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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I - Matrices – SMTA1202

Subject Name: Matrices,Calculus and Sampling

(Common to Bio groups)

Subject code: SMTA1202

Course Material

UNIT 1 MATRICES

RANK OF A MATRIX

Let A be any matrix of order mxn.The determinants of the sub square matrices of A are called the minors of A. If all the minors of order **(r+1)** are zero but there is at least one non zero minor of order **r,** then **r** is called the rank of A and is written as R(A). For an mxn matrix,

- If m is less than n then the maximum rank of the matrix is m
- If m is greater than n then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

Example

1. **Find the rank of A =** $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$|A|$ = 1(20-12) -2(5-4) +3(6-8) = 0

Hence R(A) < 3.

Let the second order minor $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0$

R(A)=2.

2. **Find the Rank of B =** $\begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix} \quad \mathbf{R_2} = R_2 - 2R_1,\ R_3 = R_3 - 3R_1,\ R_4 = R_4 - 6R_1$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix} \quad R_2 = 1/5R_2,\ R_3 = R_3,\ R_4 = R_4$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{pmatrix} \quad R_3 = R_3 - 4R_2,\ R_4 = R_4 - 9R_2$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_4 = R_4 - R_3$$

The number of Nonzero Rows is 3.Hence R(B)=3.

3. Find the Rank of the Matrix A = $\begin{pmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{pmatrix} \quad R_1 = R_2,\ R_2 = R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{pmatrix} \quad R_2 = R_2 - 2R_1,\ R_3 = R_3 - 4R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{pmatrix} \quad R_3 = R_3 - R_2$$

The number of Nonzero Rows is 3. Hence R(A)=3.

4. Find the Rank of the Matrix A = $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \quad R_3 = R_3 - R_2$$

The number of Nonzero Rows is 3. Hence R(A)=3.

5. Find the Rank of the Matrix B = $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

A possible minor of least order is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{pmatrix}$ whose determinant is non zero.

Hence it is possible to find a nonzero minor of order 3.

Hence R(B)=3.

CONSISTENCY OF LINEAR ALGEBRAIC EQUATION

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

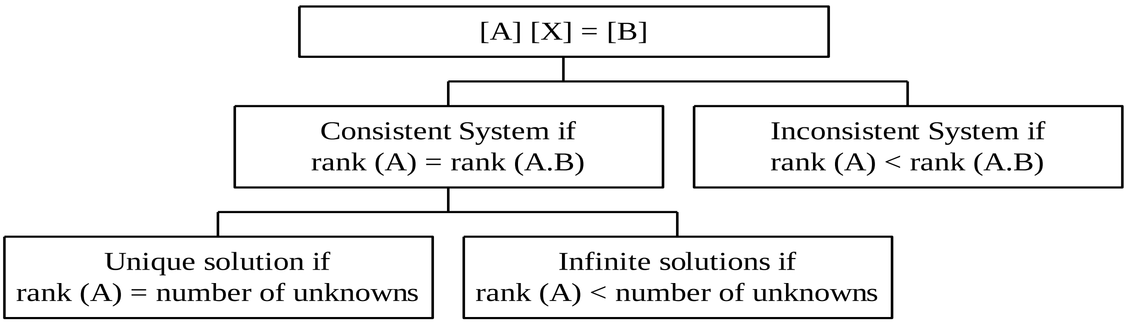
Denoting the matrices by A, X, and C, the system of equation is, AX = C where A is called the coefficient matrix, C is called the right hand side vector and X is called the solution vector. Sometimes AX=C systems of equations are written in the augmented form. That is

$$[\mathrm{A}:\mathrm{C}] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots c_m \end{bmatrix}$$

__Rouche'sTheorem__

1. A system of equations AX =C is **consistent** if the rank of A is equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$. If in addition, the rank of the coefficient matrix A is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix A is less than the number of unknowns, then infinite solutions exist.

2. A system of equations AX=C is **inconsistent** if the rank of A is not equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$.



Problems

1. **Check whether the following system of equations**

$25x_1 + 5x_2 + x_3 = 106.8$

$64x_1 + 8x_2 + x_3 = 177.2$

$89x_1 + 13x_2 + 2x_3 = 280$ is consistent or inconsistent.

__**Solution**__

The augmented matrix is

$$[A:B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :280.0 \end{bmatrix}$$

To find the rank of the augmented matrix consider a square sub matrix of order 3×3 as

$\begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$ whose determinant is 12. Hence R $[A:B]$ is 3.

So the rank of the augmented matrix is 3 but the rank of the coefficient matrix $[A]$ is 2

as the Determinant of A is zero. Hence R$[A:B] \neq$ R [A].Hence the system is inconsistent.

2. Check the consistency of the system of linear equations and discuss the nature of the solution?

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + x_2 - 2x_3 = 1$$
$$4x_1 - 3x_2 - x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 4$$

Solution

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 4R_1,\ R_4 = R_4 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 / -5$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 + 11R_1$$

Here R$[A:B]$=R[A] =3.Hence the system is consistent. Also R[A] is equal to the number of unknowns. Hence the system has an unique solution.

3. **Check whether the following system of equations is a consistent system of equations. Is the solution unique or does it have infinite solutions**

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10$$

Solution

The given system has the augmented matrix given by

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix} \quad R_2 = R_2 - R_1,\ R_3 = R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2$$

A and $[A:B]$ are each of rank r = 3, the given system is consistent but R[A] is not equal to the number of unknowns. Hence the system does not has a unique solution.

4. Check whether the following system of equations

$3x - 2y + 3z = 8$
$x + 3y + 6z = -3$
$2x + 6y + 12z = -6$

is a consistent system of equations and hence solve them.

Solution

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 3 & -2 & 3 & 8 \\ 1 & 3 & 6 & -3 \\ 2 & 6 & 12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 3 & -2 & 3 & 8 \\ 2 & 6 & 12 & -6 \end{bmatrix} \quad R_1 = R_2 \ \ R_2 = R_1$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 0 & 11 & 15 & -17 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 2R_1$$

R[A:B] = R[A] = 2**.**Therefore the system is consistent and posses solution but rank is not equal to the number of unknowns which is 3.Hence the system has infinite solution. From the upper triangular matrix we have the reduced system of equations given by

$$x + 3y + 6z = -3 \; ; \; 11y + 15z = -17 .$$

By assuming a value for y we have one set of values for z and x.For example when y=3, z = $-10/3$ and x = 8.Similarly by choosing a value for z the corresponding y and x can be calculated. Hence the system has infinite number of solutions.

5. Check whether the following system of equations

$$X + y + z = 6$$
$$3x - 2y + 4z = 9$$
$$x - y - z = 0$$

Is a consistent system of equations and hence solve them.

<u>Solution</u>

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & 4 & 9 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & 1 & -9 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2 = R_2 - 3R_1 \, , \, R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2 = R_2/-5$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & -12/5 & -12/5 \end{bmatrix} \quad R_3 = R_3 + 2R_2$$

Hence R[A B] =R[A] =3 which is equal to the number of unknowns. Hence the system is consistent with unique solution. Now the system of equations takes the form

$$x + y + z = 6; \quad y - z/5 = 9/5; \; -12/5 z = -12/5.$$

Hence z =1. Substituting z = 1 in y-z/5 =9/5 we have y-1/5 = 9/5 or y = 1/5+9/5 = 10/5.

Hence y =2. Substituting the values of y,z in x+y + z = 6 we have x= 3. Hence the system has the unique solution as x= 3 , y =2 , z =1.

<u>CHARACTERISTIC EQUATION</u>

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

If X_r be the non-zero vector satisfying AX = λX, when$\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

Working rule to find characteristic equation:

<u>For a 3 x 3 matrix:</u>

<u>Method 1:</u>

The characteristic equation is $|A - \lambda I| = 0$

<u>Method 2:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where S_1 = sum of the main diagonal elements, S_2 = sum of the minors of the main diagonal elements, S_3 = Determinant of A = $|A|$

<u>For a 2 x 2 matrix:</u>

<u>Method 1:</u>

The characteristic equation is $|A - \lambda I| = 0$

<u>Method 2:</u>

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where S_1 = sum of the main diagonal elements, S_2 = Determinant of A = $|A|$

1. **Find the characteristic equation of** $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

<u>Solution</u>: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where S_1 = sum of the main diagonal elements = 8 + 7 + 3 = 18,

S_2 = sum of the minors of the main diagonal elements=45

S_3 = Determinant of A = $|A|$=0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

2. **Find the characteristic equation of** $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

<u>Solution</u>: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equationl of A is $\lambda^2 - S_1\lambda + S_2 . S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic equation is $\lambda^2 - 5\lambda + 7$ =0.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX

Working rule to find Eigen values and Eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called Eigen values
3. To find the Eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct Eigen values, we get n independent Eigen vectors
2. If 2 or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to the repeated Eigen values
3. If X_i is a solution for an Eigen valueλ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the Eigen vector corresponding to an Eigen value is not unique but may be any one of the vectors cX_i

Problems

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sum of the main diagonal elements = 1 - 1 = 0,$

$S_2 = Determinant of A = \ |A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$**--------------- (1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$, $3x_1 + x_2 = 0$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$

i.e., we get only one equation $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 3 = 0$,
$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -5 + (-6) + (-2) = -5 - 6 - 2 = -13$
$S_3 = Determinant of A = |A|$ = 2 (-5)-2 (-6)-7(2) = -10 + 12 – 14 = -12

Therefore, the characteristic equation of A is $\lambda^3 - 13\lambda + 12 = 0$

3	1	0	-13	12
	0	3	9	− 12
	1	3	-4	0

$$(\lambda-3)(\lambda^2+3\lambda-4)=0 \Rightarrow \lambda=3\,,\lambda=\frac{-3\pm\sqrt{3^2-4(1)(-4)}}{2(1)}=\frac{-3\ \pm\sqrt{25}}{2}=\frac{-3\pm 5}{2}$$

$$=\frac{-3+5}{2}\,,\frac{-3-5}{2}=1,-4$$

Therefore, the eigen values are 3, 1, and -4

<u>To find the eigen vectors</u>: **Let** $[A-\lambda I]X=0$

$$\begin{bmatrix}2-\lambda & 2 & -7\\ 2 & 1-\lambda & 2\\ 0 & 1 & -3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Case 1: If $\lambda=1$, $\begin{bmatrix}2-1 & 2 & -7\\ 2 & 1-1 & 2\\ 0 & 1 & -3-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

i.e., $\begin{bmatrix}1 & 2 & -7\\ 2 & 0 & 2\\ 0 & 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow x_1+2x_2-7x_3=0$ --------- (1)

$2x_1+0x_2+2x_3=0$ ------- (2)

$0x_1+x_2-4x_3=0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

2 -7 1 2

0 2 2 0

$$\frac{x_1}{4}=\frac{x_2}{-16}=\frac{x_3}{-4} \Rightarrow \frac{x_1}{1}=\frac{x_2}{-4}=\frac{x_3}{-1}$$

Therefore, $X_1=\begin{pmatrix}1\\-4\\-1\end{pmatrix}$

Case 2: If $\lambda=3$, $\begin{bmatrix}2-3 & 2 & -7\\ 2 & 1-3 & 2\\ 0 & 1 & -3-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

i.e., $\begin{bmatrix}-1 & 2 & -7\\ 2 & -2 & 2\\ 0 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow -x_1+2x_2-7x_3=0$ -------- (1)

$2x_1 - 2x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 - 6x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x1 x2 x3

2 -7 -1 2

-2 2 2 -2

$$\frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = -4$, $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 6x_1 + 2x_2 - 7x_3 = 0$ -------- (1)

$2x_1 + 5x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 + x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x1 x2 x3

2 -7 6 2

5 2 2 5

$$\Rightarrow \frac{x_1}{39} = \frac{x_2}{-26} = \frac{x_3}{26} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}$$

Therefore, $X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

3.Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$
$-1 - 1 - 1 = \ -3$

$S_3 = Determinant\ of\ A = \ |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 2$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get

x_1 x_2 x_3

1 1 -2 1

-2 1 1 -2

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = -1$**,** $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 Is orthogonal to X_1 *and* X_2 .

$[1 \quad 1 \quad 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$[0 \quad 1 \quad -1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

l m n

1 1 1 1

1 -1 0 1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 and X_3.

4.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 1 = 2,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -4 - 4 + 4 = -4$

,

$S_3 = Determinant of A = |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8
	0	2	0	−8
	1	0	-4	0

$(\lambda - 2)(\lambda^2 - 4) = 0 \Rightarrow \lambda = 2, \quad \lambda = 2, -2$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$ --------- (1)

$x_1 + 3x_2 + x_3 = 0$ ------------- (2)

$x_1 + 3x_2 + x_3 = 0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
-1		1		2		-1
3		1		1		3

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} \Rightarrow \frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$$

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
-2		2		0		-2
-1		1		1		-1

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

5.Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
1		3		3		1
7	✕	1	✕	1	✕	7

$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$. Therefore, $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case 2: If $\lambda = 3$, $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
1		3		-2		1
2	✕	1	✕	1	✕	2

$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6$**,** $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

1 3 -5 1

-1 1 1 -1

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

Property 1:

(i) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal (or) The sum of the eigen values of a matrix is equal to the trace of the matrix

(ii) Product of the eigen values is equal to the determinant of the matrix

Property 2:

A square matrix A and its transpose $\boldsymbol{A^T}$ have the same eigen values (or) A square matrix A and its transpose $\boldsymbol{A^T}$ have the same characteristic values

Property 4:

If $\boldsymbol{\lambda}$ is an eigen value of a matrix A, then $\frac{1}{\lambda}$, $(\boldsymbol{\lambda} \neq \mathbf{0})$ is the eigen value of $\boldsymbol{A^{-1}}$

Property 5:

If $\boldsymbol{\lambda}$ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value

Property 6:

If $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_n}$ are the eigen values of a matrix A, then $\boldsymbol{A^m}$ has the eigen values $\boldsymbol{\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m}$(m being a positive integer)

Property 7:

The eigen values of a real symmetric matrix are real numbers

Property 8:

The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal

Property 9:

Similar matrices have same eigen values

Property 10:

If a real symmetric matrix of order 2 has equal eigen values, then the matrix is a scalar matrix

Property 11:

The eigen vector X of a matrix A is not unique.

Property 12:

If $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct eigen values of a n x n matrix, then the corresponding eigen vectors $X_1, X_2, \dots, X_n$ form a linearly independent set

Property 13:

If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots

Property 14:

Two eigen vectors $X_1 \; and \; X_2$ are called orthogonal vectors if $X_1^T X_2 = 0$

Property 15:

Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal

Property 16:

If A and B are n x n matrices and B is a non-singular matrix then A and $B^{-1}AB$ have same eigen values

Problems:

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   Solution: Sum of the eigen values = Sum of the main diagonal elements = -3.

   Product of the eigen values = |A| = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Two of the eigen values of** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **are 2 and 8. Find the third eigen value**

Solution: We know that sum of the eigen values = Sum of the main diagonal elements

= 6+3+3 = 12

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = 12 \Rightarrow 2 + 8 + \lambda_3 = 12 \Rightarrow \lambda_3 = 2$

Therefore, the third eigen value = 2

3. **If 3 and 15 are the two eigen values of A =** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$**, find |A|, without expanding the determinant**

Solution: Given $\lambda_1 = 3 \; and \; \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$

$We\ know\ that\ the\ product\ of\ the\ eigen\ values = |A|$

$\Rightarrow (3)(15)(0) = |A| \qquad \Rightarrow |A| = 0$

4. **If 2, 2, 3 are the eigen values of A =** $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$**, find the eigen values of** A^T

Solution: By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of A^T are $2,2,3$

5. **Two of the eigen values of A =** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **are 3 and 6. Find the eigen values of** A^{-1}

Solution: Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 $\Rightarrow k = 2$.Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A

(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A =$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$$

$$S_2 = |A| = 1 - (-4) = 5$$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

2. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and hence find its inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^2 - \lambda - 1$.

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 - A - I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$A^2 - A - I = 0$,

Multiplying by A^{-1} we get $A - I - A^{-1} = 0$,

$A^{-1} = A - I$

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

3. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and hence find is inverse.**

Solution: The characteristic polynomial of A is p(λ) = $\lambda^3 - 2\lambda^2 - 5\lambda + 6$.

$$A^2 = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}, A^3 = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix}$$

To verify $A^3 - 2A^2 - 5A + 6I = 0$ -------------- (1)

$A^3 - 2A^2 - 5A + 6I =$

$$\begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = 5I + 2A - A^2$

$$6A^{-1} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

4. **Verify Cayley-Hamilton theorem for the matrix A =** $\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$ **and hence find its inverse and** A^4**.**

Solution: The characteristic polynomial of A is $p(\lambda) = \lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$.

$$A^2 = \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}, A^3 = \begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix}$$

To verify $A^3 - 4A^2 - 3A + 18I = 0$ -------------- (1)

$A^3 - 4A^2 - 3A + 18I =$

$$\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} - 4\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 3\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} + 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 4A - 3I + 18A^{-1} = 0$

$18A^{-1} = 3I + 4A - A^2$

$$18A^{-1} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 4\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

$$A^{-1} = \frac{1}{18}\begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

Multiply equation (1) by A

We get $A^4 - 4A^3 - 3A^2 + 18A = 0$

$A^4 = 4A3 + 3A^2 - 18A$

$$A^4 = 4\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} + 3\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 18\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 383 & 108 & 151 \\ 260 & 81 & 130 \\ -734 & -216 & -286 \end{pmatrix}$$

5. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\, of\, the\, minirs\, of\, the\, main\, diagonal\, elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \ \Rightarrow \ A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - \ 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by A^{-1},$A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

Questions

Course code: SMTA 1202
Course Name: MATRICES, CALCULUS and SAMPLING

Unit I - Matrices

Part A

1. Define Rank of a matrix CO1(L1)
2. Find the rank of the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}$ CO1(L1)
3. What do you mean by consistent and inconsistent system of equations? Give examples. CO1(L1)
4. Find the values of 'a' and 'b' if the equations 2x-3y = 5 and ax+by = -10 have many solutions. CO1(L1)
5. Recall Cayley Hamilton theorem. CO1(L1)
6. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ CO1(L1)
7. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ CO1(L1)
8. Find the eigen values of A^{-1} and A^{3}, if A= $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$ CO1(L1)
9. Find the sum of the squares of the eigen values of $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$ CO1(L1)
10. Find the third eigen value, if The product of two eigen values of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. CO1(L1)
11. Find the sum of the eigen values of 2A if A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ CO1(L1).
12. Find the third eigen value, if two eigen values of the matrix CO1(L1).

    A= $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are 1, 1.

Part B

1. Show that the system of equations $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ has unique solutions and hence solve this system. CO1(L2).

2. Identify all the values of μ for which rank of the matrix $A = \begin{pmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{pmatrix}$ is equal to 3. CO1(L3)

3. Test for consistency of the following system of equations CO1(L4).
   and solve if consistent 2x-5y+2z = -3, -x-3y+3z = -1, x+y-z = 0

4. Test for consistency of the following system of equations CO1(L4).
   and solve if consistent: 5x+3y+7z = 4, 3x+26y+2z = 9, 7x+2y+10z = 5

5. Test for consistency of the following system of equations CO1(L4).
   and solve if consistent $x_1+2x_2+2x_3-x_4 = 3$, $x_1+2x_2+3x_3+x_4 = 1$, $3x_1+6x_2+8x_3+x_4 = 5$

6. Solve the following equations if it is consistent. CO1(L3)

$$x + 2y - 3z = -3$$
$$2x + 5y + 10z = 25$$
$$3x - 4y + 5z = 29$$

7. Determine the value of 'k' for which the equations kx-2y + z =1, x-2ky + z = -2 and x-2y + kz = 1 have i)no solution ii)one solution iii) many solutions. CO1(L5)

8. Determine the values of a and b for which the equations x+y+2z =3, 2x-y+3z =4 and 5x-y+az =b have i) no solution ii) unique solution iii) many solutions. CO1(L5)

9. Determine the values of 'μ'for which the equations 3x+y- μ z =0, 4x-2y-3z = 0 and 2μx+4y+ μz = 0 possess a non-trivial solution. For these values of μ, also find the solution. CO1(L5)

10. Determine the eigen values and eigen vectors of the following matrices CO1(L5)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

11. Determine the eigen values and eigen vectors of the following matrices CO1(L5)

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

12. Determine the eigen values and eigen vectors of the following matrices CO1(L5)

$$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

13. Determine the eigen values and eigen vectors of the following matrices CO1(L5)

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

14. Show that the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ satisfies its own characteristic CO3(L2)

equation and hence Find A^{-1}.

15. Examine Cayley – Hamilton theorem for the matrices given below and CO3(L4)
hence find A^{-1} and A^4

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

16. Examine Cayley – Hamilton theorem for the matrices given below and CO3(L4)
hence find A^{-1} and A^4

$$\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

17. Examine Cayley – Hamilton theorem for the matrices given below and CO3(L4)
hence find A^{-1} and A^4

$$\begin{pmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$$

18. Examine Cayley – Hamilton theorem for the matrices given below and CO3(L4)
hence find A^{-1} and A^4

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$$

SUBJECT NAME: MATRICES,CALCULUS and SAMPLING

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMTA1202

COURSE MATERIAL

UNIT-II DIFFERENTIAL CALCULUS

Definition 1. Differentiation

The rate at which a function changes with respect to the independent variable is called the derivative of the function.

(i.e) If y= f(x) be a function, where x and y are real variables which are independent and dependent variables respectively, then the derivative of y with respect to x is $\frac{dy}{dx}$.

Definition 2. Derivative of addition or subtraction of functions

If f(x) and g(x) are two functions of x, then $\frac{d[f(x) \pm g(x)]}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$

Definition 3. Product rule

If $y = uv$, where u and v are functions of x, then $\frac{d[uv]}{dx} = v\frac{d[u]}{dx} + u\frac{d[v]}{dx}$

Definition 4. Quotient rule

If $y = \frac{u}{v}$, where u and v are functions of x, then $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Important Derivatives Formulae

1. $\frac{d}{dx}(c) = 0$ where 'c' is any constant.
2. $\frac{d}{dx}(x^n) = nx^{n-1}$.
3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.
4. $\frac{d}{dx}(a^x) = a^x \log a$
5. $\frac{d}{dx}(e^x) = e^x$.
6. $\frac{d}{dx}(\sin x) = \cos x$.
7. $\frac{d}{dx}(\cos x) = -\sin x$.
8. $\frac{d}{dx}(\tan x) = \sec^2 x$.

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$.

11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

15. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1-x^2}}$.

17. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

Problems

I. Ordinary Differentiation Problems

1. Differentiate $\mathbf{x + \frac{1}{x}}$

Solution Let $y = x + \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{d\left(x+\frac{1}{x}\right)}{dx} \quad = \frac{d(x)}{dx} + \frac{d(x^{-1})}{dx} = 1 - \frac{1}{x^2}$

2. Differentiate $\mathbf{3\tan x + 2\cos x - e^x + 5}$

Solution:

Let $y = 3\tan x + 2\cos x - e^x + 5$

Then $\frac{dy}{dx} = \frac{d(3\tan x + 2\cos x - e^x + 5)}{dx} \quad = 3\frac{d(\tan x)}{dx} + 2\frac{d(\cos x)}{dx} - \frac{d(e^x)}{dx} + \frac{d(5)}{dx}$

$= 3sec^2 x - 2\sin x - e^x$

3. Differentiate $\mathbf{y = e^{2x}\cos 3x}$

Solution: $\frac{dy}{dx} = \frac{d(e^{2x}\cos 3x)}{dx} \quad = \cos 3x\,\frac{d(e^{2x})}{dx} + e^{2x}\,\frac{d(\cos 3x)}{dx}$

$= 2\cos 3x\, e^{2x} - 3e^{2x}\sin 3x$

4. Differentiate $\mathbf{y = e^{\sin x}x^2}$

Solution: $\frac{dy}{dx} = \frac{d(e^{\sin x}x^2)}{dx}$

$= x^2 \frac{d(e^{sinx})}{dx} + e^{sinx}\frac{d(x^2)}{dx}$
$= x^2 e^{sinx}(cosx) + 2xe^{sinx}$

5. **Differentiate y = $\mathbf{x^3 e^{-x} tanx}$**
   Solution: $\frac{dy}{dx} = \frac{d(x^3 e^{-x} tanx)}{dx}$
   $= e^{-x} tanx \frac{d(x^3)}{dx} + x^3\, tanx \frac{d(e^{-x})}{dx} + x^3 e^{-x} \frac{d(tanx)}{dx}$
   $= 3x^2 e^{-x} tanx - x^3 e^{-x} tanx + x^3 e^{-x} sec^2 x$
6. **Differentiate y $= \frac{\mathbf{e^x}}{\mathbf{cosx}}$**
   Solution: $\frac{dy}{dx} = \frac{d\left(\frac{e^x}{cosx}\right)}{dx} = \frac{cosx\, e^x - e^x\ (-sinx)}{cos^2 x}$
   $= \frac{cosx\, e^x + e^x\ (sinx)}{cos^2 x}$
7. **Differentiate y $= \frac{\mathbf{ax+b}}{\mathbf{cx+d}}$**
   Solution: $\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$ (by quotient rule)
8. **Differentiate $\frac{\mathbf{x^2+2x+3}}{\sqrt{\mathbf{x}}}$**
   Solution: $\frac{dy}{dx} = \frac{\sqrt{x}(2x+2)-(x^2+2x+3)\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2} = \frac{2\sqrt{x}(x+1)-(x^2+2x+3)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$
   $= \frac{2\sqrt{x}\times 2\sqrt{x}(x+1)-(x^2+2x+3)}{2\sqrt{x}(\sqrt{x})^2} = \frac{4x(x+1)-(x^2+2x+3)}{2x^{3/2}}$
   $= \frac{4x^2+4x-x^2-2x-3}{2x^{3/2}} = \frac{3x^2+2x-3}{2x^{3/2}}$
9. **Differentiate y $= (\mathbf{3x^2 - 1})^{\mathbf{3}}$**
   Solution: Given $y = (3x^2 - 1)^3$
   Differentiating w.r.to x, we get
   $\Rightarrow \frac{dy}{dx} = 3(3x^2 - 1)^2 6x$
   $= 3(9x^4 - 6x^2 + 1) = 27x^4 - 18x^2 + 3$
10. **Differentiate: $\mathbf{log}\left(\frac{\mathbf{1+sinx}}{\mathbf{1-sinx}}\right)$**
   Solution: Let $y = \log\left(\frac{1+sinx}{1-sinx}\right)$
   $\Rightarrow y = \log(1 + sinx) - \log(1 - sinx)$
   Differentiate y w.r.to x, we get
   $\frac{dy}{dx} = \frac{1}{1+sinx} cosx - \frac{1}{1-sinx}(-cosx)$

   $= \frac{(1-sinx)cosx+cosx(1+sinx)}{(1+sinx)(1-sinx)}$

   $= \frac{cosx-sinx\, cosx+cosx+ cosx\, sinx}{1-sin^2 x}$
   $= \frac{2\, cosx}{cos^2 x} = 2\frac{1}{cosx} = 2\, secx$

II. Differentiation Problems on Logarithmic Functions

1. **Differentiate $\mathbf{x^{sinx}}$**
   Solution: Let $y = x^{sinx}$
   Taking log on both sides, we get $\log y = sinx \log x$

Now differentiating with respect to x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x(\cos x) + \sin x\frac{1}{x}$ (Using product rule)

$\Rightarrow \frac{dy}{dx} = y\left(\log x(\cos x) + \sin x\frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y(x\cos x \log x + \sin x)}{x}$

$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{x\cos x \log x + \sin x}{x}\right)$

2. **If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$**

Solution: Given $x^y = e^{x-y}$

Taking log on both sides, we get $\log x^y = \log e^{x-y}$

$\Rightarrow y\log x = (x-y)\log_e e$

$\Rightarrow y\log x = (x-y)$………(1)

$\Rightarrow \frac{1}{x}y + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \log x\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}(\log x + 1) = \frac{x-y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\log x)}$

$\Rightarrow \frac{dy}{dx} = \frac{y\log x}{x(1+\log x)}$….(2)

Again from (1) $y + y\log x = x$

$\Rightarrow y(1+\log x) = x, \frac{y}{x} = \frac{1}{1+\log x}$

$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

3. **If y = $x^{x^{x^{\dots\infty}}}$, then find $\frac{dy}{dx}$**

Solution:

Given $y = x^{x^{x^{\dots\infty}}} = x^y$

Taking log on both sides

$\log y = y\log x$

Differentiating w. r. to x we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \left(\frac{1-y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{y}{1-y\log x}\right) = \frac{y^2}{x(1-y\log x)}$

4. **Differentiate $y = \log\left(\frac{x^2+1}{x^2-1}\right)$**

Solution:

$y = \log(x^2+1) - \log(x^2-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2+1}2x - \frac{1}{x^2-1}2x$

$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{1}{x^2+1} - \frac{1}{x^2-1}\right)$

$$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{x^2-1-(x^2+1)}{(x^2+1)(x^2-1)}\right) = 2x\left(\frac{x^2-1-x^2-1)}{x^4-1}\right) = 2x\left(\frac{-2)}{x^4-1}\right) = \frac{-4x}{x^4-1}$$

5. **Differentiate $\mathbf{y = e^{3x^2+2x+3}}$**

Solution: $\frac{dy}{dx} = e^{3x^2+2x+3}(6x+2)$

III. Differentiation of Implicit functions

If two variables x and y are connected by the relation f(x, y) = 0 and none of the variable is directly expressed in terms of the other, then the relation is called an implicit function.

Problems

1. **Find $\frac{dy}{dx}$, if $x^3+y^3 = 3axy$**

Solution:

Differentiating w.r.to x, we get

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$

$\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$

$\Rightarrow \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{(3ay-3x^2)}{3y^2-3ax} = \frac{3(ay-x^2)}{3(y^2-ax)} = \frac{(ay-x^2)}{(y^2-ax)}$

2. **Find $\frac{dy}{dx}$, if $x^2 + y^2 = 16$**

Solution:

Given $x^2 + y^2 = 16$

$\Rightarrow y^2 = 16 - x^2$

$\Rightarrow y = \sqrt{16 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(16 - x^2)^{-1/2} \times (-2x)$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{16-x^2}} = -\frac{x}{y}$

3. **Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$**

Solution: Given $x = at^2, y = 2at$

$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

Now $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$

4. **Find $\frac{dy}{dx}$, if $y^2 + x^3 - xy + \cos y = 0$**

Solution:

Given $y^2 + x^3 - xy + \cos y = 0$

$\Rightarrow 2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} = 0$

$\Rightarrow (2y - \sin y)\frac{dy}{dx} + 3x^2 - \left(x\frac{dy}{dx} + y \times 1\right) = 0$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y = 0$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} = y - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{2y-\sin y-x}$

IV.Successive Differentiation

The process of differentiating a given function again and again is called as successive differentiation and the results of such differentiation are called successive derivatives.

Notations:

i) $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$,....... n^{th} order derivative: $\frac{d^ny}{dx^n}$

ii) f ′(x) , f ″(x) , f ‴(x),….., nth order derivative: $f^n(x)$

iii) Dy,D^2y,D^3y……., nth order derivative: D^ny

iv) y′ , y″ , y‴ ,……, nth order derivative: $y^{(n)}$

v) y_1,y_2,y_3,……..,nth order derivative: y_n

Problems

1. If y=sin(sinx), prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$

Solution:

Given y=sin(sinx)…………………(1)

Differentiating (1) with respect to x we get,

$\frac{dy}{dx}$ = cos(sinx).cosx………………..(2)

Differentiating (2) with respect to x we get,

$\frac{d^2y}{dx^2}$ =cos(sinx)(-sinx)+cosx(-sin(sinx).cosx) [Product Rule]

$\frac{d^2y}{dx^2}$ = -sinxcos(sinx) – $\cos^2$xsin(sinx)……………..(3)

Therefore,

$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x =$ -sinxcos(sinx) – $\cos^2$xsin(sinx) +(tanx) cos(sinx).cosx +y $\cos^2$x

= - sinxcos(sinx) – y $\cos^2$x + sinxcos(sinx) + y $\cos^2$x

= 0

2.If x = a(cost + tsint), y = a(sint – tcost), find $\frac{d^2y}{dx^2}$

Solution:

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{1}{at\cos^3 t}.$$

3. If $ax^2+2hxy+by^2=1$ **then prove that** $\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}$

Solution:

Given $ax^2+2hxy+by^2=1$ ....(1)

Differentiating (1) partially with respect to x,we get,

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}=0$$

Then , $\frac{dy}{dx}=\frac{-(ax+hy)}{(hx+by)}$ ....(2)

Differentiating (2) with respect to x again,we get,

$$\frac{d^2y}{dx^2}=\frac{(hx+by)\left[-a-h\frac{dy}{dx}\right]+(ax+hy)\left[h+b\frac{dy}{dx}\right]}{(hx+by)^2}$$

$$=\frac{(h^2-ab)y-\frac{dy}{dx}(h^2-ab)x}{(hx+by)^2}=\frac{(h^2-ab)\left(y-x\frac{dy}{dx}\right)}{(hx+by)^2}$$

$$=\frac{(h^2-ab)\left(y+x\frac{(ax+hy)}{(hx+by)}\right)}{(hx+by)^2}=\frac{(h^2-ab)(ax^2+by^2+2hxy)}{(hx+by)^3}$$

Thus, $\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}$ (from(1))

n^th derivative of some standard functions:

S.No	Y=f(x)	$y_n=\frac{d^ny}{dx^n}=D^ny$
1.	e^{mx}	m^ne^{mx}
2.	$(ax+b)^m$	$m(m-1)(m-2).........(m-n+1)a^n(ax+b)^{m-n}$
3.	$\frac{1}{ax+b}$	$\frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$
4.	$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
5.	$\sin(ax+b)$	$a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$
6.	$\cos(ax+b)$	$a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

4. Find y_n, where $y = \dfrac{3}{(x+1)(2x-1)}$

Solution:

Resolving into partial fractions,

$$y = \frac{2}{2x-1} - \frac{1}{x+1}$$

$$\therefore y_n = \frac{2(-1)^n 2^n n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

5. Find the n^{th} derivative of $\log(9x^2-1)$.

Solution:

Let $y = \log(9x^2 - 1) = \log\{(3x+1)(3x-1)\}$

$$= \log(3x+1) + \log(3x-1)$$

Then $y_n = \dfrac{d^n}{dx^n}(\log(3x+1)) + \dfrac{d^n}{dx^n}(\log(3x-1))$

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!(3)^n}{(3x+1)^n} + \frac{(-1)^{n-1}(n-1)!(3)^n}{(3x-1)^n}$$

6. Find y_n, where $y = e^{7x+5}$

Solution:

Let $y = e^{7x+5}$

Then $y_n = \dfrac{d^n}{dx^n}\left(e^{7x+5}\right) = \dfrac{d^n}{dx^n}\left(e^5 e^{7x}\right) = e^5 \dfrac{d^n}{dx^n}\left(e^{7x}\right)$

$$\Rightarrow y_n = e^5 7^n e^{7x}$$

Leibnitz formula for the n^{th} derivative of a product

If u and v are functions of x,then

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \ldots\ldots.. + nc_r u_{n-r} v_r + \ldots\ldots. + uv_n$$

Problems

7. Find the nth differential coefficient of $x^2 \log x$

Solution:

Take u=logx , v = x^2

$$\frac{d^n}{dx^n}\left(x^2 \log x\right) = \frac{d^n}{dx^n}(\log x)x^2 + nc_1 \frac{d^{n-1}}{dx^{n-1}}(\log x)\frac{d}{dx}\left(x^2\right) + nc_2 \frac{d^{n-2}}{dx^{n-2}}(\log x)\frac{d^2}{dx^2}\left(x^2\right)$$

(since all the other terms are zero)

$$= \frac{(-1)^{n-1}(n-1)!x^2}{(x)^n} + \frac{n(-1)^{n-2}(n-2)!(2x)}{(x)^{n-1}} + \frac{n(n-1)(-1)^{n-3}(n-3)!2}{2(x)^{n-2}}$$

$$= \frac{2(-1)^{n-2}(n-3)!}{(x)^{n-2}}$$

8. If y = x^2e^x, show that $y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$ **where y_n stands for** $\frac{d^n y}{dx^n}$

Solution:

Take $u = e^x$, $v = x^2$

$$y_n = \frac{d^n}{dx^n}(x^2e^x) = \frac{d^n}{dx^n}(e^x)x^2 + nc_1\frac{d^{n-1}}{dx^{n-1}}(e^x)\ \frac{d}{dx}(x^2) + nc_2\frac{d^{n-2}}{dx^{n-2}}(e^x)\ \frac{d^2}{dx^2}(x^2)$$

(since all the other terms are zero)

$$\therefore y_n = e^x x^2 + 2nxe^x + n(n-1)e^x$$

Now, $y_1 = x^2e^x + 2xe^x$, $y_2 = x^2e^x + 4xe^x + 2e^x$

$$\therefore \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y = \frac{n(n-1)\left[x^2e^x + 4xe^x + 2e^x\right]}{2}$$

$$- n(n-2)\left[x^2e^x + 2xe^x\right] + \frac{(n-1)(n-2)x^2e^x}{2}$$

$$= x^2e^x\left[\frac{n(n-1)}{2} - n(n-2) + \frac{(n-1)(n-2)}{2}\right] + xe^x\left[2n(n-1) - 2n(n-2)\right] + n(n-1)e^x$$

$= y_n$ on simplification.

V. Partial Differentiation

Consider $z = f(x, y)$, here z is a function of two independent variables x and y. z can be differentiated with respect to x or y but when we are differentiating z with respect to x (or y) we must keep the variable y (or x) as a constant.

Notations:

Let z=f(x,y)

First order partial derivatives of f(x, y) with respect to x and y.

$\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

Second order partial derivatives of f(x, y) with respect to x and y

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

Second order mixed partial derivatives of f(x, y)

$\frac{\partial^2 f}{\partial x\,\partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y\,\partial x} = f_{yx}$

Problems:

1. **If u = $x^3 + y^3 + 3xy$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$**

   Solution: Given If u = $x^3 + y^3 + 3xy$

   $\frac{\partial u}{\partial x} = 3x^2 + 3y$, $\frac{\partial u}{\partial y} = 3y^2 + 3x$

2. **If u = log ($x^3 + y^3 + z^3 - 3xyz$), show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}$**

Solution: u = log $(x^3 + y^3 + z^3 - 3xyz)$

$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3x^2 - 3yz$,
$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3y^2 - 3xz,$
$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3z^2 - 3xy$

Now $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3yz-3xz-3xy}{x^3 + y^3 + z^3 - 3xyz}$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$$

3. **If $f(x,y) = x^2 \sin y + y^2 \cos x$, then find its all first and 2nd order partial derivatives.**
   Solution: Given $f(x,y) = x^2 \sin y + y^2 \cos x$
   $f_x = 2x \sin y - y^2 \sin x;\ f_y = x^2 \cos y + 2y \cos x.$
   $f_{xx} = 2 \sin y - y^2 \cos x;\ f_{yy} = -x^2 \sin y + 2 \cos x;$
   $f_{xy} = 2x \cos y - 2y \sin x;\ fyx = 2x \cos y - 2y \sin x.$
4. **If $f(x,y) = \frac{y}{x} \log x$, then find its all 1st and 2nd order derivatives**.
   Solution: $f_x = \frac{y}{x}\frac{1}{x} + \log x \left(\frac{-y}{x^2}\right) = \frac{y}{x^2}(1 - \log x), f_y = \frac{\log x}{x}$,
   $f_{xx} = \frac{y}{x^2}\left(-\frac{1}{x}\right) - \frac{2y}{x^3}(1 - \log x) = \frac{y}{x^3}\left(-1 - 2(1 - \log x)\right) = \frac{y}{x^3}(\log x - 3)$;
   $f_{yy} = 0,\ f_{yx} = \frac{1}{x^2}(1 - \log x);\ f_{xy} = \frac{1}{x}\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1 - \log x).$
5. **Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = \sin(ax + by + cz)$**
   Solution:
   $\frac{\partial u}{\partial x} = a \cos(ax + by + cz)$
   $\frac{\partial u}{\partial x} = b \cos(ax + by + cz)$
   $\frac{\partial u}{\partial x} = c \cos(ax + by + cz)$

VI. Euler's Theorem for Homogeneous Functions

A homogenous function of degree *n* of the variables *x*, *y*, *z* is a function in which each term has degree *n*. For example, the function $f(x, y, z) = Ax^3 + By^3 + Cz^3 + Dxy^2 + Exz^2 + Fyz^2 + Gyx^2 + Hzx^2 + Izy^2 + Jxyz$, is a homogeneous function of *x*, *y*, *z*, in which all terms are of degree three.

Note:

A function f(x ,y) of two independent variables x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for any positive quantity t.

Euler's theorem:

1). If $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

2). If $f(x, y, z)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

Result: If z is a homogeneous function of x, y of degree n and z=f(u) then

(i). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Problems on Euler's theorem

1. Verify Euler's theorem when $\mathbf{u = x^3 + y^3 + z^3 + 3xyz}$

Solution:

Given $u = x^3 + y^3 + z^3 + 3xyz$

Now $tu = (tx)^3 + (ty)^3 + (tz)^3 + 3txtytz$

$= t^3(x^3 + y^3 + z^3 + 3xyz) = t^3u$

Therefore u is a homogeneous function of degree 3.

$\frac{\partial u}{\partial x} = 3x^2 + 3yz$

$\frac{\partial u}{\partial y} = 3y^2 + 3xz$

$\frac{\partial u}{\partial z} = 3z^2 + 3xy$

Therefore $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3xz) + z(3z^2 + 3xy)$

$= 3x^3 + 3y^3 + 3z^3 + 9xyz$

$= 3(x^3 + x^3 + 3xy) = 3u$

Hence Euler's theorem is verified.

2. If $\mathbf{u = x\,log\left(\frac{y}{x}\right)}$, then prove that $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$

Solution:

Given $u = x\log\left(\frac{y}{x}\right)$

u is a homogeneous function of degree 1.

Therefore by Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \times u = u$

3. If $\mathbf{(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}}$, then prove that $\mathbf{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0}$

Solution:

$f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

Now $f(tx, ty) = \frac{1}{(tx)^2} + \frac{1}{txty} + \frac{\log tx - \log ty}{(tx)^2 + (ty)^2}$

$= \frac{1}{t^2x^2} + \frac{1}{t^2xy} + \frac{\log\frac{tx}{ty}}{t^2(x^2 + y^2)}$

$= \frac{1}{t^2}\left(\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}\right)$

$= t^{-2}\left(\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}\right)$

Therefore f(x, y) is a homogeneous function of degree -2

By Euler's theorem , $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f$

$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$

4. If u = $\mathbf{\tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)}$, show that $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u}$

Solution: Given u = $\tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$

$\Rightarrow \tan u = \left(\frac{x^3 + y^3}{x - y}\right)$

Let $z = \tan u = \left(\frac{x^3 + y^3}{x - y}\right)$

And z is a homogeneous function of order 2.

We know that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Here $f(u) = \tan u$

$\Rightarrow f'(u) = \sec^2 u$

Therefore by the result,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2\sin u \times \cos u = \sin 2u$$

(Or)

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2z$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{1}{\cos^2 u}\frac{\partial u}{\partial x} + y\frac{1}{\cos^2 u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{1}{\cos u}\frac{\partial u}{\partial x} + y\frac{1}{\cos u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{1}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u.$$

All the best

Questions
Unit II - Differential Calculus
Part A

1. Find $\frac{dy}{dx}$ given $y = x^2 + 2x$ CO1(L1)
2. Find the n^{th} derivative of e^{2x} CO1(L1)
3. Find $\frac{dy}{dx}$ given $y = \sqrt[3]{3x^2}$ CO1(L1)
4. Find $\frac{dy}{dx}$ given $y = \frac{3-2x}{3+2x}$ CO1(L1)
5. Find $\frac{dy}{dx}$ given $y = (1 - 5x)^6$ CO1(L1)
6. Find y_1 given $x^2 - y^2 - x = 1$ CO1(L1)
7. Recall Euler's theorem on homogenous functions. CO1(L1)
8. Examine Euler's theorem for the function $\varphi(x, y, z) = axy + byz + czx$ CO1(L4)
9. find y_1.,if $y = x^x$ CO4(L1)
10. Find y_2, if $y = e^x + e^{-x}$ CO1(L1)

Part – B

1. Find $\frac{dy}{dx}$, given $y = e^{\sqrt{\tan x}}$ CO4(L1)
2. Find y_2 given $y = \frac{1}{ax+b}$ CO1(L1)
3. Determine $\frac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$ CO4(L4)
4. Show that $y_2 - 2ay_1 + (a^2 + b^2)\, y = 0$, If $y = e^{ax} \sin bx$, CO4(L2)
5. Determine the n^{th} derivative of sin(ax+b) CO4(L4)
6. Show that $\frac{dy}{dx} = \frac{\cos x}{y-1}$ If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$ CO4(L2)
7. (a). Show that $\left(1 + x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$ if $y = \log\left(x + \sqrt{1+x^2}\right)$ CO4(L2)

   (b) Show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. if $x^y = e^{x-y}$ CO4(L2)
8. Find $\frac{dy}{dx}$,if $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ CO4(L2)

9. Apply Euler's theorem and show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\cot u$, CO4(L4)

given $u=\cos^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$

10. If $u=\log\left(\sqrt{x^2+y^2}\right)$ then prove that $u_{xx}+u_{yy}=0$. CO4(L2)

11. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$, If $u=\log\left(\frac{x^2+y^2}{x+y}\right)$ CO4(L2)

12. Examine Euler's theorem for the function $y=\tan^{-1}\left(\frac{y}{x}\right)$ CO4(L2)

13. Show that $x^2y_2+xy_1+y=0$ if $y=a\cos(\log x)$ CO4(L2)

14. Show that $x^2y_2-2xy_1+(x^2+2)y=0$ if y= asinx CO4(L2)

15. Show that $(1-x^2)y_2-xy_1=0$ and $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$ CO4(L2)

if $y=\sin^{-1}x$

SUBJECT NAME: ENGINEERING MATHEMATICS II

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMT1106

COURSE MATERIAL

UNIT-III INTEGRAL CALCULUS

Standard results

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

2. $\int \frac{1}{x}\, dx = \log x + c$

3. $\int e^x dx = e^x + c$

4. $\int \sin x\, dx = -\cos x + c$

5. $\int \cos x\, dx = \sin x + c$

6. $\int \tan x\, dx = \log \sec x + c$

7. $\int \cot x\, dx = \log \sin x + c$

8. $\int \sec x \tan x\, dx = \sec x + c$

9. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

10. $\int \sec^2 x\, dx = \tan x + c$

11. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

12. $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$

13. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$

14. $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + c$

Problems

1. Evaluate $\int \left(x + \frac{1}{x} \right)^3 dx$

Solution:

$$\int \left(x + \frac{1}{x} \right)^3 dx = \int \left(x^3 + \frac{1}{x^3} + 3x + \frac{3}{x} \right) dx$$

$$= \int \left(x^3 + x^{-3} + 3x + 3x^{-1} \right) dx$$

$$= \int x^3 dx + \int x^{-3} dx + 3\int x \, dx + 3\int x^{-1} dx$$

$$= \frac{x^4}{4} + \frac{x^{-2}}{-2} + \frac{3x^2}{2} + 3 \log x + c$$

$$= \frac{x^4}{4} - \frac{1}{2x^2} + \frac{3x^2}{2} + 3 \log x + c$$

2. Find $\int \frac{1}{\sin^2 x \cos^2 x} dx$

Solution:

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x \, dx + \int \cos ec^2 x \, dx$$

$$= \tan x - \cot x + c$$

3. Evaluate $\int \frac{1}{1 + \cos x} dx$

Solution:

$$\int \frac{1}{1 + \cos x} dx = \int \frac{1}{1 + \cos x} \times \frac{1 - \cos x}{1 - \cos x} dx$$

$$= \int \frac{1 - \cos x}{1 - \cos^2 x} dx = \int \cos ec^2 x \, dx - \int \cos ec \, x \cot x \, dx$$

$$= -\cot x + \cos ec \, x + c$$

Standard results

i. $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a\,(n+1)} + c \quad if \quad n \neq -1$

ii. $\int \dfrac{1}{ax+b} \, dx = \dfrac{1}{a} \log(ax+b) + c$

iii. $\int e^{ax+b} dx = \dfrac{e^{ax+b}}{a} + c$

iv. $\int \sin(ax+b) \, dx = \dfrac{-\cos(ax+b)}{a} + c$

v. $\int \cos(ax+b) \, dx = \dfrac{\sin(ax+b)}{a} + c$

vi. $\int \tan(ax+b) \, dx = \dfrac{\log[\sec(ax+b)]}{a} + c$

vii. $\int \cot(ax+b) \, dx = \dfrac{\log[\sin(ax+b)]}{a} + c$

viii. $\int \sec(ax+b) \tan(ax+b) \, dx = \dfrac{\sec(ax+b)}{a} + c$

ix. $\int \cos ec(ax+b) \cot(ax+b) \, dx = \dfrac{-\cos ec(ax+b)}{a} + c$

x. $\int \sec^2(ax+b) \, dx = \dfrac{\tan(ax+b)}{a} + c$

xi. $\int \cos ec^2(ax+b) \, dx = \dfrac{-\cot(ax+b)}{a} + c$

xii. $\int \dfrac{1}{a^2 - x^2} \, dx = \dfrac{1}{2a} \log\left(\dfrac{a+x}{a-x}\right) + c$

xiii. $\int \dfrac{1}{x^2 - a^2} \, dx = \dfrac{1}{2a} \log\left(\dfrac{x-a}{x+a}\right) + c$

xiv. $\int \dfrac{1}{a^2 + x^2} \, dx = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + c$

xv. $\int \dfrac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c$

xvi. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

xvii. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log\left(x+\sqrt{x^2+a^2}\right)+c$

xviii. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

xix. $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)+c$

xx. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+c$

Problems:

1. Evaluate

i. $\int \sin^3 x \cos^2 x\, dx$ **ii.** $\int \frac{\sin x}{\sin(x+a)}\, dx$

Solution:

i. We have $\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x(\sin x)\, dx$

$$= \int \left(1-\cos^2 x\right) \cos^2 x\, (\sin x)\, dx$$

Put $t = \cos x$ *so that* $dt = -\sin x\, dx$

Therefore, $\int \sin^2 x \cos^2 x\, (\sin x)\, dx = -\int \left(1-t^2\right) t^2 dt$

$$= -\int \left(t^2-t^4\right)dt = -\left(\frac{t^3}{3}-\frac{t^5}{5}\right)+c$$

$$= -\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+c$$

ii. *Put* $x+a=t$. *Then* $dx = dt$. *Therefore*

$$\int \frac{\sin x}{\sin(x+a)}\, dx = \int \frac{\sin(t-a)}{\sin t}\, dt$$

$$= \int \frac{\sin t\cos a-\cos t\sin a}{\sin t}\, dt$$

$$= \cos a \int dt - \sin a \int \cot t \, dt$$

$$= (\cos a)t - (\sin a)(\log \sin t) + c$$

$$= (\cos a)(x+a) - (\sin a)[\log \sin (x+a)] + c$$

$$= x \cos a + a \cos a - (\sin a)[\log \sin (x+a)] + c$$

$$\therefore \int \frac{\sin x}{\sin (x+a)} dx = x \cos a - \sin a \log[\sin (x+a)] + c_1$$

where $c_1 = a \cos a + c$ is another arbitrary constant.

2 . Evaluate $\int \frac{x^3 dx}{(x^2+1)^3}$

Solution:

Put $x^2 + 1 = t$

Then, $2x \, dx = dt$

$$\int \frac{x^3 \, dx}{(x^2+1)^3} = \int \frac{(t-1)\frac{dt}{2}}{t^3} = \frac{1}{2} \int \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$$

$$= \frac{1}{2}\left[\frac{-1}{t} + \frac{1}{2t^2} \right] + c$$

$$= \frac{1}{2}\left[\frac{-1}{x^2+1} + \frac{1}{2(x^2+1)^2} \right] + c$$

3. Evaluate $\int \frac{\sec^2 (\log x)}{x} dx$

Solution:

$$Put \ t = \log x, \quad \therefore dt = \frac{1}{x} dx$$

$$\therefore \int \frac{\sec^2 (\log x)}{x} dx = \int \sec^2 t \, dt$$

$$= \tan t + c = \tan(\log x) + c$$

Integration of rational function of the type $\frac{lx+m}{ax^2+bx+c}$

1. Evaluate $\int \frac{2x+3}{x^2+5x+7}dx$

Solution:

$Let\ 2x+3 = A(2x+5)+B$

$Equating\ the\ Coefficients\ of\ x,$

$2 = 2A \quad \therefore A=1$

Equating the constant term,

3 = 5A + B

$\therefore B = 3-5 = -2$

$$I = \int \frac{2x+3}{x^2+5x+7}dx$$

$$= \int \frac{1(2x+5)-2}{x^2+5x+7}dx$$

$$= \int \frac{2x+5}{x^2+5x+7}dx - 2\int \frac{dx}{x^2+5x+7}$$

$Put\ x^2+5x+7 = t$

$(2x+5)dx = dt$

$$\therefore I = \int \frac{dt}{t} - 2\int \frac{dx}{\left(x+\frac{5}{2}\right)^2+\frac{3}{4}}$$

$$= \log t - 2\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{5}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \log(x^2+5x+7) - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+5}{\sqrt{3}}\right)+c$$

2. Evaluate $\int \frac{5x+1}{x^2-2x-35}dx$

Solution:

Let 5x+1 = A (2x-2) + B

Equating the coefficient of x,

$$5 = 2A \quad \therefore A = \frac{5}{2}$$

Equating the Constant Terms, 1 = - 2A + B

Therefore, B = 6

$$\therefore I = \int \frac{5x+1}{x^2-2x-35}$$

$$= \int \frac{\frac{5}{2}(2x-2)+6}{x^2-2x-35}$$

$$= \frac{5}{2}\int \frac{(2x-2)}{x^2-2x-35}dx + 6\int \frac{dx}{(x-1)^2-36}$$

$$= \frac{5}{2}\log(x^2-2x-35)+6\frac{1}{2\ .\ 6}\log\left(\frac{x-1-6}{x-1+6}\right)+c$$

$$= \frac{5}{2}\log(x^2-2x-35)+\frac{1}{2}\log\left(\frac{x-7}{x+5}\right)+c$$

Integration of irrational function of the type $\frac{lx+m}{\sqrt{ax^2+bx+c}}$

1. Evaluate $\int \frac{2x+2}{\sqrt{x^2+4x+7}}dx$

Solution:

Let 2x + 2 = A (2x + 4) + B

Equating the coefficient of x, 2A = 2 $\therefore A = 1$

Equating the constant terms, 2 = 4 A + B $\therefore B = -2$

$$I = \int \frac{(2x+4-2)}{\sqrt{x^2+4x+7}}dx = \int \frac{2x+4}{\sqrt{x^2+4x+7}}dx - \int \frac{2}{\sqrt{x^2+4x+7}}dx$$

$$= 2\sqrt{x^2+4x+7} - 2\log\left((x+2)+\sqrt{(x+2)^2+3}\right)+c$$

2. Evaluate $\int \sqrt{\frac{1-x}{1+x}}\ dx$

Solution:

$$\int \sqrt{\frac{1-x}{1+x}}\ dx = \int \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x + \sqrt{1-x^2} + c$$

Integration of the function of the type $\frac{1}{(ax+b)\sqrt{lx^2+mx+n}}$

1. Find $\int \frac{1}{(1+x)\sqrt{1+x^2}} dx$

Solution:

Put $\frac{1}{1+x} = t$, x + 1 = $\frac{1}{t}$, log (1+ x) = - log t , $\frac{dx}{1+x} = -\frac{dt}{t}$

$$\therefore \int \frac{1}{(1+x)\sqrt{1+x^2}} dx = \int \frac{-dt/t}{\sqrt{1+\left(\frac{1}{t}-1\right)^2}}$$

$$= \int \frac{-dt}{t\sqrt{1+\frac{1}{t^2}+1-\frac{2}{t}}} = -\int \frac{dt}{\sqrt{2t^2-2t+1}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2-t+\frac{1}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}}$$

$$= -\frac{1}{\sqrt{2}} \log\left[\left(t-\frac{1}{2}\right)+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}\right]+c = -\frac{1}{\sqrt{2}} \log\left[(2t-1)+\sqrt{(2t-1)^2+1}\right]+c$$

where $t = \frac{1}{1+x}$

2. Evaluate $\int \frac{1}{x\sqrt{x^2+6x+109}}\,dx$

Solution:

$$Put\ t = \frac{1}{x}\ \ or\ \ x = \frac{1}{t}$$

$$\therefore dx = \frac{-1}{t^2}dt$$

$$I = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{6}{t}+109}}$$

$$= \int \frac{-dt}{\sqrt{109\,t^2+6t+1}}$$

$$= -\frac{1}{\sqrt{109}}\int \frac{dt}{\sqrt{t^2+\frac{6t}{109}+\frac{1}{109}}}$$

$$= -\frac{1}{\sqrt{109}}\int \frac{dt}{\sqrt{\left(t+\frac{3}{109}\right)^2+\frac{1}{109}-\frac{9}{109^2}}}$$

$$= -\frac{1}{\sqrt{109}}\int \frac{dt}{\sqrt{\left(t+\frac{3}{109}\right)^2+(\frac{10}{109})^2}}$$

$$= -\frac{1}{\sqrt{109}}\log\left[\left(t+\frac{3}{109}\right)+\sqrt{t^2+\frac{6t}{109}+\frac{1}{109}}\right]+c$$

$$where, t = \frac{1}{x}$$

Integrals of the type $\int \frac{a\sin x+\ b\cos x}{c\sin x+d\cos x}dx$

1. Evaluate $\int \frac{2\sin x+3\cos x}{4\sin x+5\cos x}dx$

Solution:

2 sin x + 3 cosx = A (4 sin x + 5 cos x) + B(4 cos x – 5 sin x)

Equating the coefficients of sin x and cos x, we get A= $\frac{23}{41}$, B = $\frac{2}{41}$

$$I = \int \frac{\frac{23}{41}(4\sin x + 5\cos x) + \frac{2}{41}(4\cos x - 5\sin x)}{4\sin x + 5\cos x}dx$$

$$= \frac{23}{41}\int dx + \frac{2}{41}\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx = \frac{23}{41}x + \frac{2}{41}\log(4\sin x + 5\cos x) + c.$$

2. Evaluate $\int \frac{dx}{1+\tan x}$

Solution:

$$I = \int \frac{dx}{1+\tan x} = \int \frac{\cos x}{\cos x + \sin x}dx$$

Cos x = A(cos x + sin x) + B(-sin x + cos x)

A = $\frac{1}{2}$, B = $\frac{1}{2}$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\log(\sin x + \cos x) + c = \frac{1}{2}x + \frac{1}{2}\log(\sin x + \cos x) + c$$

INTEGRATION USING PARTIAL FRACTIONS

S.No.	Form of the Rational Function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$,where x^2+bx+c cannot be factorized further.	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Problems:

1. Find $\int \frac{dx}{(x+1)(x+2)}$

Solution:

The Integrand is a proper rational function. So we write,

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x+1)$$

Equating the coefficient of x-term and the constant term, we get

A + B = 0 and 2A+B = 1

Solving we get A = 1 and B = -1

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{(x+1)} - \int \frac{dx}{(x+2)}$$

= log(x+1) - log (x+2) + c

$$= \log\left(\frac{x+1}{x+2}\right) + c$$

2: Find $\int \frac{x^2 + 1}{x^2 - 5x + 6} dx$

Solution:

Here the integrand is not a proper rational function. So we divide $x^2 + 1$ by $x^2 - 5x + 6$

$$\frac{x^2 + 1}{x^2 - 5x + 6} = 1 + \frac{5x - 5}{x^2 - 5x + 6}$$

Now $\frac{5x - 5}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3}$

5x – 5 = A (x-3) + B (x-2)

Equating the coefficient of x-term and the constant term, we get

A + B = 5 and 3A + 2B = 5 solving we get A = -5 and B = 10

$$\frac{x^2 + 1}{x^2 - 5x + 6} = 1 - \frac{5}{x-2} + \frac{10}{x-3}$$

$$\int \frac{x^2 + 1}{x^2 + 5x + 6} dx = \int dx - \int \frac{5}{x-2} dx + \int \frac{10}{x-3} dx$$

= x - 5 log (x-2) + 10 log (x-3) + c.

3. Find $\int \frac{3x-2}{(x+1)^2 (x+3)} dx$

Solution:

$$\frac{3x-2}{(x+1)^2 \ (x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$$

3x – 2 = A (x+1)(x+3) + B(x+3) + C$(x+1)^2$

Equating the coefficient of x^2 , x-term and the constant term, we get

A +C = 0 , 4A + B +2C =3, 3A + 3B + C = -2. Solving these equations we get

A = $\frac{11}{4}$, B = $\frac{-5}{2}$ and C = $\frac{-11}{4}$

$$\frac{3x-2}{(x+1)^2 \ (x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

$$= \frac{11}{4}\log(x+1) + \frac{5}{2(x+1)} - \frac{11}{4}\log(x+3) + c$$

$$= \frac{11}{4}\log\left(\frac{x+1}{x+3}\right) + \frac{5}{2(x+1)} + c \ .$$

INTEGRATION BY PARTS:

$$\int u \ dv = u \ v - \int v \ du$$

1. Find $\int x \cos x \, dx$

Solution:

Let u =x, dv =cosx dx

Then integration by parts gives,

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

2. Find $\int \log x\,dx$

Solution:

Let u = logx , dv = dx

Then, $\int \log x\,dx = (\log x)x - \int \frac{1}{x}x\,dx$

$$= x(\log x) - x + c$$

3. Find $\int xe^x\,dx$

Solution:

Let u = x , dv = e^xdx

$$\int xe^x\,dx = xe^x - \int 1e^x\,dx = xe^x - e^x + c$$

4. Find $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution:

Let u = $\sin^{-1}$x , dv = x/√(1-x^2)dx

For finding v,

Put t =1 – x^2 then dt = -2x dx

Then v = $\int \frac{-dt}{2\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$

$$\therefore \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \sin^{-1}x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + c$$

BERNOULLI'S FORMULA

$$\int u\,dv = uv - u'v_1 + u''v_2 + \ldots$$

Problems

1. Solve $\int x^2 e^x \, dx$

Solution:

$$\int x^2 e^x \, dx = x^2 e^x - 2x(e^x) + 2e^x + C$$

2. Solve $\int x \sin ax \, dx$

Solution:

$$\int x \sin ax \, dx = x\left(\frac{-\cos ax}{a}\right) - \left(\frac{-\sin ax}{a^2}\right) + C$$

3. Solve $\int (ax^2 + bx + c) \cos x \, dx$

Solution:

$$\int (ax^2 + bx + c) \cos x \, dx = (ax^2 + bx + c)(\sin x) + (2ax + b)(-\cos x) + 2a(-\sin x) + c$$

DEFINITE INTEGRAL

PROPERTIES OF DEFINITE INTEGRALS:

1. $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

2. $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

3. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)\,dx \quad if \quad f(x) \; is \; even$

$= 0 \; if \quad f(x) \; is \; odd$

4. $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, a < c < b

5. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx \quad if \quad f(2a-x) = f(x)$

$= 0 \; if \quad f(2a-x) = -f(x)$

6. $\int_0^{\pi} f(\sin x)dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

Problems:

1.Solve $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Solution:

$$I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad (1)$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}++\sqrt{\cos x}}\,dx \quad (2)$$

$$(1)+(2) \Rightarrow 2I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx=\frac{\pi}{2}$$

Hence $I=\frac{\pi}{4}$.

2. Solve $\int_0^{\frac{\pi}{2}} \log\sin x\,dx$

Solution:

$$I=\int_0^{\frac{\pi}{2}} \log\sin x\,dx$$

$$=\int_0^{\frac{\pi}{2}} \log\cos x\,dx \text{ (by property 2)}$$

$$2I=\int_0^{\frac{\pi}{2}} (\log\sin x+\log\cos x)\,dx$$

$$=\int_0^{\frac{\pi}{2}} \log(\sin x\cos x)\,dx=\int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx=\int_0^{\frac{\pi}{2}} \log\sin 2x\,dx-\int_0^{\frac{\pi}{2}} \log 2\,dx$$

$$= \int_0^{\pi} \log \sin y \left(\frac{dy}{2}\right) - \frac{\pi}{2}\log 2 = \frac{1}{2}\int_0^{\pi} \log \sin y \; dy - \frac{\pi}{2}\log 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin y \; dy - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2.$$

3. Solve $\int_0^{\frac{\pi}{2}} \log \; (\tan x + \cot x)\, dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \log \; (\tan x + \cot x)\, dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin^2 x + \cos^2 x}{\sin x . \cos x}\right) dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\sin x . \cos x}\right) dx = -\int_0^{\frac{\pi}{2}} \log \sin x \, dx - \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

$$= \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2 = \pi \log 2.$$

4. Solve $\int_0^{\frac{\pi}{4}} \log \; (1 + \tan x)\, dx$

Solution:

$$I = \int_0^{\frac{\pi}{4}} \log \; (1 + \tan x)\, dx = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + \tan x}\right) dx = \int_0^{\frac{\pi}{4}} \log 2 - \log(1 + \tan x)\; dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2 \;\; dx - I$$

$$2I = \frac{\pi}{4}\log 2 ,$$

$$I = \frac{\pi}{8}\log 2$$

Questions

Unit III - Integral Calculus

Part A

1. List any two properties of definite integrals. CO2(L1)
2. Evaluate $\int \sin^2 3x dx$ CO2(L5)
3. Evaluate $\int \frac{\sin(\log x)}{x} dx$ CO5(L5)
4. Evaluate $\int \log x \, dx$ CO5(L5)
5. Evaluate $\int \frac{dx}{\sqrt{a^2 - x^2}}$ CO5(L5)
6. Evaluate $\int e^x x \, dx$ CO5(L5)
7. Evaluate $\int \tan^2 x \, dx$ CO5(L5)
8. Evaluate $\int \sqrt{x^2 - a^2} \, dx$ CO5(L5)
9. Show that $\int_a^b f(x)dx = -\int_b^a f(x)dx$ CO2(L2)
10. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x} dx$ CO2(L5)
11. Evaluate $\int_0^1 \frac{dx}{1+x}$ CO2(L5)
12. Evaluate $\int_0^2 \frac{dx}{4+x^2}$ CO2(L5)
13. Evaluate $\int_0^1 x^2 (1-x)^9 dx.$ CO2(L5)

Part B

1. Evaluate $\int \frac{3x}{1+2x^4} dx$ CO5(L5)
2. Evaluate $\int_0^\infty x e^{-x^2} dx$ CO2(L5)
3. Evaluate $\int \frac{dx}{4+5\cos x}$ CO5(L5)
4. Evaluate $\int \frac{3x+5 \ dx}{x^2+4x+7}$ CO5(L5)

5. Evaluate i) $\int e^{3x}\sin 4x\ dx$ ii) $\int \frac{x+2\,dx}{x^2+4x-5}$ CO5(L5)

6. Evaluate $\int \frac{(2\sin x\ +\cos x)\ dx}{3\sin x\ +\cos x}$ CO5(L5)

7. Evaluate $\int \frac{(2\sin x\ +\cos x)\ dx}{3\sin x\ +\cos x}$ CO5(L5)

8. Show that $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \frac{\pi}{8}\log 2.$ CO2(L2)

9. Evaluate $\int \frac{x}{(x-2)(x+3)}dx$ CO5(L5)

10. Evaluate $\int_0^{\infty} x^4 e^{-x}\ dx$ CO2(L5)

11. .Evaluate $\int \frac{\sqrt{5-x}\ dx}{\sqrt{2-x}}$ CO5(L5)

12. .Evaluate $\int \frac{x+4\ dx}{6x-7-x^2}$ CO5(L5)

13. Evaluate $\int \frac{3x+1}{(x-1)^2(x+3)}dx$ CO5(L5)

14. Evaluate $\int_0^{\frac{\pi}{2}} \log\sin x dx$ CO2(L5)

15. Evaluate $\int \frac{\sqrt{a^2-x^2}}{x^4}dx.$ CO5(L5)

16. Show that $\int_0^{a} \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$ CO2(L5)

17. Evaluate $\int_0^{\frac{\pi}{6}} \frac{\cos x dx}{3+4\sin x}$ CO2(L5)

SUBJECT NAME: MATRICES, CALCULUS and SAMPLING

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMTA1202

COURSE MATERIAL

UNIT- IV VECTOR CALCULUS

Definitions

Scalars

The quantities which have only magnitude and are not related to any direction in space are called scalars. Examples of scalars are (i) mass of a particle (ii) pressure in the atmosphere (iii) temperature of a heated body (iv) speed of a train.

Vectors

The quantities which have both magnitude and direction are called vectors.

Examples of vectors are (i) the gravitational force on a particle in space (ii) the velocity at any point in a moving fluid.

Scalar point function

If to each point p(x,y,z) of a region R in space there corresponds a unique scalar f(p) then f is called a scalar point function.

Example

The temperature distribution in a heated body, density of a body and potential due to a gravity.

Vector point function

If to each point p(x,y,z) of a region R in space there corresponds a unique vector $\vec{f}(p)$ then $\vec{f}$ is called a vector point function.

Example

The velocity of a moving fluid, gravitational force.

Scalar and vector fields

When a point function is defined at every point of space or a portion of space, then we say that a field is defined. The field is termed as a scalar field or vector field as the point function is a scalar point function or a vector point function respectively.

Vector Differential Operator (∇)

The vector differential operator Del, denoted by ∇ is defined as

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Gradient of a scalar point function

Let $\phi(x, y, z)$ be a scalar point function defined in a region R of space. Then the vector point function given by $\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$

$= \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$ is defined as the gradient of ϕ and denoted by

grad ϕ

Directional Derivative (D.D)

The directional derivative of a scalar point function ϕ at point (x,y,z) in the direction of a vector $\vec{a}$ is given by D.D = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$ (or) D.D = $\nabla\phi\,.\,\hat{a}$

The unit normal vector

The unit vector normal to the surface $\phi(x, y, z)$ = c is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Angle between two surfaces

Angle between the surfaces $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ is given by $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Problems

1) Find $\nabla\phi$ if $\phi(x, y, z) = xy - y^2 z$ at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(xy - y^2 z)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(xy - y^2 z) + \vec{j}\,\frac{\partial}{\partial y}(xy - y^2 z) + \vec{k}\,\frac{\partial}{\partial z}(xy - y^2 z)$$

$$= y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k}$$

$$\therefore \nabla\phi == y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k}\,.$$

At (1,1,1), $\nabla\phi = \vec{i}(1) + \vec{j}(1 - (2)(1)(1)) - \vec{k}(1)^2$

$$= \vec{i} - \vec{j} - \vec{k}$$

2) Find $\nabla\phi$ if $\phi(x,y,z) = x^2 y + 2xz^2 - 8$ at the point (1,0,1)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(x^2 y + 2xz^2 - 8)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^2 y + 2xz^2 - 8) + \vec{j}\,\frac{\partial}{\partial y}(x^2 y + 2xz^2 - 8) + \vec{k}\,\frac{\partial}{\partial z}(x^2 y + 2xz^2 - 8)$$

$$= (2xy + 2z^2)\vec{i} + (x^2)\vec{j} + 4xz\vec{k}$$

At (1,0,1), $\nabla\phi = \vec{i}(2(1)(0) + 2(1^2)) + \vec{j}(1^2) + \vec{k}\,4(1)(1)$

$$= 2\vec{i} + \vec{j} + 4\vec{k}$$

3) Find the unit normal vector to the surface $\phi(x,y,z) = x^2 yz^3$

at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(x^2 yz^3) = \vec{i}\,\frac{\partial}{\partial x}(x^2 yz^3) + \vec{j}\,\frac{\partial}{\partial y}(x^2 yz^3) + \vec{k}\,\frac{\partial}{\partial z}(x^2 yz^3)$$

$= 2xyz^3\vec{i} + x^2 z^3 \vec{j} + 3x^2 yz^2 \vec{k}$

At(1,1,1), $\nabla\phi = \vec{i}\,2(1)(1)(1) + \vec{j}(1^2)(1^3) + \vec{k}\,3(1^2)(1)(1^2)$

$= 2\vec{i} + \vec{j} + 3\vec{k}$

$|\nabla\phi| = \sqrt{2^2 + 1^2 + 3^2}$

$= \sqrt{14}$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{14}}$$

4) Find the unit normal vector to the surface $\phi(x, y, z) = x^2 + y^2 - z$ **at the point (1,-1,-2)**

Solution:

$\nabla\phi = (\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z})\phi$

$\nabla\phi = (\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z})(x^2 + y^2 - z)$

$= \vec{i}\dfrac{\partial}{\partial x}(x^2 + y^2 - z) + \vec{j}\dfrac{\partial}{\partial y}(x^2 + y^2 - z) + \vec{k}\dfrac{\partial}{\partial z}(x^2 + y^2 - z)$

$= 2x\vec{i} + 2y\vec{j} - \vec{k}$

At (1,-1,-2), $\nabla\phi = \vec{i}\,2(1) + \vec{j}\,2(-1) - \vec{k}$

$= 2\vec{i} - 2\vec{j} - \vec{k}$

$|\nabla\phi| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} - 2\vec{j} - \vec{k}}{3}$$

5) Find the angle between the surfaces xyz and x^3yz at the point (1,1,-2)

Solution:

Given the surface $\phi_1(x,y,z) = xyz$

$$\nabla\phi_1 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi_1$$

$$\nabla\phi_1 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xyz)$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

At(1,1,-2), $\nabla\phi_1 = \vec{i}(1)(-2) + \vec{j}(1)(-2) + (1)(1)\vec{k} = -2\vec{i} - 2\vec{j} + \vec{k}$

$$|\nabla\phi_1| = \sqrt{(-2)^2 + (-2)^2 + 1^2}$$

= 3

Given the surface $\phi_2(x,y,z) = x^3yz$

$$\nabla\phi_2 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi_2$$

$$\nabla\phi_2 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^3yz)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^3yz) + \vec{j}\frac{\partial}{\partial y}(x^3yz) + \vec{k}\frac{\partial}{\partial z}(x^3yz)$$

$$= 3x^2yz\vec{i} + x^3z\vec{j} + x^3y\vec{k}$$

At (1,1,-2), $\nabla\phi_2 = \vec{i}\,3(1^2)(1)(-2) + \vec{j}(1^3)(-2) + (1^3)(1)\vec{k} = -6\vec{i} - 2\vec{j} + \vec{k}$

$$|\nabla\phi_2| = \sqrt{(-6)^2 + (-2)^2 + 1^2} = \sqrt{41}$$

Angle between the surfaces is given by $\cos\theta = \dfrac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(-2\vec{i} - 2\vec{j} + \vec{k}).(-6\vec{i} - 2\vec{j} + \vec{k})}{3\sqrt{41}}$$

$$= \frac{12+4+1}{3\sqrt{41}} = \frac{17}{3\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{17}{3\sqrt{41}}\right)$$

6) Find the angle between the normal to the surface xy - z² at the point (1,4,-2) and (1,2,3)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(xy - z^2) = \vec{i}\,\frac{\partial}{\partial x}(xy - z^2) + \vec{j}\,\frac{\partial}{\partial y}(xy - z^2) + \vec{k}\,\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\vec{i} + x\vec{j} - 2z\vec{k}$$

At (1,4,-2), $\nabla\phi_1 = \vec{i}\,(4) + \vec{j}(1) - 2(-2)\vec{k}$

$= 4\vec{i} + \vec{j} + 4\vec{k}$

$$|\nabla\phi| = \sqrt{4^2 + 1^2 + 4^2}$$

$= \sqrt{33}$

At (1,2,3), $\nabla\phi_2 = \vec{i}\,(2) + \vec{j}(1) - 2(3)\vec{k}$

$= 2\vec{i} + \vec{j} - 6\vec{k}$

$$|\nabla\phi| = \sqrt{2^2 + 1^2 + (-6)^2}$$

$= \sqrt{41}$

Angle between the surfaces is given by $\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(2\vec{i} + \vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{41}}$$

$$= \frac{8+1-24}{\sqrt{33}\sqrt{41}} = \frac{-15}{\sqrt{33}\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-15}{\sqrt{33}\sqrt{41}}\right)$$

7) Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at the point (2,-1,1) in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xy^2 + yz^3) = \vec{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \vec{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \vec{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

At (2,-1,1), $\nabla\phi = \vec{i}(-1^2) + \vec{j}(2(2)(-1) + 1^3) + 3(-1)(1^2)\vec{k}$

$= \vec{i} - 3\vec{j} - 3\vec{k}$

To find the directional derivative of ϕ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ find the unit vector along the direction

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (2,-1,1) = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}).\frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \frac{1-6-6}{3} = \frac{-11}{3} \text{ units.}$$

8) Find the directional derivative of $\phi(x,y,z) = xyz + yz^2$ at the point (1,1,1) in the direction of $\vec{i} + \vec{j} + \vec{k}$

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xyz + yz^2) = \vec{i}\frac{\partial}{\partial x}(xyz + yz^2) + \vec{j}\frac{\partial}{\partial y}(xyz + yz^2) + \vec{k}\frac{\partial}{\partial z}(xyz + yz^2)$$

$$= yz\vec{i} + (xz + z^2)\vec{j} + (xy + 2yz)\vec{k}$$

At (1,1,1), $\nabla\phi = \vec{i}(1)(1) + \vec{j}((1)(1) + 1^2) + ((1)(1) + 2(1)(1))\vec{k}$

$$= \vec{i} + 2\vec{j} + 3\vec{k}$$

To find the directional derivative of ϕ in the direction of the vector $\vec{i} + \vec{j} + \vec{k}$ find the unit vector along the direction

$$\vec{a} = \vec{i} + \vec{j} + \vec{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (1,1,1) = $\nabla\phi . \frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}).\frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}}$$

$$= \frac{1 + 2 + 3}{\sqrt{3}} = \frac{6}{\sqrt{3}} \text{ units.}$$

Divergence of a differentiable vector point function $\vec{F}$

The divergence of a differentiable vector point function $\vec{F}$ is denoted by div $\vec{F}$ and is defined by

div $\vec{F}$ = $\nabla \cdot \vec{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot \vec{F}$

$$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \quad \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl of a vector point function

The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and is defined by

Curl $\vec{F} = \nabla \times \vec{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \times \vec{F}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Vector Identities

Let ϕ be a scalar point function and $\vec{U}$ and $\vec{V}$ be vector point functions. Then

(1) $\nabla \cdot (\vec{U} \pm \vec{V}) = \nabla \cdot \vec{U} \pm \nabla \cdot \vec{V}$
(2) $\nabla \times (\vec{U} \pm \vec{V}) = \nabla \times \vec{U} \pm \nabla \times \vec{V}$
(3) $\nabla \cdot (\phi\vec{U}) = \nabla\phi \cdot \vec{U} + \phi\nabla \cdot \vec{U}$
(4) $\nabla \times (\phi\vec{U}) = \nabla\phi \times \vec{U} + \phi\nabla \times \vec{U}$
(5) $\nabla \cdot (\vec{U} \times \vec{V}) = \vec{V} \cdot (\nabla \times \vec{U}) - \vec{U} \cdot (\nabla \times \vec{V})$
(6) $\nabla \times (\vec{U} \times \vec{V}) = (\nabla \cdot \vec{V})\vec{U} - (\nabla \cdot \vec{U})\vec{V} + \vec{U}(\vec{V} \cdot \nabla) - \vec{V}(\vec{U} \cdot \nabla)$
(7) $\nabla(\vec{U} \cdot \vec{V}) = (\nabla \cdot \vec{V})\vec{U} + (\nabla \cdot \vec{U})\vec{V} + \vec{U} \times (\nabla \times \vec{V}) - (\nabla \times \vec{U}) \times \vec{V}$

Solenoidal and Irrotational vectors

A vector point function is solenoidal if div $\vec{F}$ = 0 and it is irrotational if curl $\vec{F}$ = 0.

Note:

If $\vec{F}$ is irrotational, then there exists a scalar function called Scalar Potential ϕ such that

$\vec{F} = \nabla\phi$

Problems

1) Find div $\vec{r}$ and curl $\vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution:

div $\vec{r} = \nabla \cdot \vec{r} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot \vec{r}$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$

$$=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3.$$

Curl $\vec{r}$ = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)=0.$$

2) Find the divergence and curl of the vector $\vec{V}=xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k}$ at the point (1,-1,1)

Solution:

Given $\vec{V}=xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k}$

div$\vec{V}$ = $\nabla\cdot\vec{V}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot\vec{V}$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k})$$

$$=\frac{\partial(xyz)}{\partial x}+\frac{\partial(3xy^2)}{\partial y}+\frac{\partial(xz^2-y^2z)}{\partial z}$$

$= yz+6xy+2xz-y^2$

At (1,-1,1), $\nabla\cdot\vec{V}=(-1).1+6(1)(-1)+2(1)(1)-(-1)^2$

= -1-6+2-1 = -6.

Curl $\vec{V}$ = $\nabla\times\vec{V}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\times\vec{V}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy^2 & xz^2-y^2z \end{vmatrix}$$

$$= \vec{i}\,(\frac{\partial}{\partial y}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(3xy^2)) - \vec{j}(\frac{\partial}{\partial x}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(xyz)) + \vec{k}(\frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial y}(xyz)) .$$

$$= \vec{i}\,(-2yz) - \vec{j}(z^2 - yx) + \vec{k}(3y^2 - xz) .$$

At (1,-1,1), $\nabla \times \vec{V} = \vec{i}\,(-2(-1)(1)) - \vec{j}(1^2 - (-1)(1)) + \vec{k}((3(-1)^2 - 1(1))$

$$= 2\vec{i} - 2\vec{j} + 2\vec{k}$$

3) Find the constants a, b, c so that $\vec{F} = (x + 2y + ax)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational**.

Solution:

Given $\nabla \times \vec{F} = 0$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \vec{i}\,(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)) - \vec{j}(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az) + \\ \vec{k}(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az) \end{bmatrix} = 0 .$$

$$\Rightarrow \vec{i}\,(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = 0 .$$

c+1 = 0, 4-a = 0, b-2 = 0

Hence c = -1, a = 4, b = 2.

4) Prove that $\vec{F} = (2x + yz)\vec{i} + (4y + zx)\vec{j} - (6z - xy)\vec{k}$ **is both solenoidal**

and irrotational.

Solution:

$$\nabla \cdot \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{V}$$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot((2x+yz)\vec{i}+(4y+zx)\vec{j}-(6z-xy)\vec{k})$$

$$=\frac{\partial(2x+yz)}{\partial x}+\frac{\partial(4y+zx)}{\partial y}-\frac{\partial(6z-xy)}{\partial z}$$

= 2+4-6 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+yz & 4y+zx & -(6z-xy)\end{vmatrix}$$

$$=\begin{bmatrix}\vec{i}(\frac{\partial}{\partial y}(-(6z-xy))-\frac{\partial}{\partial z}(4y+zx))-\vec{j}(\frac{\partial}{\partial x}(-(6z-xy))-\frac{\partial}{\partial z}(2x+yz)+\\ \vec{k}(\frac{\partial}{\partial x}(4y+zx)-\frac{\partial}{\partial y}(2x+yz)\end{bmatrix}$$

$\Rightarrow \vec{i}(x-x)-\vec{j}(y-y)+\vec{k}(z-z)=0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

5) Prove that $\vec{F}=(y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k}$ **is both solenoidal and irrotational and find its scalar potential**.

Solution:

$$\nabla\cdot\vec{F}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot\vec{F}$$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot((y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k})$$

$$=\frac{\partial(y^2-z^2+3yz-2x)}{\partial x}+\frac{\partial(3xz+2xy)}{\partial y}+\frac{\partial(3xy-2xz+2z)}{\partial z}$$

= -2+2x-2x+2 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= \left[\begin{array}{l} \vec{i}\,(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)) - \vec{j}\,(\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x) + \\ \vec{k}\,(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x) \end{array} \right]$$

$\Rightarrow \vec{i}\,(3x - 3x) - \vec{j}\,(3y - 2z + 2z - 3y) + \vec{k}\,(3z + 2y - 2y - 3z) = 0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$\Rightarrow (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k} = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = y^2 - z^2 + 3yz - 2x \quad \text{...(1)}$$

$$\frac{\partial\phi}{\partial y} = 3xz + 2xy \quad \text{...(2)}$$

$$\frac{\partial\phi}{\partial z} = 3xy - 2xz + 2z \quad \text{...(3)}$$

Integrating (1) with respect to ‘x’ treating ‘y’ and ‘z’ as constants, we get

$$\phi = xy^2 - xz^2 + 3xyz - 2\frac{x^2}{2} + f(y,z) \quad \text{...(4)}$$

Integrating (2) with respect to ‘y’ treating ‘x’ and ‘z’ as constants, we get

$$\phi = 3xyz + 2\frac{xy^2}{2} + f(x,z) \quad \text{...(5)}$$

Integrating (3) with respect to ‘z’ treating ‘x’ and ‘y’ as constants, we get

$$\phi = 3xyz - 2x\frac{z^2}{2} + 2\frac{z^2}{2} + f(x,y) \quad \text{.....(6)}$$

Hence from equations (4), (5), (6), we get

$$\phi = 3xyz + xy^2 - xz^2 - x^2 + z^2 + c$$

6) Prove that $\vec{F} = 3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k}$ is irrotational and find its scalar potential.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2 & 2x^3y + \cos z & -y\sin z \end{vmatrix}$$

$$= \begin{bmatrix} \vec{i}\left(\frac{\partial}{\partial y}(-y\sin z) - \frac{\partial}{\partial z}(2x^3y + \cos z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-y\sin z) - \frac{\partial}{\partial z}(3x^2y^2)\right) \\ \vec{k}\left(\frac{\partial}{\partial x}(2x^3y + \cos z) - \frac{\partial}{\partial y}(3x^2y^2)\right) \end{bmatrix}$$

$\Rightarrow \vec{i}(-\sin z - (-\sin z)) - \vec{j}(0 - 0) + \vec{k}(6x^2y - 6x^2y) = 0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = 3x^2y^2 \quad \text{.....(1)}$$

$$\frac{\partial\phi}{\partial y} = 2x^3y + \cos z \quad \text{.....(2)}$$

$$\frac{\partial \phi}{\partial z} = -y \sin z \quad \text{……………………………………………………………………..……(3)}$$

Integrating (1) with respect to ‘x’ treating ‘y’ and ‘z’ as constants, we get

$$\phi = 3\frac{x^3 y^2}{3} + f(y,z) \quad \text{……………………………………………………(4)}$$

Integrating (2) with respect to ‘y’ treating ‘x’ and ‘z’ as constants, we get

$$\phi = 2\frac{x^3 y^2}{2} + y\cos z + f(x,z) \quad \text{………………………………………………………………..………(5)}$$

Integrating (3) with respect to ‘z’ treating ‘x’ and ‘y’ as constants, we get

$$\phi = y\cos z + f(x,y) \quad \text{……………………………………………………………..(6)}$$

Hence from equations (4), (5), (6), we get

$$\phi = x^3 y^2 + y\cos z + c$$

Questions

Unit IV - Vector Calculus

Part-A

1. Find the directional derivative of $x^2 + 2xy$ at (1, –1, 3) in the direction of x axis CO2(L1)

2. Find $\nabla \phi$ if $\phi = x^3 y^2 z^4$. CO2(L1)

3. Show that $\nabla^2 u = 0$ if $u = x^2 - y^2$ CO2(L2)

4. Find div $\vec{F}$, if $\vec{F} = x^2 y\vec{i} + xz\vec{j} + 2yz\vec{k}$ at (2,-1,1) CO2(L1)

5. Define gradient of a vector. CO2(L1)

6. Find curl $\vec{F}$, If $\vec{F} = x^2 y\vec{i} + xz\vec{j} + 2yz\vec{k}$ CO2(L1)

7. Show that curl $\vec{F} = \vec{0}$, if $\nabla.(\vec{u} \times \vec{\mathbf{v}}) = \vec{\mathbf{v}}.(\nabla \times \vec{u}) - \vec{u}.(\nabla \times \vec{\mathbf{v}})$ = z i +x j +y k, CO2(L2)

8. Show Curl (gradϕ) = $\vec{0}$. CO2(L2)

9. Find the value of 'a' so that the vector $\overline{F} = (x+3y)\hat{i} + (ay-3z)\hat{j} + (x=3z)\hat{k}$ is solenoidal CO2(L2)

10. Show that $\nabla.(\vec{u} \times \vec{\mathbf{v}}) = \vec{\mathbf{v}}.(\nabla \times \vec{u}) - \vec{u}.(\nabla \times \vec{\mathbf{v}})$ CO2(L2)

Part-B

1. Find the maximum directional derivative of $F = 2x^2 + 3y^2 + 5z^2$ at (1,1,-4). CO2(L2)
2. Find the angle between two surfaces $Z = x^3 + y^3 - 3$ and $x^2 + y^2 + z^2 = 9$ at (2, -1, 2). CO2(L2)

3. Show that the vector $\vec{F} = (3x^2 + 2y^2 + 1)\vec{i} + (4xy - 3zy^2 - 3)\vec{j} + (2 - y^3)\vec{k}$ is irrotational and find its scalar potential. CO2(L2)

4. Find the constant c so that $\overrightarrow{F}$ is irrotational CO2(L2)
   Given $\vec{F} = (cxy - z^3)\vec{i} + (c-2)x^2\vec{j} + (1-c)xz^2\vec{k}$
5. Show that $\nabla^2 r^n = n(n+1)r^{n-2}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ CO2(L2)

6. Find the unit normal vector to the surface $x^2 - y^2 + z = 2$ at (1,1,2) CO2,L2)

7. Find (a) $\nabla.\vec{F}$ (b) $\nabla.(\nabla.\vec{F})$ (c) $\nabla\times(\nabla.\vec{F})$ (d) $(\nabla\times\vec{F})$ (e) $\nabla.(\nabla\times\vec{F})$ CO2(L2)

(f) $\nabla\times(\nabla\times\vec{F})$ at the point (1,1,1), if $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

8. show that $div\left(\frac{\vec{r}}{r^3}\right) = 0$, if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and r = $|\vec{r}|$, CO2(L2)

9. Find the constants a,b,c so that CO2(L1)
$\vec{f} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

10. Show that $\vec{F} = \left(6xy+z^3\right)\vec{i} + \left(3x^2-z\right)\vec{j} + \left(3xz^2-y\right)\vec{k}$ is irrotational CO2(L2)
vector and find the Scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

SUBJECT NAME: MATRICES, CALCULUS and SAMPLING

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMTA 1202

COURSE MATERIAL

UNIT- V THEORY OF SAMPLING AND TESTING OF HYPOTHESIS

Population: The group of individuals, under study is called population.

Sample: A finite subset of statistical individuals in a population is called Sample.

Sample size: The number of individuals in a sample is called the Sample size.

Parameters and Statistics: The statistical constants of the population are referred as Parameters and the statistical constants of the Sample are referred to as Statistics.

Standard Error: The standard deviation of sampling distribution of a statistic is known as its standard error and is denoted by (S.E)

Test of Significance: It enables us to decide on the basis of the sample results if the deviation between the observed sample statistic and the hypothetical parameter value is significant or the deviation between two sample statistics is significant.

Null Hypothesis: A definite statement about the population parameter which is usually a hypothesis of no-difference and is denoted by Ho.

Alternative Hypothesis: Any hypothesis which is complementary to the null hypothesis is called an Alternative Hypothesis and is denoted by H_1.

if $\mu = \mu_0$ is the null hypothesis H_0 then, the alternate hypothesis H_1 could be $\mu > \mu_0$ (Right tail) or $\mu < \mu_0$ (Left tail) or $\mu \neq \mu_0$ (Two tail test)

Errors in Sampling: Type I and Type II errors.

Type I error: Rejection of H_0, when it is true.

Type II error: Acceptance of H_0, when it is false.

Critical region: A region corresponding to a statistic "t" in the sample space S which leads to the rejection of H_o is called Critical region or Rejection region.

Acceptance Region: Those regions which lead to the acceptance of H_o are called Acceptance Region.

Level of Significance: The probability α that a random value of the statistic "t" belongs to the critical region is known as the level of significance.

Types of samples: Small sample and Large sample. A sample is said to be a small sample if the size is less than or equal to 30 otherwise it is a large sample.

Large Sample

Z test for mean

Test of significance for single Mean

$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$, where $\bar{x}$ the sample mean, μ is is the population mean, σ is the population standard deviation and n is the sample size.

Test of significance for difference of mean

$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}}$, where $\bar{x}_1$ is the first sample mean, $\bar{x}_2$ is the second sample mean, n_1 is the first sample size, n_2 is the second sample size, s_1^2 is the first sample variance and s_2^2 is the second sample variance.

Confidence Limits

The values of $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$ are called 95% confidence limits for the mean of the population corresponding to the given sample. The values of $\bar{x} \pm 2.58 \frac{\sigma}{\sqrt{n}}$ are called 99% confidence limits for the mean of the population corresponding to the given sample.

Z test for proportions

Test of significance for single proportion

$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$, where P is the population proportion, Q = 1 – P, p is the sample proportion and n is the sample size.

Test of significance for difference of proportion

$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$, where p_1 is the first sample proportion, p_2 is the second sample proportion, n_1 is the first sample size, n_2 is the second sample size, $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$ and Q = 1 – P

Small Sample

t -Test of significance for single Mean

$t = \dfrac{\bar{x} - \mu}{\dfrac{s}{\sqrt{n-1}}}$, where $\bar{x}$ the sample mean, μ is is the population mean, s is the sample standard deviation and n is the sample size.

If the mean and standard deviation are not given, then the following formulae are used to calculate

$$\bar{x} = \frac{\sum x}{n}, \quad s^2 = \frac{\sum (x - \bar{x})^2}{n}$$

Degrees of freedom is n - 1

Confidence Limits

Let $\bar{x}$ be the sample mean and n be the sample size. Let s be the sample standard deviation. Then the 95 % level confidence limits are given by $\bar{x} \pm t_{0.05} \dfrac{s}{\sqrt{n-1}}$. The 99 % level confidence limits are given by $\bar{x} \pm t_{0.01} \dfrac{s}{\sqrt{n-1}}$.

Test of significance for difference of mean

$Z = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$, where $\bar{x}_1$ is the first sample mean, $\bar{x}_2$ is the second sample mean, n_1 is the first sample size, n_2 is the second sample size, $s^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

Degrees of freedom is $n_1 + n_2 - 2$

F test

F= $\dfrac{Greater\ variance}{Smaller\ variance}$ i.e, $F = \dfrac{S_1^2}{S_2^2}$ if $S_1^2 > S_2^2$ (OR) $F = \dfrac{S_2^2}{S_1^2}$ if $S_2^2 > S_1^2$

If the sample variances s_1^2 and s_2^2 are given, then the following formula can be used to calculate S_1^2 and S_2^2 :

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}, S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

If the sample variances s_1^2 and s_2^2 are not given and the set of observations for both samples are given then the following formula can be used to calculate S_1^2 and S_2^2

$S_1^2 = \frac{\sum (x-\bar{x})^2}{n_1 - 1}$, $S_2^2 = \frac{\sum (y-\bar{y})^2}{n_2 - 1}$, where n_1 is the first sample size, n_2 is the second sample size, $\bar{x}$ is the first sample mean and $\bar{y}$ is the second sample mean.

χ^2 test

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

Where O is the observed frequency and E is the expected frequency.

Calculation of expected frequencies in testing independence of attributes

Expected Frequency = (Row total * Column Total) / Grand total

Explanation for the above with two classes is given below

Observed Frequencies

			Total
	a	c	a+c
	b	d	b+d
Total	a+b	c+d	a+b+c+d = N

Expected Frequencies

			Total
	E(a) = $\frac{(a+c)(a+b)}{N}$	E(c) = $\frac{(a+c)(c+d)}{N}$	a+c
	E(b) = $\frac{(b+d)(a+b)}{N}$	E(d) = $\frac{(c+d)(b+d)}{N}$	b+d
Total	a+b	c+d	a+b+c+d = N

Problems

1. **A company manufacturing electric light bulbs claims that the average life of its bulbs is 1600 hours. The average life and standard deviation of a random sample of 100 such bulbs were 1570 hours and 120 hours respectively. Test the claim of the company at 5% level of significance.**

Solution:

Null Hypothesis H_0: $\mu = 1600$. There is no significant difference between sample mean and population mean

Alternative Hypothesis H_1: $\mu \neq 1600$. There is a significant difference between sample mean and population mean.

The statistic test is $Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$$Z = \frac{1570 - 1600}{\frac{120}{\sqrt{100}}} = -2.5$$

$$|z| = 2.5$$

Calculated value z = 2.5

Tabulated value of z at 5% level of significance for a two tail test is 1.96

Calculated value >Tabulated value, H_0 is rejected.

We cannot accept the claim of the company.

2. **The breaking strength of ropes produced by a manufacturer has mean 1800N and standard deviation 100N. By introducing a new technique in the manufacturing process it is claimed that the breaking strength has increased. To test this claim a sample of 50 ropes is tested and it is found that the breaking strength is 1850N. Can we support the claim at 1% level of significance?**

Solution:

Null Hypothesis H_0: μ = 1800 N

Alternative Hypothesis H_1: μ > 1800 N (one tailed test)

n = 50, $\bar{x}$ = 1850 μ =1800 σ = 100

The statistic test is $Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$$Z = \frac{1850 - 1800}{\frac{100}{\sqrt{50}}} = 3.54$$

Calculated value z = 3.54

Tabulated value of z at 5% level of significance for a one tail test is 2.33

Calculated value >Tabulated value, H_0 is rejected.

The difference is significant and so we support the claim of the manufacturer.

3. **Measurements of the weights of a random sample of 200 ball bearings made by a certain machine during one week showed a mean of 0.824N and a standard deviation of 0.042N. Find the 95% and 99% confidence limits for the mean weight of all the ball bearings.**

Solution:

The 95% confidence limits are $\bar{x} \pm 1.96 \frac{s}{\sqrt{n}}$

n = 200, $\bar{x}$ = 0.824 s = 0.042

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}} = 0.824 \pm (1.96)\left(\frac{0.042}{\sqrt{200}}\right) = 0.824 \pm 0.0058$$

The 95% confidence interval is (0.8182, 0.8298)

The 99% confidence limits are $\bar{x} \pm 2.58 \frac{s}{\sqrt{n}}$

$$\bar{x} \pm 2.58 \frac{s}{\sqrt{n}} = 0.824 \pm (2.58)\left(\frac{0.042}{\sqrt{200}}\right) = 0.824 \pm 0.0077$$

The 99% confidence interval is (0.8163, 0.8317)

4. **In a survey of buying habits, 400 women shoppers are chosen at random in supermarket A. Their average weekly food expenditure is Rs.250 with standard deviation Rs.40. For 400 women shoppers chosen at random in supermarket B, the average weekly food expenditure is Rs.220 with standard deviation is Rs.55. Test at 1% level of significance whether the average weekly food expenditure of the populations of shoppers are equal.**

Solution:

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

$n_1 = 400 \quad n_2 = 400 \quad \bar{x} = 250 \quad \bar{y} = 220 \quad s_1 = 40 \quad s_2 = 55$

$$s = \sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)} = \sqrt{\frac{40^2}{400} + \frac{55^2}{400}} = 3.4$$

$$Z = \frac{\bar{x} - \bar{y}}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}} = \frac{250 - 220}{3.4} = 8.82$$

Calculated value z = 8.82

Tabulated value of z at 1% level of significance for a two tailed test is 2.56

Calculated value >Tabulated value, H_0 is rejected.

The difference in the weekly food expenditure is significantly different.

5. **A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Test whether the proportion of bad ones is not significantly different from 0.1 at 1% level of significance**

Solution:

Null Hypothesis H_0: P = 0.1There is no significant difference between sample and population proportion.

Alternative Hypothesis H_1: $P \neq 0.1$ There is a significant difference between sample and population proportion.

The statistic test is $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$

$$p = \frac{65}{500} = 0.13$$

P = 0.1 Q= 1 – P = 1 – 0.1= 0.9

$$Z = \frac{0.13-0.1}{\sqrt{\frac{(0.1)(0.9)}{500}}} = 2.238$$

Calculated value z = 2.238
Tabulated value of z at 5% level of significance for a two tail test is 1.96
Calculated value >Tabulated value, H_0 is rejected.
The proportion of bad ones is significantly different from 0.1

6. **In a sample of 1000 people, 540 were rice eaters and the rest were wheat eaters. Can we assume that the proportion of rice eaters is more than 50% at 1% level of significance.**

Solution:
H_0: P = 0.5
H_1: P > 0.5 (One tailed test)
P = 0.5 Q= 1 – P = 1 – 0.5 = 0.5

$$p = \frac{540}{1000} = 0.54$$

The statistic test is $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$

$$Z = \frac{0.54-0.5}{\sqrt{\frac{(0.5)(0.5)}{1000}}} = 2.532$$

Calculated value z = 2.532
Tabulated value of z at 5% level of significance for a one tail test is 2.33
Calculated value >Tabulated value, H_0 is rejected.
The rice eaters are more than 50% of the population.

7. **In a random sample of 900 votes, 55% are favored the Democratic candidate for the post of the President. Test the hypothesis that the Democratic candidate has more chances of winning the President post.**

Solution:
H_0: P = 0.5
H_1: P > 0.5 (Right tailed test)

P = 0.5 , Q= 1 – P = 1 – 0.5 = 0.5

p = $\frac{55}{100} = 0.55$

The statistic test is $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$

$$Z = \frac{0.55-0.5}{\sqrt{\frac{(0.5)(0.5)}{900}}} = 3$$

Calculated value z = 3

Tabulated value of z at 5% level of significance for a one tail test is 2.33

Calculated value >Tabulated value, H_0 is rejected.

The Democratic candidate is having more chances to win the President Post.

8. **In a random sample of 1000 persons from town A, 400 are found to be consumers of wheat. In a sample of 800 from town B, 400 are found to be consumers of wheat. Do these data reveal a significant difference between town A and town B so far as the proportion of wheat consumers is concerned?**

Solution:

H_0: Two towns do not differ much as far as the proportion of wheat consumption. $P_1 = P_2$

H_1: $P_1 \neq P_2$

The Statistic test is $Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

$p_1 = \frac{400}{1000} = 0.4$ $p_2 = \frac{400}{800} = 0.5$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{1000(0.4)+800(0.5)}{1000+800} = 0.444$$

Q= 1 – P = 1 – 0.444 = 0.556

$$Z = \frac{0.4-0.5}{\sqrt{(0.444)(0.556)\left(\frac{1}{1000}+\frac{1}{800}\right)}} = \frac{0.1}{0.024} = 4.17$$

Calculated value z = 4.17

Tabulated value of z at 5% level of significance for a two tail test is 1.96

Calculated value >Tabulated value, H_0 is rejected.

Hence the data reveal a significant difference between town A and town B so far as the proportion of wheat consumers is concerned.

9. **In the past, a machine has produced washers having a thickness of 0.050 inch. To determine whether the machine is in proper working order, a sample of 10 washers is chosen, for which the mean thickness is 0.053 inch and the standard**

deviation is 0.003 inch. Test the hypothesis that the machine is in proper working order, using 5% and 1% level of significance.

Solution:

H_0: μ = 0.050

H_1: μ $\neq$ 0.050 (two tailed test)

$n = 10 \qquad \bar{x} = 0.053 \qquad s = 0.003 \qquad \mu = 0.050$

The statistic test is $t = \dfrac{\bar{x}-\mu}{\dfrac{s}{\sqrt{n-1}}} = \dfrac{0.053-0.050}{\dfrac{0.003}{\sqrt{10-1}}} = 3.00$

Calculated value t = 3.00

Degree of freedom = n – 1 =10 – 1 = 9

At 5% LOS:

Tabulated value of t at 5% level of significance with 9 degrees of freedom for a two tailed test is 2.26

Calculated value >Tabulated value, H_0 is rejected.

The Machine is not in proper working order at 5% level of significance

Tabulated value of t at 1% level of significance with 9 degrees of freedom for a two tailed test is 3.25

Calculated value < Tabulated value, H_0 is accepted.

The Machine is in proper working order at 1% level of significance.

10. The specifications for a certain kind of ribbon call for a mean breaking strength of 180 pounds. If five pieces of the ribbon (randomly selected from the different rolls) have a mean breaking strength of 169.5 pounds with a standard deviation of 5.7 pounds. Test the null hypothesis $\mu = 180$ pounds against the alternative hypothesis $\mu < 180$ pounds at the 0.01 level of significance. Assume that the population distribution is normal.

Solution:

H_0: $\mu = 180$

H_1: $\mu < 180$ (left tailed test)

$n = 5 \qquad \bar{x} = 169.5 \qquad s = 5.7 \qquad \mu = 180$

The statistic test is $t = \dfrac{\bar{x}-\mu}{\dfrac{s}{\sqrt{n-1}}} = \dfrac{169.5-180}{\dfrac{5.7}{\sqrt{5-1}}} = -3.68$

Calculated value t = 3.68

Degree of freedom = n – 1 =5 – 1 = 4

Tabulated value of t at 1% level of significance with 4 degrees of freedom for a one tail test is 3.747.

Calculated value > Tabulated value, H_0 is accepted.

Hence the mean breaking strength can be taken as 180 pounds.

11. Ten individuals are chosen at random from a normal population and their heights are found to be 63,63,66,67,68,69,70,70,71,71 inches. Test the hypothesis that the mean height is greater than 66 inches at 5% level of significance

Solution:

H_0: μ = 66

H_1: μ > 66 (one tailed test)

$$\bar{x} = \frac{\sum x}{n} = \frac{678}{10} = 67.8$$

X	63	63	66	67	68	69	70	70	71	71	Total
$(x-\bar{x})$	- 4.8	- 4.8	- 1.8	- 0.8	0.2	1.2	2.2	2.2	3.2	3.2	
$(x-\bar{x})^2$	23.04	23.04	3.24	0.64	0.04	1.44	4.84	4.84	10.24	10.24	81.6

$$s^2 = \frac{\sum (x-\bar{x})^2}{n} = \frac{81.6}{10} = 8.16$$

s = 2.857

The statistic test is $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n-1}}} = \frac{67.8-66}{\frac{2.857}{\sqrt{9}}} = 1.89$

Calculated value t = 1.89

Degree of freedom = n – 1 =10 – 1 = 9

Tabulated value of t at 5% level of significance with 9 degrees of freedom for a one tail test is 1.833

Calculated value >Tabulated value, H_0 is rejected. Accepted H_1

The Mean is significantly higher than 66 inches.

12. Two independent samples of size 8 and 7 items had the following values

Sample I	**9**	**11**	**13**	**11**	**15**	**9**	**12**	**14**
Sample II	**10**	**12**	**10**	**14**	**9**	**8**	**10**	

Test if the difference between the mean is significant

Solution:

H_0: $\mu_1 = \mu_2$ There is no significant difference between means

H_1: $\mu_1 \neq \mu_2$ There is a significant difference between means

$$\bar{x} = \frac{\sum x}{n} = \frac{94}{8} = 11.75 \qquad \bar{y} = \frac{\sum y}{n} = \frac{73}{7} = 10.43$$

x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$
9	- 2.75	7.56	10	- 0.43	0.185
11	- 0.75	0.56	12	1.57	2.465
13	1.25	1.56	10	- 0.43	0.185
11	- 0.75	0.56	14	3.47	12.041
15	3.25	10.56	9	- 1.43	2.045
9	- 2.75	7.56	8	- 2.43	5.905
12	0.25	0.06	10	- 0.43	0.185
14	2.25	5.06			
94		33.48	73		23.011

$$s^2 = \frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2} = \frac{33.48+23.011}{8+7-2} = 4.35$$

s = 2.086

The statistic test is $t = \dfrac{\bar{x}-\bar{y}}{s\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} = \dfrac{11.75-10.43}{2.086\sqrt{\dfrac{1}{8}+\dfrac{1}{7}}} = 1.22$

Calculated value t = 1.22

Degree of freedom = n_1+n_2-2 = 8 + 7 – 2 = 13

Tabulated value of t at 5% level of significance with 13 degrees of freedom for a two tail tets is 2.16

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference between means.

13. The IQ of 16 students from one area of a city showed a mean of 107 with the standard deviation 10, while the IQ of 14 students from another area showed a mean of 112 with standard deviation 8. Is there a significant difference between the IQ's of the two groups at 1% and 5% level of significance?

Solution:

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

$n_1 = 16 \quad n_2 = 14 \quad s_1 = 10 \quad s_2 = 8 \quad \bar{x} = 107 \quad \bar{y} = 112$

$$s^2 = \frac{n_1 s_1^{\ 2} + n_2 s_2^{\ 2}}{n_1+n_2-2} = \frac{16(10)^2 + 14(8)^2}{16+14-2} = \frac{2496}{28} = 89.143$$

s = 9.44

The statistic test

$$t = \frac{\bar{x}-\bar{y}}{s\sqrt{\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} = \frac{107-112}{9.44\sqrt{\dfrac{1}{16}+\dfrac{1}{14}}} = -1.45$$

Calculated value t = 1.45

Degree of freedom = $n_1 + n_2 - 2$ = 16 + 14 – 2 = 28

At 5% LOS:

Tabulated value of t at 5% level of significance with 28 degree of freedom for a two tail test is 2.05

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference in the IQ level of the two groups.

At 1% LOS:

Tabulated value of t at 1% level of significance with 28 degree of freedom is 2.76

Calculated value < Tabulated value, H_0 is accepted.

There is no significant difference in the IQ level of the two groups.

14. A random sample of 10 parts from machine A has a sample standard deviation of 0.014 and another sample of 15 parts from machine B has a sample standard deviation of 0.08. Test the hypothesis that the samples are from a population with same variance.

Solution:

H_0: $\sigma_1^2 = \sigma_2^2$

H_1: $\sigma_1^2 \neq \sigma_2^2$

$n_1 = 10 \quad n_2 = 15 \quad s_1 = 0.014 \quad s_2 = 0.08$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10 \times (0.014)^2}{10 - 1} = 0.0002$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{15 \times (0.08)^2}{15 - 1} = 0.006$$

$$F = \frac{S_2^2}{S_1^2} = \frac{0.006}{0.0002} = 30$$

Calculated value F = 30

Tabulated Value of F at 5% level of significant with (14, 9) degrees of freedom is 3.03

Calculated value > Tabulated value, H_0 is rejected

There is a significant difference in the variances of two populations.

15. Two random samples drawn from two normal populations are

Sample I	**20**	**16**	**26**	**27**	**23**	**22**	**18**	**24**	**25**	**19**		
Sample II	**27**	**33**	**42**	**35**	**32**	**34**	**38**	**28**	**41**	**43**	**30**	**37**

Obtain the estimates of the variances of the population and test whether the two populations have the same variance.

Solution:

H_0: $\sigma_1^2 = \sigma_2^2$

H_1: $\sigma_1^2 \neq \sigma_2^2$

$$\bar{x} = \frac{\sum x}{n} = \frac{220}{10} = 22$$

$$\bar{y} = \frac{\sum y}{n} = \frac{420}{12} = 35$$

x	(x-$\bar{x}$)	$(x-\bar{x})^2$	y	(y-$\bar{y}$)	$(y-\bar{y})^2$
20	- 2	4	27	- 8	64
16	- 6	36	33	- 2	4
26	4	16	42	7	49
27	5	25	35	0	0
23	1	1	32	- 3	9
22	0	0	34	- 1	1
18	- 4	16	38	3	9
24	2	4	28	- 7	49
25	3	9	41	6	36
19	- 3	9	43	8	64
			30	- 5	25
			37	2	4
		120			314

$n_1 = 10 \qquad n_2 = 12$

$$S_1^{\,2} = \frac{\sum (x-\bar{x})^2}{n_1 - 1} = \frac{120}{9} = 13.33$$

$$S_2^{\,2} = \frac{\sum (y-\bar{y})^2}{n_2 - 1} = \frac{314}{11} = 28.54$$

$$F = \frac{S_2^2}{S_1^2} = \frac{28.54}{13.33} = 2.14$$

Calculated value F = 2.14

Tabulated Value of F at 5% level of significance with (11, 9) degrees of freedom is 3.1

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference between variances.

16. In one sample of 8 observations the sum of squares of deviations of the sample values from the sample mean was 84.4 and in another sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level.

Solution:

H₀: $\sigma_1^2 = \sigma_2^2$

H₁: $\sigma_1^2 \neq \sigma_2^2$

$n_1 = 8 \quad n_2 = 10 \quad \sum(x-\bar{x})^2 = 84.4 \quad \sum(y-\bar{y})^2 = 102.6$

$$S_1{}^2 = \frac{\sum(x-\bar{x})^2}{n_1 - 1} = \frac{84.4}{7} = 12.057$$

$$S_2{}^2 = \frac{\sum(y-\bar{y})^2}{n_2 - 1} = \frac{102.6}{9} = 11.4$$

$$F = \frac{S_1^2}{S_2^2} = \frac{12.057}{11.4} = 1.057$$

Calculated value F = 1.057

Tabulated Value of F at 5% level of significance with (7, 9) degrees of freedom is 3.29

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference between variances.

17. The mean life of a sample of 9 bulbs was observed to be 1309 hrs with standard deviation 420 hrs. A second sample of 16 bulbs chosen from a different batch showed a mean life of 1205 hrs with a standard deviation 390 hrs. Test at 5% level whether both the samples come from the same normal population.

Solution:

Both t-test and F-test has to be done to check whether they have come from the same population. Fist F-test is done and the followed by t-test.

F-test:

H₀: $\sigma_1^2 = \sigma_2^2$

H₁: $\sigma_1^2 \neq \sigma_2^2$

$n_1 = 9 \quad n_2 = 16 \quad s_1 = 420 \quad s_2 = 390 \quad \bar{x} = 1309 \quad \bar{y} = 1205$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9 \times (420)^2}{9-1} = 198450$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{16 \times (390)^2}{16-1} = 162240$$

$$F = \frac{S_1^2}{S_2^2} = \frac{198450}{162240} = 1.223$$

Calculated value F = 1.223

Tabulated Value of F at 5% level of significant with (15, 8) degree of freedom is 3.22

Calculated value < Tabulated value, H_0 is accepted .

t-test:

H0: $\mu_1 = \mu_2$

H1: $\mu_1 \neq \mu_2$

$$s^2 = \frac{n_1 s_1^{\ 2} + n_2 s_2^{\ 2}}{n_1 + n_2 - 2} = \frac{9(420)^2 + 16(390)^2}{9 + 16 - 2} = \frac{4021200}{23} = 174834.7826$$

s = 418.13

The statistic test

$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{1309 - 1205}{418.13\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{104}{174.22} = 0.596$$

Calculated value t = 0.596

Degree of freedom = $n_1 + n_2 - 2$ = 9 + 16 – 2 = 23

Tabulated value of t at 5% level of significance with 23 degree of freedom is 2.069

Calculated value < Tabulated value, H_0 is accepted

Since in both F-Test and t-Test we have accepted the null hypothesis, we conclude that the samples have come from the same normal populations.

18. A dice is tossed 120 times with the following results:

No. turned up	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Frequency	**30**	**25**	**18**	**10**	**22**	**15**	**120**

Test the hypothesis that the dice is unbiased.

Solution:

Null Hypothesis H_0: The dice is an unbiased one.

Alternative Hypothesis H_1: The dice is biased

O	E	O – E	$(O-E)^2$	$\left[\frac{(O-E)^2}{E}\right]$
30	20	10	100	5.00
25	20	5	25	1.25
18	20	- 2	4	0.20
10	20	-10	100	5.00
22	20	2	4	0.20
15	20	- 5	25	1.25
				12.90

Calculated $\chi^2 = \left[\frac{(O-E)^2}{E}\right] = 12.90$

Degree of freedom = n – 1 = 6 – 1 = 5

Calculated value of χ^2 at 5% level of significance with 5 degree of freedom is 11.07

Tabulated value = 11.07

Tabulated value > calculated value, H_0 is rejected.
The dice are biased.

19. Genetic theory states that children having one parent of blood type M and other of blood type N will always be one of the three types M, MN, N and that the ratios of these types will be 1:2:1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and remainder type N. Test the hypothesis using χ^2 test.

Solution:

H_0: There is no significant difference between the theoretical ratio and observed ratio.
H_1: There is no significant difference between the theoretical ratio and observed ratio.

If theoretical ratio is true the 300 children should be distributed as follows:

$$Type\ M = \frac{1}{4} \times 300 = 75$$

$$Type\ MN = \frac{2}{4} \times 300 = 150$$

$$Type\ N = \frac{1}{4} \times 300 = 75$$

Observed frequencies:

$$Type\ M = \frac{30}{100} \times 300 = 90$$

$$Type\ MN = \frac{45}{100} \times 300 = 135$$

$$Type\ N = \frac{25}{100} \times 300 = 75$$

Type	observed	Expected	$O - E$	$(O - E)^2$	$\left[\frac{(O-E)^2}{E}\right]$
M	90	75	15	225	3
MN	135	150	- 15	225	1.5
N	75	75	0	0	0
Total					4.5

Calculated $\chi^2 = \left[\frac{(O-E)^2}{E}\right] = 4.5$

Degree of freedom = n – 1 = 3 – 1 = 2

Calculated value of χ^2 at 5% level of significance with 2 degree of freedom is 5.99

Tabulated value = 5.99

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference between the theoretical ratio and observed ratio.

20. A certain drug was administered to 456 males, out of a total 720 in a certain locality, to test its efficacy against typhoid. To incidence of typhoid is shown below. Find out the effectiveness of the drug against the disease. (The table value of χ^2 for 1 degree of freedom at 5% level of significance is 3.84)

	Infection	**No Infection**	**Total**
Administering the drug	**144**	**312**	**456**
Without administering the drug	**192**	**72**	**264**
Total	**336**	**384**	**720**

Solution:

Null Hypothesis H_0: The drug is independent.

Alternative Hypothesis H_1: The drug is not independent

The expected frequencies are

$\frac{336 \times 456}{720} = 212.8$ ≈213	$\frac{384 \times 456}{720} = 243.2$ ≈243	456
$\frac{336 \times 264}{720} = 123.2$ ≈123	$\frac{384 \times 264}{720} = 140.8$ ≈141	264
336	384	720

O	E	O – E	$(O-E)^2$	$\left[\frac{(O-E)^2}{E}\right]$
144	213	- 69	4761	22.35
192	123	69	4761	38.71
312	243	69	4761	19.59
72	141	- 69	4761	33.77
				114.42

Calculated χ^2= $\left[\frac{(O-E)^2}{E}\right]$= 114.42

Degree of freedom = (r – 1) (c – 1)= (2-1)(2-1) = 1

Tabulated value of χ^2 at 5% level of significance with 1 degree of freedom is 3.841

Tabulated value = 3.841

Calculated value >Tabulated value, H_0 is rejected.

Therefore, the drug is definitely effective in controlling the typhoid.

21. A brand Manager is concerned that her brand's share may be unevenly distributed throughout the country. In a survey in which the country was divided into four geographical regions, a random sampling of 100 consumers in each region was surveyed, with the following results:

	Region				
	NE	**NW**	**SE**	**SW**	**TOTAL**
Purchased the brand	**40**	**55**	**45**	**50**	**190**
Did not purchase	**60**	**45**	**55**	**50**	**210**

Using χ^2 test, find out if the brand is unevenly distributed throughout the country.

Solution:

H_0: There is no significant difference between the observed and expected frequencies

H_1: There is a significant difference between the observed and expected frequencies

The expected frequencies are :

	Region				
	NE	NW	SE	SW	TOTAL
Purchased the brand	$\frac{190\times100}{400}\approx47$	$\frac{190\times100}{400}\approx48$	$\frac{190\times100}{400}\approx47$	$\frac{190\times100}{400}\approx48$	190
Did not purchase	$\frac{210\times100}{400}\approx53$	$\frac{210\times100}{400}\approx52$	$\frac{210\times100}{400}\approx53$	$\frac{210\times100}{400}\approx52$	210
	100	100	100	100	400

O	E	O – E	$(O-E)^2$	$\left[\frac{(O-E)^2}{E}\right]$
40	47	- 7	49	1.04
55	48	7	49	1.02
45	47	- 2	4	0.085
50	48	2	4	0.083
60	53	7	49	0.924
45	52	-7	49	0.942
55	53	2	4	0.075
50	52	- 2	4	0.076
				4.245

Calculated $\chi^2 = \left[\frac{(O-E)^2}{E}\right] = 4.245$

Degree of freedom = (r – 1)(c – 1)= (2-1)(4-1) = 3

Tabulated value of χ^2 at 5% level of significance with 3 degree of freedom is 7.815

Tabulated value = 7.815

Calculated value < Tabulated value, H_0 is accepted

There is no significant difference between the observed and expected frequencies.

Questions

Unit V-Theory Of Sampling And Testing Of Hypothesis

Part - A

1. What is the difference between population and sample? CO2 (L1)
2. Define Type I and Type II errors. CO2(L1)
3. Define level of significance. CO2 (L1)
4. List any two uses of ψ^2 test. CO2 (L1)
5. Find the formula for chi-square for 2 x 2 contingency table. CO2 (L1)
6. Define Null Hypothesis and Alternate Hypothesis. CO2 (L1)
7. Recall the formula for confidence limits for μ for small sample at 5% level of significance. CO2 (L1)
8. Define critical region. CO 2(L1)
9. Define critical value. CO2 (L1)
10. List two applications of student's t- test. CO2 (L1)

Part-B

1. Test the hypothesis of the following CO4(L4)

   The average breaking strength of steel rod is specified to be 18.5 thousand pounds. To test this sample of 14 rods was tested. The mean and SD obtained were 17.85 and 1.955 respectively. Is the result of the experiment significant?

2. Test the hypothesis of the following CO4(L4)

   The mean weakly sales of soap bars in departmental stores were 146.3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a SD of 17.2. Was the advertising campaign successful?

3. Test the hypothesis of the following CO4(L4)

   A sample of 26 bulbs gives a mean life of 990 hours with SD of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not upto the standard?

4. Test the hypothesis of the following CO4(L4)

   A random sample of 10 boys had the following IQ's 70,120,110,101,88,83,95,98,107,100. Do these data support the assumption of a population mean IQ of 100? Find the reasonable range in which most of the mean IQ values of samples of 10 boys lie?

5. Test the hypothesis of the following CO4(L4)

The heights of 10 males of a given locality are found to be 70,67,62,68,61,68,70,64,64,66 inches. Is it reasonable to believe that the average height is greater than 64 inches. Test at 5%.

6. Test the hypothesis of the following CO4(L4)

Samples of two types of electric light bulbs were tested for length of life and following data were obtained.

TYPE I	TYPE II
Sample size $n_1 = 8$	Sample size $n_2 = 7$
Sample mean $x_1 = 1234$ hrs	Sample mean $x_2 = 1036$ hrs
Sample S.D. $s_1 = 36$ hrs	Sample S.D. $s_2 = 40$ hrs

Is the difference in the means sufficient to warrant that type I is superior to type II regarding length of life.

7. Test whether the samples come from the same normal population CO4(L4)
Below are given the gain in weights (in N) of pigs fed on two diets A and B.

Diet A	25	32	30	34	24	14	32	24	30	31	35	25		
Diet B	44	34	22	10	47	31	40	32	35	18	21	35	29	22

8. Test whether the samples come from the same normal population CO4(L4)
The nicotine content in milligrams of two samples of tobacco were found to be as follows:

Sample A	24	27	26	21	25	
Sample B	27	30	28	31	22	36

9. Show that the die is biased from the data CO4(L2)
A die is thrown 264 times with the following results.

No. appeared on the die	1	2	3	4	5	6
Frequency	40	32	28	58	54	60

10. Use Chi square test for the following information CO4(L)
In a certain sample of 2000 families 1400 families are consumers of tea.
Out of 1800 Hindu families, 1236 families consume tea. State whether there is any significant difference between consumption of tea among Hindu and Non-Hindu families.

11. Find the value of Chi-Square and is there any good association between the two variables from the given following contingency table for hair colour and eye colour.

Hair colour / Eye colour	Fair	Brown	Black
Grey	20	10	20
Brown	25	15	20
Black	15	5	20

12. Test the claim at 5% level of significance from the following information CO4(L4)

A manufacture claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 were faulty. In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?