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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – I –DIFFERENTIAL EQUATIONS – SMTA1201

UNIT – I

DIFFERENTIAL EQUATIONS

INTRODUCTION

During the past three decades the development of non-linear analysis, dynamical systems and their applications to Science and Engineering has stimulated renewed enthusiasm for the theory of Ordinary Differential Equations (ODE).

An Ordinary Differential Equation is an equation containing a function of one independent variable and its derivatives.Differential equations have wide application in various engineering and science discipline. In general, modeling variations of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain or concentration of pollutant with the cahnge of time ‘t’ or location, or both would require differential equation . The study of differential equation began in 1675, when Gottfried Willhelm Von Leibnig (1646 - 1716) wrote the equation.

$$\int x\, dt = \frac{x^2}{2}$$

The search for general methods of integrading differential equations began when Isaac Newton (1646 - 1727) classified first order Differential equations

(i) $\frac{dy}{dx} = f(x)$

(ii) $\frac{dy}{dx} = f(x, y)$

(iii) $x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial x} = u$

The first two classes contain only ordinary derivatives of one dependent variable, with respect to single independent variable, and are known today as Differential Equation and third is partial differential equation.

A simple example is Newton’s second law of motion, the relationship between the displacement ‘x’ and time ‘t’ of the object under the force F which leads to the differential equation is

$$m\left(\frac{d^2x(t)}{dt^2}\right) = F[x(t)] \qquad ...(1)$$

for the motion of a particle with constant mass 'm'. In general, F depends on the position x(t) on the particle at time 't' and so the unknown function x(t) appears on both sides of the differential equations as in the notation *F[x(t)].*

Linear Differential equations with Constant Coefficients

The general form on nth order linear differential equation with constant coefficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad ...(2)$$

where $a_0 \neq 0, a_1, a_2, ..., a_n$ are constants and 'X' is a function of x.

put $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2} \cdots, D^n = \frac{d^n}{dx^n}$

Eqn (2) becomes

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_{n-1} D + a_n)\, y = X \qquad ...(3)$$

The general solution of (3) is

y = Complementary Function + Particular Integral

i.e., y = C.F. + P.I.

To find the Complementary Function

The Auxiliary Equation of (3) is obtained by putting $D = m$ and $X = 0$.

$\therefore$ The Auxiliary eqn is

$$a_0 m^n + a_1 m^{n-1} + \ldots + a_n = 0 \qquad ...(4)$$

Solving equation (4), we get 'n' roots for 'm'. Say $m_1, m_2, ... m_n$.

Case (i): If all the roots m_1, m_2, &m_n are real and different, then the complementary function,

$$C.F. = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + ...$$

Case (ii): If any two roots are equal say $m_1 = m_2 = m$ (say), then the complementary function is given by

$$C.F. = (Ax + B)e^{mx} \ (or) \ (A + Bx)e^{mx}$$

Case (iii): If any three roots are equal say $m_1 = m_2 = m_3 = m$, then the complementary function is given by

$$C.F. = (Ax^2 + Bx + C)e^{mx} \ (or) \ (A + Bx + Cx^2)e^{mx}$$

Case (iv): If the roots are imaginary of the form $(\alpha \pm i\beta)$ then the complementary function is

$$C.F. = e^{\alpha x}[A\cos \beta x + B\sin \beta x]$$

To find particular Integral

When the R.H.S. of the given differential equation is zero, we need not find particular Integral. When R.H.S. of a given differential equation is a function of x say e^{ax}, *sin ax* or *cos ax*, x^n, $e^{ax} f(x)$, $x f(x)$, we have to find particular Integral.

Case (i): If $f(x) = e^{ax}$, then $P.I. = \frac{1}{F(D)} e^{ax}$ Replace D by a in *F(D)*, provided $F(D) \neq 0$.

If $F(a) = 0$, then $P.I. = \frac{x}{F'(D)} e^{ax}$, provided $F'(a) \neq 0$.

If $F'(a) = 0$, then $P.I. = \frac{x^2}{F''(D)} e^{ax}$, provided $F''(a) \neq 0$ and so on.

Case (ii): If $f(x) = \sin ax$ or $\cos ax$ then

$$P.I. = \frac{1}{F(D)} \sin ax \text{ or } \cos ax$$

Replace D^2 by $-a^2$ in *F(D)*, provided $F(D) \neq 0$.

If $F(D) = 0$, when we replace D^2 by $-a^2$ then

$$P.I. = \frac{x}{F'(D)} \sin ax \text{ (or) } \cos ax$$

Again replace D^2 by $-a^2$ in $F'(D)$ provided $F'(D) \neq 0$, then $P.I. = \frac{x^2}{F''(D)} \sin ax$ (or) $\cos ax$ and the process may be repeated if $F(D) = 0$ and so on.

Case (iii): If $f(x) = x^n$, then $P.I. = \frac{1}{F(D)} x^n$

$= [F(D)]^{-1} x^n$

Expand = $[F(D)]^{-1}$ by using Binomial theorem and then operate on x^n.

Case (iv): If $f(x) = e^{ax} X$, where X is $\sin ax$ (or) $\cos ax$, then

$$P.I. = \frac{1}{F(D)} e^{ax} X = e^{ax} \frac{1}{F(D+a)} X$$

Here $\frac{1}{F(D+a)} X$ can be evaluated by using anyone of the first three types.

Case (v): If $f(x) = x^n \sin ax$ or $x^n \cos ax$, then

$$P.I. = \frac{1}{F(D)} x^n \sin ax \ \ (or)\ x^n \cos ax$$

Now, $\frac{1}{F(D)} x^n (\cos ax + i \sin ax)$

$$= \frac{1}{F(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{F(D+ia)} x^n$$

$$\therefore \frac{1}{F(D)} x^n \sin ax = \text{Imaginary part of } \ e^{iax} \frac{1}{F(D+ia)} x^n$$

$$\frac{1}{F(D)} x^n \cos ax = \text{Real part of } \ e^{iax} \frac{1}{F(D+ia)} x^n$$

Example

1. Find the complementary function of $\frac{d^2 y}{d^2 x} + 2\frac{dy}{dx} + y = 0$

Solution:The given equation can be written as $(D^2 + 2D + 1)\, y = 0$

The Auxiliary equation is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1, -1$

$C.F. = (A + Bx)\, e^{-x}$

2. Solve the equation $(D^2 + 3D + 2)\, y = 0$

Solution:The Auxiliary equation is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -2, -1$

$y = Ae^{-2x} + Be^{-x}$

3. Find the complementary function of $(D^2 + 1)\, y = 0$

Solution:The A.E. is $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

$y = e^{0x} (A \cos x + B \sin x)$

$\therefore y = A \cos x + B \sin x$

4. Solve $y'' + 4y' + 20y = 0$

Solution:The A.E. is $m^2 + 4m + 20 = 0$

$$m = \frac{-4 \pm \sqrt{16-80}}{2}$$

$$= \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm 8i}{2}$$

$$= -2 \pm 4i$$

$\therefore y = e^{-2x} (A \cos 4x + B \sin 4x)$

5. Solve the equation $(D^3 - 3D^2 + 4D - 2)\, y = 0$

Solution:The A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$(m - 1)(m^2 - 2m + 2) = 0$

$m = 1$ or $m = 1 \pm i$

The solution is $y = Ae^x + e^x (B \cos x + C \sin x)$

6. Find the complementary function of $(D^3 + 2D^2 + D)\, y = 0$

Solution:The A.E. is $m^3 + 2m^2 + m = 0$

$m\,(m^2 + 2m + 1) = 0$

$m = 0,\ (m + 1)^2 = 0$

$m = 0, –1, –1$

$C.F = A + (Bx + C)\, e^{-x}$

7. Solve the equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Solution:The given equation can be written as

$(D^2 – 5D + 6)\, y = 0$

The auxiliary equation is $m^2 – 5m = 6 = 0$

$(m – 3)\,(m – 2) = 0$

$m = 2, 3$

$y = Ae^{2x} + Be^{3x}$

8. Solve the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution:The given equation can be written as

$(D^2 + D)\, y = 0$

The auxiliary equation is $m^2 + m = 0$

$m(m + 1) = 0$

$m = 0, –1$

$y = A + Be^{-x}$

9. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

Solution:The given equation can be written as

$(D^3 + 2D^2 – D – 2)\, y = 0$

The auxiliary equation is $m^3 + 2m^2 - m - 2 = 0$

$m^2 (m + 2) - 1(m + 2) = 0$

$(m^2 - 1)(m + 2) = 0$

$m^2 = 1$ or $m = -2$

$m = -1, 1, -2$

$y = Ae^{-2x} + Be^{-x} + Ce^{x}$

10. Find the complementary function of $(D^2 - 2D + 2)\, y = 0$

Solution:The auxiliary equation is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$C.F. = e^{x}(A \cos x + \sin x)$

11. Solve $(D^2 - 4D + 13)\, y = 0$

Solution:The auxiliary equation is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$y = e^{2x}(A \cos 3x + B \sin 3x)$

12. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Solution:The auxiliary equation is $m^4 + 8m^2 + 16 = 0$

$(m^2 + 4)^2 = 0$

$m^2 + 4 = 0, m^2 + 4 = 0$

$m = \pm 2i \qquad m = \pm 2i$

$\therefore y = (A_1 + A_2 x) \cos 2x + (A_3 + A_4 x) \sin 2x$

13. Solve $(D^2 - 9)\, y = 0$

Solution:The auxiliary equation is $m^2 - 9 = 0$

$\therefore m^2 = 9$

$m = \pm 3$

$y = Ae^{-3x} + Be^{3x}$

14. Solve $(D^2 - 5D + 7)\, y = 0$

Solution:The A.E. is $m^2 - 5m + 7 = 0$

$$m = \frac{5 \pm \sqrt{25-28}}{2}$$

$$= \frac{5 \pm \sqrt{-3}}{2} = \frac{5}{2} \pm \frac{i\sqrt{3}}{2}$$

$$y = e^{\frac{5}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

15. Solve $\frac{d^2y}{dx^2} + a^2 y = 0$

Solution:The given equation can be written a

$(D^2 + a^2)\, y = 0$

The A.E. is $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$\therefore y = A \cos ax + B \sin ax.$

Exercises: (Part A)

(1) Solve $(D^2 + 2D - 15)\, y = 0$

(2) $(2D^2 + 7D + 5)\, y = 0$

(3) Solve $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 5y = 0$

(4) $(D^2 + 6D + 9)\, y = 0$

(5) $(D^2 + 8D + 16)\, y = 0$

(6) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

(7) $(D^2 + D + 1)\, y = 0$

(8) $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 3y = 0$

(9) $(D^2 + 16)\, y = 0$

(10) $\frac{d^2y}{dx^2} + y = 0$

(11) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

(12) $(D^2 + 8D + 16)\, y = 0$

(13) $(D^2 - 7D + 12)\, y = 0$

(14) $(D^3 - 6D^2 + 11D - 6)\, y = 0$

(15) $(D^2 + 7)\, y = 0$

Answers

(1) $y = Ae^{3x} + 13e^{-5x}$

(2) $y = Ae^{-x} + Be^{\frac{-5x}{2}}$

(3) $y = (Ax + B)e^{-4x}$

(4) $y = (Ax + B)e^{-3x}$

(5) $y = (Ax + B)e^{-4x}$

(6) $y = (Ax + B)e^{5x}$

(7) $y = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$

(8) $y = e^{-x}\left[A\cos\sqrt{2}x + B\sin\sqrt{2}x\right]$

(9) $y = A\cos 4x + B\sin 4x$

(10) $y = A\cos x + B\sin x$

(11) $y = Ae^{2x} + B\cos 2x + C\sin 2x$

(12) $y = (Ax + B)e^{-4x}$

(13) $y = Ae^{3x} + Be^{4x}$

(14) $y = Ae^{x} + Be^{2x} + Ce^{3x}$

(15) $y = A\cos\sqrt{7}x + B\sin\sqrt{7}x$

Problem based on P.I. = $\dfrac{1}{f(D)}e^{ax}$

1. Find the particulars integral of $(D - 1)\,y = e^x$ [s.u. May '07]

Solution: Particular Integral $= \dfrac{1}{D-1}e^{x}$

$= \dfrac{1}{1-1}e^{x}$

(replacing D by 1)

$= \dfrac{e^x}{0} = \dfrac{xe^x}{1}\ (\because Dr\ is\ 0)$

$= xe^x$

2. Find the particular integral of $(D^2 - 4D + 13)$ y = e^{2x}

Solution: Particular Integral $= \dfrac{1}{D^2 - 4D + 13}e^{2x}$

$= \dfrac{1}{2^2 - 4x^2 + 3}e^{2x}$

(replacing D by 2)

$= \dfrac{1}{4 - 8 + 13}e^{2x}$

$= \dfrac{1}{9}e^{2x}$

3. Find the particular Integral of $(D^2 - 2D + 1)\,y = e^x$ [s.u. May '10]

Solution: Particular Integral $= \frac{1}{D^2 - 2D + 1} e^x$

$= \frac{1}{1^2 - 2\times1+1} e^x$

(replacing *D* by 1)

$= \frac{1}{0} e^x$

$= \frac{xe^x}{2D-1} (\because Dr \ is \ 0)$

$= \frac{xe^x}{2-2}$ (replacing D by 1)

$= \frac{xe^x}{0}$

$= \frac{x^2e^x}{2} (\because Dr \ is \ 0)$

4. Find the particular Integral of $(D^2 - 4D + 4)\, y = \cos h2x$ [s.u. Dec '07]

Solution: Particular Integral $= \frac{1}{(D^2 - 4D + 1)} \cos h2x$

$= \frac{1}{(D^2 - 4D + 1)} \left(\frac{e^{2x} + e^{-2x}}{2} \right)$

$= \frac{1}{2} \left[\frac{1}{D^2 - 4D + 4} e^{2x} + \frac{1}{D^2 - 4D + 4} e^{-2x} \right]$

$= \frac{1}{2} \left[\frac{1}{(2)^2 - 4\times2+4} e^{2x} + \frac{1}{(-2)^2 - 4\times(-2)+4} e^{-2x} \right]$

$= \frac{1}{2} \left[\frac{1}{0} e^{2x} + \frac{1}{16} e^{-2x} \right]$

$= \frac{1}{2} \left[\frac{xe^{2x}}{2D-4} + \frac{e^{-2x}}{16} \right]$

$$= \frac{1}{2}\left[\frac{xe^{2x}}{2\times 2-4}+\frac{e^{-2x}}{16}\right]$$

$$= \frac{1}{2}\left[\frac{x^2e^{2x}}{2}+\frac{e^{-2x}}{16}\right]$$

5. Solve $(D-2)^2y = e^{2x}$ where $D = \frac{d}{dx}$ [s.u. Dec '06]

Solution: Auxiliary Equation: $(m-2)^2 = 0$

$m = 2, 2$

Complimentary Function $(C.F) = (Ax + B)\, e^{2x}$

Particular Integral $= \frac{1}{(D-2)^2}e^{2x}$

$= \frac{1}{(2-2)^2}e^{2x}$ (replacing D by 2)

$= \frac{x}{(2D-4)}e^{2x}$ (Since Dr is 0)

$= \frac{xe^{2x}}{2\times 2-4}$

$= \frac{xe^{2x}}{0}$

$= \frac{x^2e^{2x}}{2}$

Complete Solution y = Complimentary function + Particular Integral

$$y = (Ax+B)e^{2x}+\frac{x^2e^{2x}}{2}$$

6. Solve $(D^2-1)\, y = e^x$ [s.u Dec '11]

Solution: Auxiliary Equation $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

Complimentary Function = $Ae^{x} + Be^{-x}$

Particular Integral $= \dfrac{1}{(D^2-1)}e^{x}$

$= \dfrac{1}{1^2-1}e^{x}$ (Replace D by 1)

$= \dfrac{e^{x}}{0}$

$= \dfrac{xe^{x}}{2D}\ (\because Dr\ is\ Zero)$

$= \dfrac{xe^{x}}{2\times 1} = \dfrac{xe^{x}}{2}$

Complete Solution: $y = Ae^{x} + Be^{-x} + \dfrac{xe^{x}}{2}$

7. Find the particular integral of $(D^2 + D - 6)\, y = e^{2x}$ [s.u. Dec '12]

Solution: Particular Integral $= \dfrac{1}{D^2 + D - 6}e^{2x}$

$= \dfrac{1}{2^2 + 2 - 6}e^{2x}$

$= \dfrac{1}{0}e^{2x}$

$= \dfrac{xe^{2x}}{2D+1}$

$= \dfrac{xe^{2x}}{2\times 2+1}$

$= \dfrac{xe^{2x}}{5}$

8. Solve $(D^2 - 3D + 2)\, y = e^{4x}$ where $D = \dfrac{d}{dx}$ [s.u Dec'09]

Solution: AuxiliaryEquation $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

Complimentary Function = $Ae^{x} + Be^{2x}$

Particular Integral $= \dfrac{1}{(D^2 - 3D + 2)} e^{4x}$

$= \dfrac{1}{(4^2 - 3 \times 4 + 2)} e^{4x}$

(replacing D by 4)

$= \dfrac{1}{6} e^{4x}$

Complete Solution: $y = Ae^{x} + Be^{2x} + \dfrac{e^{4x}}{6}$

9. Solve $(4D^2 - 4D + 1)\, y = 4$

Solution: Auxiliary Equation $4m^{2+} - 4m + 1 = 0$

$4m^2 - 2m - 2m + 1 = 0$

$2m\,(2m - 1) - 1\,(2m - 1) = 0$

$(2m - 1)^2 = 0$

$m = \dfrac{1}{2}, \dfrac{1}{2}$

Complimentary Function = $(Ax + B)e^{\frac{1}{2}x}$

Particular Integral $= \dfrac{1}{4D^2 - 4D + 1} 4e^{0x}$

$= \dfrac{1}{4 \times 0^2 - 4 \times 0 + 1} 4e^{0x}$

$= \dfrac{4e^{0x}}{1} = 4$

Complete Solution: $y = (Ax + B)e^{\frac{x}{2}} + 4$

10. Solve $(D^3 - 3D^2 + 4D - 2)\, y = e^{x}$

Solution: Auxiliary Equation is $m^3 - 3m^2 + 4m - 2 = 0$.

$m = 1$ satisfies the equation

$\therefore m - 1$ is a factor of this equation

To find the other roots, divide the given equation by $m - 1$

$$\begin{array}{c|cccc} & 1 & -3 & 4 & -2 \\ 1 & & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{4i^2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ The roots of the equation are $m = 1, 1 + i, 1 - i$.

Complimentary Function=$Ae^x + e^x(A \cos x + B \sin x)$

Particular Integral: $$\frac{1}{(D^3 - 3D^2 + 4D - 2)} e^x = \frac{1}{1^3 - 3\times 1^2 + 4\times 1 - 2} e^x$$

$$= \frac{e^x}{0}$$

$$= \frac{xe^x}{(3D^2 - 6D + 4)} \text{ (Since the denominator is 0)}$$

$$= \frac{xe^x}{(3\times 1^2 - 6\times 1 + 4)} = xe^x$$

Complete Solution: $y = Ae^x + e^x (A \cos x + B \sin x) + xe^x$

11. $(D^2 - 3D + 2)\, y = e^{4x} (\sin hx)$

Solution: Auxiliary Equation $\Rightarrow m^2 - 3m + 2 = 0$

$(m - 1)(m - 2) = 0$

$m = 1, 2$

Complimentary Function = $Ae^x + Be^{2x}$

Particular Integral $= \dfrac{1}{(D^2-3D+2)} e^{4x}(\sin\ hx)$

$= \dfrac{1}{(D^2-3D+2)} e^{4x}\left(\dfrac{e^x-e^{-x}}{2}\right)$

$= \dfrac{1}{(D^2-3D+2)}\left(\dfrac{e^{5x}-e^{3x}}{2}\right)$

$= \dfrac{1}{2}\left[\dfrac{1}{(D^2-3D+2)}e^{5x} - \dfrac{1}{(D^2-3D+2)}e^{3x}\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{(5^2-3\times5+2)}e^{5x} - \dfrac{1}{(3^2-3\times3+2)}e^{3x}\right]$

$= \dfrac{1}{2}\left[\dfrac{e^{5x}}{12} - \dfrac{e^{3x}}{2}\right]$

$= \dfrac{e^{5x}-6e^{3x}}{24}$

Complete Solution: $y = Ae^x + Be^{2x} + \dfrac{e^{5x}-6e^{3x}}{24}$

Problems based on $P.I. = \dfrac{1}{f(D)}x^n$

Following formulae are important

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

12. Find the particular integral of $(D^2 - 9D + 20)\, y = 20x$ [s.u Dec ‘10]

Solution: Particular Integral $= \dfrac{1}{(D^2-9D+20)}20x$

$$= \frac{20}{20\left(1+\frac{D^2-9D}{20}\right)}x$$

$$= \left(1+\frac{D^2-9D}{20}\right)^{-1}x$$

$$= \left(1-\left(\frac{D^2-9D}{20}\right)+...\right)x$$

$$= x+\frac{9}{20}D(x)-D^2(x)...$$

$$= x+\frac{9}{20}$$

13. Solve $(D^2 - 1)\, y = x$

Solution: Auxiliary Equation $\Rightarrow m^2 - 1 = 0$

$m^2 = 1$

$m^2 = \pm 1$

Complimentary Function = $Ae^x + Be^{-x}$

Particular Integral $= \frac{1}{D^2-1}x$

$$= \frac{-1}{1-D^2}x$$

$$= -(1-D^2)^{-1}x$$

$$= -(1+D^2+(D^2)^2...)x$$

$$= -(x+D^2(x)+D^4(x)...)$$

$$= -x$$

Complete Solution: $y = Ae^x + Be^{-x} - x$

14. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

Solution: Auxiliary Equation $\Rightarrow m^2 - 5m + 6 = 0$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

Complimentary Function = $Ae^{3x} + Be^{2x}$

Particular Integral $= \frac{1}{D^2 - 5D + 6}(x^2+3)$

$$= \frac{1}{6\left(1 + \frac{D^2 - 5D}{6}\right)}(x^2+3)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 \ldots\right](x^2+3)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} + \frac{25D^2}{36} - \frac{10D^3}{36}\ldots\right](x^2+3)$$

$$= \frac{1}{6}\left[(x^2+3) - \frac{1}{6}D^2(x^2+3) + \frac{5}{6}D(x^2+3) + 0 + \frac{25}{36}D^2(x^2+3) - 0\ldots\right]$$

$$= \frac{1}{6}\left[(x^2+3) - \frac{1}{6}(2) + \frac{5}{6}(2x) + \frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2 + 3 - \frac{1}{3} + \frac{5}{6}x + \frac{25}{36}\right]$$

$$= \frac{1}{108}\left[18x^2 + 30x + 73\right]$$

Complete solution: $y = Ae^{3x} + Be^{2x} + \frac{1}{108}[18x^2 + 30x + 73]$

15. Solve $(D^4 - 2D^3 + D^2)\, y = x^3$

Solution: Auxiliary Equation $\Rightarrow m^4 - 2m^3 + m^2 = 0$

$$m^2(m^2 - 2m + 1) = 0$$

$m^2\ (m - 1)^2 = 0$

$m = 0, 0, 1, 1$

Complimentary Function = $(Ax + B)\ e^{0x} + (Cx + D)\ e^x$

Particular Integral $= \dfrac{1}{D^4 - 2D^3 + D^2} x^3$

$$= \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2}[1 + D^2 - 2D]^{-1} x^3$$

$$= \frac{1}{D^2}[1 - (D^2 - 2D) + (D^2 - 2D)^2 - (D^2 - 2D)^3 \ldots]x^3$$

$$= \frac{1}{D^2}[1 + 2D + 3D^2 + 4D^3]x^3$$

(omitting $D^4$and higher powers]

$$= \frac{1}{D^2}[x^3 + 2D(x^3) + 3D^2(x^3) + 4D^3(x^3)]$$

$$= \frac{1}{D^2}[x^3 + 6x^2 + 18x + 24]$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + 6\frac{x^3}{3} + \frac{18x^2}{2} + 24x\right]$$

$$= \frac{x^5}{4 \times 5} + \frac{6x^4}{3 \times 4} + \frac{18x^3}{2 \times 3} + \frac{24x^2}{2}$$

$$= \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x$$

Complete Solution: $y = (A + Bx) + (C + Dx)e^x + \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 + 12x^2$

16. Solve $(D^3 + 8) = x^4 + 2x + 1$

Solution: Auxiliary Equation $\Rightarrow m^3 + 8 = 0$

$m = -2$ satisfies the equation

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 8 \\ -2 & 0 & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

$(m+2)\ (m^2 - 2m + 4) = 0$

$m = -2$ or $m = \dfrac{2 \pm \sqrt{4-16}}{2}$

$m = -2$ or $m = \dfrac{2 \pm \sqrt{12i}}{2}$

$m = -2$ $m = \dfrac{2 \pm i2\sqrt{3}}{2}$

$m = -2$ $m = 1 \pm i\sqrt{3}$

Complimentary Function = $Ae^{-2x} + e^{x}\left[B\cos\sqrt{3}x + \sin\sqrt{3}x\right]$

Particular Integral $= \dfrac{1}{(D^3+8)}(x^4 + 2x + 1)$

$$= \frac{1}{8\left(\frac{1+D^3}{8}\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \ldots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[x^4 + 2x + 1 - \frac{1}{8}D^3\ (x^4 + 2x + 1)\right]$$

$$= \frac{1}{8}\left[x^4 + 2x + 1 - 3x\right]$$

$$= \frac{1}{8}\left[x^4 - x + 1\right]$$

Complete Solution: $y = Ae^{-2x} + e^{x}(B\cos\sqrt{3}x + \sin\sqrt{3}x) + \dfrac{1}{8}(x^4 - x + 1)$

Exercises

Solve:

1. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=e^x$

2. $(D^2+6D+9)y=5e^{3x}$

3. $(D^2+6)y=6e^{3x}$

4. $(D^2+2D+2)y=\sin\ hx$

5. $(D^2-4D+4)y=e^{2x}$

6. $(D^3-3D+4D-2)y=e^x$

7. $(D^3+3D^2+3D+1)y=e^{-x}$

8. $(D^2-4)y=x^3$

9. $(D^2+5D+4)y=x^2+7x+9$

10. $(D^2+D-6)y=x$

11. $(D^2-3D+2)y=2x^2+1$

12. $(D^3-1)y=x$

13. $(D^3-13D+12)y=x$

14. $(D^3-2D+D)y=x^2+x$

15. $(D^3-3D^2-6D+8)y=x$

16. $(D^4-2D^3+D^2)y=x^2+e^x$

17. $(D^3+8)y=x^4+2x+1+\cos h2x$

18. $(D^4-4)y=x\sin\ h2x$

19. Find the particular integral of $(D-1)^3y = 2\cos hx$

20. Find the particular integral of $(D^2 + \mathrm{a}^2)\ y = b\cos ax + c\sin ax$

Answer

1. $\left[Ans: y = C_1e^{-x} + C_2e^{-5x} + \frac{1}{21}e^{2x}\right]$

2. $\left[Ans: y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}\right]$

3. $\left[Ans: y = C_1\cos 3x + C_2\sin 3x + \frac{e^{3x}}{3}\right]$

4. $\left[Ans: y = e^{-x}(C_1\cos x + C_2\sin x) + \frac{e^{x}}{10} - \frac{e^{-x}}{2}\right]$

5. $\left[Ans: y = (C_1 \_ + C_2x)e^{2x} + \frac{x^2}{2}e^{2x}\right]$

6. $\left[Ans: y = C_1e^{x} + e^{x}(c_2\cos x + c_3\sin x) + xe^{x}\right]$

7. $\left[Ans: y = (C_1 + C_2x + C_3x^2)e^{-x} + \frac{x^3}{6}e^{-x}\right]$

8. $\left[Ans: y = C_1e^{2x} + C_2e^{-2x} - \frac{1}{8}x(2x^2 + 3)\right]$

9. $\left[Ans: y = C_1e^{-x} + C_2e^{-4x} + \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right)\right]$

10. $\left[Ans: y = C_1e^{-3x} + C_2e^{2x} - \frac{1}{36}(6x + 1)\right]$

11. $\left[Ans: y = C_1e^{x} + C_2e^{2x} + x^2 + 3x + 4\right]$

12. $\left[Ans: y = C_1 + C_2e^{x} + C_3e^{-x}\frac{1}{2}x^2\right]$

13. $\left[Ans: y = C_1e^{x} + C_2e^{-4x} + C_3e^{3x}\frac{1}{144}(12x + 13)\right]$

14. $\left[Ans: y = C_1 + (C_2 = C_3x)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]$

15. $\left[Ans: y = C_1e^{x} + C_2e^{-2x} + C_3e^{4x} + \frac{1}{8}\left(x + \frac{3}{4}\right)\right]$

16. $\left[Ans: y = (C_1x + C_2) + (C_3x + C_4)e^{x} + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + \frac{x^2}{2}e^{x} \right]$

17. $\left[Ans: y = Ae^{-2x} + e^{x}\left(B\cos\sqrt{3}x + C\sin\sqrt{3}x\right) + \frac{1}{8}\left(x^4 - x + 1\right)\frac{+1}{96}\left(3e^{2x} + 4xe^{-2x}\right)\right]$

18. $\left[Ans: y = C_1e^{2x} + C_2e^{-2x} - \frac{x}{3}\sin hx + \frac{2}{9}\cos hx \right]$

19. $\left[Ans: y = \frac{x^3}{6}e^{x} - \frac{1}{8}e^{-x} \right]$

20. $\left[Ans: y = \frac{x}{2a}(b\sin ax - c\cos ax)\right]$

1. Solve $(D^2 - 4D + 3)y = \sin 3x$

Auxiliary Equation is $m^2 - 4m = 0$

$(m - 1)(m - 3) = 0$

$m = 1, 3$

Complementary function $(C.F) = Ae^{x} + Be^{3x}$

P.I. $= \frac{1}{D^2 - 4D + 3}\sin 3x$

$D^2 \rightarrow -a^2 = -9$

$= \frac{1}{-9 - 4D + 3}\sin 3x$

$= \frac{1}{-6 - 4D}\sin 3x$

$= \frac{1}{-6 - 4D} \times \frac{-6 + 4D}{-6 + 4D}\sin 3x$

$= \frac{(-6 + 4D)}{36 - 16D^2}\sin 3x$

$D^2 \rightarrow -a^2 = -9$

$$= \frac{(-6+4D)}{36-16(-9)}\sin 3x$$

$$= \frac{1}{36+44}[-6\sin 3x + 4D(\sin 3x)]$$

$$= \frac{1}{180}[-6\sin 3x + 12\cos 3x]$$

$$= \frac{6}{180}[2\cos 3x - \sin 3x]$$

P.I. $= \frac{1}{30}[2\cos 3x - \sin 3x]$

C.S = C.F + P.I

$y = Ae^{x} + Be^{3x} + \frac{1}{30}[2\cos 3x - \sin 3x]$

2. Solve $(D^2 - 2D + 1)y = \cos 2x + x^2$

Solution: Auxiliary Equation is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1$ (twice)

C.F = $(Ax + B)\, e^{-x}$

P.I. $= \frac{1}{D^2 - 2D + 1}(\cos 2x + x^2)$

$$= \frac{1}{D^2 + 2D + 1}\cos 2x + \frac{1}{D^2 + 2D + 1}x^2$$

P.I. = $P.I._1 + P.I._2$

$P.I._1 = \frac{1}{D^2 + 2D + 1}\cos 2x$

$D^2 \longrightarrow -a^2 = -4$

$$= \frac{1}{-4 + 2D + 1}\cos 2x$$

$$= \frac{1}{-3+2D} \times \frac{-3-2D}{-3-2D} \cos 2x$$

$$= \frac{-3-2D}{9-4D^2} \cos 2x$$

$D^2 \rightarrow -a^2 = -4$

$$= \frac{-(3+2D)}{9-4(-4)} \cos 2x$$

$$= \frac{-(3+2D)}{9+16} \cos 2x$$

$$= \frac{-1}{25}[3\cos 2x + 2D(\cos 2x)]$$

$$\text{P.I.}_1 = \frac{-1}{25}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_1 = \frac{-1}{25}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_2 = \frac{1}{D^2+2D+1} x^2$$

$$= \frac{1}{1+D^2+2D} x^2$$

$$= [1+(D^2+2D)]^{-1} x^2$$

$$= [1-(D^2+2D)+(D^2+2D)^2+...]x^2$$

$= [x^2 - D^2 + 2D)x^2 + (D^2+2D)^2 x^2 + ...]$ [neglecting D^3& higher powers]

$$= [x^2 - D^2(x^2) - 2D(x^2) + 4D^2(x^2)]$$

$$= [x^2 - 2 - 4x + 4(2)]$$

$$\text{P.I.}_1 = x^2 - 4x + 6$$

C.S. = C.F. + P.I_1+ P.I_2

$$y = (Ax+B)e^{-x} - \frac{1}{25}[3\cos 2x - 4\sin 2x] + x^2 - 4x + 6$$

3. Solve $(D^2 - 8D + 9)y = 8\sin 5x$

Solution: Auxiliary Eqn is $m^2 - 8m + 9 = 0$

$$m = \frac{8 \pm \sqrt{64-36}}{2}$$

$$m = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$$

$$\text{C.F} = Ae^{(4+\sqrt{7})x} + Be^{(4-\sqrt{7})x}$$

$$\text{P.I} = 8\frac{1}{D^2 - 8D + 1}\sin 5x \quad D^2 \to -a^2 = -25$$

$$= 8\frac{1}{-25 - 8D + 9}\sin 5x$$

$$= 8\frac{1}{-16 - 8D}\sin 5x$$

$$= -\frac{8}{8}\frac{1}{2+D} \times \frac{2-D}{2-D}\sin 5x$$

$$= -\frac{(2-D)}{4-D^2}\sin 5x \qquad D^2 \to -a^2 = -25$$

$$= \frac{(2-D)\sin 5x}{4-(-25)}$$

$$= \frac{1}{29}[2\sin 5x - D(\sin 5x)]$$

$$\text{P.I} = \frac{1}{29}[2\sin 5x - 5(\cos 5x)]$$

C.S = C.F + P.I

$$y = Ae^{(4+\sqrt{7})x} + Be^{(4-\sqrt{7})x} - \frac{1}{29}[2\sin 5x - 5\cos 5x]$$

4. Solve $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Solution: Auxiliary Eqn is $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$

$m = 0;\ m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = 0,\ m = -1$ (twice)

C.F. = $Ae^{0x} + (Bx + C)\, e^{-x}$

C.F. = $A + (Bx + C)\, e^{-x}$

$$\text{P.I} \quad = \frac{1}{D^3 + 2D^2 + D}[e^{2x} + \sin 2x]$$

$$\text{P.I} \quad = \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

P.I. = $\text{P.I.}_1 + \text{P.I.}_2$

$$\text{P.I.}_1 = \frac{1}{D^3 + 2D^2 + D} e^{2x}$$

$$D \rightarrow a = 2$$

$$= \frac{1}{8 + 2(4) + 2} e^{2x}$$

$$\text{P.I.}_1 = \frac{1}{18} e^{2x}$$

$$\text{P.I.}_2 = \frac{1}{D^3 + 2D^2 + D} \sin 2x \qquad D^2 \rightarrow -a^2 = -4$$

$$= \frac{1}{-4D + 2(-4) + D} \sin 2x$$

$$= \frac{1}{-8 - 3D} \sin 2x$$

$$= \frac{1}{-8 - 3D} \times \frac{-8 + 3D}{-8 + 3D} \sin 2x$$

$$= \frac{-8 + 3D}{64 - 9D^2} \sin 2x \qquad D^2 \rightarrow -a^2 = -4$$

$$= \frac{-8 + 3D}{64 - 9(-4)} \sin 2x$$

$$= \frac{-8+3D}{64+36}\sin 2x$$

$$= \frac{1}{100}[-8\sin 2x + 3D(\sin 2x)]$$

$$= \frac{1}{100}[-8\sin 2x + 6\cos 2x]$$

$$= \frac{2}{100}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_2 = \frac{1}{50}[3\cos 2x - 4\sin 2x]$$

$$\text{C.S} = \text{C.F} + \text{P.I.}_1 + \text{P.I.}_2$$

$$y = A + (Bx + C)e^{-x} + \frac{1}{18}e^{2x} + \frac{}{50}[3\cos 2x - 4\sin 2x]$$

5. Solve $(D^2 - 4D + 3)y = \cos 3x$

Solution: Auxiliary Eqn is $m^2 - 4m + 3 = 0$

$(m - 1)(m - 3) = 0$

$m = 1, 3$

$$\text{C.F} = Ae^{x} + Be^{3x}$$

$$\text{P.I} = \frac{1}{D^2 - 4D + 3}\cos 3x \qquad D^2 \to a^2 = -9$$

$$= \frac{1}{-9 - 4D + 3}\cos 3x$$

$$= \frac{1}{-6 - 4D}\cos 3x$$

$$= \frac{1}{-6 - 4D} \times \frac{-6 + 4D}{-6 + 4D}\cos 3x$$

$$= \frac{-6 + 4D}{36 - 16D^2}\cos 3x \qquad D^2 \to -a^2 = -9$$

$$= \frac{-6+4D}{36-16(-9)}\cos 3x$$

$$= \frac{-6+4D}{36+144}\cos 3x$$

$$= \frac{1}{180}[-6\cos 3x + 4D(\cos 3x)]$$

$$= \frac{1}{180}[-6\cos 3x + 12\sin 3x]$$

$$= \frac{-6}{180}[\cos 3x + 2\sin 3x]$$

P.I $= \dfrac{-1}{30}[\cos 3x + 2\sin 3x]$

C.S = C.F + P.I

$y = Ae^{x} + Be^{3x} - \dfrac{1}{30}[\cos 3x + 2\sin 3x]$

6. Solve $(D^2 + 3D + 2)y = 2\sin^2 x$

Auxiliary equation is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -1, -2$

C.F $= Ae^{-x} + Be^{-2x}$

P.I $= \dfrac{1}{D^2 + 3D + 2} 2\sin^2 x$

$$= \frac{1}{D^2+3D+2}\, 2\left[\frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{D^2+3D+2}(1-\cos 2x)$$

$$= \frac{1}{D^2+3D+2} - \frac{1}{D^2+3D+2}\cos 2x$$

$$= \frac{1}{D^2+3D+2}e^{0x} - \frac{1}{D^2+3D+2}\cos 2x$$

$$D \to a = 0, \text{D}^2 \to \text{- a}^2 \text{= -4}$$

$$= \frac{1}{2}e^{0x} - \frac{1}{-4+3D+2}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{-2+3D} \times \frac{-2-3D}{-2-3D}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{4-9D^2}(2+3D)\cos 2x \qquad D^2 \to -a^2 = -4$$

$$= \frac{1}{2} + \frac{1}{4+36}[2\cos 2x + 3D\cos 2x]$$

$$= \frac{1}{2} + \frac{2}{40}[2\cos 2x - 6\sin 2x]$$

$$= \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$$

P.I. $= \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$

C.S = C.F. + P.I

$y = Ae^{-x} + Be^{-2x} + \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$

7. Solve $(D^2 + 6D + 8)\, y = \cos^2 x$

Solo A. Eqn is $m^2 + 6m + 8 = 0$

$(m + 2)\,(m + 4) = 0$

$m = -2, -4$

C.F $= Ae^{-2x} + Be^{-4x}$

P.I $= \frac{1}{D^2+6D+8}\cos^2 x$

$$= \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2+6D+8}(1) + \frac{1}{2}\frac{1}{D^2+6D+8}\cos 2x \qquad D^2 \to -a^2 = -4$$

$$= \frac{1}{2}\frac{1}{8} + \frac{1}{2}\frac{1}{(-4+6D+8)}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{4-6D}{4-6D} \times \frac{1}{4+6D}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{4-6D}{16-36D^2}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{(4-6D)}{16-36(-4)}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{1}{16+144}(4-6D)\cos 2x$$

$$= \frac{1}{16} + \frac{1}{2}\frac{1}{160}[4\cos 2x - 6D(\cos 2x)]$$

$$= \frac{1}{16} + \frac{1}{320}[4\cos 2x + 12\sin\ 2x]$$

$$= \frac{1}{16} + \frac{4}{320}[\cos 2x + 3\sin\ 2x]$$

P.I $= \frac{1}{16} + \frac{1}{80}[\cos 2x + 3\sin\ 2x]$

C.S = C.F. + P.I

y $= Ae^{-2x} + Be^{-4x} + \frac{1}{16} + \frac{1}{80}[\cos 2x + 3\sin\ 2x]$

8. Solve $(D^2 + 16)y = e^{-3x} + \cos 4x$

Auxiliary Eqn is $m^2 + 16 = 0$

$m^2 = -16$

$m = \pm 4i$

$\alpha = 0 \quad \beta = 4$

C.F. $= A \cos 4x + B \sin 4x$

$$\text{P.I.}_1 = \frac{1}{D^2+16}e^{-3x}$$

$$D \rightarrow a = -3$$

$$= \frac{1}{9+16}e^{-3x}$$

$$\text{P.I.}_1 = \frac{1}{25}e^{-3x}$$

$$\text{P.I.}_2 = \frac{1}{D^2+16}\cos 4x$$

$$D^2 \rightarrow -a^2 = -16$$

$$= \frac{1}{-16+16}\cos 4x$$

$$= x\frac{1}{2D}\cos 4x$$

$$= \frac{x}{2}\frac{\sin 4x}{4}$$

$$\text{P.I.}_2 = \frac{x\sin 4x}{8}$$

$$\text{C.S} = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2$$

$$y = A\cos 4x + B\sin 4x + \frac{1}{25}e^{-3x} + \frac{x\sin 4x}{8}$$

9. Solve $(D^2 + 9)\, y = \sin 3x$

Solution: Auxiliary Eqn is $m^2 + 9 = 0$

$$m^2 = -9$$

$$m = \pm 3i$$

$$\text{C.F} = A\cos 3x + B\sin 3x$$

$$\text{P.I.} = \frac{1}{D^2+9}\sin 3x \qquad D^2 \rightarrow -a^2 = -9$$

$$= \frac{1}{-9+9}\sin 3x$$

$$= x\frac{1}{2D}\sin 3x$$

P.I $= -\dfrac{x\cos 3x}{6}$

C.S = C.F + P.I.

$y = A\cos 3x + B\sin 3x - \dfrac{x\cos 3x}{6}$

10. Solve $(D^2 + 1)\, y = \cos x$

Auxiliary Eqn is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

C.F $= A\cos x + B\sin x$

P.I. $= \dfrac{1}{D^2+1}\cos x$ $\quad D^2 \rightarrow -a^2 = -1$

$$= \frac{1}{-1+1}\cos x$$

$$= x\frac{1}{2D}\cos x$$

P.I $= -\dfrac{x\sin x}{2}$

C.S = C.F + P.I.

$y = A\cos x + B\sin x - \dfrac{x\sin x}{2}$

11. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$

Auxiliary Eqn is $m^2 - 4m + 3 = 0$

$(m - 1)(m - 3) = 0$

$m = 1, 3$

$$\text{C.F} = Ae^{x} + Be^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3}\sin 3x\cos 2x$$

$$= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}(\sin(3x + 2x) + \sin(3x - 2x))\right]$$

$$\text{P.I} = \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin 5x + \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin x$$

$$\text{P.I.} = \text{P.I.}_1 + \text{P.I.}_2$$

$$\text{P.I.}_1 = \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin 5x \qquad D^2 \to -a^2 = -25$$

$$= \frac{1}{2}\frac{1}{-25 - 4D + 3}\sin 5x$$

$$= \frac{1}{2}\frac{1}{-22 - 4D}\sin 5x$$

$$= \frac{-1}{4}\frac{1}{11 + 2D} \times \frac{11 - 2D}{11 - 2D}\sin 5x$$

$$= \frac{-1}{4}\frac{(11 - 2D)}{(121 - 4D^2)}\sin 5x \qquad D^2 \to -a^2 = -25$$

$$= \frac{-1}{4}\frac{(11 - 2D)}{(121 - 49 - 25)}\sin 5x$$

$$= \frac{-1}{4}\frac{(11 - 2D)}{121 + 100}\sin 5x$$

$$= \frac{-1}{4}\frac{1}{221}(11\sin 5x - 2D\sin 5x)$$

$$= \frac{-1}{884}(11\sin 5x - 10\cos 5x)$$

$$\text{P.I.}_1 = \frac{1}{884}[10\cos 5x - 11\sin 5x]$$

$$\text{P.I.}_2 = \frac{1}{2}\frac{1}{D^2 - 4D + 3}\sin x \qquad D^2 \to -a^2 = -1$$

$$= \frac{1}{2}\frac{1}{(-1-4D+3)}\sin x$$

$$= \frac{1}{2}\frac{1}{(2-4D)}\sin x$$

$$= \frac{1}{2}\frac{1}{2-4D}\times\frac{2+4D}{2+4D}\sin x$$

$$= \frac{1}{2}\frac{1}{(4-16D^2)}(2+4D)\sin x \qquad D^2 \to -a^2 = -1$$

$$= \frac{1}{2}\frac{(2+4D)}{(4-16(-1))}\sin x$$

$$= \frac{1}{2}\frac{1}{20}(2\sin x + 4D\sin x)$$

$$= \frac{1}{\not 2}\frac{1}{20}\not 2(\sin x + 2\cos x)$$

P.I.$_2$ $= \frac{1}{20}[\sin x + 2\cos x]$

C.S = C.F + P.I.$_1$ + P.I.$_2$

y $= Ae^x + Be^{3x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[\sin x + 2\cos x]$

12. Solve $(D^2 + 4)\, y = 2\cos x \cos 3x$

The auxiliary Eqn is $m^2 + 4 = 0$

m^2 $= -4$

m $= \pm 2i$

C.F = A cos $2x$ + B sin $2x$

P.I $= \frac{1}{D^2+4}[2\cos x\cos 3x]$

$$= \frac{1}{D^2+4}\left[\not 2\,\frac{1}{\not 2}[\cos(3x+x)+\cos(3x-x)]\right]$$

$$= \frac{1}{D^2+4}[\cos 4x + \cos 2x]$$

$$= \frac{1}{D^2+4}\cos 4x + \frac{1}{D^2+4}\cos 2x$$

$D^2 \rightarrow -a^2 = -16$ $\quad\quad$ $D^2 \rightarrow -a^2 = -4$

$$= \frac{1}{-16+4}\cos 4x + \frac{1}{-4+4}\cos 2x$$

$$= \frac{1}{-12}\cos 4x + x.\frac{1}{2D}\cos 2x$$

P.I $= -\frac{1}{12}\cos 4x + \frac{x\sin 2x}{4}$

C.S $=$ C.F + P.I

$y = A\cos 2x + B\sin 2x - \frac{1}{12}\cos 4x + \frac{x\sin 2x}{4}$

Exercise

1. Solve $(D^2 - 4D + 4)y = \cos 2x$

2. Solve $(D^2 + 3D + 2)y = \sin 3x$

3. Solve $(D^2 + 1)y = \cos(2x - 1)$

4. Solve $(D^2 - 3D + 2)y = 7\cos x$

5. Solve $(D^2 - 7D + 12)y = e^{5x} + \cos 2x$

6. Solve $(D^2 - 4D + 3)y = \sin 2x$

7. Solve $(D^2 - 4D - 5)y = 3\cos 4x + e^{2x}$

8. Solve $(D^2 - 4D + 4)y = \cos x + e^x$

9. Solve $(D^3 + 2D^2 + D)y = e^{-2x} + \sin 2x$

10. Solve $(D^3 + 4D)y = \sin 2x$

11. Solve $(D^2+4)y=\cos 2x$

12. Solve $(D^2+6D+5)y=\sin^2 x$

13. Solve $(D^2+4)y=3\cos^2 x+x^2$

14. Solve $(D^2+9)y=\sin^3 x$

15. Solve $(D^2+6D+5)y=\cos^2 x$

16. Solve $(D^3-1)y=\cos\frac{x}{2}\sin\frac{x}{2}+2e^x$

17. Solve $(D^2+2D+1)y=\sin 2x\cos x$

18. Solve $(D^2+1)y=\sin 2x\sin x$

19. Solve $(D^2+1)y=2\sin x\cos 3x$

20. Solve $(D^2+D+1)y=2\cos 2x\cos x$

Answer

1. $y=(Ax+B)e^{2x}-\frac{1}{8}\sin 2x$

2. $y=Ae^{-x}+Be^{-2x}+\frac{1}{130}[9\cos 3x=7\sin 3x]$

3. $y=A\cos x+B\sin x-\frac{1}{3}\cos(2x-1)$

4. $y=Ae^{x}+Be^{2x}+\frac{7}{10}[\cos x-3\sin x]$

5. $y=Ae^{3x}+Be^{4x}+\frac{e^{5x}}{2}+\frac{1}{130}[4\cos 2x-7\sin 2x]$

6. $y=Ae^{x}+Be^{3x}+\frac{1}{65}[8\cos 2x-\sin 2x]$

7. $y=Ae^{-x}+Be^{5x}-\frac{1}{9}e^{2x}-\frac{3}{697}[21\cos 4x+16\sin 4x]$

8. $y=(Ax+B)e^{2x}+e^{x}-\frac{1}{8}\sin 2x$

9. $y=A+(Bx+C)e^{-x}-\frac{1}{2}e^{-2x}+\frac{1}{50}[3\cos 2x-4\sin 2x]$

10. $y=A+B\cos 2x+C\sin 2x-\frac{x}{8}\sin 2x$

11. $y=A\cos 2x+B\sin 2x+\frac{x\sin 2x}{4}$

12. $y=Ae^{-5x}+Be^{-x}+\frac{1}{10}-\frac{1}{290}[\cos 2x+12\sin 2x]$

13. $y=A\cos 2x+B\sin 2x+\frac{x^2}{4}-\frac{1}{4}+\frac{3x\sin 2x}{8}$

14. $y=A\cos 3x+B\sin 3x+\frac{3\sin x}{32}+\frac{x\cos 3x}{24}$

15. $y=Ae^{-5x}+Be^{-x}+\frac{1}{10}+\frac{1}{290}[\cos 2x+12\sin 2x]$

16. $y=Ae^{x}+Be^{-\frac{x}{2}}\left[B\cos\frac{\sqrt{3}}{2}x+C\sin\frac{\sqrt{3}}{2}x\right]+\frac{2xe^{x}}{3}+\frac{1}{4}[\cos x-\sin x]$

17. $y=[Ax+B]e^{-x}-\frac{1}{100}[3\cos 3x+4\sin 3x]-\frac{\cos x}{4}$

18. $y=A\cos x+B\sin x+\frac{x\sin x}{4}+\frac{1}{16}\cos 3x$

19. $y=A\cos x+B\cos x-\frac{1}{15}\sin 4x+\frac{1}{3}\sin 2x$

20. $y=e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{145}[8\cos 3x+3\sin 3x]+\sin x$

Type (4): $f(x) = e^{ax} \sin bx \ (or) \ e^{ax} \cos bx$

Method of finding P.I.

$$\text{P.I} = \frac{1}{\Phi(D)} e^{ax} \sin bx$$

Replace D by $D + a$

$$\text{P.I} = e^{ax}\left[\frac{1}{\Phi(D+a)}\right]\sin bx$$

$\left[\frac{1}{\Phi(D+a)}\right]\sin bx$ is evaluated using type (2)

Note: If $f(x) = xV$ where $V = \sin ax$ or $\cos ax$, then

$$\text{P.I} = x.\frac{V}{\Phi(D)} - \frac{\Phi'(D).V}{[\Phi(D)]^2}$$

1. Solve: $(D^2 + 5D + 4)Y = e^{-x} \sin 2x$

Solution: Given, $(D^2 + 5D + 4)y = e^{-x} \sin 2x$

(i) To find C.F.

$(D^2 + 5D + 4)y = 0$

The auxiliary equation is

$m^2 + 5m + 4 = 0$

$(m + 1)(m + 4) = 0$

$m + 1 = 0, m + 4 = 0$

$m = -1, m = -4$

The roots are real and distinct

$\therefore \text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$

$= Ae^{-x} + Be^{-4x}$

(ii) To find P.I.

$$\text{P.I} = \frac{1}{D^2+5D+4} e^{-x} \sin 2x$$

Here, $a = -1$ and $b = 2$

Replace D by $D + a = D - 1$

$$\text{P.I} = e^{-x} \frac{1}{(D-1)^2+5(D-1)+4} \sin 2x$$

$$= e^{-x} \frac{1}{D^2-2D+1+5D-5+4} \sin 2x$$

$$= e^{-x} \frac{1}{D^2+5D} \sin 2x$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-x} \frac{1}{-4+3D} \sin 2x$$

$$\text{P.I} = e^{-x} \frac{(3D+4)}{(3D+4)(3D-4)} \sin 2x$$

$$= e^{-x} \frac{(6\cos 2x + 4\sin 2x)}{9D^2-16}$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-x} \frac{(6\cos 2x + 4\sin 2x)}{9(-4)-16}$$

$$= 2e^{-x} \frac{3\cos 2x + 2\sin 2x}{(-52)}$$

$$= \frac{e^{-x}}{26}(3\cos 2x + 2\sin 2x)$$

The solution is

y = C.F + P.I

$$= Ae^{-x} + Be^{-4x} + \frac{(-e^{-x})}{26}(3\cos 2x + 2\sin 2x)$$

$$= Ae^{-x} + Be^{-4x} - \frac{(e^{-x})}{26}(3\cos 2x + 2\sin 2x)$$

2. Solve: $(D^2 - 2D + 4)y = e^x \sin x$

Solution:Given, $(D^2 - 2D + 4)y = e^x \sin x$

(i) To find C.F.

$$(D^2 - 2D + 4)y = 0$$

The auxiliary equation is

$$m^2 - 2m + 4 = 0$$

Here, a = 1, b = -2 and c = 4

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm i2\sqrt{3}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

The roots are imaginary Here α = 1 and ß = $\sqrt{3}$

$$\text{C.F} = e^{ax}(A\cos ßx + B\sin ßx)$$

$$= e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4} e^x \sin x$$

Here a = 1, and b = 1

Replace D by $D + a = D + 1$

P.I $= e^x \dfrac{1}{(D+1)^2 - 2(D+1) + 4} \sin x$

$= e^x \dfrac{1}{D^2 + 2D + 1 - 2D - 2 + 4} \sin x$

P.I $= = e^x \dfrac{1}{D^2 + 3} \sin x$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 + 3 \quad = -1 + 3$

$= 2$

$\neq 0$

$\therefore$ P.I $= e^x . \dfrac{\sin x}{2}$

The solution is

y = C.F + P.I.

$y = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \dfrac{1}{2}e^x \sin x$

3. Solve: $(D^2 - 2D + 5)y = e^{2x}\cos x$

Solution: Given, $(D^2 - 2D + 5)y = e^{2x}\cos x$

(i) To find C.F

$(D^2 - 2D + 5)y = 0$

The auxiliary equation is

$m^2 - 2m + 5 = 0$

Here $a = 1$, $b = -2$ and $c = 5$

$\therefore m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i$$

The roots are imaginary. Here α = 1 and ß = 2

$$\therefore \text{ C.F } = e^{ax}(A\cos ßx + B\sin ßx)$$

$$\text{C.F } = e^{x}(A\cos 2x + B\sin 2x)$$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 2D + 5} e^{2x} \cos x$$

Here, a = 2 and b = 1

Replace D By D + a = D + 2

$$\text{P.I} = e^{2x}.\frac{1}{(D+2)^2 - 2(D+2) + 5} \cos x$$

$$= e^{2x}.\frac{1}{D^2 + 4D + 4 - 2D - 4 + 5} \cos x$$

$$= e^{2x}.\frac{1}{D^2 + 2D + 5} \cos x$$

Replace D^2 By $-b^2$ = -1

$$\text{P.I} = e^{2x}.\frac{1}{-1 + 2D + 5} \cos x$$

$$= e^{2x}.\frac{1}{2D + 4} \cos x$$

$$= \frac{e^{x}}{2}.\frac{1}{D+2} \cos x$$

$$= \frac{e^{2x}}{2} \cdot \frac{(D-2)\cos x}{(D-2)(D+2)}$$

$$= \frac{e^{2x}}{2} \cdot \left[\frac{-\sin x - 2\cos x}{D^2 - 4}\right]$$

Replace D^2 by $-b^2 = -1$

$\therefore$ $D^2 - 4 = -1 - 4 = -5 \neq 0$

$$\therefore \text{ P.I } = \frac{e^{2x}}{2}\left[\frac{-\sin x - 2\cos x}{-5}\right]$$

$$= \frac{-e^{2x}}{(-10)}(\sin x + 2\cos x)$$

$$= \frac{e^{2x}}{10}(\sin x + 2\cos x)$$

The solution is y = C.F. + P.I

i.e., $y = e^{x}(A\cos 2x + B\sin 2x) + \frac{e^{2x}}{10}(\sin x + 2\cos x)$

4. Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x}\cos x$

Solution:Given, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x}\cos x$

$$\left(\frac{d^2}{dx^2} - 5\frac{d}{dx} + 6\right) y = e^{x}\cos x$$

$$(D^2 - 5D + 6)y = e^{x}\cos x$$

(i) To find C.F

$(D^2 - 5D + 6)\, y = 0$

The auxiliary equation is

$m^2 - 5m + 6 = 0$

$(m - 2)\,(m - 3) = 0$

$\therefore m - 2 = 0, m - 3 = 0$

$m = 2, m = 3$

The roots are real and distinct

$\therefore$ C.F $= Ae^{m_1 x} + Be^{m_2 x}$

$= Ae^{2x} + Be^{3x}$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 5D + 6} e^x \cos x$$

Here $a = 1$ and $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 5D - 5 + 6} \cos x$$

$$= e^x \frac{1}{D^2 - 3D + 2} \cos x$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \frac{1}{-1 - 3D + 2} \cos x$$

$$= e^x \frac{1}{-3D + 1} \cos x$$

$$= -e^x \frac{1}{3D - 1} \cos x$$

$$= -e^x \frac{(3D+1)\cos x}{(3D+1)(3D-1)}$$

$$= e^x \frac{[-3\sin x + \cos x]}{9D^2 - 1}$$

Replace D^2 by $-b^2 = -1$

P.I $= \dfrac{-e^{x}(-3\sin x+\cos x)}{9(-1)-1}$

$= \dfrac{e^{x}}{-10}[\cos x-3\sin x]$

The solution is

y = C.F + P.I

$= Ae^{2x}+Be^{3x}+\dfrac{e^{x}}{10}(\cos x-3\sin x)$

5. Solve: $(D^2 + 4D + 3)y = e^{-x}\sin x$

Solution:Given, $(D^2 + 4D + 3)y = e^{-x}\sin x$

(i) To find C.F.

$(D^2 + 4D + 3)\, y = 0$

The auxiliary equation is

$m^2 + 4m + 3 = 0$

$(m + 3)\,(m + 1) = 0$

$m + 3 = 0,\ m + 1 = 0$

$m = -3,\ m = -1$

The roots are real and distinct

$\therefore$ C.F $= Ae^{m_1x}+Be^{m_2x}$

$= Ae^{-3x}+Be^{(-1)x}$

(ii) To find P.I

P.I $= \dfrac{1}{D^2+4D+3}e^{-x}\sin x$

Here $a = -1;\ b = 1$

Replace D by $D + a = D - 1$

P.I $= e^{-x} \dfrac{1}{(D-1)^2+4(D-1)+3} \sin x$

$= e^{-x} \dfrac{1}{D^2-2D+1+4D-4+3} \sin x$

$= e^{-x} \dfrac{1}{D^2+2D} \sin x$

Replace $D^2$by $-b^2 = -1$

P.I $= e^{-x} \dfrac{1}{-1+2D} \sin x$

$= e^{-x} \dfrac{1}{2D-1} \sin x$

$= e^{-x} \dfrac{(2D+1)\sin x}{(2D-1)(2D+1)}$

$= e^{-x} \dfrac{[2\cos x + \sin x]}{4D^2-1}$

Replace D^2 by $-b^2 = -1$

$\therefore\ 4D^2 - 1 = 4(-1) - 1 = -5 \neq 0$

P.I $= e^{-x} \dfrac{(2\cos x + \sin x)}{-5}$

$= \dfrac{-e^{-x}}{5}(2\cos x + \sin x)$

The solution is

y = C.F + P.I

$= Ae^{-3x} + Be^{-3} - \dfrac{e^{-x}}{5}(2\cos x + \sin x)$

6. Solve: $\dfrac{d^2y}{dx^2} + 4y = e^x \sin x$

Solution:Given, $\dfrac{d^2y}{dx^2} + 4y = e^x \sin x$

$(D^2+4)\ y = e^x \sin x$

(i) To find C.F

$(D^2 + 4)\, y = 0$

The auxiliary equation is

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$= \pm 2i$

$= 0 \pm 2i$

The roots are imaginary

Here, $\alpha = 0$, $ß = 2$

$\therefore$ C.F $= e^{ax}(A\cos ßx + B\sin ßx)$

$= (A\cos 2x + B\sin 2x)$

C.F $= A\cos 2x + B\sin 2x$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2+4} e^x \sin x$$

Here $a = 1$, $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2+4} \sin x$$

$$= e^x \frac{1}{D^2+2D+1+4} \sin x$$

$$= e^x \frac{1}{D^2+2D+5} \sin x$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \frac{1}{-1+2D+5} \sin x$$

$$= e^x \frac{1}{2D+4} \sin x$$

$$= \frac{e^x}{2} \frac{1}{D+2} \sin x$$

$$= \frac{e^x}{2} \frac{(D-2)\sin x}{(D-2)(D+2)}$$

$$= \frac{e^x}{2} \frac{(\cos x - 2\sin x)}{D^2 - 4}$$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 - 4 = -1 - 4 = -5 \neq 0$

$$\text{P.I} = \frac{e^x}{2} \frac{(\cos x - 2\sin x)}{(-5)}$$

$$= \frac{e^x}{10} (\cos x - 2\sin x)$$

The solution is

y = C.F + P.I

$$y = A\cos 2x + b\sin 2x + \frac{(-e^x)}{10} (\cos x - 2\sin x)$$

7. Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x} \sin 2x$

Solution:Given, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x} \sin 2x$

$$(D^2 + 4D + 4)y = e^{-3x} \sin 2x$$

(i) To find C.F

$$(D^2 + 4D + 4)y = 0$$

The auxiliary equation is

$m^2 + 4m + 4 \quad = 0$

$(m + 2)(m + 2) \quad = 0$

$m + 2 = 0, m + 2 = 0$

$m = -2, \qquad m = -2$

$\therefore m = -2$ (twice)

The roots are real and equal

$\therefore$ C.F $= e^{mx}(Ax + B)$

$= e^{-2x}(Ax + B)$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 + 4D + 4} e^{-3x} \sin 2x$$

Here $a = -3$ and $b = 2$

Replace D by $D + a = D - 3$

$$\therefore \text{P.I.} = e^{-2x} \cdot \frac{1}{(D-3)^3 + 4(D-3) + 4} \sin 2x$$

$$= e^{-3x} \frac{1}{D^2 - 6D + 9 + 4D - 12 + 4} \sin 2x$$

$$\text{P.I} = e^{-3x} \frac{1}{D^2 - 2D + 1} \sin 2x$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-3x} \frac{1}{-4 - 2D + 1} \sin 2x$$

$$= e^{-3x} \frac{1}{-2D - 3} \sin 2x$$

$$= e^{-3x} \frac{(2D-3)\sin 2x}{(2D-3)(2D+3)}$$

$$= e^{-3x} \frac{[2(2\cos 2x) - 3\sin 2x]}{4D^2 - 9}$$

Replace D^2 by $-b^2 = -4$

$\therefore 4D^2 - 9 = 4(-4) - 9 = 16 - 9 = 25 \neq 0$

$$\therefore \text{ P.I } = -e^{-3x}\frac{(4\cos 2x - 3\sin 2x)}{-25}$$

$$= \frac{e^{-3x}}{25}(4\cos 2x - 3\sin 2x)$$

The solution is

$y \quad$ = C.F + P.I

$$y = e^{-2x}(Ax+B) + \frac{e^{-3x}}{25}(4\cos 2x - 3\sin 2x)$$

8. Solve: $(D^2 + 2D)y = e^{-x}\cos x$

Solution:Given $(D^2 + 2D)y = e^{-x}\cos x$

$(D^2 + 2D)y = 0$

The auxiliary equation is

$m^2 + 2m = 0$

$m(m + 2) = 0$

$m = 0, m + 2 = 0$

$m = 0, \quad m = -2$

$\therefore$ The roots are real and distinct

$$\text{C.F } = Ae^{m_1 x} + Be^{m_2 x}$$

$$= Ae^{0x} + Be^{-2x}$$

$$= A + Be^{-2x}$$

(ii) To find P.I

P.I $= \frac{1}{D^2+2D} e^{-x} \cos x$

Here a = –1, b = 1

Replace D by $D + a = D - 1$

P.I $= e^{-x} \frac{1}{(D-1)^2 + 2(D-1)} \cos x$

$= e^{-x} \frac{1}{D^2 - 2D + 1 + 2D - 2} \cos x$

$= e^{-x} \frac{1}{D^2 - 1} \cos x$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 - 1 = -1 - 1 = -2 \neq 0$

P.I $= e^{-x} \frac{\cos x}{-2}$

$= -\frac{e^{-x}}{2} \cos x$

The solution is

y = C.F + P.I

$= A + Be^{-2x} - \frac{e^{-x}}{2} \cos x$

9. Solve: $(D^2 - 4D + 13)y = e^{2x} \cos 3x$

Solution:Given, $(D^2 - 4D + 13)y = e^{2x} \cos 3x$

(i) To find C.F

$(D^2 - 4D + 13)y = 0$

The auxiliary equation is

$m^2 - 4m + 13 = 0$

Here $a = 1$, $b = -4$, $c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm i6}{2}$$

$$m = 2 \pm i3$$

The roots are imaginary

Here $\alpha = 2$ and $ß = 3$

$$\text{C.F} = e^{ax}(A\cos ßx + B\sin ßx)$$

$$= e^{2x}(A\cos 3x + B\sin 3x)$$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 4D + 13} e^{2x} \cos 3x$$

Here, $a = 2$ and $b = 3$

$$\text{P.I} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 13} \cos 3x$$

$$\text{P.I} = e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 13} \cos 3x$$

$$= e^{2x} \frac{1}{D^2 + 9} \cos 3x$$

Replacing D^2 by $-b^2 = -9$

$\therefore D^2 + 9 = -9 + 9 = 0$

$$\text{P.I} = e^{2x} x \frac{\cos 3x}{2D}$$

$$= \frac{x}{2} e^{2x} \left(\frac{\cos 3x}{D} \right)$$

$$= \frac{x}{2} e^{2x} \int \cos 3x dx$$

$$= \frac{x}{2} e^{2x} \frac{\sin 3x}{3}$$

$$= \frac{x}{6} e^{2x} \sin 3x$$

The solution is

y = C.F + P.I

$$= e^{2x} (A \cos 3x + B \sin 3x) + \frac{x}{6} e^{2x} \sin 3x$$

10. Solve: $(D^2 - 2D + 5)y = e^x \cos 2x$

Solution:Given, $(D^2 - 2D + 5)y = e^x \cos 2x$

(i) To find C.F

$(D^2 - 2D + 5)y = 0$

The auxiliary equation is

$m^2 - 2m + 5 = 0$

Here $a = 1$, $b = -2$ and $c = 5$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm i4}{2}$$

$= 1 \pm 2i$

The roots are imaginary. Here $\alpha = 1$ and $ß = 2$.

C.F $= e^{2x}(A\cos ßx + B\sin\ ßx)$

$= e^{x}(A\cos 2x + B\sin\ 2x)$

(ii) To find P.I

P.I $= \frac{1}{D^2 - 2D + 5} e^{x} \cos 2x$

Here $a = 1$, $b = 2$

Replace D by $D + a = D + 1$

P.I $= e^{x} \frac{1}{(D+1)^2 - 2(D+1) + 5} \cos 2x$

$= e^{x} \frac{1}{D^2 + 2D + 1 - 2D - 2 + 5} \cos 2x$

$= e^{x} \frac{1}{D^2 + 4} \cos 2x$

Replace D^2 by $-b^2 = -4$

$\therefore D^2 + 4 = -4 + 4 = 0$

$\therefore$ P.I$= e^{x}.\, x \frac{\cos 2x}{2D}$

$= \frac{xe^{x}}{2}\left(\frac{\cos 2x}{D}\right)$

$= \frac{xe}{2}\left(\frac{1}{2}\sin\ 2x\right)$

$= \frac{x}{4} e^{x} \sin\ 2x$

The solution is

y = C.F + P.I

$= e^{x}(A\cos 2x + B\sin\ 2x) + \frac{x}{4} e^{x} \sin\ 2x$

11. Solve: $(D+1)^2 y = e^{-x} \cos x$

Solution:Given, $(D+1)^2 = e^{-x} \cos x$

(i) To find C.F.

$(D + 1)^2 y = 0$

The auxiliary equation is

$(m + 1)^2 = 0$

$(m + 1)(m + 1) = 0$

$(m + 1) = 0, (m + 1) = 0$

$m = -1, m = -1$

$m = -1$ (twice)

The roots are real and equal

$\therefore$ C.F $= e^{mx}(Ax+B)$

$= e^{-x}(Ax+B)$

(ii) To find P.I

$$\text{P.I} = \frac{1}{(D+1)^2} e^{-x} \cos x$$

$$= \frac{1}{D^2+2D+1} e^{-x} \cos x$$

Here $a = -1$ and $b = 1$

Replace D by $D + a = D - 1$

$$\therefore \text{P.I} = e^{-x} \frac{1}{(D-1)^2+2(D-1)+1} \cos x$$

$$\text{P.I} = e^{-x} \frac{1}{D^2-2D+1+2D-2+1} \cos x$$

$$= e^{-x} \frac{\cos x}{D^2}$$

$= e^{-x} \frac{1}{D}\left(\frac{1}{D}\cos x\right)$

$= e^{-x} \frac{1}{D}\left(\int \cos x \, dx\right)$

$= e^{-x} \frac{1}{D}(\sin x)$

$= e^{-x} \int \sin x \, dx$

$= e^{-x}(-\cos x)$

$= -e^{-x} \cos x$

The solution is

$y \quad = \text{C.F} + \text{P.I}$

$= e^{-x}(Ax + B) - e^{-x}\cos x$

12. Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Solution:

Given $(D^2 - 2D + 1)y = xe^x \sin x$

(i) To find C.F

$(D^2 - 2D + 1)y = 0$

The auxiliary equation is

$m^2 - 2m + 1 = 0$

$(m - 1)(m - 1) = 0$

$m = 1, m = 1$

$\therefore m = 1$ (twice)

The roots are real and equal

C.F $= e^{mx}(Ax + B)$

$= e^x(Ax + B)$

(i) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 2D + 1} xe^x \sin x$$

Here $a = 1$, $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} x \sin x$$

$$\text{P.I} = e^x \left(\frac{x \sin x}{D^2} \right)$$

$$= e^x \left[x \frac{1}{D^2} \sin x \right] - e^x \frac{2D}{(D^2)^2} \sin x$$

$$= e^x \left[x \frac{1}{D^2} \sin x \right] - e^x \frac{2 \cos x}{(D^2)^2}$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \left(x \frac{\sin x}{(-1)} \right) - e^x \frac{2 \cos x}{(-1)^2}$$

$$= -xe^x \sin x - 2e^x \cos x$$

$$= -e^x (x \sin x - 2 \cos x)$$

The solution is

$$y = \text{C.F} + \text{P.I}$$

$$= e^x (Ax + B) - e^x (x \sin x - 2 \cos x)$$

Exercises

Solve the following differential equations

1. $(D^2 - 4D + 13)y = e^{2x} \cos 3x$

2. $(D^2+4D+3)y = e^x \sin\ x$

3. $(D^2-2D+5)y = e^{2x} \sin\ x$

4. $(D+1)^2 y = e^{-x} \cos x$

5. $(D^2-5D+6)y = e^x \cos 2x$

6. $(D^2-4D+3)y = 3e^x \cos 2x$

7. $(D^2-6D+13)y = 8e^{3x} \sin\ 4x$

8. $(D^2-4D+13)y = e^{2x} \cos 3x$

9. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y = e^{-x} \sin 2x$

10. $\frac{d^2x}{dx^2}-2\frac{dy}{dx}+y = e^x \cos x$

Answer

1. $y = e^{2x}[A\cos 3x + B\sin 3x] + \frac{1}{6}xe^{2x} \sin 3x$

2. $y = Ae^{-x} + Be^{-x}\ \frac{e^x}{85}(6\cos x - 7\sin\ x)$

3. $y = e^x(A\cos 2x + B\sin 2x) - \frac{e^{2x}}{20}(2\cos x - 4\sin\ x)$

4. $y = (Ax+B)e^{-x} - e^{-x} \cos x$

5. $y = Ae^{2x} + Be^{3x} = \frac{e^x}{20}(\cos 2x + 3\sin 2x)$

6. $y = Ae^x + Be^{2x} - \frac{3}{8}e^x[\sin\ 2x + \cos 2x]$

7. $y = e^{3x}(A\cos 2x + B\sin\ 2x) - xe^{3x} \cos 4x$

8. $y = e^{2x}[A\cos 3x + B\sin 3x] + \frac{xe^{2x}}{6} \sin 3x$

9 $y = e^{-2x}(Ax+B) - \frac{1}{25}e^{-x}(4\cos 2x + 3\sin\ 2x)$

10. $Y = (Ax+B)\, e^{x} - e^{x}(x\sin\ x + 2\cos x)$

Problem based on $f(x) = x^n e^{ax}$

1. Solve $(D^2 - D - 6)y = xe^{-2x}$

Solution: Given $(D^2 - D - 6)y = xe^{-2x}$

The Auxiliary Equation is

$m^2 - m - 6 = 0$

$m = 3, -2$

C.F = $Ae^{-2x} + Be^{3x}$

P.I $= \frac{1}{D^2 - D - 6} x^{e-2x}$

$= e^{-2x} \frac{1}{(D-2)^2 - (D-2) - 6} x$

$= e^{-2x} \frac{1}{D^2 - 4D + 4 - D + 2 - 6} x$

$= e^{-2x} \frac{1}{D^2 - 5D} x$

$= e^{-2x} \frac{1}{5D\left[\frac{D}{5} - 1\right]} x$

$= e^{-2x} \frac{1}{-5D\left[1 - \frac{D}{5}\right]} x$

$= e^{-2x} \frac{1}{-5D}\left[1 - \frac{D}{5}\right]^{-1} x$

$$= \frac{e^{-2x}}{-5D}\left[1+\frac{D}{5}+\left(\frac{D}{5}\right)^2+\ldots\right]x \qquad \left[\begin{array}{l}\because D(x)=1\\ D^2(x)=0\\ \vdots\\ D^n(x)=0\\ \forall n \geq R\end{array}\right]$$

$$= \frac{e^{-2x}}{-5D}\left[x+\frac{D}{5}(x)+\frac{D^2(x)}{25}+\ldots\right]$$

$$= \frac{e^{-2x}}{-5D}\left[x+\frac{1}{5}\right]$$

$$= \frac{e^{-2x}}{-5}\int\left(x+\frac{1}{5}\right)dx$$

$$= \frac{e^{-2x}}{-5}\left[\frac{x^2}{2}+\frac{1}{5}x\right]$$

$$= \frac{xe^{-2x}}{-5}\left[\frac{x}{2}+\frac{1}{5}\right]$$

$$= \frac{xe^{-2x}}{-5}\left[\frac{5x+2}{10}\right]$$

$$= \frac{xe^{-2x}}{-50}[5x+2]$$

y = C.F + P.I

$$= Ae^{-2x}+Be^{3x}-\frac{xe^{-2x}}{50}(5x+2)$$

2. Solve $(D^2-4D+4)y = x^2e^{2x}$

Solution: Given $(D^2-4D+4)y = x^2e^{2x}$

The Auxiliary Equation is

$m^2 - 4m + 4 = 0$

$m = 2, 2$

C.F $= (A + Bx)\, e^{2x}$

P.I $= \dfrac{1}{D^2 - 4D + 4} x^2 e^{2x}$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 4} x^2$$

$$= e^{2x} \frac{1}{D^2 + 4D - 4D + 8 - 8} x^2$$

$$= e^{2x} \frac{1}{D} \int x^2 dx$$

$$= e^{2x} \frac{1}{D} \left[\frac{x^3}{3} \right]$$

$$= \frac{e^{2x}}{3} \int x^3 dx$$

$$= \frac{e^{2x}}{3} \left(\frac{x^4}{4} \right)$$

$$= \frac{e^{2x} x^4}{12}$$

y = C.F + P.I

$$= (A + Bx) e^{2x} + \frac{x^4 e^{2x}}{12}$$

3. Solve: $(D^3 - D) y = e^x x$

Solution: Given $(D^3 - D) y = e^x x$

The Auxiliary Equation is

$m^3 - 1 = 0$

$m\,(m^2 - 1) = 0$

$\Rightarrow m = 0,\ m = \pm 1$

C.F = $Ae^{0x} + Be^{-x} + Ce^{x}$

$= A + Be^{-x} + Ce^{x}$

P.I $= \dfrac{1}{(D^3 - D)} e^x x$

$= e^x \dfrac{1}{(D+1)^3 - (D+1)} x$

$= e^x \dfrac{1}{D^3 + 3D^2 + 3D + 1 - D - 1} x \quad \left[\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\right]$

$= e^x \dfrac{1}{D^3 + 3D^2 + 2D} x$

$= e^x \dfrac{1}{2D\left[\dfrac{D^2}{2} + \dfrac{3D}{2} + 1\right]} x$

$= e^x \dfrac{1}{2D\left[\dfrac{D^2 + 3D}{2} + 1\right]} x$

$= \dfrac{e^x}{2D}\left(1 + \dfrac{D^2 + 3D}{2}\right)^{-1} x$

$= \dfrac{e^x}{2D}\left[1 - \left(\dfrac{D^2 + 3D}{2}\right) + \left(\dfrac{D^2 + 3D}{2}\right)^2 + \right] x$

$= \dfrac{e^x}{2D}\left[1 - \dfrac{D^2}{2} - \dfrac{3D}{2} + \dots\right] x$

$= \dfrac{e^x}{2D}\left[x - \dfrac{D^2}{2}(x) - \dfrac{3D}{2}(x)\right]$

$= \dfrac{e^x}{2D}\left[x - (0) - \dfrac{3}{2}(1)\right] \qquad \left[\because \begin{array}{c} D(x) = 1 \\ D^2(x) = 0 \end{array}\right]$

$= \dfrac{e^x}{2D}\left[x - \dfrac{3}{2}\right]$

$= \frac{e^x}{2}\int \left(x - \frac{3}{2}\right)dx$

$= \frac{e^x}{2}\left[\frac{x^2}{2} - \frac{3}{2}x\right]$

$= \frac{xe^x}{4}(x-3)$

$\therefore$ y = C.F + P.I

$= A + Be^{-x} + Ce^x + \frac{xe^x}{4}(x-3)$

4. Solve: $(D^2 - 2D + 1)y = e^x(3x^2 - 1)$

Solution: Given $(D^2 - 2D + 1)y = e^x(3x^2 - 1)$

The Auxiliary Equation is

$m^2 - 2m + 1 = 0$

$m = 1, 1$

P.I $= \frac{1}{(D^2 - 2D + 1)} e^x(3x^2 - 1)$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1}(3x^2 - 1)$

$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1}(3x^2 - 1)$

$= e^x \frac{1}{D^2}(3x^2 - 1)$

$= e^x \frac{1}{D}\int (3x^2 - 1)dx$

$= e^x \frac{1}{D}\left[\frac{3x^3}{3} - x\right]$

$= e^x \frac{1}{D}\left[x^3 - x\right]$

$= e^x \int (x^3 - x)dx$

$= e^x \left[\frac{x^4}{4} - \frac{x^2}{2} \right]$

$\therefore$ y = C.F + P.I

$= (A + Bx)e^x + \frac{e^x x^4}{4} - \frac{e^x x^2}{2}$

ALITER

Given $(D^2 - 2D + 1)y = e^x(3x^2 - 1)$

$= 3e^x x^2 - e^x$

C.F = $(A + Bx)$ e^x

P.I$_1$ $= \frac{1}{(D^2 - 2D + 1)} 3e^x x^2$

$= 3e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x^2$

$= 3e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} x^2$

$= 3e^x \frac{1}{D^2}(x^2)$

$= 3e^x \frac{1}{D} \int x^2 dx$

$= 3e^x \frac{1}{D} \left(\frac{x^3}{3} \right)$

$= e^x \frac{1}{D}(x^2) = e^x \int x^3 dx$

P.I$_1$ $= e^x \frac{x^4}{4}$

P.I$_2$ $= \frac{1}{D^2 - 2D + 1} e^x$

$$= \frac{1}{1-2+1}$$

$$= \frac{1}{0} e^x$$

$$= x \frac{1}{2D-2} e^x$$

$$= x \frac{1}{2-2} e^x$$

$$= x.\frac{1}{0} e^x$$

$$= x^2 \frac{1}{2} e^x$$

$$= \frac{x^2 e^x}{2}$$

$\therefore$ P.I = P.I$_1$ – P.I$_2$

$$= \frac{x^4 e^x}{4} - \frac{x^2}{2}$$

y = C.F + P.I

$$= (A+Bx)\, e^x + \frac{x^4 e^x}{4} - \frac{x^2}{2} e^x$$

5. Solve $(D^2 + 2D - 1)y = (x + e^x)2$

$= x^2 + 2xe^x + e^{2x}$

The Auxiliary Equation is $m^2 + 2m - 1 = 0$

(i.e) $m = \dfrac{-2 \pm \sqrt{4+4}}{2}$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

$$\text{C.F} = Ae^{(-1-\sqrt{2})} + Be^{(-1+\sqrt{2})x}$$

$$\text{P.I}_1 = \frac{1}{D^2 + 2D - 1} x^2$$

$$= \frac{-1}{[1-(D^2+2D)]} x^2$$

$$= -[1-(D^2+2D)]^{-1} x^2$$

$$= -[1+(D^2+2D)+(D^2+2D)^2+\ldots]x^2$$

$$= -[1+D^2+2D+D^4+4D^3+4D^2+\ldots]x^2$$

$$= -[x^2+D^2(x^2)+2D(x^2)+D^4(x^2)+4D^3(x^2)+4D^2(x^2)]$$

$$= -[x^2+2+2(2x)+0+0+4(2)]$$

$$= -[x^2+2+4x+8]$$

$$= -[x^2+4x+10]$$

$$\text{P.I}_2 = \frac{1}{D^2+2D-1} 2xe^x$$

$$= 2e^x \frac{1}{(D+1)^2+2(D+1)-1}$$

$$= 2e^x \frac{1}{D^2+2D+1+2D+2-1} x$$

$$= 2e^x \frac{1}{D^2+4D+2} x$$

$$= 2e^x \frac{1}{\left(\frac{D^2}{2}+\frac{4D}{2}+1\right)} x$$

$$= 2e^x \frac{1}{2\left(1+\left(\frac{D^2+4D}{2}\right)\right)} x$$

$$= e^x\left(1+\frac{D^2+4D}{2}\right)^{-1}x$$

$$= e^x\left[1-\left(\frac{D^2+4D}{2}\right)+\left(\frac{D^2+4D}{2}\right)^2-...\right]x$$

$$= e^x\left(1-\frac{D^2}{2}-\frac{4D}{2}+....\right)x \qquad \left[\begin{array}{l}\because D(x)=1\\ D^2(x)=0\end{array}\right.$$

$$= e^x\left[x-\frac{D^2(x)}{2}-\frac{4D(x)}{2}\right]$$

$$= e^x[x-(0)-2]$$

$$= e^x(x-2)$$

P.I$_3$ $= \frac{1}{D^2+2D-1}e^{2x}$

$$= \frac{1}{4+4-1}e^{2x} \qquad [\because \text{Replace D by 2}]$$

$$= \frac{1}{7}e^{2x}$$

$\therefore$ P.I = P.I$_1$ + P.I$_2$ + P.I$_3$

$$= -(x^2+4x+10)+e^x(x-2)+\frac{1}{7}e^{2x}$$

y = C.F + P.I

$$= Ae^{(-1-\sqrt{2})x}+Be^{(1+\sqrt{2})x}-(x^2+4x+10)+e^x(x-2)+\frac{1}{7}e^{2x}$$

6. Solve $(D^2-2D+2)y=e^x x^2+5+e^{-2x}$

Solution:Given $(D^2-2D+2)y=e^x x^2+5+e^{-2x}$

The Auxiliary Equation is

$m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F $= e^x[A\cos x + B\sin x]$

P.I$_1$ $= \frac{1}{D^2 - 2D + 2} e^x x^2$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2$

$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2} x^2$

$= e^x \frac{1}{D^2 + 1} x^2$

$= e^x (1 + D^2)^{-1} x^2$

$= e^x [1 - D^2 + D^4 - ...] x^2$

$= e^x [x^2 - D^2(x^2) + D^4(x^2) - ...]$

$= e^x [x^2 - 2]$

P.I$_2$ $= \frac{1}{D^2 - 2D + 2} 5e^{0x}$

$= 5\frac{1}{(0) - 2(0) + 2} e^{0x}$

$= \frac{5}{2}$

P.I$_3$ $= \frac{1}{D^2 - 2D + 2} e^{-2x}$

$= \frac{1}{(-2)^2 - 2(-2) + 2} e^{-2x}$

$= \frac{1}{4 + 4 + 2} e^{-2x}$

$= \dfrac{1}{10} e^{-2x}$

P.I $= P.I_1 + P.I_2 + P.I_3$

$= e^x (x^2 - 2) + \dfrac{5}{2} + \dfrac{1}{10} e^{-2x}$

y $=$ C.F + P.I

$= e^x (A\cos x + B\sin x) + e^x (x^2 - 2) + \dfrac{5}{2} + \dfrac{1}{10} e^{-2x}$

7. Solve $(D^3 - 7D - 6)y = (1 + x)e^{2x}$

Solution: $(D^3 - 7D - 6)y = e^{2x} + xe^{2x}$

The Auxiliary Equation is $m^3 - 7m - 6 = 0$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$(m + 1)(m^2 - m - 6) = 0$

$m = -1, -2, 3$

C.F $= Ae^{-2x} + Be^{-x} + Ce^{3x}$

$P.I_2 = \dfrac{1}{D^3 - 7D - 6} xe^{2x}$

$= e^{2x} \dfrac{1}{(D+2)^3 - 7(D+2) - 6} x$

$= e^{2x} \dfrac{1}{D^3 + 6D^2 + 12D + 8 - 7D - 14 - 6} x$

$= e^{2x} \dfrac{1}{D^3 + 6D^2 + 5D - 12} x$

$= e^{2x} \dfrac{1}{12\left[\left(\dfrac{D^3 + 6D^2 + 5D}{12}\right) - 1\right]} x$

$$= e^{2x}\frac{1}{-12\left[1-\left(\frac{D^3-6D^2+5D}{12}\right)\right]}x$$

$$= \frac{-e^{2x}}{12}\left[1-\frac{(D^3+6D^2+5D)}{12}\right]^{-1}x$$

$$= \frac{-e^{2x}}{12}\left[1-\frac{(D^3+6D^2+5D)+...}{12}\right]x$$

$$= \frac{-e^{2x}}{12}\left[x-\frac{(D^3+6D^2+5D)(x)}{12}+...\right]$$

$$= \frac{-e^{2x}}{12}\left[x-\frac{5D(x)}{12}\right]$$

$$= \frac{-e^{2x}}{12}\left[x-\frac{5}{12}\right]$$

$$P.I_1 = \frac{1}{(D^3-7D-6)}e^{2x}$$

$$= \frac{1}{8-14-6}e^{2x}$$

$$= \frac{-e^{2x}}{12}$$

$$\therefore P.I = P.I_1 + P.I_2$$

$$= \frac{-e^{2x}}{12}-\frac{e^{2x}}{12}\left(x-\frac{5}{12}\right)$$

$$= \frac{-e^{2x}}{12}\left[1+\left(x-\frac{5}{12}\right)\right]$$

$$= \frac{-e^{2x}}{12}\left(x+\frac{7}{12}\right)$$

$$\therefore\ y = C.F + P.I$$

$$= Ae^{-2x} + Be^{-x} + Ce^{3} - \frac{e^{2x}}{12}\left(x + \frac{7}{12}\right)$$

8. Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Solution: Given $(D^2 - 2D + 1)y = xe^x \sin x$

The Auxiliary Equation is

$m^2 - 2m + 1 = 0$

C.F $= (A + Bx)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} xe^x \sin x$$

$$= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 1}\right] x \sin x$$

$$= e^x \left[\frac{1}{D^2 + 2D + 1 - 2D - 2 + 1}\right] x \sin x$$

$$= e^x \left[\frac{1}{D^2 + 2D - 2D - 2 + 1}\right] x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D} \int (x \sin x)\, dx$$

$$= e^x \frac{1}{D} [-x \cos x - (-\sin x)]$$

$$= e^x \left[-\int x \cos dx + \int \sin x dx\right]$$

$$= e^x [-(x \sin x + \cos x) - \cos x]$$

$u = x$: $v = \cos x$

$u' = 1$: $v_1 = \sin x$

$u'' = 0$: $v_2 = -\cos x$

$$= e^x [-x \sin x - \cos x - \cos x]$$

$= -e^{x}[x\sin x + 2\cos x]$

$y \quad = \text{C.F + P.I}$

$= (A+Bx)e^{x} - e^{x}(x\sin x + 2\cos x)$

9. Solve: $(D^2 - 4D + 4y) = 8x^2 e^{2x} \sin 2x$

Solution:Given $(D^2 - 4D + 4y) = 8x^2 e^{2x} \sin 2x$

The Auxiliary Equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$ (twice)

$\text{C.F} = (A+Bx)e^{2x}$

$$\text{P.I} = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D}\left[\int x^2 \sin 2x dx\right] \qquad \left[\begin{array}{l} By\ Berroulli\ \ formula \\ \int uv dx = uv_1 - u'v_2 + u''v_3 \end{array}\right.$$

$$= 8e^{2x} \frac{1}{D}\left[\frac{-x^2\cos 2x}{2} + \frac{2x\sin 2x}{4} + \frac{2\cos 2x}{8}\right]$$

$$= 8e^{2x} \frac{1}{D}\left[\frac{-x^2\cos 2x}{2} + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right]$$

$$= 8e^{2x} \frac{1}{D}\left[-\frac{1}{2}\int x^2 \cos 2x dx + \frac{1}{2}\int x\sin 2x dx + \frac{1}{4}\int \cos 2x dx\right]$$

$$= 8e^{2x}\left[\frac{-1}{2}\left(\frac{x^2\sin 2x}{2} + \frac{2x\cos 2x}{4} - \frac{2\sin 2x}{8} + \frac{1}{2}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right] + \frac{1}{8}\sin 2x\right)\right]$$

$$= 8e^{2x}\left[\frac{-x^2 \sin 2x}{4} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{8} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[\frac{-x^2 \sin 2x}{4} - \frac{2x\cos 2x}{4} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\frac{-x^2 \sin 2x}{3} - \frac{x\cos 2x}{4} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\frac{-2x^2 \sin 2x}{8} - \frac{4x\cos 2x}{8} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}[-2x^2 \sin 2x - 4x\cos 2x + 3\sin 2x]$$

$$= 8e^{2x}[(3-2x^2)\sin 2x - 4x\cos 2x]$$

10. Solve: $\frac{d^2y}{dx^2} - 4y = x\sin hx$

Solution:The Auxiliary Equation is

$m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F $= Ae^{2x} + Be^{-2x}$

P.I $= \frac{1}{D^2-4} x\sin hx$

$$= \frac{1}{D^2-4} x\left[\frac{e^x - e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^x x - \frac{1}{D^2-4}e^{-x}x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D+1-4}x - e^{-x}\frac{1}{D^2-2D+1-4}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{3\left(\frac{D^2}{3}+\frac{2D}{3}-1\right)}x - e^{-x}\frac{1}{3\left(\frac{D^2}{3}-\frac{2D}{3}-1\right)}x\right]$$

$$= \frac{1}{6}\left[e^{x}\frac{1}{-\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]}x - e^{-x}\frac{1}{-\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]}x\right]$$

$$= -\frac{1}{6}\left\{e^{x}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x - e^{-x}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x\right\}$$

$$= -\frac{1}{6}\left\{e^{x}\left[1+\frac{D^2}{3}+\frac{2D}{3}+\ldots\right]x - e^{-x}\left[1+\frac{D^2}{3}-\frac{2D}{3}+\ldots\right]x\right\}$$

$$= \frac{-1}{6}\left\{e^{x}\left[x+\frac{D^2}{3}(x)+\frac{2D}{3}(x)+\ldots\right] - e^{-x}\left[x+\frac{D^2}{3}(x)-\frac{2D}{3}(x)+\ldots\right]\right\}$$

$$= \frac{-1}{6}\left\{e^{x}\left[x+\frac{2}{3}\right] - e^{-x}\left[x-\frac{2}{3}\right]\right\}$$

$$= \frac{-1}{6}\left\{x(e^{x}-e^{-x})+\frac{2}{3}(e^{x}+e^{-x})\right\}$$

$$= \frac{-1}{6}\left\{2x\sin\ hx+\frac{2}{3}(2\cos hx\right\}$$

$$= \frac{-1}{6}\left[2x\sin\ hx+\frac{4}{3}\cos hx\right]$$

$$= -\frac{x}{3}\sin\ hx-\frac{2}{9}\cos hx$$

$\therefore$ y = C.F + P.I

$$= Ae^{2x} + Be^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

11. Solve: $(D^3 - 3D^2 + 3D - 1)y = e^{-3}x^3$

Solution: Given $(D^3 - 3D^2 + 3D - 1)y = e^{-x}x^3$

The Auxiliary Equation is

$m^3 - 3m^2 + 3m - 1 = 0$

$(m - 1)^3 = 0$

$m = 1, 1, 1$ (thrice)

C.F $= (A + Bx + Cx^2)e^x$

P.I $= \dfrac{1}{(D-1)^3} e^{-x}x^3$

$$= e^{-x} \frac{1}{(D-1-1)^3} x^3$$

$$= e^{-x} \frac{1}{(D-2)^3} x^3$$

$$= e^{-x} \frac{1}{2^3\left[\frac{D}{2}-1\right]^3} x^3$$

$$= \frac{e^{-x}}{-8\left[1-\frac{D}{2}\right]^3} x^3$$

$$= \frac{e^{-x}}{-8}\left[1-\frac{D}{2}\right]^{-3} x^3$$

$$= \frac{e^{-x}}{-8}\left[1+3\frac{D}{2}+\frac{3(3+1)}{2!}\frac{D^2}{4}+\frac{3(3+1)(3+2)}{3!}\frac{D^3}{8}+\ldots\right]$$

$$\left[\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+2)(n+2)}{3!}x^3 + \ldots\right.$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{3}{2}D(x^3) + \frac{3.4}{2}\frac{D^2(x^2)}{4} + \frac{3.4(5)}{8}\frac{D^3}{8}(x^3) + ...\right]$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{3}{2}(3x^2) + \frac{3}{2}(6x)\frac{5}{4}(6)\right]$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{9}{2}x^2 + 9x + 15\right]$$

P.I $= \dfrac{e^{-x}}{-16}\left[2x^3 + 9x^2 + 18x + 15\right]$

y = C.F + P.I

$$= (A + Bx + Cx^2)e^x - \frac{e^{-x}}{16}[2x^3 + 9x^2 + 18x + 15]$$

12. Solve: $(D^2 - 4)y = x\sin\ h2x$

Solution: Given $(D^2 - 4)y = x\sin\ h2x$

$$= x\left[\frac{e^{2x} - e^{-2x}}{2}\right]$$

$$= \frac{xe^{2x}}{2} - \frac{xe^{-2x}}{2}$$

The Auxiliary Equation is

$m^2 - 4 = 0$

$\Rightarrow m^2 = 4$

$\therefore m = \pm 2$

C.F. $= Ae^{-2x} + Be^{2x}$

P.I. $= \dfrac{e^{2x}}{8}\dfrac{1}{D}\left[x - \dfrac{D}{4}(x) + \dfrac{D^2}{16}(x)....\right]$

$$= \frac{e^{2x}}{8}\frac{1}{D}\left[x-\frac{1}{4}\right]$$

$$= \frac{e^{2x}}{8}\left[\frac{x^2}{2}-\frac{x}{4}\right]$$

$$= \frac{xe^{2x}}{8}\left[\frac{2x-1}{4}\right]$$

$$= \frac{xe^{2x}}{32}[2x-1]$$

$$\text{P.I}_1 = \frac{1}{(D^2-4)}\frac{x}{2}e^{-2x}$$

$$= \frac{e^{2x}}{2}\frac{1}{(D-2)^2-4}x$$

$$= \frac{e^{-2x}}{2}\frac{1}{D^2+4D+4-4}x$$

$$= \frac{e^{-2x}}{2}\frac{1}{(D^2-4D)}x$$

$$= \frac{e^{-2x}}{8}\frac{1}{-4D\left[1-\frac{D}{4}\right]}x$$

$$= \frac{e^{-2x}}{-8}\frac{1}{D}\left[1-\frac{D}{4}\right]^{-1}x$$

$$= \frac{e^{-2x}}{-8D}\left[1+\frac{D}{4}+\left(\frac{D}{4}\right)^2+\ldots\right]x$$

$$= \frac{e^{-2x}}{-8D}\left[x+\frac{D}{4}(x)+\frac{D^2}{16}(x)+\ldots.\right]$$

$$= \frac{e^{-2x}}{-8D}\left[x+\frac{1}{4}\right]$$

$$= \frac{-e^{-2x}}{8D} \int \left(x + \frac{1}{4}\right) dx = \frac{-e^{-2x}}{8}\left[\frac{x^2}{2} + \frac{x}{4}\right]$$

$$= \frac{-xe^{-2x}}{32}[2x+1]$$

P.I $= P.I_1 + P.I_2$

$$= \frac{xe^{2x}}{32}(2x-1) - \frac{xe^{2x}}{32}(2x+1)$$

$$y = Ae^{-2x} + Be^{2x} + \frac{xe^{2x}}{32}(2x-1) - \frac{xe^{2x}}{32}(2x+1)$$

Exercise Problems

1. $(D^2 - 2D + 2)y = x^2 e^{3x}$

2. $(D^2 + 4D + 4)y = e^{-x}x^2$

3. $(D^2 - 4D - 5)y = e^{-2x}(x+1)$

4. $(D^2 + 8D + 15)y = e^{3x}x$

5. $(D^2 + 9)y = (x^2 + 1)e^{3x}$

6. $(D^2 - 2D + 1)y = x^2 3x$

7. $(D^2 - 2D + 1)y = x^2 e^x$

8. $(D^2 + 2D + 1)y = \dfrac{e^{-x}}{x^2}$

9. $(D^2 - 4)y = x^2 \cos h2x$

Answers

1. $y = e^x(A\cos x + B\sin x) + \dfrac{1}{125}e^{3x}(25x^2 - 40x + 22)$

2. $y = (Ax + B)e^{-2x} + e^{-x}(x^2 - 4x + 6)$

3. $y = Ae^{5x} + Be^{-x} + \frac{e^{-2x}}{7}\left[x + \frac{15}{7}\right]$

4. $y = Ae^{-3x} + Be^{-5x} + \frac{e^{3x}}{48}\left[x + \frac{7}{24}\right]$

5. $y = A\cos 3x + B\sin 3x + \frac{e^{3x}}{18}\left[x^2 - \frac{2x}{3} + \frac{10}{9}\right]$

6. $y = (A + Bx)e^{x} + \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$

7. $y = (A + Bx)e^{x} + \frac{x^4 e^{x}}{12}$

8. $y = (A + Bx)e^{-x} - e^{-x}\log x$

9. $y = Ae^{-2x} + Be^{2x} + \frac{x}{96}(8x^2 \sinh 2x - 6x\cosh 2x + 3\sinh 2x)$

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLES CO-EFFICIENTS

I - Cauchy Euler Type

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad \ldots(1)$$

Where $a_0, a_1, \ldots, a_n$ are constants and X is a function of x is called Euler's homogeneous linear differential equation.

Equation (1) can be reduce to constant co-efficient by means of transformation

$x = e^{x}$(or) $z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x}\frac{dy}{dz} [\because x = e^{z} \Rightarrow z = \log x]$$

$$x\frac{dy}{dx} = D'y \quad \text{where } D' = d/dz \qquad (2)$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dz^2} - \frac{d}{x}\frac{-1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$= (D'^2 - D')\,y \qquad \text{where } D' = d/dz$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y \qquad \ldots(3)$$

Similarly $x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$ …(4)

and so on. Substituting (2), (3), (4) and so on in (1), differential equation of variables co-efficients reduced to constant co-efficients and can be solved by any one of the known methods.

II – Legendre's Type

An equation of the form

$$a_0(ax+b)^n\frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_ny = X \qquad \ldots(1)$$

where $a_0, a_1, \ldots, a_n$ are constants and X is a function of x is called Legendre's linear differential equation.

Equation (1) can be reduced to linear differential equation with constant co-efficient by the substitution.

$ax + b = e^z \Rightarrow z = \log (az + b)$

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dz}$$

$(ax + b) D = aD'$, where $D' = d / dz$

Similarly $(ax + b)^2 D^2 = a^2 D (D'-1)$ and so on.

Examples

1. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ [S.U May' 09, Dec' 10, May' 12]

Solution: Given $[x^2D^2 - xD + 1]\, y = 0$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$ where $D' = d / dz$

Equation (1) reduces to

$(D'(D'-1) - D'+1)\, y = 0$

$(D'^2 - 2D' + 1)\, y = 0$

A.E is $m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

C.F $= y = (Az + B)\, e^z$

$y = (A \log x + B)\, x$

2. Solve $xy'' + y' + \frac{y}{x} = 0$ [S.U May' 07, May' 11]

Solution: Given $\left[xD^2 + D + \frac{1}{x}\right]y = 0$

[Multiply by x]

$[x^2D^2 + xD + 1]\, y = 0$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$, where $D' = d/dz$

(1) reduces to $[D'(D' - 1) + D' + 1]\, y = 0$

$[D'^2 + 1]\, y = 0$

A.E : $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

C.F = $y = A \cos z + B \sin z$

$y = A \cos (\log x) + B \sin (\log x)$

3. Solve $(x^2D^2 - 3xD + 4)\, y = 0$ [S.U Dec' 07]

Solution: Given $(x^2D^2 - 3xD + 4)\, y = 0$ ...(1)

$x^2D^2 = D'(D' - 1)$

$xD = D'$, Where $D' = d/dz$

(1) reduces to $[D'(D' - 1) - 3D' + 4]\, y = 0$

$(D'^2 - 4D' + 4)\, y = 0$

A.E : $m^2 - 4m + 4 = 0$

$m = 2, 2$ (repeated roots)

C.F = $y = (Az + B)\, e^{2z}$

$y = (A \log x + B)\, e^{2\log x}$

$= (A \log x + B)\, x^2$

4. Solve $xy'' + y' = 0$ [S.U Dec '08]

Solution: Given $xy'' + y' = 0$

multiply by $x \Rightarrow x^2y'' + xy' = 0$ …(1)

$(x^2D^2 + xD)\, y = 0$

A.E : $(D'(D' - 1) + D')$ y = 0 $[\because x^2D^2 = D'(D' - 1)\ xD = D']$

$D'^2 y = 0$

A.E: $m^2 = 0$

$m = 0, 0$

y = C.F. = $(Az + B)e^{0z}$

y = $A \log x + B$

5. Convert the Euler equation $(x^2D^2 - 7xD + 12)\, y = x^2$ into a differential equation with constant co-efficients [S.U. Dec '09]

Solution: Given $(x^2D^2 - 7xD + 12)\, y = x^2$ ...(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$, where $D' = \dfrac{dy}{dz}$

Equation (1) reduces to

$(D'(D' - 1) - 7D' + 12)\, y = (e^z)^2$

$(D'^2 - 8D' + 12)\, y = e^{2z}$ …(2)

Equation (2) is a linear differential equation with constant co-efficients.

6. Transform $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x$ into linear differential equation with constant co-efficients [S.U. Dec '11]

Solution: Given $[x^2D^2 - xD + 2]y = x$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$

Equation (1) reduces to

$[D'(D'-1) - D' + 2]y = e^z$

$[D'^2 - 2D' + 2]y = e^z$ …(2)

Equation (2) is a linear differential equation with constant co-efficients.

7. Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + \frac{2y}{x} = 0$ [S.U.Dec' 11]

Solution: Given $\left[xD^2 + 4D + \frac{2}{x}\right]y = 0$

Multiply by x

$[x^2D^2 + 4xD + 2]y = 0$

Put $x = e^z \Rightarrow z = \log x$

$[D'(D'-1) + 4D' + 2]y = 0$

$[D'^2 + 3D' + 2]y = 0$

A.E: $m^2 + 3m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -1, -2$

C.F $= y = Ae^{-z} + Be^{-2z}$

$y = \frac{A}{x} + \frac{B}{x^2}$ $[\because x = e^z]$

8. Solve $(x^2D^2 - 2xD - 4)y = 32 (\log x)^2$ [S.U May' 09, May' 11]

Solution: Given $(x^2D^2 - 2xD - 4)y = 32 (\log x)^2$

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$$xD = D', \quad D' = \frac{d}{dz}$$

$$[D'(D'-1) - 2D' - 4]y = 32z^2$$

$$[D'^2 + 3D' - 4]y = 32z^2 \qquad \ldots(2)$$

Equation (2) is a linear differential equation with constant co-efficients.

A.E: $m^2 - 3m - 4 = 0$

$(m - 4)(m + 1) = 0$

$m = 4, -1$

C.F $= Ae^{4Z} + Be^{-z}$

$$\text{P.I.} = \frac{1}{D'^2 - 3D' - 4} 32z^2$$

$$= 32 \frac{1}{-4\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]} z^2$$

$$= -8\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} z^2$$

$$= -8\left[1 + \left(\frac{D'^2 - 3D'}{4}\right) + \left(\frac{D'^2 - 3D'}{4}\right)^2 + \ldots\right] z^2 \qquad [\because (1-x)^{-1} = 1 + x + x^2 + \ldots]$$

$$= -8\left[z^2 + \frac{D'^2}{4}(z^2) - \frac{3}{4}D'(z^2) + \frac{9D'^2}{16}(z^2)\right]$$

$$\{\because D'(z^2) = 2z, \quad D'^2(z^2) = 2, \quad D'^2(z^2) = 0\}$$

$$= -8\left[z^2 + \frac{2}{4} - \frac{3}{4}(2z) + \frac{9}{16}(2)\right]$$

$$\text{P.I} = -8\left[z^2 - \frac{3}{2}z + \frac{13}{8}\right]$$

$\therefore$ General solution y = C.F + P.I

$$= Ae^{4z} + Be^{-z} - 8\left[z^2 - \frac{3z}{2} + \frac{13}{8}\right]$$

$$y = Ax^4 + \frac{B}{x} - [8(\log x)^2 - 12(\log x) + 13]$$

9. Solve $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$ [S.U.Dec '07]

Solution: Given $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$xD = D'$

(1) reduces to

$(D'(D'-1) + D' + 1)y = z \sin z$

$(D'^2 + 1)y = z \sin z$

A.E: $m^2 + 1 = 0$

$m = \pm i$

C.F $= A \cos z + B \sin z$

P.I $= \dfrac{1}{D'^2 + 1} z \sin z$

$= \dfrac{1}{D'^2 + 1}$ I.P. of $e^{iz} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{(D'+i)^2 + 1} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{D'^2 + 2iD'} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{2iD'\left(1 + \dfrac{D'}{2i}\right)} z$

$$= I.P.\ of\ e^{iz}\frac{-i}{2D'}\left(1-\frac{iD'}{2}\right)^{-1}z$$

$$= I.P.\ of\ e^{iz}\frac{-i}{2D^1}\left[1-\frac{iD'}{2}+\left(\frac{iD'}{2}\right)^2+....\right]z$$

$$= I.P.\ of\ e^{iz}\frac{-i}{2D'}\left[z+\frac{i}{2}\right] \quad \{\because D'(z)=1 \quad D'^2(z)=0\}$$

$$= I.P.\ of\ e^{iz}\frac{-1}{2}\left[\frac{z^2}{2}+\frac{iz}{2}\right] \qquad \{\because \frac{1}{D'}=Int.w.r.t.z\}$$

$$= I.P.\ of(\cos z+\sin z)\left(\frac{-iz^2}{4}+\frac{z}{4}\right)$$

P.I $= \dfrac{z^2}{4}\cos z+\dfrac{z}{4}\sin z$

y = C.F + P.I

General solution: $y = A\cos z + B\sin z - \dfrac{z^2}{4}\cos z + \dfrac{z}{4}\sin z$

$$y = A\cos(\log x) + B\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

10. Solve $(2x+3)^2\dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x$ [S.U. Dec '09]

Solution: The given eqn is a Legendre's linear differential equation

Put $(2x+3) = e^z \Rightarrow z = \log(2x+3)$

$$(2x+3)\frac{dy}{dx} = 2\frac{dy}{dz} = 2D'y$$

$$(2x+3)^2\frac{d^2y}{dx^2} = 4D'(D'-1)y$$

Equation reduces to

$$[4D'(D'-1)-2(2D')-12]y = 6\left[\frac{e^z-3}{2}\right] \quad [\because 2x+3=e^z]$$

$$[4D'^2-8D'-12]y = 3[e^z-3]$$

$$[D'^2-2D'-3]y = \frac{3}{4}(e^z-3) \qquad \ldots(1)$$

Equation (1) is a linear differential equation with constant co-efficients

A.E: $m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, -1$

$\therefore$ C.F $= Ae^{3z} + Be^{-z}$

$$\text{P.I} = \frac{1}{D'^2-2D'-3}\frac{3}{4}(e^z-3)$$

$$= \frac{3}{4}\frac{1}{D'^2-2D'-3}(e^z-3e^{0z})$$

$$= \frac{3}{4}\left[\frac{-1}{4}e^z+1\right] \quad \text{[subst. D}' = 1 \text{ in I}^{st} \text{ and D}' = 0 \text{ in II}^{nd}]$$

y = C.F + P.I

General Solution

$$y = Ae^{3z} + Be^{-z} - \frac{3}{16}e^z + \frac{3}{4}$$

$$y = A(2x+3)^3 + B(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$$

11. Solve $(x^2D^2+4xD+2)y = x\log x$ [S.U. Dec' 06, May' 11]

Solution: Given $(x^2D^2+4xD+2)y = x\log x$ …(1)

Put $x = e^Z \Rightarrow z = \log x$

$$x^2D^2 = D'(D'-1)$$

$$xD = D'$$

$$(1) \Rightarrow (D'(D'-1)+4D'+2)y = ze^z$$

$$(D'^2+3D'+2)y = ze^z$$

A.E: $m^2 + 3m + 2 = 0$

$(m+ 2)\ (m + 1) = 0$

$m = -1, -2$

C.F $= Ae^{-z} + Be^{-2z}$

$$\text{P.I.} = \frac{1}{D'^2+3D'+2} ze^z$$

$$= e^z \frac{1}{(D'+1)^2+3(D'+1)+2} z$$

$$= e^z \frac{1}{D'^2+5D'+6} z$$

$$= \frac{e^z}{6} \frac{1}{\left(1+\frac{D'^2+5D'}{6}\right)} z$$

$$= \frac{e^z}{6} \left[1+\frac{D'^2+5D'}{6}\right]^{-1} z$$

$$= \frac{e^z}{6} \left[1-\left(\frac{D'^2+5D'}{6}\right)+\left(\frac{D'^2+5D'}{6}\right)-\dots\right] z$$

$$\text{P.I} = \frac{e^z}{6}\left[z-\frac{5}{6}\right] \qquad [\because \quad D'(z)=1, D'^2(z)=0]$$

y $=$ C.F + P.I

General Solution

$$y = Ae^{-z} + Be^{-2z} + \frac{e^z}{6}\left[z - \frac{5}{6}\right]$$

$$y = \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6}\left(\log x - \frac{5}{6}\right)$$

12. Solve $(x^2D^2 - xD + 4)y = x^2 \sin(\log\ x)$ [S.U May' 10]

Solution: Given

$$(x^2D^2 - xD + 4)y = x^2 \sin(\log\ x) \qquad \ldots(1)$$

Put $x = e^z, z = \log x$

$x^2D^2 = D'(D' - 1)$

$x^2D^2 = D'(D' - 1)$

Equation (1) reduces to

$(D'(D - 1) - D' + 4)y = e^{2Z} \sin\ z$

$(D'^2 - 2D' + 4)y = e^{2Z} \sin\ z$

A.E: $m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$m = 1 \pm i\sqrt{3}$

C.F $= e^z(A\cos\sqrt{3}z + B\sin\sqrt{3}z)$

$$\text{P.I} = \frac{1}{D'^2 - 2D' + 4} e^{2z} \sin z$$

$$= I.P.\ of\ \frac{1}{D'^2 - 2D' + 4} e^{2z} e^{iz}$$

$$= I.P.\ of\ \frac{1}{D'^2 - 2D' + 4} e^{(2+i)z}$$

$$= I.P.\ of\ \frac{1}{(2+i)^2 - 2(2+i) + 4} e^{(2+i)z} \quad \text{Replace } D' = 2 + i$$

$$= I.P.\ of\ \frac{1}{3+2i} e^{2z} e^{iz}$$

$$= I.P.\ of\ \frac{(3-2i)}{13} e^{2z} (\cos z + i \sin z)$$

P.I $= \frac{e^{2z}}{13}(3\sin z - 2\cos z)$

y = C.F + P.I

$$= e^{z}(A\cos\sqrt{3}z + B\sin\sqrt{3}z) + \frac{e^{2z}}{13}(3\sin z - 2\cos z)$$

General solution

$$y = x\left(A\cos\left(\sqrt{3}\log x\right) + B\sin\left(\sqrt{3}\log x\right)\right) + \frac{x^2}{13}(3\sin(\log x) - 2\cos(\log x))$$

13. Solve $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = x\cos(\log x)$ [S.U May' 08]

Solution: Put $e^z = x \Rightarrow z = \log x$

$$x^2D^2 = D'(D'-1)$$

$$xD = D',\ D' = \frac{d}{dz}$$

Then the given equation becomes

$$[D'(D'-1) + 3D' + 5]y = e^z \cos z$$

$$xD = D',\ D' = \frac{d}{dz}$$

A.E: $m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$m = -1 \pm 2i$$

$$\text{C.F} = e^{-z}(A\cos 2z + B\sin 2z)$$

$$\text{P.I} = \frac{1}{D'^2 + 2D'^2 + 5} e^z \cos z$$

$$= R.P \frac{1}{D'^2 + 2D'^2 + 5} e^z e^{iz}$$

$$= R.P \frac{1}{D'^2 + 2D' + 5} e^{(1+i)z}$$

$$= R.P \frac{1}{(1+i)^2 + 2(1+i) + 5} e^{(1+i)z}$$

$$= R.P \frac{1}{7+4i} e^z e^{iz}$$

$$= R.P \frac{(7-4i)}{65} e^z (\cos z + i \sin z)$$

$$= \frac{e^z}{65}(7\cos z + 4\sin z)$$

ALITER

$$\text{P.I} = \frac{1}{D'^2 + 2D' + 5} e^z \cos z$$

$$= e^z \frac{1}{(D'+1)^2 + (D'+1) + 5} \cos z$$

$$= e^z \frac{1}{D'^2 + 4D' + 8} \cos z \quad \text{Replace } D'^2 = -1$$

$$= e^z \frac{(4D'-7)}{(4D'+7)(4D'-7)} \cos z$$

$$= e^z \frac{[4(-\sin z) - 7\cos z]}{(4D')^2 - 7^2}$$

$$= \frac{e^z}{65}(4\sin z + 7\cos z)$$

y = C.F + P.I

$$= e^{-z}(A\cos 2z + B\sin 2z) + \frac{e^z}{65}(7\cos z + 4\sin z)$$

$$y = \frac{1}{x}(A\cos(2(\log x)) + B\sin(2(\log x)) + \frac{x}{65}[4\sin(\log x) + 7\cos(\log x)]$$

14. Solve $(x^2D^2 - xD + 2)y = x\log x$ [S.U. Dec 11]

Solution: Given $(x^2D^2 - xD + 2)y = x\log x$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$xD = D'$

Equation (1) reduces to

$(D'(D'-1) - D' + 2)y = ze^z$

$(D'^2 - 2D' + 2)y = ze^z$

A.E: $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$m = 1 \pm i$

C.F. $= e^z (A\cos z + B\sin z)$

$$\text{P.I.} = \frac{1}{D'^2 - 2D' + 2} ze^z$$

$= e^z \dfrac{1}{(D'+1)^2 - 2(D'+1) + 2} z$

$= e^z \dfrac{1}{D'^2+1} z$

$= e^z (1 + D'^2)^{-1} z$

$= e^z (1 - D'^2 + (D'^2)^2 - ...) z$

P.I $= ze^z [\Rightarrow D'(z) = 1, D'^2(z) = 0]$

y = C.F + P.I

$= e^z (A\cos + B\sin z) + ze^z$

$y = x(A\cos(\log x) + B\sin(\log x) + x\log x$

15. Solve $x^2 y'' + xy' + y = \cos(2\log x)$ [S.U Dec 08]

Solution: Given $(x^2D^2 + xD + 1)y = \cos(2\log x)$

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$xD = D', \quad D' \dfrac{d}{dz}$

$(D'(D'-1) + D' + 1)y = \cos 2z$

$(D'^2 + 1)y = \cos 2z$

A.E. $m^2 + 1 = 0$

$m^2 = - 1$

$m = \pm i$

C.F $= A\cos z + B\sin z$

P.I. $= \dfrac{1}{D'^2+1} \cos 2z$

Replace $D'^2 = - 4$

$$= \frac{-1}{3}\cos 2z$$

y = C.F + P.I

$$= A\cos z + B\sin z - \frac{1}{3}\cos 2z$$

$$y \quad = A\cos(\log x) + B\sin(\log x) - \frac{1}{3}\cos(2\log x)$$

Exercise

1. Convert the equation $xy'' - 3y' + x^{-1}y = x^2$ as a linear equation with constant co-efficients.

2. Solve: $x^3y''' + 3x^2y' + xy' + y = 0$

3. Solve: $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$

4. Solve: $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

5. Solve: $(x^2D^2 + xD + 1)y = \sin(2\log x)\sin(\log x)$

6. Solve: $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos\log(x+1)$

7. Solve: $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

8. Solve: $(x^2D^2 + xD - 9)y = \sin^3(\log x)$

9. Solve: $(x^2D^2 - 3xD - 5)y = \sin(\log x)$

10. Solve: $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 4y = x^2 + 2\log x$

11. Solve: $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x^2+1)^2$

12. Solve: $x^2 \frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 25y = (\log x)^2$

13. Solve: $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

14. Solve: $\left(D^2 - \frac{D}{x} + \frac{1}{x^2}\right)y = \frac{2\log x}{x^4}$

15. Find Particular integral for $(x^3D^3 + 3x^2D^2 + xD + 1)y = \sin(\log x)$

16. Solve: $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

17. Solve: $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2 + \sin(5\log x)$

18. Solve: $(x^2D^2 + 4xD + 2)y = x = \frac{1}{x}$

19. Solve: $x^3y''' + 2x^2y'' - xy' + y = \log x$

20. Solve: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = \frac{5}{x^2}$

Answer

1. $(D'^2 - 4D' + 1)y = e^{3z}$

2. $y = \frac{A}{x} + \sqrt{x}\left[Bos\left(\frac{\sqrt{3}}{2}\log x\right) + C\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$

3. $y = \frac{A}{x} + \frac{B}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}$

4. $y = (A\log x + B)x + \frac{1}{9x^2}\left((\log x)^2 + \frac{4}{3}(\log x) + \frac{2}{3}\right)$

5. $y = A\cos(\log x) + B\sin(\log x) - \frac{1}{16}\sin(3\log x) - \frac{1}{4}\log x\cos(\log x)$

6. $y = A\cos(\log(x+1)) + B\sin(\log(x+1)) + 2\log(x+1)\sin\log(x+1)$

7. $y = A(3x+2)^2 + \dfrac{B}{(3x+2)^2} + \dfrac{1}{108}[(3x+2)^2\log(3x+2)+1]$

8. $y = Ax^3 + \dfrac{B}{x^3} - \dfrac{3}{40}(\sin(\log x)) + \dfrac{1}{72}\sin(3\log x)$

9. $y = Ax^5 + \dfrac{B}{x} + \dfrac{2\cos(\log x) - 3\sin(\log x)}{26}$

10. $y = \dfrac{1}{\sqrt{x}}\left[A\cos\left(\dfrac{\sqrt{15}}{2}\log x\right) + B\sin\left(\dfrac{\sqrt{15}}{2}\log x\right)\right] + \dfrac{x^2}{10} + \dfrac{1}{2}\left(\log x - \dfrac{1}{4}\right)$

11. $y = Ax^4 + \dfrac{B}{x^5} + \dfrac{x^4\log x}{9} - \dfrac{x^2}{7} - \dfrac{1}{20}$

12. $y = \dfrac{1}{x^4}[A\cos(3\log x) + B\sin(3\log x)] + \dfrac{1}{25}\left[(\log x)^2 - \dfrac{16}{25}\log x + \dfrac{78}{625}\right]$

13. $y = \dfrac{A}{x^3} + Bx^3 - \dfrac{x^2}{2}\left(\log x + \dfrac{2}{3}\right)$

14. $y = x(A\log x + B) + \dfrac{2}{9x^2}\left(\log x + \dfrac{2}{3}\right)$

15. $P.I = \dfrac{1}{2}(\sin(\log x) + \cos(\log x))$

16. $y = x^2(\log x + B) + x^2(\log x)^2$

17. $y = Ax + Bx^2 + x^2\log x + \dfrac{15\cos(5\log x) - 23\sin(5\log x)}{754}$

18. $y = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{x}{6} + \dfrac{\log x}{x}$

19. $y = \dfrac{A}{x} + (B\log x + C) + \log x + 1$

20. $y = Ax^3 + \dfrac{B}{x^3} - \dfrac{1}{x^2}$

Simultaneous linear Differential Equations with constant co-efficients

Simultaneous linear Differential Equations:

The system of differential equations which consist of one independent and two or more dependent variables.

To solve such system completely, we must have as many simultaneous equations as the number of dependent variables.

Here we consider only the 1st order simultaneous linear differential equations.

Let *x*, *y* be the two dependent variables and'*t*' be the independent variable, then consider the system like

$$f_1(D)x + g_1(D)y = \Phi(t) \qquad \ldots(1)$$

$$f_2(D)x + g_2(D)y = \Phi(t) \qquad \ldots(2)$$

$$D = \frac{d}{dt}$$

Where f_1, f_2, g_1, g_2 are polynominals in D.

Methods to solve the simultaneous equations

Solving simultaneous differential equation is based on the process of elimination of the variables which is applied in solving the simple algebraic equations.

Method 1

The method of solution is to eliminate the dependent variables *x* or *y* between the two given equations. First getting an equation in the dependent variable and then solve the equations by the methods as used already.

After getting the solution either for *x* or *y*, substitute the solution either in (1) or in (2) to get the solution for the other dependent variable.

Note (1)

The number of arbitrary constants in the solution of the system

$$f_1(D)x + g_1(D)y = \varphi_1(t)$$

$$f_2(D)x + g_2(D)y = \varphi_2(t)$$

is equal to the degree of D in the determinant $\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix}$ (i.e.) The number of arbitrary constants in the solution of the system is equal to the number of dependent variables appeared in the system.

Example:

Consider the system $\frac{dx}{dt} + y = \sin t; \frac{dy}{dt} + x = \cos t.$ Here the number of arbitraryconstants is equal to the number of dependent variables in the system

Let $D = \frac{d}{dt}$, then the above system is

$Dx + y = \sin t$ ...(1)

$Dx + x = \cos t$ ...(2)

Now $\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} = \begin{vmatrix} D & 0 \\ 0 & D \end{vmatrix} = D^2,$ Here the degree is 2

$\therefore$ We have two arbitrary constants in the solution of the system.

Method 2

Elimination of *x* in the equations (1) and (2) gives the solution of *y*, in which we have two arbitrary constants.

Likewise elimination of *y* in the equations (1) and (2) gives the solution.

But according to the rule (in Note), we must have only two arbitrary constants. Therefore we can write the relation between the arbitrary constants. (i.e) one constant can be expressed interms of other.

Note (2)

Also, we can eliminate first *y* and then using *y* we may find out the solution of *x*.

Problems

1. Solve: $\frac{dx}{dt} + y - 1 = \sin t; \frac{dy}{dt} + x = \cos t$ (S.U 2006)

Solution:Given $\dfrac{dx}{dt}+y-1=\sin t; \dfrac{dy}{dt}+x=\cos t$

Let $D=\dfrac{d}{dt}$, then the above equations are $Dx+y-1=\sin t$

(i.e) $Dx+y=\sin t+1$ …(1)

And $Dy+x=\cos t$ …(2)

First we can eliminate x from (1) and (2)

$(2)\times D \Rightarrow D^2y+Dx=D(\cos t)=-\sin t$ …(3)

(i.e) $Dx+D^2y=-\sin t$

$$\begin{array}{ll} & \not{D}x+y=\sin t+1 \\ (1)-(3) & \underline{\not{D}x+D^2y=-\sin t} \\ & (-D^2+1)y=2\sin t+1 \end{array}$$

(i.e) $-(D^2-1)y=2\sin t+1$

(i.e) $(D^2-1)y=-2\sin t-1$ which is 2nd order differential equation in y with constant coefficients.

$\therefore$ The solution of $y(t) = C.F. + P.I.$

Now the Auxiliary equation is

$m^2 - 1 = 0$

(i.e) $m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= Ae^t + Be^{-t}$

P.I. $= \dfrac{1}{D^2-1}(-2\sin t-1)$

$= \dfrac{1}{D^2-1}(-2\sin t)-\dfrac{1}{D^2-1}e^{0t}$

$$= \frac{-2}{-1-1}\sin t - \frac{1}{0-1}e^{0t}$$

$$= \frac{-2}{-2}\sin t + \frac{1}{1}e^{0t}$$

$= \sin t + 1$

$$\therefore y \quad = Ae^{t} + Be^{-t} + \sin t + 1$$

To find the solution of x, first we can find Dy.

$$Dy \quad = D[Ae^{t} + Be^{-t} + \sin t + 1]$$

$$= Ae^{t} - Be^{-t} + \cos t$$

Consider the equation (2)

$Dy + x = \cos t$

$\Rightarrow x = \cos t - Dy$

$x = \cos t - (Ae^{t} - Be^{t} + \cos t)$

$= \cancel{\cos} t - Ae^{t} + Be^{t} - \cancel{\cos} t$

$x = Be^{-t} - Ae^{t}$

$\therefore$ The solutions are

$x = Be^{-t} - Ae^{t}$

$y = Ae^{-t} + Be^{t} + \sin t + 1$

where A and B are arbitrary constants.

2. Solve $\frac{dx}{dt} + y = \sin t; \frac{dy}{dt} + x = \cos t$ where $x(0) = 2$ and $y(0) = 5$ (S.U 2007)

Solution:Given $\frac{dx}{dt} + y = \sin t; \frac{dy}{dt} + x = \cos t$

(i.e)

$Dx + y = \sin t;$ …(1)

$Dy + x = \cos t$ …(2)

First we can eliminate x from (1) and (2)

$(2)\times D \Rightarrow Dx + D^2 y = D\cos t = -\sin t$ …(3)

$$\begin{array}{r} Dx - y = \sin t \\ \underline{Dx + D^2 y = -\sin t} \\ (1)-(3) \Rightarrow (-D^2 - 1)y = 2\sin t \end{array}$$

(i.e) $(D^2-1)y = 2\sin t$, this is a 2nd order differential equation in y with constant coefficients.

y = C.F. + P.I.

Now the auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m^2 = \pm 1$

$\Rightarrow$ C.F. = $Ae^t + Be^{-t}$

P.I. $= \dfrac{1}{D^2-1}(-2\sin t) = \dfrac{-2}{-1-1}\sin t$ (Replace D^2 by – 1)

$= \dfrac{2}{2}\sin t = \sin t$

$\therefore y = Ae^t + Be^{-t} + \sin t$ …(4)

To find the solution of x, first let us find Dy and substitute the value in (2)

$Dy = D[Ae^t + Be^{-t}\sin t] = Ae^t - Be^{-t} + \cos t$

Now (2) $\Rightarrow Dy + x = \cos t$

$\Rightarrow x = \cos t - Dy$

$= \cos t - (Ae^t - Be^{-t} + \cos t)$

$= \cos t - Ae^t + Be^{-t} - \cos t$

$x = Be^{-t} - Ae^t$ …(5)

To find the value of A and B, We can use the given conditions

Given $x(0) = 2$, $y(0) = 0$

(4) $\Rightarrow y(0) = Ae^{0} + Be^{-0} + \sin(0)$ Since $t = 0$

$0 = A + B$

$\Rightarrow B = -A$

(5) $\Rightarrow x(0) = Be^{-o} - Ae^{o}$ [t = 0]

$2 = B - A$

$2 = -A - A = -2A$

$\therefore A = -1$

$\Rightarrow B = 1$

$\therefore$ The solutions are

$x = e^{-t} + e^{t} = 2\cosh t$ $\qquad \because \cos ht = \dfrac{e^{t} + e^{-t}}{2}$

$y = e^{-t} - e^{t} + \sin t$

$= -(e^{t} - e^{-t}) + \sin t$ $\qquad \because \sin\ ht = \dfrac{e^{t} - e^{-t}}{2}$

$y = -2\sinh\ t + \sin t$

3. Solve $Dx + y = \cos t\ ; x + Dy = 2\sin t$.

Solution:Given

$Dx + y = \cos t;$ …(1)

$x + Dy = 2\sin t$ …(2)

where $D = \dfrac{d}{dt}$

First we can eliminate y from (1) & (2)

$(1) \times D \Rightarrow D^{2}x + Dy = D(\cos t) = -\sin t$

(i.e)

$$D^2x + \not{D}y = -\sin t \text{ ....(3)}$$
$$x + \not{D}y = 2\sin t$$
$$(3)-(2) \Rightarrow (D^2-1)x = -3\sin t \text{...............}(I)$$

which is a 2nd order differential equation in *x* with constant coefficients the solution of

x(t) = C.F. + P.I.

Now Auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. = $Ae^t + Be^{-t}$

$$\text{P.I.} \quad = \frac{1}{D^2-1}(-3\sin t)$$

$$= \frac{-3}{-1-1}\sin t \qquad \text{Replace } D^2 \text{ by } -1$$

$$= \frac{3}{2}\sin t$$

$$\therefore x(t) = Ae^t + Be^{-t} + \frac{3}{2}\sin t \qquad \ldots(4)$$

Now to find *y* we can find *Dx* and then substitute *Dx* in (1).

$$Dx \quad = D\left[Ae^t + Be^{-t} + \frac{3}{2}\sin t\right]$$

$$= Ae^t - Be^{-t} + \frac{3}{2}\cos t$$

(1) $\Rightarrow Dx + y = \cos t$

$\therefore y \quad = \cos t - Dx$

$$= \cos t - \left[Ae^{t} - Be^{-t} + \frac{3}{2}\cos t \right]$$

$$= \cos t - Ae^{t} - Be^{-t} - \frac{3}{2}\cos t$$

$$= \frac{\cos t}{2} - Ae^{t} + Be^{-t}$$

$$y = Be^{-t} - \frac{\cos t}{2} - Ae^{t}$$

Aliter

To find y we can eliminate x from (1) & (2)

$$(2) \times D \Rightarrow Dx + D^2 y = D(2\sin t) = 2\cos t$$

$$(6) - (1) \Rightarrow Dx + D^2 y = 2\cos t \text{ ....(6)}$$

$$Dx + y = \cos t$$

$$(D^2 - 1)y = \cos t \text{ ....(II)}$$

$\therefore y = CF + PI$

Auxiliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

C.F. = $Ce^{t} + De^{-t}$

[Since already A, B are introduced in x]

$$\text{P.I.} = \frac{1}{D^2 - 1}\cos t$$

$$= \frac{1}{-1-1}\cos t \text{ Replace } D^2 \text{ by } -1$$

$$= \frac{-1}{2}\cos t$$

$$\therefore\ y(t) = Ce^{t} + De^{-t} - \frac{1}{2}\cos t \qquad ...(7)$$

Here the solutions of x and y contains four constants. But the solutions for x and y should contain as the order of equations (I) and (II).

Hence the values of C and D should be expressed interms of A and B as explained below.

Inserting the values of x and y in (1)

(i.e) $Dx + y = \cos t \quad \left[Here\ Dx = \frac{d}{dt}(x) \right]$

$$D\left[Ae^{t} + Be^{-t} + \frac{3}{2}\sin t \right] + Ce^{t} + De^{-t} - \frac{1}{2}\cos t = \cos t$$

(i.e) $\frac{d}{dt}\left[Ae^{t} + Be^{-t} + \frac{3}{2}\sin t \right] + Ce^{t} + De^{-t} - \frac{1}{2}\cos t = \cos t$

$$Ae^{t} - Be^{-t} + \frac{3}{2}\cos t + Ce^{t} + De^{-t} - \frac{1}{2}\cos t - \cos t = 0$$

$$\Rightarrow Ae^{t} + Be^{-t} + \frac{3}{2}\cos t + Ce^{t} + De^{-t} - \frac{3}{2}\cos t = 0$$

$$\Rightarrow Ae^{t} - Be^{-t} + Ce^{t} + De^{-t} = 0$$

(i.e) $(A + C)e^{t} + (D - B)e^{-t} = 0.$

$$A + C = 0, D - B = 0$$

(i.e) $-A = +C;\ D = B$ ($\therefore e^{t}$, e^{-t} will not be zero)

$\therefore$ The solutions are $x = Ae^{t} + Be^{-t} + \frac{3}{2}\sin t$

In (7), But $C = -A, D = B \quad y = Ae^{t} + Be^{-t} - \frac{1}{2}\cos t$

Where A and B are arbitrary constants.

4. Solve $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} = t^{2} - x$ (S.U 2008, 2012)

Solution: Given $Dx - y = t; \quad Dy = t^2 - x \qquad where \ \frac{d}{dt} = D$

(ie) $\therefore CF = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)$

First we eliminate x from (1) and (2)

$$Dx + D^2 y = 2t \rightarrow (3)$$

$$(3) - (1) \Rightarrow Dx - y = t \rightarrow (1)$$

$$(D^2 + 1)y = t$$

is a differential equation in y with constant coefficients.

$\therefore y = CF + PI$

Now the auxiliary equation is

$m^2 + 1 = 0$

(ie) $m^2 = -1$

$m = \pm i$

$\therefore$C.F. = $A \cos t + B \sin t$

P.I. $= \frac{1}{D^2 + 1}(t) = [1 + D^2]^{-1}(t)$

$= [1 - D^2 + D^4 - ...](t)$

$= t - D^2(t) + D^4(t) - ...$

$= t \qquad [\therefore D^2(t) = 0, D^4(t) = 0\ldots$

Neglecting D^2 and higher powers of D]

$\therefore y = A \cos t + B \sin t + t \qquad [(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots, |x|<1.]$

To find x, find Dy and substitute in equation (2)

$$Dy = \frac{d}{dt}(y) = -A\sin t + B\cos t + 1$$

$(2) \Rightarrow x + Dy = t^2$

$\Rightarrow x = t^2 - Dy$

$= t^2 - [-A\sin t + B\cos t + 1]$

$= t^2 + A\sin t - B\cos t - 1$

$x(t) = A\sin t - B\cos t + t^2 - 1$

$\therefore$ The solutions are

$x(t) = A\sin t - B\cos t + t^2 - 1$

$y(t) = A\cos t - B\sin t + t$

Where A and B are arbitrary constants.

5. Solve $(D + 2)\, x - 3y = t$; $(D + 2)\, y - 3x = e^{2t}$ (S.U 2009)

Solution: Given

$(D + 2)\, x - 3y = t \quad \rightarrow (1)$

$(D + 2)\, y - 3x = e^{2t} \quad \rightarrow (2)$

First we can eliminate x from (1) & (2)

$(1) \times 3 \Rightarrow 3(D+2)x - 3(3y) = 3t$

$(D+2)x - 9y = 3t \rightarrow (3)$

$(2) \times (D+2) \Rightarrow -3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$

$\Rightarrow -3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$

$= 2e^{2t} + 2e^{2t}$

$\Rightarrow -3(D+2)x + (D+2)^2 y = 4e^{2t} \rightarrow (4)$

$(3) + (4) \qquad 3(D+2)x - 9y = 3t$

$-3(D+2)x + (D+2)^2 y = 4e^{2t}$

$D^2 + 4D + 4 - 9)y = 4e^{2t} + 3t$

(ie) $(D^2 + 4D - 5)\, y = 4e^{2t} + 3t$ is differential equation in y with arbitrary constants.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is

$m^2 + 4m - 5 = 0$

(ie) $(m + 5)(m - 1) = 0$

$m = -5, \qquad m = 1$

$\therefore$ C.F. $= Ae^{t} + Be^{-5t}$

$$\text{P.I.} = \frac{1}{D^2 + 4D - 5}(4e^{2t} + 3t)$$

$$= \frac{4}{D^2 + 4D - 5}e^{2t} + \frac{3}{D^2 + 4D - 5}(t)$$

$$= \frac{4}{2^2 + 4(2) - 5}e^{2t} + \frac{3}{-5\left[1 + \frac{D^2 + 4D}{-5}\right]}(t)$$

$$= \frac{4}{4 + 8 - 5}e^{2t} - \frac{3}{5}\left[1 - \frac{D^2 + 4D}{5}\right]^{-1}(t)$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[1 + \left(\frac{D^2}{5} + \frac{4D}{5}\right) + \left(\frac{D^2}{5} + \frac{4D}{5}\right)^2 + \dots\right](t)$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[1 + \frac{4D}{5}\right](t)$$ [neglecting D^2 and higher power of D, since $D^2(t) = 0$]

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[t + \frac{4}{5}D(t)\right]$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left(t + \frac{4}{5}\right)$$

$$= \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$$

$$\therefore y = Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$$

Now consider $(D+2)y - 3x = e^{2t}$

(ie) $Dy + 2y - 3x = e^{2t}$

(ie) $\frac{d}{dt}\left[Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)\right]$

$$+2\left[Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)\right] - e^{2t} = 3x$$

(ie) $3x = Ae^{t} + 5Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}(3)$

$$+2Ae^{t} + 2Be^{-5t} + \frac{8}{5}e^{2t} - \frac{2}{5}(3t) - \frac{2}{5}\frac{12}{6} - e^{2t}$$

$$3x = 3Ae^{t} - 3Be^{-5t} + \frac{16}{7}e^{2t} - e^{2t} - \frac{3}{5} - \frac{24}{25} - \frac{2}{5}(3t)$$

$$= 3Ae^{t} - 3Be^{-5t} + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}$$

$$\therefore x = \frac{1}{3}\left[3Ae^{t} - 3Be^{-5t} + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}\right]$$

$$x = Ae^{t} - Be^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}$$

$\therefore$ The Solutions are $y = Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$

$x = Ae^{t} - Be^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{5}$, where A and B are arbitrary constants.

6. Solve: $\frac{dx}{dy} - 7x + y = 0;\quad \frac{dy}{dt} - 2x - 5y = 0$ (S.U 2010)

Solution:Given $Dx - 7x + y = 0;\quad Dy - 2x - 5y = 0$ where $D = \frac{d}{dy}$

(ie) $(D-7)x + y = 0 \rightarrow (1)$

$-2x + (D-5)y = 0 \rightarrow (2)$

First we can eliminate *x*from (1) and (2)

(1) ×2 ⇒ $2(D-7)\,x + 2y = 0$ ………..(3)

(2) ×$(D-7)$⇒ $-2(D-7)\,x + (D-5)\,(D-7)\,y = 0$ ………..(4)

(3) ×4⇒ $(D^2 - 5D - 7D + 35 + 2)\,y = 0$

(ie) $(D^2 - 12D + 37)y = 0$ is a differential equation is *y* with constant coefficients.

$\therefore$ y = C.F. + P.I.

Now the Auxiliary equation is $m^2 - 12m + 37 = 0$

$$m = \frac{12 \pm \sqrt{144-148}}{2} = \frac{12 \pm \sqrt{-4}}{2} = \frac{12 \pm 2i}{2}$$

$$m = 6 \pm i$$

$\therefore$ C.F. $= e^{6t}(A \cos t + B \sin t)$

P.I. = 0

$\therefore y = e^{6t}(A \cos t + B \sin t)$

Now (2) ⇒ $(D-5)\,y - 2x = 0$

$2x = (D-5)$ y

$= Dy - 5y$

$$= \frac{d}{dt}[e^{6t}(A\cos t + B\sin t)] - 5[e^{6t}(A\cos t + B\sin t)]$$

$$= e^{6t}\frac{d}{dt}[(A\cos t + B\sin t)] + \frac{d}{dt}(e^{6t})[(A\cos t + B\sin t)] - 5e^{6t}(A\cos t + B\sin t)$$

$$= e^{6t}[(-A\sin t + B\cos t)] + 6e^{6t}[(A\cos t + B\sin t)] - 5e^{6t}[(A\cos t + B\sin t)]$$

$$= e^{6t}[B\cos t - A\sin t] + e^{6t}(A\cos t + B\sin t)$$

$$2x = e^{6t}[\cos t(B+A) + \sin t(B-A)]$$

$\therefore \quad x = \frac{e^{6t}}{2}[\cos t(B+A) + \sin t(B-A)]$

The solutions are $x = \frac{e^{6t}}{2}[\cos t(B+A) + \sin t(B-A)]$

$y = e^{6t}(A\cot t + B\sin t)$ where A and B are arbitrary constants

7. Solve: $\frac{dx}{dt} + y = 0; \quad \frac{dy}{dt} + x = 2\cos t$

Solution:Given $\frac{dx}{dt} + y = 0; \quad \frac{dy}{dt} + x = 2\cos t$

(ie) $Dx + y = 0 \rightarrow (1) \qquad Dy + x = 2\cos t \rightarrow (2)$

First eliminate x from (1) and (2)

$(2)\times D \Rightarrow D^2y + Dx = D(2\cos t)$

(ie) $Dx + D^2y = -2\sin t \rightarrow (3)$

$(1) \Rightarrow \qquad Dx + 2y \qquad = 0$

$(3) \Rightarrow \quad Dx + D^2y \qquad = -2\sin t$

$(1) - (3) \Rightarrow \qquad -D^2y + y \qquad = 2\sin t$

$-(D^2 - 1)\,y = 2\sin t$

(ie) $(D^2 - 1)\,y = 2\sin t$ is a differential equation in y with constant co-efficient.

$\therefore y =$ C.F. + P.I.

Now the auxiliary equation is $m^2 - 1 = 0$

(ie) $m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= Ae^{t} + Be^{-t}$

P.I. $= \frac{1}{D^2 - 1}(-2\sin t)$

$= \frac{-2}{-1-1} \sin t$ Replace D^2 by -1

$= \sin t$

$\therefore y = Ae^t + Be^{-t} + \sin t$

To find the solution of x, first find $\frac{d}{dt}(y)$ and put this value in (2)

$Dy = \frac{d}{dy}(y) = D[Ae^t + Be^{-t} + \sin t]$

$= Ae^t - Be^{-t} + \cos t$

(2) $\Rightarrow Dy + x = 2\cos t$

(ie) $x = 2\cos t - Dy$

$= 2\cos t - (Ae^t - Be^{-t} + \cos t)$

$= 2\cos t - Ae^t + Be^{-t} - \cos t$

$x = Be^{-t} - Ae^t + \cos t$

$\therefore$ The solutions are $x = Be^{-t} - Ae^t + \cos t$

$y = Ae^t + Be^{-t} + \sin t$

Here A and B are arbitrary constants.

8. Solve: $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} + x = 1$

Solution:Given $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} + x = 1$

(ie) $Dx - y = t \rightarrow (1); \qquad Dy + x = 1 \rightarrow (2)$

First eliminate x from (1) and (2)

$(2)\times D \Rightarrow D^2y + Dx = D(1) = 0$

$Dx + D^2y = 0$……………(3)

$- Dx - y = t$

$-(1)+(3) \Rightarrow \quad (D^2+1)\,y = -t$

is a 2nd order differential equation in y with constant coefficients

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

$\therefore$ C.F. $= A\cos t + B\sin t$

P.I. $= \dfrac{1}{(D^2+1)}(-t)$

$= (1 + D^2)^{-1}\,(-t)$

$= [1 - D^2 + (D^2)^2 - \ldots](-t)$ $[\therefore (1+x)^{-1} = 1 - x + x^2 - \ldots]$

$= -t + D^2\,(t^2) + \ldots$

$= -t$ [Neglecting D^2 and higher powers of $D \because D^2(t) = 0$]

$\therefore y \quad = A\cos t + B\sin t - t$

To find x, find Dy and substitute this in (2)

$Dy \quad = \dfrac{dy}{dt} = \dfrac{d}{dt}[A\cos t + B\sin t - t]$

$= -A\sin t + B\cos t - 1$

(2) reduces $Dy = 1 + A\sin t - B\cos t + 1$

$x = 2 + A\sin t - B\cos t$

The solutions are $x = 2 + A\sin t - B\cos t$

$y = A\cos t - B\sin t - t$ where A, B are arbitrary constants,

9. Solve: $\dfrac{dx}{dt} + y = \sin 2t;\quad \dfrac{dy}{dt} - x = \cos 2t;$ (S.U 2011)

Solution:Given $Dx + y = \sin 2t \rightarrow (1);\quad Dy - x = \cos 2t\left[\because D = \dfrac{d}{dt}\right] \rightarrow (2)$

First we can eliminate x between (1) & (2)

$D\times(2) \Rightarrow D^2y - Dx = D(\cos 2t) = -2\sin 2t$

$$-Dx + D^2y = -2\sin 2t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(ie) $$Dx + y = \sin 2t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(1) +(3)$\Rightarrow$ $(D^2 + 1)\,y = -\sin 2t$

is a 2nd order differential equation in y with constant coefficients

$\therefore y =$ C.F. + P.I.

The auxiliary equation is $m^2 + 1 = 0$

(ie) $m^2 = -1$

(ie) $m = \pm i$

$\therefore$ C.F. $= A\cos t + B\sin t$

$$\text{P.I.} = \frac{1}{D^2+1}(-\sin 2t) = \frac{1}{-2^2+1}\sin 2t$$

$$= \frac{-1}{-4+1}\sin 2t = \frac{1}{-3}\sin 2t = \frac{1}{3}\sin 2t$$

$$\therefore y = A\cos t + B\sin t + \frac{1}{3}\sin 2t$$

Now let to find Dy, get x, we can use Dy in (2)

$$Dy = \frac{d}{dt}(y) = \frac{d}{dt}A\cos t + B\sin t + \frac{1}{3}\sin 2t$$

$$= A\sin t + B\cos t + \frac{2}{3}\cos 2t$$

$$(2) \Rightarrow Dy - x = \cos 2t$$

$$x = Dy - \cos 2t$$

$$= -A\sin t + B\cos t + \frac{2}{3}\cos 2t - \cos 2t$$

$$x \quad = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$$

$\therefore$ The solutions are $x = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$

$y = A\cos t + B\sin t + \frac{1}{3}\sin 2t$ where A and B are arbitrary constants.

10. Solve: $(D+4)x + 3y = t; \quad 2x + (D+5)y = e^{2t}$

Solution:

Given $(D+4)x + 3y = t \rightarrow (1); \quad 2x + (D+5)y = e^{2t} \rightarrow (2)$

First we can eliminate x from (1) and (2)

$(1) \times 2 \Rightarrow 2 \qquad 2(D + 4)\, x + \qquad 6y \;\; = 2t$

$(2) \times (D + 4) \Rightarrow 2(D + 4)\, x + (D + 5)(D + 4)\, y = (D + 4)e^{2t}$.......(4)

$(3) - (4) \Rightarrow \qquad 6y - (D + 5)\,(D + 4)\, y = 2t - D(e^{2t}) - 4e^{2t}$

(ie) $\qquad 6y - (D^2 + 5D + 4D + 20)\, y = 2t - 2e^{2t} - 4e^{2t}$

$[-(D^2 + 9D + 20) + 6]\, y = 2t - 6e^{2t}$

$-[D^2 + 9D + 20 - 6]\, y = 2t - 6e^{2t}$

$(D^2 + 9D + 14)\, y = -(2t - 6e^{2t})$

$(D^2 + 9D + 14)\, y = 6e^{2t} - 2t$, is a 2nd order differential equations in y with constant co-efficients.

$\therefore y$ = C.F. + P.I.

The auxiliary equation is $m^2 + 9m + 14 = 0$

$(m + 7)\,(m + 2) = 0$

$m = -7,\ m = -2$

$\therefore$ C.F. $= A^{-2t} + Be^{-7t}$

Now P.I. $= \dfrac{1}{D^2 + 9D + 14}(6e^{2t} - 2t)$

$$= \frac{1}{D^2+9D+14}6e^{2t} - \frac{2}{D^2+9D+14}t$$

$$= \frac{6}{2^2+9(2)+14}e^{2t} - \frac{2}{14\left(1+\frac{D^2+9D}{14}\right)}t$$

$$= \frac{6}{4+18+14}e^{2t} - \frac{2}{14}\left[1+\left(\frac{D^2}{14}+\frac{9D}{14}\right)\right]^{-1}(t)$$

$$\frac{6}{36}e^{2t} - \frac{1}{7}\left[1-\left(\frac{D^2}{14}+\frac{9D}{14}\right)+\left(\frac{D^2}{14}+\frac{9D}{14}\right)^2 - ....\right](t)$$

$= \frac{1}{6}e^{2t} - \frac{1}{7}\left[t - \frac{9}{14}D(t)\right]$ [Neglecting D^2 and higher powers of D $\because D^2(t) = 0$]

$$= \frac{1}{6}e^{2t} - \frac{1}{7}\left(t - \frac{9}{14}\right)$$

$$= \frac{1}{6}e^{2t} - \frac{t}{7} + \frac{9}{98}$$

$$\therefore y \quad = Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}$$

To find x, First find $\frac{d}{dt}(y)$ and substitute $\frac{d}{dt}(y)$ in (2)

$$Dy \quad = D\left[Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}\right]$$

$$= -2Ae^{-2t} - 7Be^{-7t} + \frac{2e^{2t}}{6} - \frac{1}{7}$$

$$= -2Ae^{-2t} - 7Be^{-7t} + \frac{e^{2t}}{3} - \frac{1}{7}$$

Now $(2) \Rightarrow 2x + (D+5)y = e^{2t}$

(ie) $2x = e^{2t} - (D+5)y = e^{2t} - Dy - 5y$

$$2x = e^{2t} - \left(-2Ae^{-2t} - 7Be^{-7t} + \frac{1}{3}e^{2t} - \frac{1}{7} \right)$$

$$-5\left(Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98} \right)$$

$$= e^{2t} + 2Ae^{-2t} + 7Be^{-7t} - \frac{1}{3}e^{2t} + \frac{1}{7}$$

$$-5Ae^{-2t} + 5Be^{-7t} - \frac{5e^{2t}}{6} + \frac{5t}{7} - \frac{45}{98}$$

$$= e^{2t} - \frac{1}{3}e^{2t} - \frac{5}{6}e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} - \frac{45}{98} + \frac{1}{7}$$

$$= \left(\frac{6-2-5}{6} \right) e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} + \left(\frac{-45+14}{98} \right)$$

$$2x = -\frac{1}{6}e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} - \frac{31}{98}$$

$$\therefore x = -\frac{1}{12}e^{2t} - \frac{3}{2}Ae^{-2t} + Be^{-7t} + \frac{5t}{14} - \frac{31}{196}$$

$\therefore$ The solutions are $x = Be^{-7t} - \frac{3}{2}Ae^{-2t} - \frac{1}{12}e^{2t} + \frac{5t}{14} - \frac{31}{136}$,

$y = Ae^{-2t} = Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}$ where A and B are arbitrary constants.

11. Solve $\frac{dx}{dt} + 2x - 2y = e^t$; $\frac{dy}{dt} + 2x + y = 0$

Solution: Given $Dx + 2x - 2y = e^t$

(i.e) $(D+2)x - 2y = e^t$ …(1)

$Dy + 2x + y = 0$

(i.e) $(D+1)y + 2x = 0$ …(2)

First eliminate x from (1) and (2)

(1) × 2 2(D + 2) x - 22 y = 2e^t……………(3)

(2) $\times(D + 2) \Rightarrow 2(D + 2)\, x + (D + 1)(D + 2)\, y = (D + 2)(0)$…….(4)

(3) –(4)$\Rightarrow$ $-4y - (D^2 + D + 2D + 2)\, y = 2e^{2t}$

(ie) $-[D^2 + 3D + 2 + 4]y = 2e^{t}$

$(D^2 + 3D + 6)y = 2e^t$ is an differential equation with constant co-efficients in y.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is

$m^2+3m + 6 = 0$

$$\therefore \text{ C.F.} = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)$$

$$\text{P.I.} = \frac{1}{D^2+3D+6}(-2e^t) = -\frac{2}{1+3(1)+6}e^t$$

$$= -\frac{2}{10}e^t = -\frac{1}{5}e^t$$

$$\therefore y = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t$$

$$Dy = D\left[e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t\right]$$

Substitute Dy in (2)

$$(2) \Rightarrow (D+1)y + 2x = 0$$

$$Dy + y + 2x = 0 \Rightarrow 2x = -Dy - y.$$

$$2x = -\left[e^{\frac{-3}{2}t}D\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) + D(e^{\frac{-3}{2}1})\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)\right]$$

$$+\frac{1}{5}D(e^t) - \left[e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2} + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t\right]$$

$$= e^{\frac{-3}{2}t}\left(-\frac{\sqrt{15}}{2}A\sin\frac{\sqrt{15}}{2}t+\frac{\sqrt{15}}{2}B\cos\frac{\sqrt{15}}{2}t\right)$$

$$+\frac{3}{2}e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)+\frac{1}{5}e^{t}$$

$$-e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)+\frac{1}{5}e^{t}$$

$$= e^{\frac{-3}{2}t}\frac{\sqrt{15}}{2}\left(B\cos\frac{\sqrt{15}}{2}t-A\sin\frac{\sqrt{15}}{2}t\right)+\frac{1}{2}e^{\frac{-3}{2}e^{t}}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)+\frac{2}{5}e^{t}$$

$$= e^{\frac{-3}{2}t}\left[\cos\frac{\sqrt{15}}{2}t\left(\frac{\sqrt{15}}{2}B+\frac{A}{2}\right)+\sin\frac{\sqrt{15}}{2}t\left(\frac{B}{2}-\frac{A\sqrt{15}}{2}\right)\right]+\frac{2}{5}e^{t}$$

$$\therefore x = \frac{1}{2}e^{\frac{-3}{2}t}\left[\cos\left(\frac{\sqrt{15}}{2}t\right)\left(\frac{A+B\sqrt{15}}{2}\right)+\sin\left(\frac{\sqrt{15}}{2}t\right)\left(\frac{B-A\sqrt{15}}{2}\right)\right]+\frac{1}{5}e^{t}$$

$\therefore$ The solution are

$$x = \frac{1}{2}e^{\frac{-3}{2}t}\left[\cos\left(\frac{\sqrt{15}t}{2}\right)\left(\frac{A+B\sqrt{15}}{2}\right)+\sin\left(\frac{\sqrt{15}t}{2}\right)\left(\frac{B-A\sqrt{15}}{2}\right)\right]+\frac{1}{5}e^{t}$$

$y = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)-\frac{1}{5}e^{t}$, where A and B are arbitrary constants.

12. Solve $D^2x + y = \sin t$; $D^2y + x = \cos t$

Solution: Given $D^2x + y = \sin t \rightarrow (1)$

$x + D^2y = \cos t \rightarrow (2)$

First we can be eliminate x from (1) and (2)

(2) $\times D^2 \Rightarrow$ $D^2x + D^4y = D^2(\cos t) = D(-\sin t) = -\cos t$

$\not{D^2x} + D^4y = -\cos t$…………………. (3)

$\not{D^2x} + y = \sin t$………………………. (1)

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(3) – (1) ⇒ $(D^4 - 1)\, y = \cos t - \sin t$

is a differential equation in y with constant co-efficients.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is

$m^4 - 1 = 0$

(i.e) $(m^2 + 1)(m^2 - 1) = 0$

$\Rightarrow m^2 = -1,\ m^2 = 1$

$\Rightarrow m = \pm i,\ \ m = \pm 1.$

$\therefore$ C.F. $= C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

$$\text{P.I.} = \frac{1}{D^4 - 1}(-\cos t - \sin t)$$

$$= \frac{-1}{D^4 - 1}\cos t - \frac{1}{D^4 - 1}(\sin t)$$

$$= \frac{t}{4D^3}\cos t - \frac{1}{4D^3}(\sin t)$$

$$= \frac{t}{4}\int \left\{\int \left(\int \cos t\,dt\right)dt\right\}dt - \frac{t}{4}\int \left\{\int \left(\int \sin t\,dt\right)dt\right\}dt$$

$$= \frac{t}{4}\int \left\{\int \sin t\,dt\right\}dt - \frac{t}{4}\int \left\{\int -\cos t\,dt\right\}dt$$

$$= -\frac{t}{4}\int -\cos t\,dt - \frac{t}{4}\int -\sin t\,dt$$

$$= +\frac{t}{4}\sin t - \frac{t}{4}\cos t = \frac{t}{4}(\sin t - \cos t)$$

$$\therefore y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4}(\sin t - \cos t)$$

First we can find $Dy = \dfrac{d}{dt}(y)$

$$Dy = C_1e^t - C_2e^{-t} + C_3\sin t - C_4\cos t + \frac{t}{4}(-\cos t - \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$\therefore D^2y = C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(-\sin t - \cos t)$$

$$+\frac{1}{4}(-\cos t - \sin t) + \frac{1}{4}(-\cos t - \sin t)$$

Now (2) $\Rightarrow x = \cos t - D^2 y$

$$x = \cos t - \left[C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(\cos t - \sin t) - \frac{2}{4}(\cos t + \sin t)\right]$$

$$= \cos t - C_1e^t - C_2e^{-1} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t) + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{3}{2}\cos t + \frac{1}{2}\sin t - C_1e^t - C_2e^{-t} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t)$$

$\therefore$ The solutions are

$$x = -\frac{3}{2}\cos t + \frac{1}{2}\sin t - C_1e^t - C_2e^{-t} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t)$$

$$y = C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(\sin t - \cos t)$$

Where C_1, C_2, C_3 and C_4 are arbitrary constants.

Exercise

1. Solve $\frac{dx}{dt} + 2dy = -\sin t; \quad \frac{dy}{dt} - 2x = \cos t$

2. Solve $Dx - (D-2)y = \cos t; (D-2)x + Dy = \sin 2t$

3. Solve $(2D+1)x + (3D+1)y = e^t; (D+5)x + (D+7)y = 2e^t$

4. Solve $\frac{dx}{dt} + 2x + 3y = 0; \quad 3x + \frac{dy}{dt} + 2y = 2e^{2t}$

5. Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}; \quad \frac{dy}{dt} + 3x + 2y = 0$

6. Solve $\frac{dx}{dt} + y = -\sin t; \quad \frac{dy}{dt} + x = \cos t$

7. Solve $\frac{dx}{dt} + 2x - 3y = 5t; \quad \frac{dy}{dt} - 3x + 2y = 2e^{2t}$

8. Solve $\frac{dx}{dt} + 2y = \sin 2t; \quad \frac{dy}{dt} - 2x = \cos 2t$

9. Solve $\frac{dx}{dt} + 2y = -5e^{t}; \quad \frac{dy}{dt} - 2x = 5e^{t}$

10. Solve $Dx + 3y = \cos t; \quad 2x + 5Dy = \sin t$

11. Solve $\frac{dx}{dt} + 5x - 2y = t; \quad \frac{dy}{dt} + 2x + y = 0$ $given\ x = y = 0\ \ when\ \ t = 0$

12. Solve $\frac{dx}{dt} + 2y = 5e^{t}; \quad \frac{dy}{dt} - 2x = 5e^{t},$ $given\ x = y = 0\ \ when\ \ t = 0$

13. Solve $(D+4)x + 3y = 0; \ 2x + (D+5)y = 0$

14. Solve $\frac{dx}{dt} + 2x - 3y = t; \quad \frac{dy}{dx} - 3x + 2y$

15. Solve $\frac{dx}{dt} + 5x + y = e^{t}; \quad \frac{dy}{dt} + 3y - x = e^{2t}$

16. Solve $\frac{dx}{dt} - 2x + Dy = \sin 2t; \ Dx - (D-2)y = \cos 2t$

Answers

1. $x = A\cos 2t + B\sin 2t - \cos t; \ \ y = A\sin 2t - B\cos 2t - \sin t$

2. $x = e^{t}(A\sin t - B\cos t) - \frac{1}{2}\sin 2t$

$y = e^{t}(A\sin t - B\cos t) - \frac{1}{2}\sin 2t$

3. $x = Ae^{-2t} + Be^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{2}e^{2t}$

$y = -\frac{2}{3}Ae^{-2t} + Be^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{1}{6}e^{2t}$

4. $x = Ae^{t} + Be^{-5t} + \frac{6}{7}e^{2t}$; $y = Be^{-5t} - Ae^{t} + \frac{8}{7}e^{2t}$

5. $x = Ae^{-5t} + Be^{t} + \frac{8}{7}e^{2t}$; $y = Ae^{-5t} - Be^{t} - \frac{6}{7}e^{2t}$

6. $x = Ae^{t} + Be^{-t}; + y = Be^{-t} - Ae^{t} + \sin t$

7. $x = Ae^{t} - Be^{-5t} + \frac{6}{7}e^{2t} - 2t - \frac{13}{5}$

$y = Ae^{t} - Be^{-5t} + \frac{8}{7}e^{2t} - 3t - \frac{12}{5}$

8. $x = -A\sin 2t + B\cos 2t - \frac{1}{2}\cos 2t$

$y = A\cos 2t + B\sin 2t$

9. $x = B\cos 2t - A\sin 2t - e^{t}$

$y = A\cos 2t - B\sin 2t - 3e^{t}$

10. $x = \left[\frac{1}{2}\sin t - 5\left(A\frac{\sqrt{6}}{5}e^{\frac{\sqrt{6}}{5}t} - B\frac{\sqrt{6}}{5}e^{\frac{\sqrt{6}}{5}t} - \frac{1}{11}\sin t\right)\right]$

$y = Ae^{\frac{\sqrt{6}}{5}t} + Be^{\frac{\sqrt{6}}{5}t} + \frac{1}{11}\cos t$

11. $x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t)$

$y = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t)$

12. $x = -3\cos 2t - \sin 2t + 3e^{t}$

$y = \cos 2t + 3\sin 2t - e^{t}$

13. $x = Ae^{-2t} + Be^{-7t}$

$$y=\frac{1}{3}\left[3Be^{-7t}-2Ae^{-2t}\right]$$

14. $$x=Ae^{-5t}+Be^{t}+\frac{1}{5}-\frac{2}{25}[5t-4]+\frac{1}{6}te^{t}$$

$$y=\frac{1}{3}\left[-3Ae^{-5t}+3Be^{t}+\frac{1}{2}te^{t}-\frac{4}{25}(5t-4)-t\right]$$

15. $$x=(At+B)e^{-4t}+\frac{4}{25}e^{t}-\frac{e^{2t}}{36}$$

$$y=-At^{-4t}-e^{-4t}(At+B)+\frac{e^{t}}{25}+\frac{7}{36}e^{2t}$$

16. $$x=e^{t}(A\cos t+B\sin t)-\frac{1}{2}\cos 2t$$

$$y=e^{4}(A\sin t-B\cos t)-\frac{1}{2}\sin 2t$$

17. $$x=-\frac{3}{2}Ae^{-2t}+Be^{-7t}-\frac{1}{12}e^{2t}$$

$$y=Ae^{-2t}+Be^{-7t}+\frac{1}{6}e^{2t}$$

18. $$x=(A+Bt)e^{t}+Ce^{-\frac{3t}{2}}-\frac{t}{2}$$

$$y=(-2A+6B-2Bt)e^{t}-\frac{c}{3}e^{\frac{3t}{2}}-\frac{1}{3}$$

19. $$x=(At+B)\cos t+(ct+D)\sin t+\frac{1}{25}e^{t}(4\sin t-3\cos t)$$

$$y=-(At+B)\sin t+(ct+D)\cos t-\frac{1}{25}e^{t}(3\sin t-4\cos t)$$

20. $$x=Ae^{2t}+Be^{-2t}+c\cos 2t+D\sin 2t-\frac{4}{15}\sin t+\frac{3}{16}t$$

$$y=\frac{A}{3}e^{2t}+\frac{B}{3}e^{-2t}+3c\cos 2t+3D\sin 2t-\frac{1}{5}\sin t+\frac{5}{16}t$$

METHOD OF VARIATION OF PARAMETERS

This method is very useful in finding the general solution of the second order equation.

$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=X$ where a_1, a_2 are constants and 'X' is a function of x .

The complementary function is

C.F $= Af_1 + Bf_2$

Where A, B are constants and f_1 and f_2 are functions of x . Then particular Integral,

$P.I. = Pf_1 + Qf_2$

Where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx, \quad Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx$

The complete solution is

$y = Af_1 + Bf_2 + P.I.$

1. Solve $\frac{d^2y}{dx^2}+y=\operatorname{cosec} x$ by using method of variation of parameters.

Solution:Given $\frac{d^2y}{dx^2}+y=\operatorname{cosec} x$

(i.e) $(D^2+1)y=\operatorname{cosec} x$

The Auxiliary Equation is

$m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$ [imaginary roots]

$\therefore$ C.F. $= A\cos x + B\sin x$

Here $f_1=\cos x$; $f_2=\sin x$;

$f_1'= -\sin x$; $f_2'=\cos x$

$f_1f_2' - f_1'f_2= \cos^2x + \sin^2x = 1$

P.I. $= Pf_1 + Qf_2$

where

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\sin x \operatorname{cosec} x}{1} dx \qquad [\because x = \operatorname{cosec} x]$$

$$= -\int \sin x . \frac{1}{\sin x} dx$$

$$= -\int dx$$

$$= -x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x . \operatorname{cosec} x ds}{1}$$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \int \cot x dx$$

$$= \log(\sin x)$$

$\therefore P.I = Pf1 + Qf2$

$= -x \cos x + \sin x \log(\sin x)$

$\therefore y = C.F + P.I$

$= A \cos x + B \sin x - x \cos x + \sin x \log(\sin x)$

2. Solve $\frac{d^2y}{dx^2} + y = \tan x$ by the method of variation of parameters.

Solution:Given $(D^2 + 1)y = \tan x$

The Auxiliary eqn is

$m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

C.F = $A \cos x + B \sin x$

$f_1 = \cos x;$ $f_2 = \sin x;$

$f_1' = -\sin x;$ $f_2' = \cos x;$

$f_1 f_2' - f_1' f_2 = 1$

$P.I. = Pf_1 + Qf_2$

$$P = \int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx = -\int \frac{\sin x . \tan x}{1} dx$$

$$= -\int \frac{\sin x . \sin x}{\cos x} dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx$$

$$= -\int \frac{1}{\cos x} dx + \int \frac{\cos^2 x}{\cos x} dx$$

$$= \int -\sec x + \int \cos x dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \tan x}{1} dx$$

$$= \int \cos x \frac{\sin x}{\cos x} dx$$

$$= \int \sin x \, dx$$

$= -\cos x$

P.I $= Pf_1 + Qf_2$

$= [-\log(\sec x + \tan x) + \sin x] \cos x + (-\cos x) \sin x$

$= -\cos x \log(\sec x + \tan x) + \sin x \cos x - \sin x \cos x$

$\therefore$ P.I $= -\cos x \log(\sec x + \tan x)$

$\therefore y \quad = \text{C.F} + \text{P.I}$

$= A \cos x + B \sin x - \cos x \log (\sec x + \tan x)$

3. Solve $(D^2 + 4)\, y = \tan 2x$ using the method of variation of parameters.

Solution:Given $(D^2 + 4)\, y = \tan 2x$

The Auxiliary Equation is $m^2 + 4 = 0$

$m = \pm\, 2i$

$\text{C.F} = A \cos 2x + B \sin 2x$

$f_1 = \cos x;$ $\qquad f_2 = \sin x;$

$f_1' = -2 \sin x;$ $\qquad f_2' = 2\cos x;$

$f_1 f_2' - f_1' f_2 = 2 \cos^2 2x + 2 \sin^2 2x = 2$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1^1 f_2} dx$$

$$= -\int \frac{\sin 2x . \tan 2x}{2} dx$$

$$= \frac{1}{2}\int \sin 2x \times \frac{\sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{1}{\cos 2x} dx + \frac{1}{2}\int \frac{\cos^2 2x dx}{\cos 2x} dx$$

$$= -\frac{1}{2}\left[\frac{1}{2}\log(\sec 2x + \tan 2x)\right] + \frac{1}{2}.\frac{1}{2}\sin 2x$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x$$

$$Q = \int \frac{f_1 X}{f_1 f' - f f_2} dx$$

$$= \int \frac{\cos 2x}{2} \tan 2x dx$$

$$= \frac{1}{2} \int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= \frac{1}{2} \int \sin 2x dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{4}\cos 2x$$

P.I $= Pf_1 + Qf_2$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x) + \frac{1}{4}\cos 2x \sin 2x$$

$$\frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

y = C.F. + P.I

$$= A\cos 2x + B\sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

4. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ using method of variation of parameters.

Solution:Given $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

i.e $(D^2 + 4)y = 4\tan 2x$

$m^2 + 4 = 0$

$m = \pm 2i = 0 \pm 2i$ [imaginary root $\alpha = 0$, $ß = 2$]

C.F = e^{0x} (A cos $2x$ + B sin $2x$)

Now C.F = A cos $2x$ + B sin $2x$

Here f_1 = cos $2x$; f_2 = sin $2x$;

f' = –2 sin $2x$; f' = 2cos $2x$;

$f_1 f_2' - f' f_2$ = 2 cos2$x$cos2x+ 2 sin $2x$ sin $2x$

= 2 (cos^2 $2x$ + sin^2 $2x$) = 2

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin 2x 4 \tan 2x}{2} dx \qquad \because x = 4 \tan 2x$$

$$= -2 \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= 2\left[\int \frac{1}{\cos 2x} dx - \int \frac{\cos^2 2x}{\cos 2x} dx\right]$$

$$= -2\left[\int \sec 2x dx - \int \cos 2x dx\right]$$

$$= -2\left[\frac{1}{2}\log(\sec 2x + \tan 2x\right] + 2\frac{\sin 2x}{2}$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$= \sin 2x - \log(\sec 2x + \tan 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f'_2 - f_1' f_2} dx$$

$$= \int \frac{\cos 2x 4 \tan 2x}{2} dx$$

$= 2\int \cos 2x \dfrac{\sin 2x}{\cos 2x} dx$

$= 2\int \sin 2x dx$

$= 2\left(\dfrac{-\cos 2x}{2}\right)$

$= -\cos 2x$

P.I $= Pf_1 + Qf_2$

$= \cos 2x[\sin 2x - \log(\sec 2x + \tan 2x)]$

$= -\cos 2x \sin 2x$

$= -\cos 2x \log(\sec 2x + \tan 2x)]$

$\therefore y$ = C.F + P.I

$= (A\cos 2x + B\sin 2x) - \cos 2x \log(\sec 2x + \tan 2x)$

5. Solve $(D^2 + 2D + 5)y = e^{-x} \tan x$, by method of variation of parameters.

Solution:Given $(D^2 + 2D + 5)y = e^{-x} \tan x$,

The Auxiliary Equation is

$m^2 + 2m + 5 = 0$

$m = \dfrac{-2 \pm \sqrt{4-20}}{2}$

$= \dfrac{-2 \pm 4i}{2}$

$= -1 \pm 2i$

C.F $= e^{-x} (A \cos 2x + B \sin 2x)$

Here $f_1 = e^{-x} \cos 2x$

$f'_1 = -2e^{-x} \sin 2x - e^{-x} \cos 2x$

$f_2 = e^{-x} \sin 2x$

$f'_2 = 2e^{-x} \cos 2x - e^{-x} \sin 2x$

$\therefore f_1 f'_2 - f_2 f_1' = 2e^{-2x} \cos^2 2x - e^{-x} \sin 2x \cos 2x$

$+ 2e^{-2x} \sin^2 2x + e^{-2x} \sin 2x \cos 2x$

$= 2e^{-2x} (\cos^2 x + \sin^2 x)$

$= 2e^{-2x}$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f_1' f_2} dx$$

$$= -\int \frac{e^{-x} \sin 2x}{2e^{-2x}} e^{-x} \tan x dx \qquad [\because x = e^{-x} \tan z]$$

$$= -\frac{1}{2} \int 2\sin x \cos x \frac{\sin x}{\cos x}$$

$$= -\int \sin^2 x dx$$

$$= -\int \frac{1 - \cos 2x}{2} dx$$

$$= -\frac{1}{2}\left[x - \frac{1 \sin 2x}{2}\right]$$

$$= -\frac{1}{2} x + \frac{\sin 2x}{4}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{e^{-x} \cos 2x}{2e^{-2x}} e^{-x} \tan x dx$$

$$= \frac{1}{2} \int \cos 2x \tan x dx$$

$$= \frac{1}{2} \int (2\cos^2 x - 1) \frac{\sin x}{\cos x} dx \qquad \because \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$= \frac{1}{2} \int 2\cos^2 x \frac{\sin x}{\cos x} - \frac{1}{2} \int \frac{\sin x}{\cos x} dx \qquad 2\cos^2 x - 1 = \cos 2x$$

$$= \frac{1}{2}\int 2\sin x\cos x dx - \frac{1}{2}\int \frac{\sin x}{\cos x} dx$$

$$= \frac{1}{2}\int \sin 2x dx - \frac{1}{2}\int \frac{\sin x}{\cos x} dx$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + \frac{1}{2}\log\cos x$$

$$= -\frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)$$

P.I $= Pf_1 + Qf_2$

$$= -\frac{x}{2} + \frac{\sin 2x}{4} - \frac{\cos 2x}{3} + \frac{1}{2}\log(\cos x)$$

y $= C.F + P.I$

$$= Ae^{-x}\cos 2x + Be^{-x}\sin 2x - \frac{x}{2} + \frac{\sin 2x}{4} - \frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)$$

6. Solve $y'' - 2y' + 2y = e^{x}\tan x$, using method of variation of parameters.

Solution:Given $(D^2 - 2D + 2)y = e^{-x}\tan x$,

The A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

The A.E is $= \frac{2 \pm \sqrt{-4}}{2}$

$$= \frac{2 \pm \sqrt{2i}}{2}$$

$= 1 \pm i$

C.F $= e^{x}(A\cos x + B\sin x)$

$= Ae^{x}\cos x + Be^{x}\sin x$

f_1 $= e^{-x}\cos x - e^{x} \sin x$

f'_2 $= e^{x}\sin x + e^{-x}\cos x$

$f_1f'_2 - f_1'f_2 \quad = e^{2x}\cos x\sin x + e^{2x}\cos^2 x$

$-e^{2x}\cos x\sin x + e^{2x}\sin^2 x$

$= e^{2x}(\cos^2 x + \sin^2 x)$

$= e^{2x}$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f_1' f_2} dx$$

$$= -\int \frac{e^x \sin x}{e^{2x}} e^x \tan x dx$$

$$= -\int \sin x \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx$$

$$= -\int \frac{1}{\cos x} dx + \int \frac{\cos^2 x}{\cos x} dx$$

= -log(secx +tanx) +sinx

7. Solve $(D^2 - 4D + 4)y = e^{2x}$, by the method of variation of

Solution:Given $(D^2 - 4D + 4)y = e^{2x}$

The A.E. is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$ (Equal roots)

C.F. $= (Ax + B)e^{2x}$

$= Axe^{2x} + Be^{2x}$

$= Af_1 + Bf_2$

Here

$f_1 = xe^{2x}$ $f_2 = e^{2x}$

$f'_1 = 2xe^{2x} + e^{2x}$ $f'_2 = 2e^{2x}$

$f_1 f'_2 - f_1' f_2 = 2xe^{4x} - 2xe^{4x} - e^{4x}$

$= -e^{4x}$

P.I. $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{e^{2x}.e^{2x}}{-e^{4x}} dx$$

$$= -\int dx$$

$$= x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{xe^{2x}e^{2x}}{-e^{4x}} dx$$

$$= -\int xdx$$

$$= -\frac{x^2}{2}$$

$$\text{P.I} = x(xe^{2x}) + \frac{-x^2}{2} e^{2x}$$

$$= x^2xe^{2x} - \frac{x^2}{2} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

y = C.F + P.I

$$= (Ax + B)e^{2x} + \frac{x^2}{2} e^{2x}$$

8. Solve $(D^2+4)y = \sec 2x$ by using method of variation of parameters.

Solution:

Given $(D^2+4)y = \sec 2x$

The Auxiliary Equation is $m^2 + 4 = 0$

$m = \pm 2i$

C.F. $= e^{ex}(A\cos 2x + B\sin 2x)$

$= A\cos 2x + B\sin 2x$

Here

$f_1 = \cos 2x$ $\qquad$ $f_2 = \sin 2x$

$f'_1 = 2\sin 2x$ $\qquad$ $f'_2 = 2\cos 2x$

$f_1 f'_2 - f_1' f_2 = 2$

P.I. $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\,dx$$

Put $t = \cos 2x$

$dt = -2\sin 2x dx$

$$\therefore p = -\frac{1}{2}\int \frac{dt}{2t}$$

$$= -\frac{1}{4}\int \log t$$

$p \quad = -\frac{1}{4}\log(\cos 2x)$

$= \frac{1}{4}\log(\cos 2x)$

$Q \quad = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$

$= \int \frac{\cos 2x \sec 2x}{2} dx$

$= \frac{1}{2}\int dx$

$= \frac{1}{2}x$

P.I. $= Pf_1 + Qf_2$

$= \frac{1}{4}\cos 2x \log(\cos 2x) + \frac{1}{2}x\sin 2x$

$\therefore y \quad$ = C.F + P.I

$= A\cos 2x + B\sin 2x + \frac{1}{4}\cos 2x \log(\cos 2x) + \frac{1}{2}x\sin 2x$

9. Solve $(D^2+1)y = \sec x$ by the method of variation of parameters.

Solution:Given $y'' + y = \sec x$

(ie) $(D^2+1)y = \sec x$

The Auxiliary Equation is $m^2 + 1 = 0$

$m = \pm i$

C.F. $= A\cos x + B\sin x$

$= Af_1 + Bf_2$

Here $f_1 = \cos x$ $\qquad f_2 = \sin x$

$f_1' = -\sin x$ $\qquad f_2' = \cos x$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x \sec x}{1} dx$$

$$= -\int \frac{\sin x}{\cos x} dx$$

$= \log(\cos x)$

put $t = \cos x$

$P = \log t$

P.I $= Pf_1 + Qf_2$

$$\therefore P = -\int -\frac{dt}{t}$$

$= \log t$

$= \log(\cos x)$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \sec 2x}{1} dx$$

$\therefore$ P.I $= \cos x \log(\cos x) + x \sin x$

y = C.F + P.I

$= A\cos x + B\sin x + \cos x \log(\cos x) + x \sin x$

10. Solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Solution:Given $(D^2 + a^2)y = \tan ax$

The Auxiliary Equation is $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

C.F. $= e^{ax}(A \cos ax + B \sin ax)$

$f_1 = \cos ax$ $f_2 = \cos ax$

$f'_1 = - a \sin ax$ $\qquad\qquad$ $f'_2 = a \cos ax$

$f_1 f'_2 - f_1' f_2 = a \cos^2 ax = a \sin^2 ax = a$

P.I $=$ $Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin ax \tan ax}{a} dx$$

$$= -\frac{1}{a}\int \sin ax \times \frac{\sin ax}{\cos ax} dx$$

$$= -\frac{1}{a}\int \frac{(1-\cos^2 ax)}{\cos ax} dx$$

$$= -\frac{1}{a}\int \left[\frac{1}{\cos ax} - \frac{\cos^2 ax}{\cos ax}\right] dx$$

$$= -\frac{1}{a}\int [\sec ax - \cos ax] dx$$

$$= -\frac{1}{a}\int \sec ax dx \times \frac{1}{a}\int \cos ax dx$$

$$= -\frac{1}{a}\left(\frac{1}{a}\log(\sec ax + \tan ax)\right) + \frac{1}{a}\left[\frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos ax \tan ax}{a} dx$$

$$= \frac{1}{a}\int \sin ax \, dx$$

$$= \frac{1}{a} - \left[\frac{\cos ax}{a}\right] = -\frac{1}{a^2}\cos ax$$

P.I $= Pf_1 + Qf_2$

$$= \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax \cos ax - \frac{1}{a^2}\sin ax \cos ax$$

$$= \frac{1}{a^2}[\cos ax \log(\sec ax + \tan ax)]$$

y = C.F + P.I

$$= A\cos ax + B\sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

11. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = x\sin x$

Solution:Given $\frac{d^2y}{dx^2} + y = x\sin x$

(ie) $(D^2+1)y = x\sin x$

The Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

C.F. = $A \cos x + B \sin x$

$f_1 = \cos x$ $\quad f_2 = \sin x$

$f'_1 = -\sin x$ $\quad f'_2 = \cos x$

$f_1f'_2 - f_1'f_2 = 1$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x(x\sin x)}{1} dx$$

$$= -\int x\sin^2 x \, dx$$

$\because u = x \qquad d_v = \cos 2x$

$du = dx \qquad v = \frac{\sin 2x}{2}$

$$= -\int x \sin^2 x\, dx$$

$$= -\int \frac{x}{2} dx \int \frac{x \cos 2x}{2} dx$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right]$$

$$= -\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x (x \sin x)}{1} dx$$

$$= \int x \frac{\sin 2x}{2} dx$$

$\therefore u = x \qquad dv = \sin 2x$

$du = dx \qquad v = -\dfrac{\cos 2x}{2}$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2} + \int \frac{\cos 2x dx}{2} \right]$$

$$= \frac{-x \cos 2x}{4} + \frac{1}{8} \sin 2x$$

P.I $= Pf_1 + Qf_2$

$$= \left[\frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} \right] \cos x + \left[\frac{-x \cos 2x}{4} + \frac{1}{8} \sin 2x \right] \sin x$$

$$= \frac{-x^2}{4} \cos x + \frac{x}{4} \sin 2x \cos x + \frac{\cos x \cos 2x}{8}$$

$$\frac{-x \cos 2x \sin x}{4} + \frac{1}{8} \sin 2x \sin x$$

$$= \frac{-x^2}{4} \cos x + \frac{x}{4} [\sin 2x \cos x - \cos 2x \sin x]$$

$$+\frac{1}{8}[\cos 2x \cos x + \sin 2x \sin x]$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}\sin(2x - x) + \frac{1}{8}\cos(2x - x)$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

$\therefore$ y = C.F + P.I

$$= A\cos x + B\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

$$= \left(A + \frac{1}{8}\right)\cos x + B\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

$$= c_1 \cos x + c_2 \sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

where $c_1 = A + \frac{1}{8} c_2 = B$

12. Solve $\frac{d^2y}{dx^2} + y = \cos ec\, x \cot x$ using the method of variation of parameters.

Solution:Given $(D^2 + 1)y = \cos ec\, x \cot x$

The Auxiliary Equation is $m^2 + 1 = 0$

$m = \pm i$

C.F = $A \cos x + B \sin x$

$f_1 = \cos x$ $f_2 = \sin x$

$f_1 f'_2 - f_1' f_2 = \cos^2 x + \sin^2 x = 1$

P.I = $Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x \cos ecx \cot x}{1} dx$$

$$= -\int \sin x \frac{1}{\sin x} \frac{\cos x}{\sin x} dx$$

$$= \log(\sin x)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \cos x \cos ecx \cot x dx$$

$$= \int \cos x \frac{1}{\sin x} \frac{\cos x}{\sin x} dx$$

$$= \int \frac{1 - \sin^2 x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int dx$$

$$= \int \cos ec^2 x dx - x$$

$$= -\cot x - x$$

P.I $= Pf_1 + Qf_2$

$= -\cos x \log(\sin x) - \sin x \cot x - x \sin x$

$= -\cos x \log(\sin x) - [\cot x + x] \sin x$

y = C.F + P.I

$= A \cos x + B \sin x - \cos x \log(\sin x) - [\cot x + x] \sin x$

Exercie Problems

1. Using method of variation of parameters solve. $(D^2 + 9)y = \sec 3x$
2. Solve $(D^2 + 1)y = \cot x$
3. Solve $(D^2 + 25)y = \tan 5x$
4. Solve $(D^2 + 16)y = \text{cosec } 4x$
5. Solve $(D^2 + 25)y = \sec 5x$
6. Solve $(D^2 + 9)y = \cot 3x$

7. Solve $(D^2-6D+9)y=\frac{e^{3x}}{x}$

8. Solve $(D^2+36)y=\cos ec6x$

9. Solve $(D^2-1y=\frac{1}{1+e^x}$

10. Solve $(D^2-1)y=\frac{2}{1+e^x}$

11. Solve $y''+y=x$

12. Solve $y''-3y'+2y=x^2$

13. Solve $(2D^2-D-3)y=25e^{-x}$

Answers

1. $y=A\cos 3x+B\sin 3x+\frac{1}{9}\cos 3x\log(\cos 3x)+\frac{1}{3}x\sin 3x$

2. $y=A\cos x+B\sin x-\sin x\log(\cos ecx+\cot x)$

3. $y=A\cos 5x+B\sin 5x-\frac{1}{25}[\log(\sec 5x+\tan 5x])-\sin 5x]\cos 5x-\frac{\cos 5x}{25}\sin 5x$

4. $y=A\cos 4x+B\sin 4x-\frac{1}{9}x\cos 4x+\frac{1}{16}[\log(\sin 4x)]\sin 4x$

5. $y=A\cos 5x+B\sin 5x-\frac{1}{25}\cos 5x\log(\sec 5x)+\frac{1}{5}x\sin 5x$

6. $y=A\cos 3x+B\sin 3x-\frac{\sin 3x\cos 3x}{3}+\frac{\cos 3x\sin 3x}{3}$

7. $y=(Ax+B)e^{3x}+\log x(xe^{3x})-xe^{3x}$

8. $y=A\cos 6x+B\sin 6x-\frac{1}{6}x\cos 6x+\frac{1}{36}\sin 6x.\log(\sin 6x)$

9. $y=Ae^x+Be^{-x}+\frac{e^x}{2}[-e^{-x}+\log(1+e^x)-x]-\frac{1}{2}e^{-x}\log(1+e^x)$

10. $y=Ae^x+Be^{-x}-1+e^x\log(1+e^{-x})-e^{-x}\log(1+e^x)$

11. $y = A\cos x + B\sin x + x$

12. $y = Ae^{x} + Be^{-x} + \frac{1}{2}x^{2} - \frac{3}{2}x + \frac{7}{4}$

13. $y = Ae^{\frac{3}{2}x} + Be^{-x} - 2e^{-x} - 5xe^{-x}$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II –VECTOR CALCULUS – SMTA1201

UNIT – II

VECTOR CALCULUS

Scalars

The quantities which have only magnitude and are not related to any direction in space are called scalars. Examples of scalars are (i) mass of a particle (ii) pressure in the atmosphere (iii) Temperature of a heated body (iv) speed of a Train.

Vectors

The quantities which have both magnitude and direction are called Vectors.

Examples of vectors are (i) The gravitational force on a particle in space (ii) The velocity at any point in a moving fluid.

Representation and notation of a Vector

A vector is often denoted by two letters with an arrow over them ie., $\overrightarrow{AB}$, A is called the origin (initial point) and B is the terminus (end point). Its magnitude is given by the length AB and direction is from A to B as indicated by the arrow. We write vector quantities also in single letter like $\vec{a}, \vec{b}, \vec{c},$ and the corresponding letters a, b, c denote their magnitudes.

The magnitude $\left|\vec{a}\right|$ of a vector $\vec{a}$ is called its modulus or module.

Collinear or Parallel Vectors

Two or more vectors are said to be collinear or parallel when they act along the same line or along parallel lines.

Coplanar Vectors

Three or more vectors are said to be coplanar when they are parallel to the same plane or lie in the same plane whatever their magnitude may be.

Unit Vectors

A vector whose magnitude is of unit length is called a unit vector. If $\vec{a}$ is a vector whose magnitude is 'a' then the unit vector in the direction of $\vec{a}$ is denoted by $\hat{n}$ and is obtained by dividing the vector $\vec{a}$ by its magnitude 'a' i.e. $\hat{n} = \frac{\vec{a}}{|\vec{a}|}$

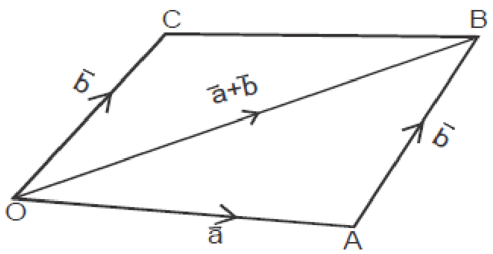
Position Vector

If O be a fixed origin and P any point, then the vector $\overrightarrow{OP}$ is called the position vector of the point P(*x*,*y*,*z*) with respect to the origin *O*(0,0,0).

Addition of Vectors

Let $\vec{a}$ and $\vec{b}$ be any two vectors. Choose any point O as origin and draw the vectors $\vec{a}$ and $\vec{b}$ so that the terminals of $\vec{a}$ coincides with the origin of $\vec{b}$ ie., $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Then the vector given by $\overrightarrow{OB}$ is defined as the sum of vectors $\vec{a}$ and $\vec{b}$.

The above law is called triangle law of addition.



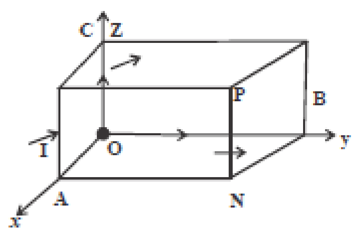
The Unit Vectors $\vec{i}, \vec{j}, \vec{k}$ (Orthonormal system of unit vectors)

Let OX, OY and OZ be three mutually perpendicular straight lines in the right handed orientation. These three mutually perpendicular lines can uniquely determine the position of a point. Hence these lines can be taken as the co-ordinate axes with O as origin. The planes XOY, YOZ and ZOX are called co-ordinate planes.

Let $\overrightarrow{OP}$ represents a vector $\vec{r}$. With OP as diagonal construct a rectangular paralleloiped whose three coterminous edges OA, OB, OC lie along OX, OY and OZ respectively. Let OA = x, OB = y, OC = z. Then $\overrightarrow{OA} = x\vec{i}$, $\overrightarrow{OB} = y\vec{j}$ and $\overrightarrow{OC} = z\vec{k}$. Now, we have

$$\vec{r} = \overrightarrow{op} = \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OA} + \overrightarrow{AN} + \overrightarrow{NP}$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\vec{i} + y\vec{j} + z\vec{k}$$



Thus $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Here x, y, z are called the co-ordinate of the point P referred to the axes OX, OY and OZ. Also $x\vec{i}, y\vec{j}$ and $z\vec{k}$ are called resolved parts of the vector $\vec{r}$ in the direction of $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively. The modulus of $\vec{r}$ is given by $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

Scalar Product or dot Product

Let a $\vec{a}$ and $\vec{b}$ be two vectors. The scalar product or dot product of $\vec{a}$ and $\vec{b}$ is defined to be $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \times \cos\theta$, where θ is the angle between the two vectors when drawn from a common origin.

Note:

(i) $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

(ii) $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$

(iii) $\vec{a}\,.\,\vec{b} = 0$ *if* $\vec{a}$ *and* $\vec{b}$ are perpendicular vectors.

(iv) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

(v) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1.$

(vi) $\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0.$

since $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular vectors.

(vii) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector product or Cross product

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then the vector product or cross product of $\vec{a}$ and $\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ with magnitude ab sin θ. Here $0 \le \theta \le \pi$ is the angle between $\vec{a}$ and $\vec{b}$. The direction is along a unit vector $\hat{n}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ form right handed system. Thus $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$.

Note

(i) $\left|\vec{a} \times \vec{b}\right|$ = area of the parallelogram with sides a and b

(ii) $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$

(iii) $\vec{a} \times \vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are parallel.

(iv) $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right)$

(v) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$ ($\therefore$ Parallel Vector)

(vi) $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$

(vii) $\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}$

(viii) $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $b = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$,

then $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Definition

The scalar triple product or box product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined to be the scalar $\vec{a}.\left(\vec{b} \times \vec{c}\right)$. It is usually denoted by $\left[\vec{a}, \vec{b}, \vec{c}\right]$.

It can be easily verified that

$$\vec{a}.\vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ and $c = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Note

(i) $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ represents the volume of the parallelepiped formed by the co-terminus edges $\vec{a}, \vec{b}$ and $\vec{c}$.

(ii) $[\vec{a},\vec{b},\vec{c}]=[\vec{b},\vec{c},\vec{a}]=[\vec{c},\vec{a},\vec{b}]$

(iii) $[\vec{a},\vec{b},\vec{c}]=-[\vec{b},\vec{a},\vec{c}]=-[\vec{c},\vec{b},\vec{a}]=-[\vec{a},\vec{c},\vec{b}]$

(iv) The vector $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c}]=0$.

Results

(i) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

(ii) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

(iii) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

(iv) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\vec{c}-(\vec{a},\vec{b}.\vec{c})\vec{d}$

In this chapter, we introduce a vector differential operator which is used to obtain gradient of a scalar valued function, divergence and curl of a vector valued function and discuss briefly the properties arising out of these concepts. We see the general rules for differentation of a vector functions.

Rules

If $\vec{a},\vec{b}$ are vector functions of a scalar 't' and 'ϕ' is a scalar function of 't', then

(i) $\frac{d}{dt}(\vec{a}\pm\vec{b})=\frac{d\vec{a}}{dt}+\frac{d\bar{b}}{dt}$

(ii) $\frac{d}{dt}(\vec{a}.\vec{b})=\vec{a}.\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}.\vec{b}$

(iii) $\frac{d}{dt}(\vec{a}\times\vec{b})=\vec{a}\times\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}\times\vec{b}$

(iv) $\frac{d}{dt}(\phi\vec{a})=\phi\frac{d\vec{a}}{dt}+\frac{d\phi}{dt}\vec{a}$

Scalar Point function

If to each point P (x,y,z) of a region R in space there corresponds a unique scalar f(P) then f is called a scalar point function.

Example

The temperature distribution in a heated body, density of a body and potential due to a gravity.

Vector point function

If to each point P (x,y,z) of a region R in space there corresponds a unique vector $\vec{f}$ (P), then $\vec{f}$ is called a vector point function.

Example

The velocity of a moving fluid, Gravitational force.

Vector differential operator (∇)

The vector differential operator Del, denoted by ∇is defined as

$$\nabla = \vec{i}\,\frac{\delta}{\delta x} + \vec{j}\,\frac{\delta}{\delta y} + \vec{k}\,\frac{\delta}{\delta z}$$

Level Surface

Let the surface ϕ(x,y,z) = c passes through a point P. If the value of the function at each point on the surface is the same as at P, then such a surface is called a level surface through P.

Example

ϕ (x,y,z) represents potential at the point P, then equipotential surface ϕ(x,y,z) =c is a level surface.

Gradient of a scalar point function

Let ϕ(x,y,z) be a scalar point function defined in a region R of space. Then the vector point function given by

$$\nabla\phi = \left(\vec{i}\,\frac{\delta}{\delta x} + \vec{j}\,\frac{\delta}{\delta y} + \vec{k}\,\frac{\delta}{\delta z}\right)\phi$$

$= \vec{i}\,\frac{\delta\phi}{\delta x} + \vec{j}\,\frac{\delta\phi}{\delta y} + \vec{k}\,\frac{\delta\phi}{\delta z}$ is defined as the gradient of $\phi$and denoted as grad ϕ.

Directional Derivative : (D.D)

The directional derivative of a scalar point function ϕ at point (x,y,z) in the direction of a vector $\vec{a}$ is given by

$$D.D = \nabla\phi.\frac{a}{|\vec{a}|}$$

Definition

The unit vector normal to the surface ϕ (x,y,z) = c is given by $\frac{\nabla\phi}{|\nabla\phi|}.\hat{n}$

Definition

The Directional derivative at a point is maximum in the direction of the normal to the level surface at P and its magnitude is $|\nabla\phi|$ (ie) maximum of $D.D = |\nabla\phi|$

Definition

Angle between the normal to surface is given by $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Example: 1

If ϕ $(x,y,z) = x^2y - 2y^2z^3$ find $\nabla\phi$ at the point (1, -1, 2)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy, \frac{\partial\phi}{\partial y} = x^2 - 4yz^3, \frac{\partial\phi}{\partial z} = -6y^2z^2$$

$$\therefore \nabla\phi = 2xy\vec{i} + (x^2 - 4yz^3)\vec{j} - 6y^2z^2\vec{k}$$

$$\nabla\phi_{(1,-1,2)} = 2(1)(-1)\vec{i} + (1 - 4(-1)(2)^3\vec{j} - 6 - (-1)^2(2)^2\vec{k}$$

$$= 2\vec{i} + 33\vec{j} - 24\vec{k}$$

Example: 2

If $\phi(x,y,z) = x^2yz^3$ find $\nabla\phi$ at the point (1, 1, 1)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3, \frac{\partial\phi}{\partial y} = x^2z^3, \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\therefore \nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} - 3x^2yz^2\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

Example: 3

Find the unit vector normal to the surface $x^2y + 2xz^2 = 8$ at (1,0,2).

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2, \quad \frac{\partial\phi}{\partial y} = x^2, \quad \frac{\partial\phi}{\partial z} = 4xz$$

$$\nabla\phi = (2xy + 2z^2)\vec{i} + x^2\vec{j} + 4xyz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2 + 1^2 + 8^2} = \sqrt{129}$$

Since unit vector normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{|\sqrt{129}|}$$

Example: 4

Find the unit vector normal to the surface $z = x^2 + y^2$ at the point (-1,-2,-5).

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Given $z = x^2 + y^2 \Rightarrow \phi(x, y, z) = x^2 + y^2 - z$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} - 1\vec{k}$$

$$\nabla\phi_{(-1,-2,-5)} = 2(-1)\vec{i} + 2(-2)\vec{j} - \vec{k}$$

$$= 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (-4)^2 + (-1)^2} = \sqrt{21}$$

$\therefore$ Unit vector normal to the surface is $\hat{n} = \dfrac{-2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$

Example: 5

Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2,-1,2).

Solution:

Given the surface $\phi_1(x,y,z) = x^2 + y^2 + z^2 - 9$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi_1}{\partial x} = 2x, \quad \frac{\partial\phi_1}{\partial y} = 2y, \quad \frac{\partial\phi_1}{\partial z} = 2z$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} + 2(2)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$$

Given the surface $\phi_2(x,y,z) = x^2 + y^2 - z^2 - 9$

$$\frac{\partial\phi_2}{\partial x} = 2x, \quad \frac{\partial\phi_2}{\partial y} = 2y, \quad \frac{\partial\phi_2}{\partial z} = -1$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_{2(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} - \vec{k} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_2| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$$

Since the angle between the surfaces

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}).(4\vec{i} - 2\vec{j} - \vec{k})}{6\times 3}$$

$$= \frac{16+4-4}{18} = \frac{16}{18}$$

$$\theta = \cos^{-1}\left(\frac{8}{9}\right)$$

Example 6

Find the angle between the normal to the surface $xy - z^2 = 0$ at the points (1,4,-2) and (-3,-3,3).

Solution:

Given the surface $\phi(x,y,z) = xy - z^2$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \frac{\partial\phi}{\partial z} = -2z$$

$$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\nabla\phi_1 = \nabla\phi_{(1,4,-2)} = 4\vec{i} + \vec{j} - 2(-2)\vec{k}$$

$$\nabla\phi_1 = 4\vec{i} + \vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$$

$$\nabla\phi_2 = \nabla\phi_{(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 2(3)\vec{k}$$

$$\nabla\phi_2 - 3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$|\nabla\phi_2| = \sqrt{(-3)^2 + (-3)^2 + (-6)^2} = \sqrt{54}$$

Since the angle between the normal is

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{-12 - 3 - 24}{9\sqrt{22}} \quad \Rightarrow \cos\theta = \frac{-39}{9\sqrt{22}}$$

$$\Rightarrow \cos\theta = \frac{-13}{3\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{-13}{3\sqrt{22}}\right)$$

$$\Rightarrow \theta = \cos^{-1} = \left(\frac{13}{3\sqrt{22}}\right)$$

Example 7

Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point (1,-2,-1) in the directional of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given the surface $\phi(x,y,z) = x^2yz + 4xz^2$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2 z, \quad \frac{\partial\phi}{\partial z} = x^2 y + 8xy$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2 z\vec{j} + (x^2 y + 8xy)\vec{k}$$

$$\nabla\phi_{(1,-2,-1)} = (2(1)(-2)(-1) + 4(-1)^2\vec{i} + 1^2(-1)\vec{j} + [1^2(-2) + 8(1)(-1)]\vec{k}$$

$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

To find the Directional Derivative of ϕ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$

Find the unit vector along the direction

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k} \Rightarrow |\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

Directional Derivative along the direction $\vec{a}$ at the point (1,-2,-1) = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

$$= (8\vec{i} - \vec{j} - 10\vec{k}).\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{16 + 1 + 20}{3}$$

$$= \frac{37}{3} \text{ units.}$$

Example 8

Find the directional derivative of $\phi\ (x,y,z) = xy^2 + yz^3$ at (1,-1,2) towards the point (2,1,-1).

Solution:

Given the surface $\phi\ (x,y,z) = xy^2 + yz^3$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y^2, \quad \frac{\partial \phi}{\partial y} = 2xy + z^3, \quad \frac{\partial \phi}{\partial z} = 3yz^2$$

$$\nabla \phi = y^2 \vec{i} + (2xy + z^3)\vec{j} + 3yz^2 \vec{k}$$

$$\nabla \phi_{(1,-1,2)} = (-1)^2 \vec{i} + [2(1)(-1) + 2^3]\vec{j} + 3(-1)2^2 \vec{k}$$

$$\nabla \phi = \vec{i} + 6\vec{j} - 12\vec{k}$$

To find the Directional derivative along the point P (1,-1,2) towards the point Q(2,1,-1)

Find the Vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= (2\vec{i} + \vec{j} - \vec{k}) - (\vec{i} - \vec{j} + 2\vec{k})$$

$$\overrightarrow{PQ} = \vec{i} + 2\vec{j} - 3\vec{k}$$

$$\overrightarrow{PQ} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

Directional Derivative = $\nabla \phi . \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}$

$$= (\vec{i} + 6\vec{j} - 12\vec{k}) . \frac{(\vec{i} + 2\vec{j} - 3\vec{k})}{\sqrt{14}}$$

$$= \frac{1 + 12 + 36}{\sqrt{14}}$$

$$= \frac{49}{\sqrt{14}} \text{ Units.}$$

Example 9

Find the directional derivative of the scalar function $\phi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3,1,3).

Solution:

Given the surface $\phi = (x,y,z) = xyz$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = xz, \quad \frac{\partial\phi}{\partial z} = xy$$

$$\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\nabla\phi_{(3,1,3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$$

Given the surface $\phi_1\ (x,y,z) = xy - z$

$$\frac{\partial\phi_1}{\partial x} = y, \frac{\partial\phi_1}{\partial y} = x, \frac{\partial\phi_1}{\partial z} = -1$$

The normal to the surface is

$$\nabla\phi_1 = y\vec{i} + x\vec{j} - \vec{k}$$

$$\nabla\phi_{(3,1,3)} = \vec{i} + 3\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$$

Directional derivative of $\phi(x,y,z)$ along the direction of outward to the surface $\phi_1(x, y, z)$ at (3,1,3) is

Directional Derivative = $\nabla\phi . = \dfrac{\nabla\phi_1}{|\nabla\phi_1|}$

$$= (3\vec{i} + 9\vec{j} + 3\vec{k}).\frac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$$

$$= \frac{3 + 27 - 3}{\sqrt{11}}$$

$$= \frac{27}{\sqrt{11}}$$ Units.

Example 10

Find the maximum value of the directional derivative of $\phi = x^3yz$ at the point (1,4,1).

Solution:

Given the surface $\phi_1(x, y, z) = x^3yz$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2yz \quad \frac{\partial\phi}{\partial y} = x^3z \quad \frac{\partial\phi}{\partial z} = x^3y$$

$$\nabla\phi = 3x^2yz\vec{i} + x^3z\vec{j} + x^3y\vec{k}$$

$$\nabla\phi_{(1,4,1)} = 3(1)^2(4)(1)\vec{i} + (1)^3(1)\vec{j} + (1)^3(4)\vec{k}$$

$$\nabla\phi_{(1,4,1)} = 3(1)^2(4)(1)\vec{i} + (1)^3(1)\vec{j} + (1)^3(4)\vec{k}$$

$$\nabla\phi = 12\vec{i} + \vec{j} + 4\vec{k}$$

Since the maximum of the Directional derivative $|\nabla\phi|$

$\therefore$ Maximum Directional Derivative at (1,4,1) = $\sqrt{12^2 + 1^2 + 4^2}$

= $\sqrt{161}$

Example 11

In what direction from the point (1, 1, -2) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum? Also find the value of the maximum directional derivative.

Solution:

Given the surface $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -4y, \quad \frac{\partial\phi}{\partial z} = 8z$$

$$\nabla\phi = 2x\vec{i} - 4y\vec{j} + 8z\vec{k}$$

$$\nabla\phi_{(1,1,-2)} = 2\vec{i} - 4\vec{j} - 16\vec{k}$$

Maximum of the directional derivative at the point (1,1,-2) = $|\nabla\phi|$

$$= \sqrt{2^2 + (-4)^2 + (-16)^2}$$

$$= \sqrt{4 + 16 + 256}$$

$$= \sqrt{276}$$

Example 12

Find the Directional Derivative of $\phi = xy + yz + zx$ at the point (1,2,3) along the x-axis.

Solution:

Given the surface $\phi(x, y, z) = xy + yz + zx$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y + z, \quad \frac{\partial\phi}{\partial y} = x + z, \quad \frac{\partial\phi}{\partial z} = x + y$$

$$\nabla\phi = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = (2+3)\vec{i} + (1+3)\vec{j} + (1+2)\vec{k}$$

$$= 5\vec{i} + 4\vec{j} + 3\vec{k}$$

Directional Derivative of ϕ along the direction of x-axis at the point (1,2,3)

$$= \nabla\phi.\frac{\vec{i}}{|\vec{i}|}$$

$$= (5\vec{i} + 4\vec{j} + 3\vec{k}).\vec{i}$$

$$= 5$$

Example 13

If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$, , find the scalar potential ϕ.

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating like coefficients we get

$$\frac{\partial\phi}{\partial x} = 2xyz \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = x^2z \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = x^2y \qquad \ldots (3)$$

Partially integrating (1), (2) and (3) with respect to x, y and z respectively, we get

$$\phi = x^2yz + f(y,z) \qquad \ldots (4)$$

$$\phi = x^2zy + f(x,z) \qquad \ldots (5)$$

$$\phi = x^2yz + f(x,y) \qquad \ldots (6)$$

From (4), (5) and (6) we get

$\phi = x^2yz + C$ (union of all the three results)

Example: 14

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 - z = 0$ at the point (2,-1,5).

Solution:

Given the surface $\phi(x, y, z) = x^2 + y^2 - z$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1$$

$\nabla\phi = 2x\vec{i} + 2y\vec{j} - \vec{k}$

$\nabla\phi_{(2,-1,5)} = 4\vec{i} - 2\vec{j} - \vec{k}$ which is normal to the surface.

$\therefore$ Direction ratio of the normal to the surface at the point (2,-1,5) are (4,-2,-1)

$\therefore$ Equation of the tangent plane is

$4(x - 2) - 2(y + 1) - (z - 5) = 0$

$4x - 8 - 2y - 2 - z + 5 = 0$

$4x - 2y - z - 5 = 0$

$4x - 2y - z = 5$ which is a tangent plane.

Equation of normal line passing through point (2,-1,5) and having Direction ratio (4,-2,-1) is

$$\frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-5}{-1}$$

Exercise

1. If $\phi(x, y, z) = x^2y + y^2x + z^2$ find $\nabla\phi$ at the point (1,1,1).
2. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find grad ϕ at the point (1,-2,-1).
3. Find the unit vector normal to the surface $x^3+y^3+3xyz = 3$ at the point (1,2,-1).
4. Find the unit vector normal to the surface $x^2y + 2xz = 4$ at the point (2,-2,3)
5. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2+2z = 1$ at the point (1,-2,1)
6. Find the angle between the normals to the surface $xy^3z^2=4$ at the point (-1,-1,2) and (4,1,-1)
7. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6,4,3)
8. Find the directional derivative of $\phi = x^3+y^3+z^3$ at the point (1,-1,2) in the direction of the vector $\vec{i} + 2\vec{j} + \vec{k}$

9. Find the directional derivative of $\phi = (x, y, z) = \mathbf{x}^2 - 2y^2 + 4z^2$ at the point (1,1,-1) in the direction $2\vec{i} - \vec{j} - \vec{k}$

10. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P(1,2,3) in the direction of the line PQ where Q (5,0,4)

11. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point P (1,-2,-1) along the direction of PQ where Q(3,-3,-2).

12. Find the directional derivative of $\phi = xy^2 + yz^2$ at the point (2,-1,1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point (-1,2,1).

13. Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (1,-1,2) in the direction of the normal to the surface $x^2 + y^2 + z^2 = 9$ at the point (1,2,2).

14. Find the maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point (1,1,-4).

15. Find the maximum directional derivative of $\phi = x^3y^2z$ at the point (1,1,1).

16. In what direction is the directional derivative of the function $\phi = x^2 - 2y^2 + 4z^2$ from the point (1,1,-1) is maximum and what is its value?

17. Find the direction along which the directional derivative of the function $\phi = xy + 2yz + 3xz$ is greatest at the point (1,1,1). Also find the greatest directional derivative.

18. Find the function ϕ if grad $\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$

19. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, find <j)(x, y, z) if $\phi(l, -2, 2) = 4$

20. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 25$ at the point (4,0,3)

Answer

1.	$3\vec{i} + 3\vec{j} + 2\vec{k}$	11.	$\frac{27}{\sqrt{6}}$
2.	$-(16\vec{i} + 9\vec{j} + 4\vec{k})$	12.	$\frac{15}{\sqrt{17}}$

3. $\dfrac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{14}}$

4. $\dfrac{1}{3}(\vec{i}+2\vec{j}+2\vec{k})$

5. $\cos^{-1}\left(\dfrac{3}{7\sqrt{6}}\right)$

6. $\cos^{-1}\left(\dfrac{45}{\sqrt{2299}}\right)$

7. $\cos^{-1}\left(\dfrac{-24}{\sqrt{5246}}\right)$

8. $\dfrac{7\sqrt{6}}{2}$

9. $\dfrac{8}{\sqrt{6}}$

10. $\dfrac{4\sqrt{21}}{3}$

13. -7

14. $\sqrt{1652}$

15. $\sqrt{14}$

16. $2\vec{i}-4\vec{j}-4\vec{k}, \quad 2\sqrt{21}$

17. $4\vec{i}+3\vec{j}+5\vec{k}, \quad 5\sqrt{21}$

18. $\phi(x,y,z)=xy^2-x^2yz^3+3y+\dfrac{3}{2}z^4+C$

19. $\phi(x,y,z)=x^2yz^3+20$

20. $4x+3z=25$

$$\frac{x-4}{4}=\frac{z-3}{3}, y=0$$

Divergence of a vector point function

The divergence of a differentiable vector point function $\vec{F}$ is denoted by div $\vec{F}$

$$\text{div } \vec{F}=\nabla.\vec{F}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\vec{F} \qquad \text{If } \vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$$

$$=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Curl of a vector point function

The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and is defined bycurl

$$\vec{F} = \nabla \times \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \times \vec{F}$$

if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Definition

A vector point function $\vec{F}$ is said to be solenoidal if div $\vec{F}$ = 0 and it is said to be irrotational if curl $\vec{F}$ = 0.

Example

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{div } \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

i.e., $\nabla.\vec{r} = 3$

$$\text{curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

Hence $\nabla \times \vec{r} = 0$.

1. Find the divergence and curl of the vector $\overline{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (2,-1,1)

Solution:

Given $\overline{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$

$$\text{div } \overline{V} = \nabla.\overline{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3xy^2) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 6xy + 2xz - y^2$$

At (2,-1,1), $\nabla.\overline{V} = (-1).1 + 6(2)(-1) + 2(2)(1) - (-1)^2$

$= -1 + 12(-1) + 4 - 1$

$= -1 + 12 + 4 - 1$

= (-10)

$$\text{curl } \overline{V} = \nabla \times \overline{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy^2 & xz^2 - y^2 z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(3xy^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(xyz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(xy)\right]$$

$$= \vec{i}[-2yz] - \vec{j}[z^2 - xy] + \vec{k}[3y^2 - xy]$$

At (2,-1,1), $\nabla \times \overline{V} = \vec{i}[-2(-1)(1)] - \vec{j}[1 - 2(-1)] + \vec{k}[3(-1)3^2 - 2(1)]$

$$= 2\vec{i} - 3\vec{j} + \vec{k}$$

2. If $\vec{F} = (x^2 - y^2 + 2xy)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, find

$\nabla.\vec{F}, \nabla(\nabla.\vec{F}), \nabla \times \vec{F}, \nabla.(\nabla \times \vec{F})$ and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

$$\vec{F} = (x^2 - y^2 + 2xy)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xy) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$$

$$= (2x + 2z) + (-x + z) + 2z = x + 5z$$

$$\nabla(\nabla.\vec{F}) = \frac{\partial}{\partial x}(x + 5z)\vec{i} + \frac{\partial}{\partial y}(x + 5z)\vec{j} + \frac{\partial}{\partial z}(x + 5z)\vec{k}$$

$$= \vec{i} + 0\vec{j} + 5\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(xz - xy + yz)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(x^2 - y^2 + 2xz)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial y}(xz - xy + yz) - \frac{\partial}{\partial z}(x^2 - y^2 + 2xz)\right]$$

$$= \vec{i}[0 - (x + y)] - \vec{j}[2x - 2x] + \vec{k}[z - y + 2y]$$

$$= -(x + y)\vec{i} + (y + z)\vec{k}$$

$$\nabla.(\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x + y)] + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}[y + z]$$

$$= 1 + 0 + 1 = 0$$

$$\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x + y) & 0 & y + z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y + z) - 0\right] - \vec{j}\left[\frac{\partial}{\partial x}(y + z) - \frac{\partial}{\partial z}(-x + y))\right] + \vec{k}\left[0 + \frac{\partial}{\partial x}(x + y)\right]$$

$$= \vec{i} + \vec{k}$$

$$\therefore (\nabla.\vec{F})_{(1,1,1)} = 6$$

$$[\nabla(\nabla.\vec{F})_{(1,1,1)} = \vec{i} + 5\vec{k}$$

$$(\nabla \times \vec{F})_{(1,1,1)} = 2\vec{i} + 2\vec{k}$$

$$[\nabla.(\nabla \times \vec{F})_{(1,1,1)} = 0$$

$$[\nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{i} + \vec{k}$$

3. Show that $\vec{F}=(y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k}$ is both solenoidal and irrotational.

Solution:

$$\nabla.\vec{F}=\frac{\partial}{\partial x}(y^2-z^2+3yz-2x)+\frac{\partial}{\partial y}(3xz+2xy)+\frac{\partial}{\partial z}(3xy-2xy+2z)$$

$$=-2+2x-2x+2$$

= 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (y^2-z^2+3yz-2x) & (3xy+2xy) & (3xy-2xz+2z)\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}(3xy-2xy+2z)-\frac{\partial}{\partial z}(3xz+2xy)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(3xy-2xz+2z)-\frac{\partial}{\partial z}(y^2-z^2+3yz-2x)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(3xz+2xy)-\frac{\partial}{\partial y}(y^2-z^2+3yz-2x)\right]$$

$$=\vec{i}[3x-3x]-\vec{j}[3y-2z+2z-3y]+\vec{k}[3z+2y-2y-3z]$$

= 0 for all points (x, y, z)

$\therefore \vec{F}$ is an irrotational vectors.

4. Find the constants a, b, c so that

$\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+Cy+2z)\vec{k}$ is irrotational.

Solution:

Given $\nabla\times\vec{F}=0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+Cy+2z) \end{vmatrix} = 0$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+Cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(4x+Cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$(C+1)\vec{i} - (4-a)\vec{j}(b-2)\vec{k}$$

= 0

i.e. $C + 1 = 0,\quad \therefore C = -1$

$4 - a = 0,\quad \therefore a = 4$

$b - 2 = 0,\quad \therefore b = 4$

$\therefore a = 4,\quad b = 2,\ C = -1.$

5. Determine the constant m so that the vector

$\vec{F} = (x+y)\vec{i} + (3x+my)\vec{j} + (x-5z)\vec{k}$ is such that its divergence is zero.

Solution:

div $\vec{F} = 0$

i.e., $\nabla.\vec{F} = 0$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(3x+my) + \frac{\partial}{\partial z}(x-5z)$$

$\Rightarrow 1 + m - 5 = 0$

$\Rightarrow m - 4 = 0$

$\Rightarrow m = 4$

Laplacian operator ∇^2

The operator $\nabla^2$is called the laplacian operator. If ϕ is a scalar function of *x, y, z* then

$$\nabla^2\phi == \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

6. Prove that $\nabla.\nabla\phi = \nabla^2\phi$

Solution:

$$\nabla = \vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$$

$$\nabla.\nabla\phi = \left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right).\left(\vec{i}\,\frac{\partial\phi}{\partial x}+\vec{j}\,\frac{\partial\phi}{\partial y}+\vec{k}\,\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$= \nabla^2\phi$$

7. Prove that curl (grad ϕ) = 0

Solution:

Curl (grad ϕ) = $\nabla\times(\nabla\phi)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]-\vec{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)\right]+\vec{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]$$

8. Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2) + \vec{k}$ is irrotational and fine its scalar potential.

Solution:

$$\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2) + \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2 \cos x + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2 \cos x + z^3)\right]$$

$$= \vec{i}[0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2z\cos x - 2y\cos x]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0.$$

Hence $\vec{F}$ is irrotational

$$\vec{F} = \nabla\phi$$

i.e., $(y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2)\vec{k}$

$$= \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating the coefficients $= \vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = y^2 \cos x + z^3 \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \qquad \ldots (3)$$

Integrating (1) with respect to '*x*' treating '*y*' and '*z*' as constants we get.

$$\phi = y^2 \sin x + z^3 x + f(y,z) \qquad \ldots (4)$$

Integrating (2) with respect to '*y*' treating '*x*' and '*z*' as constants we get.

$$\phi = 2\left(\frac{y^2}{2}\right)\sin x - 4y + f(x,z) \qquad \ldots (5)$$

Integrating (3) with respect to '*z*' treating '*x*' and '*y*' as constants we get.

$$\phi = \frac{3xz^3}{3} + f(x,y) \qquad \ldots (6)$$

from equations (4), (5) and (6) we get

$$\phi = y^2 \sin x + xz^3 - 4y + C$$

9. If $r = |\bar{r}|$, where $\bar{r}$ is the position vector of the point (*x*, *y*, *z*), prove that $\nabla^2(r^n) = (n+1).r^{n-2}$

Solution:

$$\nabla^2(r^n) = \nabla.(\nabla r^n)$$

$$\nabla r^n = \vec{i}\,\frac{\partial}{\partial x}(r^n) + \vec{j}\,\frac{\partial}{\partial y}(r^n) + \vec{k}\,\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\left[nr^{n-1}\frac{\partial r}{\partial x}\right] + \vec{j}\left[nr^{n-1}\frac{\partial r}{\partial y}\right] + \vec{k}\left[nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \vec{i}\left[nr^{n-1}.\frac{x}{r}\right] + \vec{j}\left[nr^{n-1}.\frac{y}{r}\right] + \vec{k}\left[nr^{n-1}.\frac{z}{r}\right]$$

$$\because r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$= nr^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= nr^{n-2}\vec{r}$$

$$\nabla.\nabla r^n = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(nr^{n-2}x\vec{i} + y\vec{j} + 2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2)}y) + \frac{\partial}{\partial z}(nr^{n-2)}z)$$

$$= n\left[r^{n-2} + x.(n-2)r^{n-3}\left(\frac{x}{r}\right)\right] + n\left[r^{n-2} + y.(n-2)r^{n-3}\left(\frac{y}{r}\right)\right] + n\left[r^{n-2} + z.(n-2)r^{n-3}\left(\frac{z}{r}\right)\right]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}.r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= nr^{n-2}[3 + n - 2] = n(n-2)r^{n-2}$$

10. Find f® if the vector $f(r)\ \vec{r}$ is both solenoidal and irrotational.

Solution:

$f(r)\ \vec{r}$ is solenoidal. $\nabla.f(r)\vec{r} = 0$

i.e., $\nabla.f(r).\vec{r} + f(r)\nabla.\vec{r} = 0$

since $\nabla.f(r) = \dfrac{f'(r)}{r}\vec{r}$ we get

i.e, $\dfrac{f'(r)}{r}\vec{r}.\vec{r} + 3f(r) = 0$

since $\vec{r}.\vec{r} = r^2$ we get

i.e. $rf'(r) = 3f(r) = 0$

i.e. $\dfrac{f'(r)}{f(r)} + \dfrac{3}{r} = 0$ (on division by 3r)

Integrating both sides with respect to r log $f(r)$ + 3 log r = log C

log $r^3 f(r)$ = log C

$$f(r) = \frac{C}{r^3} \qquad \ldots (1)$$

$f(r)\ \vec{r}$ is also irrotational

$\therefore \nabla \times (f(r)\vec{r}) = 0$

i.e, $\nabla f(r) \times \vec{r} + f(x) \nabla \times \vec{r} = 0$

i.e., $\dfrac{f'(r)}{r}(\vec{r} \times \vec{r}) + 0 = 0$ since $(\nabla \times \vec{r} = 0)$

since $\vec{r} \times \vec{r} = 0$

$$\frac{f'(r)}{r}(0) + 0 = 0 \qquad \ldots (2)$$

This is true for all values of $f(r)$

From (1) & (2) $f(r) = \dfrac{C}{r^3}$ we get $f(r)\vec{r}$ is both solenoidal and irrotational.

Exercise

1. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ find div $\bar{r}$ and curl $\bar{r}$.

2. If $\bar{r} = x^2 y\bar{i} + y^2 z\bar{j} + z^2 x\bar{k}$, find curl $\bar{F}$.

3. If $\phi = x^2 + y^2 + z^2$ prove that curl (grad ϕ) = 0.

4. For what value of 'a' the vector $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal.

5. Prove that div (curl $\bar{F}$) = 0

6. If $\bar{A}$ and $\bar{B}$ are irrotatioanl, prove that $\bar{A} \times \bar{B}$ is solenoidal.

7. Find the value of 'c' so that the vector $\bar{F} = (cxy - z^3)\bar{i} - (c-2)x^2\bar{j} + (1-c)xz^2\bar{k}$ is irrotational.

8. Find the values of the constants a, b, c so that $\overline{F} = (axy + bz^3)\overline{i} + (3x^2 - cz)\overline{j} + (3xz^2 - y)\overline{k}$ may be irrotational for these values of a, b, c. Also find the scalar potential of $\overline{F}$.

9. Find div $\overline{F}$ and curl $\overline{F}$,where $\overline{F}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$.

10. Prove that $f = (2x + yz)\overline{i} + (4y + zx)\overline{j} - (6z - xy)\overline{k}$ is solenoidal as well as irrotational. Also find the scalar potential.

11. $\overline{F}$ is solenoidal, prove that curl curl curl $\overline{F}$ = $\nabla^4 F$.

12. If $\overline{F} = 3x^2\overline{i} + 5xy^2\overline{j} + xyz^3\overline{k}$, find $\nabla.\overline{F}, \nabla(\nabla.\overline{F}), \nabla\times\overline{F}, \nabla.(\nabla\times\overline{F})$ and $\nabla\times(\nabla\times\overline{F})$ at the point (1, 2, 3).

13. Show that $\overline{F} = (z^2 + 2x + 3y)\overline{i} + (3x + 2y + z)\overline{j} + (y + 2zx)\overline{k}$ is irrotational, but not solenoidal. Find also its scalar potential.

14. Prove that $f(r)\overline{r}$ is irrotational.

15. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Answers

1. 3, 0

2. $-y^2\overline{i} - z^2\overline{j} - x^2\overline{k}$

3. - 2

7. 4

8. $a = 6$, $b = 1$, $c = 1$

9. $6(x + y + z)$, 0

10. $\phi = x^2 + 2y^2 + 3z^2 + xyz + k$

12. $80, 80\overline{i} + 37\overline{j} + 36\overline{k}, 27\overline{i} - 54\overline{j} + 20\overline{k}, 0, 74\overline{i} + 27\overline{j}$

13. $x^2 + y^2 + 3xy + yz + z^2x + C$

LINE INTEGRALS

Any integral which is to be evaluated along a curve is called a line integral.

Let $\overline{F}(x, y, z)$ be a vector point function defined at all points in some region of space and let C be a curve in thatregion. The integral $\int_C \overline{F}.d\bar{r}$ is defined as the line integral of $\overline{F}$ along the curve C.

Note

(1) Physically $\int_C \overline{F}.d\bar{r}$ denotes the total work done by the force $\overline{F}$ in displacing a particle from A to B along the curve C.

(2) $\int_A^B \overline{F}.d\bar{r}$ depends not only on the curve C but also on the terminal points A and B.

(3) If the path of integration C is a closed curve, the line integral is denoted as $\oint_C \overline{F}.d\bar{r}$.

(4) If the value of $\int_A^B \overline{F}.d\bar{r}$ does not depend on the curve C, but only on the terminal points A and B, than $\overline{F}$ is called a conservative vector or conservative force.

(5) If $\overline{F}$ is irrotational (conservative) and C is a closed curve then $\int_C \overline{F}.d\bar{r} = 0$.

(6) If $\int_C \overline{F}.d\bar{r}$ is independent of the path C then curl $\overline{F} = \overline{0}$.

(1) If $\overline{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \overline{F}.d\bar{r}$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution:

Given $\overline{F} = 3xy\vec{i} - y^2\vec{j}$

$$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\overline{F}.d\bar{r} = 3xydx - y^2dy$$

Given $y = 2x^2$

$$dy = 4xdx$$

$$\therefore \overline{F}.d\bar{r} = 3x(2x^2)dx - (2x^2)^2(4xdx)$$

$= (6x^3 - 16x^5)dx$

$\int_C \overline{F}.d\overline{r} = \int_0^1 (6x^3 - 16x^5)dx$

$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_{.0}^1$

$= \frac{6}{4} - \frac{16}{6}$

$= \frac{-7}{6}$

(2) If $\overline{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$. Evaluate $\int_C \overline{F}.d\overline{r}$ from (0, 0, 0) to (1, 1, 1) along the curve $x = t$, $y = t^2$, $z = t^3$.

Solution:

Given $\overline{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

$d\overline{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overline{F}.d\overline{r} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$

Given $x = t$ $\quad y = t^2$ $\quad z = t^3$

$dx = dt$ $\quad dy = 2tdt$ $\quad dz = 3t^2\text{dt}$

$\int_C \overline{F}.d\overline{r} = \int_0^1 (3t^2 + 6t^2)dt - 14(t^2)(2tdt) + 20t(t^3)^2(3t^2dt)$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt$

$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$

$= 3 - 4 + 6$

= 5 Units.

(3) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xy - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, y-=t, $z = 4t^3$.

Solution:

Work done $= \int_C \bar{F}.d\bar{r}$

Given $\vec{F} = 3x^2\vec{i} + (2xy - y)\vec{j} - z\vec{k}$

$d\bar{r} = 2x\vec{i} + 2y\vec{j} + dz\vec{k}$

$\bar{F}.d\bar{r} = 3x^2 d + (2xz - y)dy - zdz$

Given $x = 2t^2$ $y = t$ $z = 4t^3$

$dx = 4tdt$ $dy = dt$ $dz = 12t^2dt$

$$\int_C \bar{F}.d\bar{r} = \int_0^1 [48t^5 + (16t^5 - t) - 48t^5]dt$$

$$= \int_0^1 (16t^5 - t)dt$$

$$= \left[16.\frac{t^6}{6} - \frac{t^2}{2}\right]_0^1$$

$$= \frac{16}{6} - \frac{1}{2}$$

$$= \frac{13}{6} \text{ Units}$$

(4) Find the work done by the force $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ along the curve C where C is the rectangle in the xy-plane bounded by $x = 0$, $x = a$, $y = 0$, $y = b$.

Solution:

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\bar{F}.d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

The curve C is the rectangle *OABC* and *C* consists of four different paths *OA, AB, BC, CO*.

$$\therefore \int_C \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$$

Along *OA*,

$y = 0 \qquad dy = 0$

Along *AB*,

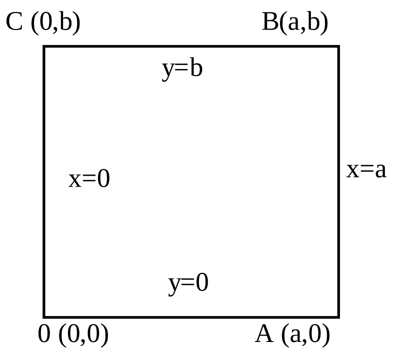
$x = a \qquad dx = 0$

Along *BC*,

$y = b \qquad dy = 0$

Along CO,

$x = 0 \qquad dx = 0$



$$\therefore \int_C \overline{F}.d\bar{r} = \int_{OA} x^2 dx + \int_{AB} -2ay dy + \int_{BC} (x^2 + b^2) dx + 0.$$

$$= \int_0^a x^2 dx - 2a\int_0^b y dy + \int_0^b (x^2 + b^2) dx$$

$$= \left(\frac{x^3}{3}\right)_0^a - 2a\left(\frac{y^2}{2}\right)_0^b + \left(\frac{x^3}{3} + b^2 3\right)_a^0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -2ab^2$$

5. If $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$. Evaluate $\int_C \overline{F}.d\bar{r}$ where *C* is the straight line from *A* (0,0,0) to *B*(2,1,3).

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}.d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$$

Equation of the straight line AB is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where $(x_1, y_1, z_1) = (0, 0, 0)$ and $(x_2, y_2, z_2) = (2, 1, 3)$

$$\therefore \frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-3}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\therefore x = 2t$ $\quad$ $y = t$ $\quad$ $z = 3t$

$dx = 2dt$ $\quad$ $dy = dt$ $\quad$ $dz = 3dt$

$$\int_C \bar{F}.d\bar{r} = \int_0^1 3(2t)^2(2dt) + (12t^2 - t)dt + 9tdt$$

$$= \int_0^1 24t^2dt + (12t^2 + 8t)dt$$

$$= \int_0^1 (36t^2 + 8t)dt$$

$$= \left(36.\frac{t^3}{3} + 8\frac{t^2}{2}\right)_0^1$$

$$= \frac{36}{3} + \frac{8}{2}$$

$= 12 + 4$

$= 16.$

6. Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t 2\vec{i} + \sin t\vec{j} + t\vec{k}$ from t=0 to $t = 2\pi$.

Solution:

Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}.d\vec{r} = zdx + xdy + ydz$$

work done by $\vec{F} = \int_C \bar{F}.d\bar{r}$

$$= \int_C zdx + rdy + ydz$$

From the vector equation of the curve *C*,

$x = \cos t \qquad y = \sin t \qquad z = t$

$dx = -\sin t dt \qquad dy = \cos t dt \qquad dz = dt$

$$\therefore \int_C \vec{F}.d\bar{r} = \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]dt$$

$$= \left[t\cos t - \sin t - \frac{1}{2}\left(t + \frac{\sin 2t}{2} \right) - \cos t \right]_0^{2\pi}$$

$$= (2\pi + \pi - 1) - (-1)$$

$$= 3\pi$$

(7) Find the work done by the force $\vec{F} = y(3x^2 - y - z^2)\vec{i} + x(2x^2 y - z^2)\vec{j} - 2xyzy\vec{k}$ when it moves a particle around a closed curve *C*.

Solution:

To evaluate the work done by a force, the equation of the path C and the terminal points must be given.

Since C is a closed curve and the particle moves around this curve completely, any point (x_0, y_0, z_0) can be taken as the initial as well as the final point.

But the equation of C is not given. Hence we verify when the given force $\vec{F}$ is conservative, ie., irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\vec{i} - (-2yz + 2yz)\vec{j} + (6x^2 y - 6x^3 y + z^3 - z^2)\vec{k}$$

$$= 0$$

Since $\nabla \times \vec{F} = \bar{0}$

$\Rightarrow \vec{F}$ is irrotatonal

$\Rightarrow \oint_C \vec{F}.d\bar{r} = \bar{0}$

(8) If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$ check whether the integral $\int_C \vec{F}.d\bar{r}$ is independent of the path C.

Solution:

$\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$

$\int_C \vec{F}.d\bar{r}$ is independent of the path of integration, if $\nabla \times \vec{F} = \bar{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^2z \end{vmatrix}$$

$= \vec{i}(0-0) - \vec{j}(-6x^2 + 6x^2z) + \vec{k}(4x - 4x)$

$= \bar{0}.$

Hence the line integral is independent of the path C.

(9) If $\vec{F} = x\vec{j} - y\vec{i}$, find $\int_C \vec{F}.d\bar{r}$ along the arc of the circle $x^2 + y^2 = 1$ from (1,0) to (0,1).

Given $\vec{F} = x\vec{j} - y\vec{i}$

$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\bar{F}.d\bar{r} = -ydx + xdy$

Given $x^2 + y^2 = 1$ … (1)

$2xdx + 2ydy = 0$

$2xdx = -2ydy$

$xdx = -ydy$ … (2)

$\bar{F}.d\bar{r} = -ydx + xdy\left(\dfrac{-x}{y}dx\right)$ (from 2))

$$= -y dx - \frac{x^2}{y} dx$$

$$= \left[y + \frac{x^2}{y} \right] dx$$

$$= -\left[\frac{y^2 + x^2}{y} \right] dx$$

$$= \frac{-dx}{y} \qquad \text{(from (1))}$$

$$\overline{F}.d\overline{r} = \frac{-1}{\sqrt{1-x^2}} dx \qquad (\because x^2 + y^2 = 1)$$

$$\int_C \overline{F}.d\overline{r} = -\int_1^0 \frac{dx}{\sqrt{1-x^2}}$$

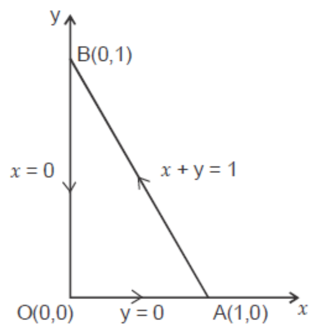
$$= \int_1^0 \frac{dx}{\sqrt{1-x^2}}$$

$$= (\sin^{-1} x)_0^1$$

$$= \frac{\pi}{2}.$$

(10) Evaluate $\int_C \overline{F}.d\overline{r}$ where C is the boundary of the region given by $x = 0$, $y = 0$, x +y = 1 and $\vec{F} = (3x^2 - 8y^2)\vec{i} + (4y - 6xy)\vec{j}$

Solution:



Given $\vec{F} = (3x^2 - 8y^2)\vec{i} + (4y - 6xy)\vec{j}$

$$d\overline{r} = dx\vec{i} + dy\vec{j} + dx\vec{k}$$

$$\overline{F}.d\overline{r} = (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Here C consists of the lines $x = 0$, $x + y = 1$.

Along AB,

$$x + y = 1$$

$$\Rightarrow y = 1 - x$$

$$dy = -dx$$

Along BO,

$$x = 0, \quad dx = 0$$

$$\int_C \bar{F}.d\bar{r} = \int_{0A} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BO} \bar{F}.d\bar{r}$$

$$= \int_0^1 3x^2 dx + \int_0^1 [3x^2 - 8(1-x)^2 dx + (4(1-x) - 6x(1-x))(-dx)] + \int_1^0 4y dy$$

$$= 3.\left(\frac{x^3}{3}\right)_0^1 + \int_0^1 (-11x^2 + 26x - 12)dx + 4.\left(\frac{y^2}{2}\right)_1^0$$

$$= 1 + \left[-11.\frac{x^3}{3} + 26.\frac{x^2}{2} - 12x\right]_1^0 + 2(0-1)$$

$$= 1 + \frac{11}{3} - 13 + 12 - 2$$

Exercise

(1) Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\vec{i} + y\vec{j}$ and C is $y^= 4x$ in the xy-plane from (0, 0) to (4, 4).

(2) If $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$, find $\int_C \bar{F}.d\bar{r}$ along C, where C is the straight line joining the points (0, 0, 0) to (1, 1, 1).

(3) Find the work done by the force $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j}$ when it moves a particle from (1, -2, 1) to (3, 1, 4) along any path.

(4) Find the total work done in moving a particle in a force field given by $\vec{F} = (2x - y + z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$ along a circle C in the xy-plane $x^2 + y^2 = 9$, $z = 0$.

(5) Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

(6) Given the vector field $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$, evaluate the work done in moving a particle from the point (0, 0, 0) to (1, 1, 1) along the curve C, $x = t$, $y = t^2$, $z = t^3$.

(7) If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F}.d\bar{r}$ from (0, 0) to (1, 1) along the line $y = x$.

(8) If $\vec{F} = 3x(x+2y)\vec{i} + (3x^2 - y^3)\vec{j}$, show that $\int_C \vec{F}.d\bar{r}$ is independent of the path C.

(9) Find the work done by the force $\vec{F} = y^2\vec{i} + 2(xy+z)\vec{j} + 2y\vec{k}$, when it moves a particle around a closed curve C.

(10) Find the work done by $\vec{F} = xy\vec{i} + (y-z)\vec{j} + 2x\vec{k}$, when the particle moves along the curve $x = t$, $y = t^2$, $z = t^3$, from $t = 1$ to $t = 2$.

SURFACE INTEGRAL

Introduction

A surface integral is a definite integral taken over a surface. It can be thought of as the double integral analogue of the line integral. Given the surface, one may integrate over its scalar field (ie, functions which return scalars as value) and vector field ((ie) functions which return vectors as value). Surface integrals have applications in physics, particularly with the classical theory of electromagnetism. Various useful results for surface integrals can be derived using differential geometry and vector calculus, such as the divergence theorem and its generalization Stokes theorem.

Consider any surface (planar, curved, closed or open) and let $\vec{F} = \vec{F}(x, y, z)$ be a vector point function, defined and continuous on a region S of the surface. Then $\iint_S \vec{F}.d\vec{s}$ where ds denotes an element of the surface S is called the surface integral of $\vec{F}$ over S.

We define it as the limit of a sum as follows.

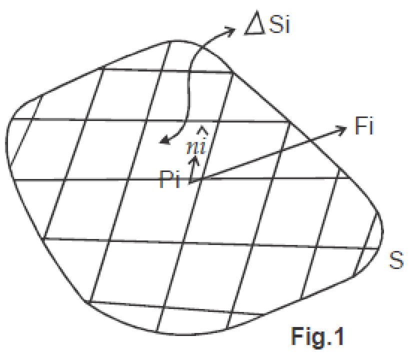


Fig.1

Subdivide *S*, in any manner into n elements of areas ΔS_i,I = 1, 2, ...n. Let $\vec{F_i}$ be the value of $\vec{F}$ at some point, P_i inside or on the boundary of the sub-region ΔS_i.

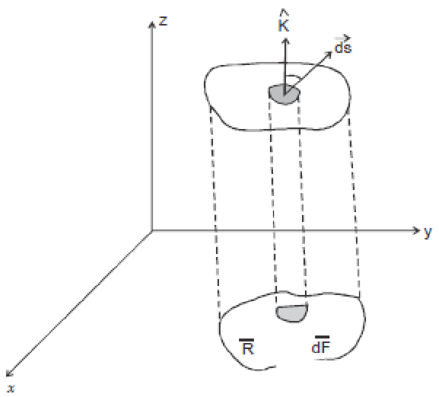
From the vector sum $In = \sum_{i=1}^{n} F_i \Delta S_i$. If the limits of the above sum exists, as $n \to \infty$ in such a way that each ΔS_i collapses ((ie) shrinks) to a point, and is independent of the mode of the sub-division of S, then this limit is called the surface integral of $\vec{F}$ over *S* and is denoted by $\iint_S \vec{F}.d\vec{s}$.

Normal surface Integral of $\vec{F}$ over the parts of a given surface

Consider the above Fig. 1 let P_i be any point of *S* and let $\hat{n}$,a unit normal vector at P_i, pointing outwardly (called the outward unit normal at P_i) to the surface ΔS_i. Then $\vec{F}.\hat{n}_i$ is the scalar complement of $\vec{F_i}$, in the direction of $\hat{n}_i$.

The limiting value of the sum $\sum_{i=1}^{n} F_i.\vec{n}_i$ as $n \to \infty$ where n is the number of subregions ΔSi, such that each ΔS_i shrinks to a point, if it exists and is independent of the manner of division of *S* into sub-regions, is called the normal surface integral of $\vec{F}$ over S and is denoted by $\iint_S \vec{F}.\hat{n}\, d\vec{s}$.

Evaluation of double integral



The surface *S* is projected onto a region *R* of the xy-plane, so that an element of surface area ds at point *P* projects onto the area element. We see that $d\vec{F} = |\cos\alpha|\, d\vec{S}$, where α is the angle between the unit vector $\hat{k}$ in the z-direction and the unit normal $\hat{n}$ to the surface at *P*. So at any given point of *S*, we have simply $d\vec{s} = \frac{d\vec{F}}{|\cos\alpha|} = \frac{d\vec{F}}{|\hat{n}.\hat{k}|}$

Now if the surface S is given by the equation $f(x,y,z) = 0$ then the unit normal at any point of the surface is simply given by $\hat{n} = \frac{\nabla \vec{f}}{|\nabla \vec{f}|}$ evaluated at that point. The scalar element of the surface area then becomes $d\vec{s} = \frac{d\vec{F}}{\hat{n}.\hat{k}} = \frac{|\nabla \vec{f}|\, d\vec{F}}{|\nabla \vec{f}|.\hat{k}} = \frac{|\nabla \vec{f}|\, d\vec{f}}{\frac{d\vec{f}}{dz}}$

Where $|\nabla \vec{f}|$ and $\frac{\partial \vec{f}}{\partial z}$ and are evaluated on the surface S. we can therefore express any surface integral over S as a double integral over the region R in the xy-plane.

Note

The projection of the elementary surface ds on the xy palne is $dxdy$. Also the projection of the vector $\hat{n}\, ds$ of magnitude ds on the XOY plane to which $\hat{k}$ is the unit normal vector.

Thus $dy = |\hat{n}.ds.\hat{k}| = |\hat{n}.\hat{k}|\, d\vec{s}$ giving $ds = \frac{dxdy}{|\hat{n}.\hat{k}|}$. Hence $\iint_S \vec{F}.\hat{n}d\vec{s} = \iint_{S'} \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$.

where S is the projection of S on the XOY plane. Similarly $\int_S \vec{F}.\hat{n}d\vec{s} = \iint_{S'} \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{i}|}$ where $s'$is the projection of S on the YOZ plane and $\int_S \vec{F}.\hat{n}d\vec{s} = \iint_{S'}, \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{j}|}$ where $s'$is the projection S on the XOZ plane.

Flux

In physical applications the integral $\int_S \vec{F}.\hat{n}d\vec{s}$ is called the flux of $\vec{F}$ through S.

Cylindrical and Spherical polar co-ordinates

In evaluating surface and volume integrals, in certain cases, it will be advantage to change the variable x, y, z into cylindrical or spherical polar co-ordinates. So it is better to recall the relations between these coordinates and the respective Jacobian of transformation.

Polar Co-ordinates

We know that in two dimensions, the relation between x, y and the polar co-ordinates r, θ are $x = r \cos \theta$; $y = r \sin \theta$ and $0 \le r < \infty$, $0 \le \theta \le 2\pi$. And the jacobian of the transformation is r so that $dxdy = rdrd\theta$.

Cylindrical Co-ordinates

If P is (x, y, z) (refer Fig.3) and if PL is the perpendicular from P to the XOY plane then OL angle XOL, LP are the cylindrical polar co-ordinates which are denoted by r, θ, z where $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$.

The relations between x, y, z and r, θ, z and that between

$$dxdydz \;\&\; drd\theta dz \text{ are } \left.\begin{array}{l} x = r\cos\theta \\ y = r\cos\theta \\ x = z \end{array}\right\} \quad dxdydz = rdrd\theta dz$$

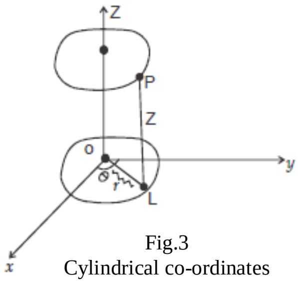


Fig.3
Cylindrical co-ordinates

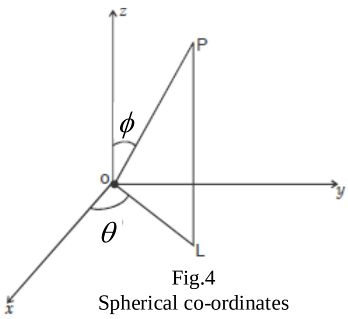


Fig.4
Spherical co-ordinates

Spherical Co-ordinates

If P is (x, y, z) (refer Fig.4) and if PL is perpendicular from P to the XOY plane, then OP, angle ZOP, angle XOL are the spherical polar co-ordinates which are denoted by r, θ, ϕ where $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The relations between x, y, z and r, θ, ϕ.

are $\quad x = r \sin \theta \cos \phi$

$y = r \sin \theta \cos \phi$

$z = r \cos \theta$

The relation between $dxdydz$ and $drd\theta d\phi$ is $dxdydz = r^2 \sin \theta drd\theta d\phi$. For a hemisphere the limits of θ will be from 0 to $\frac{\pi}{2}$ and the limits for ϕ will be 0 to 2π.

Note

(i) If S is a closed surface, the outer surface is usually chosen as the positive side

(ii) $\int_S \phi d\vec{s}$ and $\int_S \bar{F}\times d\vec{s}$, where ϕ is a scalar point function, are also surface integrals.

(iii) To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int_S \bar{F}.d\vec{s}$ is also denoted as $\iint_S \bar{F}.d\vec{s}$.

(iv) The area of the region S is $\iint_S ds$.

Solved Problems

1. Obtain $\int_S \vec{F}.\hat{n}\, ds$, where $\vec{F} = (x^2+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ over the surface of the plane $2x + y + 2z = 5$ in first octant.

Solution

Let the given surface be $\phi = 2x + y + 2z - 6$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

Let s' be the projection is S in the XOY plane

$$\int_S \vec{F}.\hat{n}ds = \iint_S \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\vec{k}|}$$

Now $\vec{F}.\hat{n} = ((x^2+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}).\dfrac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

$$= \frac{2(x^2+y^2) - 2x + 4yz}{3}$$

$$= \frac{2}{3}\left(x^2 + y^2 - x + 2y\left(\frac{6-2x-y}{2}\right)\right) \qquad \text{since } z = \frac{6-2x-y}{2}$$

$$= \frac{2}{3}\left(x^2 + y^2 - x + 6y - 2xy - y^2\right)$$

$$= \frac{2}{3}\left(x^2 - 2xy^2 - x + 6y\right)$$

Since the equation of the line AB is $2x+y=6$ (or) $y=6-2x$. In the region s' as x varies from 0 to 3, y varies from 0 to $6-2x$.

$$\therefore \int_S \vec{F}.\hat{n}d\vec{s} \quad = \iint_S \frac{2}{3}(x^2 - 2xy - x + 6y)\frac{dxdy}{\frac{2}{3}}$$

$$= \int_0^3 \int_0^{-2x} (x^2 - 2xy - x + 6y)dxdy$$

$$= \int_0^3 (x^2y - xy^2 - xy + 3y^2)_{y=0}^{y=6-2x} dx$$

$$= \int_0^3 (108 - 114x + 44x^2 - 6x^3)dx$$

$$= \frac{171}{2}$$

2. Given that $\vec{F} = x\hat{i} + y\hat{j} - 2z\hat{k}$ find $\int_S \vec{F}.d\vec{s}, S$ being the surface of the sphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

Solution:

Let $\phi = x^2 + y^2 + z^2$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$ since $x^2 + y^2 + z^2 = a^2$

$$\int_S \vec{F}.d\vec{s} = \iint_s \vec{F}.\hat{n}ds$$

$= \iint_s \vec{F}.\hat{n}\frac{dxdy}{\hat{n}.\hat{k}}$ where s' is the projection of the spherical surface above the XY-plane on the XOY plane.

$$\vec{F}.\hat{n} \quad = (x\hat{i} + y\hat{j} + 2z\hat{k}).\frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{1}{a}(x^2 + y^2 - 2z^2)$$

$$= \frac{1}{a}(x^2 + y^2 - 2a^2 - x^2 - y^2) \text{ since } x^2 + y^2 + z^2 = a^2 \quad \therefore z^2 = a^2 - x^2 - y^2$$

$$= \frac{1}{a}(3x^2 + 3y^2 - 2a^2)$$

$$\hat{n}.\hat{k} \qquad = (x\hat{i} + y\hat{j} + 2z\hat{k}).\hat{k}$$

Since $s'$is the circle x^2 $+y^2$ $=a^2$on the XOY plane. x varies from -a to +a and y varies from

$$\sqrt{a^2 - x^2} \ to + \sqrt{a^2 + x^2}$$

Hence $\int\limits_S \vec{F}.d\vec{s} = \iint\limits_S \frac{1}{a}(3x^2 + 3y^2 - 2a^2).\frac{a}{z} dxdy$

$$= \int\limits_{x=-a}^{a} \int\limits_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2) - 2a^2}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

Taking x $=r$ cos θ, y $=r$ sin θ, we get x^2 $+y^2$ $+r^2$and $dxdy$ $=rdrd\theta$. For the circle s', r varies from 0to a and θvaries from 0 to 2π.

$$\int\limits_S \vec{F}.d\vec{s} = \int\limits_0^{2\pi} \int\limits_0^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} rdrd\theta = \int\limits_0^{2\pi} \int\limits_0^{a} \frac{-3(a^2 - r^2) + a^2}{\sqrt{a^2 - r^2}} rdrd\theta$$

$$= 3\int\limits_0^{2\pi} \int\limits_0^{a} r\sqrt{a^2 - r^2} drd\theta + a^2 = \int\limits_0^{2\pi} \int\limits_0^{a} \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$$

$$= I_1 + I_2 \text{ (say)} \qquad \ldots (1)$$

Consider $I_1 \quad -3\int\limits_0^{2\pi} \int\limits_0^{a} \sqrt{a^2 - r^2} drd\theta$

Let $\quad a^2 - r^2 = t^2$

$2rdr = -2tdt$

$rdr = -tdt$

If $r = 0, t = a$

$r = a, t = 0$

$$\int_0^a r\sqrt{a^2 - r^2}\,dr = \int_0^a \sqrt{t^2}\,(-t)dt$$

$$= -\int_0^a t^2 dt$$

$$= \int_0^a t^2 dt$$

$$= \left(\frac{t^3}{3}\right)_0^a$$

$$= \frac{a^3}{3}$$

$$\therefore I_1 = -3\int_0^{2\pi}\int_0^a r\sqrt{a^2 - r^2}\,dr d\theta$$

$$= -3\int_0^{2\pi}\left(\frac{a^3}{3}\right)d\theta$$

$$= -a^3(\theta)_0^{2\pi}$$

$$= -a^3 2\pi$$

Consider $I_2 = a^2\int_0^{2\pi}\int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$

Let $a^2 - r^2 = t^2$ If $r = 0$ $t =$ a

$2rdr = -2tdt$ $r = 0$ $t =$ a

$rdr = -tdt$

$$\therefore \int_0^a \frac{rdr}{\sqrt{a^2 - r^2}} = \int_a^0 \frac{1}{\sqrt{t^2}}(-tdt)$$

$$= \int_a^0 -dt$$

$$= \int_a^0 dt$$

$$= (t)_0^a$$

$$= a$$

$$\therefore I_2 = a^2 \int_0^{2\pi} \int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$$

$$= a^2 \int_0^{2\pi} ad\theta$$

$$= a^3 \int_0^{2\pi} d\theta$$

$$= a^3 (\theta)_0^{2\pi}$$

$$= a^3 (\theta)_0^{2\pi}$$

$$= a^3 2\pi$$

$$\therefore (1) \Rightarrow \int_0 \vec{F}.d\vec{s} = -a^3 2\pi + a^3 2\pi$$

$$= 0$$

3. Obtain $\int \vec{F}.\hat{n}d\vec{s}$ over the surface of the cylinder $x^2 + y^2 = 16$ in the first octant between z = 0 &z =5 where $\vec{F} = z\vec{i} + x\vec{j} - 3y^2 z\hat{k}$

Solution:

Let $\phi = x^2 + y^2 - 16$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{4} \qquad \text{since } x^2 + y^2 = 16$$

Now $\vec{F}.\hat{n} = (z\vec{i} + x\vec{j} - 3y^2 zk).\frac{1}{4}(x\vec{i} + y\vec{j})$

$$= \frac{1}{4}(xz + xy)$$

$$= \frac{1}{4}x(z + y)$$

The surface S of the cylinder in the first octant can be projected onto YOZ (or ZOX) plane into a surface s'

Hence $\int_S \vec{F}.\hat{n}ds = \int_{S'} \frac{1}{4}x(y+z)\frac{dydz}{|\hat{n}.\hat{i}|}$

$= \int_{S'} \frac{1}{4}x(y+z)\frac{dydz}{\frac{x}{4}}$ since $\hat{n}.\hat{i} = \frac{x}{4}$

$= \int_{z=0}^{5}\int_{y=0}^{4}(y+z)dydz$, S' is a rectangle of sides 4 & 5 units.

$= \int_{z=0}^{5}\left(\frac{y^2}{2}+zy\right)_{y=0}^{4} dx$

$= \int_{z=0}^{5}(8+4z)dz$

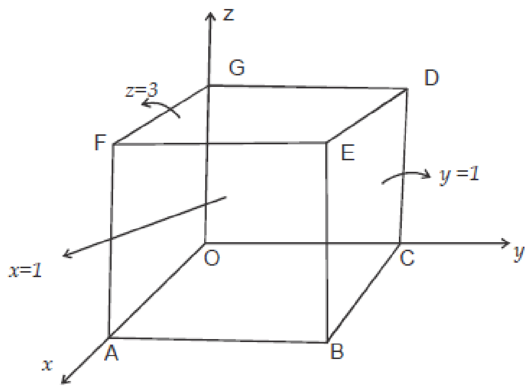
$= \left(8z+\frac{4z^2}{2}\right)_0^5$

$= 40+50$

$= 90$

4. If $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and S is the rectangle parellelopiped bounded $x = 0$, $y = 0$, $z = 0$, calculate $\iint_S \vec{F}.\hat{n}d\vec{s}$

Solution:



There are six faces of the parellelepiped and we calculate the integral over each of these faces. We denote the values of $\vec{F}$ on these faces by $\vec{F}_1, \vec{F}_2 ....\vec{F}_6$

Face	$\hat{n}$	Equation	$\overrightarrow{ds}$	$\vec{F}$
ABEF	$\hat{i}$	$x = 1$	$dydz$	$\vec{F}_1 = 2y\hat{i} + yz^2\hat{j} + z\hat{k}$
COGD	$-\hat{i}$	$x = 0$	$dydz$	$\vec{F}_2 = yz^2\hat{j}$
BCDE	$\hat{j}$	$y = 2$	$Dzdx$	$\vec{F}_3 = 4x\hat{i} + 2z^2\hat{j} + xz\hat{k}$
GOAE	$-\hat{j}$	$y = 1$	$dzdx$	$\vec{F}_4 = xz\hat{k}$
EDGE	$\hat{k}$	z = 3	$dxdy$	$\vec{F}_5 = 2xy\hat{i} + 9y\hat{j} + 3x\hat{k}$
AOCB	$-\hat{k}$	$z = 0$	$Dxdy$	$\vec{F}_6 = 2xy\hat{i}$

$$\therefore \int \vec{F}.ds = \iint_{ABEF} \vec{F}_1.\hat{n}ds + \iint_{COGD} \vec{F}_2.\hat{n}ds + \iint_{BCDE} \vec{F}_3.\hat{n}ds$$

$$+ \iint_{GOAE} \vec{F}_4.\hat{n}ds + \iint_{EDGE} \vec{F}_5.\hat{n}ds + \iint_{AOCB} \vec{F}_6.\hat{n}ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \qquad \text{(Say)} \qquad \ldots (1)$$

Consider, $I_1 = \iint_{ABEF} \vec{F}_1.\hat{n}ds$

$$\vec{F}_1.\hat{n} = (2y\hat{i} + yz^2\hat{j} + z\hat{k}).\hat{i}$$

$$= 2y$$

on the surface ABEF, z varies from 0 to 3 and y varies from 0 to 2

$$\therefore I_1 = \int_{y=0}^{2} \int_{z=0}^{3} 2ydydz$$

$$= 2\int_0^2 ydy(z)_0^3$$

$$= 2 \times 3\int_0^2 ydy$$

$$= 6\left(\frac{y^2}{2}\right)_0^2$$

$$= 12$$

Consider $\therefore I_2 = \iint_{COGD} \vec{F}_2.\hat{n}ds$

$$\vec{F}_2.\hat{n} = yz^2\vec{j}.(-\vec{i})$$

$$= 0$$

on the surface COGD, z varies from 0 to 3 and y varies from 0 to 2.

$$\therefore I_2 = \int_{y=0}^{2}\int_{z=0}^{3} 0\, dydz$$

$$= 0$$

Consider $\therefore I_3 = \iint_{BCDE} \vec{F}_3.\hat{n}ds$

$$\vec{F}_3.\hat{n} = (4x\hat{i} + 2z^2\hat{j} + xz\hat{k}).\hat{j}$$

$$= 2z^2$$

on the surface BCDE, z varies from 0 to 3 and x varies from 0 to 1

$$\therefore I_3 = \int_{x=0}^{1}\int_{z=0}^{1} 2z^2 dxdz$$

$$= \int_{x=0}^{1}\left(\frac{2z^3}{3}\right)_0^3 dx$$

$$= \frac{2}{3}\int_{x=0}^{1}\left(z^3\right)_0^3 dx$$

$$= 18\int_0^1 dx$$

$$= 18$$

Consider $\quad I_4 = \iint_{GOAE} \vec{F}_4.\hat{n}ds$

$$\vec{F}_3.\hat{n} = xz\hat{k}.(-\hat{j})$$

$$= 0$$

on the surface GOAE, z varies from 0 to 3, x varies from 0 to 1.

$$\therefore I_4 = \int_{x=0}^{1} \int_{z=0}^{3} 0.dxdz$$

$$= 0$$

Consider $$I_5 = \iint_{EDGF} \vec{F}_5.\hat{n}ds$$

$$\vec{F}_5.\hat{n} = (2xy\hat{i} + 9y\hat{j} + 3x\hat{k}).\hat{k}$$

$$= 3x$$

on the surface EDGE, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_5 = \int_{x=0}^{1} \int_{y=0}^{2} 3xdxdz$$

$$= 3\int_{0}^{1} (y)_0^2 xdx$$

$$= 3 \times 2\int_{0}^{1} xdx$$

$$= 6\left(\frac{x^2}{2}\right)_0^1$$

$$= 3$$

Consider $$I_6 = \iint_{AOCD} \vec{F}_6.\hat{n}ds$$

$$\vec{F}_6.\hat{n} = (2xy\hat{i}.(-\hat{k})$$

$$= 0$$

on the surface AOCB, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_6 = \int_{x=0}^{1} \int_{y=0}^{2} 0dxdz$$

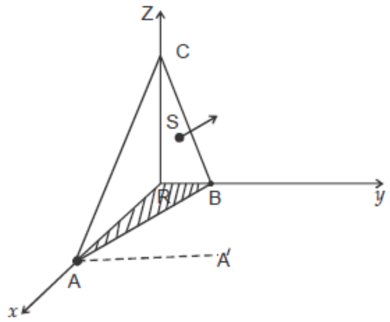
$$= 0$$

$\therefore (1) \Rightarrow \int_S \vec{F}.d\vec{s} = 12+0+18+0+3+0$

$= 13$

5. Evaluate the integral $\iint_S \vec{A}.\hat{n}ds$ if $A = 4y\hat{i} + 18z\hat{j} - x\hat{k}$ and S is the surface of the portion of the plane 3x + 2y + 6z = 6 contained in the first octant.

Solution:



Let OABC be the given surface S. Then the projection R of S on the xoy plane is OAB.

$\phi = 3x + 2y + 6z - 6$

$\nabla\phi = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$|\nabla\phi| = \sqrt{9+4+36}$

$= 7$

$\vec{A} = 4y\hat{i} + 18z\hat{j} - x\hat{k}$

$\hat{n} = \dfrac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7}$

$\hat{n}.\hat{k} = \dfrac{6}{7}$

$\vec{A}.\hat{n} = \dfrac{1}{7}(12y + 36z - 6x)$

$\iint_S \vec{A}.\hat{n}ds = \iint_S \vec{A}.\hat{n}\dfrac{dxdy}{|\hat{n}.\hat{k}|}$

$$= \iint_R \frac{1}{7}\frac{12y + 36z - 6x}{\frac{6}{7}} dxdy$$

$$= \iint_R (2y + 6z - x)dxdy$$

$$= \iint_R (2y + (-3x - 2y + 6 - x)dxdy \qquad \text{Since } 3x + 2y + 6z = 6$$

$$\therefore 6z = 6 - 3x - 2y$$

$$= \iint_R (2y - 3x - 2y + 6 - x)dxdy$$

$$= \iint_R (6 - 4xdxdy$$

Let AA'parellel to the y axis. Then R lies between the y-axis &AA', where A is (2,0,0). Thus $0 \le x \le 2$ with this restriction on the x-co-ordinate of a point of R, the y-co-ordinate varies from y =0 to $\frac{3(2-x)}{2}$. Since R is bounded by OA and AB,

$$\text{Thus } = \iint_S \vec{A}.\hat{n}ds = \int_0^2 \int_{y=0}^{\frac{3(2-x)}{2}} (6 - 4x)dydx$$

$$= \int_0^2 (6 - 4x)(y)_0^{\frac{3(2-x)}{2}} dx$$

$$= \int_0^2 (6 - 4x)\left(\frac{3(2-x)}{2}\right)dx$$

$$= \int_0^2 \frac{3}{2}(12 - 6x - 8x + 4x^2)dx$$

$$= 3\int_0^2 (2x^2 - 7x + 6)dx$$

$$= 3\left(\frac{2x^3}{3} - \frac{7x^2}{2} + 6x\right)_0^2$$

$$= 3\left(\frac{16}{3} - \frac{28}{2} + 12\right)$$

$$= 3\left(\frac{32-84+72}{6}\right)$$

$$= \frac{1}{2}(20)$$

$$= 10$$

6. Evaluate $= \iint_S \vec{F}.d\vec{s}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes z=0 &z=2.

Solution:

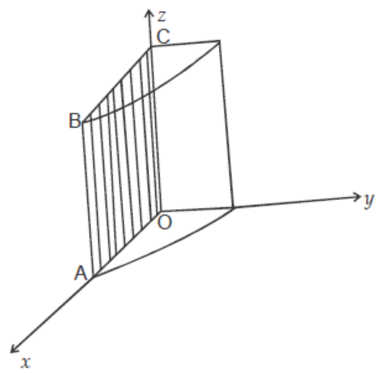
Let $\phi = x^2 + y^2 - 9$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$|\nabla\phi| = \sqrt{4(x^2+y^2)} \qquad \therefore x^2 + y^2 = 9$$

$$= 6$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{6}$$



$$= \frac{x\hat{i} + y\hat{j}}{3}$$

$$\vec{F}.\hat{n} = \frac{xyz + 2y^3}{3}$$

$$\iint_S \vec{F}.d\,\vec{s} = \iint_S \vec{F}.\hat{n}ds$$

$$= \iint_S (xyz + 2y^3)ds$$

$$= \frac{1}{3}\iint_{OR} (xyz + 2y^3)\frac{dxdz}{\hat{n}.\hat{j}}$$

where R is the rectangular region OABC in the xoz-plane, got by projecting the cylindrical surface S on the xoy plane and z varies from 0 to 2, x varies from 0 to 3.

$$\therefore \iint_S \vec{F}.\hat{n}ds = \frac{1}{3}\iint_R (xyz+2y^3)\frac{dxdz}{\frac{y}{3}}$$

$$= \iint_R (xy+2y^2)dxdz$$

$$= \int_{y=0}^{2}\int_{x=0}^{3} (xz+2(9-x^2)dxdz \qquad \because x^2+y^2=9 \Rightarrow y^2=9-x^2$$

$$= \int_0^2 \left(\frac{x^2}{2}z+18x-\frac{2x^3}{3}\right)_0^3 dt$$

$$= \int_0^2 \left(\frac{9}{2}z+18\times 3-2\times 9\right)dz$$

$$= \left(\frac{9}{2}\frac{z^2}{2}+36z\right)_0^2$$

$$= 9+72$$

$$= 81$$

Volume Integral

In multivariable calculus, a volume integral refers to an integral over a 3-dimensional domain. Let *V* denote the volume enclosed by some closed surfaces and $\vec{F}$, a vector function defined throughout *V*. Then, $\iiint_v \vec{F}.d\vec{V}$, where $d\vec{V}$ denotes an element of the volume *V*, is called the volume integral $\vec{F}$ over *V*.

We define it as the limit of a sum as follows.

Sub-divide V into n regions of elementary volumes $\Delta V_i, i = 1, 2, \ldots. n$. Let $\vec{F_i}$ be the value of $\vec{F}$ at some point P_i inside (or) on the boundary of the region, enclosing the volume ΔV_i.

Form the vector sum $I_n = \sum_{i=1}^{n} F_i \Delta V_i$. If the limit of I_n exists as $n \to \infty$, in such a way that each ΔV_i shrinks into a point, and is independent of the manner of division of *V* into these elementary volumes, then the limit is called the volume integral of $\vec{F}$ over *V* and is denoted by $\iiint_v \vec{F}.d\vec{V}$.

Remark

A volume integral is a triple integral of the constant function 1 which gives the volume of the region D (ie) the integral $Vol(D) = \iiint dxdydz.$

A triple integral within a region D in $R^3$of a function $f(x, y, z)$ is usually written as $\iiint f(x, y, z)dxdydz.$

Note

A volume integral in cylindrical co-ordinates is $\iiint\limits_D f(r, \theta, z)rdrd\theta dx$ and a volume integral in spherical co-ordinates has the form $\iiint\limits_D f(r, \theta, \phi)r^2 \sin\phi drd\theta d\phi$

Remarks

Integrating the function $f(x, y, z) = 1$ over a unit cube yields the following result $\int_0^1 \int_0^1 \int_0^1 1 \times dxdydz = 1$so, the volume of the unit cube is 1 as expected. That is, rather trivial however a volume integral is far more powerful. For instance if we have a scalar function $f : R^3 \to R$ describing the density of the cube at a given point (x,y,z) by $f = x+y+z$ then performing the volume integral will give the total mass of the cube

$$\int_0^1 \int_0^1 \int_0^1 (x+y+z)dxdydz = \frac{3}{2}.$$

Solved Problems

1. If $\vec{F} - 2z\hat{i} - x\hat{j} + y\hat{k},$ evaluate $\iiint\limits_D \vec{F}.dV$ where V is the region bounded by the surfaces $x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2$

Solution:

$$\iiint \vec{F}.dV = \iiint\limits_V (2z\hat{i} - x\hat{j} + y\hat{k})dxdydz$$

$$= \int_{x=0}^{2} \int_{y=0}^{4} \int_{z=x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})dzdydx$$

$$= \int_0^2 \int_0^4 (z^2\hat{i} - xz\hat{j} + yz\hat{k})^2 dydx$$

$$=\int_0^2\int_0^4 (4\hat{i}-2x\hat{j}+2y\hat{k}-x^4\hat{i}+x^3\hat{j}-x^2y\hat{k})dydx$$

$$=\int_0^2 \left(4y\hat{i}-2xy\hat{j}+y^2\hat{k}-x^4y\hat{i}+x^3y\hat{j}-\frac{x^2y^2}{2}\hat{k}\right)_0^4 dx$$

$$=\int_0^2 \left(16\hat{i}-8x\hat{j}+16\hat{k}-4x^4\hat{i}+4x^3\hat{j}-8x^2\hat{k}\right)dx$$

$$=\left(16x\hat{i}-4^2\hat{j}+16x\hat{k}-\frac{4x^5}{5}\hat{i}+x^4\hat{j}-\frac{8x^3}{3}\hat{k}\right)_0^2$$

$$=32\hat{i}-16\hat{j}+32\hat{k}-\frac{128}{5}\hat{i}+16\hat{j}-\frac{64}{3}\hat{k}$$

$$=\frac{32}{5}\hat{i}+\frac{32\hat{k}}{3}$$

$$=\frac{32}{15}(\hat{i}+5\hat{j})$$

2. Evaluate $\iiint_v (\nabla.\vec{F})$ if $\vec{F}=x^2\hat{i}+y^2\hat{i}+z^2\hat{i})$ and if V is the volume of the region enclosed by the cube $\theta \le x, y, z \le 1$

Solution:

$$\iiint_v (\nabla.\vec{F})d\vec{V}=2\iiint_v (x+y+z)dV$$

$$=2\int_{x=0}^1\int_{y=0}^1\int_{z=0}^1 (x+y+z)dzdydx$$

$$=2\int_0^1\int_0^1 \left(xz+yz+\frac{z^2}{2}\right)_0^1 dydx$$

$$=2\int_0^1\int_0^1 \left(x+y+\frac{1}{2}\right)dydx$$

$$=2\int_0^1 \left(xy+\frac{y^2}{2}+\frac{y}{2}\right)_0^1 dx$$

$$=2\int_0^1 \left(x+\frac{1}{2}+\frac{1}{2}\right)dx=2\left(\frac{x^2}{2}+\frac{1}{2}x+\frac{1}{2}x\right)_0^1$$

$$= 2\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$$

$$= 3$$

3. Evaluate $\iiint_{v} \nabla.\vec{A}dV$ if $\vec{A} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and S is the surface of the parellelopiped formed by the planes x = 0, x = 2, y = 0, y = 1, z = 0, z = 3.

Solution:

$$\iiint_{v} (\nabla.\vec{A})dV = \iiint_{S} (2y + z^2 + x)dV$$

$$= \int_0^2 \int_0^1 \int_0^3 (2y + z^2 + x)dzdydx$$

$$= \int_0^2 \int_0^1 \left(2yz + \frac{z^3}{3} + xz\right)_0^3 dydx$$

$$= \int_0^2 (3y^2 9y + 3x)_0^1 dx$$

$$= \int_0^2 (6y + 9 + 3x)dydx$$

$$= \int_0^2 (3 + 9 + 3x)dx$$

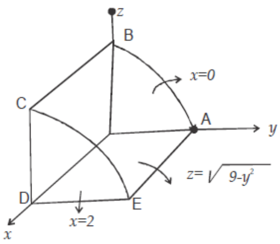
$$= \left(12x + \frac{3x^2}{2}\right)_0^2$$

$$= 24 + 6$$

$$= 30$$

4. Find $\int_{v} (\nabla.\vec{A})dV$ where $\vec{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}, V$ being the region in the first octant bounded by $y^2 + z^2 = 9$ &$x = 2$.

Solution:



$$\nabla.\vec{A} = 4xy - 2y + 8xz$$

To cover the volume of the region shown in the figure, we take x = 0 to x = 2, y = 0 to y = 3 and z = 0 to z = $\sqrt{9-y^2}$

$$\int_v \nabla.\vec{A}dV = \int_{x=0}^{2}\int_{y=0}^{3}\int_{0}^{2z\sqrt{9-y^2}} (4xy - 2y + 8xz)dzdydx$$

$$= \int_0^2\int_0^3 (4xyz - 2yz + 4xz^2)_0^{\sqrt{9-y^2}}\, dxdy$$

$$= \int_0^2\int_0^3 \left((4x-2)y\sqrt{9-y^2} + 4x(9-y^2)\right)dxdy$$

$$= \int_0^3 \left(y\sqrt{9-y^2}\left(\frac{4x^2}{2} - 2x\right) + (9-y^2)\frac{4x^2}{2}\right)_0^2 dy$$

$$= \int_0^3 \left(y\sqrt{9-y^2}(2x^2 - 2x) + (9-y^2)2x^2\right)_0^2 dy$$

$$= 4\int_0^3 y\sqrt{9-y^2}dy + 8\int_0^3 y(9-y^2)dy$$

Let $9 - y^2 = t^2$

$-2\, ydy = 2tdt$

$ydy = -tdt$

If $y = 0,\ \ t = 3$

$y = 3,\ \ t = 0$

$$\int_v (\nabla.\vec{A})dV = 4\int_3^0 \sqrt{t^2}(-tdt) + 8\left(9y - \frac{y^3}{3}\right)_0^3$$

$$= 4\int_0^3 t^2 dt + 8(27-9)$$

$$= 4\left(\frac{t^3}{3}\right)_0^3 + 144 = 4\left(\frac{27}{3}\right) + 144 = 36 + 144 = 180$$

5. Find $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^3\hat{k}$ and V is the volume enclosed by $x^2 + y^2 = a^2$, $z = h$ and prove that $\iiint_v (\nabla.\vec{F})dV = \pi a^2(4h + h^3)$

Solution:

Given $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^3\hat{k}$

$$\nabla.\vec{F} = 4 - 4y + 3z^2$$

Also, on the circle $x^2 + y^2 = a^2$, as y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$, x varies from $-a$ to a

$$\iiint_v \nabla.\vec{F}dV = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h (4 - 4y + 3z^2)dzdydx$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4z - 4zy + z^3)_0^h dydx$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4h - 4yh + h^3)dydx$$

$$= \int_{-a}^{a} (4hy - 2y^2h + h^3y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} (4y + h^3)y - 2y^2h)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} [4h + h^3)\sqrt{a^2 - x^2} - 2(a^2 - x^2)h - (4h + h^3)(-\sqrt{a^2 - x^2}) + 2(a^2 - x^2)h]dx$$

$$= \int_{-a}^{a} [4h + h^3)\left(2\sqrt{a^2 - x^2}\right) - 2h(a^2 - x^2 - a^2 + x^2)]dx$$

$$= 2\int_{-a}^{a} (4h + h^3)\sqrt{a^2 - x^2}dx$$

$$= 2(4h+h^3)\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$$

$$= 2(4h+h^3)\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_{-a}^{a}$$

$$= 2(4h+h^3)\left(\frac{a^2}{2}\sin^{-1}1-\frac{a^2}{2}\sin^{-1}(-1)\right)$$

$$= 2(4h+h^3)\left(\frac{a^2}{2}.\frac{\pi}{2}-\frac{a^2}{2}\left(\frac{-\pi}{2}\right)\right)$$

$$= \pi a^2(4h+h^3)$$

6. *S.T* if $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, $\iiint_v (\nabla.\vec{A})dV = \frac{12}{5}\pi R^5$ where *V* is the volume enclosed by the sphere of radius *R* with origin as centre.

Solution:

$$\iiint_v (\nabla.\vec{A})dV = \iiint_v (3x^2+3y^2+3z^2)dV$$

$$= 3\iiint_v (x^2+y^2+z^2)dV$$

To evaluate the integral, we consider the transformation $x = r \sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$ and $z = r\cos\theta$. Then using the Jacobian, we obtain $dV = r^2 \sin\theta dr d\theta d\phi$, where r changes from 0 to R, θ from 0 to π and ϕ from 0 to 2π.

$$\iiint_v \nabla.(\vec{A}d\vec{V}) = 3\int_0^R \int_0^\pi \int_0^{2\pi} r^2 r^2 \sin\theta d\phi d\theta dr$$

$$= 3\int_0^R \int_0^\pi r^4 \sin\theta (\phi)_0^{2\pi} d\theta dr$$

$$= 3\int_0^R \int_0^\pi r^4 \sin\theta (2\pi-0) d\theta dr$$

$$= 6\pi\int_0^R \int_0^\pi r^4 \sin\theta \, d\theta dr$$

$$= 6\pi\int_0^R r^4 (-\cos\theta)_0^\pi dr$$

$$= -6\pi \int_0^R r^4 (\cos\pi - \cos 0)\, dr$$

$$= -6\pi \times -2 \int_0^R r^4 dr$$

$$= 12\pi \left(\frac{r^5}{5} \right)_0^R$$

$$= \frac{12\pi R^5}{5}$$

Exercise

1. Show that $\iint_S \vec{F}.\hat{n}d\vec{s} = \frac{3}{2}$, where $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the x=0, x=1, y=0, y=1, z=0, and z=1.

2. Evaluate $\iint_S \vec{F}.\hat{n}d\vec{s}$ where, $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

3. Evaluate $\iint_S (z\hat{i} + x\hat{j} - y^2 z\hat{k}) d\,\vec{s}$ where S is the surface of the cylinder $x^2 + y^2 = 1$ in the first octant between the planes z=0 &z=2.

4. Find the area of the surface of the portion of the plane 3x+2y+6z=6 contained in the first octant.

5. Evaluate $\iint_S \vec{A}.\hat{n}ds$ if $\vec{A} = (x^2 + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane 2x+y+2z=6 in the first octant.

6. Evaluate $\iiint_V (\nabla.\vec{F}) d\vec{V}$ where $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ and V is the region bounded by x=0, y=0, z=0 & 2x+2y+z = 4.

7. Evaluate $\iiint_V \vec{F}.d\vec{V}$ where $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ and V is the volume of the region enclosed by the cylinder $x^2 + y^2 = a^2$ between the planes z=0, z=c.

8. Evaluate $\iiint_V \nabla.\vec{A}d\vec{V}$ if $\vec{A} = 2x^2 Y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ where V is the region bounded by the cylinder $y^2 + z^2 = 9$& the plane $x = 2$.

9. Evaluate $\iiint_V \nabla.\vec{F}d\vec{V}$ where $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped bounded by x=0, x=a, y=0, y=b, z=0, z=c.

10. Evaluate $\iiint_V 45x^2 ydV$ where V is the surface bounded by the plane x=0, y=0, z=0, 4x+2y+z=8.

Answers

1. 264

2. $\frac{25}{6}$

3. 202

4. 18π

6. $\frac{86}{105}$

7. $\frac{2}{3}$

9. 0

10. $\frac{427}{20}$

12. $\frac{3}{8}$

13. 3

14. $\frac{7}{2}$

15. 81

16. $\frac{8}{3}$

17. $\frac{a^4 c\pi}{4}\hat{k}$

18. 180

19. abc (a + b + c)

20. 128

Theorem

Gauss Divergence Theorem

If $\vec{F}$ be a vector point function having continuous partial derivation in the region bounded by a closed surface S, then where $\iiint_V (\nabla.\vec{F})dV = \iint_s \vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit outward normal at any point of the surface.

Proof

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\iiint_v (\nabla.\vec{F})dV = \iiint_v \left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})dV$$

$$= \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz \qquad \ldots (1)$$

Assume that a closed surface S is such that any line parallel to the coordinate axis intersects S at the most at two points. Divide the surface S into two parts S_1 the lower and S_2, the upper part.

Let $Z_1 = F_1(x, y)$ and $Z_2 = F_2(x, y)$ be the equation and $\hat{n}_1$ and $\hat{n}_2$ be the normals to the surface S_1 and S_2respectively. Let R be the projection of the surface S on xy-plane.

$$\iiint_v \frac{\partial F_3}{\partial F}dxdydz = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \left(\frac{\partial F_3}{\partial z}\right)dzdxdy\right]$$

$$= \iint_R [F_3(x, y, z)]_{f_1}^{f_2} dxdy$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]dxdy$$

$$= \iint_R [F_3(x, y, f_2)dxdy - \iint_R [F_3(x, y, f_1)dxdy \qquad \ldots (2)$$

$dxdy$ = projection of ds on xy-plane = $\hat{n}.\hat{k}\ ds$

For Surface S_2: $Z = F_2(x, y)$ $dxdy = \hat{n}.\hat{k}\ dS_2$

For Surface S_1: $Z = F_1(x, y)$ $dxdy = \hat{n}.\hat{k}\ ds$

Substituting in eqn (2)

$$= \iiint_v \frac{\partial F_3}{\partial z} dxdydz \qquad = \iint_{S_2} F_3(\hat{n}_2.\hat{k})ds_2 - \iint_{S_2} F_3.(-\hat{n}_1.\hat{k})ds_1$$

$$= \iint_{S_2} F_3(\hat{n}_2.\hat{k})ds_2 + \iint_{S_2} F_3(\hat{n}_1.\hat{k})ds_1$$

$$= \iint_{S_2} F_3(\hat{n}.\hat{k})ds \qquad \ldots (3)$$

Projecting the surface S on yz and zx plane, we get

$$\iiint_v \frac{\partial F_1}{\partial z} dxdydz = \iint_S F_1(\hat{n}.\hat{i})ds \qquad \ldots (4)$$

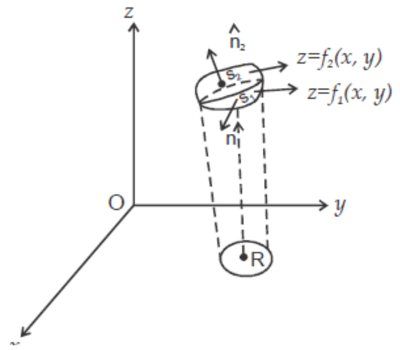
$$\iiint_v \frac{\partial F_2}{\partial z} dxdydz = \iint_S F_2(\hat{n}.\hat{i})ds \qquad \ldots (5)$$

Sub (3), (4) & (5) in (1)

$$\iiint_v (\nabla.\vec{F})dV \quad = \iint_S F_1(\hat{n}.\vec{i})ds + \iint_S F_2(\hat{n}.\vec{j})ds_2 + \iint_S F_3(\hat{n}.\vec{k})ds$$

$$= \iint_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{i}).\hat{n}ds$$

$$= \iint_S \vec{F}.\hat{n}ds$$

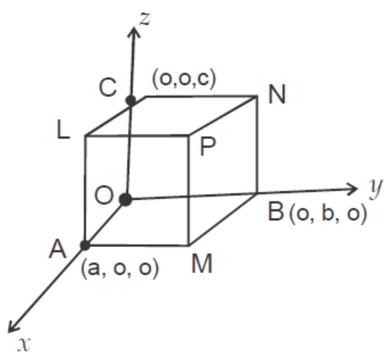


Problems

1. Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 0xy)\vec{k}$ taken over the rectangular parallopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

Solution:

For verification of divergence theorem, we shall evaluate the volume and surfaceseparately and show that they are equal.



Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 0xy)\vec{k}$

$$\nabla.\vec{F} = div\ F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).\vec{F}$$

$= 2x + 2y + 2z$

$= 2(x + y + z)$

$dV = dxdydz$ or $dV = dzdydx$

x varies from 0 to a

y varies from 0 to b

z varies from 0 to c

$$\therefore \iiint_V (\Delta.\vec{F})dV = \int_0^a \int_0^b \int_0^c 2(x + y + z)dzdydx$$

$$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right] dydx$$

$$= 2c\int_0^a \int_0^b \left[x + y + \frac{c}{2} \right] dydx$$

$$= 2c\int_0^a \left[xz + \frac{y^2}{2} + \frac{c}{2}y \right]_0^b dx$$

$$= 2bc\int_0^a \left[x\frac{b}{2} + \frac{c}{2} \right] dx$$

$$= 2bc\left[\frac{x^2}{2} + \frac{bx}{2} + \frac{cx}{2} \right]_0^a = abc[a + b + c]$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallopiped into 6 parts.

S_1 = face OAMB S_2 = face CLPN S_3 = face OBNC

S_4 = face AMPL S_5 = face OALC S_6 = face BNPM

$$\therefore \iint_C \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds \iint_{S_4} \vec{F}.\hat{n}ds + \iint_{S5} \vec{F}.\hat{n}ds \iint_{S6} \vec{F}.\hat{n}ds$$

Face $S_1 : z = 0; ds = dxdy; \hat{n} = -\vec{k}$

$$\vec{F} = x^2\vec{i} + y^2\vec{j} - xy\vec{k}$$

$$\vec{F}.\hat{n} = xy$$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}ds \int_0^a \int_0^b xydxdy = \left[\frac{x^2}{2}\right]_0^a \left[\frac{x^2}{2}\right]_0^b$$

$$= \frac{1}{4}a^2b^2$$

Face $S_2 : z = c; \hat{n} = \vec{k}; ds = dxdy$

$$\vec{F} = (x^2 - cy)\vec{i} + (y^2 - cx)\vec{j} + (c^2 - xy)\vec{k}$$

$$\vec{F}.\hat{n} = (c^2 - xy)$$

$$\therefore \iint_{S_2} \vec{F}.\hat{n}ds = \int_0^a \int_0^b (c^2 - xy)dydx$$

$$= \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx$$

$$= \int_0^a \left[c^2b - \frac{xb^2}{2}\right]dx$$

$$= b\int_0^a \left[c^2 - \frac{xb}{2}\right]dx$$

$$= b\left[c^2x - \frac{x^2}{4}b\right]_0^a$$

$$= b\left[ac^2 - \frac{a^2b}{4}\right]$$

$$= abc^2 - \frac{a^2b^2}{4}$$

Face $S_3 : \hat{n} = -\vec{i}\,; ds = dydz; x = 0$

$$\vec{F} = -yz\vec{i} + y^2\vec{j} + z^3\vec{k}$$

$$\vec{F}.\hat{n} = yz$$

$$\therefore \iint_{S_3} \vec{F}.\hat{n}ds = \int_0^b \int_0^c yzdydz$$

$$= \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{1}{4}b^2c^2$$

Face $S_4 : x = 0; \hat{n} = -\vec{i}\,; ds = dydz$

$$\vec{F} = -(a^2yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$$

$$\vec{F}.\hat{n} = a^2 - yz$$

$$\iint_{S_4} \vec{F}.\hat{n}ds = \int_0^c \int_0^b (a^2 - yz)dydz$$

$$= \int_0^c \left[a^2y - \frac{y^2}{2}z\right]_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2}{2}z\right] dz$$

$$= \left[a^2bz - \frac{b^2}{4}z^2\right]_0^c$$

$$= bc\left[a^2 - \frac{1}{4}bc\right]$$

Face $S_5 : y = 0; \hat{n} = -\vec{i}\,; ds = dzdx$

$$\vec{F} = x^2\vec{i} - zx\vec{j} + z^2\vec{k}$$

$$\vec{F}.\hat{n} = zx$$

$$\iint_{S_5} \vec{F}.\hat{n}ds = \int_0^a \int_0^c dxdzdx = \int_0^a xdx.\int_0^c zdz$$

$$= \frac{1}{2}a^2.\frac{1}{2}c^2 = \frac{1}{4}a^2c^2$$

Face $S_6 : y = b; \hat{n} = \vec{j}; ds = dzdx$

$$\vec{F} = (x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 = bx)\vec{k}$$

$$\vec{F}.\hat{n} = b^2 - zx$$

$$\iint_{S_6} \vec{F}.\hat{n}ds = \int_0^a \int_0^c (b^2 - zx)dzdx$$

$$= \int_0^a \left(b^2 z - \frac{z^2}{2}x\right)_0^c dx$$

$$= \int_0^a \left(b^2 c - \frac{c^2}{2}x\right) dx$$

$$= \left(b^2 cx - \frac{c^2}{2}x^2\right)_0^a$$

$$= ac\left(b^2 - \frac{1}{4}ac\right)$$

$$\iint_S \vec{F}.\hat{n}ds = abc^2 + ab^2c + a^2bc$$

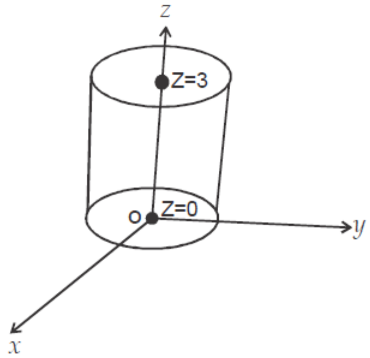
$$= abc(a+b+c)$$

$$\iint_{S_6} \vec{F}.\hat{n}ds = \iiint_V dV$$

Hence Gauss divergence theorem is verified.

2. Verify divergence theorem for $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by the cylinder $x^2 + y^2 + 4$, $z = 0$, $z = 3$

Solution:



Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

Gauss divergence theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V div\vec{F}dV$$

$$div\vec{F} = \nabla.\vec{F}$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial x}(z^2)$$

$$= 4 - 4y + 2z$$

Also given $x^2 + y^2 = 4$

$$\Rightarrow y^2 = 4 - x^2$$

$$y = \pm\sqrt{4-x^2}$$

And when $y = 0 \quad \Rightarrow 0 = \sqrt{4-x^2}$

(ie) $x = \pm 2$

$$\therefore \iiint_V \nabla.\vec{F} = \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{2=3} (4-4y+2z)dzdydx$$

$$= \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-4y) + \frac{2z^2}{2}\right]_0^3 dydx$$

$$= \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4-4y)3 + 9]dydx$$

$$= \int_{-2}^{2} \left[21y - \frac{12y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6\left((4-x^2-(4-x^2)\right)\right]dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}dx = 84\int_{0}^{2} \sqrt{4-x^2}dx \qquad \left[\therefore \sqrt{4-x^2} is\ even\ function\right]$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84[0 + 2\sin^{-1}1 - 0]$$

$$= 84 \times 2 \times \frac{\pi}{2}$$

$$= 84\pi.$$

We shall now compute the surface integral $\iint_S \vec{F}.\hat{n}ds$. S consists of the bottom surface S_1, top surface S_2 and the curved surface S_3 of the cylinder.

On $\quad S_1 : z = 0; \hat{n} = -\vec{k}$

$$\vec{F}.\hat{n} = -z^2 = 0$$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}ds = 0$$

On $\quad S_2 : z = 3; \hat{n} = \vec{k}$

$$\vec{F}.\hat{n} = z^2 = 9$$

$$ds = \frac{dxdy}{|\vec{n}.\vec{k}|} = d\frac{dxdy}{|\vec{k}.\vec{k}|} = dxdy$$

$$\iint_{S2} \vec{F}.\hat{n}ds = \iint_{S2} 9dxdy = 9\iint_{S2} dxdy$$

= 9 × area of circle S_2

= 9 ($\pi.2^2$) (radius of circle = 2)

= 36π

On $\quad S_3 : x^2 + y^2 = 4$

Let $\quad \phi = x^2 + y^2 - 4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial x} \qquad \left[\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}}\right.$$

$= 2(x\vec{i} + y\vec{j})$ $\qquad\qquad = \frac{1}{2}(x\vec{i} + y\vec{j})(\because x^2 + y^2 = 4, \sqrt{x^2 + y^2 = 2)}\ ]$

$$\vec{F}.\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}).\frac{1}{2}(x\vec{i} - y\vec{j})$$

$$= 2x^2 - y^3$$

Since S_3 is the surface of cylinder $x^2 + y^2 + 4$, we use cylindrical polar co-ordinates to evaluate $\iint_{S_3} \vec{F}.\hat{n}ds$

$x = 2\cos\theta, y = 2\sin\theta, z = z$ $\qquad (\therefore x = r\cos\theta$

$ds = 2d\theta dz$ $\qquad y = r\sin\theta$ where $r=2$

$ds = rd\theta dz)$

θ varies from 0 to 2π

z varies from 0 to 3.

$$\therefore \iint_{S_3} \vec{F}.\hat{n}ds = \int_0^3 \int_0^{2\pi} (2\times 4\cos^2\theta - 8\sin^2\theta)d\theta dz$$

$$= 16\int_0^3 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)d\theta dz$$

$$= 16\int_0^3 \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}(3\sin\theta - \sin 3\theta)\right]d\theta dz$$

$$= 16\int_0^3 \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right]_0^{2\pi} dz$$

$$= 16\int_0^3 \left[\frac{1}{2}\left(2\pi + \frac{\sin 4\pi}{2} - 0\right) - \frac{1}{4}\left(-3\cos 2\pi + \frac{\cos 6\pi}{3} - 3\cos 0 + \frac{\cos 0}{3}\right)\right]dz$$

$$= 16\int_0^3 \left(\pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12}\right)dz$$

$$= 16\pi\int_0^3 dz = 16\pi[z]_0^3$$

$$= 16\pi\times 3$$

$$= 48\pi$$

$$\therefore \iint_S \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi \qquad \ldots (2)$$

Thus from (1) & (2) $\therefore \iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$

Hence Gauss divergence theorem is verified.

Ex 3: Verify Gauss divergence theorem for $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$ over the region bounded by the upper hemisphere $x^2 + y^2 + z^2 = a^2$ and the plane z=0.

Solution:

Gauss divergence theorem is $\iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$

Given $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(a(y-x)) + \frac{\partial}{\partial z}(z^2)$$

$$= a + a + 2z + 2(a+z)$$

$$\therefore \iiint_V (\nabla.\vec{F})dV = \iiint_V 2(a+z)dV$$

$$= 2a\iiint_V dv + 2\iiint_V zdv$$

$$= 2aV + 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} zdzdydx$$

$$= 2aV + 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 2aV + \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)dydx$$

$$= 2aV + 2\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)dydx \quad \text{(since } (a^2 - x^2 - y^2) \text{ is even)}$$

$$= 2aV + 2\int_{-a}^{a} \left((a^3 - x^2)y - \frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 2aV + 2\int_{-a}^{a} \left[(a^2 - x^2)\sqrt{a^2 - x^2} - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}\right]dx$$

$$= 2aV + 2\int_{-a}^{a} \left[(a^2 - x^2) - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}\right]dx$$

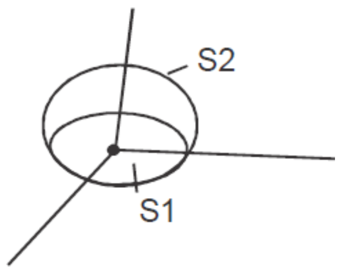
$$= 2aV + 2\int_{-a}^{a} \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} dx$$

$$= 2aV + \frac{8}{3}\int_{-a}^{a} (a^2 - x^2)^{3/2} dx \qquad (\therefore (a^2 - x^2) \text{ is even}).$$

$$= 2aV + \frac{8}{3}I$$

where $I = \int_{-a}^{a} (a^2 - x^2)^{\frac{3}{2}} dx$

Limits of x : $x = 0$ to $x = a$.



Let $x = a \sin \theta \qquad dx = a \cos \theta d\theta$

when $x = 0$, $\sin \theta = 0 \Rightarrow \theta = 0$

$x = a$, $\sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{\frac{3}{2}} a \cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^3 \cos^3 \theta a \cos\theta = a^4 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

$$= a^4 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

$$= a^4 . \frac{4-1}{4} . \frac{4-3}{4-2} . \frac{\pi}{2}$$

$$= a^4 . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$

$$I = \frac{3\pi a^4}{16}$$

$$\therefore \iiint_V \nabla . \vec{F} dV = 2aV + \frac{8}{3} \times \frac{3\pi a^4}{16}$$ (Volume of hemisphere = $\frac{2\pi}{3} a^4$)

$$= \frac{4\pi a^4}{3} + \frac{\pi a^4}{2}$$

$$= \frac{11}{6} \pi a^4 \qquad \dots (1)$$

Now we shall compute the double integral

$$\iint_S \vec{F} . \hat{n} ds$$

S consists of S_1 and S_2 ((ie) flat bottom surface & curved surface)

$$\therefore \iint_S \vec{F} . \hat{n} ds = \iint_{S_1} \vec{F} . \hat{n} ds + \iint_{S_2} \vec{F} . \hat{n} ds$$

On $S_1 : z = 0, \hat{n} = -\vec{k}$

$$\vec{F} . \hat{n} = (a(x+y)\vec{i} + a(y-x)\vec{j} + z^2 \vec{k}) . (-\vec{k})$$

$$= -z^2 = 0 \quad (\because z = 0)$$

$$\therefore \iint_S \vec{F} . \hat{n} ds = 0$$

On S_2 : Surface in $x^2 + y^2 + z^2 = a^2$

Let $\phi = x^2 + y^2 + z^2 - a^2$

$$\nabla \phi = 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\vec{F}.\hat{n} = \left[a(x+y)\vec{i} + a)(y-x)\vec{j} + z^2\vec{k}\right]\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right]$$

$$= (x+y)x + (y-x)y + \frac{z^3}{a}$$

$$= x^2 + y^2 + \frac{z^3}{a}$$

$\therefore \iint \vec{F}.\hat{n}ds = \iint_R \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\vec{k}|}$ where R is the projection of S_2 on the xy plane

$$\therefore \iint \vec{F}.\hat{n} = \iint_R \left(x^2 + y^2 + \frac{z^3}{a}\right)\frac{dxdy}{(z/a)}$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + z^2\right)dxdy$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + (a^2 - x^2 - y^2\right)dxdy$$

Change to polar co-ordinates

$x = r\cos\theta$ $y = r\sin\theta$ $r^2 = x^2 + y^2$ $dxdy = rdrd\theta$

$$\therefore \iint_{S_2} \vec{F}.\hat{n}ds = \int_0^a \int_0^{2\pi} \left[\frac{ar^2}{\sqrt{a^2 - r^2}} + (a^2 - r^2)\right]rdrd\theta$$

$$= \int_0^a \int_0^{2\pi} \left[\frac{-a(a^2 - r^2) + a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2)\right]rdrd\theta$$

$$= \int_0^a \int_0^{2\pi} \left[-a\sqrt{a^2 - r^2} + \frac{a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2)\right]rdrd\theta$$

$$= \int_0^{2\pi} d\theta . \int_0^a \left[-a\sqrt{a^2 - r^2} + \frac{a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2)\right]rdr$$

$$= [\theta]_0^{2\pi} . \int_0^a \left[-ar\sqrt{a^2 - r^2} + a^3 (a^2 - r^2)^{-1/2} . r + (a^2 - r^2) . r \right] dr$$

$$= 2\pi . \int_0^a \frac{a}{2} (a^2 - r^2)^{1/2} (-2r) . dr - \frac{a^3}{2} \int_0^a (a^2 - r^2)^{-1/2} (-2r) dr + \int_0^a (a^2 - r^2) r dr$$

$$= 2\pi . \left(\left(\frac{a}{2} . \frac{(a^2 - r^2)^{3/2}}{3/2} \right)_0^a - \frac{a^3}{2} . \left(\frac{(a^2 - r^2)^{1/2}}{1/2} \right)_0^a + \left(\frac{a^2 \times r^2}{2} - \frac{r^4}{4} \right)_0^a \right)$$

$$= 2\pi \left[\frac{-a^4}{3} + a^4 + \frac{a^4}{4} \right]$$

$$= 2\pi \left[\frac{11a^4}{12} \right] = \frac{11\pi a^4}{6}$$

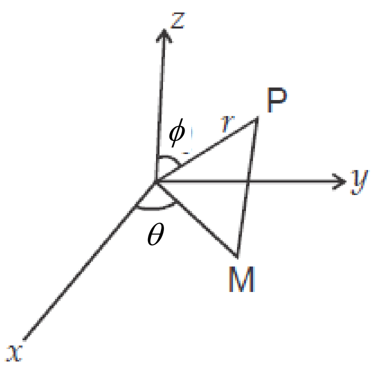
$$\therefore \iint_{S_2} \vec{F} . \hat{n} ds = \frac{11\pi a^4}{6} \qquad \ldots (2)$$

From (1) & (2) $\iint_{S_2} \vec{F} . \hat{n} ds = \iiint_V \nabla . \vec{F} dV$

Hence Gauss divergence theorem is verified.

4. Using divergence theorem evaluate $\iint_S \vec{F} . \hat{n} ds$ where $\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$ and S is the surface $x^2 + y^2 + z^2 = a^2$

Solution:



Gauss divergence theorem is

$$\iint_S \vec{F} . \hat{n} ds = \iiint_V \nabla . \vec{F} dV$$

$$\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$$

$$\nabla . \vec{F} = \frac{\partial}{\partial x}(x^3) \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(z^3)$$

$$= 3(x^2 + y^2 + z^2)$$

$$\therefore \iint_S \vec{F} . \hat{n} ds = \iiint_V 3(x^2 + y^2 + z^2) dx dy dz$$

We shall evaluate this triple integral by using spherical polar co-ordinates..

$x = r \sin\theta\cos\phi$ $\quad$ $y = r\sin\theta\sin\phi$ $\quad$ $z = r\cos\theta$

then $dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| drd\theta d\phi$

$= r^2 \sin\theta drd\theta d\phi$

$x^2 + y^2 + z^2 = r^2$

r varies from 0 to a

θ varies from 0 to π

ϕ varies from 0 to 2π

$$\therefore \iint_S \vec{F}.\hat{n}ds = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3r^4 \sin\theta drd\theta d\phi$$

$$= 3\int_0^{2\pi} d\phi.\int_0^{\pi} \sin\theta d\theta.\int_0^{a} r^4 dr$$

$$= 3(\phi)_0^{2\pi}.(-\cos\theta)_0^{\pi}.\left(\frac{r^4}{5}\right)_0^a$$

$$= 3\times 2\pi(2).\frac{a^5}{5}$$

$$= \frac{12\pi a^4}{5}$$

Note

Here $div\ \vec{F}$ $=3(x^2 + y^2 + z^2)$. Since the equation of the surface $x^2 + y^2 + z^2 = a^2$, we cannot replace $x^2 + y^2 + z^2 = a^2$ in div F, since $x^2 + y^2 + z^2 = a^2$ is true only for points on S but $\vec{F}$ is defined inside and on S.

5. If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ a,b,c are constant. Show that $\iint_S \vec{F}.\hat{n}ds = \frac{4\pi}{3}(a+b+c)$ where S is the surface of the unit sphere.

Solution:

By Gauss divergence theorem we have

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right)dV$$

$$= \iiint_V (a+b+c)dV$$

$= (a+b+c)\times$ Volume of unit sphere.

$$= (a+b+c)\frac{4\pi}{3}\times 1^3$$

$$= \frac{4\pi}{3}((a+b+c).$$

6. Using divergence theorem evaluate $\iint_S \nabla r^2.\hat{n}ds$ where S in a closed surface.

Let $\vec{F} = \nabla r^2$ where $r = x\vec{i} + y\vec{j} + z\vec{k}$

$\& r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

By Gauses divergence theorem

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V \nabla.(\nabla.r^2)dV$$

$$= \iiint_V \nabla^2 r^2 dV = \iiint_V \left(\Sigma\frac{\partial^2}{\partial x^2}\right)(x^2+y^2+z^2)dV$$

$$= \iiint_V (2+2+2)dV$$

$$= 6\iiint_V dV$$

$= 6\times$ volume of closed surfaces.

Stoke's Theorem

If S be an open surface bounded by a closed curve C and $\vec{F}$ be a continuous and differentiable vector function then $\int_C \vec{F}.d\bar{r} = \iint_S Curl\vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit outward normal at any point of the surfaces.

Proof

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\iint_S (\nabla \times \vec{F}).\hat{n}ds = \iint_S \nabla \times (F_1\vec{i} + F_2\vec{j} + F_3 k).\hat{n}ds$$

$$= \iint_S (\nabla \times F_1\vec{i}).\hat{n}ds + \iint_S (\nabla \times F_2\vec{j})\hat{n}ds + \iint_S (\nabla \times F_3\vec{k}).\hat{n}ds \qquad \ldots (1)$$

Consider, $\iint_S (\nabla \times F_1\vec{i}).\hat{n}ds + \iint_S \left[\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times F_1\vec{i}\right].\hat{n}ds$

$$= \iint_S \left(-\vec{k}\frac{\partial F_1}{\partial y} + \vec{j}\frac{\partial F_1}{\partial z}\right).\hat{n}ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\vec{j}.\hat{n} - \frac{\partial F_1}{\partial y}\vec{k}.\hat{n}\right).ds \qquad \ldots (2)$$

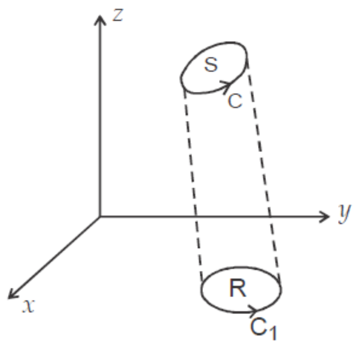
Let equation of the surface S be $z=f(x,y)$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= x\vec{i} + y\vec{j} + f(x,y)\vec{k}$

differentiating partially with respect to y.

$$\frac{\partial \vec{r}}{\partial y} = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$$



Taking dot product with $\hat{n}$

$$\frac{\partial \vec{r}}{\partial y}.\hat{n} = \vec{j}.\hat{n} + \frac{\partial f}{\partial y}\vec{k}.\hat{n} \qquad \ldots (3)$$

$\frac{\partial \vec{r}}{\partial y}$ is tangential and $\hat{n}$ is normal to the surface S, $\frac{\partial \vec{r}}{\partial y}.\hat{n} = 0$ substituting in equation (3)

$$0 = \vec{j}.\hat{n} + \frac{\partial f}{\partial y}\vec{k}.\hat{n}$$

$$\vec{j}.\hat{n} = \frac{-\partial f}{\partial y}\vec{k}.\hat{n} = \frac{-\partial z}{\partial y}\vec{k}.\hat{n} \quad [\because z = f(x,y)]$$

Substituting in equation (2):

$$\iint_S (\nabla \times F_1 i).\hat{n}ds = \iint_S \left[\frac{\partial F_1}{\partial z}\left(\frac{-\partial z}{\partial y}\vec{k}.\hat{n}\right) - \frac{\partial F_1}{\partial y}\vec{k}.\hat{n}\right]ds$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right)\vec{k}.\hat{n}ds \qquad \ldots (4)$$

Equation of the surface is $z = f(x,y)$

$F_1(x,y,z) = F_1(x,y,f(x,y)) = G(x,y)$ say

differentiating partially with respect to y,

$$\frac{\partial G}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}.\frac{\partial z}{\partial y}$$

Substituting in equation (4).

$$= \iint_S (\nabla \times F_1\vec{i}).\hat{n}ds = -\iint_S \frac{\partial G}{\partial y}\vec{k}.\hat{n}ds$$

Let R is the protection of S on xy plane and dxdy is the protection of ds on xy plane then $\vec{k}.\hat{n} = dxdy$

$$= \iint_S (\nabla \times F_1\vec{i}).\hat{n}ds = -\iint_S \frac{\partial G}{\partial y}dxdy$$

$$= \iint_{C_1} Gdx$$

Since the value of G at each point (x,y) of 1 C is some as the value of F_1at the each point (x,y,z) of C and dx is same for both the $C_1$and C we get $\iint_S (\nabla \times F_1\vec{i}).\hat{n}ds = \int_C F_1dx$ … (5)

Similarly, by projecting surface Son to yz and zx planes.

$$\iint_S (\nabla \times F_2\vec{j}).\hat{n}ds = \int_C F_2dy \qquad \ldots (6)$$

$$\iint_S (\nabla \times F_3\vec{k}).\hat{n}ds = \int_C F_3dz \qquad \ldots (7)$$

Substituting equations (5), (6) and (7) in equation (1)

$$\iint_S \nabla\times\vec{F}.\hat{n}ds = \int_C F_1dx + F_2dy + F_3dz$$

$$= \int_C \vec{F}.dr$$

Problems

1) Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines x=0, x=a, y=0, y=b

Solution:

By Stoke's theorem $\iint_S Curl\ \vec{F}.\hat{n}ds = \int_C \vec{F}.dr$

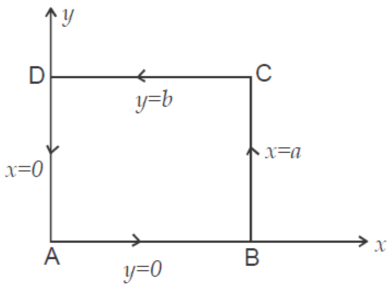
To find $\int_C \vec{F}.dr$

$$\int_C \vec{F}.dr = \int_{AB} \vec{F}.\overrightarrow{dr} + \int_{BC} \vec{F}.\overrightarrow{dr} + \int_{CD} \vec{F}.\overrightarrow{dr} \int_{DA} \vec{F}.\overrightarrow{dr}$$

Now

$$\vec{F}.dr = [(x^2 - y^2)\vec{i} + 2xy\vec{j}].[dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$= (x^2 - y^2)dx + 2xydz$$



Along AB:

y =0, dy = 0

$$\int_{AB} \vec{F}.\overrightarrow{dr} = \int_0^a x^2dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along BC:

$x = a$, dx = 0

$$\int_{BC} \vec{F}.\overrightarrow{dr} = \int_0^b 2aydy = 2a\left(\frac{y^2}{2}\right)_0^b = ab^2$$

Along CD:

$y = b$, dy = 0

$$\int_{CD} \vec{F}.\overrightarrow{dr} = \int_a^a (x^2 - b^2)dx$$

$$= \left(\frac{x^3}{3} - xb^2\right)_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$= \frac{-a^3}{3} + ab^2$$

Along DA:

$x = 0$, $dx = 0$

$$\int_{DA} \vec{F}.\overrightarrow{dr} = 0$$

$$\int_C \vec{F}.\overrightarrow{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \qquad \ldots (1)$$

To find $\iint_S Curl\ \vec{F}.\hat{n}ds$

$$\text{Now, } Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y)$$

$$= 4\vec{k}$$

Surface S is the rectangle ABCD in xy plane

$$\hat{n} = \vec{k} \text{ and } ds = \frac{dxdy}{|\hat{n}.\vec{k}|} = \frac{dxdy}{|\vec{k}.\vec{k}|} = dxdy$$

$$\iint_S curl\vec{F}.\hat{n}ds = \iint_S 4y\vec{k}.\vec{k}\ dxdy$$

$$= \int_0^a \int_0^b 4y \, dx dy$$

$$= \int_0^a 4\left(\frac{y^2}{2}\right)_0^b dx$$

$$= 2b^2 (x)_0^a$$

$$= 2ab^2 \qquad \ldots (2)$$

From equation (1) and (2).

$$\iint_S curl \, \vec{F}.\hat{n}ds = \iint_S \vec{F}.\overrightarrow{dr}$$

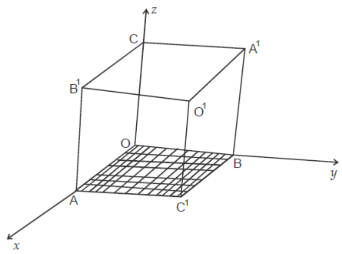
Hence Stoke's theorem is verified.

2. Verify Stoke's theorem for $\vec{F} = (xy\vec{i} - 2yz\vec{j} - xz\vec{k})$ where S is the open surface of the rectangular parallelopiped formed by the planes x=0, x=1, y=0, y=2 and z=3 above the xoy plane.

Solution:

Stoke's Theorem is, $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

To find $\int_C \vec{F}.\overrightarrow{dr}$



$$\vec{F}.\overrightarrow{dr} = (xy\vec{i} - 2yz\vec{j} - xz\vec{k}).(dx\vec{i} - dy\vec{j} - dz\vec{k})$$

$$= xydx - 2yzdy - xzdz$$

The boundary C lies on the plane z=0, $\vec{F}.\overrightarrow{dr} = xydx$

Along OA

y=0, dy=0

$$\int_{OA} \vec{F}.\overrightarrow{dr} = 0$$

Along AC^1

$x=1$, $dx=0$

$$\int_{AC^1} \vec{F}.\overrightarrow{dr} = 0$$

Along C^1B

$y=2$, $dy=0$

$$\int_{C^1B} \vec{F}.\overrightarrow{dr} = \int_1^0 2x\,dx = 2\left(\frac{x^2}{2}\right)_1^0 = -1$$

Along BO

$x=0$, $dx=0$

$$\int_{BO} \vec{F}.\overrightarrow{dr} = 0$$

$$\int_C \vec{F}.\overrightarrow{dr} = \int_{OA} \vec{F}.\overrightarrow{dr} + \int_{AC^1} \vec{F}.\overrightarrow{dr} + \int_{C^1B} \vec{F}.\overrightarrow{dr} + \int_{BO} \vec{F}.\overrightarrow{dr}$$

$$= 0 + 0 - 1 + 0 = 1 \qquad \ldots (1)$$

To find $\iint_S curl\ \vec{F}.\hat{n}ds$

Now $\quad Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$

$$= 2y\vec{i} + z\vec{j} - x\vec{k}$$

$$\iint_S curl\ \vec{F}.\hat{n}ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{x=1\\ \hat{n}=\vec{i}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{y=0\\ \hat{n}=\vec{j}}} curl\ \vec{F}.\hat{n}ds$$

$$+ \iint_{\substack{y=2\\ \hat{n}=\vec{j}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{z=0\\ \hat{n}=\vec{k}}} curl\ \vec{F}.\hat{n}ds$$

$$= \int_0^3 \int_0^2 2y\,dy\,dz + \int_0^3 \int_0^2 2y\,dy\,dz - \int_0^1 \int_0^3 z\,dz\,dx + \int_0^1 \int_0^3 z\,dz\,dx - \int_0^2 \int_0^1 x\,dx\,dy$$

$$= -\int_0^2 \int_0^1 x\,dx\,dy$$

$$= \int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy$$

$$= -\frac{1}{2}\int_0^3 dy$$

$$= -\frac{1}{2}(y)_0^2$$

$$= -\frac{1}{2}(2) = -1 \qquad \ldots (2)$$

From equations (1) and (2), $\int_C \vec{F}.\overrightarrow{dr} = \int_S curl\vec{F}.\hat{n}ds$

Hence Stoke's theorem is verified.

3. Verify Stoke's theorem for $\vec{F} = -y\vec{i} + 2yz\vec{j} + y^2\vec{k})$ where S is the upper half of the sphere$x^2 + y^2 + z^2 = a^2$and C is the circular boundary on the xoy plane.

Solution:

Stoke's Theorem is $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

To find $\int_C \vec{F}.\overrightarrow{dr}$

Here C is the circle in the xoy plane whose equation is $x^2 + y^2 = a^2$and whose parametric equations are $x = a \cos \theta$, $y = a \sin \theta$, $z = 0$, $dz = 0$.

$$\therefore \int_C \vec{F}.d\vec{r} = \int_C (-y\vec{i} + 2yz\vec{j} + y^2\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (-ydx + 2yzdy + y^2dz)$$

$$= \int_C -ydx$$

$$= \int_{x^2+y^2=a^2} -ydx$$

$$= \int_0^{2\pi} (-a\sin\theta)(-a\sin\theta)d\theta$$

$$= a^2 \int_0^{2\pi} \sin^2 \theta d\theta = a^2 \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \frac{a^2}{2}.2\pi$$

$$= \pi a^2 \qquad \ldots (1)$$

To find $\iint_S curl\ \vec{F}.\hat{n}ds$

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix}$$

$$= \vec{i}[2y - 2y] - \vec{j}[0-0] + \vec{k}[0-(-1)]$$

$$curl\ \vec{F} = \vec{k}$$

Now, $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = x^2 + y^2 + z^2 - a^2 = 0$

Here $\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-a^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-a^2) + \vec{k}\frac{\partial}{\partial x}(x^2+y^2+z^2-a^2)$

$$= \vec{i}\,2x + \vec{j}\,2y + 2z\vec{k}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

and $|\nabla\phi| = \sqrt{(2x^2 + (2y)^2 + (2z)^2} = \sqrt{4(x^2+y^2+z^2} = 2\sqrt{a^2} = 2a$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2a} + \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\iint_S cur\vec{F}.\hat{n}dt = \iint_s \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right).\vec{k}\ ds$$

$$= \iint_{s} \frac{z}{a} ds$$

$$= \iint_{R} \frac{z}{a} \frac{dx.dy}{\hat{n}.\vec{k}}$$ where R is the projection of S on the xoy plane.

$$= \iint_{R} \frac{z}{a} \frac{dx.dy}{z/a}$$

$$= \iint_{R} dxdy$$ where R is the region enclosed by $x^2 + y^2 = a^2$

$$= \pi a^2 \qquad \ldots (2)$$

From (1) and (2); Stoke's theorem is verified.

4. Evaluate by Stoke's theorem $\int_{C} (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 + 4$, $z = 2$.

Solution:

By Stoke's Theorem $\iint_{C} \vec{F}.\overrightarrow{dr} = \iint_{S} \vec{F}.\hat{n}ds$

Here $\vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$$curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

$$\therefore \int_{C} \vec{F}.\overrightarrow{dr} = \iint_{S} curl\vec{F}.\hat{n}ds$$

$$\int_{C} (e^x dx + 2ydy - dz) = 0$$

5. Evaluate $\int_{C}$ (sin zdx – cosxdy + sinydz)by using Stoke's theorem where C is the boundary of the rectangle defined by $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $z = 3$

Solution:

By Stoke's Theorem $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

Here $\vec{F} = \sin z\vec{i} - \cos x\vec{j} + \sin y\vec{k}$

Now $curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$

$$= \vec{i}[\cos y - 0] - \vec{j}[0 - \cos z] + \vec{k}[\sin x - 0]$$

$$= \cos y\vec{i} + \cos z\vec{j} + \sin x\vec{k}$$

Since S is the rectangle in the z = 3 plane, $\therefore \hat{n} = \vec{k}$

$curl\ \vec{F}.\hat{n} = (\cos y\vec{i} + \cos z\vec{j} + \sin x\vec{k}).\vec{k}$

$= \sin x$

$\int_C (\sin z dx - \cos x dy + \sin y dz)$

$= \iint_S \sin x\, dxdy$

$= \int_0^1 \int_0^\pi \sin x\, dxdy$

$= (-\cos x)_0^\pi (y)_0^1$

$= (-\cos\pi - (-\cos 0))(1-0)$

$= (1+1)(1)$

$= 2.$

Green's Theorem in the Plane

If C is a regular closed curve in the xy-plane and R be the region bounded by C, then

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$$

Where $F_1(x, y)$ and $F_2(x, y)$ are continuously differentiable functions inside and on C.

Proof

From Stoke's theorem, we have

$$\int_C \vec{F}.\overrightarrow{dr} = \iint_R Curl\vec{F}.\hat{n}ds \quad \dots (1)$$

Let $\vec{F} = F_1\vec{i} + F_2\vec{j}$, then

$$curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= \vec{i}\left(0 - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(0 - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)$$

$$= \vec{i}\left(-\frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)$$

Also as C is a closed curve in the *xy*-plane, we have.

$$\hat{n} = \vec{k}$$

$$\therefore curl\ \vec{F}.\hat{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Also on xy-plane, we have

$$ds = \frac{dxdy}{|\hat{n}.\vec{k}|} = dxdy$$

$$\therefore \iint_S curl\ \vec{F}.\hat{n}ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy \quad \dots (1)$$

Where R is the projection of S on xy-plane.

Also, $\int_c \vec{F}.\overrightarrow{dr} = \int_c (F_1\vec{i} + F_2\vec{j}).(dx\vec{i} + dy\vec{j})$

$$= \int_c F_1 dx + F_2 dy \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Corollary: If $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, the value of the integral $\int_C (F_1 dx + F_2 dy)$ is independent f the path of integration.

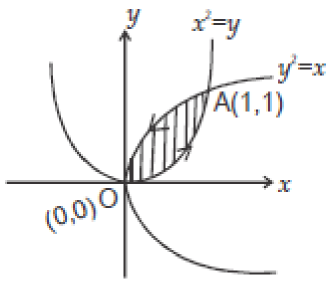
ILLUSTRATIVE EXAMPLES

1. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of theregion defined by (a) $y = \sqrt{x}$ $y = x^2$ (b) $x=0$, $y=0$, $x+y=1$

Solution:

The Green's theorem is

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$



Here $F_1 = 3x^2 - 9y^2$. $F_2 = 4y - 6x$

(a) C is $y = \sqrt{x}$, $y = x^2$

(i.e) $y^2 = x$, $y = x^2$

$$\therefore \int_C F_1 dx + F_2 dy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= I_1 + I_2 \qquad \ldots (1)$$

$$I = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along OA, $y = x^2$

$dy = 2xdx$

x varies from 0 to 1

$$\therefore I_1 = \int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)(2xdx)$$

$$= \int_0^1 (3x^2 + 8x^4 - 20x^4)dx$$

$$= (x^2 + 2x^4 - 4x^5)_0^1$$

$$= 1 + 2 - 4$$

$$\therefore I_1 = -1$$

$$I_2 = \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along AO, $x = y^2$

$dx = 2ydy$

y varies from 1 to 0

$$\therefore I_2 = \int_1^0 (3y^4 - 8y^2)(2ydy) + (4y - 6y^3)dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= 6\left(\frac{y^6}{6}\right) - 22\left(\frac{y^4}{4}\right) + 4\left(\frac{y^2}{2}\right)_1^0$$

$$= \left(y^6 - \frac{11}{2}y^4 + 2y^2\right)_1^0$$

$$= -1 + \frac{11}{2} - 2$$

$$\therefore I_2 = \frac{5}{2}$$

$\therefore$ from (1),

$$\int_C F_1 dx + F_2 dy = I_1 + I_2$$

$$= -1 + \frac{5}{2}$$

$$= \frac{3}{2} \qquad \ldots (2)$$

Now, $F_1 = 3x^2 - 8y^2, F_2 = 4y - 6xy$

$$\frac{\partial F_1}{\partial y} = 1 - 16y, \quad \frac{\partial F_2}{\partial x} = -6y$$

$$\because \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} (-6y + 16y) dxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} 10ydxdy$$

$$= 10 \int_{y=0}^{1} y(x)_{x=y^2}^{\sqrt{y}} dy$$

$$= 10 \int_{y=0}^{1} y\left(\sqrt{y} - y^2\right) dy$$

$$= 10 \int_{y=0}^{1} \left(y^{\frac{3}{2}} - y^3 \right) dy$$

$$= 10 \left[\frac{y^{5/2}}{\frac{5}{2}} - \frac{y^4}{4} \right]_{y=0}^{1}$$

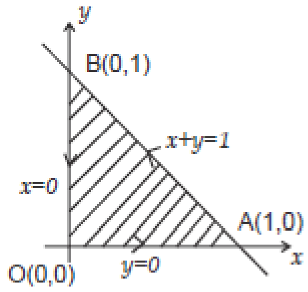
$$= 10 \left[\frac{2}{5} - \frac{1}{4} \right]$$

$$= 10 \left(\frac{3}{20} \right) = \frac{3}{2} \qquad \ldots (3)$$

From (2) and (3), we see that

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

(i.e) Green's theorem is verified.



(b) C is $x = 0, y = 0, x + y = 1$

$$\int_C F_1dx + F_2dy = \int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{OA} [3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$+ \int_{AB} [(3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{BO} [(3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= I_1 + I_2 + I_3 \text{ (say)} \qquad \ldots (1)$$

$$I_1 = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along OA, $y = 0$

$dy = 0$

x varies from 0 to 1

$$I_1 = \int_0^1 3x^2dx = 3\left(\frac{x^3}{3}\right)_0^1 = 1$$

$$I_2 = \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along AB, $x + y = 1$

$y = 1 - x$

$dy = -dx$

x varies from 1 to 0

$$\therefore I_2 = \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 (-11x^2 + 26x - 12)dx$$

$$= \left[-11\left(\frac{x^3}{3}\right) + 26\left(\frac{x^2}{2}\right) - 12(x)\right]_1^0$$

$$= 0 - \left[\frac{-11}{3} + 13 - 12\right]$$

$$= \frac{8}{3}$$

$$I_3 = \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along *BO*, $x = 0$

$dx = 0$

y varies from 1 to 0

$$\therefore I_3 = \int_1^0 4y dy = \left(\frac{y^2}{2}\right)_1^0$$

$$= 2(0-1) = -2$$

From (1),

$$\int_C (F_1 dx + F_2 dy) = I_1 + I_2 + I_3$$

$$= 1 + \frac{8}{3} - 2$$

$$= \frac{5}{3} \qquad \dots (2)$$

Now, $F_1 = 3x^2 - 8y^2$, $F_2 = 4y - 6xy$

$$\frac{\partial F_1}{\partial y} = -16y, \quad \frac{\partial F_2}{\partial x} = -6y$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = \int_{y=0}^{1} \int_{x=0}^{1-y} (-6y + 16y) dx dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1-y} 10y dx dy$$

$$= 10 \int_{y=0}^{1} y(x)_{x=0}^{1-y} dy$$

$$= 10 \int_{y=0}^{1} y(1-y)dy$$

$$= 10 \int_{y=0}^{1} y - y^2 dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{1}$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 10\left(\frac{1}{6}\right) = \frac{5}{3} \qquad \dots (3)$$

From (2) and (3), we see that

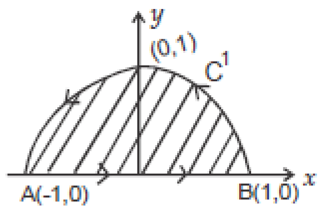
$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

(ie) Green's theorem is verified.

2) Verify Green's theorem in a plane with respect to $\oint_C (2x^2 - y^2)dx + (x^2 + y^2)dy$, where C is the boundary of the region in the xoy-plane enclosed by the x-axis and the upper half of the circle $x^2 + y^2 + 1$.

Solution:

The Green's theorem is,



$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

Here $F_1 = 2x^2 - y^2$, $F_2 = x^2 + y^2$

$$\int_C (F_1 dx + F_2 dy) = \int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$$

$$= \int_{AB} (2x^2 - y^2)dx + (x^2 + y^2)dy + \int_{C^1} (2x^2 - y^2)dx + (x^2 + y^2)dy$$

$$= I_1 + I_2 \text{ (say)} \qquad \dots (1)$$

Along AB, $\quad y = 0$

$dy = 0$

x varies from -1 to 1

$$\therefore I_1 = \int_{x=-1}^{1} (2x^2)dx$$

$$= 2\left(\frac{x^3}{3}\right)_{-1}^{1}$$

$$= \frac{2}{3}(1+1) = \frac{4}{3}$$

$$I_2 = \int_{c^1} (2x^2 - y^2)dx + (x^2 + y^2)dy$$

Along the upper half of the circle $C^1 : x^2 + y^2 = 1$,

The parametric equations are,

$x = \cos\,\theta, y = \sin\theta$

$dx = -\sin\,\theta d\theta, dy = \cos\theta d\theta$

θ varies from 0 to π

$$\therefore I_2 = \int_0^{\pi} (2\cos^2\theta - \sin^2\theta)(-\sin\theta d\theta) + (\cos^2\theta + \sin^2\theta)\cos\theta d\theta$$

$$= \int_0^{\pi} (-2\cos^2\theta \sin\theta + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2(1-\sin^2\theta)(\sin\theta) + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2\sin\theta + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2\sin\theta + \frac{3}{4}(3\sin\theta - \sin 3\theta) + \cos\theta)d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{4}\sin\theta - \frac{3}{4}\sin 3\theta + \cos\theta\right]d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{4}(-\cos\theta) - \frac{3}{4}\left(\frac{-\cos 3\theta)}{3}\right) + \sin\theta\right]_0^{\pi}$$

$$= \left(\frac{1}{4} - \frac{1}{4} + 0\right) - \left(-\frac{1}{4} + \frac{1}{4} + 0\right)$$

$$= 0.$$

From (1),

$$\int_C F_1 dx + F_2 dy = I_2 + I_2$$

$$= \frac{4}{3} + 0$$

$$= \frac{4}{3} \qquad \dots (2)$$

Now, $F_1 = 2x^2 - y^2 \quad F_2 = x^2 + y^2$

$$\frac{\partial F_1}{\partial y} = -2y, \qquad \frac{\partial F_2}{\partial x} = 2x$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} [2x - (-2y)] dx dy$$

$$= 2 \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y) dx dy$$

$$= 2 \int_{y=0}^{1} \left[\frac{x^2}{2} + yx\right]_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= 2 \int_{y=0}^{1} \left[\frac{1-y^2}{2} + y\sqrt{1-y^2}\right] - \left[\frac{1-y^2}{2} - y\sqrt{1-y^2}\right] dy$$

$$= 2 \int_{y=0}^{1} \left[\frac{1}{2} - \frac{y^2}{2} + y\sqrt{1-y^2} - \frac{1}{2} + \frac{y^2}{2} + y\sqrt{1-y^2}\right] dy$$

$$= 2 \int_{y=0}^{1} 2y\sqrt{1-y^2} dy$$

Put $\quad 1 - y^2 = t \qquad$ when $\quad y = 0, \quad t = 1$

$\quad -2y dy = dt \qquad\qquad y = 1, \quad t = 0$

$2ydy = -dt$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 2\int_{t=1}^{1} \sqrt{t}(-dt)$$

$$= -2\left[\frac{t^{3/2}}{3/2}\right]_1^0$$

$$-2\left(\frac{2}{3}\right)[0-1] = \frac{4}{3} \qquad \dots (3)$$

From (2) and (3), we see that

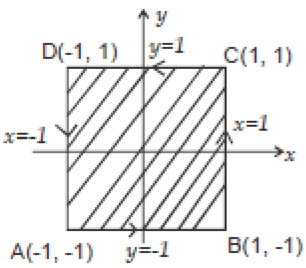
$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

(ie) Green's theorem is verified.

3. Verify Green's theorem in a plane to evaluate $\int_C x^2(1+y)dx + (x^3+y^3)dy$, where C is the square formed by $x = \pm 1$ and $y = \pm 1$

Solution

The Green's theorem is



$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

Here $F_1 = x^2(1+y),\ F_2 = x^3 + y^3$

$$\int_C (F_1 dx + F_2 dy) = \int_C x^2(1+y)dx + (x^3+y^3)dy$$

$$= \int_{AB} x^2(1+y)dx + (x^3+y^3)dy + \int_{BC} x^2(1+y)dx + (x^3+y^3)dy$$

$$+ \int_{CD} x^2(1+y)dx + (x^3+y^3)dy + \int_{DA} x^2(1+y)dx + (x^3+y^3)dy$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{(say)} \qquad \dots (1)$$

$$I_1 = \int_{AB} x^2(1+y)dx + (x^3+y^3)dy$$

Along AB, y = -1

dy = 0

x varies from -1 to 1

$$I_1 = \int_{-1}^{1} [x^2(1-1)dx + 0]$$

$$= 0$$

$$I_2 = \int_{BC} x^2(1+y)dx + (x^3 + y^3)dy$$

Along BC, x = 1

dx = 0

y varies from -1 to 1

$$\therefore I_2 = \int_{-1}^{1} (1+y^3)dy$$

$$= \left(y + \frac{y^4}{4}\right)_{-1}^{1}$$

$$= \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$= 1 + \frac{1}{4} + 1 - \frac{1}{4} = 2$$

$$I_3 = \int_{CD} x^2(1+y)dx + (x^3 + y^3)dy$$

Along CD, y = 1

dy = 0

x varies from 1 to -1

$$I_3 = \int_{1}^{-1} [x^2(1+1)dx + 0]$$

$$= 2\int_{1}^{-1} x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right)_1^{-1}$$

$$= \frac{2}{3}(-1-1) = \frac{-4}{3}$$

$$I_4 = \int_{DA} x^2(1+y)dx + (x^3+y^3)dy$$

Along *DA*, $x = -1$

$dx = 0$

y varies from 1 to -1

$$I_4 = \int_1^{-1} [0(-1+y^3)dy]$$

$$= \left[-y + \frac{y^4}{4}\right]_1^{-1}$$

$$= \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$= 1 + \frac{1}{4} + 1 - \frac{1}{4} = 2$$

From (1).

$$\int_C F_1dx + F_2dy = I_1 + I_2 I_3 + I_4$$

$$= 0 + 2 - \frac{4}{3} + 2$$

$$= \frac{8}{3} \qquad \dots (2)$$

Now $F_1 = x^2(1+y), \quad F_2 = x^3 + y^3$

$$\frac{\partial F_1}{\partial y} = x^2, \qquad \frac{\partial F_2}{\partial x} = 3x^2$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 4\int_{y=0}^{1} \int_{x=0}^{1} (3x^2 - x^2)dxdy$$

$$= 4\int_{y=0}^{1}\int_{x=0}^{1} 2x^2 dxdy$$

$$= 8\int_{y=0}^{1}\left(\frac{x^3}{3}\right)_{x=0}^{1} dy$$

$$= \frac{8}{3}\int_{y=0}^{1} dy$$

$$= \frac{8}{3}(y)_0^1$$

$$= \frac{8}{3}(1-0)$$

$$= \frac{8}{3} \qquad \ldots (3)$$

From (2) and (3), we se that

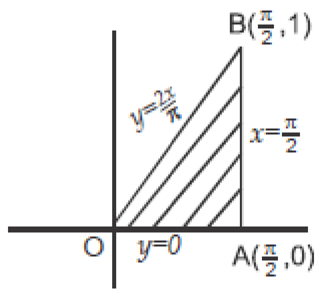
$$\int_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$ (ie) Green's theorem is verified.

4. Evaluate using Green theorem $\int_C [(y - \sin x)dx + \cos x dy]$, where C is the triangle formed by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$.

Solution:

Here $F_1 = y - \sin x$, $F_2 = \cos x$

$$\frac{\partial F_1}{\partial y} = 1, \frac{\partial F_2}{\partial x} = -\sin x$$



Using Green's theorem,

$$\int_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

(ie) $$\int_C (y - \sin x)dx + \cos x dy = \int_{y=0}^{1}\int_{x=\pi y/2}^{\pi/2} (-\sin x - 1)dxdy$$

$$= -\int_{y=0}^{1}\int_{x=\pi y/2}^{\pi/2} (\sin x + 1)dxdy$$

$$= -\int_{y=0}^{1} (-\cos x + x)_{x=\pi y/2}^{\pi/2} dy$$

$$= \int_{y=0}^{1} \left(0 + \frac{\pi}{2}\right) - \left(-\cos\left(\frac{\pi v}{2}\right) dy + \frac{\pi y}{2}\right) dy$$

$$= -\int_{y=0}^{1} \left(\frac{\pi}{2} + \cos\left(\frac{\pi v}{2}\right) - \frac{\pi y}{2}\right) dy$$

$$= \left[-\frac{\pi y}{2} + \frac{\sin \frac{\pi y}{2}}{\frac{\pi}{2}} - \frac{\pi}{2}\frac{y^2}{2} \right]_{y=0}^{1}$$

$$= -\left(\frac{\pi}{2} + \frac{2}{\pi} - \frac{\pi}{4}\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

5. By the use of Green's theorem, show that area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (xdy - ydx)$. Hence find the area of an ellipse.

Solution:

By Green's theorem is planes,

$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

Put F_1 = -y and F_2 = x

$$\frac{\partial F_1}{\partial y} = -1 \; and \; \frac{\partial F_2}{\partial x} = 1$$

Hence from (1),

$$\int_C -ydx + xdy = \iint_R (1+1) dxdy$$

$$= 2\iint_R dxdy$$

$$= 2A$$

Where A is the required area.

$\therefore A = \frac{1}{2} \int_C (xdy - ydx)$

Any point (x,y) on the ellipse is given by

$x = a \cos \theta$, $y = b \sin \theta$ where θ is the parameter.

$dx = -a \sin \theta d\theta$ $dy = b \cos \theta d\theta$

$$\therefore \text{Area of the ellipse} = \frac{1}{2} \int_C (xdy - ydx)$$

$$= \frac{1}{2} \int_0^{2\pi} a \cos\theta(b \cos \theta d\theta) - b \sin \theta(-a \sin \theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (ab\cos^2 \theta + ab\sin^2 \theta) d\theta$$

$$= \frac{1}{2}(ab) \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d\theta$$

$$= \frac{ab}{2} \int_0^{2\pi} 2\theta$$

$$= \frac{ab}{2} (\theta)_0^{2\pi} = \pi ab$$

Problems for practice

1. Verify Gauss divergence theorem for the function $\vec{F} = (x^3 - yz)\vec{i} - 2x^2 y\vec{j} + 2\vec{k}$ over the cube bounded by $x = 0$, $y = 0$, $x = a$, $y = a$, $z = 0$, $z = a$.

2. Verify Gauss divergence theorem for $F = y\vec{i} + x\vec{j} + z^2\vec{k}$ for the cylindrical region S given by $x^2 + y^2 = a^2$, $z = 0$, $z = h$.

3. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by planers $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

4. Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

5. Using divergence theorem, evaluate $\iint_S \vec{F}.\hat{n} ds$ and S is the closed surface enclosing a volume V and $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$.

6. Find $\iint_S \vec{F}.\hat{n}ds$ where $\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}$ and S is the surface of the sphere having centre at (3,-1,2) and radius 3.

7. Verify Stoke's theorem for $\vec{F} = (xy+y^2)\vec{i} - x^2\vec{j}$ the region in the xoy plane bounded by $y = x$ and $y = x^2$.

8. Verify Stoke's theorem for the function $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

9. Verify Stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ integrated around the square in the plane $z = 0$ whose sides are along the lines $x = 0$, $x = a$, $y = 0$, $y = a$.

10. Verify Stoke's theorem for $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where S is the open surface of the cube formed by $x = 0$, $x = 2$, $y = 0$, $y = 2$ and $z = 2$.

11. Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circular boundary in the xoy plane.

12. Evaluate $\int_C \vec{F}.\overrightarrow{dr}$ by Stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

13. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ prove that $\int_C \vec{r}.\overrightarrow{dr} = 0$ by using Stoke's theorem.

14. Use Stoke's theorem to find $\int_C \vec{F}.\overrightarrow{dr}$ when $\vec{F} = (xy - x^2)\vec{i} + x^2\vec{j}$ and C is the boundary of the triangle in the xoy plane formed by $x = 1$, $y = 0$ and $y = x$.

15. Verify Green's theorem is a plane to find the finite area enclosed by parabola $y^2 = 4ax$ and $x^2 = 4ay$.

16. Verify Green's theorem in a plane for $\int_C ((xy+y^2)dx + x^2dy)$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

17. Verify Green's theorem in a plane for $\int_C [(x^2+2xy)dx + (y^2+x^3y)dy]$ where C is a square with vertices A(0,0), B(1,0), D(1,1) and E(0,1).

18. Verify Green's theorem for the integral $\int_C [(x^2-2xy)dx + (x^2y+3)dy]$ along the boundary of the region defined be $y^2 = 8x$ and $x = 2$.

19. Verify Green's theorem for $\int_C [e^{-x}\sin y dx + e^{-x}\cos y dy]$ where C is the rectangle with vertices $(0,0), (\pi,0), \left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$

20. Show that the integral $\int_{(0,0)}^{(1,1)} [(x^2 + y^2)dx + 2xydy]$ is independent of the path of integration.

21. Evaluate by Green theorem in the plate $\int_C [(\cos x \sin y - xy)dx + \sin x \cos y dy]$ which C is the circle $x^2 + y^2 = 1$.

22. Using Green's theorem, evaluate $\int_C [x^2 y dx + x^2 dy]$ where C is the boundary of the triangle with vertices (0,0), (1,0), (1,1)

23. Evaluate by Green's theorem $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ which C is the rectangle with vertices (0,0), (π,0), (π,1), (0,1).

24. A vector field $\vec{F}$ is given $\vec{F} = \sin y\vec{i} + x(1+\cos y)\vec{j}$ Evaluate the integral $\int_C \vec{F}.\overrightarrow{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

25. Find the area of the loop of folium of Descartes $x^3 + y^3 = 3axy$, $a > 0$.

[Hint: Area $= \frac{1}{2}\int_C xdy - ydx$

$$= \frac{1}{2}\int_C x^2 d\left(\frac{y}{x}\right) = \frac{1}{2}\int_C x^2 dt; x = \frac{3at}{1+t^3}, y = tx$$ Limits of t are 0 to 1.]

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III –LAPLACE TRANSFORMATION– SMTA1201

UNIT – III

LAPLACE TRANSFORMS

1. Introduction

A transformation is mathematical operations, which transforms a mathematical expressions into another equivalent simple form. For example, the transformation logarithms converts multiplication division, powers into simple addition, subtraction and multiplication respectively.

The Laplace transform is one which enables us to solve differential equation by use of algebraic methods. Laplace transform is a mathematical tool which can be used to solve many problems in Science and Engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1790, in his work on probability theory. This technique became very popular when Heavisidefunctions was applied to the solution of ordinary differential equation in electrical Engineering problems.

Many kinds of transformation exist, but Laplace transform and fourier transform are the most well known. The Laplace transform is related to fourier transform, but whereas the fourier transform expresses a function or signal as a series of mode of vibrations, the Laplace transform resolves a function into its moments.

Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In Physics and Engineering it is used for analysis of linear time invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation form the time domain in which inputs and outputs are functions of time, to the frequency domain, where the same inputs and outputs are functions of complex angular frequency in radius per unit time. Given a simple mathematical or functional discription of an input or output to a system, the Laplace transform provides an alternative functional discription that often simplifies the process of analyzing the behaviour of the system or in synthesizing a new system based on a set of specification. The Laplace transform belongs to the family of integral transforms. The solutions of mechanical or electrical problems involving discontinuous force function are obtained easily by Laplace transforms.

1.1 Definition of Laplace Transforms

Let $f(t)$ be a functions of the variable t which is defined for all positive values of t. Lets be the real constant. If the integral $\int_0^\infty e^{-st} f(t)dt$ exist and is equal to F(s), then F(s) is called the Laplace transform of $f(t)$ and is denoted by the symbol L[*f*(t)].

i.e. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F[s]$

The Laplace Transform of $f(t)$ is said to exist if the integral converges for some values of s, otherwise it does not exist.

Here the operator L is called the Laplace transform operator which transforms the functions $f(t)$ into F(s).

Remark: $\lim_{S\to\infty} F(s) = 0$.

1.2 Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous in any interval $[a,b]$ if it is defined on that interval, and the interval can be divided into a finite number of sub intervals in each of which $f(t)$ is continuous.

In otherwords piecewise continuous means *f(t)* can have only finite number of finite discontinuities.

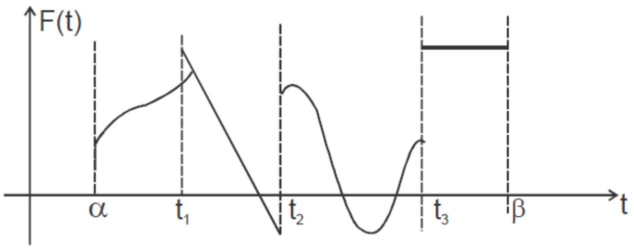


Figure 1.1

An example of a function which is periodically or sectional continuous is shown graphically in Fig 1.1 above. This function has discontinuities at t_1, $t_2$and t_3.

1.3 Definition of Exponential order

A function $f(t)$ is said to be of exponential order if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

1.4 Sufficient conditions for the existence of the Laplace Transforms

Let $f(t)$ be defined and continuous for all positive values of t. The Laplace Transform of $f(t)$ exists if the following conditions are satisfied.

(i) $f(t)$ is piecewise continuous (or) sectionally continuous.

(ii) $f(t)$ should be of exponential order.

1.5 Seven Indeterminates

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \times \infty$
4. $\infty \times \infty$
5. 1^{∞}
6. ∞^{o}.
7. 0^{o}.

Example

Check whether the following functions are exponential or not $(a)\ f(t) = t^2$ $(b)\ f(t) = e^{t^2}$

Solution:

(a) $f(t) = t^2$

By the definition of exponential order

$$\lim_{S \to \infty} e^{-st} f(t) = 0$$

$$\Rightarrow \lim_{t \to \infty} e^{-st} . t^2$$

$$\Rightarrow \lim_{t \to \infty} \frac{t^2}{e^{st}} \Rightarrow \left(\frac{\infty}{\infty}\right) \text{ which is indeterminate form}$$

Apply L – Hospital Rule

$$\lim_{t \to \infty} e \frac{2t}{e^{st} \times S} \Rightarrow \left(\frac{\infty}{\infty}\right) \text{ which is indeterminate form}$$

Again apply L – Hospital Rule.

$$\Rightarrow \lim_{t \to \infty} \frac{2}{S^2 e^{st}} \Rightarrow \lim_{t \to \infty} \frac{2}{S^2} . e^{-st} = 0 \text{ (finite)}$$

$$\therefore \lim_{t \to \infty} e^{-st} . t^2 = 0 \text{ (finite numbers)}$$

Hence $f(t) = t^2$ is exponential order.

(b) $f(t) = e^{t^2}$

Solution:

By the definition of exponential order.

$$\Rightarrow \lim_{t\to\infty} e^{-st} f(t) = 0$$

$$\Rightarrow \lim_{t\to\infty} e^{-st} . e^{t^2} \Rightarrow \lim_{t\to\infty} e^{-st+t^2} = e^{\infty} = \infty$$

$\therefore f(t) = e^{t^2}$ is not of exponential order.

2. Laplace Transform of Standard functions

1. Prove that $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$ or $s > -a$

Proof:

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{-at}] = \int_0^\infty e^{-st} . e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)}\, dt$$

$$= \left[\frac{-e^{-(s+a)}}{s+a}\right]_0^\infty = \frac{-1}{s+a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s+a}$$

Hence $L[e^{-at}] = \frac{1}{s+a}$

2. Prove that $L[e^{at}] = \frac{1}{s-a}$ where $s > a$

Proof:

By the defn of $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{+at}] = \int_0^\infty e^{-st}.e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^0\right]$$

$$= \frac{1}{s+a}$$

Hence $L[e^{at}] = \dfrac{1}{s-a}$

3. $L(\cos at) = \int_0^\infty e^{-st}\,\cos at\,dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin\,at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-S)$$

$$= \frac{s}{s^2+a^2}$$

$$\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - b\sin bx]$$

Hence $L(\cos at) = \dfrac{s}{s^2+a^2}$

4. $L(\sin at) = \int_0^\infty e^{-st}\,\sin at\,dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin\ at + a\cos at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

L(sin at) $= \dfrac{a}{s^2+a^2}$

5. *L(cos hat)* $= \dfrac{1}{2}L\left(e^{at}+e^{-at}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)+\frac{1}{2}\left(\frac{s+a+s-a}{(s+a)(s-a)}\right)$$

$$= \frac{s}{s^2-a^2}$$

L(cos hat) $= \dfrac{s}{s^2-a^2}$

6. *L(sin hat)* $= \dfrac{1}{2}L\left(e^{at}-e^{-at}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right)$$

$$= \frac{a}{s^2-a^2}$$

L(sin hat) $= \dfrac{a}{s^2-a^2}$

7. *L(1)* $= \int_0^\infty e^{-st}\,.1\,.\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left(0-\frac{1}{-s}\right) = \frac{1}{s}$$

L(1) $= \dfrac{1}{s}$

8. $L(t^n) = \int_0^\infty e^{-st} t^n \, dt$

$$= \left[(t^n) \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty nt^{n-1} \left(\frac{e^{-st}}{-s} \right) dt$$

$$= (0-0) + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$

8. $L(t^t) = \int_0^\infty e^{-st} t^n dt$

$$= \left[(t^n) \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty nt^{n-1} \left(\frac{e^{-st}}{-s} \right) dt$$

$$= (0-0) + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s} L\left(t^{n-1}\right)$$

$$L(t^n) = \frac{n}{s} L\left(t^{n-1}\right)$$

$$L(t^{n-1}) = \frac{n-1}{s} L\left(t^{n-2}\right)$$

$$L(t^3) = \frac{3}{s} L\left(t^2\right)$$

$$L(t^2) = \frac{2}{s} L(t)$$

$$L(t^n) = \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} ..... \frac{3}{s} . \frac{2}{s} . \frac{1}{s} . L(1).$$

$$= \frac{n!}{s^n} L[1] = \frac{n!}{s^n} . \frac{1}{s}$$

$$L(t^n) = \frac{n!}{s^{n+1}} \; or \; \frac{\sqrt{(n+1)}}{s^{n+1}}$$

In particular $n = 1,2,3....$

we get $L(t) = \frac{1}{s^2}$

$L(t^2) = \frac{2!}{s^3}$

$L(t^3) = \frac{3!}{s^4}$

2.1 Linear property of Laplace Transform

1. $L(f(t) \pm g(t)) = L(f(t) \pm L(g(t))$

2. $L(Kf(t) \pm g(t)) = KL(f(t)$

Proof (1): By the defn of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L(f(t) \pm g(t)] = \int_0^\infty e^{-st}\left[f(t) \pm g(t)\right]dt$$

$$= \int_0^\infty e^{-st} f(t)dt \pm \int_0^\infty e^{-st} g(t)dt$$

$$= L\left[f(t)\right] \pm L\left[g(t)\right]$$

Hence $L[f(t) \pm g(f)] = L[f(t)] \pm L[g(t)]$

(2) $L[Kf(t)] = KL[f(t)]$

By the defn of L.T

$$L[Kf(t)] = \int_0^\infty e^{-st} Kf(t)dt$$

$$= K\int_0^\infty e^{-st} f(t)dt$$

$$= KL[f(t)]$$

Hence $L[K(t)] = KL[f(t)]$

2.2 Recall

1. 2 sin A cos B= sin$(A+B)$ + sin$(A-B)$

2. $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

3. $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

4. $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

5. $\sin^2 A = \dfrac{1-\cos 2A}{2}$

6. $\cos^2 A = \dfrac{1+\cos 2A}{2}$

7. $\sin 3A = 3\sin A - 4 \sin^3 A$

8. $\cos 3A = 4\cos^3 A - 3 \cos A$

9. $\sin (A+B) = \sin A \cos B + \cos A \sin B$

10. $\sin (A-B) = \sin A \cos B - \cos A \sin B$

11. $\cos (A-B) = \cos A \cos B + \sin A \sin B$

12. $\cos (A+B) = \cos A \cos B - \sin A \sin B$

3.1 Problems

1. Find Laplace Transform of $\sin^2 t$

Solution:

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2} L(1-\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

2. Find $L(cos^3 t)$

Solution:

We know that $\cos^3 A = 4 \cos^3 A - 3 \cos A$

hence $\cos^2 A = \dfrac{3}{4}\cos A + \dfrac{1}{4}\cos 3A$

$L(\cos^2 t) \quad = \quad \frac{1}{4} L(3\cos t + \cos 3t)$

$$= \frac{1}{4}\left(\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right)$$

3. Find $L(\sin 3t \cos t)$

Solution:

We know that $\sin A \cos B \quad = \quad \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

hence $\sin 3t \cos t \quad = \quad \frac{1}{2}(\sin 4t + \sin 2t)$

$$L(\sin 3t \cos t) = \frac{1}{2} L(\sin 4t + \sin 2t)$$

$$= \frac{1}{2}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right)$$

$$= \frac{2}{s^2+16} + \frac{1}{s^2+4}$$

4. Find $L(\sin t \sin 2t \sin 3t)$

Solution:

We know that $\sin t \sin 2t \sin 3t = \sin t \frac{1}{2}(\cos t - \cos 5t)$

$$= \frac{1}{2}\sin t \cos t - \frac{1}{2}(\sin t \cos 5t)$$

$$= \frac{1}{4}\sin 2t \cos 2t - \frac{1}{4}(\sin 6t - \sin 4t)$$

$$L(\sin t \sin 2t \sin 3t) = \frac{1}{4} L(\sin 2t + \sin 4t - \sin 6t)$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2 36}\right]$$

5. Find $L(1 + e^{-3t} - 5e^{4t})$

Solution:

$$L[1+e^{-3t} - 5e^{4t}] = L[1]\ L[e^{-3t}] + 5L\ (e^{4t}]$$

$$= \frac{1}{s} + \frac{1}{s+3} - \frac{5}{s-4}$$

6. Find $L(3 + e^{6t} + \sin 2t - 5 \cos 3t)$

Solution:

$$L(3 + e^{6t} + \sin 2t - 5 \cos 3t) = 3L(1) + L(e^{6t}) + L(\sin 2t) - 5L(\cos 3t)$$

$$= 3.\frac{1}{s} + \frac{1}{s-6} + \frac{2}{s^2+4} - \frac{5s}{s^2+9}$$

7. Find $L(\sin (2t + 3))$

Solution:

$$L(\sin(2t + 3)) = L(\sin 2t \cos 3 + \sin 3 \cos 2t)$$

$$= \cos 3L\ (\sin 2t) + \sin 3L(\cos 2t)$$

$$= \cos 3\frac{2}{s^2+4} + \sin\ 3\frac{s}{s^2+4}$$

8. Find $L(\sin 4t + 3 \sin h2t - 4 \cos h5t + e^{-5t})$

Solution:

$$L(\sin 4t + 3 \sin h2t - 4 \cos h5t + e^{-5t})$$

$$= L(\sin 4t) + 3L\ (\sin h2t) - 4L\ (\cos h5t) + \mathrm{L}(e^{-5t})$$

$$= \frac{4}{s^2+16} + 3.\frac{2}{s^2-4} - 4.\frac{s}{s^2-25} + \frac{1}{s+5}$$

$$= \frac{4}{s^2+16} + \frac{6}{s^2-4} - \frac{4s}{s^2-25} + \frac{1}{s+5}$$

9. Find $L((1 + t)^2)$

Solution:

$$L((1 + t)^2) = L(1 + 2t + t^2)$$

$$= L(1) + 2L(t) + L(t^2)$$

$$= \frac{1}{s} + 2.\frac{1}{s^2} + \frac{2!}{s^3}$$

10. Find the Laplace Transform *f(t)* = $\begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Solution:

By definition,

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt$$

$$= \int_0^\pi e^{-st} \sin t\ dt + \int_\pi^\infty e^{-st}(0)dt$$

$$= \int_0^\pi e^{-st} \sin t\ dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + 1^2}(-s\sin - \cos t) \right]_0^\pi \qquad \because \int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2 + b^2}(\sin bx - b\cos bx)$$

$$= \frac{e^{-s\pi}}{s^2 + 1}(-s \sin \pi - \cos \pi) - \frac{e^0}{s^2 + 1}(0 - 1)$$

$$= \frac{e^{-s\pi}}{s^2 + 1}(1) + \frac{1}{s^2 + 1}$$

$$= \frac{1}{s^2 + 1}\left(e^{-s\pi} + 1\right)$$

11. Find the Laplace Transform *f(t)* = $\begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution:

By definition, $L(f(t)) = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} e^t dt + \int_1^\infty e^{-st} 0\, dt$$

$$= \int_0^1 e^{(-st+1)t} dt$$

$$= \left[\frac{e^{(1-s)t}}{1-s} \right]_0^1$$

$$= \frac{1}{1-s}\left(e^{1-s} - 1\right)$$

3.2 Note

1. $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$ (By definition)

$\Gamma(n+1) = n!$, $n = 1,2,3,\ldots$

$\Gamma(n+1) = n\Gamma(n)$, $n > 0$

12. Find $L\left(\frac{1}{\sqrt{t}} + t^{3/2}\right)$

Solution:

$$L\left(\frac{1}{\sqrt{t}} + t^{3/2}\right) = L(t^{-1/2}) + L(t^{3/2})$$

$$= \frac{\Gamma\left(-\tfrac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} + \frac{\Gamma\left(\tfrac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}$$

$$= \frac{\Gamma\left(\tfrac{1}{2}\right)}{s^{1/2}} + \frac{3}{2}.\frac{1}{2}\frac{\Gamma\left(\tfrac{1}{2}\right)}{s^{5/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}$$

4. First Shifting Theorem (First translation)

1. If $L(f(t)) = F(s)$. then $L(e^{-at} f(t)) = F(s+a)$

Proof

By definition, $L[f(t)] \quad = \quad \int_0^\infty e^{-st} f(t)dt$

$L[e^{-at}f(t)] \quad = \quad \int_0^\infty e^{-st}.e^{-at} f(t)dt$

$= \quad \int_0^\infty e^{-t(s+a)} f(t)dt$

$= \quad F(s+a)$

Hence $L[e^{-at} f(t)] \quad = \quad F(s+a)$

4.1 Corollary: $L(e^{at} f(t)) = F(s\text{-}a)$

4.2 Note

1. $L(e^{-at}f(t)) \quad = \quad L[f(t)]_{s\to s+a}$

$= \quad [F(s)]_{s\to s+a}$

$= \quad F(s+a)$

2. $L(e^{at}f(t)) \quad = \quad L[f(t)]_{s\to s-a}$

$= \quad [F(s)]_{s\to s-a}$

$= \quad F(s\text{-}a)$

4.3 Problems

1. Find $L(te^{2t})$

Solution:

$L(te^{2t}) \quad = \quad [L(t)]_{s\to s-2}$

$= \left(\frac{1}{s^2}\right)_{s\to s-2} \quad = \quad \frac{1}{(s-2)^2}$

2. Find $L(t^5e^{-t})$

Solution:

$L(t^5e^{-t}) \quad = \quad [L(t^5)]_{s\to s+1}$

$$= \left(\frac{5!}{s^6}\right)_{s\to s+1}$$

$$= \frac{5!}{(s+1)^6}$$

3. Find $L(e^{-2t} \sin 3t)$

Solution:

$$L(e^{-2t} \sin 3t) = L(\sin 3t)]_{s\to s+2}$$

$$= \left(\frac{3}{s^2+9}\right)_{s\to s+2}$$

$$= \frac{3}{(s+2)^2+9}$$

4. Find $L(e^{-t} \cos h4t)$

Solution:

$$L(e^{-t} \cos h4t) = L(\cos h4t)]_{s\to s+1}$$

$$= \left(\frac{3}{s^2+16}\right)_{s\to s+1}$$

$$= \frac{s+1}{(s+1)^2-16}$$

5. Find $L(e^{3t} \sin^2 4t)$

Solution:

$$L(e^{3t} \sin^2 4t) = L(\sin^2 4t)_{s\to s-3}$$

$$= L\left(\frac{1-\cos 8t}{2}\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(L(1)-L(\cos 8t)\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+64}\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{(s-3)^2+64}\right)$$

6. Find $L(e^{-2t} \sin 4t \cos 6t)$

Solution:

$$L(e^{-2t} \sin 4t \cos 6t) = L(\sin 4t \cos 6t)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(2\sin 4t \cos 6t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(\sin(4t+6t) + (\sin 4t - 6t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(\sin 10t - \sin 2t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(\frac{10}{s^2+100} - \frac{2}{s^2+4}\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(\frac{10}{(s+2)^2+100} - \frac{2}{(s+2)^2+4}\right)$$

7. Find $L(e^{4t} (\sin^3 3t + \cosh^3 3t)$

Solution:

$$L(e^{4t} (\sin^3 3t + \cosh^3 3t) = L(\sin^3 3t + \cosh^3 3t)_{s\to s-4}$$

$$= L\left(\frac{3\sin 3t - \sin 9t}{4} + \frac{3\cosh 3t + \cosh 9t}{4}\right)_{s\to s-4}$$

$$\because \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}, \cosh^3\theta = \frac{3\cosh\theta + \cosh 3\theta}{4}$$

$$= \left[\frac{3}{4}L(\sin 3t) - \frac{1}{4}L(\sin 9t) + \frac{3}{4}L(\cosh 3t) + \frac{1}{4}(\cosh 9t)\right]_{s\to s-4}$$

$$= \left(\frac{3}{4}.\frac{3}{s^2+9} - \frac{1}{4}\frac{9}{s^2+81} + \frac{3}{4}.\frac{s}{s^2-9} + \frac{1}{4}\frac{s}{s^2-81}\right)_{s\to s-4}$$

$$= \frac{3}{4}.\frac{3}{(s-4)^2+9} - \frac{1}{4}\frac{9}{(s-4)^2+81} + \frac{3}{4}.\frac{s-4}{(s-4)^2-9} + \frac{1}{4}\frac{s-4}{(s-4)^2-81}$$

8. Fine $L(\cos ht \cos 2t)$

Solution:

$$L(\cos ht \cos 2t) = \left(\left(\frac{e^t + e^{-t}}{2}\right)\cos 2t\right)$$

$$= \frac{1}{2}L(e^t \cos 2t + e^{-t}\cos 2t)$$

$$= \frac{1}{2}L(\cos 2t)_{s\to s-1} + L(\cos 2t)_{s\to s+1}$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left(\frac{s-1}{(s-1)^2+4} + \frac{s+1}{(s+1)^2+4}\right)$$

5. Theorem

If $L(f(t)) = F(s)$, then $L(tf(t)) = \frac{-d}{ds}(F(s))$

Proof:

Given $F(s) = L(f)(t))$

differentiate both sides, w.r. to 's'

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}(L(f)(t)))$$

$$= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)dt\right)$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)dt$$

$$= -\int_0^\infty tf(t)^{-st} dt$$

$$\frac{d}{ds}(F(s)) \quad = \quad -L(tf(t))$$

$$\therefore L(tf(t)) \quad = \quad \frac{-d}{ds}F(s)$$

or $L(tf(t)) \quad = \quad F'(s)$ where $F(s) = L(f(t))$

similarly we can show that,

$$L\left(t^2 f(t)\right) \quad = \quad (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$L\left(t3 f(t)\right) \quad = \quad (-1)^3 \frac{d^3}{ds^3} F(s)$$

In general, $L\left(t^n f(t)\right) \quad = \quad (-1)^n \frac{d^n}{ds^n} F(s)$

5.1 Problems

1. Fine $L\left(te^{3t}\right)$

Solution:

We know that $L\left(tff(t)\right) = \frac{-d}{ds} L(f(t))$

Here $f(t) = e^{3t}$

$$\text{L}\left(te^{3t}\right) = \frac{-d}{ds} L\left(e^{3t}\right)$$

$$= \frac{-d}{ds}\left(\frac{1}{s-3}\right)$$

$$= \left(-\frac{(s-3)(0)-(1)}{(s-3)^1}\right)$$

$$= \frac{1}{(s-3)^2}$$

2. Find $L\left(t \sin 3t\right)$

Solution:

$$L(tf(t)) = \frac{-d}{ds} L(f(t))$$

$$L(tf(t)) = \frac{-d}{ds} L(\sin 3t)$$

$$= \frac{-d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \left(\frac{-(s^2+9)(0)+3(2s)}{(s^2+9)^2}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

3. Find $L(t\cos^2 3t)$

Solution:

$$L(t\cos^2 3t) = \frac{-d}{ds} L(\cos^2 3t)$$

$$= \frac{-d}{ds} L\left(\frac{1+\cos 6t}{2}\right)$$

$$= \frac{-1}{2}\frac{d}{ds}\left(L(1)+L(\cos 6t)\right)$$

$$= \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{(s^2+16).1-s(2s)}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{16-s^2}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right)$$

4. Find $L(te^{-2t}\sin 3t)$

Solution:

$$L\left(e^{-2t}(t\sin 3t\right) = L(t\sin 3t)_{s\to s+2}$$

$$= \left\{\frac{-d}{ds}L(\sin 3t)\right\}_{s\to s+2}$$

$$= \left\{\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right\}_{s\to s+2}$$

$$= \left\{\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right\}_{s\to s+2}$$

$$= \frac{6(s+2)}{\left((s+2)^2+9\right)^2}$$

5. Find $L\left(te^{-2t}\sin 2t\sin 3t\right)$

Solution:

$$L\left(te^{-2t}\sin 2t\sin 3t\right) = L\left(t\sin 2t\sin 3t\right)_{s\to s+2}$$

$$= \left[\frac{1}{2}\times L(t.2\sin 2t\sin 3t)\right]_{s\to s+2}$$

$$= \left[\frac{1}{2}\times L(t(\cos(2t-3t)-\cos(2t+3t)))\right]_{s\to s+2}$$

$$= \frac{1}{2}\times L(t\cos t-t\cos 5t)_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L(\cos t)+\frac{d}{ds}L(\cos 5t)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)+\frac{d}{ds}\left(\frac{s}{s^2+25}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[-\left(\frac{\left(s^2+1\right).1-s(2s)}{\left(s^2+1\right)^2}\right)+\frac{d}{ds}\left(\frac{\left(s^2+25\right).1-s(2s)}{\left(s^2+25\right)^2}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[-\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)+\left(\frac{25-s^2}{\left(s^2+25\right)^2}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{s^2-1}{(s^2+1)^2}+\frac{25-s^2}{(s^2+25)^2}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{(s+2)^2-1}{((s+2)^2+1)^2}+\frac{25-(s+2)^2}{((s+2^2)+25)^2}\right]$$

6. Find $L\left(t^2e^{-t}\cos h2t\right)$

Solution:

$$L\left(e^{-t}\left(t^2\cos h2t\right)\right) = L\left(t^2\cos h2t\right)_{s\to s+1}$$

$$= \left((-1)^2\frac{d^2}{ds^2}L(\cos h2t)\right)_{s\to s+1}$$

$$= \left(\frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{d}{ds}\left(\frac{(s^2-4)\ .\ 1-s(2s)}{(s^2-4)^2}\right)\right)_{s\to s+1}$$

$$= \frac{d}{ds}\left(\frac{-4-s^2}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= \frac{-d}{ds}\left(\frac{4+s^2}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= \left(\frac{(s^2-4)^2(2s)-(4+s^2)2\,(s^2-4).(2s)}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= \left(-2s(s^2-4)\left(\frac{s^2-4-2(4+s^2)}{(s^2-4)^4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{-2s(s^2-4-8-2s^2)}{(s^2-4)^3}\right)_{s\to s+1}$$

$$= \left(\frac{2s\left(s^2+12\right)}{\left(s^2-4\right)^3}\right)_{s\to s+1}$$

$$= \frac{2(s+1)\left((s+1)^2+12\right)}{\left((s+1)^2-4\right)^3}$$

6. Theorem

If $L(f(t)) = F(s)$ and if $\underset{t\to 0}{Lt}\,\frac{f(t)}{t}$ exist then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty e^{-st} f(t)ds$

Proof:

By definition, $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt$

Integrate both sides w.r.t. 'S' from S → ∞

$$\int_s^\infty F(s)dt = \int_s^\infty \int_0^\infty e^{-st} f(t)dtds$$

$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)ds\right]dt$ (Changing the order of integration since 's' and 't' are independent variable)

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-st}ds\right)dt$$

$$= \int_0^\infty f(t)dt\left(\frac{e^{-st}}{-t}\right)_s^\infty$$

$$= \int_0^\infty f(t)dt\left(\frac{-1}{t}\left(0-e^{-st}\right)\right)$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt$$

$$= L\left(\frac{f(t)}{t}\right)$$

$$\therefore L\left(\frac{f(t)}{t}\right) = \int_s^\infty L(f(t))ds$$

Similarly we can provide that $L\left(\frac{f(t)}{t^2}\right) = \int_s^\infty \int_s^\infty L(f(t))ds\,ds$

In general $L\left(\frac{f(t)}{t_n}\right) = \underbrace{\int_s^\infty \int_s^\infty \cdots\cdots \int_s^\infty}_{n \quad times} L(f(t)) \underbrace{ds\ ds \ldots ds}_{n\,times}$

Recall

1. $\log(AB) = \log A + \log B$

2. $\log\left(A/_B\right) = \log A - \log B$

3. $\log A^B = B \log A$

4. $\log 1 = 0$

5. $\log 0 = -\infty$

6. $\log \infty = \infty$

7. $\int \frac{1}{x} dx = \log x$

8. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

9. $\tan^{-1}(\infty) = \frac{\pi}{2}$

10. $\cot^{-1}\left(s/_a\right) = \frac{\pi}{2} - \tan^{-1}\left(s/_a\right)$

Problems

1. Find $L\left(\frac{1-e^{2t}}{t}\right)$

Solutions:

$$\underset{t\to 0}{Lim} \frac{1-e^{2t}}{t} = \frac{0}{0} \text{ (Interminate form)}$$

Apply L – Hospital Rule

$$\underset{t\to 0}{Lim} \frac{-2e^{2t}}{1} = -2$$

$\therefore$ the given function exists in the limit t $\rightarrow$ 0

$$L\left(\frac{1-e^{2t}}{t}\right) = \int_s^\infty L\left(1-e^{2t}\right)ds$$

$$= \int_s^\infty \left(L(1) - L\left(e^{2t}\right)\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-2}\right)ds$$

$$= \left(\log s - \log(s-2)\right)_s^\infty$$

$$= \left[\log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s\left(1-\frac{2}{s}\right)}\right)\right]_s^\infty = \log\left(\frac{1}{1-\frac{2}{s}}\right)_s^\infty$$

$$= 0 - \log\frac{s}{s-2}$$

$$= \log\left(\frac{s}{s-2}\right)^{-1}$$

$$= \log\left(\frac{s-2}{s}\right)$$

2. Find $L\left(\frac{1-\cos at}{t}\right)$

Solution:

$$\lim_{t\to 0}\frac{1-\cos at}{t} = \frac{0}{0} \text{ (Indeterminte form)}$$

Apply L – Hospital Rule.

$$\lim_{t\to 0}\frac{a\ \sin at}{1} = 0 \text{ (finite)}$$

$\therefore$ the given function exists in the limit t $\rightarrow$ 0

$$L\left(\frac{1-\cos at}{t}\right) = \int_s^\infty L(1-\cos at)\,ds$$

$$= \int_s^\infty \left(L(1) - L(\cos at)\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right)_s^\infty$$

$$= \left(\log s - \log\left(s^2+a^2\right)^{1/2}\right)_s^\infty$$

$$= \left(\log \frac{s}{\sqrt{s^2+a^2}}\right)_s^\infty$$

$$= \left(\log \frac{s}{s\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log \frac{1}{\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log 1 - \log \frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{s}{\sqrt{s^2+a^2}}\right) = \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)^{-1} = \log\left(\frac{\sqrt{a^2+s^2}}{s}\right)$$

3. Find $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)$

Solution:

$$\underset{t\to 0}{Lim}\frac{e^{-at}-e^{-bt}}{t}=\frac{0}{0}$$ (Indeterminte form)

Apply L – Hospital Rule.

$$\underset{t\to 0}{Lim}\frac{-ae^{-at}+be^{-bt}}{1}=b-a$$

$\therefore$ the given function exists in the limit t $\rightarrow$ 0

$$L\left(\frac{e^{-at}-e^{-bt}}{t}\right) = \int_s^\infty L\left(e^{-at}-e^{-bt}\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$= \left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{(s+a)}{(s+b)}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1+a/s)}{1+b/s}\right)\right]_s^\infty$$

$$= \log 1-\log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= \log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+a}{s+b}\right)^{-1}$$

$$= \log\left(\frac{s+a}{s+b}\right)$$

4. Find $L\left(\frac{\cos at-\cos bt}{t}\right)$

Solution:

$$\underset{t\to 0}{Lim}\frac{\cos at-\cos bt}{t}=\frac{0}{0} \text{ (Interminate form)}$$

Apply L – Hospital Rule

$$\underset{t\to 0}{Lim}\frac{-a\sin at+b\sin bt}{1}=0 \text{ (finite)}$$

$\therefore$ the given function exists in the limit $t\to 0$

$$L\left(\frac{\cos at-\cos bt}{1}\right) = \int_s^\infty L(\cos at-\cos bt)ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log\left(s^2+a^2\right)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2\left(1+a^2/s^2\right)}{s^2\left(1+b^2/s^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{\left(1+a^2/s^2\right)}{\left(1+b^2/s^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1-\log\left(\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right)$$

5. Find $L\left(\frac{e^{at}-\cos bt}{t}\right)$

Solution:

Since $\underset{t\to 0}{Lim}\dfrac{e^{at}-\cos bt}{t}$ exists

$$L\left(\frac{e^{at}-\cos bt}{t}\right) = \int_s^\infty L\left(e^{at}-\cos bt\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s-a}-\frac{1}{s^2+b^2}\right)ds$$

$$= \left[\log(s-a)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\log(s-a)-\log\sqrt{s^2+b^2}\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s\left(1-a/s\right)}{s\sqrt{1+b^2/s^2}}\right)\right]_s^\infty$$

$$= \log 1-\log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= -\log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right)$$

6. Find $L\left(\dfrac{\sin^2 t}{t}\right)$

Solution:

Since $\underset{t\to 0}{Lim}\dfrac{\sin^2 t}{t}$ exists

$$L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty L\left(\sin^2 t\right)ds$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(L(1) - L(\cos 2t)\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s}{s\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{1}{s\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log \frac{s}{\sqrt{1+4/s^2}}\right]$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

7. Find $L\left(\frac{\sin 3t\cos 2t}{t}\right)$

Solution:

$\lim_{t\to 0}\left(\frac{\sin 3t\cos 2t}{1}\right)$ exists

$$L\left(\frac{\sin 3t\cos 2t}{t}\right) = \int_s^\infty L(\sin 3t\cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(2\sin 3t\cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(\sin 5t + \sin t)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)ds$$

$$= \frac{1}{2}\left[5.\frac{1}{5}\tan^{-1} s/5 + \tan^{-1} s/1\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1} s/5 + \tan^{-1} s/1\right]_s^\infty$$

$$= \frac{1}{2}\left(\tan^{-1}(\infty) + \tan^{-1}(\infty) - \tan^{-1}(s/5) - \tan^{-1}(s/1)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\tan^{-1}(s/5) - \tan^{-1}(s/1)\right)$$

$$= \frac{1}{2}\left(\pi\tan^{-1}(s/5) - \tan^{-1} s\right)$$

8. Find $L\left(\frac{\sin at}{t}\right)$. Hence find the value of $\int_0^\infty \frac{\sin t}{t}dt$

Solution:

Since $\underset{t\to 0}{Lim}\frac{\sin at}{t}$ exists

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at)\, ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

$$= \left(a.\frac{1}{a}\tan^{-1}\frac{s}{a}\right)_s^\infty$$

$$= \left(\tan^{-1}\frac{s}{a}\right)_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= L\left(\frac{\sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Deduction:

By definition

$$\int_0^{\infty} e^{-st}\frac{\sin at}{t} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Put $s = 0$, $a = 1$

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$= \frac{\pi}{2}$$

9. Find $L\left(\frac{\cos at}{t}\right)$

Solution:

$$\underset{t\to 0}{Lt}\frac{\cos at}{t} = \frac{1}{0} = \infty$$

$\therefore \underset{t\to 0}{Lt}\frac{\cos at}{t}$ does not exist.

Hence $L\left(\frac{\cos at}{t}\right)$ does not exist.

10. Find $L\left(\frac{e^{at}}{t}\right)$

Solution:

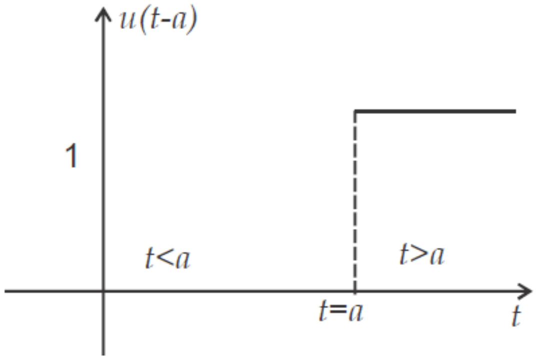
$$\underset{t \to 0}{Lt} \frac{e^{at}}{t} = \frac{1}{0} = \infty$$

$\therefore L\left(\frac{e^{at}}{t}\right)$ does not exist.

7. Unit Step function (or) heavisides unit step function

The unit step function about the point $t = a$ is defined as $U(t\text{-}a) = \begin{cases} 0 & for\ t < a \\ 1 & for\ t > 1 \end{cases}$

It can also be denoted by *H(t-a)*



7.1 Find the Laplace transform of unit step function.

Solution:

The Laplace transform of unit step function is

$$L(U(t-a)) = \int_0^\infty e^{-st} U(t-a)dt$$

$$= \int_0^a e^{-st} 0.dt + \int_a^\infty e^{-st}(1)dt$$

$$= \int_a^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{-1}{s}\left(e^{-\infty} - e^{-as}\right)$$

$$L(U(t-a)) = \frac{-1}{s}\left(0 - e^{-at}\right) = \frac{e^{-as}}{s}$$

$$\therefore L(U(t-a)) = \frac{e^{-as}}{s}$$

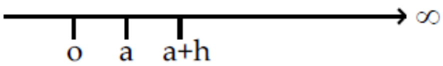
8. Dirac delta function (or) Unit Impulse function

8.1 Dirac delta function or unit impulse function about the point t = a is defined as

$$\delta(t-a) = \begin{cases} \underset{h\to 0}{Lt}\,\dfrac{1}{h} & a < t < a+h \\ 0 & otherwise \end{cases}$$

Find the Laplace transform of Dirac delta function.

Solution:



$$L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)dt$$

$$= \int_0^\infty e^{-st}0dt + \underset{h\to 0}{Lt}\frac{1}{h}\int_a^{a+h} e^{-st}dt + \int_{a+h}^\infty e^{-st}0dt$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\int_a^{a+h} e^{-st}dt$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{-1}{s}\left(e^{-(a+h)s} - e^{-as}\right)\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{e^{-as}}{s}\frac{e^{-(a+h)s}}{s}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{e^{-as}\left(1-e^{-hr}\right)}{sh} = \frac{0}{0} \text{ (Indeterminate form)}$$

Applying L′ Hospital Rule.

$$= \underset{h\to 0}{Lt}\frac{e^{-as}\left(e^{-hs}s\right)}{s} = e^{-as}$$

$$L(\delta(t-a)) = e^{-as} \text{ when } a = 0, L(\delta(t)) = 1$$

8.2 Note

The dirac delta function is the derivative of unit step function.

9. Second shifting Theorem (Second Translation)

If $L(f(t)) = F(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0 & t > a \end{cases}$,

Then $L(G)(t)) = e^{-as}F(s)$

Proof:

$$L(G)(t)) = \int_0^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}0\,.\,dt + \int_a^\infty e^{-st}f(t-a)dt$$

$$= \int_0^\infty e^{-st}f(t-a)dt$$

Put $t - a = u$ When $t = a,\quad u = 0$

$dt = du$ $t = \infty,\quad u = \infty$

$$\therefore L(G(t)) = \int_0^\infty e^{-st(u+a)}f(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su}f(u)du$$

In $\int_0^a e^{-su}f(u)du, u$ is a dummy variable. Hence we can replace it by the variable t.

$$\therefore L(G)(t)) = e^{-sa}\int_0^\infty e^{-st}f(t)dt$$

$$= e^{-sa}L(f(t))$$

$$= e^{-as}F(S)$$

Another form of second shifting theorem

If $L(f)(t)) = F(s)$ and $a > 0$ then

$L(f(t-a)U(t-a) = e^{-as}F(s)$ where $U(t-a)$ is the unit step function.

Proof:

We know that by the definition of unit step function.

$$U(t-a)=\begin{cases}1 & t>a\\ 0 & t<a\end{cases}$$

$$\therefore\ f(t-a)U(t-a)=\begin{cases}f(t-a) & t>a\\ 0 & t<a\end{cases} \qquad \ldots(1)$$

Let $f(t\text{-}a)U(t\text{-}a) = G(t)$

$$\therefore (1) \text{ becomes, } G(t)=\begin{cases}f(t-a) & t>a\\ 0 & t<a\end{cases}$$

which is precisely the same as the first form of second shifting theorem, as discussed above

$$\therefore L(G)(t)) = e^{-as}F(s)$$

9.1 Problems

1. Find the Laplace transform of $G(t)$, where

$$G(t)=\begin{cases}\cos\left(t-\dfrac{2\pi}{3}\right) & if \quad t>\dfrac{2\pi}{3}\\ 0 & if \quad t<\dfrac{2\pi}{3}\end{cases}$$

Solution:

We know that by second shifting if $L(f(t)) = F(s)$ and $G(t)$ $\begin{cases}f(t-a) & t>a\\ 0 & t<a\end{cases}$

then $L(G)(t)) = e^{-as-} F(s)$ $\qquad \ldots(1)$

Here $f(t\text{-}a) = cos \left(t = \dfrac{2\pi}{3}\right)$

(ie) $f(t) = cos\ t\ \&\ a\ \dfrac{2\pi}{3}$ $\qquad \ldots (2)$

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} \qquad \ldots (3)$$

Submitting (2) & (3) in (1), we get

$$\therefore L(G(t)) = e^{\frac{2\pi}{3}s} . \frac{s}{s^2+1}$$

2. Find the Laplace transform using second shifting theorem for

$$G(t) = \begin{cases} (t-a)^3; & t > a \\ 0 & t < a \end{cases}$$

Solution:

Here a = 2, $f(t\text{-}a)$ = t-2$)^3$

$f(t) = t^3$

$L(f(t)) = L(t^3) =) \frac{3!}{s^4} = F(s)$

$\therefore L(G)(t)) = e^{-as} F(s)$

$= e^{-as} \frac{3!}{s^4}$

3. Using second shifting theorem, find the Laplace transform of

$$G(t) = \begin{cases} \sin t - \pi/3; & t > \pi/3 \\ 0 & t < \pi/3 \end{cases}$$

Solution:

Here $a = \frac{\pi}{3}$, $f(t-a) = \sin\left(t - \pi/3\right)$

$\therefore f(t) = \sin t$

$\therefore L(f(t)) \quad = \quad L(\sin t)$

$= \quad \frac{1}{s^2+1} = F(s)$

$\therefore L(G(t)) \quad = \quad e^{-as}F(s)$

$= \quad e^{-\pi/3 s} . \frac{1}{s^2+1}$

$$= e^{-\pi/3 s} \frac{1}{s^2+1}$$

10. Change of Scale Property

If $L(f(t)) = F(s)$, Then $L(f)at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof:

By definition, $L(f(t) = \int_0^{\infty} e^{-st} f(t)dt$

$\therefore L(f)(at) = \int_0^{\infty} e^{-st} f(a\,t)dt$

Put $at = y$ when $t = 0, \ y = 0$

$adt = dy$ $t = \infty, \ y = \infty$

$$L(f(at)) = \int_0^{\infty} e^{-s(y/a)} f(y)\frac{dy}{a}$$

$$= \frac{1}{a}\int_0^{\infty} e^{-(s/a)y} f(y)dy$$

$$= \frac{1}{a}\int_0^{\infty} e^{-(s/a)y} f(t)dt$$ (Replacing the dummy variable y by t)

$$L(f(at)) = \frac{1}{a} F\left(s/a\right)$$

10.1 Corollary

$$L\left[f\left(t/a\right)\right] = aF(as)$$

10.2 Problems

1. Assuming $L(\sin t)$. Find $L(\sin 2t)$ and $L\left(\sin t/2\right)$

Solution:

We know that $L(\sin t) = \frac{1}{s^2+1}$ … (1)

$\therefore L(\sin 2t) = \dfrac{1}{2} \cdot \dfrac{1}{\left(s/2\right)^2 + 1}$ Using (1) (Replace S by s/2)

$L(\sin 2t) = \dfrac{1}{2}\left(\dfrac{4}{s^2+4}\right)$

$= \dfrac{2}{s^2+4}$ … (2)

$\therefore L\left(\sin t/2\right) = 2\dfrac{1}{(2s)^2+1} = \dfrac{2}{4s^2+1}$ Using (2) (Replace s by 2s)

2. Give that $L(t\cos t) = \dfrac{s^2-1}{(s^2+1)^2}$

Find (i) $L(t\cos at)$ and (ii) $L\left(t\cos\dfrac{t}{a}\right)$

Solution:

(i) Given $L(t\cos t) = \dfrac{s^2-1}{(s^2+1)^2}$

Replacing t by at

$\therefore L(at\cos at) = \dfrac{1}{a}\dfrac{\left(s/a\right)^2 - 1}{\left(\left(s/a\right)^2+1\right)}$ ($\because$ Replacing s by s/a)

$L(at\cos at) = \dfrac{a^4(s^2-a^2)}{a^3(s^2+a^2)^2}$

$\therefore L(t\cos at) = \dfrac{a^4(s^2-a^2)}{a^4(s^2+a^2)^2} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

(ii) Given $L\left(t\cos\dfrac{t}{a}\right) = \dfrac{s^2-1}{(s^2+1)^2}$

Replacing by $t/a, L\left(t/a \cos t/a\right) = a\left(\dfrac{(as)^2-1}{\left((as)^2+1\right)^2}\right)$

$$L\left(t\cos\frac{t}{a}\right)=a^2\left(\frac{a^2s^2-1}{\left(a^2s^2+1\right)^2}\right)$$ Replace s by as.

11. Laplace Transform of Derivations

Here, we explore how the Laplace transform interacts with the basic operators of calculus differentation and integration. The greatest interest will be in the first identity that we will derive. This relates the transform of a derivative of a function to the transform of the original function, and will allow to convert many initial - value problems to easily solved algebraic Equations. But there are useful relations involving the Laplace transform and either differentiation (or) integration. So we'll look at them too.

11.1. Theorem

If $L(f(t)) = F(s)$ Then

(i) $L(f'(t)) = sL(f(t)) - f$

(ii) $L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$

and in genereal

$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots\ldots f^{n-1}(0)$

Proof:

(i) By definition,

$$
\begin{aligned}
L(f'(t)) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty e^{-st} d(f(t) \\
&= \left(e^{-st} f(t)\right)_0^\infty - \int_0^\infty f(t)d\left(e^{-st}\right) \\
&= \left(0 - f(0)\right) - \int_0^\infty f(t)e^{-st}(-s)dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)dt \\
&= -f(0) + sL\left(f(t)\right)
\end{aligned}
$$

$\therefore L(f'(t)) = SL(f(t)) - f(0)$ …. (1) which proves (i)

(ii) Again by definition,

$$L(f''(t)) = \int_0^\infty e^{-st} f''(t)dt$$

$$= \int_0^\infty e^{-st} d(f'(t))$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty f'(t)e^{-st}(-s)dt$$

$$= \left[0 - f'(t)\right] + S\int_0^\infty e^{-st} f'(t)dt$$

$$= -f'(0) + sL(f'(t))$$

$$= sL(f'(t)) - f'(0)$$

$$= s(sL(f(t)) - f(0)) - f'(0) \qquad \text{Using (1)}$$

$$L(f''(t)) = s^2 Lf(t) - sf(0) - f'(0) \qquad \text{…(2)}$$

Similarly proceeding like this, we can show that

$$L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots f^{n-1}(0) \qquad \text{…(3)}$$

The above results (1), (2) and (3) are very useful in solving linear differential equations with constant coefficients.

11.2 Note

We have, $L(f'(t)) = sL(f(t)) - f(0)$ …. (1)

and

$L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)$ ….(2)

When $f(0) = 0$ and $f'(0) = 0$

(1) & (2) becomes

$Lf'(t) = sLf(t)$ and $Lf''(t) = s^2 Lf(t)$

This shows that under certain conditions, the process of Laplace transform replaces differentiation bymultiplication by the factor s and *s* 2 respectively.

12. Laplace Transform of integrals

Analogous to the differentiation identities $L[f'(t)] = sF(s) - f(0)$ and $L[tf(t) = -F'(s)$are a pair of identities concerning transforms of integrals and integrals of transforms. These identities will not be nearly as important to us as the differentiation identities, but they do have their uses and are considered to be part of the standard set of identities for the Laplace Transform.

Before we start, however, take another look at the above differentiation identities. They show that, under the Laplace transform, the differentiation of one of the functions, f(t) or F(S) corresponds to the multiplication of the other by the approprate variable.

This may lead to suspect that the analogous integrations identities. They show that, under Laplace transform integration of one of the functions f(t) or F(S), corresponds to the division of the other by the appropriate variables.

12.1 Theorem: If $L[f(t)] = F(s)$ then $L\left[\int_0^r f''(t)dt\right] = \frac{1}{s}L[f(t)]$

Proof:

Let $\int_0^t f(t)dt = \phi(t)$ …(1)

Differentiate both sides with respect to 't'

$\therefore f(t) = \phi(t)$ …(2)

and $\phi(0) = \int_0^t f(t)dt = 0$

We know that $L[\phi(t)] = sL[\phi(t)] - \phi(0)$

$L[\phi(t)] = sL[\phi(t)]$ $\therefore \phi(0) = 0$

$\therefore L[f(t)] = sL\left[\int_0^t f(t)dt\right]$ by (1) & (2)

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

Similarly we can prove that

$$L\left[\int^t \int^t f(t)dt\right] = \frac{1}{s^2}L[f(t)]$$

$$\therefore \text{ In general } L\left[\underbrace{\int^{t}\int^{t}\cdots\cdots\int^{t}}_{n\ items} f(t)dt\right] = \frac{1}{sn} L[f(t)]$$

12.2 Note

The above result expresses that the integral between the limits from '0' to '*t*' is transformed into simple division by the factor '*S*' using Laplace transform.

12.3 Problems

1. Find $L\left(e^{-t}\int_0^t t\cos t dt\right)$

Solution:

$$L\left(e^{-t}\int_0^t t\cos dt\right) = \left[L\left(\int_0^t t\cos t dt\right)\right]_{s\to s+1}$$

$$= \left(\frac{1}{s}L(t\cos t)\right)_{s\to s+1}$$

$$= \left(\frac{1}{s}\left(\frac{-d}{ds}(L(\cos t))\right)\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{s^2+1-s(2s)}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left(\frac{s^2-1}{s(s^2+1)^2}\right)_{s\to s+1}$$

$$= \left(\frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}\right)_{s\to s+1}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

2. Find $L\left(e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right)$

Solution:

$$L\left(e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right) = \left[L\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}L\left(\frac{\sin t}{t}\right)\right]_{s\to s+1}$$

Since $\lim_{t\to 0}\frac{\sin t}{t}$ exist

$$= \left[\frac{1}{s}\int_s^\infty L(\sin t)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\tan^{-1} s\right)_s^\infty\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\tan^{-1}\infty - \tan^{-2}(s)\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}(s)\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\right]_{s\to s+1} = \frac{\cot^{-1}(s+1)}{s+1}$$

3. Find the Laplace Transform of $\int_0^t te^{-t}\sin t dt$

Solution:

$L(te^{-t}\sin tdt) = (L(t\sin t))_{s\to s+1}$

$$= \left(\frac{-d}{ds} L(\sin t)\right)_{s \to s+1}$$

$$= \left(\frac{-d}{ds}\left(\frac{1}{s^2+1}\right)\right)_{s \to s+1}$$

$$= \left(\frac{(s^2+1)0 - 2S}{(s^2+1)^2}\right)_{s \to s+1}$$

$$= \left(\frac{2s}{(s^2+1)^2}\right)_{s \to s+1}$$

$$= \frac{2(s+1)}{((s+1)^2+1)^2}$$

$$= \frac{2(s+1)}{s^2+2\,s+2}$$

4. Find $L\left(\int_0^t \frac{e^{-t}\sin t}{t} dt\right)$

Solution:

$$L\left(\int_0^t \frac{e^{-t}\sin t}{t} dt\right) = \frac{1}{s} L\left(\frac{e^{-t}\sin t}{t}\right)$$

Since $\lim_{t \to 0} \frac{e^{-t}\sin t}{t}$ exist.

$$= \frac{1}{s}\left[\int_s^\infty L(e^{-t}\sin t)\right] ds$$

$$= \frac{1}{s}\left[\int_s^\infty L(\sin t)\right]_{s \to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{1}{s^2+1}\right)\right]_{s \to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{1}{(s+1)^2+1}\right)\right] ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{ds}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{s}\left(\tan^{-1}(s+1)\right)_s^\infty$$

$$= \frac{\cot^{-1}(s+1)}{s}$$

Problems

1. Find $L\left(\int_0^t e^{2t}dt\right)$

Solution:

$$L\left(\int_0^t e^{2t}dt\right) = \frac{1}{s}L\left(e^{2t}\right)$$

$$= \frac{1}{s}\cdot\frac{1}{s-2}$$

$$= \frac{1}{s(s-2)}$$

2. Find $L\left(\int_0^t \sin 3t dt\right)$

Solution:

$$L\left(\int_0^t \sin 3t dt\right) = \frac{1}{s}L(\sin 3t)$$

$$= \frac{1}{s}\cdot\frac{3}{s^2+9}$$

$$= \frac{3}{s(s^2+9)}$$

3. Find $L\left(\int_0^t e^{-2t}\cos 3t dt\right)$

Solution:

$$L\left(\int_0^t e^{-2t}\cos 3t dt\right) = \frac{1}{s}L(e^{-2t}\cos 3t)$$

$$= \frac{1}{s} L(\cos 3t)_{s \to s+2} \qquad \text{(Using first shifting theorem)}$$

$$= \frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s \to s+2}$$

$$= \frac{1}{s}\left(\frac{s+2}{(s+2)^2+9}\right)$$

4. Find $L\left(\int_0^t e^{-t} \sin h2t dt\right)$

Solution:

$$L\left(\int_0^t e^{-t} \sin h2t dt\right) = \frac{1}{s} L\left(e^{-t} \sin h2t\right)$$

$$= \frac{1}{s} L(\sin h2t)_{s \to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{s^2-4}\right)_{s \to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{(s+1)^2-4}\right)$$

5. Find $L\left(\int_0^t \sin 3t \cos 2t dt\right)$

Solution:

$$L\left(\int_0^t \sin 3t \cos 2t dt\right) = \frac{1}{s} L(\sin 3t \cos 2t)$$

$$= \frac{1}{2s} L(2\sin 3t \cos 2t)$$

$$= \frac{1}{2s} L(\sin 5t + \sin t)$$

$$= \frac{1}{2s}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$$

6. Find $L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt$

Solution:

$$L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt = L\left(\int_0^t t\sin^2 tdt\right)_{s\to s+3}$$

$$= \left[\frac{1}{s}L\left(t\sin^2 t\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}L\left(\sin^2 t\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}L\left(\frac{1-\cos 2t}{2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\frac{d}{ds}L(1-\cos 2t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\left(\frac{-1}{s^2}-\frac{\left(s^2+4\right).1-s(2s)}{\left(s^2+4\right)^2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{+1}{2s}\left(\frac{+1}{s^2}-\frac{4-s^2}{\left(s^2+4\right)^2}\right)\right]_{s\to s+3}$$

$$= \frac{1}{2(s+3)}\left(\frac{+1}{(s+3)^2}+\frac{4-(s+3)^2}{((s+3)^2+4)^2}\right)$$

$$= \frac{1}{2(s+3)^3}\left(\frac{4-(s+3)^2}{2(s+3)(s^2+6s+13)^2}\right)$$

7. Find $L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)\right)$

Solution:

$$L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}\,dt\right)\right) = \frac{1}{s}L\left(\frac{\sin 3t\cos 2t}{t}\right)_{s\to s-4}$$

$$= \left[\frac{1}{s}L\left(\frac{\sin 3t\cos 2t}{t}\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{s}\int_s^\infty L(\sin 3t\cos 2t)dt\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(\sin 3t\cos 2t)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(\sin 5t+\sin t)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty \left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(5.\frac{1}{5}\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)_s^\infty\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left((\tan^{-1}\infty+\tan^{-1}\infty)-\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\pi-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right)\right]_{s\to s-4}$$

$$= \frac{1}{2(s-4)}\left(\pi-\tan^{-1}\frac{s-4}{5}-\tan^{-1}(s-4)\right)$$

13. Periodic Functions

Laplace transform of periodic functions have a particular structure. In many applications the nonhomogeneous term in a linear differential equation is a periodic function. In this section, we desire a formulafor the Laplace transform of such periodic functions.

13.1 Definition of Periodic functions

A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T)=f(t) where T is a positive constant. The least value of T>0 is called the period of f(t).

Example 1

Consider $f(t) = \sin t$

$$f(t + 2\pi) = \sin(t + 2\pi)$$

$$= \sin t$$

(ie) $$f(t) = f(t + 2\pi)$$

$$= \sin t$$

sin *t* t is a periodic function with period 2π.

Example 2

tan *t* is a periodic function with period π.

13.2 Laplace Transform of Periodic functions

Let $f(t)$ be a periodic function with period a

$$f(t) = f(t + a) = f(t + 2a) = f(t + 3a)\ldots\ldots$$

Now $$L(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty e^{-st} f(t)dt + \int_a^{2a} e^{-st} dt + \int_{2a}^{3a} e^{-st} f(t)dt$$

$$+ \int_{3a}^{4a} e^{-st} f(t)dt + \ldots\ldots..$$

Put in the second integral $t = T + a;\ dt = dT$

in the Third integral $t = T + 2a;\ dt = dT$

in the Fourth integral $t = T + 3a;\ dt = dT$

When	*t = a,*	*T = 0*
	t = 2a,	*T = a*
When	*t = 2a,*	*T = 0*
	t = 3a,	*T = a*
When	*t = 3a,*	*T = 0*
	t = 4a,	*T = a*

$$\therefore L(f(t)) = \int_0^a e^{-st} f(t)dt + e^{-as}\int_0^a e^{-sT} f(T+a)dT$$

$$+ e^{-2as}\int_0^a e^{-st} f(T+2a)dt + ...$$

$$= \int_0^a e^{-st} f(t)dt + e^{-sa}\int_0^a e^{-st} f(t+a)dt + e^{-2as}\int_0^a e^{-st} f(t+2a)dt$$

$$= (1+e^{-as}+(e^{-as})+...)\int_0^a e^{-st} f(t)dt$$

$$= (1-e^{-as})^{-1} + \int_0^a e^{-st} f(t)dt \qquad \left(\because (1-x)^{-1} = 1+x+x^2+...\right)$$

$$L(f(t)) = \frac{1}{1-e^{-as}} + \int_0^a e^{-st} f(t)dt$$

13.3 Problems

1. Find the Laplace Transform of the square wave given by

$$f(t) = \begin{matrix} E & for & 0 < t < a/2 \\ -E & for & a/2 < t < a \end{matrix}$$

and $f(t+a) = f(t)$

Solution:

Given that $f(t+a) = f(t)$

Hence $f(t)$ is a periodic function with period $p = a$

$L(f(t))$ $= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} E dt + \int_{a/2}^{a} e^{-st}(-E)dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[E\int_0^{a/2} e^{-st} dt - E\int_{a/2}^{a} e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a} \right]$$

$$= \frac{E}{s(1-e^{-as})} \left[\left(-e^{-sa/2}+1\right) + \left(e^{-sa} - e^{-sa/2}\right) \right]$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-sa/2} - e^{sa/2} + e^{-sa}\right)$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-2sa/2} - e^{sa}\right)$$

$$= \frac{E}{s(1-e^{-\frac{as}{2}})(1+e^{-sa/})} \left(1 - e^{\frac{-sa}{2}}\right)^2$$

$$= \frac{E\left(1-e\frac{-sa}{2}\right)}{s(1-e^{-sa/2})}$$

$$= \frac{E}{s} \tan h\left(\frac{sa}{4}\right)$$

2. Find the Laplace transform of the function $f(t) = \begin{matrix} t & 0 < t < b \\ 2b - t & b < t < 2b \end{matrix}$

Solution:

The given function is a periodic function with period 2b

$\therefore L(f(t))$ $= \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2bs}} \int_0^b e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} t dt + \int_b^{2b} e^{-st}(2b-t)dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \begin{array}{c} \left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right) \right]_0^b + \\ \left[(2b-1)\left(\frac{e^{-st}}{-s}\right) - (-1)\frac{e^{-st}}{s^2} \right]_b^{2b} \end{array} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{-be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^{-2bs}}{s^2} + \frac{b}{s} e^{-bs} \frac{-e^{-bs}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left(\frac{1-2e^{bs} + e^{-2bs}}{s^2} \right)$$

$$= \frac{(1-e^{-bs})^2}{s^2(1+e^{-bs})(1-e^{-bs})}$$

$$= \frac{1-e^{-bs}}{s^2(1+e^{-bs})}$$

$$= \frac{1}{s^2} \cdot \left(\frac{\left(1-e\frac{-bs}{2}\right) . e^{\frac{-bs}{2}}}{\left(1+e^{\frac{-bs}{2}}\right) . e^{\frac{-bs}{2}}} \right)$$

$$= \frac{1}{s^2} \cdot \frac{e^{\frac{bs}{2}} - e^{\frac{-bs}{2}}}{e^{\frac{bs}{2}} + e^{\frac{-bs}{2}}}$$

$$= \frac{1}{s^2} \tan h\left(\frac{bs}{2}\right)$$

3. Find the Laplace transform of $f(s)$ $\begin{matrix} \sin t & in & 0 < t < \pi \\ 0 & in & \pi < t < 2\pi \end{matrix}$ and $f(t+2\pi) = f(t)$.

Solution:

Given that $f(t + 2\pi) = f(t)$

Hence $f(t)$ is a periodic function with period $P = 2\pi$.

$$L(f(t)) = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi}} \left[\int_0^\pi e^{-st} \sin t dt + \int_\pi^{2\pi} e^{-st}.0dt \right]$$

$$= \frac{1}{1-e^{-2s\pi}} \left[\frac{1}{s^2+1} \left(e^{-st}(s \sin t - 1.\cos t) \right)_0^\pi \right]$$

$$= \frac{1}{s^2+1} . \frac{1}{1-e^{-2s\pi}} \left(e^{-s\pi}(0+1) - 1(0-1) \right)$$

$$= \frac{1}{s^2+1} \frac{1}{(1-e^{-2s\pi})} (e^{-s\pi}+1)$$

$$= \frac{1}{s^2+1} . \frac{1}{(1-e^{-s\pi)}} \frac{(1+s^{-s\pi)}}{(1+e^{-s\pi})}$$

$$= \frac{1}{s^2+1} . \frac{1}{1-e^{-s\pi}}$$

4. Find the Laplace transform of the Half-wave rectifier function

$$f(t) = \begin{cases} \sin wt, & 0 < t < \frac{\pi}{w} \\ 0, & \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$$

Solution:

$$\text{Given } f(t) = \begin{cases} \sin wt, & 0 < t < \frac{\pi}{w} \\ 0, & \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$$

This ia a periodic function with period $\frac{2\pi}{w}$ in the interval $\left(0, \frac{2\pi}{w}\right)$.

$$\therefore L(f(t)) = \frac{1}{1-e^{\frac{-2\pi s}{w}}} \int_0^{\frac{2\pi}{w}} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\int_0^{\frac{\pi}{w}} e^{-st} f(t)dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\int_0^{\frac{\pi}{w}} e^{-st}\sin wt dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\frac{e^{-st}(-s\sin wt - w\cos wt)}{s^2+w^2}\right]_0^{\frac{\pi}{w}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\frac{e^{\frac{-s\pi}{w}}(w)+w}{s^2+w^2}\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}}\frac{w\left(1+e^{-s\pi/w}\right)}{s^2+w^2}$$

$$= \frac{w}{(1+e^{-s\pi/2})(1-e^{-s\pi/w})}\cdot\frac{(1+e^{-s\pi/w})}{s^2+w^2}$$

$$= \frac{w}{\left(1-e^{-s\pi/w}\right)s^2+w^2}$$

5. Find the Laplace transform of the periodic function

$$f(t) = \begin{matrix} t, & for & 0<t<1 \\ 2-t & for & 1<t<2 \end{matrix} \quad and \quad f(t+2) = f(t)$$

Solution:

The given function is a periodic function with period 2.

$$\therefore L(f(t)) = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt + \int_1^2 (2-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left(\frac{te^{-st}}{-s} - 1.\frac{e^{-st}}{s^2}\right)_0^1 + \left(\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right)_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1-2e^{-s}+e^{-2s}}{s^2}\right)$$

$$= \frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})s^2} = \frac{1}{s^2}\left(\frac{(1-e^{-s})}{(1+e^{-s})}\right)$$

$$= \frac{1}{s^2}\frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/}} = \frac{1}{s^2}\tan h\left(s/2\right)$$

6. Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < \frac{\pi}{2} \\ \pi - t, & \frac{\pi}{2} < t < \pi \end{cases} \quad f(\pi + r) = f(t)$$

Solution:

$$\therefore L(f(t)) = \frac{1}{1-e^{-s\pi}}\left[\int_0^{\pi/2} te^{-st}tdt + \int_{\pi/2}^{\pi}(\pi - t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_0^{\pi/2} + \left((\pi-1)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\frac{\pi/2\, e^{-s\pi/2}}{-s} - \frac{e^{-s\pi/s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s\pi}}{s^2} - \frac{\pi/2\, e^{-s\pi/2}}{-s} + \frac{s^{-s\pi/s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\frac{1-2e^{-s\pi/s}+e^{-s\pi}}{s^2}\right]$$

$$= \frac{(1-e^{-s\pi/2})^2}{s^2(1-e^{-s\pi/2})(1+e^{-s\pi/2})}$$

$$= \frac{1-e^{-s\pi/2}}{s^2(1+e^{-s\pi/2})}$$

7. Find the Laplace transform of the rectangular wave given by

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

Solution:

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic the interval (0, 2b) with period 2b.

$$\therefore L(f(t)) = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} (1)dt + \int_b^{2b} e^{-st} (-1)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} dt + \int_b^{2b} e^{-st} dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-S}\right)_0^{b} - \left(\frac{e^{-st}}{-S}\right)_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{e^{-sb}}{-S} + \frac{1}{S} + \frac{e^{-2sb}}{S} - \frac{e^{-sb}}{S}\right]$$

$$= \frac{1}{S}\left[\frac{1-2e^{-sb}+e^{-2sb}}{1-e^{-2bs}}\right]$$

$$= \frac{1}{S}\left[\frac{(1-e^{-sb})^2}{(1+e^{-sb})(1-e^{-sb})}\right]$$

$$= \frac{1}{S}\frac{1-e^{-sb}}{1+e^{-sb}}$$

$$= \frac{1}{S}\frac{(1-e^{-sb})e^{-sb/2}}{(1+e^{-sb})(e^{-sb2})}$$

$$= \frac{1}{S}\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}$$

$$= \frac{1}{S} \tan h\left(\frac{sb}{2}\right)$$

14. Initial value theorem

If *L(f(t))* = *F(s),* then $\underset{t\to 0}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ sF(s)$

Proof:

We know that *L[f(t)] = sL[f(t)] – f (0))*

Take the limit as S → ∞ on both sides, we have

$$\underset{s\to\infty}{Lt}\ L(f'(t)) = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$

$$\underset{s\to\infty}{Lt}\int_0^\infty e^{-st}dt = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$ (∵ By definition of Laplace Transform)

$$\int_0^\infty \underset{s\to\infty}{Lt}\ e^{-st} f'(t)dt = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$ (∵s is independent of t, we can take the limit in the L.H.S. before integration)

$$0 = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = f(0)$$

$$= \underset{t\to 0}{Lt}\ f(t)$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = \underset{t\to 0}{Lt}\ f(t)$$

15. Final value Theorem

If *L(f(t)) = F(s),* then $\underset{s\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

Proof:

We know that *L(f′(t)) = sL[f(t)] – f (0)*

L(f′(t)) = sF(s) – f (0)

$$\int_0^\infty e^{-st} f'dt = sF(s) - f(0)$$

Take the limit as *s* → 0 on both sides,

$$\underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$ ($\because$ s is independent of t, we can take the limit in the L.H.S. before integration)

$$\int_0^\infty f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$(f(t))_0^\infty = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\underset{t\to\infty}{Lt}\, f(t) - f(0) = \underset{s\to 0}{Lt}\, sF(s) - f(0)$$

Since $f(0)$ is n ot a function of 's' (or) 't' it can be cancelled both sides,

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

15.1 Problems

1. If $L(f)(t)) = \dfrac{1}{s(s+a)}$ find $\lim_{t\to\infty} f(t)$ and $\lim_{t\to 0} f(t)$

Solution:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s\times\frac{1}{s(s+a)}$$

$$= \lim_{s\to\infty}\frac{1}{(s+a)}$$

$$= 0$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s\times\frac{1}{s(s+a)}$$

$$= \lim_{s\to 0}\frac{1}{(s+a)}$$

$$= \frac{1}{a}$$

2. If $L(e^{-t} \cos^2 t) = F(s)$. Find $\lim_{s\to 0}(sF(s))$ and $\lim_{s\to\infty}(sF(s))$

Solution:

$L(e^{-t}\cos^2 t) = F(s)$

(ie)., $f(t) = e^{-t} \cos^2 t$

By final value theorem,

$$\lim_{s\to 0}(sF(s)) = \lim_{t\to\infty}(e^{-t}\cos^2 t) = 0$$

By initial value theorem,

$$s\lim_{s\to 0} sF(s) = \lim_{t\to 0}(e^{-t}\cos^2 t) = 1$$

3. Verify the initial and final value theorem for the function $f(t) = 1 - e^{-at}$

Solution:

Given that $f(t) = 1 - e^{-at}$ …(1)

$L(f(t)) = L(1-e^{-at})$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$SF(s) = s\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

$$= 1 - \frac{1}{s+a} \quad \text{… (2)}$$

From (1), $\underset{t\to 0}{Lt}\, f(t) = \underset{t\to 0}{Lt}\, 1 - e^{-at}$

$= 1 - 1$

$= 0$ …(3)

$\underset{t\to\infty}{Lt}\, f(t) = \underset{t\to\infty}{Lt}\, 1 - e^{-at}$

$$= \quad 1 - 0$$

$$= \quad 1 \qquad \ldots(4)$$

From (2), $\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ 1 - \dfrac{s}{s+a} = 1$ … (5)

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ 1 - \frac{s}{s+a}$$

$$= \underset{s\to\infty}{Lt}\ 1 - \frac{s}{s\left(1 + a/s\right)} = 0 \qquad \ldots (6)$$

From (3) & (6), we have

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

and from (4) & (5)

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$$

4. Verify initial and final value theorem for the function $f(t) = e^{-2t}\cos 3t$

Solution:

Given $f(t) = e^{-2t}\cos 3t$

$$L(f(t)) = L(e^{-2t}\cos 3t)$$

$$= L(\cos 3t)_{s\to s+2}$$

$$F(s) = \left(\frac{s}{s^2+9}\right)_{s\to s+2} = \frac{s+2}{(s+2)^2+9}$$

$$SF(s) = \frac{s(s+2)}{s^2+4s+13} = \frac{s^2+2s}{s^2+4s+13}$$

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ e^{-2t}\cos 3t = 1 \qquad \ldots(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ e^{-2t}\cos 3t = 0 \qquad \ldots(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{s^2+2s}{s^2+4s+13} = 0 \qquad \ldots(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{s^2(1+2/s)}{s^2(1+4/s+13/s^2)} = 1 \qquad \ldots(4)$$

From (1) and (4), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$

From (2) and (3), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

5. Verify initial and final value theorem for $f(t) = t^2e^{-3t}$

Solution:

$f(t) = t^2e^{-3t}$

$L(f(t)) = [L(t^2)]_{s\to s+3}$

$$= \left(\frac{2!}{s^3}\right)_{s\to s+3} = \frac{2}{(s+3)^3}$$

$$sF(s) = \frac{2}{(s+3)^3}$$

$$\underset{s\to 0}{Lt}\ f(s) = \underset{t\to 0}{Lt}\ t^2e^{-3t} = 0 \qquad \ldots(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ t^2e^{-3t} = 0 \qquad \ldots(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{2s}{(s+3)^3} = 0 \qquad \ldots(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{2s}{(s+3)^3} = \underset{s\to 0}{Lt}\ \frac{2s}{\left(1+\frac{3}{s}\right)^3}$$

$$= \underset{s\to\infty}{Lt}\ \frac{2}{s^2\left(1+\frac{3}{s}\right)^3} = 0 \qquad \ldots(4)$$

From (1) & (4)

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

From (2) & (3)

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s).$$

Exercise - 1 (a)

Find the Laplace transform of the following

1. $5 - 3t - 2\,e^{-t}$ Ans: $\dfrac{3s^2 + 2s - 3}{s^2(s+1)}$

2. $6 \sin 2t - 5 \cos 2t$ Ans: $\dfrac{12 - 5s}{s^2 + 4}$

3. e^{3t-5} Ans: $\dfrac{e^5}{s-3}$

4 $\cos(wt+\infty)$ Ans: $\dfrac{s\cos\infty - w\sin\infty}{s^2 + w^2}$

5. $7e^{2t} + 9e^{-2t} + 5 \cos t + 7t^3 + 5 \sin 3t + 2$

Ans: $\dfrac{7}{s-2} + \dfrac{9}{s+2} + \dfrac{5s}{s^2+1} + \dfrac{42}{s^4} + \dfrac{15}{s^2+9}\,\dfrac{2}{s}$

6. $\sin 2t \cos 3t$ Ans: $\dfrac{2(s^2 - 5)}{(s^2+1)(s^2+25)}$

7. $\cos h2t - \cos h3t$ Ans: $\dfrac{-5s}{(s-4)(s-9)}$

8. $\sin^2 at$ Ans: $\dfrac{2a^2}{s(s^2+4a^2)}$

9. $(t^2 + 1)^2$ Ans: $\dfrac{24}{s^5} + \dfrac{4}{s^3} + \dfrac{1}{s}$

10. $a + bt + \dfrac{c}{vt}$ Ans: $\dfrac{a}{s} + \dfrac{b}{s^2} + c\sqrt{\pi/s}$

11. $\sin^3 2t$ Ans: $\dfrac{48}{(s^2+4)(s^2+36)}$

12. $(\sin t - \cos t)^2$ Ans: $\dfrac{s^2 - 2s + 4}{s(s^2+4)}$

13. $\cos \pi t + 4e^{2t/3}$ Ans: $\dfrac{s}{s^2 + \pi^2} + \dfrac{12}{3s^2}$

Exercise - 1 (b)

Find the Laplace transform of the following functions.

1. t^3e^{-3t} Ans: $\dfrac{6}{(s+3)^4}$

2. $e^{-2t}(\cos 4t + 3 \sin 4t)$ Ans: $\dfrac{s+10}{s^2-4s+20}$

3. $e^t(t+2)$ Ans: $\dfrac{2}{(s-1)^3}+\dfrac{4}{(s-1)^2}+\dfrac{4}{s-1}$

4. $e^{-at}t^2$ Ans: $\dfrac{2}{(s+a)^3}$

5. $e^{-t}\cos^2 t$ Ans: $\dfrac{1}{2s+2}+\dfrac{s+1}{2s^2+4s+10}$

6. $e^{-2t}(1-2t)$ Ans: $\dfrac{6}{(s+2)^2}$

7. $e^{-2t}\cos t$ Ans: $\dfrac{s+2}{s^2+4s+5}$

8. $e^t \sin t \cos t$ Ans: $\dfrac{1}{(s-1)^2+4}$

9. $e^{-t} \cos ht$ Ans: $\dfrac{s+1}{s^2+2s}$

10. $e^{at}t^n$ Ans: $\dfrac{n!}{(s-a)^{n+1}}$

11. $t^2 \sin h2t$ Ans: $\dfrac{1}{(s-2)^3}+\dfrac{1}{(s+2)^3}$

12. $\sin h2t \sin 3t$ Ans: $\dfrac{1}{2}\left[\dfrac{3}{s^2-4s+13}-\dfrac{3}{s^2+4s+13}\right]$

13. $\cosh t \cos 3t \cos 4t$

Ans: $\dfrac{1}{4}\left[\dfrac{s-2}{s^2-4s+53}-\dfrac{s+2}{s^2+4s+53}+\dfrac{s-2}{s^2-4s+5}-\dfrac{s+2}{s^2+4s+53}\right]$

14. sin $h2t$ $\sin^2 t$ Ans: $\frac{1}{4}\left[\frac{1}{s-2}-\frac{1}{s+2}-\frac{S-2}{s^2-4s+8}+\frac{S+2}{s^2+4s+18}\right]$

15. sin $h3t$ sin $3t$ sin $4t$

Ans: $\frac{1}{4}\left[\frac{s+3}{s^2+6s+10}-\frac{s+3}{s^2+6s+58}+\frac{s-3}{s^2-6s+58}-\frac{s-3}{s^2-6s+10}\right]$

Exercise - 1 (c)

Find the Laplace transform of the following functions.

1. $t\cos at$ Ans: $\frac{s^2-a^2}{(s^2+a^2)^2}$

2. $t^3\sin at$ Ans: $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

3. $t^3\sin t$ Ans: $\frac{24s(1-s)^2}{(1-s^2)^4}$

4. t^3e^{-3t} Ans: $\frac{3!}{(s+3)^4}$

5. $t^3\cos hat$ Ans: $\frac{2s(s^2+3a^2)}{(s^2-a^2)^3}$

6. $(1+te^{-t})^3$ Ans: $\frac{1}{s}+\frac{3}{(s+1)^3}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$

7. $te^{at}\sin at$ Ans: $\frac{2a(s-a)}{(s^2-2as+2a^2)^2}$

8. $te^{-t}\sin^2 t$ Ans: $\frac{1}{2}\frac{1}{(s+2)^2}+\frac{(s+1)^2+4+-2(s+1)^2}{((s+1)^2+4)^2}$

9. $t\cos t\cos 2t$ Ans: $\frac{1}{2}\left[\frac{s^2-9}{(s^2+9)^2}+\frac{s^2-1}{(s^2+1)^2}\right]$

10. $t\cos^2 2t$ Ans: $\frac{1}{2}\left[\frac{1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right]$

11. $r \cos h2t \sin 2t$ Ans: $\frac{1}{2}\left[\frac{4(s-2)}{(s^2-4s+8)^2}+\frac{4(s+2)}{(s^2+4s+8)^2}\right]$

12. $r \cos ht \sin 3t$ Ans: $\frac{1}{2}\left[\frac{s^2-2s-8}{(s^2-2s+10)^2}+\frac{s^2+2s-8}{(s^2+2s+10)^2}\right]$

13. $r^2e^{-t}\cos t$ Ans: $\frac{2(s+1)(s^2+2s-2)}{(s^2+2s+2)^3}$

14. $te^{-t}\cos ht$ Ans: $\frac{s^2+2s+2}{(s^2+2s)^3}$

15. $\frac{t\sin 2t}{e^{-2t}}$ Ans: $\frac{4s-8}{(s^2-4s-8)^2}$

Exercise 1 - (d)

Find the Laplace transform of the following functions

1. $\frac{\sin t}{t}$ Ans: $\cos^{-1}s$

2. $\frac{e^{2t}-e^{bt}}{t}$ Ans: $\log\frac{s-b}{s-a}$

3. $\frac{e^{2t}-e^{-3t}}{t}$ Ans: $\log\frac{s+3}{s-2}$

4. $\frac{1-\cos at}{t}$ Ans: $\frac{1}{2}\log\left(\frac{s^2+a^a}{s^2}\right)$

5. $\frac{\sin^2 t}{t}$ Ans: $\frac{1}{4}\log\frac{s^2+4}{s^2}$

6. $\frac{\sin t\sin 2t}{t}$ Ans: $\frac{1}{4}\log\frac{s^2+9}{s^2+1}$

7. $\frac{e^t-\cos 2t}{t}$ Ans: $\log\frac{\sqrt{s^2+4}}{s-1}$

8. $\frac{\sin 3t\cos t}{t}$ Ans: $\frac{1}{2}\left[\pi-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$

9. $\dfrac{e^{-t}-e^{-2t}}{t}$ Ans: $\log\dfrac{s+2}{s+1}$

10. $\dfrac{e^{-at}-e^{-bt}}{t}$ Ans: $\log\dfrac{s+b}{s+a}$

11. $\dfrac{\cos 4t\sin 2t}{t}$ Ans: $\dfrac{1}{2}\left[\tan^{-1}\left(\dfrac{s}{2}\right)-\tan^{-1}\left(\dfrac{s}{6}\right)\right]$

12. $\dfrac{\cos 2t-\cos 3t}{t}$ Ans: $\log\dfrac{\sqrt{s^2+9}}{\sqrt{s^2+4}}$

13. $\dfrac{\sin ht}{t}$ Ans: $\dfrac{\log\sqrt{s+1}}{\log\sqrt{s-1}}$

14. $\dfrac{1-e^{-2t}}{t}$ Ans: $\log\dfrac{s+2}{s}$

15. $\dfrac{e^{at}-\cos bt}{t}$ Ans: $\dfrac{1}{2}\log\left(\dfrac{s^2+s^2}{(s-a)^2}\right)$

Exercise 1 (e)

Find the Laplace transform of the following functions.

1. $\int_0^t e^t\cos^2 t\,dt$ Ans: $\dfrac{s^2-2s+3}{s(s-1)(s^2-2s+5)}$

2. $\int_0^t t\sin t\sin 2t\,dt$ Ans: $\dfrac{1}{2s}\left[\dfrac{s^2-1}{(s^2+1)^2}+\dfrac{s^2-9}{(s^2+9)^2}\right]$

3. $\int_0^t \dfrac{\sin ht}{t}\,dt$ Ans: $\dfrac{1}{2}\log\left(\dfrac{s+1}{s-1}\right)$

4. $\int_0^t e^{2t}\sin 3t\,dt$ Ans: $\dfrac{1}{s}\left(\dfrac{3}{s^2-4s+13}\right)$

5. $\int_0^t e^{-2t}\sin^3 t\,dt$ Ans: $\dfrac{3}{2s}\left[\dfrac{s+2}{(s^2+4s+5)^2}+\dfrac{3(s+2)}{(s^2+4s+13)^2}\right]$

6. $\int_0^t \dfrac{\sin^2 t}{t}\,dt$ Ans: $\dfrac{1}{2}\log\dfrac{\sqrt{s^2+4}}{s}$

7. $\int_0^t \frac{e^{-t}\sin t dt}{t}$ Ans: $\cot^{-1}(s+1)$

8. $e^t \int_0^t \frac{\sin t}{t} dt$ Ans: $\frac{\cot^{-1}(s-1)}{s-1}$

9. $\int_0^t te^t \sin t dt$ Ans: $\frac{1}{s}\cdot\frac{2(s+1)}{s^2+2s+2}$

10. $e^{-t}\int_0^t t\cos t dt$ Ans: $\frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$

Exercise - 1 (f)

Find the Laplace transform of the following

1. $f(t) = t$ for $0 < t < 4$, $f(t+4) = f(t)$ Ans: $\frac{1-4Se^{-4s}-e^{-4s}}{(1-e^{-4s})s^2}$

2. $f(t)=\begin{cases} t & 0<t<1 \\ 2-t & 1<1<2 \end{cases}$ $and\ f(t+2)=f(t)$ Ans: $\frac{1}{s^2}\tan h\left(s/2\right)$

3. $f(t)=\begin{cases} 1 & 0<t<a/2 \\ -t & a/2<t<a \end{cases}$ $and\ f(a+t)=f(a)$ Ans: $\frac{1}{1-e^{-as}}\left(\frac{1+e^{-as}-2e^{-as/2}}{s}\right)$

4. $f(t)=\begin{cases} \sin t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$ $and\ f(t+2\pi)=f(t)$ Ans: $\frac{1}{1-e^{-\pi/s}}\frac{1}{s^2+1}$

5. $f(t)=\begin{cases} t & 0<t<1 \\ 0 & 1<1<2 \end{cases}$ $and\ f(t+2)=f(t)$ Ans: $\frac{1-e^{-s}(s+1)}{s^2(1-e^{-2s})}$

6. $f(t)=\begin{cases} 0 & 0<t<w/2 \\ -\sin wt & \frac{\pi}{w}<t<\frac{2\pi}{w}, \end{cases}$ $f\left(t+\frac{2\pi}{w}\right)=f(t)$ Ans: $\frac{w}{(w^2+S^2)(e^{\pi S/w}-1)}$

7. $f(t)=e^{-t}, 0\le t<2,\ f(t+2)=f(t)$ Ans: $\frac{1-e(s+1)}{(s+1)(1-e^{-2s})}$

8. $f(t)=\begin{cases} 1 & 0<t<1 \\ -1 & a<t<2a \end{cases}$ $given\ f(t+2a)=f(t)$

9. $f(t)=\begin{cases}\sin wt & 0<t<\pi/w \\ 0 & \pi/w<t<2\pi/w\end{cases}$ given that $f\left(t+2\pi/w\right)=f(t)$

10. $f(t)=\sin wt0<1<\pi/w\; f\left(t+\pi/w\right)=f(t)$

16.1. Definition

If the Laplace transform of a function *f* (*t*) is *F*(*S*) (ie) *L*(*f* (*t*)) = *F*(*S*) then *f* (*t*) is called an inverse laplace transform of *F* (*s*) and is denoted by

$f(t) = L^{-1}(F(s))$

Here L^{-1}1 is called the inverse Laplace transform operator.

17. Standard results in inverse Laplace transforms

Laplace Transform	**Inverse Laplace Transform**
$L(1)=\frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right)=1$
$L(e^{at})=\frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right)=e^{at}$
$L(e^{-at})=\frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right)=e^{-at}$
$L(t)=\frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right)=t$
$L(t^2)=\frac{2!}{s^3}$	$L^{-1}\left(\frac{2!}{s^3}\right)=t^2$
$L(t^3)=\frac{3!}{s^4}$	$L^{-1}\left(\frac{3!}{s^4}\right)=t^3$
$L(t^n)=\frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right)=t^n$

where n is a +ve integer

$L(\sin at)=\frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin at$

$$L(\cos at) = \frac{s}{s^2 + a^2} \qquad L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$$

$$L(\sin\ hat) = \frac{a}{s^2 - a^2} \qquad L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sin\ hat$$

$$L(\cos\ hat) = \frac{s}{s^2 - a^2} \qquad L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cos\ hat$$

$$L(t \sin\ at) = \frac{2as}{(s^2 + a^2)^2} \qquad L^{-1}\left(\frac{2as}{(s^2 + a^2)^2}\right) = t \sin\ at$$

$$L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \qquad L^{-1}\left(\frac{s^2 - a^2}{(s^2 + a^2)^2}\right) = t \cos at$$

$$L(t \sin\ hat) = \frac{2as}{(s^2 - a^2)^2} \qquad L^{-1}\left(\frac{2as}{(s^2 + a^2)^2}\right) = t \sin\ hat$$

$$L(t \cos hat) = \frac{s^2 + a^2}{(s^2 - a^2)^2} \qquad L^{-1}\left(\frac{s^2 + a^2}{(s^2 - a^2)^2}\right) = t \cos hat$$

$$L(e^{at} \sin\ bt) = \frac{b}{(s - a)^2 + b^2} \qquad L^{-1}\left(\frac{b}{(s + a)^2 + b^2}\right) = e^{at} \sin bt$$

$$L(e^{at} \cos bt) = \frac{s - a}{(s - a)^2 + b^2} \qquad L^{-1}\left(\frac{s - a}{(s - a)^2 + b^2}\right) = e^{at} \cos bt$$

$$L(e^{at} \sin\ hbt) = \frac{b}{(s - a)^2 + b^2} \qquad L^{-1}\left(\frac{b}{(s - a)^2 + b^2}\right) = e^{at} \sin\ hbt$$

$$L(e^{at} \cos hbt) = \frac{s - a}{(s - a)^2 + b^2} \qquad L^{-1}\left(\frac{s - a}{(s - a)^2 - b^2}\right) = e^{at} \cos hbt$$

$$L(te^{-at}) = \frac{1}{(s + a)^2} \qquad L^{-1}\left(\frac{1}{(s + a)^2}\right) = te^{-at}$$

$$L(t^2 e^{-at}) = \frac{2!}{(s + a)^3} \qquad L^{-1}\left(\frac{2!}{(s + a)^3}\right) = t^2 e^{-at}$$

18. Properties of Inverse Laplace Transforms

18.1 Linear Property

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$L^{-1}(c_1F_1(s) + c_2F_2(s)) = c_1L^{-1}(F_1(s)) + c_2L^{-1}(F_2(s))$ where c_1 & c_2 are constants.

Proof:

We know that

$$L(c_1f_1(t) + c_2f_2(t)) = c_1L(f_1(t)) + c_2L(f_2(t))$$

$$= c_1F_1(s) + c_2F_2(s)$$

$$= [\because L(f_1(t)) + F_1(s) \text{ and } L(f_2(t)) = F_2(s)]$$

$$c_1f_1(t) + c_2f_2(t) = L^{-1}(c_1F_1(s)) + c_2F(s))$$

$$= L^{-1}(c_1F_1(s)) + L^{-1}(c_2F(s))$$

$$= c_1L^{-1}(F_1(s)) + c_2L^{-1}(F_2(s))$$

Problems

1. Find $L^{-1}\left(\frac{1}{s-3} + s + \frac{s}{s^2-4}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s-3} + s + \frac{s}{s^2-4}\right) = L^{-1}\left(\frac{1}{s-3}\right) + L^{-1}(s) + L^{-1}\left(\frac{s}{s^2-4}\right)$$

$$= e^{3t} + 1 + \cos h2t$$

$$= e^{3t} + \cos h2t + 1$$

2. Find $L^{-1}\left(\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right)$$

$$= L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left(\frac{s}{s^2-9}\right)$$

$$= \quad t + e^{-4t} + \frac{\sin 2t}{2} + \cos h3t$$

3. Find $L^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{3s}{s^2+4} + \frac{4}{s^2+16}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{3s}{s^2+4} + \frac{4}{s^2+16}\right)$$

$$= \quad L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{2}{s^2}\right) - L^{-1}\left(\frac{3s}{s^2+4}\right) + L^{-1}\left(\frac{4}{s^2+16}\right)$$

$$= \quad 1 + 2t - 3\cos 2t + \sin 4t$$

4. Find $L^{-1}\left(\frac{4}{s^6} - \frac{2}{s^{10}} + \frac{2}{s^2-9} + \frac{3s}{s^2+25}\right)$

Solution:

$$L^{-1}\left(\frac{4}{s^6} - \frac{2}{s^{10}} + \frac{2}{s^2-9} + \frac{3s}{s^2+25}\right)$$

$$= \quad \frac{4}{5!}L^{-1}\left(\frac{5!}{s^6}\right) - \frac{2}{9!}L^{-1}\left(\frac{9!}{s^{10}}\right) + \frac{2}{3}L^{-1}\left(\frac{3}{s^2-9}\right) + 3L^{-1}\left(\frac{s}{s^2+25}\right)$$

$$= \quad \frac{1}{36}t^5 - \frac{1}{181440}t^9 + \frac{2}{3}\sin\ h3t + 3\cos 5t$$

5. Find $L^{-1}\left(\frac{2}{s^5} - \frac{3}{s^4} + \frac{2}{s^2-3} + \frac{5}{s^2+100} + \frac{s}{s^2+10}\right)$

Solution:

$$L^{-1}\left(\frac{2}{s^5} - \frac{3}{s^4} + \frac{2}{s^2-3} + \frac{5}{s^2+100} + \frac{s}{s^2+10}\right)$$

$$= \frac{2}{4!}L^{-1}\left(\frac{4!}{s^s}\right) - \frac{3}{3!}L^{-1}\left(\frac{3!}{s^4}\right) + \frac{3}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2-\sqrt{3^2}}\right) + \frac{5}{10}L^{-1}\left(\frac{10}{s^2-100}\right) + L^{-1}\left(\frac{s}{s^2+10}\right)$$

$$= \frac{1}{12}t^4\ \frac{1}{2}t^3\sqrt{3}\sin\sqrt{3t} + \frac{1}{2}\sin\ h10t + \cos\sqrt{10t}$$

6. Find $L^{-1}\left(\frac{5}{s^5-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$

Solution:

$$L^{-1}\left(\frac{5}{s^5-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$$

$$= \quad L^{-1}\left(\frac{5}{s^2-25}\right)+4L^{-1}\left(\frac{s}{s^2-16}\right)+L^{-1}\left(\frac{s}{s^2-9}\right)+L^{-1}\left(\frac{s}{s^2-25}\right)$$

$=$ sin $h5t$ + 4 cos $h4t$ + cos $3t$ – cos $h5t$

7. Find $L^{-1}\left(\frac{1}{2s+3}\right)$

Solution:

$$L^{-1}\left(\frac{1}{2s+3}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+3/2}\right)$$

$$= \frac{1}{2}e^{-3/2t}$$

19. First Shifting Property

(i) If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(F(s-a)) = e^{at}L^{-1}(F(s))$

Proof:

We know that $L(f(s)) = f(t)$ then $L^{-1}(F(s-a)) = e^{at}L^{-1}(F(s))$

Hence $e^{at}f(t) = L^{-1}(F(s-a))$

$e^{at}L^{-1}(F(s)) = L^{-1}(F(s-a))$

(ii) If $L^{-1}(F(s)) = f(t)$ Then $L^{-1}(F(s+a)) = e^{-at}L^{-1}(F(s))$

Proof:

We know that $L(f(f)) = F(s)$ Then $L(e^{-at}f(t)) = F(s+a)$

Hence $e^{-at}f(t) = L^{-1}(F(s+a))$

$$e^{-at}L^{-1}(F(s)) = L^{-1}(F(s+a))$$

19.1 Problems

1. Find $L^{-1}\left(\frac{1}{(s+1)^2}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t}t$$

2. Find $L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{-t}\sin t$$

3. Find $L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right)$

Solution:

$$L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) = e^{3t}L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= e^{3t}\cos 2t$$

4. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+2)}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-2t} - 2e^{-2t} . t$$

$$= e^{-2t} (1-2t)$$

5. Find $L^{-1}\left(\frac{s}{(s-1)^2+3} + \frac{3s}{(s+2)^2-5}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s-1)^2+3} + \frac{3s}{(s+2)^2-5}\right) = L^{-1}\left(\frac{s}{(s-1)^2+3}\right) + 3L^{-1}\left(\frac{s}{(s+2)^2-5}\right)$$

$$= L^{-1}\left(\frac{s-1+1}{(s-1)^2+3}\right) + 3L^{-1}\left(\frac{s+2-2}{(s+2)^2-5}\right)$$

$$= L^{-1}\left(\frac{s-1}{(s-1)^2+3}\right) + L^{-1}\left(\frac{1}{(s-2)^2+3}\right)$$

$$+3L^{-1}\left(\frac{s+2}{(s+2)^2-5}\right) - 6L^{-1}\left(\frac{1}{(s+2)^2-5}\right)$$

$$= e^t L^{-1}\left(\frac{s}{s^2+3}\right) + e^t L^{-1}\left(\frac{1}{s^2+3}\right) + 3e^{-2t} L^{-1}\left(\frac{s}{s^2-5}\right)$$

$$- 6e^t L^{-1}\left(\frac{1}{s^2-5}\right)$$

$$= e^t L^{-1}\left(\frac{s}{s^2+\sqrt{3}^2}\right) + \frac{e^t}{\sqrt{3}} L^{-1}\left(\frac{\sqrt{3}}{s^2+\sqrt{3}^2}\right)$$

$$= 3e^{-2t} L^{-1}\left(\frac{s}{s^2+\sqrt{5}^2}\right) + \frac{6}{\sqrt{5}} e^{-2t} L^{-1}\left(\frac{\sqrt{5}}{s^2+\sqrt{5}^2}\right)$$

$$= e^t \cos\sqrt{3}t + \frac{e^t}{\sqrt{3}} \sin\sqrt{3}t + 3e^{-2t} \cos h\sqrt{5}t$$

$$= \frac{6}{\sqrt{5}} e^{-2t} \sin h\sqrt{5t}$$

6. Find $L^{-1}\left(\frac{3s-4}{s^2-8s+65}\right)$

Solution:

$$L^{-1}\left(\frac{3s-4}{s^2-8s+65}\right) = L^{-1}\left(\frac{3s-4}{(s-4)^2+49}\right)$$

$$= L^{-1}\left(\frac{3\left(s-4/3\right)}{(s-4)^2+49}\right) = 3L^{-1}\left(\frac{3-4+4-4/3}{(s-4)^2+49}\right)$$

$$= 3L^{-1}\left(\frac{s-4+8/3}{(s-4)^2+49}\right)$$

$$= 3L^{-1}\left(\frac{s-4}{(s-4)^2+49}\right) + 3.8/3\, L^{-1}\left(\frac{1}{(s-4)^2+49}\right)$$

$$= 3e^{4t}L^{-1}\left(\frac{s}{s^2+49}\right) + 8e^{4t}L^{-1}\left(\frac{1}{s^2+49}\right)$$

$$= 3e^{4t}\cos 7t + \frac{8}{7}e^{4t}L^{-1}\left(\frac{7}{s^2+49}\right)$$

$$= 3e^{4t}\cos 7t + \frac{8}{7}e^{4t}\sin 7t$$

20. Change of Scale Property

If $L(f(t)) = F(s)$, then $L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$

Proof:

$F(s) = L(f(t))$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$F(as) \quad = \quad \int_0^\infty e^{-ast} f(t)dt$

Let $\quad at = t_1$ When $\quad t = 0, t_1 = 0$

$dt = \dfrac{dt_1}{a}$ $\quad t = \infty, t_1 = \infty$

$$F(as) = \int_0^\infty e^{-st} f\left(\frac{t_1}{a}\right)\frac{dt_1}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-st_1} f\left(\frac{t_1}{a}\right)dt_1$$

$$= \frac{1}{a}\int_0^\infty e^{-st} f\left(\frac{t}{a}\right)dt_1 \qquad \left(\because \int_a^b f(t)dt = \int_a^b f(t_1)dt_1\right)$$

$$= \frac{1}{a}L\left(f\left(t/a\right)\right)$$

$$\therefore L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right)$$

20.1 Problems

1. If $L^{-1}\left(\dfrac{s^2-1}{(s^2+1)^2}\right) = t\cos t$, then find $L^{-1}\left(\dfrac{9s^2-1}{(9s^2+1)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right) = t\cos t$$

writing as for S,

$$L^{-1}\left(\frac{a^2s^2-1}{(a^2s^2+1)^2}\right) = \frac{1}{a}.\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

Put $\quad a = 3,\ L^{-1}\left(\dfrac{9s^2-1}{(9s^2+1)^2}\right) = \dfrac{1}{3}.\dfrac{t}{3}\cos\left(\dfrac{t}{3}\right)$

$$= \frac{t}{9}\cos\left(\frac{t}{3}\right)$$

2. Find $L^{-1}\left(\frac{s}{(2s^2-8)}\right)$

Solution:

We know that $L^{-1}\left(\frac{s}{(s^2-4^2)}\right) = \cos h4t$

Putting as for S,

$$L^{-1}\left(\frac{2s}{(2s)^2-4^2)}\right) = \frac{1}{2}\cos h\left(\frac{4t}{2}\right)$$

$$L^{-1}\left(\frac{2s}{4s^2-16}\right) = \frac{1}{2}\cos h\ 2t$$

(ie)

$$L^{-1}\left(\frac{s}{2s^2-18}\right) = \frac{1}{2}\cos h\ 2t$$

3. Find $L^{-1}\left(\frac{s}{s^2a^2+b^2}\right)$

Solution:

$$\frac{s}{s^2a^2+b^2} = \frac{1}{a}\frac{as}{s^2a^2+b^2}$$

$$= \frac{1}{a}F(as) \text{ where } F(as) = \frac{1}{s^2+b^2}$$

$$\therefore L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = \frac{1}{a}L^{-1}\frac{sa}{s^2a^2+b^2}$$

$$= \frac{1}{a}L^{-1}(F(as))$$

$$= \frac{1}{a}.\frac{1}{a}f\left(\frac{t}{a}\right)$$

where $f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$

$$\therefore f\left(\frac{t}{a}\right) = \cos\left(\frac{bt}{a}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^2+b^2}\right) = \frac{1}{a}.\frac{1}{a}\cos\left(\frac{bt}{a}\right)$$

$$= \frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$$

21. Result

We know that if $L(f(t)) = F(S)$, then $L(tf(t)) = \frac{-d}{ds}F(s)$

$$L(tf(t)) = -F'(s)$$

Hence $L^{-1}(F'(s)) = tf(t)$

$$= tL^{-1}(F(s))$$

$$\therefore L^{-1}(F'(s)) = -tL^{-1}(F(s))$$

21.1 Problems

1. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Solution:

Let $F'(s) = \frac{s}{(s^2+a^2)^2}$

$$\frac{d}{ds}F(s) = \frac{s}{(s^2+a^2)^2}$$

$$\therefore F(s) = \int \frac{s}{(s^2+a^2)^2}ds$$

put $s^2 + a^2 = u$

$$2sds = du$$

$$\therefore \int \frac{s}{(s^2+a^2)^2}\,ds = \int \frac{\frac{du}{2}}{u^2}$$

$$= \frac{-1}{2u} = \frac{-1}{2(s^2+a^2)}$$

$$\therefore F(s) = \frac{-1}{2(s^2+a^2)}$$

We know that $L(F'(s)) = -tL^{-1}(F(s))$

$$\therefore L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = -tL^{-1}\left(\frac{1}{2(s^2+a^2)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+a^2)}\right)$$

$$= \frac{t}{2}\frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{t}{2a}\sin at$$

2. Find $L^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right)$

Solution:

Let $\left(\frac{s+3}{(s^2+6s+13)^2}\right) = F'(s)$

$$\frac{dF(s)}{ds} = \frac{s+3}{(s^2+6s+13)^2}$$

$$\therefore F(s) = \frac{(s+3)ds}{(s^2+6s+13)^2}$$

Put $s^2 + 6s + 13 = u$

$(2s + 6)ds = du$

$2(s + 3)ds = du$

(ie) $F(s) = \int \frac{\frac{du}{2}}{u^2} = \frac{-1}{2u}$

$$= \frac{-1}{2(s^2+6s+13)}$$

We know that $L^{-1}(F'(s)) = -tL^{-1-}(F(s))$

$$\therefore L^{-1}\frac{s+3}{(s^2+6s+13)^2} = -tL^{-1}\left(\frac{-1}{2(s^2+6s+13)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{-1}{(s^2+6s+13)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+3)^2+2^2}\right)$$

$$= \frac{t}{2}e^{-3t}L^{-1}\left(\frac{1}{(s^2+2^2)}\right)$$

$$= \frac{t}{2}e^{-3t}\frac{1}{2}L^{-1}\left(\frac{2}{(s^2+2^2)}\right)$$

$$= \frac{t}{4}e^{-3t}\sin 2t$$

3. Find $L^{-1}\left(\frac{2(s+1)}{(s^2+2s+2)^2}\right)$

Solution:

$$F'(s) = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$\frac{dF(s)}{ds} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$F(s) = \int \frac{2(s+1)}{(s^2+2s+2)^2}ds$$

Put $s^2 + 2s + 2 = u$

$(2s + 2)ds = du$

$2(s + 2)ds = du$

$$\therefore F(s) = \int \frac{du}{u^2}$$

$$= \frac{-1}{u}$$

$$= \frac{-1}{s^2+2s+2}$$

$$\therefore L^{-1}\left(\frac{2(s+1)}{(s^2+2s+2)^2}\right) = -tL^{-1}\left(\frac{1}{s^2+2s+2}\right)$$

$$= -tL^{-1}\left(\frac{1}{s^2+2s+2}\right) = tL^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= te^{-t}L^{-1}\left(\frac{1}{s^2 1}\right)$$

$$= te^{-t}\sin^{t}$$

4. Find $L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right)$

Solution:

$$\text{Let } F'(s) = \left(\frac{s+2}{(s^2+4s+5)^2}\right)$$

Integrate both sides w.r.t. 'S'

$$F'(s) = \frac{s+2}{(s^2+4s+5)^2}$$

$$\int F'(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2}$$

$$F(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2}$$ Let $y = s^2 + 4s + 5$

$F(s) \quad = \int \frac{dy/2}{y^2}$ $\qquad dy = (2s+4)\,ds$

$= \frac{1}{2}\int \frac{dy}{y^2}$ $\qquad = \frac{dy}{2} = (s+2)ds$

$= \frac{1}{2}\int y^{-2}dy$

$F(s) \quad = \frac{1}{2}\left(\frac{y^{-2+1}}{-2+1}\right)$

$= \frac{-1}{2y}$

$= \frac{-1}{2(s^2+4s+5)}$

We know that

$L^{-1}(F'(s)) \quad = -tL^{-1}(F(s))$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = tL^{-1}\left(\frac{-1}{2(s^2+4s+5)}\right)$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+4s+5)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= \frac{t}{2}e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{t}{2}e^{-2t}\sin t$$

5. Find $L^{-1}\left(\tan^{-1}\left(1/s\right)\right)$

Solution:

Let $F(s) = \tan^{-1}\left(1/s\right)$s

$$F'(s) = \frac{1}{1+\left(1/s\right)^2}\left(\frac{-1}{s^2}\right) \quad \left[\because \frac{d(\tan^{-1} x)}{dzx} = \frac{1}{1+x^2}\right]$$

$$F'(s) = \frac{s^2}{s^2+1}\left(\frac{-1}{s^2}\right)$$

$$= \frac{-1}{s^2+1}$$

We know that $L^{-1}(F'(s)) = -tL^{-1}(F(s))$

or

$$L^{-1}(F(s)) = \frac{-1}{t}L^{-1}(F'(s)) \qquad \ldots(1)$$

$\therefore$(1) becomes, $L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{-1}{t}L^{-1}F(s)$

$$= \frac{1}{t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{1}{t}\sin t$$

6. Find $L^{-1}\left(\tan^{-1}\left(a/s\right) + \cot^{-1}\left(s/b\right)\right)$

Solution:

Let $F(s) = \tan^{-1}\left(a/s\right)s + \cot^{-1}\left(s/b\right)$

$$F(s) = \frac{1}{1+\left(a/s\right)^2}\left(\frac{-a}{s^2}\right) + \frac{-1}{1+\left(s/b\right)^2}\left(\frac{1}{b}\right)$$

$$F'(s) = \frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right) - \frac{b^2}{b^2+s^2}\left(\frac{1}{b}\right)$$

$$F'(s) = \frac{-a}{s^2+a^2} - \frac{b}{b^2+s^2}$$

We know that $L^{-1}(F(s)) = \frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\tan^{-1}\left(a/s\right)+\cot^{-1}\left(s/b\right)\right) = \left(\frac{-a}{s^2+a^2}-\frac{b}{b^2+s^2}\right)$$

$$= \frac{1}{t}L^{-1}\left(\frac{a}{s^2+a^2}-\frac{b}{b^2+s^2}\right)$$

$$= \frac{1}{t}L^{-1}\left(L^{-1}\left(\frac{a}{s^2+a^2}\right)-L^{-1}\left(\frac{b}{b^2+s^2}\right)\right)$$

$$= \frac{1}{t}(\sin at+\sin bt)$$

7. Find $L^{-1}\left(\log\left(1+a^2/s^2\right)\right)$

Solution:

Let $F(s) = \log\left(1+a^2/s^2\right)$

$$\therefore F(s) = \log\left(\frac{s^2+a^2}{s^2}\right)$$

$$F(s) = \log\left(s^2+a^2\right)-\log s^2$$

$$F(s) = \log\left(s^2+a^2\right)-2\log s^2$$

$$\therefore F(s) = \frac{2s}{s^2+a^2}-\frac{2}{s}$$

We know that $L^{-1-}(F(s)) = \frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log(1+\left(a^2/s^2\right)\right) = \frac{-1}{t}L^{-1}\left(\frac{2s}{s^2+a^2}-\frac{2}{s}\right)$$

$$= \frac{-2}{t}\left(L^{-1}\left(\frac{s}{s^2+a^2}\right)-L^{-1}\left(\frac{1}{s}\right)\right)$$

$$= \frac{-2}{t}(\cos at-1)$$

$$= \frac{2}{t}(1-\cos at)$$

8. Find $L^{-1}\left(\log\dfrac{(s+a)}{(s+b)}\right)$

Solution:

Let $F(s) = \log\dfrac{(s+a)}{(s+b)}$

$= \log(s+a) - \log(s+b)$

$F'(s) = \dfrac{1}{s+a} - \dfrac{1}{s+b} \qquad \because L^{-1}(F(s)) = \dfrac{-1}{t}L^{-1}(s))$

$$L^{-1}\left(\log\frac{(s+a)}{(s+b)}\right) = \frac{-1}{t}L^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$= \frac{-1}{t}(e^{-ar} - e^{-bt})$$

9. Find $L^{-1}\left(\log\dfrac{s(s^2+a^2)}{(s^2+b^2)}\right)$

Solution:

Let $F(s) = \log\dfrac{s(s^2+a^2)}{(s^2+b^2)}$

$F(s) = \log(s(s^2+a^2) - \log(s^2+b^2))$

$F(s) = \log s + \log(s(s^2+a^2) - \log(s^2+b^2))$

$F'(s) = \dfrac{1}{s} + \dfrac{2s}{(s^2+a^2)} - \dfrac{2s}{(s^2+^2)}$

We know that $L^{-1}(F(s)) = \dfrac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\log\frac{s(s^2+a^2)}{s(s^2+b^2)} = \frac{-1}{t}L^{-1}\left(\frac{1}{s} + \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right)$$

$$= \frac{-1}{t}\left(L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{2s}{s^2+a^2}\right) - L^{-1}\left(\frac{2s}{s^2+b^2}\right)\right)$$

$$= \frac{-1}{t}[1 + 2\cos at - 2\cos bt]$$

10. Find $L^{-1}\left(\log \frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

Solution:

Let $F(s) = \log\left(\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

$$= \log(s(s^2+1)(s-4)^2 - \log((s^2-9)(s^2+4))$$

$$F(s) = \log s + \log(s^2+1) + \log(s-4)^2 - \log(s^2-9) - \log(s^2+4)$$

$$F'(s) = \frac{1}{s} + \frac{2s}{s^2+1} + \frac{2(s-4)}{(s-4)^2} - \frac{2s}{s^2-9} - \frac{2s}{s^2+4}$$

we know that, $L^{-1}(F(s) = \frac{-1}{1} L^{-1}(F'(s))$

$$L^{-1}\log \frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)} = \frac{-1}{1} L^{-1}\left(\frac{1}{s} + \frac{2s}{s^2+1} + \frac{2}{s-4} - \frac{2s}{s^2-9} - \frac{2s}{s^2+4}\right)$$

$$= \frac{-1}{1}(1 + 2\cos t + 2e^{4t} - 2\cos h3t - 2\cos 2t)$$

11. Find $L^{-1}\left(\log \frac{s-a}{(s^2+a^2)}\right)$

Solution:

Let $F(s) = \log \frac{s-a}{s^2+a^2}$

$$= \log(s-a) - \log(s^2+a^2)$$

$$F'(s) = \frac{1}{s-a} - \frac{2s}{s^2+a^2}$$

We know that $L^{-1}(F(s)) = \frac{-1}{t} L^{-1}(F'(s))$

$$L^{-1}\left(\log\frac{s-a}{(s^2+a^2)}\right) = \frac{-1}{t}L^{-1}\left(\frac{1}{s-a}-\frac{2s}{s^2+a^2}\right)$$

$$= \frac{-1}{t}L^{-1}\left(\frac{2s}{s^2-a^2}-\frac{1}{s-a}\right)$$

$$= \frac{-1}{t}\left(L^{-1}\left(\frac{2s}{s^2+a^2}\right)-L^{-1}\left(\frac{1}{s-a}\right)\right)$$

$$= \frac{1}{t}(2\cos at - e^{at})$$

22. Theorem

If $L(f(t)) = F(s)$ and $\varphi(t)$ is a function such that $L(\varphi(t)) = F(s)$ and $\varphi(0) = 0$, then $f(t) = \varphi'(t)$, (ie) $L^{-t}(sf(s)) = \frac{d}{dt}L^{-1}(F(s))$.

Proof:

We know that

$$L(\varphi'(t)) = sL(\varphi(t))) - \varphi(0)$$

$$= sF(s) \qquad (\because \varphi(0) = 0$$

(ie) $L(\varphi'(t)) = L(f(t))$

$$\therefore \varphi'(t) = f(t)$$

From this result, we get

$$L^{-1}(s(s)) = f(t)$$

$$= \varphi'(t)$$

$$= \frac{d}{dt}\varphi(t)$$

$$= \frac{d}{dt}L^{-1}(F(s)) \qquad (\because L\varphi(t)) = F(s))$$

Provided $L^{-1}(F(s)) = 0$ as $t \to 0$

Problems

1. Find $L^{-1}\left(\frac{s}{(s+2)^2+4}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2+4}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2+4}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{(s+2)^2+4}\right) \quad \text{(using the above result)}$$

$$= \frac{d}{dt}e^{-2t}L^{-t}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{d}{dt}e^{-2t}L^{-t}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\frac{1}{2}\sin 2t\right)$$

$$= \frac{1}{2}\left(2e^{-2t}\cos 2t + \sin 2te^{-2t}(-2)\right)$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

Aliter:

$$L^{-1}\left(\frac{s}{(s+2)^2+4}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2+4}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+4} - \frac{s}{(s+2)^2+4}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+4}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2+4}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{s}{s^2+2^2}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= e^{-2t}\cos 2t - 2e^{-2t}\frac{1}{2}\sin 2t$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

2. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s}{(s+2)^2}\right)$$

$$= L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-2t} + t(e^{-2t}(-2))$$

$$= e^{-2t}(1-2t)$$

Aliter:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2}\right) - L^{-1}\left(\frac{2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+2)}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-2t} - 2e^{-2t}t$$

$$= e^{-2t}(1-2t)$$

3. Find $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right) = L^{-1}\left(s.\frac{s}{(s^2+a^2)}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$= \frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$$

(By the Previous Section 21.1 Problem No.1)

$$= \frac{1}{2a}(at\cos at + \sin at)$$

4. Find $L^{-1}\left(\dfrac{s^2}{(s-1)^4}\right)$

Solution:

$$L^{-1}\left(\frac{s^2}{(s-1)^4}\right) = L^{-1}\left(s.\frac{s}{(s-1)^4}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{s}{(s-1)^4}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{s-1+1}{(s-1)^4}\right)$$

$$= \frac{d}{dt}\left(L^{-1}\left(\frac{s-1}{(s-1)^4}\right) + L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$= \frac{d}{dt}\left(L^{-1}\left(\frac{1}{(s-1)^3}\right) + L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$= \frac{d}{dt}\left(e^t L^{-1}\left(\frac{1}{s^3}\right) + e^t L^{-1}\left(\frac{1}{s^4}\right)\right)$$

$$= \frac{d}{dt}\left(e^t \frac{t^2}{2} + e^t \frac{t^3}{6}\right)$$

$$= \frac{1}{2}(e^t 2t + t^2 e^t) + \frac{1}{6}(e^t 3t^2 + t^3 e^t)$$

$$= te^t + e^t t^2 + \frac{t^3 e^t}{6}$$

5. Find $L^{-1}\left(\frac{s-3}{s^2+4s+13}\right)$

Solution:

$$L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) = L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) - L^{-1}\left(\frac{3}{s^2+4s+13}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{s^2+4s+13}\right) - 3L^{-1}\left(\frac{1}{s^2+4s+13}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2+9}\right) - 3L^{-1}\left(\frac{1}{(s+2)^2+3^2}\right)$$

$$= \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2+3^2}\right) - 3e^{-2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\frac{\sin 3t}{3}\right) - 3^{-2t}\left(\frac{\sin 3t}{3}\right)$$

$$= \frac{1}{3}(3e^{-2t}\cos 3t - 2\sin 3te^{-2t}) - 3^{-2t}\sin 3t$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

23. Theorem

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t L^{-1}(F(s))dt$$

Proof:

We know that,

$$L\left(\int_0^t f(x)dx\right) = \frac{1}{s}L(f(t))$$

$$\therefore \int_0^t f(x)dx = L^{-1}\left(\frac{1}{s}L(f(t))\right)$$

(ie) $$L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(t)dt \qquad s[\therefore F(s) = L(f)(t))]$$

$$= \int_0^t L^{-1}(F(s))dt$$

$$\therefore L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t L^{-1}(F(s))dt$$

Note:

Similarly $$L^{-1}\left(\frac{1}{s^2}F(s)\right) = \int_0^t \int_0^t L^{-1}(F(s))dtdt$$

$$L^{-1}\left(\frac{1}{s^3}F(s)\right) = \int_0^t \int_0^t \int_0^t L^{-1}(F(s))dtdtdt$$

$$L^{-1}\left(\frac{1}{s^n}F(s)\right) = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n\,times} L^{-1}(F(s))\underbrace{dtdt\cdots dt}_{n\,times}$$

23.1 Problems

1. Find $L^{-1}\left(\frac{1}{s(s+1)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s+1)}\right) = \int_0^t L^{-1}\left(\frac{1}{(s+1)}\right)dt \quad \text{(by the above theorem)}$$

$$= \int_0^t e^{-t}dt$$

$$= \left(-e^{-t}\right)_0^t$$

$$= -\left(e^{-t} - 1\right)$$

$$= 1 - e^{-t}$$

2. Find $L^{-1}\left(\frac{1}{s(s+2)^3}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s+2)^3}\right) = \int_0^t \left(\frac{1}{(s+2)^3}\right)dt$$

$$= \int_0^t e^{-2t} L^{-1}\left(\frac{1}{s^3}\right)dt$$

$$= \int_0^t \frac{e^{-2t}}{2} L^{-1}\left(\frac{2}{s^3}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-2t}t^2 dt$$

$$= \frac{1}{2}\left[(t^2)\left(\frac{e^{-2t}}{2}\right) - (2t)\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$\left[\because \int u dv = uv - u'v_1 + u''v_2 \ldots\right]$$

$$= \frac{1}{2}\left[\frac{-t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{-e^{-2t}}{2}\left(t^2 + t + \frac{1}{2}\right) + \frac{1}{4}\right]$$

$$= \frac{1}{8}\left(1 - \left(2t^2 + 2t + 1\right)e^{-2t}\right)$$

3. Find $L^{-1}\left(\frac{54}{s^3(s-3)}\right)$

Solution:

$$L^{-1}\left(\frac{54}{s^3(s-3)}\right) = 54\int_0^t \int_0^t \int_0^t L^{-1}\left(\frac{1}{(s-3)}\right)dt\,dt\,dt$$

$$= 54\int_0^t \int_0^t \int_0^t e^{3t}\,dt\,dt\,dt$$

$$= 54\int_0^t \int_0^t \left(\frac{3^{3t}}{(3)}\right)_0^t dt\,dt$$

$$= 18\int_0^t \int_0^t \left(e^{3t} - 1\right) dt\,dt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t\right)_0^t dt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t\right) - \left(\frac{1}{3} - 0\right) dt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t - \frac{1}{3}\right) dt$$

$$= 18\left(\frac{e^{3t}}{9} - \frac{t^2}{2} - \frac{t}{3} - \frac{1}{9}\right)$$

$$= 2e^{3t} - 9t^2 - 6t - 2$$

4. Find $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right) dt$$

$$= \int_0^t \frac{1}{a} L^{-1}\left(\frac{a}{s^2+a^2}\right) dt$$

$$= \frac{1}{a}\int_0^t \sin at\, dt$$

$$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t$$

$$= \frac{-1}{a^2}(\cos at - 1)$$

$$= \frac{+1}{a^2}(\cos at)$$

5. Find $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s}{s(s^2+a^2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}.\frac{s}{(s^2+a^2)^2}\right)$$

$$= \int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)dt$$

$$= \int_0^t \frac{t\sin at}{2a}dt$$

$$= \frac{1}{2a}\left(t\left(\frac{-\cos at}{a}\right)-1\left(\frac{-\sin at}{a^2}\right)\right)_0^t$$

$$= \frac{1}{2a}\left(\frac{-t\cos at}{a}+\frac{\sin at}{a^2}\right)$$

(By the previous section 21.1 Problem no.1)

6. Find $L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right) = L^{-1}\left(\frac{1}{s}.\frac{1}{s^2-2s+5}\right)$$

$$= \int_0^t L^{-1}\left(\frac{1}{s^2-2s+5}\right)dt$$

$$= \int_0^t L^{-1}\left(\frac{1}{(s-1)^2+2^2}\right)dt$$

$$= \int_0^t e^t L^{-1}\left(\frac{1}{s^2+2^2}\right)dt$$

$$= \int_0^t e^t \frac{\sin 2t}{2} t$$

$$= \frac{1}{2}\int_0^t e^t \sin 2t \, dt$$

$$= \frac{1}{2}\left[\frac{e^t}{1^2+2^2}(\sin 2t - 2\cos 2t)\right]_0^t$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t\right]_0^t$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t - 0 + 2\right]$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t + 2\right]$$

7. Find $L^{-1}\left(\frac{1}{s(s^2-6s+13)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-6s+13)}\right) = L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2+6s+13}\right)$$

$$= \int_0^t L^{-1}\left(\frac{1}{(s+3)^2+4}\right)dt$$

$$= \int_0^t e^{-3t} L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-3t} L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-3t} \sin 2t \, dt$$

$$= \frac{1}{2}\left\{\frac{e^{-3t}}{(-3)^2+2^2}(-3\sin 2t - 2\cos 2t)\right\}_0^t$$

$$= \frac{-1}{26}\left\{e^{-3t}(3\sin 2t + 2\cos 2t) - 2\right\}$$

8. Find $L^{-1}\left(\frac{1}{s(s^2+a^2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2+a^2)^2}\right) = L^{-1}\left(\frac{1}{s^2(s^2+a^2)^2}\right)$$

$$= \int_0^t \int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) dtdt$$

$$= \int_0^t \int_0^t \frac{t}{2a}\sin at\,dtdt \quad \text{(refer the above problem)}$$

$$= \frac{1}{2a}\int_0^t \int_0^t t\sin at\,dtdt$$

$$= \frac{1}{2a}\int_0^t \left(\left(t\frac{-\cos at}{a}\right) - (1)\left(\frac{-\sin at}{a^2}\right)\right)dt$$

$$= \frac{1}{2a}\int_0^t \left(\frac{\sin at}{a^2} - \frac{t\cos at}{a}\right)_0^t dt$$

$$= \frac{1}{2a^3}\int_0^t (\sin at - at\cos at)dt$$

$$= \frac{1}{2a^3}\left[\left(\frac{-\cos at}{a}\right)_0^t - a\left(t\left(\frac{\sin at}{a}\right) - (1)\left(\frac{-\cos at}{a^2}\right)\right)_0^t\right]$$

$$= \frac{1}{2a^3}\left[\frac{-\cos at}{a} - t\sin at - \frac{-\cos at}{a}\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at - \frac{2}{a}\right]$$

$$= \frac{-1}{2a^4}(2 - 2\cos at - at\sin at)$$

Inverse Laplace Transform using Second Shifting Theorem

If $L(f(t)) = F(s)$, then $L(f(t - a)) = U(tc - a)) = e^{-as} F(s)$ where 'a' is a positive constant and $U(t - a)$ isthe unit step function.

The above property can be written in terms of inverse Laplace operator as,

If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(e^{-as}F(s)) = f(t - a)U(t - a)$

$\therefore L^{-1}(e^{-as}F(s)) = L^{-1}(F(s))_{t\to t-a}.\ U(t-a)$ where U is the unit step function.

Thus we want to find the Laplace inverse transform of the product of two factors one of which is e^{-as}, ignore e^{-as}, find the inverse transform of the other function and then replace t by $t - a$ in it and multiply by $U(t - a)$

Problems

1. Find $L^{-1}\left(\frac{e^{-s}}{s+2}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-s}}{s+2}\right) = L^{-1}\left(\frac{1}{s+2}\right)_{t\to t-1} U(t-1).$$

$$= \left(e^{-2t}\right)_{t\to t-1} U(t-1) \text{ where U is the unit step function.}$$

$$= e^{-2(t-1)}U(t-1).$$

2. Find $L^{-1}\left(\frac{e^{-2s}}{s-1}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{s-1}\right) = \left\{L^{-1}\left(\frac{1}{s-1}\right)\right\}_{t\to t-2} .U(t-2)$$

$$= (e^{t})_{t\to t-2}U(t-2) \text{ where U is the unit step function}$$

$$= e^{t-2}U(t-2)$$

3. Find $L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left\{L^{-1}\left(\frac{1}{(s+1)^{5/2}}\right)\right\}_{t\to t=1} U(t-1) \qquad \ldots (1)$$

Now, $L^{-1}\left(\frac{1}{(s+1)^{5/2}}\right) = e^{-t}L^{-1}\left(\frac{1}{s^{5/2}}\right)$ Using first shifting property.

$$= e^{-t}\frac{1}{\Gamma\left(5/2\right)}t^{3/2} \qquad \left(\because L^{-1}\left(\frac{1}{s^{n}}\right) = \frac{1}{\Gamma(n)}t^{n-1}\right)$$

$$= e^{-t}\frac{1}{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}t^{3/2}$$

$$= \frac{4}{3.\sqrt{\pi}}3^{-t}t^{3/2} \qquad \ldots (2)$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left(\frac{4}{3\sqrt{\pi}}e^{-t}t^{3/2}\right)_{t\to t-1} U(t-1)$$

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left(\frac{4}{3\sqrt{\pi}}\right).e^{-(t-1)}(t-1)^{3/2}.U(t-1)$$

4. Find $L^{-1}\left(\frac{se^{-as}}{s^{2}-w^{2}}\right), a > 0$

Solution:

$$L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \left\{L^{-1}\left(\frac{s}{s^2 - w^2}\right)\right\}_{t \to t-a} U(t-a)$$

$$= (\cosh wt)_{t \to t-a} . U(t-a)$$

$$= \cosh wt(t-a).U(t-a)$$

5. Find $L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right) = \left\{L^{-1}\left(\frac{1}{(s+1)^3}\right)\right\}_{t \to t-2} .U(t-2) \quad \ldots(1)$$

Now, $$L^{-1}\left(\frac{1}{(s+1)^3}\right) = e^{-r} L^{-1}\left(\frac{1}{s^3}\right)$$

$$= \frac{e^{-t}}{2!} L^{-1}\left(\frac{2!}{s^2}\right)$$

$$= \frac{e^{-t}}{2} t^2 \quad \ldots(2)$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right) = \left(\frac{e^{-t}}{2} t^2\right)_{t \to t-2} U(t-2)$$

$$= \frac{e^{-(t-2)}.(t-2)^2 U(t-2)}{2}$$

6. Find $L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)e^{-5s}\right)$

Solution:

$$L^{-1}\left(e^{-5s}\left(\frac{3a-4s}{s^2+a^2}\right)\right) = L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)\right)_{t \to t-s} .U(t-5)$$

$$= \left[3L^{-1}\left(\frac{a}{a^2+s^2}\right) - 4L^{-1}\left(\frac{s}{a^2+s^2}\right)\right]_{t \to t-5} .U(t-5)$$

$$= (3\sin at - 4\cos at)_{t\to t-5} U(t-5)$$

$$= 3\sin a(t-5) - 4\cos a(t-5).U(t-5)$$

7. Find $L^{-1}\left(\dfrac{e^{-\pi S}}{(s-2)(S+5)}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-\pi S}}{(s-2)(S+5)}\right) = L^{-1}\left(\frac{1}{(s-2)(S+5)}\right)_{t\to t-\pi}$$

Now, $$\frac{1}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}$$

1 = A(s+5) + B(s-2)

Put s = -5 Put s = 2

$$\therefore B = \frac{-1}{7} \qquad \therefore A = \frac{1}{7}$$

$$\therefore L^{-1}\left(\frac{1}{(s-2)(s+5)}\right) = \frac{1}{7}L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{7}L^{-1}\left(\frac{1}{s+5}\right)$$

$$= \frac{1}{7}e^{2t} - \frac{1}{7}e^{-5t}$$

$$\therefore L^{-1}\left(\frac{e^{-\pi s}}{(s-2)(s+5)}\right) = \left(\frac{e^{2t}}{7} - \frac{e^{5t}}{7}\right)_{t\to t-\pi} U(t-\pi)$$

$$= \left(\frac{e^{2(t-\pi)}}{7} - \frac{e^{-5(t-\pi)}}{7}\right) U(t-\pi)$$

Exercise - 1(g)

Find the inverse Laplace transform of the following functions.

1. $\dfrac{e^{-as}}{s^2}, a > 0$ Ans: $\begin{cases} 0 & if \quad t < a \\ \dfrac{t-a}{1!} & if \quad t > a \end{cases}$

2. $\dfrac{e^{-2s}-e^{-3s}}{s}$ Ans: $\begin{Bmatrix} 0 & if & t<2 \\ 1 & if & t>2 \end{Bmatrix} + \begin{Bmatrix} 0 & if & t<3 \\ 1 & if & t>3 \end{Bmatrix}$

3. $\dfrac{e^{-3s}}{s-2}$ Ans: $\begin{cases} 0 & if \quad t<3 \\ e^{2(t-3)} & if \quad t>2 \end{cases}$

4. $\dfrac{se^{-s}}{s^2-9}$ Ans: $\begin{cases} 0 & if \quad t<1 \\ \cos 3(t-1) & if \quad t>1 \end{cases}$

5. $\dfrac{1+e^{-\pi s}}{s^2-1}$ Ans: $\sin t + \begin{cases} 0 & if \quad t<\pi \\ \sin(t-\pi) & if \quad t>\pi \end{cases}$

6. $\dfrac{1}{(s+1)^3}$ Ans: $e^{-t}\dfrac{t^2}{2!}$

7. $\dfrac{s^2+2s+3}{s^3}$ Ans: $1+2t+\dfrac{t^2}{2!}$

8. $\dfrac{s}{(s-2)^3}$ Ans: $e^{2t}\dfrac{t^3}{3!}$

9. $\dfrac{2s+3}{s^2+5}$ Ans: $2\cos 2t + 6\sin 2t$

10. $\dfrac{s+6}{s^2-16}$ Ans: $\cos h4t + 24\sin\ h4t$

Exercise - 1 (h)

Find the inverse Laplace transform of the following functions.

1. $\dfrac{1}{s^2-6s+10}$ Ans: $e^{3t}\sin t$

2. $\dfrac{1}{s^2-8s+16}$ Ans: te^{-4t}

3. $\dfrac{3s-2}{s^2-4s+20}$ Ans: $3e^{2t}\cos 4t + e^{2t}\sin 4t$

4. $\dfrac{3s+7}{s^2-4s+20}$ Ans: $4e^{3t} = e^{-t}$

5. $\dfrac{s+a}{(s+a)^2+a^2}$ Ans: $e^{-at}(b\cos bt-(d-ca)\sin bt)$

6. $\dfrac{s}{(s-a)^2+a^2}$ Ans: $e^{bt}\cos at$

7. $\dfrac{s+1}{s^2+6s+25}$ Ans: $e^{-3t}\left(\cos 4t-\dfrac{1}{2}\sin 4t\right)$

8. $\dfrac{1}{s^2+8s+16}$ Ans: te^{-4t}

9. $\dfrac{s}{(s+3)^2}$ Ans: $e^{-3t}(1-2t)$

10. $\dfrac{s}{(s^2+1)^2}$ Ans: $\dfrac{t}{2}\sin t$

Exercise - 1(i)

Find the inverse Laplace transform of the following functions.

1. $\dfrac{s}{(s-4)^5}$ Ans: $\dfrac{e^{4t}t^3(4-3t)}{24}$

2. $\dfrac{1}{(s^2+9)^2}$ Ans: $\dfrac{\sin 3t-3t\cos 3t}{54}$

3. $\dfrac{s+2}{(s^2+4s+5)^2}$ Ans: $\dfrac{t}{2}e^{-2t}\sin t$

4. $\dfrac{s^2+2s}{(s^2+2s+2)^2}$ Ans: $te^{-t}\cos t$

5. $\dfrac{1}{s(s+2)^3}$ Ans: $\dfrac{1}{S}(1-(1+2t+2t^2)e^{-2t}$

6. $\dfrac{s^2-s+2}{s(s-3)(s+2)}$ Ans: $\dfrac{1}{3}+\dfrac{8}{15}e^{st}+\dfrac{4}{5}e^{-2t}$

7. $\dfrac{2s-1}{s^2(s-1)^2}$ Ans: $t(e^t-1)$

8. $\dfrac{1}{s^2(s^2+a^2)^2}$ Ans: $\dfrac{at-\sin at}{a^3}$

9. $\dfrac{s+1}{s(s+2)}$ Ans: $\dfrac{1+e^{-t}}{2}$

10. $\dfrac{1}{(s*s^2+2s+2)}$ Ans: $\dfrac{1}{2}(1-\sin t+\cos t)e^t$

Exercise - 1(j)

Find the inverse Laplace Transform of the following functions.

1. $\log\dfrac{s-1}{s}$ Ans: $\dfrac{1-e^t}{t}$

2. $\log\dfrac{1+s}{s^2}$ Ans: $\dfrac{2-e^t}{t}$

3. $\log\left(1-\dfrac{a}{s}\right)$ Ans: $\dfrac{1-e^{at}}{t}$

4. $\log\dfrac{s^2+a^2}{s^2+b^2}$ Ans: $\dfrac{2}{t}(\cos bt-\cos at)$

5. $\log\dfrac{s^2+1}{s(s+1)}$ Ans: $\dfrac{1}{t}(1+e^{-r}-2\cos t)$

6. $\dfrac{1}{2}\log\dfrac{s^2+b^2}{(s-a)^2}$ Ans: $\dfrac{1}{t}(e^{at}-\cos bt)$

7. $\dfrac{1}{2}\log\dfrac{s^2+1}{(s+1)^2}$ Ans: $\dfrac{1}{t}(e^{-t}-\cos t)$

8. $\log\dfrac{s+3}{s(s-2)}$ Ans: $\dfrac{1}{t}(1+et^{2t}-e^{-3t})$

9. $\cot^{-1}(as)$ Ans: $\frac{1}{t}\sin\left(t/a\right)$

10. $\cot^{-1}\left(\frac{2}{s+1}\right)$ Ans: $\frac{-1}{t}(e^{-t}\sin 2t)$

11. $\cot^{-1}(1+s)$ Ans: $\frac{1}{t}e^{-t}\sin t$

12. $\tan^{-1}\left(\frac{s+a}{b}\right)$ Ans: $\frac{-1}{t}e^{at}\sin bt$

24. Partial Fraction

The rational fraction P(x)/Q(x) is said to be resolved into partial fraction if it can be expressed as the sum ofdifference of simple proper fractions.

Rules for resolving a Proper Fraction P(x) / Q(x) into partial fractions

Rule 1

Corresponding to every non repeated, linear factor (ax+b) of the denomiator Q(x), there exists a partial fraction of the form $\frac{A}{ax+b}$ where A is a constant, to be determined.

For Example

(i) $$\frac{2x-7}{(x-2)(3x-5)}=\frac{A}{x-2}+\frac{B}{3x-5}$$

(ii) $$\frac{5x^2+18x+22}{(x-1)(x+2)(2x+3)}=\frac{A}{x-1}+\frac{A}{x+2}+\frac{C}{2x+3}$$

Rule 2

Corresponding to every repeated linear factor $(ax\ b)^k$of the denominator Q(x), there exist k partial fractions of the forms,

$$\frac{A_1}{ax+b},\frac{A_2}{(ax+b)^2},\frac{A_3}{(ax+b)^3},\cdots\frac{A_k}{(ax+b)^k}$$

where $A_1, A_2,$ ..... A_k are constants to be determined.

For example

(i) $$\frac{4x-3}{(x+2)(2x-3)^2}=\frac{A}{x+2}+\frac{B}{2x-3}+\frac{C}{(2x-3)^2}$$

(ii) $$\frac{x+2}{(x-1)(2x-1)^3}=\frac{A}{x-1}+\frac{B}{(2x+1)}+\frac{C}{(2x-1)^2}+\frac{D}{(2x+1)^3}$$

Rule 3

Corresponding to every non-repeated irreducible quadratic factor $ax^{2\,+}\,bx + c$ of the denominator Q(x)there exists a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

($ax^2 + bx + c$) is said to be an irreducible quadratic factor, if it cannot be factorized into two linear factors with real coefficients.

Example

(i) $$\frac{x^2+1}{(x^2+4)(x^2+9)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+9}$$

(ii) $$\frac{8x^3-5x^2+2x+4}{(2x-1)^2(3x^2+4)}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{Cx+D}{3x^2+4}$$

In the case of an improper fraction, by division, it can be expressed as the sum of integral function and a proper fraction and then proper fraction is resolved into partial fractions.

Inverse Laplace Transform using Partial Fractions

1. Find $L^{-1}\left(\frac{1}{(s+1)(s+3)}\right)$

Solution:

Let $F(s)$ $= \left(\frac{1}{(s+1)(s+3)}\right)$

Let us split $F(S)$ into partial fractions,

$$\frac{1}{(s+1)(s+3)} = \frac{A}{(s+1)}+\frac{B}{(s+3)}$$

$$1 = A(S+3) + B(S+1)$$

Putting $S = -1$ Putting $S = -3$

$A = \frac{1}{2}$ $B = -\frac{1}{2}$

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{1/2}{(s+1)} + \frac{-1/2}{(s+3)}$$

$$\therefore \left(\frac{1}{(s+1)(s+3)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$$= \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

2. Find $L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)$

Solution:

Consider, $\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A(s+3)(s-2) + Bs(s-2) + Cs(s+3)}{s(s+3)(s-2)}$$

$$s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3)$$

put s = -3   put s = 2   put s = 0

9 – 3 – 2 -= B(-3)(5)   4 + 2 – 2 -= C(2)(5)   -2 = A(3)(-2)

4=15B 4 = 10C $A = \frac{1}{3}$

$B = \frac{4}{15}$ $\therefore C = \frac{4}{10}$ $C = \frac{2}{5}$

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{1}{3}.\frac{1}{s} + \frac{4}{15}.\frac{1}{s+3} + \frac{2}{5}.\frac{1}{s-2}$$

$$\therefore L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

3. Find $L^{-1}\left(\frac{s}{s^2+5s+6}\right)$

Solution:

Consider, $\frac{s}{s^2+5s+6}=\frac{s}{(s+2)(s+3)}=\frac{A}{(s+2)}+\frac{B}{(s+3)}$

$S = A(s + 3) + B(s + 2)$

Put $s = -3$ | Put $s = -2$

$-3 = A(0) + b(-1)$ | $-2 = A(1) + B(0)$

$-3 = -B$ | $A = -2$

$B = 3$

$$\frac{s}{(s+2)(s+3)}=\frac{-2}{(s+2)}+\frac{3}{(s+3)}$$

$$\therefore L^{-1}\left(\frac{1}{(s+2)(s+3)}\right)=2L^{-1}\left(\frac{1}{(s+2)}\right)+3L^{-1}\left(\frac{B}{(s+3)}\right)$$

$$= -2e^{-2t} + 3e^{-3t}$$

4. Find $L^{-1}\left(\frac{s}{(s+1)^2}\right)$

Solution:

Consider, $\frac{s}{(s+1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}$

$$\frac{s}{(s+1)^2}=\frac{A(s+1)+B}{(s+1)^2}$$

$s = A(s + 1) + B$

Put s = -1   Put s = 0

B = -1   0 = $A + B$

0 = $A - 1$

$A = 1$

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$L^{-1}\left(\frac{s}{(s+1)^2}\right) = L^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+1)}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= e^{-t} - e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t} - e^{-t}(t) = e^{-t}(1-t)$$

5. Find $L^{-1}\left(\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right)$

Solution:

$$\frac{5s^2 - 15S - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

put s = -1

-27A = 9

$A = \frac{-9}{27}$

$A = \frac{-1}{3}$

Put s = 2

3D = -21

D = -7

Equating the coefficient of s^3

$A + B = 0$

$B = \frac{1}{3}$

Equating the constant coefficient

$-8A + 4B - 2C + D = -11$

$$\frac{8}{3} + \frac{4}{3} - 2C - 7 = -11$$

-2C = -8

C = 4

$$\therefore \frac{5s^2-15S-11}{(s+1)(s-2)^2} = \frac{\frac{-1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left(\frac{5s^2-15S-11}{(s+1)(s-2)^2}\right) = \frac{-1}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{-1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$+4L^{-1}\left(\frac{1}{(s-2)^2}\right) - 7L^{-1}\left(\frac{1}{(s-2)^3}\right)$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left(\frac{1}{s^2}\right) - 7e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}.t - \frac{7}{2}e^{2t}t^2$$

6. Find $L^{-1}\left(\frac{2s^2+5s+2}{(s-3)^4}\right)$

Solution:

To resolve $\frac{2s^2+5s+2}{(s-3)^4}$ into partial fraction

we substitute $s-3=y$ (or) $s=y+3$

$$\therefore \frac{2s^2+5s+2}{(s-3)^4} = \frac{2(y+3)^2+5(y+3)+2}{y^4}$$

$$= \frac{2(y^2+6y+9)+5y+15+2}{y^4}$$

$$= \frac{2y^2+17y+35}{y^4}$$

$$= \frac{2}{y^2} + \frac{17}{y^3} + \frac{35}{y^4}$$

$$\frac{2s^2+5s+2}{(s-3)^4} = \frac{2}{(s-3)^2} + \frac{17}{(s-3)^3} + \frac{35}{(s-3)^4}$$

$$\therefore L^{-1}\left(\frac{2s^2+5s+2}{(s-3)^4}\right) = 2L^{-1}\left(\frac{1}{(s-3)^2}\right)+17L^{-1}\left(\frac{1}{(s-3)^3}\right)+35L^{-1}\left(\frac{1}{(s-3)^4}\right)$$

$$= 2e^{3t}L^{-1}\left(\frac{1!}{s^2}\right)+\frac{17}{2}e^{3t}L^{-1}\left(\frac{2!}{s^3}\right)+\frac{35}{6}e^{3t}L^{-1}\left(\frac{3}{s^4}\right)$$

$$= 2e^{3t}.t+\frac{17}{2}e^{3t}t^2+\frac{35}{6}t^3e^{3t}$$

7. Find $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s+b^2)}\right)$

Solution:

$$\frac{s^2}{(s^2+a^2)(s+b^2)} = \frac{A}{(s^2+a^2)}+\frac{B}{(s^2+b^2)}$$

$$s^2 = A(s^2+b^2)+B(s^2+a^2)$$

Put $s^2=-a^2$, $\quad -a^2 = A(-a^2+b^2)$

$$A=\frac{-a^2}{b^2-a^2}=\frac{a^2}{a^2-b^2}$$

Put $s^2=-b^2$, $\quad -b^2 = B(-b^2+a^2)$

$$B=\frac{-b^2}{a^2-b^2}$$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{\frac{a^2}{a^2-b^2}}{(s^2+a^2)}+\frac{\frac{-b^2}{a^2-b^2}}{(s^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2}-\frac{b^2}{s^2+b^2}\right)$$

$$L^{-1}\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}L^{-1}\left(\frac{a^2}{s^2+a^2}-\frac{b^2}{s^2+b^2}\right)$$

$$= \frac{1}{a^2 - b^2}\left(L^{-1}\left(\frac{a^2}{s^2 + a^2}\right) - L^{-1}\frac{b^2}{s^2 + b^2}\right)$$

$$= \frac{1}{a^2 - b^2}(a\sin at - b\sin bt)$$

8. Find $L^{-1}\left(\dfrac{1-s}{(s+1)^2(s^2+4s+13)}\right)$

Solution:

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{S+1} + \frac{Bs+C}{s^2+4s+13}$$

$$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$$

Putting	s = -1	Equating coefficient of s^2	Equating constant coefficient
	2=10A	$A + B = 0$	$13A + C = 1$
	$A = \frac{1}{5}$	$A = \frac{-1}{5}$	$C = 1 - \frac{13}{5}$
			$C = \frac{-8}{5}$

(ie), $$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{\frac{1}{5}}{s+1} + \frac{\frac{-1}{5}s - \frac{8}{5}}{s^2+4s+13}$$

$$L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right) = \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{5}L^{-1}\left(\frac{s+8}{s^2+4s+13}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left(\frac{s+2+6}{(s+2)^2+9}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right) - \frac{1}{5}L^{-1}\left(\frac{6}{(s+2)^2+3^2}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t = \frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$= \quad \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t = \frac{2}{5}e^{-2t}\sin 3t$$

9. Find $L^{-1}\left(\dfrac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right)$

Solution:

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-3s+2}$$

$$4s^2-3s+5 = A(s^2-3s+2) + (Bs+C)(s+1)$$

Putting	$s = -1$	Equating coefficient s^2	Equating constant coefficients
	$6A = 12$	$4 = A + B$	$5 = 2A + C$
	$A = 2$	$B = 2$	$C = 5 - 2A$
			$C = 1$

$$\therefore \frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{2}{s+1} + \frac{2s+1}{s^2 3s+2}$$

$$L^{-1}\left(\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right) = L^{-1}\left(\frac{2}{s+1}\right)L^{-1}\left(\frac{2s+1}{s^2-3s+2}\right)$$

$$= 2L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{2s+1}{(s-3/2)^2 - 1/4}\right)$$

$$= 2e^{-t} + 2L^{-1}\frac{s+1/2}{\left(s-3/2\right)^2 - 1/4}$$

$$= 2e^{-t} + 2L^{-1}\left(\frac{s+1/2-2+2}{\left(s-3/2\right)^2 - 1/4}\right)$$

$$= 2e^{-t} + 2L^{-1}\left(\frac{s+3/2}{\left(s-3/2\right)^2 - 1/4}\right) + 4L^{-1}\left(\frac{1}{\left(s-3/2\right)^2 - 1/4}\right)$$

$$= \quad 2e^{-t} + 2e^{\left(\frac{3}{2}\right)} L^{-1}\left(\frac{s}{s^2 - \left(\frac{1}{2}\right)^2}\right) + 4e^{\left(\frac{3}{2}\right)t} \sin\ h\left(\frac{t}{2}\right).2$$

$$= \quad 2e^{-t} + 2e^{\left(\frac{3}{2}\right)} \cosh\left(\frac{t}{2}\right) + 8e^{\left(\frac{3}{2}\right)t} \sin\ h\left(\frac{t}{2}\right)$$

Exercise - 1 (c)

Find the inverse Laplace transform of the following by Partial fraction method.

1. $\dfrac{86s - 78}{(s+3)(s-4)(5s-1)}$ Ans: $-3e^{-3t} + 2e^{4t} + e^{\left(\frac{1}{5}\right)t}$

2. $\dfrac{2 - 5s}{(s-6)(s^2+11)}$ Ans: $\dfrac{1}{45}\left(-28e^{-6t} + 28\cos\cos\sqrt{11}t - \dfrac{67}{\sqrt{11}}\sin\sqrt{11}r\right)$

3. $\dfrac{25}{s^3(s^2+4s+5)}$ Ans: $\dfrac{1}{5}\left(11 - 20t\,\dfrac{25}{2}t^2 - 11e^{-2t}\cos t - 2e^{-2t}\sin t\right)$

4. $\dfrac{1}{(s+1)(s^2+2s+2)}$ Ans: $e^{-1}(1 - \cos t)$

5. $\dfrac{1}{(s-1)(s+3)}$ Ans: $\dfrac{1}{4}(e^t - e^{-3t})$

6. $\dfrac{1}{(s+1)(s^2+1)}$ Ans: $\dfrac{1}{2}(\sin t - \cos t + e^{-t})$

7. $\dfrac{1}{(P+2)^2(P-2)}$ Ans: $\dfrac{1}{16}(e^{2t} - (4t+1)e^{-2t})$

8. $\dfrac{1}{s(s+1)^3}$ Ans: $1 - e^{-t} - \left(\dfrac{t^2}{2} + t + 1\right)$

9. $\dfrac{3s+1}{(s-2)(s^2+1)}$ Ans: $\dfrac{1}{5}(7e^{2t} - 7\cos t + \sin t)$

10. $\dfrac{1}{(s+1)^2(s^2+4)}$ Ans: $\dfrac{e^{-t}}{50}(te^{-t} - 3\sin 2t - 4\cos 2t)$

11. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ Ans: $\dfrac{1}{2}e^t - e^{2t} + \dfrac{5}{2}e^{3t}$

12. $\dfrac{19s+37}{(s+1)(s-2)(s+3)}$ Ans: $5e^{2t} - 3e^{-t} - 2e^{-3t}$

13. $\dfrac{1}{s^2(s^2+1)}$ Ans: $t - \sin t$

14. $\dfrac{1}{s^2(s^2+1)(s^2+9)}$ Ans: $\dfrac{t}{9} - \dfrac{\sin t}{8} + \dfrac{1}{72}\left(\dfrac{\sin 3t}{3}\right)$

15. $\dfrac{2s^2+5s+4}{s^3+s^2-2s}$ Ans: $2 + e^t - e^{2t}$

25. Convolution of two functions

If $f(t)$ and $g(t)$ are given functions, then the convolution of $f(t)$ and $g(t)$ is defined as $\int_0^t f(u)g(t-u)du.$ It is denoted by $f(t) * g(t)$.

25.1 Convolution Theorem

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$, then $L(f(t) * g(t)) = L(f(t))L(g(t))$

(ie) $L(f(t) * g(t)) = F(s).\ G(s)$

where $F(s) = L(f(t)),\ G(s) = L(g(t))$

Proof:

By definition of Laplace Transform,

We have $L(f(t)) * g(t) = \int_0^\infty \{e^{-st} f(t) * g(t)\} dt$

$$= \int_0^\infty e^{-st} \left\{ \int_0^t f(u)(t-u)du \right\} dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)dudt$$

on changing the order of integration,

$$= \int_0^\infty f(u)\left\{\int_u^\infty e^{-st} g(t-u)du\right\}dt$$

Put $t - u = v$ When $t = u, v = 0$

$dt = dv$ When $t = \infty, v = \infty$

$$L(f(t)) * g(t) = \int_0^\infty f(u)\left\{\int_0^\infty e^{-s(u+v)} g(v)dv\right\}du$$

$$= \int_0^\infty f(u)e^{-su}\left\{\int_0^\infty e^{sv} g(v)dv\right\}du$$

$$= \int_0^\infty e^{su} f(u)du \int_0^\infty e^{-sv} g(v)dv$$

$$= \int_0^\infty e^{-st} f(t)dt \int_0^\infty e^{-st} g(t)dv$$

$$= L(f)(t))L(g(t))$$

$$\therefore L(f(t)) * g(t) = F(s).G(s)$$

Corollary

Using the above theorem

We get,

$$L^{-1}(F(s).G(s)) = f(t) * g(t)$$

$$= L^{-1}(F(s) * L^{-1}(G(s))$$

Note

$$f(t) * g(t) = g(t) * f(t)$$

1. Find the value of $1*e^{-t}$

Solution:

Let $f(t) = 1, g(t) = e^{-t}$

$f(u) = 1, g(t-u) = e^{-(t-u)}$

$= e^{-t}e^{u}$

By definition, *f(t) * g(t)* $= \int_0^t f(u)g)t-u)du$

1*e^t $= \int_0^r 1e^{-t}e^u du$

$= e^{-t}(e^u)_0^t$

$= e^{-t}(e^t-1)$

$= 1-e^{-t}$

2. Evaluate 1*sin *t*

Solution:

Let *f(t)* = sin *t* *g(t)* = 1

f(t) = sin *u* *g(t-u)* = 1

By definition, *f(t) * g(t)* $= \int_0^t f(u)g)t-u)du$

t*e^t $= \int_0^r \sin u 1.du$

$= (\cos u)_0^t$

$= (\cos t - 1)$

$= 1 - \cos t$

3. Evaluate e^t * cos *t*

Solution:

Let *f(t) = cos t* *g(t) = e^t*

f(t) = cos u *g(t-u) = e-u^{t-u}*

$= e^t.e^{-u}$

*f(t)*g(t)* $= \int_0^t f(u)g)t-u)du$

$$e^t * \cos t = \int_0^t \cos u e^t e^{-u} du$$

$$e^t * \cos t = e^t \int_0^t e^{-u} \cos du$$

$$= e^t \left(\frac{e^{-u}}{(-1)^2 + 1^2} (-\cos u + \sin u) \right)_0^t$$

$$\left[\therefore \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

$$= e^t \left[\frac{e^{-t}}{2} (-\cos t + \sin t) - \frac{1}{2}(-1) \right]$$

$$= \frac{1}{2} (\sin t - \cos t) + \frac{1}{2} e^t$$

$$= \frac{1}{2} (\sin t - \cos t + e^t)$$

4. Use convolution theorem to find $L^{-1}\left(\frac{1}{(s+a)(s+b)} \right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+a)(s+b)} \right) = L^{-1}\left(\frac{1}{(s+a)} \right) * L^{-1}\left(\frac{1}{(s+b)} \right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} e^{-bt+bu} du$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)} \right]_0^t$$

$$= \frac{e^{-bt}}{-(a-b)} (e^{-(a-b)t} - 1)$$

$$= \frac{e^{-bt}}{-(a-b)} + \frac{e^{-bt}}{(a-b)}$$

$$= \frac{1}{(a-b)}(e^{-bt} e^{-at})$$

5. Use convolution theorem to find $L^{-1} \frac{1}{s(s^2+1)}$

Solution:

$$L^{-1} \frac{1}{s(s^2+1)} = L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 1 * \sin t$$

$$= \int_0^t \sin(t-u)du$$

$$= \left[\frac{-\cos(t-u)}{-1}\right]_0^t$$

$$= \cos 0 - \cos \text{t}$$

$$= 1 - \cos t$$

6. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$ using convolution theorem

Solution:

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2+a^2)}\right) * L^{-1}\left(\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right)$$

$$= \cos at * \frac{1}{a} \sin at$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$

$$= \frac{1}{a}\int_0^t \left(\frac{\sin a(t-u+u)+\sin a(t-u-u)}{2}\right)du$$

$$= \frac{1}{2a}\int_0^t (\sin at + \sin a(t-du))du$$

$$= \frac{1}{2a}\left[u\sin at + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{t\sin at}{2a}$$

7. Find $L^{-1}\left(\frac{1}{s(s^2-a^2)}\right)$ using convolution theorem.

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-a^2)}\right) = L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2-a^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2-a^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) * \frac{1}{a}L^{-1}\left(\frac{1}{s^2-a^2}\right)$$

$$= 1 * \frac{1}{a}\sin hat$$

Let $f(t) = \sin hat$; $g(t) = 1$

$f(u) = \sin hau$; $g(t-u) = 1$

$$1 * \frac{1}{a}\sin hat = \frac{1}{a}\int_0^t \sin hau.1du$$

$$= \frac{1}{a}\left(\frac{\cos hau}{a}\right)_0^t$$

$$= \frac{1}{a^2}(\cosh at - 1)$$

$$\therefore L^{-1}\left(\frac{1}{s(s^2-a^2)}\right) = \frac{1}{a^2}(\cos hat - 1)$$

8. Find $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$ using convolution theorem.

Solution:

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = L^{-1}\left(\frac{s}{(s^2+a^2)}\cdot\frac{s}{s^2+b^2}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2+a^2)}\right) * L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au.\cos b(t-u)du$$

$$= \int_0^t \left(\frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2}\right)du$$

$$= \frac{1}{2}\int_0^t \left(\cos((a-b)u+bt)) + \cos((a+b)u-bt)\right)du$$

$$= \frac{1}{2}\int_0^t \left[\frac{\sin(bt+(a-b)u}{a-b} + \frac{\sin(a+b)u-bt}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

9. Using convolution theorem find $L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right) = L^{-1}\left(\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) * L^{-1}\left(\frac{1}{s^2+b^2}\right)$$

$$= \frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right) * \frac{1}{b}L^{-1}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{1}{a}\sin at * \frac{1}{b}\sin bt$$

Let $f(t) = \frac{1}{a}\sin at;$ $\qquad g(t) = \frac{1}{b}\sin bt$

$$f(u) = \frac{1}{a}\sin au; \quad g(t-u) = \frac{1}{b}\sin b(t-u) = \frac{1}{b}\sin(bt-bu)$$

$$\frac{1}{a}\sin at * \frac{1}{b}\sin bt = \int_0^t \frac{1}{a}\sin au \frac{1}{b}\sin(bt-bu)du$$

$$= \frac{1}{ab}\int_0^t \sin au \sin(bt-bu)du$$

$$= \frac{1}{2ab}\int_0^t 2\sin au \sin(bt-bu)du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu)-\cos(au+bt-bu))du$$

$$= \frac{1}{2ab}\left[\frac{\sin(au-bt+bu}{a+b} - \frac{\sin(au+bt-bt}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at-bt+bt)}{a+b} - \frac{\sin(at+bt-bt)}{a-b} - \left(\frac{\sin bt}{a+b} - \frac{\sin bt}{a-b}\right)\right]$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{1}{a+b} - \frac{1}{a-b}\right) + \sin bt\left(\frac{1}{a+b} + \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{-2b}{a^2-b^2}\right) + \sin bt\left(\frac{2a}{a^2+b^2}\right)\right]$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{-2a}{a^2+b^2}\right)+\sin bt\left(\frac{2a}{a^2-b^2}\right)\right]$$

$$= \frac{2[a\sin bt - b\sin at]}{2aba(a^2-b^2)}$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}$$

$$\therefore L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right) = \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}$$

10. Find $L^{-1}\left(\frac{1}{s^2(s+1)}\right)$ using convolution theorem

Solution:

$$L^{-1}\left(\frac{1}{s^2(s+1)}\right) = L^{-1}\left(\frac{1}{s^2}.\frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2}\right) * L^{-1}\left(\frac{1}{s+1}\right)$$

$$= t * e^{-t}$$

$$= \int_0^t ue^{-(t-u)}du$$

$$= \int_0^t ue^{-t}e^{u}du$$

$$= e^{-t}\int_0^t ueudu$$

$$= e^{-t}\left[ue^u-(1)(e^u)\right]_0^t$$

$$= e^{-t}\left[(te^t-et)-(0-1)\right]$$

$$= e^{-t}\left[(te^t-e^t+1\right]$$

$$= \quad t - 1 + e^{-t}$$

Exercise - 1 (l)

Find the inverse Laplace transforms using convolution theorem.

1. $\dfrac{1}{s(s^2+4)^2}$ Ans: $\dfrac{1}{16}(1-\cos 2t - t\sin\ 2t)$

2. $\dfrac{1}{s(s^2+9)}$ Ans: $\dfrac{1}{6}(1-\cos 3t)$

3. $\dfrac{s^2}{(s^2+4)^2}$ Ans: $\dfrac{1}{2}\left(t\cos 2t + \dfrac{1}{2}\sin\ 2t\right)$

4. $\dfrac{1}{(s^2+4)(s+2)}$ Ans: $\dfrac{1}{8}(\sin\ 2t - \cos 2t + e^{-2t})$

5. $\dfrac{1}{s^2(s^2+a^2)}$ Ans: $\dfrac{1}{a^3}(at - \sin\ at)$

6. $\dfrac{4s^2}{(s^2+a^2)^2}$ Ans: $\dfrac{t}{2}\sin\ t$

7. $\dfrac{1}{(s^2-a^2)^2}$ Ans: $\dfrac{1}{2a^3}(at\cos hat - \sin\ hat)$

8. $\dfrac{1}{(s^2+4)^2}$ Ans: $\dfrac{1}{s}\left(\dfrac{\sin\ 2t}{2} - t\cos 2t\right)$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV –APPLICATIONS OF LAPLACE TRANSFORMATION– SMTA1201

UNIT – IV

APPLICATIONS OF LAPLACE TRANSFORM

1.1 INTRODUCTION

The Laplace Transform is a powerful integral transform,introduced by Laplace a French mathematician, astronomer, and physicist who applied the Newtonian theory of gravitation to the solar system (an important problem of his day). He played a leading role in the development of the metric system.

The Laplace Transform is widely used in solving linear Differential equations with initial conditions such as those arising in the analysis of electronic circuits. It can be greatly used to find the solution of problems of both ordinary and partial differential equations, system of simultaneous differential equations, and it is applied to evaluate some definite integrals.

Ordinary and partial differential equations describe the way certain quantities vary with time such as thecurrent in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through aninsulated conductor these equations are generally coupled with initial conditions that describe the state ofthe system at time t = 0. A very powerful technique for solving these problems is that of Laplace transformwhich transform the differential equation into an algebraic equation from which we get the solution.

Solutions of Differential Equations using Laplace Transform

The following results will be used in solving differential and integral equations using Laplace transforms.

Theorem

If $f(t)$ is continuous in $t \geq 0$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$and$f(t)$ and $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$

By definition $L[f'(t)] = \int_0^\infty e^{-st} f(t)dt$

$= \int_0^\infty e^{-st} d(f(t))$

$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt$,on integration by parts

$= \lim_{t\to\infty}[e^{-st} f(t)] - f(0) + s.L(f(t))$

$= 0 - (f(0) + sL(f(t))$ [$\because$ *f(t)* is of exponential order]

$= sL(f(t)) - f(0)$

Corollary 1

In the above theorem if we replace *f(t)* by *f′(t)* we get,

$L(f''(t)) = sL(f'(t)) - f'(0)$

$= s[sL(f(t)) - f(0)] - f'(0)$

$= s^2L(\text{f(t)}) - sf(0) - f'(0)$

Repeated application of the above theorem gives the following result:

$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f^1(0) - \ldots - f^{n-1}(0)$

Solved Problems

1. Using Laplace transform, solve $y' - y = t$, $y(0) = 0$.

Solution:

Given $y' - y = t$, $y(0) = 0$

Taking Laplace transform on both sides,

$L(y') - L(y) = L(t)$

$sL(y) - y(0) - L(Y) = \frac{1}{s^2}$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1}\right) dt \ \ dt$$

$$y = \int_0^t \int_0^t e^t dt \ \ dt$$

$$= \int_0^t [e^t]_0^t \ \ dt$$

$$= \int_0^t [e^t - 1] dt$$

$$= (e^t - 1)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$

Solution:

Taking Laplace transform on both sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$[s^2 L(y) - sy(0) - y^1(0)] - 4[sL(y) - y(0)] + 8L(y) = \frac{1}{s-2}$$

$$\text{i.e.,} [s^2 - 4s + 8]L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

Solving we get $A = \frac{1}{4}, \quad B = \frac{-1}{4}, \quad C = \frac{1}{2}$

$$= \frac{\frac{1}{4}}{s-2} + \frac{\frac{-1}{4}s + \frac{1}{2}}{s^2 - 4s + 8} + \frac{2s - 10}{s^2 - 4s + 8}$$

$$= \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}s - \frac{19}{2}}{s^2 - 4s + 8}$$

$$= \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2) - 6}{(s-2)^2 + 4}$$

$$y = \frac{1}{4} L^{-1}\left(\frac{1}{s-2}\right) + e^{2t}\left(\frac{\frac{7}{4}s - 6}{s^2 + 4}\right)$$

$$= \frac{1}{4} e^{2t} + e^{2t}\left(\frac{7}{4}\cos 2t - 3\sin 2t\right)$$

$$= \frac{1}{4} e^{2t}\left(1 + 7\cos 2t - 12\sin 2t\right)$$

3. Use Laplace transform to solve $y' - y = e^t$ given $y(0) = 1$

Solution:

$$y' - y = e^t$$

Taking Laplace transform on both sides of the equation,

we get $L(y') - L(y) = L(e^t)$, $y(0) = 1$

$$sL(y) - y(0) - L(y) = \frac{1}{s-1}$$

$$L(y)[s-1] = \frac{1}{s-1} + 1$$

$$L(y)=\frac{s}{(s-1)^2}$$

$$y \;=\; L^{-1}\left[\frac{s}{(s-1)^2}\right]$$

$$=\; L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$

$$=\; L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\frac{1}{(s-1)^2}$$

$$=\; e^t+te^t$$

$$=e^t(1+t)$$

4. Solve $\frac{d^2y}{dt^2}+9y=18t$ given that $y(0)=0=y\left(\frac{\pi}{2}\right)$

Solution:

$$y''+9y=18t \quad \text{where } y''=\frac{d^2y}{dt^2}$$

Taking Laplace transform on both sides of the equation, we get

$$L(y'')+9L(y)=18L(t)$$

$$\left[s^2L(y)-sy(0)-y'(0)\right]+9L(y)=\frac{18}{s^2}$$

$L(y)[s^2+9] \;=\; \frac{18}{s^2}+y'(0)$ [$\because y'(0)$ is not given we can take it to be a constant a]

$$=\; \frac{18}{s^2}+a$$

$$=\; \frac{as^2+18}{s^2}$$

$$L(y) = \frac{as^2 + 18}{s^2(s^2+9)}$$

$$= \frac{a}{s^2+9} + \frac{18}{s^2(s^2+9)}$$

$$y = L^{-1}\left(\frac{a}{s^2+9}\right) + L^{-1}\left(\frac{18}{s^2(s^2+9)}\right)$$

$$= L^{-1}\left(\frac{a}{s^2+9}\right) + L^{-1}\left(\frac{2}{s^2} - \frac{2}{(s^2+9)}\right) \qquad \text{(using partial fractions)}$$

$$= \frac{a\sin 3t}{3} + 2t - \frac{2\sin 3t}{3}$$

Now, using the conditions $t = 0$ and $t = \frac{\pi}{2}$ we have

$$0 = \frac{a}{3}\sin\left(\frac{3\pi}{2}\right) + \pi - \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$= -\frac{a}{2} + \pi + \frac{2}{3}$$

$$\frac{a}{3} = \frac{3\pi + 2}{3}$$

Hence $a = 3\pi + 2$

$$\therefore y = \frac{(3\pi+2)\sin 3t}{3} + 2t - \frac{2\sin 3t}{3}$$

$$= \pi \sin 3t + 2t$$

5. Using Laplace transform, $y'' + 4y' + 3y = \sin t, y(0) = y'(0) = 0$

Solution:

Given $y'' + 4y' + 3y = \sin t$

Taking Laplace transform on both sides

$$L(y'') + 4L(y') + 3L(y) = L(\sin t)$$

$$[s^2 L(y) - sy(0) - y'(0)] + 4(sL(y) - y(0)] + 3L(y) = \frac{1}{s^2+1}$$

$$L(y)[s^2 + 4s + 3] = \frac{1}{s^2+1}$$

$$L(y) = \frac{1}{(s^2+4s+3)(s^2+1)}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+3)(s^2+1)}\right) \quad \ldots (1)$$

Now, $$\frac{1}{(s+1)(s+3)(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+3)} + \frac{C_2+D}{(s^2+1)}$$

$$1 = A(s+3)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s+3) \quad \ldots (2)$$

Put S = -3 in (2)

$$1 = B(-2)(10) \Rightarrow B = \frac{-1}{20}$$

Put S = -1 in (2)

$$1 = A(2)(2) \Rightarrow A = \frac{1}{4}$$

Comparing the coefficient of s^3,

$$0 = A + B + C$$

$$\therefore C = -A - B = -\frac{1}{4} + \frac{1}{20} = -\frac{4}{20} = -\frac{1}{5}$$

$$\therefore C = -\frac{1}{5}$$

Put $S = 0$ in (2)

$$1 = 3A + B + 3D$$

$\therefore 3D = 1 - 3A - B$

$$= 1 - \frac{3}{4} + \frac{1}{20} = \frac{3}{10}$$

$$\therefore \frac{1}{(s+1)(s+3)(s^2+1)} = \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{20}}{s+3} + \frac{\frac{-1}{5}s + \frac{3}{10}}{(s^2+1)}$$

$$\therefore L^{-1}\left(\frac{1}{(s+1)(s+3)(s^2+1)}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{20}L^{-1}\left(\frac{1}{s+3}\right)$$

$$\frac{-1}{5}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{3}{10}L^{-1}\left(\frac{1}{(s^2+1)}\right)$$

$$= \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} - \frac{1}{5}\cos t + \frac{3}{10}\sin t$$

6. Using Laplace transform solve $y'' - 3y' + 2y = 4$ given that $y(0) = 2, y'(0) = -3$.

Solution:

$$y'' - 3y' + 2y = 4$$

Taking Laplace transform on both sides

$$L(y'') - 3L(y') + 2L(y) = L(4)$$

$$[s^2 L(y) - sy(0) - y^1(0)] - 3[sL(y) - y(0)] + 2L(y) = \frac{4}{s}$$

$$s^2 L(y) - 2s + 3 - 3sL(y) + 6 + 2L(y) = \frac{4}{s}$$

$$L(y)[s^2 - 3s + 2] = \frac{4}{s} + 2s - 3 = \frac{4 + 2s^2 - 3s}{s}$$

$$L(y) = \frac{4 + 2s^2 - 3s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \qquad \ldots(1)$$

$$2s^2 - 3s + 4 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

Put $s = 1$ in (1), $3 = -B \Rightarrow B = -3$

Put $s = 2$ in (1), $6 = 2C \Rightarrow C = 3$

Put $s = 0$ in (1), $4 = -2A \Rightarrow A = 2$

$$Y = L^{-1}\left(\frac{2s^2 - 3s + 4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right) = 2 - 3e^{t} + 3e^{2t}$$

7. Solve using Laplace transform the differented equation $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 5y = 0$ where $y = 2, \frac{dy}{dt} = -4$ at $t = 0$.

Solution:

$y'' + 2y' + 5y = 0$ where $y(0) = 2, y'(0) = -4$

Taking Laplace transform on both sides

$$L(y'') + 2L(y') + 5L(y) = 0$$

$$[s^2L(y) - sy(0) - y^1(0)] + 2[sL(y) - y(0)] + 5L(y) = 0$$

$$L(y)[s^2 + 2s + 5] - 2s + 4 - 4 = 0$$

$$L(y) = \frac{2s}{s^2 + 2s + 5}$$

$$y = 2L^{-1}\left(\frac{(s+1)-1}{(s+1)^2 + 4}\right)$$

$$= 2L^{-1}\left(\frac{(s+1)}{(s+1)^2 + 4}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2 + 4}\right)$$

$$= 2e^{-1}L^{-1}\left(\frac{s}{s^2 + 4}\right) - e^{-t}L^{-1}\left(\frac{2}{s^2 + 4}\right)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

$$= e^{-t}(2\cos 2t - \sin 2t)$$

$L(y'') + 2L(y') + 5L(y) = 0$

$[s^2L(y) - sy(0) - y^1(0)] + 2[sL(y) - y(0)] + 5L(y) = 0$

$L(y)[s^2 + 2s + 5] - 2s + 4 - 4 = 0$

$$L(y) \quad = \frac{2s}{s^2 + 2s + 5}$$

$$y \quad = 2L^{-1}\left(\frac{(s+1)-1}{(s+1)^2 + 4}\right)$$

$$= 2L^{-1}\left(\frac{(s+1)}{(s+1)^2 + 4}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2 + 4}\right)$$

$$= 2e^{-1}L^{-1}\left(\frac{s}{s^2 + 4}\right) - e^{-t}L^{-1}\left(\frac{2}{s^2 + 4}\right)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

$$= e^{-t}(2\cos 2t - \sin 2t)$$

8. Using Laplace transform, solve $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = te^{-t}$ given $y = (0) = 1, y'(0) = -2.$

Solution:

$y'' + 2y' + y = te^{-t}$

Taking Laplace transform on both sides

$L(y'') + 2L(y') + L(y) = L(te^{-t})$

$$s^2L(y) - sy(0) - y^1(0) + 2[sL(y) - y(0)] + L(y) = \frac{1}{(s+1)^2}$$

$$L(y)[s^2 + 2s + 1] = \frac{1}{(s+1)^2} + s$$

$$L(y) \quad = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{s+1}{(s+1)^2}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= \frac{e^{-t}}{3!}L^{-1}\left(\frac{3!}{s^4}\right) + e^{-t} - e^{-t}t$$

$$= \frac{e^{-t}t^3}{6} + e^{-t} - e^{-t}t$$

$$= e^{-1}\left[\frac{t^3}{6} + t + 1\right]$$

9. Solve using L.T $y'' - 2y' + y = (t+1)^2$ given $y(0) = 4$ and $y'(0) = -2$.

Solution:

Given $y'' - 2y' + y = (t+1)^2$

$$L(y'') - 2L(y') + L(y) = L(t+1)^2$$

$$[s^2L(y) - sy(0) - y^1(0)] - 2[sL(y) - y(0)] + L(y) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

$$s^2L(y) - 4s + 2 - 2sL(y) + 8 + L(y) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

$$L(y)(s-1)^2 = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} + 4s - 10$$

$$L(y) = \frac{2}{s^3(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{4s}{(s-1)^2} - \frac{10}{(s-1)^2}$$

$$y = 2L^{-1}\left(\frac{1}{s^3(s-1)^2}\right) + 2L^{-1}\left(\frac{1}{s^2(s-1)^2}\right) + L^{-1}\left(\frac{1}{s(s-1)^2}\right) +$$

$$4L^{-1}\left(\frac{s}{(s-1)^2}\right)-10L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= 2\int_0^t\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dtdtdt+2\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dtdt+\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dt$$

$$+4L^{-1}\left(\frac{s-1+1}{(s-1)^2}\right)-10e^tL^{-1}\left(\frac{1}{s^2}\right)$$

$$y \quad = 2\int_0^t\int_0^t\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dtdtdt+2\int_0^t\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dtdt+\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dt+$$

$$4L^{-1}\left(\frac{s-1}{(s-1)^2}\right)+4L^{-1}\left(\frac{1}{(s-1)^2}\right)-10e^t.t$$

$$= 2\int_0^t\int_0^t\int_0^t e^t.tdtdtdt+2\int_0^t\int_0^t e^tt\,dtdt+\int_0^t e^tt\,dt+4L^{-1}\left(\frac{1}{(s-1)^2}\right)+4e^tL^{-1}\left(\frac{1}{s^2}\right)-10e^tt$$

$$= 2\int_0^t\int_0^t (te^t-e^t)_0^t\,dtdt+2\int_0^t (e^tt-e^t)_0^t\,dt+(te^t-e^t)_0^t+4e^t+4e^tt-10e^tt$$

$$= 2\int_0^t\int_0^t (te^t-e^t+1)dtdt+2\int_0^t te^t-e^t+1)dt+(te^t-e^t+1)+4e^t-6e^t.t$$

$$= 2\int_0^t (te^t-e^t-e^t+t)_0^t\,dt+2(te^t-e^t-e^t+t)_0^t+(te^t-e^t+1)+4e^t-6e^tt$$

$$y \quad = 2\int_0^t (te^t-2e^t+t+2)dt+2(te^t-2e^t+t+2)+(te^t-e^t+1)+4e^t-6e^tt$$

$$y \quad = 2\left[te^t-e^t-2e^t+\frac{t^2}{2}+2t\right]_0^t-3e^tt-e^t+2t+5$$

$$y \quad = 2\left[te^t-3e^t+\frac{t^2}{2}+2t+3\right]-3e^tt-e^t+2t+5$$

$$y \quad = -te^t-7e^t+t^2+6t+11$$

10. Using Laplace Transform, solve $\frac{d^2y}{dt^2}-\frac{4dy}{dt}+8y=e^{2t}$ $\quad y(0)=2, y'(0)=-2$

Solution:

Given $y'' - 4y' + 8y = e^{2t}$

Taking Laplace Transform on both sides,

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$[s^2 L(y) - sy(0) - y'(0)] - 4[sL(y) - y(0)] + 8L(y) = \frac{1}{s-2}$$

$$[s^2 - 4s + 8]L(y) - 2s + 10 = \frac{1}{s-2}$$

$$L(y)[s^2 - 4s + 8] = \frac{1}{s-2} + 2s - 10 = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$y = L^{-1}\left[\frac{1}{(s-2)(s^2-4s+8)}\right] + 2L^{-1}\left[\frac{s-5}{(s-2)^2+4}\right] \qquad \ldots(1)$$

$$\frac{1}{(s-2)(s^2-4s+8)} + \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8}$$

$$1 = A(s^2 - 4s + 8) + (s-2)(Bs + C) \qquad \ldots(2)$$

Put $S = 2$ in (2)

$1 = 4A \therefore A = 1/4$

Compare the coefficient of s^2,

$0 = A + B \therefore B = -1/4$

Compare the constant terms, we have

$1 = 8A - 2C$

$$\therefore 2C = 8A - 1 = 8\left(\frac{1}{4}\right) - 1 = 1$$

$C = 1/2$

$$y = \frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+L^{-1}\left(\frac{\frac{-s}{4}+\frac{1}{2}}{s^2-4s+8}\right)+2L^{-1}\left(\frac{s-2-3}{(s-2)^2+4}\right)$$

$$=\frac{1}{4}e^{2t}-\frac{1}{4}L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right)+2L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right)-6L^{-1}\left(\frac{1}{(s-2)^2+4}\right)$$

$$=\frac{1}{4}e^{2t}-\frac{1}{4}e^{2t}L^{-1}\left(\frac{s}{s^2+4}\right)+2e^{2t}L^{-1}\left(\frac{s}{s^2+4}\right)-6e^{2t}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$=\frac{e^{2t}}{4}-\frac{1}{4}e^{2t}\cos 2t+2e^{2t}\cos 2t-3e^{2t}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$y = \frac{e^{2t}}{4}+\frac{7}{4}e^{2t}\cos 2t-3e^{2t}\sin 2t$$

$$y = \frac{e^{2t}}{4}(1+7\cos 2t-12\sin 2t)$$

11. Solve $\frac{d^2y}{dt^2} = f(t)$ with $y(0)=0\ y'(0)=1$ and $f(t)=\begin{cases}1\ for\ 0<t<1\\ 0\ for\ t>1\end{cases}$

Solution:

Given $y''(t)=f(t)$

Taking Laplace transform on both sides we get

$$L(y'')=L(f(t))$$

$$s^2L(y)-sy(0)-y'(0)=L(f(t))$$

$$s^2L(y)-1=L(f(t))$$

Now

$$L(f(t))=\int_0^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}dt+\int_1^\infty e^{-st}f(t)dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^1 = \frac{1-e^{-s}}{s}$$

$$\therefore s^2 L(y) - 1 = \frac{1-e^{-s}}{s}$$

$$L(y) \quad = \frac{1-e^{-s}+s}{s^3} = \frac{1}{s^3} + \frac{1}{s^2} - \frac{e^{-s}}{s^3}$$

$$y \qquad = L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{e^{-s}}{s^3}\right)$$

$$= \frac{t^2}{2!} + t - L^{-1}\left(\frac{e^{-s}}{s^3}\right)$$

By second shifting theorem $L^{-1}(e^{-as}F(s) = f(t-a)U_a(t)$ where $U_a(t) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$

$$\therefore L^{-1}\left(\frac{e^{-s}}{s^3}\right) \quad = f(t-1)U_1(t)$$

$$= \frac{(t-2)^2}{2!}U_1(t) \qquad \left(\because f(t) = \frac{t^2}{2!} \right)$$

$$\therefore y = \frac{t^2}{2!} + t - \frac{(t-1)^2}{2!}U_1(t)$$

12. Using Laplace transform, solve the following equation $L\frac{di}{dt} + Ri = Ee^{-at}; i(0) = 0,$ where L, R, E and a are constants.

Solution:

Taking Laplace transform on both sides of the equation

$$L(L(i'(t) = RL(i(t)) = EL(e^{-at})$$

$$L(sL(i(t)) - i(0)) + RL(i(t)) = \frac{E}{s+a}$$

$$(Ls+R)L(i(t)) = \frac{E}{s+a}$$

$$L(i(t)) = \frac{E}{(s+a)(Ls+R)}$$

$$= \frac{A}{s+a} + \frac{B}{Ls+R} \qquad \dots (1)$$

$$E = A(Ls+R) + B(s+a)$$

Put $S = -a$

$$E = A(-aL + R)$$

$$\Rightarrow A = \frac{E}{R-aL}$$

Comparing the coefficient of S on both sides

$$0 = AL + B$$

$$B = \frac{-EL}{R-aL}$$

Substituting the values of A and B in (1)

$$L(i(t)) = \frac{\frac{E}{R-aL}}{s+a} - \frac{\frac{EL}{R-aL}}{Ls+R}$$

$$L(i(t)) = \frac{E}{R-aL}\left[\frac{1}{s+a}\frac{L}{L(s+R/L)}\right]$$

$$i(t) = \frac{E}{R-aL}\left[L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+R/L}\right)\right]$$

$$= \frac{E}{R-aL}\left[e^{-at} - e^{\frac{-R}{L}t}\right]$$

Exercise

1. Solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$ and $y'(0) = -2$

2. Solve $y'' + 4y = \sin wt, y(0) = 0$ and $y'(0) = 0$

3. Solve $y'' + y' - 2y = 3\cos 3t - 11\sin 3t, y(0) = 0$ and $y'(0) = 6$

4. Solve $(D^2 + 4D + 13)y = e^{-t}\sin t, y = 0$ and $Dy = 0$ at $t = 0$ where $D = \frac{d}{dt}$

5. Solve $(D^2 + 6D + 9)x = 6t^2 e^{-3t}, x = 0$ and $Dx = 0$ at $t = 0$

6. Solve $x'' + 3x' + 2x = 2(t^2 + t + 1), x(0) = 2, x'(0) = 0$

7. Solve $y'' - 3y' - 4y = 2e^t, y(0) = y'(0) = 1$

8. Solve $x'' + 9x = 18t,\ x(0) = 0,\ x\left(\frac{\pi}{2}\right) = 0$

9. $y'' + 4y' = \cos 2t, y(\pi) = 0, \quad y'(\pi) = 0$

10. $x'' - 2x + x = t^2 e^{-3t},\ x(0) = 2,\ x'(0) = 3$

Answers

1. $y = \frac{1}{4}e^{2t}(1 + 7\cos 2t - 12\sin 2t)$

2. $y = \frac{1}{8}(\sin 2t - 2t\cos 2t)$

3. $y = \sin 3t - e^{-2t} + e^t$

4. $y = \frac{1}{85}[e^{-t}\{-2\cos t + 9\sin t\}] + e^{-st}\left\{2\cos 3t = -\frac{7}{3}\sin 3t\right\}$

5. $x = \frac{1}{2}t^4 e^{-3t}$

6. $x = t^2 - 2t + 3 - e^{-2t}$

7. $y = \frac{1}{25}(13e^{-t} - 10te^{-t} + 12e^{4t})$

8. $x = 2t + \pi\sin 3t$

9. $y = \frac{1}{4}(t-\pi)\sin 2t$

10. $x = \left(\frac{t^4}{12} + t + 2\right)e^t$

Solution of Integral equations using Laplace transform

Theorem

If *f*(*t*) is a piecewise continuous in everyfinite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Proof

Let $g(t) = \int_0^t f(t)dt$

$\therefore g'(t) = f(t)$

$\therefore L(g'(t)) = sL(g(t)) - g(0)$

i.e $L(f)(t)) = sL\left(\int_0^t f(t)dt\right) - \int_0^0 f(t)dt$

$$\therefore L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Corollary:

$$L\left[\int_0^t \int_0^t f(t)dtdt\right] = \frac{1}{s^2}L(f(t))$$

In general

$$L\left[\int_0^t \int_0^t \ldots\ldots\int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L(f(t))$$

Problems

1. Solve $y+\int_0^t y\,dt = t^2+2t$

Solution:

Given $y+\int_0^t y\,dt = t^2+2t$

Taking Laplace Transform on both sides

$$L(y)+L\left(\int_0^t y\,dt\right) = L(t^2)+L(2t)$$

$$L(y)+\frac{1}{s}L(y) = \frac{2}{s^3}+\frac{2}{s^2}$$

$$L(y)\left[1+\frac{1}{s}\right] = 2\left[\frac{1+s}{s^3}\right]$$

$$L(y)\left[\frac{s+1}{s}\right] = 2\left[\frac{s+1}{s^3}\right]$$

$$L(y) = 2\left[\frac{s+1}{s^3}\right]\left[\frac{s}{s+1}\right]$$

$$= \frac{2}{s^2}$$

$$y = L^{-1}\left(\frac{2}{s^2}\right) = 2t$$

2. Solve $\frac{dy}{dt}+2y+\int_0^t y\,dt = 2\cos t, \quad y(0)=1$

Solution:

Given $y'+2y+\int_0^t y\,dt = 2\cos t$

Taking Laplace Transform on both sides

$$L(y') + 2Ly + L\left(\int_0^t ydt\right) = 2L(\cos t)$$

$$sL(y) - y(0) + 2L(y) + \frac{1}{s}L(y) = \frac{2s}{s^2+1}$$

$$L(y)\left[s + 2 + \frac{1}{s}\right] - 1 = \frac{2s}{s^2+1}$$

$$L(y)\left[\frac{s^2+2s+1}{s}\right] = \frac{2s}{s^2+1} + 1$$

$$L(y) = \left[\frac{s^2+2s+1}{s^2+1}\right]\left[\frac{s}{s^2+2s+1}\right]$$

$$= \frac{s}{s^2+1}$$

$$(y) = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

3. Using Laplace Transform solve $y + \int_0^t y(t)dt = e^{-t}$

Solution:

Given $y + \int_0^t y(t)dt = e^{-t}$

Taking Laplace transform on both sides,

$$L(y) + L\left(\int_0^t y(t)dt\right) = L(e^{-t})$$

$$L(y) + \frac{1}{s}L(y) = \frac{1}{s+1}$$

$$L(y)\left[1 + \frac{1}{s}\right] = \frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y=L^{-1}\left(\frac{s}{(s+1)^2}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$=L^{-1}\left(\frac{1}{(s+1)}\right)-e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y=e^{-t}-e^{-t}t$$

$$y=e^{-t}(1-t)$$

4. Using Laplace transform, solve $x+\int_0^t x(t)dt=\cos t+\sin t$

Solution:

$$x+\int_0^t x(t)dt=\cos t+\sin t$$

Taking Laplace transform on both sides,

$$L(x)+L\left(\int_0^t x(t)dt\right)=L(\cos t+\sin t)$$

$$L(x)\left[1+\frac{1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)\left[\frac{s+1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)=\left(\frac{s+1}{s^2+1}\right)\left(\frac{s}{s+1}\right)$$

$$L(x)=\frac{s}{s^2+1}$$

$$\therefore x = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

5. Solve using Laplace transform $y' + 3y + 2\int_0^t y\,dt = t, \quad y(0) = 0$

Solution:

$$y' + 3y + 2\int_0^t y\,dt = t$$

Taking Laplace Transform on both sides,

$$L(y') + L(3y) + 2L\left(\int_0^t y\,dt\right) = L(t)$$

$$sL(y) - y(0) + 3L(y) + 2\frac{1}{s}L(y) = \frac{1}{s^2}$$

$$L(y)\left[s + 3 + \frac{2}{s}\right] = \frac{1}{s^2}$$

$$L(y)\left[\frac{s^2+3s+2}{s}\right] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2}\cdot\frac{s}{s^2+3s+2}$$

$$= \frac{1}{s(s+1)(s+2)}$$

$$y = L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) \qquad \ldots(1)$$

Now, $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1) \qquad \ldots(2)$$

Put $s = -1$ in (2)

$B = -1$

Put $\quad s = -2$ in (2)

$$C = \frac{1}{2}$$

Put $\quad s = 0$ in (2)

$$A = \frac{1}{2}$$

$\therefore$ (1) becomes

$$y = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$y = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}$$

Solving Integral Equations using convolution

Theorem

By the definition of convolution, we have $f(t)*g(t) = \int_0^t f(u)g(t-u)du$ and by convolution theorem $L(f(t)*g(t)) = L(f)(t)L(g)(t))$

Problems

1. Solve $y = 1 + 2\int_0^t e^{-2t}y(t-u)du$ … (1)

Solution:

$\int_0^t e^{-2t}y(t-u)du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t) = e^{-2t}, g(t) = y(t)$

Taking Laplace Transform on both sides of (1),

$$L(y) \quad = \quad L(1) + 2L\left[\int_0^t e^{-2u}y(t-u)du\right]$$

$$= \frac{1}{s} + 2L\left[e^{-2t} * y(t)\right] \quad \text{(Definition of convolution)}$$

$$= \frac{1}{s} + 2L\left(e^{-2t}\right)L(y) \quad \text{(Convolution theorem)}$$

$$= \frac{1}{s} + 2\left(\frac{1}{s+2}\right)L(y)$$

$$L(y) = \frac{1}{s} + \frac{2}{s+2}L(y)$$

$$L(y) = \left[1 - \frac{2}{s+2}\right] = \frac{1}{s}$$

$$L(y) = \left[\frac{s}{s+2}\right] = \frac{1}{s}$$

$$L(y) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}$$

$$y = L^{-1}\left(\frac{1}{s} + \frac{2}{s^2}\right)$$

$$y = 1 + 2t$$

2. Using Laplace transform solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

Solution:

Given $y = 1 + \int_0^t y(u)\sin(t-u)du$

Taking Laplace transform on both sides,

$$L(y) = L(1) + L\left[\int_0^t y(u)\sin(t-u)du\right] \quad \ldots(1)$$

Now the integral $\int_0^t y(u)\sin(t-u)du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t) = y(t), g(t) = \sin t$

$\therefore$ (1) becomes

$$L(y) = \frac{1}{s} + L(y(t) * \sin t)$$

$$L(y) = \frac{1}{s} + L(y).\frac{1}{s^2+1}$$

$$L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s}$$

$$L(y) = \frac{s^2+1}{s^3}$$

$$= \frac{1}{s} + \frac{1}{s^3}$$

$$y = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^3}\right)$$

$$y = 1 + \frac{1}{2}t^2$$

3. Using Laplace transform solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Solution:

Given $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$ … (1)

Taking Laplace transform on both sides of (1),

$$L(f)(t)) = L(\cos t) + \left[L\int_0^t e^{-u} f(t-u)du\right]$$

$$= \frac{s}{s^2+1} + L(e^{-t} * f(t))$$

$$= \frac{s}{s^2+1} + L(e^{-t})\ Lf(t))$$

$$= \frac{s}{s^2+1} + \frac{1}{s+1} L(f(t))$$

$$L(f(t))\left[1 - \frac{1}{s+1}\right] = \frac{s}{s^2+1}$$

$$L(f(t))\left[\frac{1}{s+1}\right] = \frac{s}{s^2+1}$$

$$L(f(t)) = \frac{s+1}{s^2+1}$$

$$f(t) = L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$f(t) = \cos t + \sin t$$

4. Solve the integral equation $y(t) = t^2 + \int_0^t y(t)\sin(t-u)du$

Solution:

$$y(t) = t^2 + \int_0^t y(t)\sin(t-u)du$$

Taking Laplace transform obn both sides,

$$L(y(t)) = L(t^2) + L\left[\int_0^t y(t)\sin(t-u)du\right]$$

$$L(y) = \frac{2}{s^3} + L(y(t) * \sin t)$$

$$= \frac{2}{s^3} + L(y)L(\sin t)$$

$$= \frac{2}{s^3} + L(y)\left(\frac{1}{s^2+1}\right)$$

$$L(y)\left(1-\frac{1}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)\left(\frac{s^2}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y) \quad = \frac{2(s^2+1)}{s^5}=\frac{2}{s^3}+\frac{2}{s^5}$$

$$(y) \quad = L^{-1}\left(\frac{2}{s^3}\right)+\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)$$

$$(y) \quad = t^2+\frac{1}{12}t^4$$

5. Using Laplace transform solve the integral equation $y+\int_0^t y(u)du=e^{-t}$

Solution:

$$y+\int_0^t y(u)du=e^{-t}$$

$$\therefore y+y(t)*1=e^{-t}$$

Applying Laplace transform on both sides we get

$$L(y)+L[y(t)*1]=L(e^{-t})$$

$$\therefore L(y)+L(y)L(1)=L(e^{-t})$$

$$\therefore L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y = L^{-1}\left(\frac{s}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$= e^{-t} - e^{-t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t} - e^{-t} . t$$

$$= e^{-t}(1-t)$$

Exercise

1. Solve $x' + 3x + 2\int_0^t x\, dt = t, \quad x(0) = 0$

2. Solve $y' + 4y + 5\int_0^t y\, dt = e^{-t}, \quad y(0) = 0$

3. Solve $x' + 2x + \int_0^t x\, dt = \cos t, \quad x(0) = 1$

4. Solve $y' + 4y + 13\int_0^t y\, dt = 3e^{-t}, \sin 3t \quad y(0) = 3$

5. Solve $x(t) = 4t - 3\int_0^t x(u)\sin(t-u)du$

6. Solve $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)du$

7. Solve $\int_0^t y(u)y(t-u)du = 2y(t) + t - 2$

8. Solve $y(t) = t + \int_0^t \sin u \; y(t-u)du$

9. Solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

10. Solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Answers

1. $x = \frac{1}{2}(1 + e^{-2t}) - e^{-t}$

2. $y = \frac{-1}{2}e^{-t} + \frac{1}{2}e^{-t}(\cos t + 3\sin t)$

3. $x = \frac{1}{2}[(1-t)e^{-t} + \cos t]$

4. $y = e^{2t}\left[3\cos 3t - \frac{7}{3}\sin 3t + \frac{3}{2}t\sin 3t + t\cos 3t\right]$

5. $x = t + \frac{3}{2}\sin 2t$

6. $y(t) = e^{-t}(1-t)^2$

7. $y(t) = 1$

8. $y = t + \frac{t^3}{6}$

9. $y = 1 + \frac{t^2}{2}$

10. $f(t) = \cos t + \sin t$

Simultaneous differential equations

1. Using Laplace transform solve

$$\frac{dx}{dt} + y = \sin t$$

$$\frac{dy}{dt} + x = \cos t$$

given$x(0) = 2$ and $y(0) = 0$

Solution:

Applying Laplace transform to the given equations

We get, $L(x') + L(y) = L(\sin t)$

$$L(y') + L(x) = L(\cos t)$$

$$\therefore sL(x) - x(0) + L(y) = \frac{1}{s^2+1}$$

$$sL(y) - y(0) + L(x) = \frac{s}{s^2+1}$$

$$\therefore sL(x) + L(y) = \frac{1}{s^2+1} + 2$$

$$= \frac{2s^2+3}{s^2+1} \qquad \ldots(1)$$

Also $sL(y) + L(y) = \dfrac{s}{s^2+1} \qquad \ldots(2)$

$$(1) \times s \Rightarrow s^2 L(x) + sL(y) = \frac{(2s^2+3)s}{s^2+1} \qquad \ldots(3)$$

$$(2) \Rightarrow \quad L(x) + sL(y) = \frac{s}{s^2+1} \qquad \ldots(4)$$

$$(3)-(4)(s^2-1)L(x) = \frac{(2s^3+3s)}{s^2+1} - \frac{s}{s^2+1}$$

$$= \frac{2s^3+2s}{s^2+1}$$

$$L(x) = \frac{2s}{s^2 - 1} \qquad \ldots(5)$$

Substituting (5) in (2), we get

$$sL(y) = \frac{s}{s^2+1} - \frac{2s}{s^2-1} = \frac{s(s^2-1) - 2s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$= \frac{-s^3 - 3s}{(s^2+1)(s^2-1)}$$

$$= \frac{-s(s^2+3)}{-(s^2+1)(1-s^2)}$$

$$L(y) = \frac{(s^2+3)}{(s^2+1)(1-s^2)} \qquad \ldots(6)$$

From (5), $\quad x = L^{-1}\left(\frac{2s}{s^2-1}\right)$

$$= 2\cosh t$$

$$y = L^{-1}\left(\frac{(s^2+3)}{(1-s^2)(s^2+1)}\right)$$

Consider $\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{A}{1-s} - \frac{B}{1+s} + \frac{Cs+D}{s^2+1} \qquad \ldots(7)$

$$s^2 + 3 = A(1+s)(s^2+1) + B(1-s)(s^2+1) + (Cs+D)(1-s)(1+s)$$

Put $\quad s = 1, \quad 4 = A(2)(2)$

$\Rightarrow 4 = 4A \Rightarrow A = 1$

Put $\quad s = -1, \; 4 = B(2)(2)$

$\Rightarrow B = 1$

Put $\quad s = 0, \quad 3 = A + B + D$

$3 = 1 + 1 + D$

$\Rightarrow D = 1$

Comparing the coeffcient of S,

$0 = A - B + C$

$\Rightarrow C = 0$

Substituting the values of A, B, C, D in (7) we get

$$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{1}{1-s} - \frac{1}{1+s} + \frac{1}{s^2+1}$$

$$\therefore y = L^{-1}\left(\frac{1}{1-s}\right) + L^{-1}\left(\frac{1}{1+s}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$y = e^t + e^{-t} + \sin t$$

Hence the solution is $x = 2\cos ht$ and $y = e^t + e^{-t} + \sin t$

2. Solve $\dfrac{dx}{dt} + ax = y$

$$\frac{dy}{dt} + ay = x$$

given that $x = 0$ and $y = 1$ when $t = 0$

Solution:

Applying Laplace transform we get

$$L(x') + aL(x) = L(y)$$

$$L(y') + aL(y) = L(x)$$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$

$$sL(y) - y(0) + aL(y) = L(x)$$

Given that $x(0) = 0, \quad y(0) = 1$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$

$$sL(y) - y(0) + aL(y) = L(x)$$

$$\therefore sL(x) + aL(x) = L(y)$$

$$sL(y) - 1 + aL(y) = L(x)$$

$$\therefore (s+a)L(x) = L(y)$$

$$(s+a)L(x) - L(y) = 0 \qquad \ldots(1)$$

$$-L(x) + (s+a)L(y) = 1 \qquad \ldots(2)$$

$$(1) + (s+a) \times (2) \Rightarrow L(y)[(s+a)^2 - 1] = s + a$$

$$\therefore L(y) = \frac{s+a}{(s+a)^2 - 1}$$

Also by (1) $L(x) = \dfrac{1}{(s+a)^2 - 1}$

$$\therefore x = L^{-1}\left(\frac{1}{(s+a)^2 - 1}\right)$$

$$= e^{-at} L^{-1}\left(\frac{1}{s^2 - 1}\right)$$

$$= e^{-at} \sin\ ht$$

$$y = L^{-1}\left(\frac{s+a}{(s+a)^2 - 1}\right)$$

$$= e^{-at} L^{-1}\left(\frac{s}{s^2 - 1}\right)$$

$$= e^{-at} \cos ht$$

3. Solve $\dfrac{dy}{dt} + 2x = \sin 2t$ and

$$\frac{dx}{dt} - 2y = \cos 2t, \qquad x(0) = 1 \qquad y(0) = 0$$

Solution:

Taking Laplace Transform on both sides

$L(y') + 2L(x) = L(\sin\ 2t)$

$L(x') - 2L(y) = L(\cos 2t)$

$sL(y) - y(0) + 2L(x) = \frac{2}{s^2 + 4}$

$sL(x) - x(0) + 2L(y) = \frac{s}{s^2 + 4}$

$sL(x) - 2L(y) = \frac{s}{s^2 + 4} + 1$ … (1)

$2L(x) + sL(y) = \frac{2}{s^2 + 4}$ … (2)

$(1) \times (2) \Rightarrow 2sL(x) - 4L(y) = \frac{2s}{s^2 + 4} + 2$ … (3)

$(2) \times s \Rightarrow 2sL(x) + s^2 L(y) = \frac{2s}{s^2 + 4}$ … (4)

$(3) - (4) \Rightarrow -(s^2 + 4)L(y) = 2$

$$L(y) = \frac{-2}{s^2 + 4}$$

$\Rightarrow y = -\sin\ 2t$

$\therefore y' = -2\ \cos\ 2t$

Substituting y' in $y' + 2x = \sin\ 2t$

$$2x = \sin 2t + 2\cos 2t$$

$$x = \frac{1}{2}[\sin 2t + 2\cos 2t]$$

4. Solve $\frac{dx}{dt} + 3x - 2y = 1, \quad \frac{dy}{dt} - 2x + 3y = e^t, \quad x(0) = 0, \quad y(0) = 0$

Solution:

Taking Laplace transform on both sides

$$L(x') + 3L(x) - 2L(y) = L(1)$$

$$L(y') - 2L(x) + 3L(y) = L(e^t)$$

$$sL(x) - x(0) + 3L(x) - 2L(y) = \frac{1}{s}$$

$$sL(y) - y(0) - 2L(x) + 3L(y) = \frac{1}{s-1}$$

$$(s+3)L(x) - 2L(y) = \frac{1}{s} \qquad \ldots (1)$$

$$-2L(x) + (s+3)L(y) = \frac{1}{s-1} \qquad \ldots (2)$$

$$(1)\times 2 \Rightarrow 2(s+3)L(x) - 4L(y) = \frac{2}{s}$$

$$(s+3)\times(2) \Rightarrow -2(s+3)L(x) + (s+3)^2 L(y) = \frac{s+3}{s-1}$$

Adding $\left[(s+3)^2 - 4\right]L(y) = \frac{s+3}{s-1} + \frac{2}{s}$

$$(s^2 + 6s - 5)L(y) = \frac{s^2 + 5s - 2}{s(s-1)}$$

$$L(y) = \frac{s^2 + 5s - 2}{(s^2 + 6s + 5)(s(s-1))} \qquad \ldots (3)$$

Now, $$\frac{s^2+5s-2}{s(s-1)(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s+5}$$

$$s^2+5s-2 = A(s-1)(s+1)(s+5) + Bs(s+1)(s+5)$$

$$+C(s(s-1)(s+5)) + D(s(s-1)(s+1))$$

when, $s=1,\quad 4=12B \Rightarrow B=\frac{1}{3}$

when, $s=-1,\quad -6=8C \Rightarrow C=\frac{-3}{4}$

when, $s=-5,\quad -2=-120D \Rightarrow D=\frac{1}{60}$

when, $s=0,\quad -2=-5A \Rightarrow A=\frac{2}{5}$

$\therefore$ From (3) $$y = \frac{2}{5}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) - \frac{3}{4}L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{60}L^{-1}\left(\frac{1}{s+5}\right)$$

$$\therefore y(t) = \frac{2}{5} + \frac{1}{3}e^{t} - \frac{3}{4}e^{-t} + \frac{1}{60}e^{-5t} \qquad \ldots (4)$$

$$\therefore y'(t) = \frac{1}{3}e^{t} + \frac{3}{4}e^{-t} - \frac{1}{12}e^{-5t}$$

Substituting (4) and (5) in

$y' - 2x + 3y = e^t$ we get,

$2x = y' + 3y - e^t$

$$2x = \frac{1}{3}e^{t} + \frac{3}{4}e^{-t} - \frac{1}{12}e^{-5t} + 3\left(\frac{2}{5} + \frac{e^{t}}{3} - \frac{3}{4}e^{-t} + \frac{e^{-5t}}{60}\right) - e^{t}$$

$$= \frac{6}{5} + e^{t}\left(\frac{1}{3} + 1 - 1\right) + e^{-t}\left(\frac{3}{4} - \frac{9}{4}\right) + e^{-5t}\left(\frac{-1}{12} + \frac{1}{20}\right)$$

$$= \frac{6}{5} + \frac{1}{3}e^{t} - \frac{6}{4}e^{-t} - \frac{2}{60}e^{-5t}$$

$$x(t) = \frac{3}{5} + \frac{1}{6}e^{t} - \frac{3}{4}e^{-t} - \frac{1}{60}e^{-5t}$$

5. Using Laplace transform solve

$Dx + Dy = t$ and $D^2x - y = e^{-t}, x = 3, Dx = -2$ and $y = 0$ at $t = 0$

Solution:

Taking Leplace transform on both sides, we get

$$L(x') + L(y') = L(t)$$

$$L(x'') - L(y) = L(e^{-t})$$

$$sL(x) - x(0) + sL(y) - y(0) = \frac{1}{s^2}$$

$$s^2L(x) - sx(0) - x'(0) - L(y) = \frac{1}{s+1}$$

i.e., $$sL(x) + sL(y) = \frac{1}{s^2} + 30$$

$$\Rightarrow L(x) + L(y) = \frac{1}{s^3} + \frac{3}{s} \quad \ldots (1)$$

and $$s^2L(x) - L(y) = \frac{1}{s+1} + 3s - 2 \quad \ldots (2)$$

$$(1) + (2) \Rightarrow (s^2+1)L(x) = \frac{1}{s^3} + \frac{3}{s} + \frac{1}{s+1} + 3s - 2$$

$$L(x) = \frac{1}{s^3(s^2+1)} + \frac{3}{s(s^2+1)} + \frac{1}{(s^2+1)(s+1)} + \frac{3s-2}{s^2+1} \quad \ldots (3)$$

Consider $$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2 = 1) + (Bs + C)(s + 1)$$

Put, $s = -1, \quad 1 = 2A \Rightarrow A = 1/2$

Put, $s = 0, \quad 1 = A + C \Rightarrow C = 1/2$

Comparing coefficients of S;

$$0 = B + C \Rightarrow B = -1/2$$

(3) becomes

$$x(\mathrm{t}) = \int_0^t \int_0^t \int_0^t \sin t \, dt \, dt \, dt + 3\int_0^t \sin t \, dt + \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$$

$$+ 3L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$x(\mathrm{t}) = \int_0^t \int_0^t (-\cos t)_0^t \, dt \, dt + 3(-\cos t)_0^t + \frac{1}{2} e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t + 3\cos t - 2\sin t$$

$$= \int_0^t \int_0^t (-\cos t + 1) \, dt \, dt + 3(-\cos + 1) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t - \frac{5}{2}\cos t$$

$$= \int_0^t (t - \sin t)_0^t dt + 3(-\cos + 1) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t + \frac{5}{2}\cos t$$

$$= \int_0^t (t - \sin t) dt + 3(1 - \cos t) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t - \frac{5}{2}\cos t$$

$$= \left(\frac{t^2}{2} + \cos t\right)_0^t + 3(1 - \cos t) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t + \frac{5}{2}\cos t$$

$$= \frac{t^2}{2} + \frac{1}{2}\cos t + 2 + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t$$

$$\therefore x'' = \frac{1}{2} e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t + 1$$

Substituting x'' in $y = x''(\mathrm{t}) - e^{-t}$

$$y = 1 - \frac{1}{2}e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t$$

6. Solve $x' - 2x + 3y = 0$

$y' - y + 2x = 0$

given that $x(0) = 8$ and $y(0) = 3$

Solution:

Applying Laplace transform to the given equations we get,

$$L(x') - 2L(x) + 3L(y) = 0$$

$$L(y') - L(y) + 2L(x) = 0$$

i.e., $sL(x) - x(0) - 2L(x) + 3L(y) = 0$

$$sL(y) - y(0) - L(y) + 2L(x) = 0$$

The above equations reduce to

$$(s-2)L(x) + 3L(y) = 8 \qquad \dots (1)$$

$$2L(x) + (s-1)L(y) = 3 \qquad \dots (2)$$

$$(1)\times 2 \Rightarrow 2(s-2)L(x) + 6L(y) = 16 \qquad \dots (3)$$

$$(2)\times(s-2) \Rightarrow 2(s-2)L(x) + (s-1)(s-2)L(y) = 3(s-2) \qquad \dots (4)$$

$$(3)-(4) \Rightarrow [6-(s-1)(s-2)]L(y) = 16 - 3(s-2)$$

$$\Rightarrow -[s^2 - 3s - 4]L(y) = -[3s - 22]$$

$$L(y) = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)}$$

$$= \frac{A}{s+1} + \frac{B}{s-4}$$

$3s - 22 = A(s-4) + B(s+1)$ … (5)

Put $s = 4$ in (5),

$-10 = 5B \Rightarrow B = -2$

Put $s = -1$ in (5),

$-25 = -5A \Rightarrow A = 5$

$$\therefore L(y) = \frac{5}{s+1} + \frac{2}{s-4}$$

$$y = L^{-1}\left(\frac{5}{s+1}\right) - L^{-1}\left(\frac{2}{s-4}\right)$$

$$y = 5e^{-t} - 2e^{4t}$$

$$\Rightarrow y' = -5e^{-t} - 8e^{4t}$$

Substituting y and y' in $y' - y + 2x = 0$

we get $2x = y - y'$

$= (5e^{-t} - 2e^{4t}) - (-5e^{-t} - 8e^{4t})$

$= 10e^{-t} + 6e^{4t}$

$$x = \frac{1}{2}[10e^{-t} + 6e^{4t}] = 5e^{-t} + 3e^{4t}$$

7. Solve $x'' + y = -5 \cos 2t$

$y'' + x = 5 \cos 2t$

given that $x(0) = 0$, $x'(0) = 0$, $y'(0) = 0$, $y(0) = 0$

Solution:

Applying Laplace transform to the given equations

$$L(x'') + L(y) = -5L(\cos 2t)$$

$$L(y'') + L(x) = -5L(\cos 2t)$$

$$\therefore s^2 L(x) - sx(0) - x'(0) + L(y) = \frac{-5s}{s^2+4}$$

$$s^2 L(y) - sy(0) - y'(0) + L(x) = \frac{5s}{s^2+4}$$

Given that $x(0) = x'(0) = y'(0) = y(0) = 0$

$$\Rightarrow s^2 L(x) + L(y) = \frac{-5s}{s^2+4} \qquad \ldots (1)$$

$$L(x) + s^2 L(y) = \frac{5s}{s^2+4} \qquad \ldots (2)$$

$$(1)\times 1 \Rightarrow s^2 L(x) + L(y) = \frac{-5s}{s^2+4} \qquad \ldots (3)$$

$$(2)\times s^2 \Rightarrow s^2 L(x) + s^4 L(y) = \frac{5s^3}{s^2+4} \qquad \ldots (4)$$

$$(3)-(4) \Rightarrow (1-s^4)L(y) = \frac{-5s}{s^2+4} - \frac{5s^3}{s^2+4}$$

$$= \frac{-5s-5s^3}{s^2+4}$$

$$L(y) = \frac{5s(s^2+1)}{(s^4-1)(s^2+4)}$$

$$= \frac{5s(s^2+1)}{(s+1)(s-1)(s^2+1)(s^2+4)}$$

$$y = L^{-1}\left(\frac{5s}{(s+1)(s-1)(s^2+4)}\right)$$

Now $\quad \dfrac{5s}{(s+1)(s-1)(s^2+4)} = \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{Cs+D}{s^2+4}$

$$5s = A(s-1)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-1)$$

Put $s = 1$, $5 = B(2)(5)$

$$\Rightarrow B = \frac{1}{2}$$

Put $s = -1$, $-5 = A(-2)(5)$

$$\Rightarrow A = \frac{1}{2}$$

Put $s = 0$, $0 = -4A + 4B - D$

$$0 = -4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) - D$$

$$\Rightarrow D = 0$$

Comparing the coefficient of s^3,

$$A + B + C = 0$$

$$\frac{1}{2} + \frac{1}{2} + C = 0$$

$$\Rightarrow C = -1$$

$$\therefore y = \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{1}{2} e^{-t} + \frac{1}{2} e^{t} - \cos 2t$$

$$= \cos ht - \cos 2t$$

$$y' = \sin ht + 2 \sin 2t$$

$$y'' = \cos ht + 4 \cos 2t$$

From the given equation

$$x = 5 \cos 2t - y''$$

$= 5 \cos 2t - (\cosh t + 4 \cos 2t)$

$x = \cos 2t - \cosh t$

Exercise

1. Solve the simultaneous equations

   $2x' - y' + 3x = 2t$ and $x' + 2y' - 2x - y = t^2 - t$, $x(0) = 1$, $y(0) = 1$

2. Solve the simultaneous equations

   $D^2x - Dy = \cos t$ and $Dx + D^2y = -\sin t$; $x = 1$, $Dx = 0$, $y = 0$, $Dy = 1$ at $t = 0$

3. Solve $x' - y = e^t$ and $y' + x = \sin t$; $x(0) = 1$, $y(0) = 0$.

4. Solve $x' - y = \sin t$, $y' - x = -\cos t$; $x = 2$ and $y = 0$ at $t = 0$.

5. Solve $D^2x + y = -5 \cos 2t$, $D^2y + x = 5 \cos 2t$, $x = Dx = Dy = 1$ and $y = -1$ and $t = 0$.

Answer

1. $x = -1 + \frac{9}{8}e^{-t} + \frac{7}{8}e^{\frac{3t}{5}}$

   $y = -\frac{9}{8}e^{-t} + \frac{49}{8}e^{\frac{3t}{5}} - t^2 - 3t - 4$

2. $x = 1 + t\sin t, \quad y = t\cos t$

3. $x = \frac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)$

   $x = \frac{1}{2}(-e^t - \sin t + \cos t - t\sin t)$

4. $x = 2\cos ht, \quad y = 2\sin ht - \sin t$

5. $x = \sin t + \cos 2t, \quad y = \sin t - \cos 2t$

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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – V –FOURIER TRANSFORMATION – SMTA1201

UNIT – V

FOURIER TRANSFORM

Introduction

Jean-Baptiste Joseph Fourier (21st March 1768 – 16th May1830)was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law of conduction are also named in his honour. **Joseph Fourier** introduced the **transform**in his study of heat transfer, where Gaussian functions appear as solutions of the heat equation.

In the study of **Fourier** series, complicated but periodic functions are written as the sum of **simple** waves mathematically represented by sine and cosine functions. The **Fourier transform** is an extension of the **Fourier** series that results when the period of the represented function is lengthened and allowed to approach infinity. Fourier Transform maps a time series (eg. audio samples) into the series of frequencies (their amplitudes and phases) that composed the time series. Inverse Fourier Transform maps the series of frequencies (their amplitudes and phases) back into the corresponding time series. The two functions are inverses of each other. Shortly, The Fourier Transform is a mathematical technique that transforms a function of tim*e*, $f(t)$, to a function of frequency, *F(s)*.

Applications

- The Fourier transform has many applications, in fact any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier series and Fourier transforms. Here are some examples from physics, engineering, and signal processing.
  - Communication
  - Astronomy
  - Geology
  - Optics

- Fourier Transforms helps to analyze spectrum of the signals, helps in find the response of the LTI systems. (Continuous Time Fourier Transforms is for Analog signals and Discrete time Fourier Transforms is for discrete signals)
- Discrete Fourier Transforms are helpful in Digital signal processing for making convolution and many other signal manipulation.

Integral Transform

The integral of a function f(x) is defined by

$$I[f(x)] = \int_a^b f(x)k(s,x)dx$$

Where *k(s,x)* is the kernel of the integral transform and s is the parameter. If *k(s,x)* = e^{-sx}, the integral transform leads to Laplace transform of f(x).

$$L[f(x)] = \int_a^\infty f(x)e^{-sx}dx$$

When *k(s,x)* = e^{isx}, then the Integral transform become complex form of Fourier transform.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

If we replace k(s,x) by sine and cosine functions, we get Fourier Sine and Cosine Transform.

Fourier Integral Theorem

A function f(x) which is piece-wise continuous in every finite interval in (-∞,∞) and is absolutely integrable in (-∞,∞) can be expressed as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda \qquad \ldots(1)$$

This integral is known as Fourier integral of the function f(x).

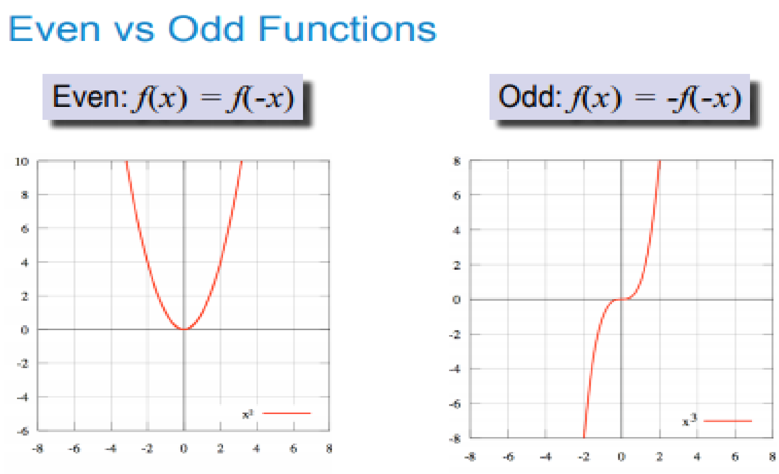
Definition of odd and even function

Odd Function

A function *f*(*x*) is said to odd if *f*(-*x*)= -*f*(*x*). Ex: *f*(*x*)=*x*, sin *x*, tan *x*, x^3.

Even Function

A function $f(x)$ is said to even if $f(-x)= f(x)$. Ex: $f(x)=x^2, x^4, \cos x, \sec x$.



Fourier Sine and cosine Integrals

The Fourier integral of f(x) is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda \qquad \ldots(2)$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda + \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\sin\lambda t\sin\lambda x\,dt\,d\lambda$$

Case (i)

If $f(t)$ is an odd function, then $f(t)\cos\lambda t$ is also an odd function and hence the first integral in equation (2) becomes zero.

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin\lambda t\sin\lambda x\,dt\,d\lambda$$

This is known as Fourier sine integral.

Case (ii)

If $f(t)$ is an even function, then $f(t)\sin\lambda t$ is odd function and hence the second integral in equation (2) becomes zero.

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \lambda t \cos \lambda x dt d\lambda$$

This is known as Fourier cosine integral.

Let us look at the definition of Fourier transform and some basic properties of it without getting into mathematical rigor.

Fourier Transforms

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty,\infty)$ $f : R \to C$ and be piece-wise continuous in each finite partial interval then the complex Fourier transform of $f(x)$ is defined by

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Inverse Fourier Transform

Inverse complex Fourier transform of F(s) is given by

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Properties of Fourier Transforms

1. Linearity property

If $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, then

$F[af(x) + bg(x)] \quad = aF[f(x)] + bF[g(x)]$

$= aF(s) + bG(s)$, where a and b are constants.

Proof: Given $F(s) = F[f(x)]$, $G(s) = F[g(x)]$

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{isx} dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= aF[f(x)] + bF[g(x)] = aF(s) + bG(s)$$

2. Shifting property

If $F[f(x)] = F(s)$ then $F(f(x-a) = e^{ias} F[f(x)] = e^{ias} F(s)$

Proof: Given $F[f(x)] = F(s)$

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

Put $u = x - a \therefore du = dx$. When $x = -\infty$, $u = -\infty$ and when $x = \infty$, $u = \infty$

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{is(u+a)}du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{isu}.e^{isa}du$$

$$= e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{isu}du = e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Changing the dummy variable from u to x

$\Rightarrow F[f(x-a) \quad = e^{isa} F[f(x)] = e^{isa}F(s)$

3. Changeof scale property

If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{|a|}F\left(\frac{s}{a}\right)$ where $a \neq 0$

Proof: Given $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax).e^{isx}dx$

$$Consider \quad u = ax$$

$$\therefore du = a\,dx \Rightarrow dx = \frac{du}{a}$$

Case (i) If a>0, then when $x = -\infty$, $u = -\infty$ and $x = \infty$, $u = \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{\frac{isu}{a}}\frac{du}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{i\left(\frac{s}{a}\right)u} du = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad ...(i)$$

Case (ii) If a<0, then when $x = -\infty$, $u = \infty$ and $x = \infty$, $u = -\infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(u).e^{\frac{isu}{a}} \frac{du}{a}$$

$$= -\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{i\left(\frac{s}{a}\right)u} du = -\frac{1}{a} F\left(\frac{s}{a}\right) \qquad ...(ii)$$

From (i) and (ii), we get

$$F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right). \; if \; a \neq 0$$

Note: Put $a = -1$, then $F[f(-x) = F(-s)$

It can be seen that, if $f(x)$ is even, then $F(s)$ is even and if $f(x)$ is odd, then $F(s)$ is odd.

4. Shifting in s

If $F[f(x)] = F(s)$ then $F(e^{iax} f(x)) = F(s+a)$

Proof: Given $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx} dx$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} e^{iax} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx = F(s+a)$$

5. Modulation Property

If $F(f(x)) = F(s)$ then $F[\cos ax f(x)] = \frac{1}{2}[F(s+a) + F(s-a)]$

Proof Given $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\therefore F[\cos axf(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos axf(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{e^{iax}+e^{-iax}}{2}f(x)e^{isx}\right]dx$$

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}e^{iax}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}e^{-iax}dx\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right\}$$

$$F[\cos axf(x)] = \frac{1}{2}\{F(s+a)+F(s-a)\}$$

6. Fourier transform of Derivative

If $F[f(x)] = F(s)$ and derivative $f'(x)$ is continuous, absolutely integrable on $(-\infty, \infty)$, then $F[f'(x)] = -(is)F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

Proof Given $F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(e^{isx}f(x)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(is)e^{isx}f(x)dx\right\}$$

Applying integration by parts, taking $u = e^{isx}$, $dv = f'(x)dx$

$$\therefore \quad du = sx\, e^{isx}\, dx,\ v = f(x)$$

We have $\left|e^{isx}\right| = \left|\cos sx + i\sin sx\right| = 1$

Since $f(x) \to 0$ as $x \to \pm\infty$, we have $e^{isx}f(x) \to 0$ as $x \to \pm\infty$

$$\therefore \quad F[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[0 - is\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]$$

$$= (-is)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = -isF[f(x)] = -isF(s)$$

Note: Similarly, we can prove that

$F(f''(x)] \quad = -isF[f'(x)]$

$= -is(-is)\ F[f(x)] = (-is)^2F[f(x)]$

Generally, for any positive integer n, $F[f^{(n)}(x)] = (-is)^n\ F(f(x))$

if $f(x), f'(x)\ldots.f^{n-1}(x)$ approaches 0 as $x \to \pm\infty$.

7. Derivative of transform

If $F[f(x)] = F(s)$, then $F(x^n\ f(x)) = (-i)^n \frac{d^nF(s)}{ds^n}$

Proof Given $F[f(x)] = F(s) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

Differentiating w.r to s we get,

$$\frac{dF(s)}{ds} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial s}(e^{isx})dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}ixdx$$

$$\frac{dF(s)}{ds} = i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{isx}dx \qquad \ldots(1)$$

We again differentiating (1) w.r. to s, we get

$$\frac{d^2F(s)}{ds^2} = i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{isx}ixdx$$

$$= i^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 f(x)e^{isx}dx = i^2F[x^2 f(x)]$$

$$F[x^2 f(x)] = (-i)^2\frac{d^2F(s)}{ds^2}$$

Continuing this way, $F[x^n f(x)] = (-i)^n\frac{d^nF(s)}{ds^n}$

8. Fourier transform of an integral function

If *f(x)* is an integral function with $F(f(x)) = F(s)$, then $F\left[\int_a^x f(x)dx\right] = \frac{i}{s}F(s)$

Proof Given $F[f(x)] = F(s)$ and *f(x)* is integrable.

Let $\int_a^x f(x)dx = g(x)$, then $f(x) = g'(x)$ by fundamental theorem of integral calculus.

$$F[f(x)] = F[g'(x)]$$

$$= -is\,F[g(x)] = -is\,F\left[\int_a^x f(x)dx\right] \qquad \text{[by property 6]}$$

$$F\left[\int_a^x f(x)dx\right] = \frac{1}{-is}F[f(x)] = \frac{i}{s}F[f(x)]$$

9. If $F[f(x)] = F(s)$, then $F[\overline{f(x)}] = \overline{F(-s)}$ where bar denotes complex conjugate.

Proof Given $F[f(x)] = F(s) \quad \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$\therefore F(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx \qquad [\because \overline{\overline{z}} = z]$$

$$\therefore \overline{F(-s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)e^{-isx}}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}\,e^{isx}dx = F[\overline{f(x)}]$$

$$F[\overline{f(x)}] = \overline{F(-s)}$$

Note: $F[\overline{f(-x)}] = \overline{F(s)}$

Definition: Convolution of two functions.

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

PROBLEMS

Problem 1. Find the Fourier transform of $f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$. Hence evaluate $\int\limits_0^\infty \frac{\sin s}{s}ds$.

Solution: Fourier transform of $f(x)$ is $F(f(x))=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-a}^{a} e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-a}^{a} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-a}^{a} \cos sx\, dx \qquad (\therefore \sin sx \text{ is an odd } fn.)$$

$$= \frac{2}{\sqrt{2\pi}}\int\limits_{0}^{a} \cos sx\, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin sx}{s}\right]_0^a$$

$$F(s)=\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin as}{s}\right]$$

By inverse Fourier transforms,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\sin as}{s}\cos sx\, ds \qquad \left[\because \frac{\sin as}{s}\sin sx \; is\ odd\right]$$

$$\text{f(x)} = \frac{2}{\pi}\int\limits_{0}^{\infty} \left(\frac{\sin as}{s}\right)\cos sx\, ds$$

Put $a = 1$, $x = 0$

$$f(0) = \frac{2}{\pi} \int_0^\infty \frac{\sin s}{s} \, ds$$

$$\frac{\pi}{2} \times 1 = \int_0^\infty \frac{\sin s}{s} ds \; (\because \; f(x) = 1, -a \le x \le a)$$

$$\therefore \int_0^\infty \frac{\sin s}{s} ds = \frac{\pi}{2}$$

Problem 2: Find the Fourier transform of $f(x) = \begin{cases} e^{ikx,} & a < x < b; \\ 0, & x < a \; and \; x > b \end{cases}$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b$$

$$F[f(x)] = \frac{-i}{(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

Definition: If the fourier transform of f(x) is equal to f(s) then the function f(x) is called **self-reciprocal**. i.e. F(f(x)) = f(s)

Problem3: Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2)+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} dx \qquad \ldots (1)$$

Consider $a^2x^2 - isx = (ax)^2 - 2(ax)\frac{(is)}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad \ldots (2)$$

Substitute (2) in (1), we get

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \qquad \text{Let } t = ax - \frac{is}{2a}, dt = adx$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{\frac{s^2}{4a^2}} \qquad \ldots (3)$$

Put $a = \frac{1}{\sqrt{2}}$ in (3)

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$\therefore e^{-s^2/2}$ is self-reciprocal with respect to Fourier Transform.

Problem 4: State and Prove convolution theorem on Fourier transform.

Solution:

Statement: If *F*(*s*) and *G*(*s*) are Fourier transform of *f*(*x*) and *g*(*x*) respectively, Then the Fourier transform of the convolutions of *f*(*x*) and *g*(*x*) is the product of their Fourier transforms.

i.e. $F[f(x)*g(x)] = F[f(x)]F[g(x)] = F(s)G(s)$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f*g)(x)e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt\; e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}dxdt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right)dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)F(g(x-t)dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, F(g(t))dt \qquad [\therefore f(g(x-t) = e^{ist}F(g(t))]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dtG(s) \qquad [\therefore F(g(t)) = G(s)]$$

$$F(f*g) = F(s).G(s). \qquad [\therefore F(f(t)) = F(s)].$$

Problem5: State and Prove Parseval's Identity in Fourier Transform.

Solution:

Statement: If *F*(*s*) is the Fourier transform of *f*(*x*), then $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{\infty}^{\infty} |F(s)|^2$

Proof by convolution theorem $F[f(x)*g(x)] = F(s)G(s)$

$f(x) * g(x) = F^{-1}[F(s)G(s)]$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)e^{isx}ds$$

Put $x = 0$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(-t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)e^{0}ds ---(1)$$

Let $g(-t) = \overline{f(t)}$, then it follows that $G(s) = \overline{F(s)}$ (by property 9)

$\therefore$ (1) becomes

$$\int_{-\infty}^{\infty}\left[f(t)\overline{f(t)}\right]dt = \int_{-\infty}^{\infty}\left[F(s)\overline{F(s)}\right]ds \quad \because z\bar{z} = |z|^2$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$

Problem 6: Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ and hence evaluate

(i) $\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt$ **(ii)** $\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$

Solutions:

Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a}(a^2 - x^2)e^{isx}dx + 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2 - x^2)(\cos sx + i \sin x)\, dx\right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2)\cos sx\, dx \left[\because (a^2 - x^2)\sin sx \textit{ is an odd fn.}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (2)\left(\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-2as\cos as + 2\sin as}{s^3}\right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \qquad \ldots (1)$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\, dx \qquad \text{(the second term is on odd function)}$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\, dx$$

Put $a = 1$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\cos sx\, dx \qquad \left[f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases}\right]$$

Put $x = 0$

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, dx \qquad \left[\begin{aligned} f(0) &= 1 - 0 \\ &= 1 \end{aligned}\right]$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$$

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4} \quad \text{[by changing s } \rightarrow \text{t]}$$

Using Parseval's identify

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-\infty}^{\infty} |a^2 - x^2|^2\,dx$$

$$\int_{-\infty}^{\infty} \frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1} (1-x^2)^2\,dx \,(\text{put a = 1})$$

$$2\times\frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2\,dx$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{16}\times 2\left(\frac{8}{15}\right) = \frac{\pi}{15}$$

Put s = t, $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

Problem 7: Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ and hence find the value of **(i)** $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$. **(ii)** $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$.

Solution:

The Fourier transform of $F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\,(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx \; [\therefore (1-|x|) \sin \; sx \; is \; an \; odd \; fn.]$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ (1-x)\left(\frac{\sin \; sx}{s}\right) - (-1)\left(\frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ -\frac{\cos s}{s^2} + \frac{1}{s^2} \right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] \quad (1)$$

(i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] (\cos \; sx - i \sin \; sx) ds \; (by \; (1))$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx \, ds \quad \text{(Second term is odd)}$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx \, ds$$

Put $x = 0$

$$\text{f}(0) = 1-|0| = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds$$

$$\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{2\sin^2(s/2)}{s^2} ds = \frac{\pi}{2}$$

Put $t = s/2 \qquad ds = 2dt$

$$\int_{0}^{\infty} \frac{2\sin^2 t}{(2t)^2} 2dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right]^2 ds = \int_{-1}^{1} (1-|x|)^2\,dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2\,dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1} (1-x)^2\,dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^2 ds = \left[2\left(\frac{1-x}{-3}\right)^3\right]_0^1$$

$$\frac{16}{\pi}\int_{0}^{\infty} \left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds = \frac{2}{3}; Let\ t = s/2, dt = \frac{ds}{2}$$

$$\frac{16}{\pi}\int_{0}^{\infty} \left(\frac{\sin t}{2t}\right)^4 2dt = \frac{2}{3}$$

$$\frac{16}{16\pi}\int_{0}^{\infty} \left(\frac{\sin t}{2t}\right)^4 dt = \frac{1}{3}$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$

Problem 8: Find the Fourier transform of $f(x)=\begin{cases} a-|x|, & |x|<a \\ 0, & |x|\geq a \end{cases}$ and hence prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0\, dx + \int_{-a}^{-a} (a-|x|) e^{isx} dx + \int_{a}^{\infty} 0 dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{-a} (a-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) \cos sx\, dx + 0 \qquad \left[\therefore \int [a-|x|] \sin sx\, dx = 0\ odd\ \ function\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^a (a-x) \cos sx\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[(a-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cos as}{s^2} + \frac{1}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos as}{s^2} \right]$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 \frac{as}{2}}{s^2} \right] \qquad \ldots (1)$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} ds.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 \frac{as}{2}}{s^2} \right] e^{-isx} ds \text{ , Put } x = 0$$

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds$$

$$\frac{\pi a}{4} = \int_0^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} ds \quad [f(0) = a\text{-}0 = a]$$

Put $a = 2$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin^2 s}{s^2} ds \qquad [\therefore s \text{ is a dummy variable, we can replace it by ‘}t\text{’}]$$

$$\text{i.e} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Problem 9: Find the Fourier transform of $e^{-a|x|}, a > 0$ and hence deduce that

(a) $\int_0^{\infty} \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ **(b)** $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}.$

Using Parseval’s Theorem find the value of $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx, a > 0.$

Solution: $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-a|x|} (\cos sx\, dx) \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx = 0, \textit{odd function}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F(e^{-a|x|}) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + s^2} \right).$$

(a) Using Fourier inverse transform,

$$\text{f(x)} = e^{-a|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] (\cos sx - i \sin sx) ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx}{a^2 + s^2} ds + 0 \qquad \left[\therefore \frac{\sin sx}{s^2 + a^2} \, is \; an \; odd \;\; fn. \right]$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace ‘}s\text{’ by ‘}t\text{’)}$$

$$= \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

(b) To prove $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$

Property:

$$F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$$

$$F[xf(x)] = (-i) \frac{dF(s)}{ds}$$

$$F[e^{-a|x|}] = (-i) \frac{dF(e^{-a|x|}\;)}{ds}$$

$$= -i \frac{d}{ds} \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right)$$

$$= ia\sqrt{\frac{2}{\pi}}\left(\frac{2s}{(a^2+s^2)^2}\right) = i\sqrt{\frac{2}{\pi}}\,\frac{2as}{(a^2+s^2)^2}$$

Parseval's identity is $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$

Result: $F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$

$$\int_{-\infty}^{\infty}\left[e^{-|ax|}\right]^2 dx = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}\right)^2 ds$$

$$2\int_0^{\infty}(e^{-ax})^2\,dx = 2\int_0^{\infty}\left(\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^{\infty} = \frac{2}{\pi}\int_0^{\infty}\frac{s^2}{(s^2+a^2)^2}\,ds$$

i.e., $\int_0^{\infty}\frac{s^2}{(s^2+a^2)^2}\,ds = \frac{\pi}{2}\left(\frac{0+1}{2a}\right) = \frac{\pi}{4a}.$

$$\therefore \int_0^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}.$$

Problem 10: Derive the relation between Fourier transform and Laplace transform.

Solution:

Consider $f(t) = \begin{cases} e^{-xt}g(t), & t \ge 0 \\ 0, & t < 0 \end{cases}$ … (1)

The Fourier transformer of $f(x)$ is given by

$$F[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-xt}g(t)e^{ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(is-x)t} g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-pt} g(t)\, dt \text{ where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L(g(t)) \quad \left[\because L\left[f(t)\right] = \int_0^\infty e^{-st} f(t) dt\right]$$

$\therefore$ Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \times$ Laplace transform of $g(t)$ where $g(t)$ is defined by (1).

Fourier Sine and Cosine Transform

Fourier sine and cosine transform are related to Fourier sine and cosine integrals. The Fourier transform applies to the problems concerning the real axis or the interval $(-\infty,\infty)$ whereas sine and cosine transform apply to the problem concerning the interval $(0,\infty)$.

The Fourier Sine Integral of f(x) in $(0,\infty)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx\, dt\, ds \text{ [Replace } \lambda \text{ by s]}$$

$$= \frac{2}{\pi} \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st\, dt \right] \sin sx\, ds$$

We denote $F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

This is known as **Infinite Fourier Sine Transform** of f(x).

Inverse Fourier Sine Transform is

$$f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds$$

The Fourier cosine Integral of f(x) in $(0,\infty)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx\, dt\, ds \text{ [Replace } \lambda \text{ by s]}$$

$$= \frac{2}{\pi}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt\right]\cos sx\,ds$$

We denote $F_C[f(x)] = F_C(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

This is known as **Infinite Fourier Cosine Transform** of f(x).

The **Inverse Fourier Cosine Transform** is

$$f(x) = F^{-1}[F_C(s)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(s)\cos sx\,ds$$

Properties of Fourier Sine and Cosine Transforms

1. Linearity Property

(i) $F_c[af(x) + bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

(ii) $F_s[af(x) + bg(x)] = aF_s[f(x)] + bF_s[g(x)]$ where a and b are constants.

Proof: (i) By definition, $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = F_c(s)$

$$F_c[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty (af(x)+bg(x))\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty (a\,f(x))\cos sx\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty (bg(x))\cos sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty (f(x))\cos sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty (g(x))\cos sx\,dx$$

$$= aF_c[f(x)] + bF_c[g(x)]$$

(ii) By definition, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = F_s(s)$

$$F_s[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty (af(x)+bg(x))\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty (a\ f(x))\sin sxdx + \sqrt{\frac{2}{\pi}}\int_0^\infty (bg(x))\sin sxdx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty (f(x))\sin sxdx + b\sqrt{\frac{2}{\pi}}\int_0^\infty (g(x))\sin sxdx$$

$$= aF_s[f(x)] + bF_s[g(x)]$$

2. Modulation property

If $F_c[f(x)] = F_c[s]$ and $F_s[f(x)] = F_s[s]$, then

(i) $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii) $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iii) $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(iv) $F_s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(i) To Prove: $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Proof: We have, $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sxdx$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \cos sxdx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\{\cos(s+a)x + \cos(s-a)x\}dx$$

$$= \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s+a)x + f(x)\cos(s-a)x]dx$$

$$= \frac{1}{2}.\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s+a)xdx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)xdx\right\}$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(ii) **To prove:** $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

Proof: We have, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$\therefore F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\{\sin(s+a)x + \sin(s-a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\sin(s+a)x + f(x)\sin(s-a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)x\,dx\right\}$$

$$= \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

(iii) **To prove:** $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

Proof We have, $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$\therefore F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty f(x)\{\sin(s+a)x - \sin(s-a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\sin(s+a)x - f(x)\sin(s-a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)x\,dx\right\}$$

$$= \frac{1}{2}[F_S(s+a) - F_S(s-a)]$$

(iv) **To prove:** $F_s[f(x)\sin ax] = \frac{1}{2}[F_C(s-a) - F_C(s+a)]$

Proof We have, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$\therefore F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^\infty f(x)\{\cos(s-a)x - \cos(s+a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s-a)x - f(x)\cos(s+a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right\}$$

$$= \frac{1}{2}[F_C(s-a) - F_C(s+a)]$$

3. Change of Scale Property

(i) $F_C[f(ax)] = \frac{1}{a}F_C\left(\frac{s}{a}\right)$ if $a > 0$ (ii) $F_S[f(ax)] = \frac{1}{a}F_S\left(\frac{s}{a}\right)$ if $a > 0$

(i) To prove: $F_C[f(ax)] = \frac{1}{a}F_C\left(\frac{s}{a}\right)$ **if** $a > 0$

Proof We have $F_C[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx\,dx$

Put $t = ax$. $\therefore dt = a\,dx \Rightarrow dx = \frac{dt}{a}$

when $x = 0$, $t = 0$ and when $x = \infty$, $t = \infty$

$$\therefore F_C[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\left(\frac{st}{a}\right) \frac{dt}{a} = \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\left(\frac{s}{a}\right) t dt = \frac{1}{a} F_c\left(\frac{s}{a}\right) \quad [\because a > 0]$$

(i) To prove: $F_S[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad if \ a > 0$ **if**a> 0

Proof We have $F_S(f(ax)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \sin sx dx$

Put $t = ax$ $\therefore dt = adx \Rightarrow dx = \frac{dt}{a}$

when $x = 0$, $t = 0$ and when $x = \infty$, $t = \infty$

$$\therefore F_S[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a} = \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{s}{a}\right) t dt = \frac{1}{a} F_S\left(\frac{s}{a}\right)$$

4. Differentiation of sine and cosine transform

(i) $F_C[xf(x)] = \frac{d}{ds}[F_S(s)] = \frac{d}{ds}[F_S(f(x))]$

(ii) $F_S[xf(x)] = -\frac{d}{ds}[F_C(s)] = -\frac{d}{ds}[F_C(f(x))]$

(i) To prove: $F_C[xf(x)] = \frac{d}{ds}[F_S(s)]$

Proof: We know $F_S(s) = F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx$

Differentiating w.r to s, we get $\frac{d}{ds}[F_S(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s}(\sin sx) dx$

$$\frac{d}{ds}[F_S(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x).(x \cos sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx dx = F_C[xf(x)]$$

$$F_C[xf(x)] = \frac{d}{ds}[F_S(s)]$$

(ii) **To prove:** $F_S[xf(x)] = -\frac{d}{ds}[F_C(s)]$

Proof: We know $F_C(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx dx$

Differentiating w.r to s, we get $\frac{d}{ds}[F_C(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s}(\cos sx) dx$

$$\frac{d}{ds}[F_C(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x).(-\sin sx) x\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin sx dx = -F_s[xf(x)]$$

$$F_s[xf(x)] = -\frac{d}{ds}[F_C(s)]$$

5. Cosine and sine transforms of derivative

If *f(x)* is continuous and absolutely integrable in (-∞, ∞) and if *f(x)* → 0 as *x* → ∞, then

(i) $F_C[f'(x)] = sF_S[f(x)] - \sqrt{\frac{2}{\pi}} f(0)$

(ii) $F_S[f'(x)] = -sF_C[f(x)]$

Proof (i) by definition of Fourier cosine transform

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx dx \quad \therefore F_C[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\cos sx dx$$

Applying integration by parts, taking $u = \cos sx$, $dv = f'(x)dx$

$\therefore$ $du = -\sin sx.\, s\, dx.$ $v = f(x)$

We get,

$$\therefore F_C[f'(x)] = \sqrt{\frac{2}{\pi}}\left\{[\cos sx.f(x)]_0^\infty - \int_0^\infty f(x)(-s\sin sx)dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\{[0-\cos 0.f(0)]\} + s\int_0^\infty f(x)\sin sx dx \quad [\because f(x)\to 0, as\ x\to\infty]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + s\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

$$F_C[f'(x)] = sF_S[f(x)] - \sqrt{\frac{2}{\pi}}f(0)$$

(ii) $$F_S[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin sx dx$$

Applying integration by parts, taking $u = \sin sx$, $dv = f'(x)dx$, we get

$$F_S[f'(x)] = \sqrt{\frac{2}{\pi}}\left\{[\sin sx.f(x)]_0^\infty - \int_0^\infty f(x)s\cos sx dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{0 - s\int_0^\infty f(x)\cos sx dx\right\} \qquad [\text{as } x\to\infty,\ f(x)\to 0]$$

$$= -s\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$F_S[f'(x)] = -sF_C[f(x)]$$

Note:

1. $$F_C[f''(x)] = sF_s[f'(x)] - \sqrt{\frac{2}{\pi}}f'(0) = s(-sF_C[(x)]) - \sqrt{\frac{2}{\pi}}f'(0)$$

2. $$F_C[f''(x)] = -s^2F_C[f(x)] - \sqrt{\frac{2}{\pi}}f'(0)$$

3. $$F_S[f''(x)] = -sF_C[f'(x)]$$

$$= -s\left\{sF_S[f(x)] - \sqrt{\frac{2}{\pi}}f(0)\right\} = -s^2F_S[f(x)] + s\sqrt{\frac{2}{\pi}}f(0)$$

These formulae are useful in solving differential equations.

6. Identities

If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively then

i) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds$$

ii) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)ds$$

iii) $$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds$$

Problem 1: Find the Fourier Sine Transform of e^{-3x}**.**

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s(e^{-3x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-3x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-3x}}{s^2+9}(-3\sin sx - s\cos sx)\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+9}\right)\quad\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]\right].$$

Problem 2: Find the Fourier cosine transform of $f(x) = \begin{cases}\cos x, & 0 < x < a\\ 0, & x \geq a\end{cases}$

Solution:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right], \text{ provided } s \neq 1, s \neq -1.$$

Problem 3: Find the Fourier cosine transform of $e^{-2x} + 3e^{-x}$.

Solution:

Let $f(x) = e^{-2x} + 3e^{-x}$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

$$F_c[e^{-2x} + 3e^{-x}] = \sqrt{\frac{2}{\pi}} \left\{\int_0^\infty e^{-2x} \cos sx\, dx + \int_0^\infty 3e^{-x} \cos sx\, dx\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{s^2+4} + \frac{3}{s^2+1}\right].$$

Problem 4: Find the Fourier cosine transform of f(x) defined as

$$f(x) = \begin{cases} x & for \quad 0 < x < 1 \\ 2 - x & for \quad 1 < x < 2 \\ 0 & for \quad x > 2 \end{cases}$$

Solution: By definition of Fourier Cosine Transform

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty f(x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx\, dx + \int_1^2 (2-x)\cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(x\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right)_0^1+\left((2-x)\frac{\sin sx}{s}+\frac{\cos sx}{s^2}\right)_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin s}{s}-\frac{\cos s-\cos 0}{s^2}\right)+\left(0-(1)\frac{\sin s}{s}+\frac{\cos 2s-\cos s}{s^2}\right)\right]$$

$$F_c(s)= \sqrt{\frac{2}{\pi}}\left[\frac{\cos 2s-2\cos s+1}{s^2}\right]$$

Problem 5: Find the Fourier sine transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

Let $sx=\theta$, $sdx=d\theta$;

X	0	∞
θ=sx	0	∞

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{\theta}\sin\theta\frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta}d\theta\left[\because\int_0^\infty \frac{\sin\theta}{\theta}d\theta=\frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}}.$$

Problem 6: Find the Fourier cosine and sine transformation of $f(x)=e^{-ax}$, $a>0$. Hence deduce that $\int_0^\infty \frac{x\sin\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}$. and $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt=\frac{\pi}{2a}e^{-a|x|}$

Solution:

The Fourier cosine transform is $F_C[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$\Rightarrow F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a+0) \right] = \sqrt{\frac{2}{\pi}} . \frac{a}{a^2+s^2}$$

The Fourier sine transform is $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (0-s) \right] = \sqrt{\frac{2}{\pi}} . \frac{s}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right)$$

By inverse Sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} ds$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

Put $a = 1$, $x = \alpha$

$$\frac{\pi}{2} e^{-a} = \int_0^\infty \frac{s \sin sx}{s^2 + 1} ds$$

Replace 's' by 'x' and 'x' by 's'

$$\int_0^\infty \frac{x \sin sx}{1 + x^2} dx = \frac{\pi}{2} e^{-\alpha}.$$

Using Fourier inverse cosine transform,

f(x) = $e^{-a|x|} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx ds$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \cos sx\ ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2 + s^2} ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace 's' by 't')}$$

$$\int_0^\infty \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

Problem 7: Find Fourier cosine transform of (i) $e^{-ax} \sin ax$ (ii) $e^{-ax} \cos ax$

Solution: (i) $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\ dx$

$$F_c[e^{-ax} \sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin ax \cos sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^\infty e^{-ax} [\sin(s + a)x - \sin(s - a)x]\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s-a)^2} \right\} \qquad \left[\therefore \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{(a^2+(s-a)^2)(s+a)-(s-a)(a^2+(s+a)^2}{(a^2+(s+a)^2)(a^2+(s-a)^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2a^2s+s^3-2as^2+2a^3+as^2-2a^2s-2s^2-s^3-2as^2+2a^3+s^2a+2sa^2}{4a^4+2a^2s^3-4a^3s+2a^2s^2+s^4-2as^3+4a^3s+2as^2-4a^2s^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2a^3-as^2}{4a^4+s^4} \right\} = \sqrt{\frac{2}{\pi}} \left(\frac{a(2a^2-s^2}{4a^4+s^4} \right)$$

(ii) $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right)$$

By Modulation Theorem.

$$F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a+s)+F_c(a-s)]$$

$$F_c[e^{-ax} \cos ax] = \frac{1}{2} \left[\sqrt{\frac{2}{\pi}} \left\{ \frac{a}{a^2+(a+s)^2} + \frac{a}{a^2+(a-s)^2} \right\} \right]$$

$$= \frac{1}{2} \times \sqrt{\frac{2}{\pi}} \times a \left\{ \frac{a^2+(a-s)^2+a^2+(a+s)^2}{[a^2+(a+s)^2](a^2+(a-s)^2]} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{4a^2+2s^2}{s^4+4a^4} \right]$$

$$F_c[e^{-ax} \cos ax] = \frac{2a}{\sqrt{2\pi}} \left[\frac{2a^2+s^2}{s^4+4a^4} \right]$$

Problem 8: Find $F_c(xe^{-ax})$ and $F_s(xe^{-ax})$

Solution:

$$F_c(xe^{-ax}) = \frac{d}{ds} F_s[f(x)]$$

$$F_c(xe^{-ax}) = \frac{d}{ds} F_s[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$F_c(xe^{-ax}) = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$F_s[xe^{-ax}] = -\frac{d}{ds}[F_c e^{-ax}] \quad \left(\because F_s(xf(x)) = -\frac{d}{ds}\big(F_c(f(x))\big)\right)$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

$$F_s[xe^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

Problem9. Find (i) $Fs\left(\frac{e^{-ax}}{x}\right)$ and (ii) $F_c\left(\frac{e^{-ax}}{x}\right)$

(i) To find $F_s\left[\frac{e^{-ax}}{x}\right]$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx \qquad \ldots (1)$$

Diff. on both sides w. r. to 's' we get

$$\frac{d}{ds}(F_s(s)) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx \quad \left[\therefore \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{b^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left\{\frac{e^{-ax}}{x}\sin sx\right\}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{xe^{-ax}\cos sx}{x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating w. r. to 's' we get

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int \frac{a}{s^2+a^2}\,ds + c$$

$$= \sqrt{\frac{2}{\pi}}a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$$

But $F_s(s) = 0$ When $s = 0$ $\therefore c = 0$ from (1)

$$\therefore F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

(ii) To find $F_c\left[\frac{e^{-ax}}{x}\right]$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx \qquad \dots (1)$$

Diff. on both sides w. r. to 's' we get

$$\frac{d}{ds}(F_c(s)) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx \quad \left[\therefore \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{b^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s} \left\{ \frac{e^{-ax}}{x} \cos sx \right\} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-xe^{-ax} \sin sx}{x} dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + a^2} \right)$$

Integrating w. r. to ‘s’ we get

$$F_c(s) = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2 + a^2} ds$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log \left(s^2 + a^2 \right)$$

$$\therefore F_c \left(\frac{e^{-ax}}{x} \right) = -\frac{1}{\sqrt{2\pi}} \log \left(s^2 + a^2 \right)$$

Problem 10: Find (i) $Fs\left(\frac{e^{-ax} - e^{-bx}}{x} \right)$ and (ii) $F_c\left(\frac{e^{-ax} - e^{-bx}}{x} \right)$

Solution: (i) $Fs\left(\frac{e^{-ax} - e^{-bx}}{x} \right) = Fs\left(\frac{e^{-ax}}{x} \right) - Fs\left(\frac{e^{-bx}}{x} \right)$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a} \right) - \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{b} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left[\tan^{-1}\left(\frac{s}{a} \right) - \tan^{-1}\left(\frac{s}{b} \right) \right] \quad \ldots\text{(ii)}$$

$$Fc\left(\frac{e^{-ax} - e^{-bx}}{x} \right) = Fc\left(\frac{e^{-ax}}{x} \right) - Fc\left(\frac{e^{-bx}}{x} \right)$$

$$= -\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)+\frac{1}{\sqrt{2\pi}}\log\left(s^2+b^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Problem11: Using Parseval's Identity calculate

(a) $\int_0^\infty \frac{1}{(a^2+x^2)^2}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$

Solution: (a) By Parseval's identity.

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds$$

$$\int_0^\infty e^{-2ax}dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}a^2\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{a^2+s^2}$$

i.e. $\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$ [Replace, *s* by *x*]

(b) By Parseval's identity.

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(f(x))|^2\, ds$$

$$\int_0^\infty (e^{-ax})^2dx = \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)^2 ds$$

i.e $\int_0^\infty \frac{s^2}{(a^2+s^2)^2}ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\pi}{2}\times\frac{1}{2a}$

$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx = \frac{\pi}{4a}$ [Replace, *s* by *x*]

Problem 12. Evaluate **(a)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$, using Fourier cosine and sine transform.

Solution: **(a)** Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

$$F_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{s^2+1}(-\cos x + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right] \qquad \ldots (1)$$

$$F_c(e^{-2x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2}{s^2+4}\right) \qquad \ldots (2)$$

$$\therefore \int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(f(x))F_c(g(x))ds$$

$$\int_0^\infty e^{-x}e^{-2x}dx = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{s^2+1}.\frac{2}{s^2+4}\right)ds \text{ (from (1) \& (2))}$$

$$\int_0^\infty e^{-3x}dx = \frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)}ds$$

$$\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{\pi}{4}\left[\frac{e^{-3x}}{-3}\right]_0^\infty = \frac{\pi}{4}\left(\frac{1}{3}\right)$$

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12} \text{ [Replace s to x]}$$

(b) To find $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx.$

Let

$f(x)=e^{-ax}, g(x)=e^{-bx}$

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \quad \ldots (1)$$

$$F_s(g(x))=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx\,dx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right) \quad \ldots (2)$$

$\int_0^\infty f(x)g(x)dx=\int_0^\infty F_s[f(x)].F_s[g(x)]ds$ From (1) and (2)

$$\int_0^\infty e^{-ax}e^{-bx}dx=\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds=\frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

i.e $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx=\frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty=\frac{\pi}{2(a+b)}$ [Replace s to x]

Problem 13: Find the Fourier sine and cosine transform of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under sine and cosine transform. Also find $F\left(\frac{1}{\sqrt{|x|}}\right)$.

Solution: We know that gamma function is given by $\Gamma(n)=\int_0^\infty e^{-y}y^{n-1}dy$ ….(1)

Put $y = ax$, we get $\int_0^\infty e^{-ax}(ax)^{n-1}a\,dx=\Gamma(n)$

$$\int_0^\infty e^{-ax}x^{n-1}dx=\frac{\Gamma(n)}{a^n}$$

Put a = is

$$\therefore \int_0^\infty e^{-isx}x^{n-1}dx=\frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i \sin sx)x^{n-1}dx = \frac{\Gamma(n)}{s^n} i^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-n}$$

$$\int_0^\infty (\cos sx - i \sin sx)x^{n-1}dx = \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1}\cos sx dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad \dots (2)$$

$$\int_0^\infty x^{n-1}\sin sx dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \dots (3)$$

X^{ly} $\sqrt{\frac{2}{\pi}}$ on both sides of equation (2) and (3)

$$F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\, dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad \dots (3)$$

$$F_s(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \dots (4)$$

Put $n = \frac{1}{2}$ in (3) and (4)

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\sin\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\,\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier sine and cosine transform.

To find $F\left\{\frac{1}{\sqrt{1x1}}\right\}$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e\,\frac{isx}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{x}}(\cos sx + i\sin sx)\,dx$$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{x}}\cos sx\,dx \qquad \text{....(5)}[\therefore \text{The second term odd}]$$

Put n = 1/2 in (2), we get

$$\int_{0}^{\infty}\frac{\cos sx}{\sqrt{x}}\,dx = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2s}} \qquad \text{...(6)}$$

Substitute (6) in (5)

$$\therefore F\left\{\frac{1}{\sqrt{|x|}}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{2s}} = \frac{1}{\sqrt{s}}$$

Problem 14: Find $f(x)$ if its sine transform is e^{-as}, $a > 0$.

Solution:

$$F_s(f(x)) = F(s)$$

Given that $F_s(f(x)) = e^{-as}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin\ x\ ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \sin\ sx\ ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-as}}{a^2 + s^2} (-a \sin sx - x \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{x}{a^2 + x^2} \right).$$

Problem 15: Find $f(x)$ if its Fourier sine Transform is $\frac{e^{-as}}{s}$.

Solution:

Let $F_s(f(x)) = \frac{e^{-as}}{s}$

Then $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{s} \sin\ sx\ ds$ … (1)

$$\therefore \frac{df}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial x} \left(\frac{e^{-as}}{s} \sin\ sx \right) ds =$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{s} s \cos sx\ ds = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx\ ds = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2}$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{dx}{a^2 + x^2}$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) + c \qquad \ldots (2)$$

At $x = 0$, $f(0) = 0$ using (1)

$$(2) \Rightarrow f(0) = \sqrt{\frac{2}{\pi}} \tan^{-1}(0) + c \quad \therefore c = 0$$

Hence $f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$

Problem 16. Find the Fourier Cosine Transform of e^{-x^2} and hence Show that $xe^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier sine transform.

Solution:

The Fourier Cosine Transform of f(x) is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} . \frac{1}{2} . 2 \int_0^\infty e^{-x^2} \cos sx dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^\infty \frac{e^{-x^2+isx}}{e^{\frac{-s^2}{4}}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^\infty e^{-x^2+isx+\frac{s^2}{4}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^\infty e^{-\left(x-\frac{is}{2}\right)^2} dx$$

Put $x-\frac{is}{2}=y; \quad dx=dy$

When $x=-\infty, \quad y=-\infty$

$x=\infty, \quad y=\infty$

$$=\frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$=\frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} 2\int_{0}^{\infty} e^{-y^2} dy$$

$$=\frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} 2\frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$F_c\left(e^{-x^2}\right)=\frac{1}{\sqrt{2}} e^{\frac{-s^2}{4}}$$

Result : $\quad F_s\left[xe^{\frac{-x^2}{2}}\right]=-\frac{d}{ds}F_c\left[e^{\frac{-x^2}{2}}\right]$

But $\quad F_c\left[e^{\frac{-x^2}{2}}\right]=e^{\frac{-s^2}{2}}$

$$F_s\left[xe^{\frac{-x^2}{2}}\right]=-\frac{d}{ds}\left(e^{\frac{-s^2}{2}}\right)$$

$$=-e^{\frac{-s^2}{2}}.\left(\frac{-2s}{2}\right)$$

$$=se^{\frac{-s^2}{2}}$$

$\therefore \quad xe^{\frac{-s^2}{2}}$ is self reciprocal with respect to sine transform

Exercise 1.

1. Find the Fourier sine and cosine transforms for $f(x) = \begin{cases} 1, & if\ 0 < x < 1 \\ 0, & if\ x \geq 1 \end{cases}$

2. Find the Fourier cosine transform of $2e^{-5x} + 5e^{-2x}$.

3. Find the Fourier sine and cosine transform of e^{-x} and hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$ and $\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$

4. Find the Fourier cosine transform of $f(x) = \begin{cases} \cos sx & if\ 0 < x < a \\ 0 & if\ x \geq a \end{cases}$

5. Find the Fourier cosine transform of e^{-x^2}.

6. Find the Fourier sine transform of $\frac{x}{1+x^2}$.

7. Find the Fourier cosine transform of $\frac{x}{1+x^2}$.

8. If $F_S[f(x)] = \frac{e^{-as}}{s}$, find f(x) and $F_S^{-1}\left[\frac{1}{s}\right]$.

9. If $F_S[f(x)] = \begin{cases} 1, & 0 \leq s \leq 1 \\ 2, & 1 \leq s \leq 2, \\ 0, & if\ s \geq 2 \end{cases}$ find $f(x)$.

10. Show that $xe^{-x^2/2}$ is self reciprocal with respect to Fourier sine transform.

11. Find the Fourier sine transform of $e^{-|x|}$, $x \geq 0$, and hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.

12. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

13. Find the Fourier sine and cosine transform of $e^{-ax} \cos ax$, $a > 0$.

14. Find the Fourier sine transform of the function $f(x) = \begin{cases} \sin x, & 0 \leq x < a \\ 0, & x > a \end{cases}$

15. Find the Fourier cosine transform of $\frac{1}{x^2+a^2}$.

Answer

1. $F_s(s)=\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right), \quad F_C(s)\sqrt{\frac{2}{\pi}}\frac{\sin\ s}{s}.$

2. $\frac{\sqrt{2}}{\pi}\left[\frac{10}{s^2+25}+\frac{10}{s^2+4}\right]$

4. $\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right], s\neq\pm 1.$

5. $\frac{1}{\sqrt{2}}e^{\frac{s^2}{4}}$

6. $\sqrt{\frac{\pi}{2}e^{-s}}$

7. $\sqrt{\frac{\pi}{2}e^{-s}}$

8. $\sqrt{\frac{2}{\pi}}\tan^{-1}\frac{x}{a};\sqrt{\frac{\pi}{2}}$

9. $\frac{2}{\pi x}[1+\cos x-2\cos 2x]$

11. $\frac{\pi}{2}e^{-m}$

12. $\frac{\sqrt{2}}{\pi}\tan^{-1}\frac{s}{a}$

13. $\frac{1}{\sqrt{2\pi}}\left\{\frac{s+a}{a^2+(s+a)^2}+\frac{s-a}{a^2+(s-a)^2}\right\}, \frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(s+a)^2}+\frac{a}{a^2+(s-a)^2}\right]$

14. $\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)a}{s-a}-\frac{\sin(s+1)a}{s+1}\right], \quad s\neq\pm 1.$

15. $\sqrt{\frac{\pi}{2}}\frac{e^{-as}}{a}, a > 0$

Exercise 2.

1. Using the Fourier transform $e^{-|x|}$, prove that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

2. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using transform methods.

   [Hint: Consider $f(x) = e^{-x}, g(x) = e^{-2x}$]

3. If $f(x) = \begin{cases} \cos x, & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$ using parseval's Identity evaluate $\int_0^\infty \frac{-\cos^2 \frac{\pi x}{2}}{(1-x)^2} dx$

4. Using Parseval's identity, evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx.$

   $\left(\text{Hint: } f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}. \text{ Find } F_c[f(x)] \text{ and use Parseval's identity}\right)$

5. If $F_s[f(x)] = \frac{e^{-as}}{s}, a > 0$ find $f(x)$ and $F_S^{-1}\left[\frac{1}{s}\right]$

6. Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 & for\,|x| < 2 \\ 0 & for\,|x| > 2 \end{cases}$ and hence evaluate

   $\int_0^\infty \frac{\sin x}{x} dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$

7. Using Parseval's identity evaluate $\int_0^\infty \frac{x^2}{(a^2+x^2)} dx.$

8. Using Parseval's identities, prove that $\int_0^\infty \frac{\sin ax}{x(a^2+x^2)} dx = \frac{\pi}{2} . \frac{(1-e^{-a^2})}{a^2}$

9. Prove that $\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}, a>0, b>0$

10. Solve the integral equation $\int_0^{\infty} f(\theta)\cos\alpha\theta d\theta = \begin{cases} 1-\alpha, & if\ 0\le\alpha\le 1 \\ 0, & if\ \alpha>1 \end{cases}$

Answer

1. $\frac{\pi}{2}$ 3. $\frac{\pi^2}{8}$ 4. $\frac{\pi}{2}$ 5. $\sqrt{\frac{2}{\pi}}\tan^{-1}\frac{x}{a}, \sqrt{\frac{\pi}{2}}$

6. $\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ 7. $\int_0^{\infty}\frac{x^2}{(a^2+x^2)^2}dx = \frac{\pi}{2}$ 10. $\int_0^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$

Finite Fourier Transforms

If *f(x)* is a function defined in the interval $(0,l)$then **the finite Fourier sine transform** of *f(x)* in $0 < x < l$ is defined as

$F_S[f(x)] = \int_0^l f(x).\sin\frac{n\pi x}{l}dx$ where '*n*' is an integer.

The **inverse finite Fourier sine transform** of $F_s[f(x)]$ *f(x)* and given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s[f(x)]\sin\frac{n\pi x}{l}$$

The **finite Fourier cosine transform** of *f(x)* in $0 < x < l$ is defined as

$$F_C[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

where *'n'* is an integer.

The **inverse finite Fourier cosine transform** of $F_c[f(x)]$ is *f(x)* and is given by

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c[f(x)]\cos\frac{n\pi x}{l}$$

Example 1. Find the finite Fourier sine and cosine transform of $f(x) = x^2$ in $0 < x < 1$.

Solution:

The finite Fourier sine transform is

$$F_S[f(x)] = \int_0^l f(x).\sin\frac{n\pi x}{l}\,dx$$

Here $f(x) = x^2$

$\therefore$ $F_s[x^2] = \int_0^l x^2 \sin\frac{n\pi x}{l}\,dx$

$$= \left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right] - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + \left(2\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)_0^l$$

$$= \frac{-l^3}{n\pi}\cos n\pi + \frac{2l^3}{n^3\pi^3}\cos n\pi - \frac{2l^3}{n^3\pi^3}$$, $\cos n\pi = (-1)^n$, $\sin n\pi = 0$

$$= \frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}[(-1)^n - 1]$$

The finite Fourier cosine transform is

$$F_c[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Here $f(x) = x^2$

$\therefore$ $F_c[x^2] = \int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$

$$= \left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l^3}{n^2\pi^2}\cos n\pi$$, $\cos n\pi = (-1)^n$, $\sin n\pi = 0$

$$= \frac{2l^3}{n^2\pi^2}(-1)^n$$

Example 2: Find the finite Fourier sine and cosine transform of $f(x) = x$ in $(0, \pi)$

Solution:

The finite Fourier sine transform of *f(x)* in (0, π) is

$$F_S[f(x)] = \int_0^{\pi} f(x).\sin nx\, dx$$

Here *f(x)* = x in (0, π)

$\therefore$ $F_s[x]$ $= \int_0^{\pi} x \sin nx\, dx$

$$= \left[x\left(\frac{-\cos nx}{n}\right) - 1\frac{-\sin nx}{n^2} \right]_0^{\pi}$$

$$= -\frac{\pi}{n}\cos n\pi = (-)^{n+1}.\frac{\pi}{n}$$

The finite Fourier cosine transform of *f(x)* in (0, π) is

$F_C[f(x)]$ $= \int_0^{\pi} f(x).\cos nx\, dx$

$\therefore F_C[x]$ $= \int_0^{\pi} (x)\cos nx\, dx$

$$= \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{1}{n^2}[\cos n\pi - 1]$$

$$= \frac{1}{n^2}[(-1)^n - 1]$$

Example 3: Find the finite Fourier sine and cosine transforms of *f(x)* = e^{ax} in (0,l)

Solution:

We know that $F_S[f(x)] = \int_0^{l} f(x)\sin\frac{n\pi x}{l}\, dx$

Here *f(x)* = e^{ax}

$\therefore F_s[e^{ax}]$ $= \int_0^{l} e^{ax}\sin\frac{n\pi x}{l}\, dx$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(a \sin \frac{n\pi x}{l} - \frac{n\pi}{l} . \cos \frac{n\pi x}{l} \right) \right\}_0^1$$

$$= \frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(\frac{-n\pi}{l} \cos n\pi \right) + \frac{\frac{n\pi}{l}}{a^2 + \frac{n^2\pi^2}{l^2}}$$

$$= \frac{n\pi l}{a^2 l^2 + n^2\pi^2} [(-1)^{n+1} e^{al} + 1]$$

$$F_C\,[e^{ax}] = \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$$

$$= \int_0^l e^{ax} \cos \frac{n\pi x}{l} \, dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left[a \cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right] \right\}_0^l$$

$$= \frac{e^{al} . l^2}{a^2 l^2 + n^2\pi^2} (a \cos n\pi) - \frac{al^2}{a^2 l^2 + n^2\pi^2}$$

$$= \frac{al^2}{al^2 + n^2\pi^2} [e^{al} . (-1)^n - 1]$$

Example 4: Find the finite Fourier cosine transform of $f(x) = \sin ax$ in $(0, \pi)$.

Solution:

$$F_C[\sin ax] = \int_0^\pi \sin ax . \cos nx \, dx$$

$$= \frac{1}{2} \int_0^\pi [\sin(a+n)x + \sin(a-n)x] dx$$

$$= \frac{1}{2}\left[\frac{-\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{a-n}\right]_0^{\pi}$$

$$= \frac{-1}{2}\left[\frac{\cos(a+n)\pi}{a+n} + \frac{\cos(a-n)\pi}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

$$= \frac{-1}{2}\left[\frac{(-1)^{a+n}}{a+n} + \frac{(-1)^{a-n}}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

if both *n* and *a* are even

$$F_c(\sin ax) = \begin{cases} 0, & \text{if both } n \text{ and } a \text{ are even} \\ \frac{1}{2}\left[\frac{2}{a+n} + \frac{2}{a-n\_}\right], & \text{if } n \text{ or } a \text{ is odd} \end{cases}$$

$$F_c(\sin ax) = \begin{cases} 0, & \text{if both } n \text{ and } a \text{ is odd} \\ \frac{2n}{a^2 - n^2}, & \text{if } n \text{ or } a \text{ is odd} \end{cases}$$

Example 5: Find *f(x)* if its finite sine transform is given by $\frac{2\pi(-)^{p-1}}{p^3}$, where *p* is positive integer 2 <*x*< π.

Solution:

We know that the inverse Fourier since transform is given by

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty} F_S[f(x)]\sin nx \qquad \dots (1)$$

Here $F_S[f(x)] = \frac{2\pi(-1)^{p-1}}{p^3}$ … (2)

Substituting (2) in (1) we get

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty} \frac{2\pi(-1)^{p-1}}{p^3}\sin px$$

$$= 4\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^3}\sin px$$

Example 6: If $f(p) = \dfrac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}$ find F_c^{-1} *[f(p)]* if 0 <*x*< 1.

Solution:

$$F_C^{-1}[f(p)] = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_C[f(x)] \cos\frac{n\pi x}{l}$$

Here $\quad f(p) = \dfrac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}$

Let $\quad F_C[f(x)] = f(p)$

$$\therefore Fc^{-1}[f(p) = \frac{1}{1} f_c(0) \frac{2}{1} \sum_{n=1}^{\infty} f(p).\cos\frac{n\pi x}{l} \qquad [\because l = 1]$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}.\cos n\pi x$$

Exercise

1. Find finite Fourier sine and cosine transform of

1. $f(x) = x in (0, l)$ [**Ans.** $\dfrac{1-\cos n\pi}{n}$,0]

2. $f(x) = x^3 in\ (0, l)$ [**Ans.** $\dfrac{l^4}{n\pi}(-l)^{n+1} + \dfrac{6l^4}{n^3\pi^3}(-l)^n; \dfrac{3l^4}{n^2\pi^2}(-1)^n - \dfrac{6l^4}{n^4\pi^4}[(-1)^n - 1]$

3. $f(x) = \begin{cases} 1\ in\ 0 < x < \dfrac{\pi}{2} \\ -1\ in \dfrac{\pi}{2} < x < \pi \end{cases}$ [**Ans.** $\dfrac{1}{n}[\cos n\pi - 2\cos\dfrac{n\pi}{2} + 1]; \dfrac{2}{n}\sin\dfrac{n\pi}{2}$]

4. $f(x) = x^3, 0 < x < 4$ [**Ans.** $\dfrac{-64}{n\pi}\cos n\pi + \dfrac{128}{n^2\pi^2}(\cos n\pi - 1); \dfrac{128}{n^2\pi^2}\cos n\pi$]

2. Find the finite cosine transform of $\left(1-\frac{x}{\pi}\right)^2$ [**Ans.** $\left\{\begin{matrix} \frac{2}{\pi^{2,}} s > 0 \\ \frac{\pi}{3}, s = 0 \end{matrix}\right\}$]