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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I MATRICES

INTRODUCTION

MATRICES

CHARACTERISTIC EQUATION:

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A

Working rule to find characteristic equation:

<u>For a 3 x 3 matrix:</u>

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of\ A = |A|$

Problems:

1. **Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 = sumofthemaindiagonalelements = 1 + 2 = 3$,

$S_2 = DeterminantofA = |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the characteristic equation of $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$**

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where
$S_1 = sumofthemaindiagonalelements$ = 8 + 7 + 3 = 18,
$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$, $S_3 = DeterminantofA = |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

3. **Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A =** $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by $A^{-1}, A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4.

Using Cayley-Hamilton theorem, find A^{-1} when A = $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 + 1 = 3$

$$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (1-1) + (1-3) + (1-0)$$
$$= 0 - 2 + 1 = -1$$

$S_3 = |A| = 1(1-1) + 0(2+1) + 3(-2-1) = 1(0) + 0 - 9 = -9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

Pre-multiplying by A^{-1}, we get, $A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-A^2 = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix}; 3A = \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

1.

Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 3 + 2 = 7$,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 =$

11,

$S_3 = Determinant of A =$ $|A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

To find the eigen vectors:

$$[A-\lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + 2x_2 - 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} 2 & 1 & -3 & 2 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 . Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

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DEPARTMENT OF MATHEMATICS

UNIT – II

INTRODUCTION

Derivative

The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x .

Differentiation of a Function

Let f(x) is a function differentiable in an interval [a, b]. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function g: [a, b] → R, such that, ∀ x ∈ [a, b], g(x) = f'(x).

This new function is said to be differentiation (differential coefficient) of the function f(x) with respect to x and it is denoted by df(x) / d(x) or Df(x) or f'(x).

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Differentiation 'from First Principle

Let f(x) is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. d / d(x) (x^n) = nx^{n-1}, x ∈ R, n ∈ R

2. d / d(x) (k) = 0, where k is constant.

3. d / d(x) (e^x) = e^x

4. d / d(x) (a^x) = $a^x \log_e$ a > 0, a ≠ 1

Fundamental Rules for Differentiation

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ (sum and difference rule)

(iii) $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (product rule)

Generalization If $u_1, u_2, u_3, \ldots, u_n$ be a function of x, then

$$\frac{d}{dx}(u_1\,u_2\,u_3 \ldots u_n) = \left(\frac{du_1}{dx}\right)[u_2u_3 \ldots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \ldots u_n] + u_1u_2\left(\frac{du_3}{dx}\right)[u_4u_5 \ldots u_n] + \ldots + [u_1u_2 \ldots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)^2\}}$ (quotient rule)

(v) if d / d(x) f(x) = φ(x), then d / d(x) f(ax + b) = a φ(ax + b)

(vi) Differentiation of a constant function is zero i.e., d / d(x) (c) = 0

Different Types of Differentiable Function

1. **Differentiation of Composite Function** (Chain Rule)

If f and g are differentiable functions in their domain, then fog is also differentiable and

(fog)’ (x) = f’ {g(x)} g’ (x)

More easily, if y = f(u) and u = g(x), then dy / dx = dy / du * du / dx.

If y is a function of u, u is a function of v and v is a function of x. Then,

dy / dx = dy / du * du / dv * dv / dx.

2. **Differentiation Using Substitution**

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below .

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
(iv)	$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a \cos 2\theta$
(v)	$a \sin x + b \cos x$	$a = r \cos\alpha, b = r \sin\alpha$
(vi)	$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$	$x = \alpha \sin^2\theta + \beta \cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

3. **Differentiation of Implicit Functions**

If f(x, y) = 0, differentiate with respect to x and collect the terms containing dy / dx at one side and find dy / dx.

Shortcut for Implicit Functions For Implicit function, put d /dx {f(x, y)} = – ∂f / ∂x / ∂f / ∂y, where ∂f / ∂x is a partial differential of given function with respect to x and ∂f / ∂y means Partial differential of given function with respect to y.

4. **Differentiation of Parametric Functions**

If x = f(t), y = g(t), where t is parameter, then

dy / dx = (dy / dt) / (dx / dt) = d / dt g(t) / d / dt f(t) = g' (t) / f' (t)

Logarithmic Differentiation Function

(i) If a function is the product and quotient of functions such as y = $f_1(x)\ f_2(x)\ f_3(x)\ldots / g_1(x)\ g_2(x)\ g_3(x)\ldots$, we first take algorithm and then differentiate.

(ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function

Let y = f(x) and z = g(x), then the differentiation of y with respect to z is

dy / dz = dy / dx / dz / dx = f' (x) / g' (x)

Successive Differentiations

If the function $y = f(x)$ be differentiated with respect to x, then the result dy / dx or $f'(x)$, so obtained is a function of x (may be a constant).

Hence, dy / dx can again be differentiated with respect of x.

The differential coefficient of dy / dx with respect to x is written as $d/dx\,(dy / dx) = d^2y / dx^2$ or $f'(x)$. Again, the differential coefficient of d^2y / dx^2 with respect to x is written as

$d / dx\,(d^2y / dx^2) = d^3y / dx^3$ or $f'''(x)$……

Here, dy / dx, d^2y / dx^2, d^3y / dx^3,… are respectively known as first, second, third, … order differential coefficients of y with respect to x. These alternatively denoted by $f'(x)$, $f''(x)$, $f''(x)$, … or y_1, y_2, y_3…., respectively.

Note $dy / dx = (dy / d\theta) / (dx / d\theta)$ but $d^2y / dx^2 \neq (d^2y / d\theta^2) / (d^2x / d\theta^2)$

Partial Differentiation

The partial differential coefficient of $f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is a written as $\partial f / \partial x$ or D_xf or f_x.

e.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^4y + x + 2y$

Then, $\partial z / \partial x$ or $\partial f / \partial x$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (here, y is consider as constant)

Higher Partial Derivatives

Let $f(x, y)$ be a function of two variables such that $\partial f / \partial x$, $\partial f / \partial y$ both exist.

(i) The partial derivative of $\partial f / \partial y$ w.r.t. 'x' is denoted by $\partial^2 f / \partial x^2$ / or f_{xx}.

(ii) The partial derivative of $\partial f / \partial y$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y^2$ / or f_{yy}.

(iii) The partial derivative of $\partial f / \partial x$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y\, \partial x$ / or f_{xy}.

(iv) The partial derivative of $\partial f / \partial x$ w.r.t. 'x' is denoted by $\partial^2 f / \partial y\, \partial x$ / or f_{yx}.

Note $\partial^2 f / \partial x\, \partial y = \partial^2 f / \partial y\, \partial x$

These four are second order partial derivatives.

(vii) $\cos^{-1} x + \sin^{-1} x = \pi/2$

(viii) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(ix) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$

(x) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(xi) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$

(xii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III

INTRODUCTION

Definite Integral

An integral that contains the upper and lower limits then it is a definite integral. On a real line, x is restricted to lie. Riemann Integral is the other name of the Definite Integral.

A definite Integral is represented as:

$\int baf(x)dx$

Indefinite Integral

Indefinite integrals are defined without upper and lower limits. It is represented as:

$\int f(x)dx = F(x) + C$

Where C is any constant and the function f(x) is called the integrand.

- $\int 1\, dx = x + C$
- $\int a\, dx = ax + C$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C;\ n \neq -1$
- $\int \sin x\, dx = -\cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \csc^2 x\, dx = -\cot x + C$
- $\int \sec x(\tan x)\, dx = \sec x + C$
- $\int \csc x(\cot x)\, dx = -\csc x + C$
- $\int \frac{1}{x}\, dx = \ln |x| + C$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = \frac{a^x}{\ln a} + C;\ a > 0, a \neq 1$
- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$
- $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
- $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
- $\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) + \int \tan^{n-2}(x)dx$
- $\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
- $\int \csc^n(x)dx = \frac{-1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)dx$

Example 1: Find the integral of the function: ∫30*x*2*dx*

Solution:

Given ∫30*x*2*dx*

$= (x33)30$

$=(333)–(033)$

$= 9$

Example 2: Find the integral of the function: $\int x^2$ dx

Solution:

Given $\int x^2$ dx

$= (x^3/3) + C.$

Example 3:

Integrate $\int (x^2-1)(4+3x)dx$.

Solution:

Given: $\int (x^2-1)(4+3x)dx$.

Multiply the terms, we get

$\int (x^2-1)(4+3x)dx = \int 4x^2+3x^3-3x-4\ dx$

Now, integrate it, we get

$\int (x^2-1)(4+3x)dx = 4(x^3/3) + 3(x^4/4)- 3(x^2/2) – 4x + C$

The antiderivative of the given function $\int (x^2-1)(4+3x)dx$ is $4(x^3/3) + 3(x^4/4)- 3(x^2/2) – 4x + C$.

The various indefinite integral formulae which must be remembered as they are extremely helpful in solving problems include:

1) $\int e^x dx = e^x + c$

2) $\int 1/x\ dx = \ln |x| + C$

3) $\int a^x\ dx = a^x/ \ln a + c \ (a > 0)$

4) $\int \cos x\ dx = \sin x + c$

5) $\int \sin x\ dx = - \cos x + c$

6) $\int \sec^2 x\ dx = \tan x + c$

7) $\int$ cosec x cot x dx = - cosec x + c

8) $\int$sec x tan x dx = sec x + c

9) $\int$ cosec^2 x dx = - cot x + c

Example 1: Evaluate $\int x^4\,dx.$

Using formula (4) from the preceding list, you find that $\int x^4\,dx = \frac{x^5}{5} + C$.

Example 2: Evaluate $\int \frac{1}{\sqrt{x}}\,dx.$.

Because $1/\sqrt{x} = x^{-1/2}$, using formula (4) from the preceding list yields

$$\begin{aligned}\int \frac{1}{\sqrt{x}}\,dx &= \int x^{-1/2}\,dx \\ &= \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= 2x^{1/2} + C\end{aligned}$$

Example 3: Evaluate $\int (6x^2 + 5x - 3)\,dx$

Applying formulas (1), (2), (3), and (4), you find that

$$\begin{aligned}\int (6x^2 + 5x - 3)\,dx &= \frac{6x^3}{3} + \frac{5x^2}{2} - 3x + C \\ &= 2x^3 + \frac{5}{2}x^2 - 3x + C\end{aligned}$$

Example 4: Evaluate $\int \frac{dx}{x+4}$.

Using formula (13), you find that $\int \frac{dx}{x+4} = \ln|x+4| + C$

Example 5: Evaluate $\int \frac{dx}{25+x^2}$.

Using formula with a = 5, you find that

$$\int \frac{dx}{25+x^2} = \frac{1}{5}\arctan\frac{x}{5} + C$$

Integration by parts:

This method is used to integrate the product of two functions. If f(x) and g(x) are two integrable functions, then∫f(x) g(x) dx = f(x) ∫ g(x) dx – ∫ {d/dx (f(x)) . ∫g(x) dx} dx.

Example 6: Evaluate $\int x^2(x^3+1)^5\,dx$.

Because the inside function of the composition is $x^3 + 1$, substitute with

$$u = x^3 + 1$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

hence, $$\int x^2(x^3+1)^5\,dx = \frac{1}{3}\int u^5\,du$$
$$= \frac{1}{3}\cdot\frac{u^6}{6} + C$$
$$= \frac{1}{18}u^6 + C$$
$$= \frac{1}{18}(x^3+1)^6 + C$$

Example 7: Evaluate $\int \sin(5x)\,dx$.

Because the inside function of the composition is 5 *x*, substitute with

$$u = 5x$$
$$du = 5dx$$
$$\frac{1}{5}du = dx$$

hence, $$\int \sin(5x)\,dx = \frac{1}{5}\int \sin u\,du$$
$$= -\frac{1}{5}\cos u + C$$
$$= -\frac{1}{5}\cos(5x) + C$$

Example 8: Evaluate $\int \frac{3x}{\sqrt{9-x^2}}\,dx$.

Because the inside function of the composition is $9 - x^2$, substitute with

$$u = 9 - x^2$$
$$du = -2x\,dx$$
$$-\frac{1}{2}\,du = x\,dx$$

hence,

$$\int \frac{3x}{\sqrt{9-x^2}}\,dx = -\frac{3}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= -\frac{3}{2}\int u^{-1/2}\,du$$
$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}} + C$$
$$= -3u^{1/2} + C$$
$$= -3\sqrt{9-x^2} + C$$

Integration by parts

Another integration technique to consider in evaluating indefinite integrals that do not fit the basic formulas is **integration by parts.** You may consider this method when the integrand is a single transcendental function or a product of an algebraic function and a transcendental function. The basic formula for integration by parts is

$$\int u\,dv = uv - \int v\,du$$

where *u* and *v* are differential functions of the variable of integration.

A general rule of thumb to follow is to first choose *dv* as the most complicated part of the integrand that can be easily integrated to find *v*. The *u* function will be the remaining part of the integrand that will be differentiated to find *du*. The goal of this technique is to find an integral, $\int v\,du$, which is easier to evaluate than the original integral.

Example 9: Evaluate $\int x \sec^2 x\,dx$.

$$\text{Let } u = x \text{ and } dv = \sec^2 x\, dx$$
$$du = dx \quad v = \tan x$$

hence,
$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx$$
$$= x \tan x - \left(-\ln|\cos x|\right) + C$$
$$= x \tan x + \ln|\cos x| + C$$

Example 10: Evaluate $\int x^4 \ln x\, dx$.

$$\text{Let } u = \ln x \text{ and } dv = x^4\, dx$$
$$du = \frac{1}{x}\, dx \quad v = \frac{x^5}{5}$$

hence,
$$\int x^4 \ln x\, dx = \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \cdot \frac{1}{x}\, dx$$
$$= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4\, dx$$
$$= \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C$$

Example 11: Evaluate $\int \arctan x\, dx$.

$$\text{Let } u = \arctan x \text{ and } dv = dx$$
$$du = \frac{1}{1 + x^2}\, dx \quad v = x$$

hence,
$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1 + x^2}\, dx$$
$$= x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$$

Integrals involving powers of the trigonometric functions must often be manipulated to get them into a form in which the basic integration formulas can be applied. It is extremely important for you to be familiar with the basic trigonometric identities, because you often used these to rewrite the integrand in a more workable form. As in integration by parts, the goal is to find an integral that is easier to evaluate than the original integral.

Example 12: Evaluate $\int \cos^3 x \sin^4 x\, dx$

$$\int \cos^3 \sin^4 x\,dx = \int \cos^2 x \sin^4 x \cos x\,dx$$
$$= \int (1 - \sin^2 x) \sin^4 x \cos x\,dx$$
$$= \int (\sin^4 x - \sin^6 x) \cos x\,dx$$
$$= \int \sin^4 x \cos x\,dx - \int \sin^6 x \cos x\,dx$$
$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$$

Example 13: Evaluate $\int \sec^6 x\, dx$

$$\int \sec^6 x\,dx = \int \sec^4 x \sec^2 x\,dx$$
$$= \int (\sec^2 x)^2 \sec^2 x\,dx$$
$$= \int (\tan^2 x + 1)^2 \sec^2 x\,dx$$
$$= \int (\tan^4 x + 2\tan^2 x + 1) \sec^2 x\,dx$$
$$= \int \tan^4 x \sec^2 x\,dx + \int 2\tan^2 x \sec^2 x\,dx + \int \sec^2 x\,dx$$
$$= \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$$

Example 14: Evaluate $\int \sin^4 x\, dx$

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$
$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right) dx$$
$$= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx$$
$$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$$
$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

If an integrand contains a radical expression of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, a specific **trigonometric substitution** may be helpful in evaluating the indefinite integral. Some general rules to follow are

1. If the integrand contains $\sqrt{a^2 - x^2}$

let $x = a\sin\theta$
$dx = a\cos\theta\,d\theta$
and $\sqrt{a^2 - x^2} = a\cos\theta$

2. If the integrand contains $\sqrt{a^2 + x^2}$

let $x = a\tan\theta$
$dx = a\sec^2\theta\,d\theta$
and $\sqrt{a^2 + x^2} = a\sec\theta$

3. If the integrand contains $\sqrt{x^2 - a^2}$

let $x = a\sec\theta$
$dx = a\sec\theta\tan\theta\,d\theta$
and $\sqrt{x^2 - a^2} = a\tan\theta$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –IV CORRELATION &REGRESSION

INTRODUCTION

Basic terms and concepts

1. A **scatter plot** is a graphical representation of the relation between two or more variables. In the scatter plot of two variables x and y, each point on the plot is an x-y pair.

2. We use regression and correlation to describe the variation in one or more variables.
A. The **variation** is the sum of the squared deviations of a variable.

N 2 i=1 Variation= x-x

B. The variation is the numerator of the **variance** of a sample:

N 2 i=1 x-x Variance= N-1

C. Both the variation and the variance are **measures of the dispersion** of a sample.

3. The **covariance** between two
random variables is a statistical measure of the degree to which the two variables move together.

A. The covariance captures how one variable is different from its mean as the other variable is different from its mean.

B. A positive covariance indicates that the variables tend to move together; a negative covariance indicates that the variables tend to move in opposite directions.

C. The covariance is calculated as the ratio of the **covariation** to the sample size less one:

where N is the sample size xi is the ith observation on variable x, is the mean of the variable x observations, yi is the ith observation on variable y, and N i i i=1 (x -x)(y -y) Covariance = N-1 x
is the mean of the variable y observations. y

D. The actual value of the covariance is not meaningful because it is affected by the scale of the two variables. That is why we calculate the correlation coefficient – to make something interpretable from the covariance information.

E. The **correlation coefficient**, r, is a measure of the strength of the relationship between or among variables.

Calculation:

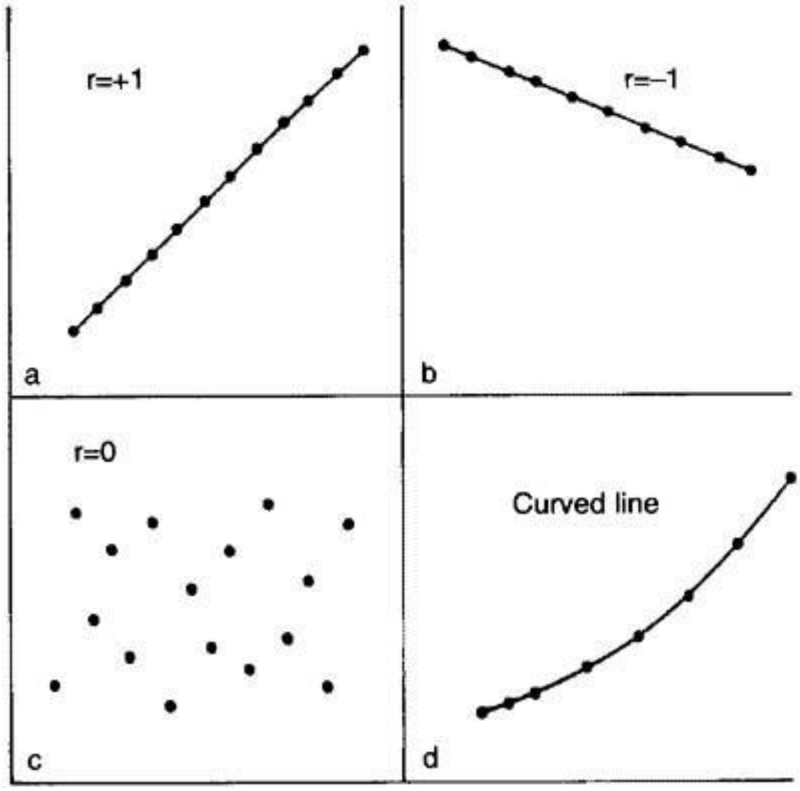
Correlation and regression

The word correlation is used in everyday life to denote some form of association. We might say that we have noticed a correlation between foggy days and attacks of wheeziness. However, in statistical terms we use correlation to denote association between two quantitative variables. We also assume that the association is linear, that one variable increases or decreases a fixed amount for a unit increase or decrease in the other. The other technique that is often used in these circumstances is regression, which involves estimating the best straight line to summarise the association.

Correlation coefficient

The degree of association is measured by a correlation coefficient, denoted by r. It is sometimes called Pearson's correlation coefficient after its originator and is a measure of linear association. If a curved line is needed to express the relationship, other and more complicated measures of the correlation must be used.

The correlation coefficient is measured on a scale that varies from + 1 through 0 to - 1. Complete correlation between two variables is expressed by either + 1 or -1. When one variable increases as the other increases the correlation is positive; when one decreases as the other increases it is negative. Complete absence of correlation is represented in Figure



The calculation of the correlation coefficient is as follows, with x representing the values of the independent variable (in this case height) and y representing the values of the dependent variable (in this case anatomical dead space). The formula to be used is:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{[\Sigma(x-\bar{x})^2(y-\bar{y})^2]}}$$

which can be shown to be equal to:

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{(n-1)SD(x)SD(y)}$$

Calculator procedureed to our partners and will not affect browsing data.

Note:
i. The type of relationship is represented by the correlation coefficient:
r =+1 perfect positive correlation +1 >r > 0 positive relationship r = 0 no relationship 0 > r > 1 negative relationship r = 1 perfect negative correlation
ii. You can determine the degree of correlation by looking at the scatter graphs.
If the relation is upward there is **positive correlation**.
If the relation downward there is **negative correlation**.

1. Recall the principle of least square method

   Ans: The sum of the square of the residue is minimum.

2. Define curve fitting.
   Ans: It is the process of constructing a curve that has the best fit to a series of data point.

3. Obtain the normal equation of straight line.

   Ans:
   The normal equation of straight line is

$$a\sum x + nb = \sum y$$

$$a\sum x^2 + n\sum x = \sum xy$$

4. Obtain the normal equation for the curve Y = Y = $aX^2 + bX + c$

Ans: the normal equation for the curve Y = $aX^2 + bX + c$ is

$$a\sum X^2 + b\sum X + nc = \sum Y \quad (1)$$
$$a\sum X^3 + b\sum X^2 + c\sum X = \sum XY \quad (2)$$
$$a\sum X^4 + b\sum X^2 + c\sum X^2 = \sum X^2Y \quad (3)$$

5. Obtain the normal equation for the curve z=ax +by +c.

Ans: the normal equation for the curve z=ax +by +c. is

$$a\sum x + b\sum y + nc = \sum z \quad (1)$$
$$a\sum x^2 + b\sum xy + c\sum x = \sum zx \quad (2)$$
$$a\sum xy + b\sum y^2 + c\sum y = \sum zy \quad (3)$$

Problems:

1. Fitting a straight line - Curve fitting

1. Fit a straight line y = a + bx using the following data

x 5 4 3 2 1

y 1 2 3 4 5

2. Fit a straight line y = a + bx using the following data

x 3 5 7 9 11

y 2.3 2.6 2.8 3.2 3.5

3. Fit a straight line to the following data on production.

Year	1996	1997	1998	1999	2000
Production	40	50	62	58	60

4. Fit a straight line to the following data on profit.

Year	1992	1993	1994	1995	1996	1997	1998	1999
Profit	38	40	65	72	69	60	87	95

5. Fit second degree parabola equation $y=a+bx+cx^2$ using the following data.

x	12	10	8	6	4	2
y	6	5	4	3	2	1

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –V

INTRODUCTION

Probability theory has applications in many branches of Science and Engineering. Probability theory as a matter of fact, is study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a **Deterministic experiment**.

Although all possible outcomes of an experiment may be known in advance the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a **Random experiment**.

(e.g.) Whenever a fair dice is thrown, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be.

Sample Space

The set of all possible outcomes which are assumed equally likely.

Event

A sub-set of S consisting of possible outcomes.

Mathematical definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A. then the probability of event A occurring is denoted as P(A), is denoted by

$$P(A) = \frac{n(A)}{n(S)}$$

Note: 1. It is obvious that $0 \le P(A) \le 1$.

2. If A is an impossible event, P(A) = 0.

3. If A is a certain event , P(A) = 1.

A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$ A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$$

Axiomatic definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, P(A) is defined as a real number satisfying the following axioms.

1. $0 \le P(A) \le 1$
2. P(S) = 1
3. If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and
4. If $A_1, A_2\ A_3, \ldots, A_n, \ldots$ are mutually exclusive events,
   $$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n, \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) \ldots$$

Important Theorems

Theorem 1: Probability of impossible event is zero.

Proof: **Let S be sample space (certain events) and ϕ be the impossible event.**

Certain events and impossible events are mutually exclusive

$P(S \cup \phi) = P(S) + P(\phi)$ (Axiom 3)

$S \cup \phi = S$

$P(S) = P(S) + P(\phi)$

$P(\phi) = 0$, hence the result.

Theorem 2: **If** $\overline{A}$ **is the complementary event of** ***A*****,** $P(\overline{A}) = 1 - P(A) \leq 1$**.**

Proof: **Let** ***A*** **be the occurrence of the event**

$\overline{A}$ **be the non-occurrence of the event .**

Occurrence and non-occurrence of the event are mutually exclusive.

$$P(A \cup \overline{A}) = P(A) + P(\overline{A})$$

$$A \cup \overline{A} = S \quad \Rightarrow \quad P(A \cup \overline{A}) = P(S) = 1$$

$$\therefore \quad 1 = P(A) + P(\overline{A})$$

$$P(\overline{A}) = 1 - P(A) \leq 1.$$

Theorem 3: (Addition theorem)

If A and B are any 2 events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Proof: We know, $A = A\overline{B} \cup AB$ and $B = \overline{A}B \cup AB$

$$\therefore \quad P(A) = P(A\overline{B}) + P(AB) \text{ and } P(B) = P(\overline{A}B) + P(AB) \qquad \text{(Axiom 3)}$$

$$P(A) + P(B) = P(A\overline{B}) + P(AB) + P(\overline{A}B) + P(AB)$$

$$= P(A \cup B) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Note: The theorem can be extended to any 3 events, A,B and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Theorem 4: If $B \subset A$, $P(B) \leq P(A)$.

Proof: A and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$

$\therefore \quad P(B \cup A\overline{B}) = P(A)$

$P(B) + P(A\overline{B}) = P(A)$ (Axiom 3)

$P(B) \leq P(A)$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Product theorem of probability

Rewriting the definition of conditional probability, We get

$$P(A \cap B) = P(A)P(A/B)$$

The product theorem can be extended to 3 events, A, B and C as follows:

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

Note: 1. If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$.

2. If $B \subset A$, $P(B/A) \geq P(B)$, since $A \cap B = B$, and $\frac{P(B)}{P(A)} \geq P(B)$,

As $P(A) \leq P(S) = 1$.

3. If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$.
4. If $P(A) > P(B)$, $P(A/B) > P(B/A)$.
5. If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$.

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

If the two events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$, Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events are said to be independent (pair wise independent).

The product theorem can be extended to any number of independent events, If A_1 A_2 A_3 ..... A_n are *n* independent events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \times P(A_2) \times P(A_3) \times \ldots \times P(A_n)$$

Theorem 4:

If the events A and B are independent, the events $\overline{A}$ and B are also independent.

Proof:

The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\overline{A} \cap B) = B$

$$\therefore \quad P(A \cap B) + P(\overline{A} \cap B) = P(B)$$

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)\,P(B) \qquad (\therefore A \text{ and } B \text{ are independent})$$

$$= P(B)\,[1 - P(A)]$$

$$= P(\overline{A})\,P(B).$$

Theorem 5:

If the events A and B are independent, the events $\overline{A}$ and $\overline{B}$ are also independent.

Proof:

$$P(\overline{A} \cap \overline{B}) = P\left(\overline{A \cup B}\right) = 1 - P(A \cup B)$$

$$= 1 - [\,P(A) + P(B) - P(A \cap B)] \qquad \text{(Addition theorem)}$$

$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$

$$= P(\overline{A})P(\overline{B}).$$

Problem 1:

From a bag containing 3 red and 2 balck balls, 2 ball are drawn at random. Find the probability that they are of the same colour.

Solution :

Let A be the event of drawing 2 red balls

B be the event of drawing 2 black balls.

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

$$= \frac{3C_2}{5C_2} + \frac{2C_2}{5C_2} = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$$

Problem 2:

When 2 card are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit?

Solution :

Let A be the event of drawing 2 spade cards

B be the event of drawing 2 claver cards

C be the event of drawing 2 hearts cards

D be the event of drawing 2 diamond cards.

$$\therefore P(A \cup B \cup C \cup D) = 4\frac{13C_2}{52C_2} = \frac{4}{17}.$$

Problem 3:

When A and B are mutually exclusive events such that P(A) = 1/2 and P(B) = 1/3, find $P(A \cup B)$ and $P(A \cap B)$.

Solution :

$P(A \cup B) = P(A) + P(B) = 5/6$; $P(A \cap B) = 0$.

Problem 4:

If P(A) = 0.29, P(B) = 0.43, find $P(A \cap \overline{B})$, if A and B are mutually exclusive.

Solution :

We know $A \cap \overline{B} = A$

$P(A \cap \overline{B}) = P(A) = 0.29$

Problem 5:

A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution :

Let A be the event of drawing a spade

B be the event of drawing a ace

$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}.$$

Problem 6:

If P(A) = 0.4, P(B) = 0.7 and $P(A \cap B) = 0.3$, find $P(\overline{A} \cap \overline{B})$.

Solution :

$\mathbf{P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)}$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 0.2$

Problem 7:

If P(A) = 0.35, P(B) = 0.75 and P(A ∪ B) = 0.95, find P($\bar{A} \cup \bar{B}$).

Solution :

$P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - [P(A) + P(B) - P(A \cup B)] = 0.85$

Problem 8:

A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random(with out replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.

Solution :

(i) **P(both are good) =** $\frac{10C_2}{16C_2} = \frac{3}{8}$

(ii) **P(both have major defects) =** $\frac{2C_2}{16C_2} = \frac{1}{120}$

(iii) **P(at least 1 is good) =** $\frac{10C_1 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$

(iv) **P(at most 1 is good) =** $\frac{10C_0 6C_2 + 10C_1 6C_1}{16C_2} = \frac{5}{8}$

(v) **P(exactly 1 is good) =** $\frac{10C_1 6C_1}{16C_2} = \frac{1}{2}$

(vi) **P(neither has major defects) =** $\frac{14C_2}{16C_2} = \frac{91}{120}$

(vii) **P(neither is good) =** $\frac{6C_2}{16C_2} = \frac{1}{8}$.

Problem 9:

If A, B and C are any 3 events such that P(A) = P(B) = P(C) = 1/4, P(A ∩ B) = P(B ∩ C) = 0; P(C ∩ A) = 1/8. Find the probability that at least 1 of the events A, B and C occurs.

Solution :

Since P(A ∩ B) = P(B ∩ C) = 0; P(A ∩ B ∩ C) = 0

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}.$$

Problem 10:

A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution :

Let A be a good tube drawn and B be an other good tube drawn.

P(both tubes drawn are good) = P(A ∩ B) = $\frac{6C_2}{10C_2} = \frac{1}{3}$

P(B/A) = $\frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$ **(By conditional probability)**

Problem 11:

In shooting test, the probability of hitting the target is 1/2, for a, 2/3 for B and ¾ for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.

Solution :

Let A, B and C be the event of hitting the target .

P(A) = 1/2, P(B) = 2/3, P(C) = 3/4

P($\bar{A}$**) = 1/2, P(**$\bar{B}$**) = 1/3, P(**$\bar{C}$**) = 1/4**

P(none of them hits) = P($\bar{A} \cap \bar{B} \cap \bar{C}$**) = P(**$\bar{A}$**) × P(**$\bar{B}$**) × P(**$\bar{C}$**) = 1/24**

P(at least one hits) = 1 – P(none of them hits)

= 1 – (1/24) = 23/24.

Problem 12:

A and B alternatively throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.

Solution:

Let A be the event of throwing 6

B be the event of throwing 7.

P(throwing 6 with 2 dice) = 5/36 P(throwing 7 with 2 dice) = 1/6

P(not throwing 6) = 31/36 P(not throwing 7) = 5/6

A plays in I, III, V,……trials.

A wins if he throws 6 before Be throws 7.

P(A wins) = P(A $\cup\ \bar{A}\ \bar{B}$ **A** $\cup\ \bar{A}\ \bar{B}\ \bar{A}\ \bar{B}$ **A** $\cup$ **……)**

= P(A) + P($\bar{A}\ \bar{B}$ **A) + P(**$\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}$ **A) + ……**

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \frac{5}{36} + \dots$$

$$= \frac{30}{61}$$

Problem 13:

A and B toss a fair coin alternatively with the understanding that the first who obtain the head wins. If A starts, what is his chance of winning?

Solution :

P(getting head) = 1/2 , P(not getting head) = 1/2

A plays in I, III, V,……trials.

A wins if he gets head before B.

P(A wins) = P(A $\cup$ $\bar{A}\ \bar{B}$ A $\cup$ $\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}$ A $\cup$ ……)

= P(A) + P($\bar{A}\ \bar{B}$ A) + P($\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}$ A) + ……

$$= \frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)^2\frac{1}{2} + \cdots$$

$$= \frac{2}{3}$$

Problem 14:

A problem is given to 3 students whose chances of solving it are 1/2 , 1/3 and 1/4 . What is the probability that (i) only one of them solves the problem and (ii) the problem is solved.

Solution :

P(A solves) = 1/2 P(B) = 1/3 P(C) = 1/4

P($\bar{A}$) = 1/2, P($\bar{B}$) = 2/3, P($\bar{C}$) = 3/4

P(none of them solves) = P($\bar{A} \cap \bar{B} \cap \bar{C}$) = P($\bar{A}$) × P($\bar{B}$) × P($\bar{C}$) = 1/4

P(at least one solves) = 1 – P(none of them solves)

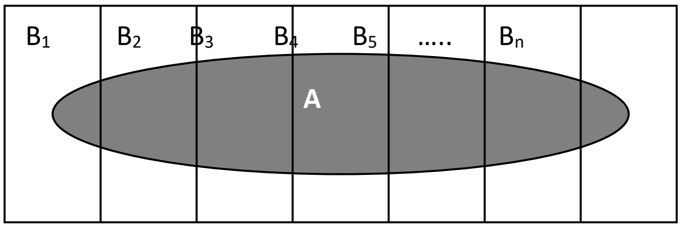
= 1 – (1/4) = 3/4 .

Baye's Theorem

Statement: If B_1, B_2, B_3, ….B_n be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with B_i, then

$$P(B_i / A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}$$

Proof :



The shaded region represents the event A, A can occur along with B_1, B_2, B_3, ….B_n that are mutually exclusive.

∴ **AB_1, AB_2, AB_3, …, AB_n are also mutually exclusive.**

Also A = $AB_1 \cup AB_2 \cup AB_3 \cup \ldots \cup AB_n$

$P(A) = P(AB_1) + P(AB_2) + P(AB_3) + \ldots + P(AB_n)$

$$= \sum_{i=1}^{n} P(AB_i)$$

$$= \sum_{i=1}^{n} P(B_i) \times P(A/B_i) \quad \textbf{(By conditional probability)}$$

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)} .$$

Problem 15:

Ina bolt factory machines A, B, C manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the produce and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C.

Solution :

Let B_1 be bolt produced by machine A

B_2 be bolt produced by machine B

B_3 be bolt produced by machine C

Let A/B_1 be the defective bolts drawn from machine A

A/B_2 be the defective bolts drawn from machine B

A/B_3 be the defective bolts drawn from machine C.

P(B_1) = 0.25 **P(A/B_1) = 0.05**

P(B_2) = 0.35 **P(A/B_2) = 0.04**

P(B_3) = 0.40 **P(A/B_3) = 0.02**

Let B_1/A be defective bolts manufactured by machine A

B_2/A be defective bolts manufactured by machine B

B_3/A be defective bolts manufactured by machine C

$$P(A) = \sum_{i=1}^{3} P(B_i) \times P(A/B_i) = \mathbf{(0.25) \times (0.05) + (0.35) \times (0.04) + (0.4) \times (0.02)}$$

$$= \mathbf{0.0345}$$

P(B_1/A) = $\dfrac{P(B_1) \times P(A/B_1)}{P(A)}$ = **0.3623**

P(B_2/A) = $\dfrac{P(B_2) \times P(A/B_2)}{P(A)}$ = **0.405**

P(B_3/A) = $\dfrac{P(B_3) \times P(A/B_3)}{P(A)}$ = **0.231 .**

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