of sign in $f(x) = 0$, and therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2$, and there are three changes of sign, therefore the given equation has at most three negative roots.

Obviously 0 is not a root of the given equation.

Hence the given equation has at most 2 + 3 + 0 = 5 real roots. Thus the given equation has at least four imaginary roots.

UNIT – IV– ELEMENTARY CONCEPTS

INTRODUCTION

1.1 Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $$\begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{12} \cdots\cdots a_{2n} \\ \cdots\cdots\cdots & \\ \cdots\cdots\cdots & \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{mxn} \quad \text{where } 1 \le i \le m,\ 1 \le j \le n.$$

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by mxn, where m is number of rows and n is number of columns.

1.2 Types of Matrices:

Row Matrix: A Matrix having only one row is called a "Row Matrix".

Eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

Column Matrix: A Matrix having only one column is called a "Column Matrix" .

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

Null Matrix: A= $\left[a_{ij}\right]_{mxn}$ such that $a_{ij}=0$ $\forall$ i and j. Then A is called a "Zero Matrix". It is denoted by Omxn.

Eg: $O_{2x3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rectangular Matrix: If A=$\left[a_{ij}\right]_{mxn}$, and $m \neq n$ then the matrix A is called a "Rectangular Matrix"

Eg : $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

Square Matrix: If A=$\left[a_{ij}\right]_{mxn}$ and m = n then A is called a "Square Matrix".

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is a 2x2 matrix

Lower Triangular Matrix: A square Matrix $A_{nXn} = \left[a_{ij}\right]_{nxn}$ is said to be lower Triangular of $a_{ij} = 0$ if i <j i.e. if all the elements above the principle diagonal are zeros.

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is a Lower triangular matrix

Upper Triangular Matrix: A square Matrix $A = \left[a_{ij}\right]_{nxn}$ is said to be upper triangular of $a_{ij} = 0$ if i.>j. i.e. all the elements below the principle diagonal are zeros.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an Upper triangular matrix

Triangle Matrix: A square matrix which is either lower triangular or upper triangular is called a triangle matrix.

Principal Diagonal of a Matrix: In a square matrix, the set of all a_{ij}, for which i = j are called principal diagonal elements. The line joining the principal diagonal elements is called principal diagonal.

Note: Principal diagonal exists only in a square matrix.

Diagonal elements in a matrix: A= $[a_{ij}]_{n\times n}$, the elements a_{ij} of A for which i = j.

i.e. a_{11}, a_{22}….a_{nn} are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

Diagonal Matrix: A Square Matrix is said to be diagonal matrix, if $a_{ij} = 0$ for $i \neq j$ i.e. all the elements except the principal diagonal elements are zeros.

Note: 1. Diagonal matrix is both lower and upper triangular.

2. If d_1, d_2……d_n are the diagonal elements in a diagonal matrix it can be represented as **diag** $[d_1, d_2, \ldots\ldots, d_n]$

Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Scalar Matrix: A diagonal matrix whose leading diagonal elements are equal is called a "Scalar Matrix". Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit/Identity Matrix: If A = $[a_{ij}]_{n\times n}$ such that a_{ij}=1 for i = j, and a_{ij}=0 for $i \neq j$ then A is called a "Identity Matrix" or Unit matrix. It is denoted by I_n

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.3 Trace of Matrix: The sum of all the diagonal elements of a square matrix A is called Trace of a matrix A, and is denoted by Trace A or tr A.

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then tr A = a+b+c

1.4 Singular & Non Singular Matrices: A square matrix A is said to be "Singular" if the determinant of $|A| = 0$, Otherwise A is said to be "Non-singular".

Note: 1. Only non-singular matrices possess inverse.

2. The product of non-singular matrices is also non-singular.

1.5 Inverse of a Matrix: Let A be a non-singular matrix of order n if there exist a matrix B such that AB=BA=I then B is called the inverse of A and is denoted by A^{-1}.

If inverse of a matrix exist, it is said to be invertible.

Note: 1. The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$.

2 .Every Invertible matrix has unique inverse.

3. If A, B are two invertible square matrices then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

4. $A^{-1} = \dfrac{AdjA}{\det A}$ $\quad where\ detA \neq 0$,

1.6 Transpose of a Matrix: The matrix obtained by interchanging rows and columns of the given matrix A is called as transpose of the given matrix A. It is denoted by A^T or A^1

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of transpose of a matrix: If A and B are two matrices and A^T, B^T are their transposes then

1) $(A^T)^T = A$; 2) $(A+B)^T = A^T + B^T$; 3) $(KA)^T = KA^T$; 4) $(AB)^T = B^T A^T$

1.7 Symmetric Matrix: A square matrix A is said to be symmetric if $A^T = A$

If $A = [a_{ij}]_{nxn}$ $then\ A^T = [a_{ji}]nxn$ where $a_{ij} = a_{ji}$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

1.8 Skew-Symmetric Matrix: A square matrix A is said to be Skew symmetric If $A^T = -A$.

If $A = \left[a_{ij}\right]_{m\times n}$ then $A^T = \left[a_{ji}\right]_{n\times m}$ where $a_{ij} = -a_{ji}$

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Note: All the principle diagonal elements of a skew symmetric matrix are always zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

1.9 Theorem: Every square matrix can be expressed uniquely as the sum of symmetric and skew symmetric matrices.

Proof: Let A be a square matrix, $A = \frac{1}{2}(A + A) = \frac{1}{2}\left(A + A^T + A - A^T\right) =$

$\frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = P + Q$, where $P = \frac{1}{2}\left(A + A^T\right)$; $Q = \frac{1}{2}\left(A - A^T\right)$

Thus every square matrix can be expressed as a sum of two matrices.

Consider $P^T = \left[\frac{1}{2}\left(A + A^T\right)\right]^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left(A^T + \left(A^T\right)^T\right) = \frac{1}{2}\left(A + A^T\right) = P$, since $P^T = P$,

P is symmetric

Consider $Q^T = \left[\frac{1}{2}\left(A - A^T\right)\right]^T = \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left(A^T - \left(A^T\right)^T\right) = -\frac{1}{2}\left(A - A^T\right) = -Q$

Since $Q^T = -Q$, Q is Skew-symmetric.

To prove the representation is unique: Let $A = R + S \rightarrow (1)$ be the representation, where R is symmetric and S is skew symmetric. i.e. $R^T = R, S^T = -S$

Consider $A^T = (R + S)^T = R^T + S^T = R - S \rightarrow (2)$

$(1) - (2) \Rightarrow A - A^T = 2S \Rightarrow S = \frac{1}{2}\left(A - A^T\right) = Q$

Therefore every square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix

Ex 1. Express the given matrix A as a sum of a symmetric and skew symmetric matrices where $A = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 3 & 11 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} \Rightarrow P = \frac{1}{2}\left(A + A^T\right) = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}; P \text{ is symmetric}$$

$$A - A^T = \begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} \Rightarrow Q = \frac{1}{2}\left(A - A^T\right) = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}; Q \text{ is skew} - \text{symmetric}$$

Now A =P+Q = $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

1.10 Orthogonal Matrix: A square matrix A is said to be an Orthogonal Matrix if $AA_T=A_TA=I$, Similarly we can prove that $A=A^{-1}$; Hence A is an orthogonal matrix.

Note: 1. If A, B are orthogonal matrices, then AB and BA are orthogonal matrices.

2. Inverse and transpose of an orthogonal matrix is also an orthogonal matrix.

Result: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Result: The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal

Solved Problems :

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.**

Sol: Given $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A\,.A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow A.A^T = I$

Similarly $A^T\,.A = I$

Hence A is orthogonal matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol: - For orthogonal matrix $AA^T = I$

So, $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$ $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ (since $c^2=2b^2$) $\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}$; $b = \pm\frac{1}{\sqrt{6}}$; $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

4. Is matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ Orthogonal?

Sol:- Given $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$\Rightarrow AA^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$

$AA^T \neq A^TA \neq I_3$

$\therefore$ Matrix is not orthogonal.

1.11 Complex matrix: A matrix whose elements are complex numbers is called a complex matrix.

1.12 Conjugate of a complex matrix: A matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called conjugate of a complex matrix. It is denoted by $\overline{A}$

If $A = [a_{ij}]_{m\times n}$, $\overline{A} = [\overline{a_{ij}}]_{m\times n}$, where $\overline{a_{ij}}$ is the conjugate of a_{ij}.

Eg: If A= $\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Note: If $\overline{A}$ and $\overline{B}$ be the conjugate matrices of A and B respectively, then

(i) $\overline{(\overline{A})}$=A (ii) $\overline{A+B} = \overline{A}+\overline{B}$ (iii) $\overline{(KA)} = \overline{K}\,\overline{A}$

1.13 Transpose conjugate of a complex matrix: Transpose of conjugate of complex matrix is called transposed conjugate of complex matrix. It is denoted by A^θ or A^*.

Note: If A^θ and B^θ be the transposed conjugates of A and B respectively, then

(i) $(A^\theta)^\theta = A$ (ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$

(iii) $(KA)^\theta = \overline{K}A^\theta$ (iv) $(AB)^\theta = A^\theta B^\theta$

1.14 Hermitian Matrix: A square matrix A is said to be Hermitian Matrix iff $A^\theta = A$

Eg: A= $\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $A^\theta = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Note: 1. In Hermitian matrix the principal diagonal elements are real.

2. The Hermitian matrix over the field of Real numbers is nothing but real symmetric matrix.

3. In Hermitian matrix A= $[a_{ij}]_{n\times n}$, $a_{ij} = \overline{a_{ji}}\ \forall i, j$.

1.15 Skew-Hermitian Matrix: A square matrix A is said to be Skew-Hermitian Matrix if $A^\theta = -A$.

Eg: Let A= $\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $(\overline{A})^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore (\overline{A})^T = -A$ $\therefore$ A is skew-Hermitian matrix.

Note: 1. In Skew-Hermitian matrix the principal diagonal elements are either Zero or Purely Imaginary.

2. The Skew- Hermitian matrix over the field of Real numbers is nothing but real Skew - Symmetric matrix.

3. In Skew-Hermitian matrix A= $\left[a_{ij}\right]_{n\times n}$, $a_{ij}=-\overline{a_{ji}}\ \forall i,j$.

1.16 Unitary Matrix: A Square matrix A is said to be unitary matrix if

$AA^{\theta}=A^{\theta}A=I \text{ or } A^{\theta}=A^{-1}$

Eg: $B=\frac{1}{6}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}$

1.17 Theorem 1: Every square matrix can be uniquely expressed as a sum of Hermitian and skew – Hermitian Matrices.

Proof: - Let A be a square matrix write
$$A=\frac{1}{2}(2A)=\frac{1}{2}(A+A)=\frac{1}{2}(A+A^{\theta}+A-A^{\theta})$$
$$A=\frac{1}{2}(A+A^{\theta})+\frac{1}{2}(A-A^{\theta}) \text{ i.e } A=P+Q$$

Let $P=\frac{1}{2}\left(A+A^{\theta}\right);Q=\frac{1}{2}\left(A-A^{\theta}\right)$

Consider $P^{\theta}=\left[\frac{1}{2}(A+A^{\theta})^{\theta}\right]=\frac{1}{2}(A+A^{\theta})^{\theta}=(A+A^{\theta})=P$

I.e. $P^{\theta}=P, P$ is Hermitian matrix.

$Q^{\theta}=\left[\frac{1}{2}(A-A^{\theta})\right]^{\theta}=\frac{1}{2}\left(A^{\theta}-A\right)=-\frac{1}{2}(A-A^{\theta})=-Q$

Ie $Q^{\theta}=-Q, Q$ is skew – Hermitian matrix.

Thus every square matrix can be expressed as a sum of Hermitian & Skew Hermitian

Solved Problems :

1. If A= $\begin{bmatrix}3 & 7-4i & -2+5i\\ 7+4i & -2 & 3+i\\ -2-5i & 3-i & 4\end{bmatrix}$ **then show that A is Hermitian and iA is skew-Hermitian.**

Sol: Given A= $\begin{bmatrix}3 & 7-4i & -2+5i\\ 7+4i & -2 & 3+i\\ -2-5i & 3-i & 4\end{bmatrix}$ then

$\overline{A}=\begin{bmatrix}3 & 7+4i & -2-5i\\ 7-4i & -2 & 3-i\\ -2+5i & 3+i & 4\end{bmatrix}$ And $\overline{(A)}^{T}=\begin{bmatrix}3 & 7-4i & -2+5i\\ 7+4i & -2 & 3+i\\ -2-5i & 3-i & 4\end{bmatrix}$

$\therefore A=\left(\overline{A}\right)^{T}$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^T$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^T = A$ And $\left(\overline{B}\right)^T = B$------------- (1)

$$\text{Now } \overline{(AB-BA)}^T = \left(\overline{AB}-\overline{BA}\right)^T$$

$$= \left(\overline{A}\overline{B}-\overline{B}\overline{A}\right)^T$$

$$= \left(\overline{A}\overline{B}\right)^T - \left(\overline{B}\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right.$$

$$= BA - AB \text{ (By (1))}$$

$$= -(AB - BA)$$

Hence AB-BA is a skew-Hemitian matrix.

3. Show that A= $\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Sol: Given A= $\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^\theta = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

4. Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I-A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$=\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$ And

$I+A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$

$\therefore (I+A)^{-1}=\dfrac{1}{1-(4i^2-1)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$=\dfrac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

Let $B=(I-A)(I+A)^{-1}$

$B=\dfrac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\dfrac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$

$B=\dfrac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

Now $\overline{B}=\dfrac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\left(\overline{B}\right)^T=\dfrac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$B\left(\overline{B}\right)^T=\dfrac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$\left(\overline{B}\right)^T=B^{-1}$

i.e. B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

1.18 Rank of a Matrix:

Let A be mxn matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that '**r**' is the rank of A if

i. Every $(r+1)^{th}$ order minor of A is '0' (zero) &
ii. At least one r^{th} order minor of A which is not zero.

It is denoted by ρ (A) and read as rank of A.

Note: 1. Rank of a matrix is unique.

2. Every matrix will have a rank.
3. If A is a matrix of order mxn, then Rank of A $\le$ min (m,n)
4. If ρ (A) = r then every minor of A of order r+1, or minor is zero.
5. Rank of the Identity matrix I_n is n.
6. If A is a matrix of order n and A is non-singular then ρ (A) = n
7. If A is a singular matrix of order n then ρ (A) < n

Important Note:

1. The rank of a matrix is $\le$ r if all minors of $(r+1)^{th}$ order are zero.
2. The rank of a matrix is $\ge$ r, if there is at least one minor of order 'r' which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

det A = 1(48-40)-2(36-28)+3(30-28) = 8-16+6 = -2 $\neq$ 0

We have minor of order 3 $\therefore \rho$ (A) =3

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\le$ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56) + 3(35-48) = -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

1.19 Elementary Transformations on a Matrix:

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow kR_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + kR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \rightarrow k\,c_j$ $c_j \rightarrow c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

1.20 Equivalance of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

1.21 Elementary Matrix or E-Matrix: A matrix is obtained from a unit matrix by a single elementary transformation is called elementary matrix or E-matrix.

Notations: We use the following notations to denote the E-Matrices.

1) $E_{ij} \rightarrow$ Matrix obtained by interchange of i^{th} and j^{th} rows (columns).

2) $E_{i(k)} \rightarrow$ Matrix obtained by multiplying ith row (column) by a non- zero number k.

3) $E_{ij(k)} \rightarrow$ Matrix obtained by adding k times of j^{th} row (column) to ith row (column).

1.22 Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i) Zero rows, if any exists, they should be below the non-zero row.

(ii) The first non-zero entry in each non-zero row is equal to '1'.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note : 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

1. The rank of the transpose of a matrix is the same as that of original matrix.
2. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **is a row echelon form.**

2. $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **is a row echelon form.**

1. Determine rank of the matrix. A if

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 - 7R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_1 - R_2,\ R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$R_3 \times \frac{1}{28}$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 +\ 32\,R_3,\ \ R_2 -\ 33\,R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sim [I_3 \quad o]$

$\therefore$ **Rank of A=3**

2. Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim [I_2 \quad 0]$$

$\therefore$ **Rank of A= 2**

3). **Find the rank of the matrix using echelon form** $A=\begin{bmatrix}2&1&3&5\\4&2&1&3\\8&4&7&13\\8&4&-3&-1\end{bmatrix}$

Sol: Given $A=\begin{bmatrix}2&1&3&5\\4&2&1&3\\8&4&7&13\\8&4&-3&-1\end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1\,; R_3 \to R_3 - 4R_1\,; R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix}2&1&3&5\\0&0&-5&-7\\0&0&-5&-7\\0&0&-15&-21\end{bmatrix}$

$R_1 \to \dfrac{R_1}{-1}, R_2 \to \dfrac{R_2}{-1}, R_3 \to \dfrac{R_3}{-3}$ $A \sim \begin{bmatrix}2&1&3&5\\0&0&5&7\\0&0&5&7\\0&0&5&7\end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ $A \sim \begin{bmatrix}2&1&3&5\\0&0&5&7\\0&0&0&0\\0&0&0&0\end{bmatrix}$

$\Rightarrow$ A is in echelon form $\therefore$ Rank of A = 2

5. **Reduce the matrix** $\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$ **into echelon form and hence find its rank.**

Sol: Let us consider given matrix to be A

$$\Rightarrow A=\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$ $\begin{bmatrix}1&2&3&0\\0&0&-3&2\\0&-4&-8&3\\0&-4&-11&5\end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&-4&-11&5\end{bmatrix}$

$R_4 \to R_4 - R_2$ $\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&0&-3&2\end{bmatrix}$

$R_4 \to R_4 - R_3 \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, this is in Echelon form and the number of non-zero rows is 3

Hence, $\rho(A) = 3$

1.23 Normal form/Canonical form of a Matrix:

Every non-zero Matrix can be reduced to any one of the following forms. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}; [I_r \quad 0]; \begin{bmatrix} I_r \\ 0 \end{bmatrix}; [I_r]$ Known as normal forms or canonical forms by using Elementary row or column or both transformations where I_r is the unit matrix of order 'r' and 'O' is the null matrix.

Note: 1.In this form "the rank of a matrix is equal to the order of an identity matrix.

2. Normal form another name is "canonical form"

Solved Problems :

1. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **into normal form, find its rank.**

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$

$R_3 \to R_3/-2 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$

$R_3 \to R_3+R_2 \qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_2 \to c_2 - 2c_1,\ c_3 \to c_3 - 3c_1,\ c_4 \to c_4 - 4c_1$ $\quad A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4 - 5c_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

$c_2 \to c_2/-3,\ c_4 \to c_4/18$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$c_4 \leftrightarrow c_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is in normal form $[I_3\ 0]$, $\therefore$ Hence Rank of A is '3'.

UNIT – V– MATRICES

MATRICES

RANK OF A MATRIX

Let A be any matrix of order mxn.The determinants of the sub square matrices of A are called the minors of A. If all the minors of order **(r+1)** are zero but there is at least one non zero minor of order **r,** then **r** is called the rank of A and is written as R(A). For an mxn matrix,

- If m is less than n then the maximum rank of the matrix is m
- If m is greater than n then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

Example

1. **Find the rank of A =** $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$|A|$ = 1(20-12) -2(5-4) +3(6-8) = 0

Hence R(A) < 3.

Let the second order minor $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ =2 $\neq$ 0

R(A)=2.

2. **Find the Rank of B =** $\begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$ $R_2 = R_2 - 2R_1$, $R_3 = R_3 - 3R_1$, $R_4 = R_4 - 6R_1$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix} \qquad R_2 = 1/5R_2,\ R_3 = R_3,\ R_4 = R_4$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{pmatrix} \qquad R_3 = R_3 - 4R_2,\ R_4 = R_4 - 9R_2$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_4 = R_4 - R_3$$

The number of Nonzero Rows is 3.Hence R(B)=3.

3. Find the Rank of the Matrix A = $\begin{pmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{pmatrix} \quad R_1 = R_2,\ R_2 = R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{pmatrix} \quad R_2 = R_2 - 2R_1,\ R_3 = R_3 - 4R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{pmatrix} \quad R_3 = R_3 - R_2$$

The number of Nonzero Rows is 3.   Hence R(A)=3.

4. Find the Rank of the Matrix A = $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \quad R_3 = R_3 - R_2$$

The number of Nonzero Rows is 3. Hence R(A)=3.

5. Find the Rank of the Matrix B = $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

A possible minor of least order is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{pmatrix}$ whose determinant is non zero.

Hence it is possible to find a nonzero minor of order 3.

Hence R(B)=3.

<u>CONSISTENCY OF LINEAR ALGEBRAIC EQUATION</u>

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

Denoting the matrices by A, X, and C, the system of equation is, AX = C where A is called the coefficient matrix, C is called the right hand side vector and X is called the solution vector. Sometimes AX=C systems of equations are written in the augmented form. That is

$$[\mathrm{A}:\mathrm{C}] = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} & \vdots\, c_1 \\ a_{21} & a_{22} & \dots\dots & a_{2n} & \vdots\, c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} & \vdots\, c_m \end{bmatrix}$$

<u>Rouche'sTheorem</u>

1. A system of equations AX =C is **consistent** if the rank of A is equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$. If in addition, the rank of the coefficient matrix A is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix A is less than the number of unknowns, then infinite solutions exist.

2. A system of equations AX=C is **inconsistent** if the rank of A is not equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$.

[A] [X] = [B]

- Consistent System if rank (A) = rank (A.B)
  - Unique solution if rank (A) = number of unknowns
  - Infinite solutions if rank (A) < number of unknowns
- Inconsistent System if rank (A) < rank (A.B)

Problems

1. **Check whether the following system of equations**

   $25x_1 + 5x_2 + x_3 = 106.8$

   $64x_1 + 8x_2 + x_3 = 177.2$

   $89x_1 + 13x_2 + 2x_3 = 280$ is consistent or inconsistent.

<u>Solution</u>

The augmented matrix is

$$[A:B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :280.0 \end{bmatrix}$$

To find the rank of the augmented matrix consider a square sub matrix of order 3×3 as

$$\begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$$ whose determinant is 12. Hence R $[A:B]$ is 3.

So the rank of the augmented matrix is 3 but the rank of the coefficient matrix $[A]$ is 2 as the Determinant of A is zero. Hence R$[A:B] \neq$ R [A].Hence the system is inconsistent.

2. Check the consistency of the system of linear equations and discuss the nature of the solution?

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Solution

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 4R_1,\ R_4 = R_4 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 / -5$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 + 11R_1$$

Here R$[A:B]$=R[A] =3.Hence the system is consistent. Also R[A] is equal to the number of unknowns. Hence the system has an unique solution.

3. Check whether the following system of equations is a consistent system of equations. Is the solution unique or does it have infinite solutions

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10$$

Solution

The given system has the augmented matrix given by

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix} \quad R_2 = R_2 - R_1,\ R_3 = R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2$$

A and $[A:B]$ are each of rank r = 3, the given system is consistent but R[A] is not equal to the number of unknowns. Hence the system does not has a unique solution.

4. Check whether the following system of equations

$3x - 2y + 3z = 8$
$x + 3y + 6z = -3$
$2x + 6y + 12z = -6$

is a consistent system of equations and hence solve them.

Solution

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 3 & -2 & 3 & 8 \\ 1 & 3 & 6 & -3 \\ 2 & 6 & 12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 3 & -2 & 3 & 8 \\ 2 & 6 & 12 & -6 \end{bmatrix} \quad R_1 = R_2 \ \ R_2 = R_1$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 0 & 11 & 15 & -17 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 2R_1$$

R[A:B] = R[A] = 2**.**Therefore the system is consistent and posses solution but rank is not equal to the number of unknowns which is 3.Hence the system has infinite solution. From the upper triangular matrix we have the reduced system of equations given by

$$x + 3y + 6z = -3 \; ; \; 11y + 15z = -17 .$$

By assuming a value for y we have one set of values for z and x.For example when y=3, z = $-10/3$ and x = 8.Similarly by choosing a value for z the corresponding y and x can be calculated. Hence the system has infinite number of solutions.

5. Check whether the following system of equations

$$\begin{aligned} X+y+z &= 6 \\ 3x-2y+4z &= 9 \\ x-y-z &= 0 \end{aligned}$$

Is a consistent system of equations and hence solve them.

<u>Solution</u>

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & 4 & 9 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & 1 & -9 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2=R_2-3R_1 , R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2 = R_2/-5$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & -12/5 & -12/5 \end{bmatrix} \quad R_3 = R_3+2R_2$$

Hence R[A B] =R[A] =3 which is equal to the number of unknowns. Hence the system is consistent with unique solution. Now the system of equations takes the form

$$x+y+z = 6; \quad y-z/5 = 9/5; \quad -12/5\,z = -12/5.$$

Hence z =1. Substituting z = 1 in y-z/5 =9/5 we have y-1/5 = 9/5 or y = 1/5+9/5 = 10/5.

Hence y =2. Substituting the values of y,z in x+y + z = 6 we have x= 3. Hence the system has the unique solution as x= 3 , y =2 , z =1.

<u>CHARACTERISTIC EQUATION</u>

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

If X_r be the non-zero vector satisfying AX = λX, when$\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

Working rule to find characteristic equation:

For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where S_1 = sum of the main diagonal elements, S_2 = sum of the minors of the main diagonal elements, S_3 = Determinant of A = $|A|$

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where S_1 = sum of the main diagonal elements, S_2 = Determinant of A = $|A|$

1. **Find the characteristic equation of** $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where S_1 = sum of the main diagonal elements = 8 + 7 + 3 = 18,

S_2 = sum of the minors of the main diagonal elements=45

S_3 = Determinant of A = $|A|$=0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

2. **Find the characteristic equation of** $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equationl of A is $\lambda^2 - S_1\lambda + S_2 . S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic equation is $\lambda^2 - 5\lambda + 7$ =0.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX

Working rule to find Eigen values and Eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called Eigen values
3. To find the Eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct Eigen values, we get n independent Eigen vectors
2. If 2 or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to the repeated Eigen values
3. If X_i is a solution for an Eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the Eigen vector corresponding to an Eigen value is not unique but may be any one of the vectors cX_i

Problems

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

   Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sum of the main diagonal elements = 1 - 1 = 0$,

$S_2 = Determinant of A =$ $|A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$**--------------- (1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$, $3x_1 + x_2 = 0$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$

i.e., we get only one equation $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 3 = 0$,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -5 + (-6) + (-2) = -5 - 6 - 2 = -13$

$S_3 = Determinant of A = |A|$ = 2 (-5)-2 (-6)-7(2) = -10 + 12 – 14 = -12

Therefore, the characteristic equation of A is $\lambda^3 - 13\lambda + 12 = 0$

3	1	0	-13	12
	0	3	9	-12
	1	3	-4	0

$$(\lambda-3)(\lambda^2+3\lambda-4)=0 \Rightarrow \lambda=3\,,\lambda=\frac{-3\pm\sqrt{3^2-4(1)(-4)}}{2(1)}=\frac{-3\ \pm\sqrt{25}}{2}=\frac{-3\pm 5}{2}$$

$$=\frac{-3+5}{2}\,,\frac{-3-5}{2}=1,-4$$

Therefore, the eigen values are 3, 1, and -4

<u>To find the eigen vectors</u>: **Let** $[A-\lambda I]X=0$

$$\begin{bmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & -7 \\ 2 & 1-1 & 2 \\ 0 & 1 & -3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1+2x_2-7x_3=0$ --------- (1)

$2x_1+0x_2+2x_3=0$ ------- (2)

$0x_1+x_2-4x_3=0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 2 & & -7 & & 1 & & 2 & \\ 0 & & 2 & & 2 & & 0 & \end{array}$$

$$\frac{x_1}{4}=\frac{x_2}{-16}=\frac{x_3}{-4} \Rightarrow \frac{x_1}{1}=\frac{x_2}{-4}=\frac{x_3}{-1}$$

Therefore, $X_1=\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$

Case 2: If $\lambda = 3$, $\begin{bmatrix} 2-3 & 2 & -7 \\ 2 & 1-3 & 2 \\ 0 & 1 & -3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1+2x_2-7x_3=0$ -------- (1)

$2x_1 - 2x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 - 6x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x1 x2 x3

2 -7 -1 2

-2 2 2 -2

$$\frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

Case 3: If $\boldsymbol{\lambda = -4}$, $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 6x_1 + 2x_2 - 7x_3 = 0$ -------- (1)

$2x_1 + 5x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 + x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x1 x2 x3

2 -7 6 2

5 2 2 5

$$\Rightarrow \frac{x_1}{39} = \frac{x_2}{-26} = \frac{x_3}{26} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}$$

Therefore, $X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

3.Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$.

Solution: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$
$-1 - 1 - 1 = \ -3$

$S_3 = Determinant\ of\ A = \ |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$$\begin{array}{r|rrr|r} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 2$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get

x_1 x_2 x_3

1 1 -2 1

-2 1 1 -2

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = -1$**,** $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 Is orthogonal to X_1 *and* X_2 .

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

l m n

1 1 1 1

1 -1 0 1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 and X_3.

4.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sumofthemaindiagonalelements = 2 + 1 - 1 = 2,$

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$
$-4 - 4 + 4 = -4$

,

$S_3 = DeterminantofA = |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8
	0	2	0	−8
	1	0	-4	0

$(\lambda - 2)(\lambda^2 - 4) = 0 \Rightarrow \lambda = 2, \qquad \lambda = 2, -2$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$ --------- (1)

$x_1 + 3x_2 + x_3 = 0$ ------------- (2)

$x_1 + 3x_2 + x_3 = 0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & 1 & & 2 & & -1 \\ 3 & & 1 & & 1 & & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} \Rightarrow \frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$$

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If$\lambda = 2$, $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & & 2 & & 0 & & -2 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

5.Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

<u>**Solution**</u>: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

<u>**To find the characteristic equation:**</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

<u>**To find the eigen vectors:**</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$. Therefore, $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case 2: If $\lambda = 3$, $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6$**,** $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

Property 1:

(i) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal (or) The sum of the eigen values of a matrix is equal to the trace of the matrix

(ii) Product of the eigen values is equal to the determinant of the matrix

Property 2:

A square matrix A and its transpose $\boldsymbol{A^T}$ have the same eigen values (or) A square matrix A and its transpose $\boldsymbol{A^T}$ have the same characteristic values

Property 4:

If $\boldsymbol{\lambda}$ is an eigen value of a matrix A, then $\frac{1}{\lambda}$, $(\boldsymbol{\lambda \neq 0})$ is the eigen value of $\boldsymbol{A^{-1}}$

Property 5:

If $\boldsymbol{\lambda}$ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value

Property 6:

If $\boldsymbol{\lambda_1, \lambda_2, \dots, \lambda_n}$ are the eigen values of a matrix A, then $\boldsymbol{A^m}$ has the eigen values $\boldsymbol{\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m}$(m being a positive integer)

Property 7:

The eigen values of a real symmetric matrix are real numbers

Property 8:

The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal

Property 9:

Similar matrices have same eigen values

Property 10:

If a real symmetric matrix of order 2 has equal eigen values, then the matrix is a scalar matrix

Property 11:

The eigen vector X of a matrix A is not unique.

Property 12:

If $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct eigen values of a n x n matrix, then the corresponding eigen vectors $X_1, X_2, \dots, X_n$ form a linearly independent set

Property 13:

If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots

Property 14:

Two eigen vectors $X_1\ and\ X_2$ are called orthogonal vectors if $X_1^T X_2 = 0$

Property 15:

Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal

Property 16:

If A and B are n x n matrices and B is a non-singular matrix then A and $B^{-1}AB$ have same eigen values

Problems:

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   <u>Solution</u>: Sum of the eigen values = Sum of the main diagonal elements = -3.

   Product of the eigen values = $|A|$ = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Two of the eigen values of** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **are 2 and 8. Find the third eigen value**

Solution: We know that sum of the eigen values = Sum of the main diagonal elements

= 6+3+3 = 12

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = 12 \Rightarrow 2 + 8 + \lambda_3 = 12 \Rightarrow \lambda_3 = 2$

Therefore, the third eigen value = 2

3. **If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find $|A|$, without expanding the determinant**

   Solution: Given $\lambda_1 = 3 \text{ and } \lambda_2 = 15, \lambda_3 = ?$

   We know that sum of the eigen values = Sum of the main diagonal elements

   $\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

   $\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$

   $\textit{We know that the product of the eigen values} = |A|$

   $\Rightarrow (3)(15)(0) = |A| \qquad \Rightarrow |A| = 0$

4. **If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T**

   Solution: By the property "A square matrix A and its transpose A^Thave the same eigen values", the eigen values of A^T are 2,2,3

5. **Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}**

   Solution: Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

   Given 3,6 are two eigen values of A. Let the third eigen value be k.

   Then, 3 + 6 + k = 11 $\Rightarrow k = 2$.Therefore, the eigen values of A are 3, 6, 2

   By the property "If the eigen values of A are$\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A

(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation**

<u>**Solution**</u>:Let A =$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$$S_1 = Sum\,of\,the\,main\,diagonal\,elements = 1 + 1 = 2$$

$$S_2 = |A| = 1 - (-4) = 5$$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

2. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and hence find its inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^2 - \lambda - 1$.

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 - A - I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$A^2 - A - I = 0$,

Multiplying by A^{-1} we get $A - I - A^{-1} = 0$,

$A^{-1} = A - I$

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

3. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and hence find is inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^3 - 2\lambda^2 - 5\lambda + 6$.

$$A^2 = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}, A^3 = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix}$$

To verify $A^3 - 2A^2 - 5A + 6I = 0$ -------------- (1)

$A^3 - 2A^2 - 5A + 6I =$

$$\begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = 5I + 2A - A^2$

$$6A^{-1} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

4. Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$ **and hence find its inverse and A^4.**

Solution: The characteristic polynomial of A is $p(\lambda) = \lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$.

$$A^2 = \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}, A^3 = \begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix}$$

To verify $A^3 - 4A^2 - 3A + 18I = 0$ -------------- (1)

$A^3 - 4A^2 - 3A + 18I =$

$$\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} - 4\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 3\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} + 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 4A - 3I + 18A^{-1} = 0$

$18A^{-1} = 3I + 4A - A^2$

$$18A^{-1} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 4\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

$$A^{-1} = \frac{1}{18}\begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

Multiply equation (1) by A

We get $A^4 - 4A^3 - 3A^2 + 18A = 0$

$A^4 = 4A3 + 3A^2 - 18A$

$$A^4 = 4\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} + 3\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 18\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 383 & 108 & 151 \\ 260 & 81 & 130 \\ -734 & -216 & -286 \end{pmatrix}$$

5. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \left|A\right| = 2(4-1) + 1(-2+1) + 2(1-2) = 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by A^{-1},$A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$