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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Differential Calculus – SMTA1106

UNIT 1 - DIFFERENTIAL CALCULUS

SUCCESSIVE DIFFERENTIATION

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $\boldsymbol{f(x)}$ be a differentiable function and let its successive derivatives be denoted by $\boldsymbol{f'(x), f''(x), \ldots, f^{(n)}(x)}$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

$\vdots$

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

1.2 Calculation of n^{th} derivative

i. n^{th} Derivative of e^{ax+b}

Let $y = e^{ax+b}$

$y_1 = ae^{ax+b}$

$y_2 = a^2e^{ax+b}$

$\vdots$

$y_n = a^ne^{ax+b}$

ii. n^{th} Derivative of $y = \log(ax+b)$

Let $y = \log(ax+b)$

$y_1 = \frac{a}{(ax+b)}$

$y_2 = \frac{-a^2}{(ax+b)^2}$

$y_3 = \frac{2!\,a^3}{(ax+b)^3}$

$\vdots$

$y_n = (-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$

iii. n^{th} Derivative of $(ax+b)^m$

Case 1: When m is a positive integer i.e., when $m > 0, m \geq n$

Let $y = (ax+b)^m$

$y_1 = m\,a(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

$\vdots$

$$y_n = m(m-1)\dots(m-n+1)a^n(ax+b)^{m-n}$$

$$y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

Case 2: When m is a negative integer i.e.,when $m > 0, m < n$

Put $m = -p$ where p is a positive integer in the above result.

$D^n(ax+b)^m = a^n m(m-1)(m-2)\dots\big(m-(n-1)\big)(ax+b)^{m-n}$

$D^n(ax+b)^{-p} = a^n(-p)(-p-1)(-p-2)\dots\big(-p-(n-1)\big)(ax+b)^{-p-n}$

$D^n(ax+b)^{-p} = a^n(-1)^n(p)(p+1)(p+2)\dots(p+n-1)(ax+b)^{-p-n}$

Multiply and divide by $1\,.\,2\,.\,3\,.\dots\,(p-1)$ to the rhs of above equation

Then $D^n(ax+b)^{-p} = (-1)^n\frac{(p-n+1)!}{(p-1)!}\,a^n(ax+b)^{-p-n}$

Change p to m we get

$$\boldsymbol{D^n(ax+b)^{-m} = (-1)^n\frac{(m-n+1)!}{(m-1)!}\,a^n(ax+b)^{-m-n}}$$

iv. n^{th} Derivative of $y = \sin(ax+b)$

Let $y = \sin(ax+b)$

$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$

$\vdots$

$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$

Similarly if $y = \cos(ax+b)$

$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

v. n^{th} Derivative of $y = e^{ax}\sin(ax+b)$

Let $y = e^{ax}\sin(bx+c)$

$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$

$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$

$= e^{ax}\,[\,r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)]$

Putting $a = r\cos\alpha,\; b = r\sin\alpha$

$= e^{ax}\;r\;\sin(bx+c+\alpha)$

Similarly $y_2 = e^{ax}\;r^2\;\sin(bx+c+2\alpha)$

$\vdots$

$y_n = e^{ax}\;r^n\;\sin(bx+c+n\alpha)$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$
$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	n^th^ derivative
e^{ax+b}	$a^n e^{ax+b}$
$\log(ax+b)$	$(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$
$(ax+b)^m$	$\frac{m!}{(m-n)!}a^n(ax+\mathrm{b})^{m-n}$ when $m>0$ and $m>n$ 0 when $0<m<n$
$(ax+b)^{-m}$	$(-1)^n\frac{(m-n+1)!}{(m-1)!}\ a^n(ax+b)^{-m-n}$
$\sin(ax+b)$	$a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$
$\cos(ax+b)$	$a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$
$e^{ax}\sin(ax+b)$	$e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$e^{ax}\cos(ax+b)$	$e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y=\frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y=\frac{1}{1-5x+6x^2}=\frac{1}{(1-3x)(1-2x)}=\frac{3}{1-3x}-\frac{2}{1-2x}$$

$$\therefore y_n=\frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}}-\frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow y_n=(-1)^{n+1}\,n!\left[\left(\frac{3}{1-3x}\right)^{n+1}-\left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x\cos 4x$

Solution: Let y $=\sin 6x\cos 4x$

$$=\frac{1}{2}(\sin 10\,x+\cos 2\,x)$$

$$\therefore y_n=\frac{1}{2}\left[10^n\sin\left(10x+\frac{n\pi}{2}\right)+2^n\cos\left(2x+\frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of sin^2xcos^3x

Solution: Let y = sin^2xcos^3x

$= sin^2x cos^2x\ cosx$

$= \frac{1}{4} sin^2 2x\ cosx \ = \frac{1}{8}(1 - cos4x)cosx$

$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$

$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$

$= \frac{1}{16}(2cosx - cos3x - cos5x)$

$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$

$= \left(\frac{1}{2} 2\ sin^2x\right)^2$

$= \frac{1}{4}((1 - \cos 2x)^2$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$

$= \frac{3}{8} - \frac{1}{2} cos2x + \frac{1}{8} cos4x$

$\therefore y_n = -\frac{1}{2} 2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} 4^n cos\left(4x + \frac{n\pi}{2}\right)$

Example 5 Find the n^{th} derivative of $e^{3x} cosx \sin^2 2x$

Solution: Let $y = e^{3x} cosx \sin^2 2x$

Now $\cos x \sin^2 2x = \frac{1}{2}(\cos x - \cos x \cos 4x)$

$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x + \cos 3x\right)$

$\Rightarrow y = e^{3x} cosx \sin^2 2x = \frac{1}{2} e^{3x} \cos x - \frac{1}{4} e^{3x} \cos 5x - \frac{1}{4} e^{3x} \cos 3x$

$\therefore y_n = \frac{1}{2} e^{3x} (9 + 1)^{\frac{n}{2}} \cos\left(x + n \tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x} (9 + 25)^{\frac{n}{2}} \cos\left(5x + n \tan^{-1}\frac{5}{3}\right)$
$- \frac{1}{4} e^{3x} (9 + 9)^{\frac{n}{2}} \cos\left(3x + n \tan^{-1}\frac{3}{3}\right)$

$= \frac{1}{2} e^{3x}\ 10^{\frac{n}{2}} \cos\left(x + n \tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\ 34^{\frac{n}{2}} \cos\left(5x + n \tan^{-1}\frac{5}{3}\right)$
$- \frac{1}{4} e^{3x}\ 18^{\frac{n}{2}} \cos(3x + n \tan^{-1} 1)$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$\therefore y_n = a^n \left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$

$= a^n \left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$

$= a^n \left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2 \sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$

$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$

$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\ ]^{\frac{1}{2}}$

$= a^n[1 + (-1)^n \sin 2ax\ ]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n$ and $\sin n\pi = 0$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\,cos\theta,\; a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\;\sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1}n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1}n!}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

Substituting $2x+1 = r\cos\theta$, $\sqrt{3} = r\sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1}n!}{\sqrt{3}\;r^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1}n!}{\sqrt{3}\left(\frac{\sqrt{3}}{\sin\theta}\right)^{n+1}}\,[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta]$$

$$\because \sqrt{3} = r\sin\theta$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1}n!}{(\sqrt{3})^{n+2}}\,2i\sin(n+1)\theta\,\sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2}n!}{\sqrt{3}^{n+2}}\,\sin(n+1)\theta\,\sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + \sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x\,(\sec x\tan x) = 2\,\sec^2 x\tan x$$

$$\therefore \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\sec^2 x\tan x - 2(x+\tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \left(\sqrt{1+x^2}\right)\frac{dy}{dx} = 1$$

Differentiating both sides w.r.t. x, we get

$$\left(\sqrt{1+x^2}\right)\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1} \frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n]$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$

$$= e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x} \sin 4x\right)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) +$$

$$2nx e^{3x}\, 5^{n-1} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) +$$

$$n(n-1) e^{3x}\, 5^{n-2} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right) + 0$$

$$= e^{3x} 5^n \left[x^2 \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) + \right.$$

$$\left. \frac{2nx}{5} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) + \frac{n(n-1)}{25} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right)\right]$$

Example 13 If $y = a \cos(\log x) + b\, sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + n(n+1) y_n = 0$$

Solution: Here $y = a \cos(\log x) + b \sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x} \sin(\log x) + \frac{b}{x} \cos(logx)$$

$$\Rightarrow x y_1 = -a \sin(logx) + b\, cos(logx)$$

Differentiating both sides w.r.t. x, we get

$$x y_2 + y_1 = -\frac{a}{x} \cos(logx) + \frac{-b}{x}\, sin(logx)$$

$$\Rightarrow x^2 y_2 + x y_1 = -\{acos(logx) + b\, sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + \left(y_{n+1}x + n_{C_1}y_n.1\right) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log\left(x + \sqrt{1+x^2}\right)$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log\left(x + \sqrt{1+x^2}\right.$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$[y_{n+2}(1+x^2) + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2] + \left(y_{n+1}x + n_{C_1}y_n.1\right) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

Example 15 If $y = \sin(m\,sin^{-1}x)$, show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y = \sin(m\,sin^{-1}x)$ ......①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\,sin^{-1}x) \quad ......②$$

$$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2\cos^2(m\,sin^{-1}x)$$

$$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2[1 - \sin^2(m\,sin^{-1}x)]$$

$$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)......③$$

$$\Rightarrow (1-x^2){y_1}^2 + m^2y^2 = m^2$$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^22yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n1\right) + m^2y_n = 0$$

$$[y_{n+2}(1-x^2) + n_{C_1} y_{n+1}(-2x) + n_{C_2} y_n(-2)] - (y_{n+1}x + n_{C_1} y_n 1) + m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n \ldots\ldots④$$

Putting $x = 0$ in ①, ②and ③

$y(0) = 0, y_1(0) = m$ and $y_2(0) = 0$

Putting $x = 0$ in ④

$y_{n+2}(0) = (n^2 - m^2)y_n(0)$

Putting $n = 1,2,3 \ldots \ldots$ in the above equation, we get

$$y_3(0) = (1^2 - m^2)y_1(0)$$

$$=(1^2 - m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 - m^2)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = (3^2 - m^2)y_3(0)$$

$$=m(1^2 - m^2)(3^2 - m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ even \\ m(1^2 - m^2)(3^2 - m^2)\ldots[(n-2)^2) - m^2], & if\ n\ is\ odd \end{cases}$$

Example 16 If $y = e^{msin^{-1}x}$, show that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{msin^{-1}x}$ …①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} e^{m\,sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad \ldots\ldots②$$

$$\Rightarrow (1-x^2){y_1}^2 = m^2y^2$$

Differentiating above equation w.r.t. x , we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0 \qquad \ldots\ldots③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)]-\left(y_{n+1}x+n_{C_1}y_n1\right)-m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)-m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0\ldots\ldots④$$

To find $y_n(0)$: Putting$x=0$ in ①, ②and ③

$y(0)=1,\ y_1(0)=m$ and $y_2(0)=m^2$

Also putting $x=0$ in , we get

$$y_{n+2}(0)=(n^2+m^2)y_n(0)$$

Putting $n=1,2,3\ldots$ in the above equation, we get

$$y_3(0)=(1^2+m^2)y_1(0)$$

$$=(1^2+m^2)m \qquad \because y_1(0)=m$$

$$y_4(0)=(2^2+m^2)y_2(0)$$

$$=m^2(2^2+m^2) \qquad \because y_2(0)=m^2$$

$$y_5(0)=\left(3^2+m^2\right)y_3(0)$$

$$=m(1^2+m^2)(3^2+m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0)=\begin{cases} m^2(2^2+m^2)\ldots[(n-2)^2+m^2], & if\ n\ is\ even \\ m(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2], & if\ n\ is\ odd\end{cases}$$

Example 17 If $y=tan^{-1}x$, show that

$(1-x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$. Also find $y_n(0)$

Solution: Here $y=tan^{-1}x\ldots.①$

$$\Rightarrow y_1=\frac{1}{1+x^2} \quad \ldots\ldots②$$

$$y_2=\frac{-2x}{1+x^2}$$

$$\Rightarrow (1+x^2)y_2+2xy_1=0\ \ldots\ldots③$$

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1+x^2)+n_{C_1}y_{n+1}(2x)+n_{C_2}y_n(2)]+2\left(y_{n+1}x+n_{C_1}y_n1\right)=0$$

$$\Rightarrow y_{n+2}(1+x^2)+y_{n+1}2nx+n(n-1)y_n+2(y_{n+1}x+ny_n)=0$$

$$\Rightarrow (1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0\ldots\ldots④$$

To find $y_n(0)$: Putting $x=0$ in ①, ②and ③, we get

$y(0)=0\,,y_1(0)=1$ and $y_2(0)=0$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = -n(n+1)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = -1(2)y_1(0)$$

$$= -2 \qquad \because y_1(0) = 1$$

$$y_4(0) = -2(3)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = -3(4)y_3(0)$$

$$= -3(4)(-2) = 4!$$

$$y_6(0) = -4(5)y_4(0) = 0$$

$$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$$

$$\vdots$$

$$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)! \text{ and } \quad y_{2n}(0) = 0$$

Example18 If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}} \quad \ldots\ldots ②$$

Squaring both the sides, we get

$$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$$

$$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$$

Differentiating the above equation w.r.t. x, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$$

$$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0 \quad \ldots\ldots ③$$

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n1) = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$$y(0) = 0\,, y_1(0) = 0 \text{ and } y_2(0) = 2$$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.2^2.4^2 \ldots\ldots\ldots.(n-2)^2, & if\ n\ is\ even \end{cases}$$

Angle of intersection of two Curves

Let y = f (x) and y = g (x) be two given intersecting curves. Angle of intersection of these curves is defined as the acute angle between the tangents that can be drawn to the given curves at the point of intersection.

Let (x_1, y_1) be the point of intersection

Slope of the tangent drawn to the curve y = f (x) at (x_1, y_1)

i.e. $m_1 = \dfrac{df(x)}{dx}_{(x_1,y_1)}$

Similarly slope of the tangent drawn to the curve y = g(x) at (x_1, y_1)

i.e. $m_2 = \dfrac{dg(x)}{dx}_{(x_1,y_1)}$

The angle of intersection between two curves = The angle of intersection between the tangents to these curves. Therefore If α is the acute angle of intersection between two curves, then $\alpha = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

PROBLEMS:

1. For the curves x^2 =4y and y^2=4x, find the angle of intersection.

Solution:

To find the point of intersection of the curves, solve the equation x^2=4y andy^2=4x.

Consider the curve x^2=4y.

On squaring both sides of the equation, we have

$x^4 = 16y^2 \Rightarrow x^4 = 16(4x) \Rightarrow x^4 - 64x = 0 \quad \Rightarrow x(x^3 - 64) = 0$

⇒x=0 or

$x^3 - 64 = 0 \Rightarrow x^3$=64 $\quad \Rightarrow$ x=4.

The corresponding values of y are:

x=0⇒ $y^2 = 4(0) \quad \Rightarrow$ y=0

x=4⇒ $y^2 = 4(4) \quad \Rightarrow y^2 = 16 \quad \Rightarrow$y=4.

The points of intersection are (0,0) and (4,4) .

(i) Consider the point (4,4).

To find m_1, consider the curve $x^2 = 4y$.

On differentiating, we get

2x dx= 4dy $\quad \Rightarrow$dx=$\frac{2\,dy}{x} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$ ... (1)

$m_1$is the value of$\frac{dy}{dx}$ at the point (4,4), that is, $\frac{4}{2} = 2$.

To find m_2, consider the curve$y^2 = 4x$.

Ondifferentiating, we get

$\frac{dy}{dx} = \frac{2}{y}$ ... (2)

$m_2$is the value of$\frac{dy}{dx}$ at the point (4,4), that is,$\frac{2}{4} = \frac{1}{2}$

We know that, the angle between the two straight lines y=m_1x+c_1 and y=m_2x+c_2 is

$$\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right).$$

At the point (4,4),$tan\theta = \frac{m_1-m_2}{1+m_1m_2} \quad = \frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}} = \frac{2-\frac{1}{2}}{2} = \frac{3}{2} \times \frac{1}{2}$

$\Rightarrow \quad \tan\theta = \frac{3}{4} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right).$

(ii) Consider the point (0, 0).

$m_1$is obtained by substituting (0, 0) in (1).

i.e., $m_1 = \frac{0}{2} = 0 \Rightarrow \tan\theta = 0 \qquad \Rightarrow \theta = \tan^{-1}(0) = 0$

$m_2$is obtained by substituting (0, 0) in (2)

i.e., $m_2 = \frac{2}{0} = \infty \Rightarrow \tan\theta = \infty \qquad \Rightarrow \theta = \tan^{-1}(\infty) = \frac{\pi}{2}.$

Therefore, the values of ψ for the two curves are 0 and$\frac{\pi}{2}$.

Hence the angle of intersection is$\frac{\pi}{2}$.

That is, the curves cut orthogonally.

2. Find the angle at which the curves (1)x^2=ay and (2)$x^3 + y^3 = 3axy$ cut eah other.

Solution:

Substituting the value of y from $x^2 = ay \; in \; x^3 + y^3 = 3axy$

$\Rightarrow x^2 = ay \Rightarrow \qquad y = \frac{x^2}{a}.$

On substituting the value of y in $x^3 + y^3 = 3axy$, we have

$$\Rightarrow x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right) \quad \Rightarrow x^3 + \frac{x^5}{a^3} = 3x^3 \quad \Rightarrow \frac{x^5}{a^3} = 2x^3 \quad \Rightarrow \frac{x^5}{a^3} - 2x^3 = 0$$

$$\Rightarrow x^3\left(\frac{x^2}{a^3} - 2\right) = 0$$

$\Rightarrow \quad x^3 = 0 \;\; or \;\; \left(\frac{x^2}{a^3} - 2\right) = 0$

$\Rightarrow$ x= 0 $\;\; or \;\; x^3 = 2a^3 \;\; \Rightarrow \;\; x = (2)^{\frac{1}{3}} \; a$ z

On substituting these values of x in $x^2 = ay$, we have y=0 or y=a $(4)^{\frac{1}{3}}$

Hence, the curve cut at the points (0,0) and $\left\{a(2)^{\frac{1}{3}}, a\,(4)^{\frac{1}{3}}\right\}$.

On differentiating $x^2 = ay$, we get,

$2x\,dx = a\,dy. \Rightarrow 2x = a\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{a}$.

On differentiating $x^3 + y^3 - 3axy = 0$, we get,

$$3x^2dx + 3y^2\,dy - 3ax\,dy - 3aydy = 0$$

$$\Rightarrow 3y^2\,dy - 3ax\,dy = 3aydy - 3x^2dx$$

$\Rightarrow$ dy($3y^2 - 3ax$)=(3ay-$3x^2$) dx

$\Rightarrow$ dy($y^2 - ax$)=(ay-x^2)dx

$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$.

(i) The value of $\frac{dy}{dx}$ at $\left\{a(2)^{\frac{1}{3}}, a(4)^{\frac{1}{3}}\right\}$ for the curve is $2^{\frac{4}{3}}$

and for the second curve at the same point is 0.

$\therefore tan\theta = 2^{\frac{4}{3}} \Rightarrow \theta = \tan^{-1}\left\{2\left(2^{\frac{1}{3}}\right)\right\}$

(ii) The value of $\frac{dy}{dx}$ at (0,0) for both of the two curves is 0

That is, the two curves touch at the origin, y=0.

That is, the tangent is common to both the curves.

3. Find the condition that the curves at $ax^2 + by^2 = 1$, $a_1x^2 + b_1y^2 = 1$ shall cut orthogonally.

Solution:

Let the curves intersect at the point whose co-ordinates are (x_1, y_1)

$\therefore ax_1^2 + by_1^2 - 1 = 0$ and $a_1x_1^2 + b_1y_1^2 - 1 = 0$

$\therefore \frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a - a_1} = \frac{1}{ab_1 - a_1b} \ldots(1)$

On differentiating the equations of the curve, we get

$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by}$

$2a_1x + 2b_1y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{a_1x}{b_1y}$

The gradients of the tangents of the two curves at the points of intersection are:

$$-\frac{ax_1}{by_1}, -\frac{a_1x_1}{b_1y_1}.$$

These curves cut each other orthogonally.

Then, we know that,

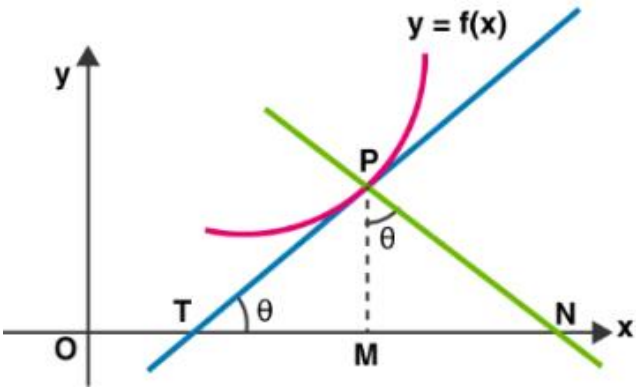
$$-\frac{ax_1}{by_1}\cdot\frac{-a_1x_1}{b_1y_1} = -1 \quad \Rightarrow \frac{aa_1x_1^2}{bb_1y_1^2} = -1 \quad \Rightarrow \frac{x_1^2}{y_1^2} = -\frac{bb_1}{aa_1}$$

But the value of $\frac{x_1^2}{y_1^2}$ from (1) is $\frac{b_1-b}{a-a_1}$

$$\Rightarrow \frac{b_1-b}{a-a_1} = -\frac{bb_1}{aa_1} \quad \Rightarrow \frac{b_1-b}{bb_1} = \frac{a_1-a}{aa_1} \Rightarrow \frac{1}{b}-\frac{1}{b_1} = \frac{1}{a}-\frac{1}{a_1} \quad \Rightarrow \frac{1}{a}-\frac{1}{b} = \frac{1}{a_1}-\frac{1}{b_1}$$

LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUB NORMAL

Let y = f (x) be the curve that is differentiable at a point P. Let the tangent and normal at P(x, y) to the curve meet at the x-axis at points T and N. M is the projection of P on the x-axis. In the figure below,



- PT is the length of the tangent
- PN is the length of the normal
- TM is the length of subtangent
- MN is the length of the subnormal

Let $\angle PTN = \theta$ and $\angle MPN = \theta$

Then $PM = y,$ $\tan\theta = \frac{dy}{dx}$, $\cot\theta = \frac{dx}{dy}$

Now length of the tangent $PT = |y \operatorname{cosec}\theta| = |y|\sqrt{1+cot^2\theta}$

$$= |y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$\therefore$ Length of the tangent $=|y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Now length of the normal $PN = |y\sec\theta| = |y|\sqrt{1+tan^2\theta} = |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\therefore$ Length of the normal $= |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Now length of the sub tangent $TM = |y\cot\theta| = \left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

$\therefore$ Length of the sub tangent = $\left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

Now length of the sub normal $MN = |y\tan\theta| = \left|y\frac{dy}{dx}\right|$

$\therefore$ Length of the sub normal = $\left|y\frac{dy}{dx}\right|$

<u>Formulae</u>:

Length of the tangent =$|y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Length of the normal $= |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Length of the sub tangent = $\left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

Length of the sub normal = $\left|y\frac{dy}{dx}\right|$

<u>Problems</u>

1.Show that, in the parabola $y^2 = 4ax$, the subtangent at any point is double the abscissa and the subnormal is constant.

<u>Proof:</u>

Differentiating the equation of the parabola,$y^2 = 4ax$, we have

$$2ydy = 4adx \qquad \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}.$$

The subtangent= $\frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{2a}{y}} = \frac{y^2}{2a} = 2x$

= double the abscissa

The subnormal= $y\,\frac{dy}{dx} = y\,\frac{2a}{y} = 2a$ (constant)

2. Show that the length of sub-normal at any point on the curve $xy = a^2$ varies as the cube of the ordinate of the point.

Proof: Equation of the curve is $xy = a^2 \Rightarrow y = \frac{a^2}{x}$

Differentiating w.r.t. x we get $\frac{dy}{dx} = -\frac{a^2}{x^2}$

Length of the sub normal = $\left|y\frac{dy}{dx}\right| = \left|y\left(-\frac{a^2}{x^2}\right)\right| = y.\frac{a^2}{\left(\frac{a^4}{y^2}\right)} = a^2y^3$

Thus the length of sub-normal at any point on the curve $xy = a^2$ varies as the cube of the ordinate

3. Show that at any point (x, y) on the curve $y = be^{x/3}$, the length of the sub-tangent is constant and the length of the sub-normal is $\frac{y^2}{a}$.

Equation of the curve is $y = be^{x/3} \Rightarrow \frac{dy}{dx} = b.e^{x/3}.\frac{1}{a} = \frac{y}{a}$

Length of the sub tangent = $\left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

$$= \frac{y}{\left(\frac{y}{a}\right)} = a = \text{constant}$$

Length of the sub normal = $\left|y\frac{dy}{dx}\right|$

$$= y.y/a = y^2/a$$

Find the length of sub-tangent and sub normal at a point of the curve $y = b.\sin\frac{x}{a}$.

Sol: Equation of the curve is $y = b.\sin\frac{x}{a} \Rightarrow \frac{dy}{dx} = b.\cos\frac{x}{a}.\frac{1}{a} = \frac{b}{a}.\cos\frac{x}{a}$

Length of the sub tangent = $\left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

$$= \frac{b.\sin\frac{x}{a}}{\frac{b}{a}.\cos\frac{x}{a}} = \left|a.\tan\frac{x}{a}\right|$$

Length of the sub normal = $\left|y\frac{dy}{dx}\right| = \left|bsin\frac{x}{a}\left(bcos\frac{x}{a}\right)\right| = \left|\frac{b^2}{2a}sin\frac{2x}{a}\right|$

5. At any point t on the curve x = a (t + sin t), y = a(1 – cos t). Find the lengths of the tangent, normal, sub-tangent and sub-normal.

Sol: Equation of the curve is x = a (t + sin t), y = a(1 – cos t)

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{a\sin t}{a(1+\cos t)} \qquad =\frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}$$

$$\frac{dy}{dx}=\tan\frac{t}{2}$$

Length of the tangent $=|y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$=\left|a(1-\cos t)\sqrt{1+\cot^2\frac{t}{2}}\right|$$

$$=\left|2a.\sin^2\frac{t}{2}.\operatorname{cosec}\frac{t}{2}\right| =\left|2a.\sin^2\frac{t}{2}.\frac{1}{\sin\frac{t}{2}}\right| =\left|2a\sin\frac{t}{2}\right|$$

Length of the normal $=|y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$=a(1-cost)\sqrt{1+tan^2\frac{t}{2}}=a\left(2sin^2\frac{t}{2}\right)sec\frac{t}{2}=2asin\frac{t}{2}tan\frac{t}{2}$$

Length of the sub tangent = $\left|y\frac{dx}{dy}\right| = \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|$

$$=\frac{a(1-cost)}{tan\frac{t}{2}}= 2asin\frac{t}{2}\,cos\frac{t}{2}=asint$$

Length of the sub normal = $\left|y\frac{dy}{dx}\right|$

$$=a(1-cost)\,tan\frac{t}{2}$$

6. Find the length of normal and sub-normal at a point on the curve $y=\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right)$.

Sol: Equation of the curve is $y=\frac{a}{2}\left(e^{x/a}+e^{-x/a}\right) \quad =a.\cosh\left(\frac{x}{a}\right)$

$$\Rightarrow\frac{dy}{dx}=a.\sinh\left(\frac{x}{a}\right)\frac{1}{a}=\sinh\frac{x}{a}$$

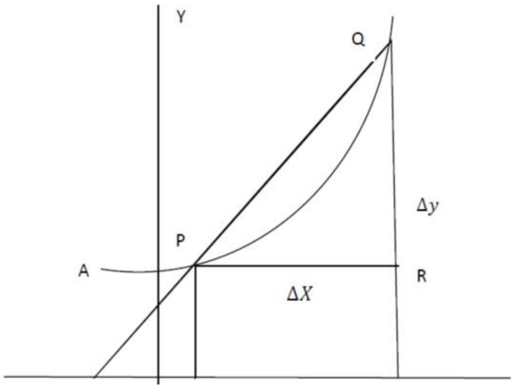
Length of the normal = $|y|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = acosh\frac{x}{a}\sqrt{1+sinh^2\frac{x}{a}} = acosh\frac{x}{a}cosh\frac{x}{a}$

$$= acosh^2\frac{x}{a}$$

Length of the sub normal = $\left|y\frac{dy}{dx}\right| = acosh\frac{x}{a}sinh\frac{x}{a} = \frac{a}{2}\left(2cosh\frac{x}{a}sinh\frac{x}{a}\right) = \frac{a}{2}sinh\frac{2x}{a}$

THE LENGTH OF AN ARC

Consider the following diagram:



Let P be any point (x,y) on the curve y=f(x). Let Q be a point very near P, so that the coordinates of Qare $(x+\Delta x, y+\Delta y)$.

Now, let S be the length of the arc AP,where A is a fixed point on the curve. Then, s+Δs is the length of the arc AQ, so that arc PQ=Δs.

As Δx tends to zero, Q approaches P.Then, the chord PQ and arc PQ become almost equal. Thus, the ultimate ratio of the arc PQ to the chord PQis unity,as $\Delta s \to 0$.

Now, from the right-angled triangle PQR,

$$(chord\ PQ)^2 = (PR)^2 + (QR)^2$$

$\Rightarrow (chord\ PQ)^2 = (\Delta x)^2 + (\Delta y)^2$

$\Rightarrow \left(\frac{chord\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$ [On dividing both sides by $(\Delta x)^2$]

$\Rightarrow \left(\frac{chord\ PQ}{arc\ PQ}\right)^2\left(\frac{arc\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$ [multiplyand divide by $(arc\ PQ)^2$ in L.H.S]

Taking the limits as $\Delta x \to 0$, we get

$\text{Lim}_{\Delta x\to 0}\frac{chord\ PQ}{arc\ PQ} = 1$, $\lim_{\Delta x\to 0}\frac{arc\ PQ}{\Delta x} = \frac{ds}{dx}$,and$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$.

$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

Simlarly, it may be shown that $\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$

NOTE:

ψ is the angle between OX and the tangent at the point (x,y)
From the figure, it easily follows that $\psi = \frac{dy}{ds}$

PROBLEM:

For the cycloid $x = a(1 - cos\theta), y = a(\theta + sin\theta)$, find $\frac{ds}{dx}$.

Solution:

Given that, $x = a(1 - cos\theta)$ *and* $y = a(\theta + sin\theta)$

$$\frac{dx}{d\theta} = asin\theta \ ; \ \frac{dy}{d\theta} = a(1 + cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{2cos^2(\theta/2)}{2sin(\theta/2)cos(\theta/2)} = \frac{cos(\theta/2)}{sin(\theta/2)} = \cot(\theta/2)$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \cot^2 \theta/2 = \text{cosec}^2 \theta/2$$

$$\Rightarrow \frac{ds}{dx} = \text{cosec}(\theta/2).$$

2. Find $\frac{ds}{dx}$ in the curve $y = a\ cosh\left(\frac{x}{a}\right)$.

Solution :

Given that, $y = a\ cosh\left(\frac{x}{a}\right)$ ...(1)

We know that, $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$.

On differentiating (1), we get

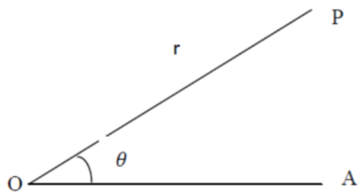
$$\frac{dy}{dx} = a\left(sin\,h\left(\frac{x}{a}\right) \times \left(\frac{x}{a}\right)\right) = sin\,h\left(\frac{x}{a}\right)$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(sin\,h\left(\frac{x}{a}\right)\right)^2 = 1 + sin^2h\left(\frac{x}{a}\right) = cosh^2\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{ds}{dx} = cosh\left(\frac{x}{a}\right)$$

POLAR COORDINATES

Consider the following diagram



The position of a point P on a plane can be indicated by stating:

(1) Its distance r from a fixed point 'O'.
(2) The inclination θ of OP to a fixed straight line through 'O'.

Here, r is called the radius vector and θ the vertorial angle, O the pole and OA the initial line, where r and θ are called the polar coordinates of P.

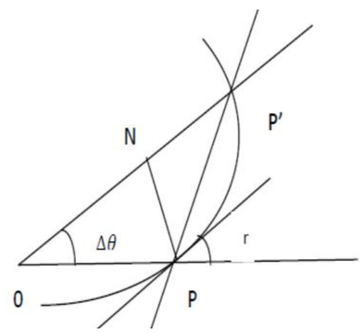
r is considered to be positive when measured away from O along the line bounding the vertorial angle and θ is considered to be positive when measured in the anticlock wise direction.

When converting polar co-ordinates to Cartesian or vice versa, it is customary to take the pole as the origin and initial line as the x-axis.

Then, the formulae for conversion are $x = r\cos\theta$ and $y = r\sin\theta$.

ANGLE BETWEEN THE RADIUS VECTOR AND THE TANGENT

Consider the following diagram. Let P, P′ be two neighbouring points on a curve. Let (r, θ) be the polar coordinates of P and $(r + \Delta r,\ \theta + \Delta\theta)$ be the polar coordinates of P′.



If we join P, P′ and draw PN perpendicular to OP′, we have

$PN = OP \sin \angle PON = r \sin \Delta\theta.$

Again, $PN = OP' - ON = r + \Delta r - r\cos\Delta\theta$

$\Rightarrow \quad = \Delta r + r(1 - \cos\Delta\theta)$

$\Rightarrow \quad = \Delta r + 2r \sin^2\left(\frac{\Delta\theta}{2}\right).$

Denote by φ the angle between the radius vector OP and the tangent at P. If we now let $\Delta\theta$ approach the limit zero, then

(1) The point P' will approach P

(2) The secant PP' will become the tangent PT in the limiting positions.

(3) The angle $PP'N$ will approach φ as a limit.

From the above diagram, we have

$$\tan PP'O = \frac{r\sin\Delta\theta}{\Delta r + 2r\sin^2\left(\frac{\Delta\theta}{2}\right)} \qquad = r\,\frac{\frac{\sin\Delta\theta}{\Delta\theta}}{\frac{\Delta r}{\Delta\theta} + \frac{r\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\sin\frac{\Delta\theta}{2}}$$

$$\lim_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta} = 1\ ,\ \lim_{\Delta\theta\to 0}\sin\frac{\Delta\theta}{2}\ ,\ \lim_{\frac{\Delta\theta}{2}\to 0}\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} = 1$$

$$\text{and} \lim_{\frac{\Delta\theta}{2}\to 0}\frac{\Delta r}{\Delta\theta} = \frac{dr}{d\theta}$$

$$\therefore \tan\varphi = \lim_{\Delta\theta\to 0}\tan PP'O = r\ \frac{1}{\frac{dr}{d\theta} + r.1.0} = r\frac{d\theta}{dr}$$

PROBLEMS:

1. Find the angle at which the radius vector cuts the curve $\frac{1}{r} = 1 + e\cos\theta$.

Solution:

Let $\emptyset$ be the angle between the radius vector and the tangent at the point at which the radius vector meets the curve.

On differentiating $\frac{1}{r} = 1 + e\cos\theta$ with respect to θ, we get

$$\frac{-1}{r^2}\frac{dr}{d\theta} = -e\sin\theta \Longrightarrow \frac{dr}{d\theta} = \frac{er^2}{1}\sin\theta.$$

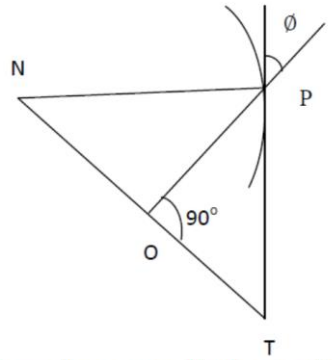
We know that, $\tan\emptyset = r\,\frac{d\theta}{dr}$.

$$\therefore \quad \tan\emptyset = \frac{r\,1}{e\sin\theta . r^2} = \frac{1}{r.e\sin\theta} = \frac{1+e\cos\theta}{e\sin\theta}.$$

$\therefore$ The required angle, $\emptyset = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$.

POLAR SUBTANGENT AND POLAR SUBNORMAL

Consider the following diagram



Draw a line NT through the pole perpendicular to the radius vector of the point P on the curve. If PT is the tangent and PN the normal to the curve at P, then

OT= Length of the polar sub tangent.

ON=Length of the subnormal of the curve at P.

Polar subtangent= $OT = OP\ tan\,\emptyset = r.r\ \frac{d\theta}{dr} = r^2\frac{d\theta}{dr}$.

Polar subnormal = ON $= \text{OP} \tan \angle\, \text{OPN} = \text{OP} \tan\, (\angle\text{TPN} - \angle\text{TPO})$

$$= \text{OP}\ \tan\,(\pi/_2 - \emptyset) = r\ \cot\,\emptyset = \frac{r}{\tan\,\emptyset} = \frac{r}{r\frac{d\theta}{dr}} = \frac{dr}{d\theta}.$$

Hence, Polar subtangent is$r^2\frac{d\theta}{dr}$ and Polar subnormal is $\frac{dr}{d\theta}$.

Problems:

1. Show that in the curve r=ae$^{\theta \cot \alpha}$

(i) The polar subtangent = r tan α.

(ii)The polar subnormal = r cot α.

Solution

Here, $r = ae^{\theta \cot \alpha}$

$\therefore \frac{dr}{d\theta} = ae^{\theta \cot \alpha} = r \cot \alpha$

Hence, the polar subnormal is(rcot α)

Also,$\frac{d\theta}{dr} = \frac{1}{r\ \cot\,\alpha}$

$\therefore\ r^2\frac{d\theta}{dr} = \dfrac{r^2}{r\ \cot\alpha} = r\ \tan\alpha$

Hence, the polar subtangentis $(r\ \tan\alpha)$

2. Show that in the curve r = aθ, the polar subtangent varies as the square of the radius vector and the polar subnormal is constant.

Solution;

Given that, r = aθ.

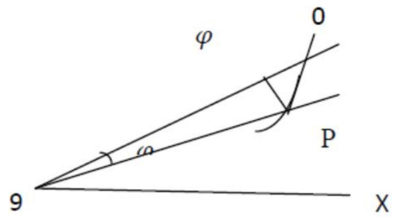
$\therefore \frac{dr}{d\theta} = a$,which is constant.

Again,$\frac{d\theta}{dr} = \frac{1}{a} \Rightarrow r^2\frac{d\theta}{dr} = \frac{r^2}{a}$.

Thus, the polar subtangent varies asther^2.

THE LENGTH OF ARC IN POLAR CO ORDINATES

Consider the following diagram



Let the coordinates of a point P on the curve be (r,θ).

Then, OP = r; and ∠AOP=θ.

Let the coordinates of a point Q on the curve very close to P be $(r + \Delta r, \theta + \Delta\theta)$.

Then, OQ= r+Δr, ∠QOA = θ+Δθ and $\angle POR = \Delta\theta$.

Let s be the length of the arc BP, where B is a fixed point on the curve. Then, the length of the arc BQ is $s + \Delta s$ and the length of the arc PQ is Δs.

Now, $PR = OP\ sin\Delta\theta = r\ sin\Delta\theta$.

$$OR = OP\ \cos\Delta\theta = r\ \cos\Delta\theta.$$

Also,$QR = r + \Delta r - r\cos\Delta\theta = r(1 - \cos\Delta\theta) + \Delta r = 2r\sin^2\frac{\Delta\theta}{2} + \Delta r$.

$PQ^2 = PR^2 + RQ^2$

$\Rightarrow (r\,sin\,\delta\theta)^2 + \{2r\,sin^2\frac{\Delta\theta}{2} + \Delta r\}^2$.

$$\therefore \left(\frac{\text{chord PQ}}{\Delta\theta}\right)^2 = (r\,\frac{\sin\Delta\theta}{\Delta\theta})^2 + \left\{\frac{2r\sin^2\frac{\Delta\theta}{2}}{\Delta\theta} + \frac{\Delta r}{\Delta\theta}\right\}^2 \quad = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left\{\frac{r\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\ \sin\frac{\Delta\theta}{2} + \frac{\Delta r}{\Delta\theta}\right\}^2$$

Passing to the limit as $\Delta\theta$ tends to zero, we get

$\frac{\sin\Delta\theta}{\Delta\theta} \to 1, \frac{\Delta r}{\Delta\theta} \to \frac{dr}{d\theta}, \sin\frac{\Delta\theta}{2} \to 0.$

$$\therefore \frac{\text{chord PQ}}{\Delta\theta} = \frac{\text{chord PQ}}{\text{arc PQ}} \cdot \frac{\text{arc PQ}}{\Delta\theta} \to 1 \cdot \frac{ds}{d\theta}$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

In the same way, it may be shown that $\left(\frac{ds}{dr}\right)^2 = \left(r\frac{d\theta}{dr}\right)^2 + 1.$

It is easily seen that $\cos\emptyset = \frac{dr}{ds}$ and $\sin\emptyset = r\,\frac{d\theta}{ds}$.

Problem:

1. Find $\frac{ds}{d\theta}$ and $\frac{ds}{dr}$ for the cardioid r = a (1+cos θ).

Solution;

Given that r = a (1+cos θ)

On differentiating the above equation, we have

$$\frac{dr}{d\theta} = a\,(0 - \sin\theta) = -a\sin\theta.$$

Also, $r.\frac{d\theta}{dr} = \frac{r}{-a\ \sin\theta} = \frac{a\,(1+\cos\theta)}{-a\ \sin\theta} = -\frac{1+\cos\theta}{\sin\theta} = -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = -\cot\theta/2$

We know that, $\left(\frac{ds}{dr}\right)^2 = 1 + \left(r\frac{d\theta}{dr}\right)^2$

$= 1 + \cot^2(\theta/2) = \operatorname{cosec}^2\theta/2$.

$\therefore \frac{ds}{dr} = \operatorname{cosec}\ \theta/2.$

We know that, $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

$$= a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta$$

$$= a^2 + a^2(\cos^2\theta + \sin^2\theta) + 2a^2\cos\theta$$

$$= a^2 + a^2 + 2a^2\cos\theta = 2a^2(1+\cos\theta)$$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 \qquad = 2a^2[2\cos^2(\theta/2)] = 4\,a^2\cos^2(\theta/2).$$

$$\Rightarrow \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$$

2. Find $\frac{ds}{d\theta}$ and $\frac{ds}{dr}$ for the curve $r = a\,(1\text{-}\cos\theta)$

Solution;

Given that, $r = a\,(1 - \cos\theta) \Rightarrow \frac{dr}{d\theta} = a\sin\theta.$

Consider, $r.\frac{d\theta}{dr} = \frac{a\,(1-\cos\theta)}{a\sin\theta}$

$$= \frac{2a\sin^2\theta/2}{2a\sin\theta/2\,\cos\theta/2} = \tan\theta/2$$

We know that, $\left(\frac{ds}{dr}\right)^2 = 1 + \left(r\,\frac{d\theta}{dr}\right)^2$

$$= 1 + \tan^2\theta/2 = \sec^2\theta/2.$$

$$\therefore \left(\frac{ds}{dr}\right) = \sec\theta/2$$

We know that, $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$

$$= a^2 + a^2\cos^2\theta + a^2\sin^2\theta - 2a^2\cos\theta$$

$$= a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a^2\cos\theta$$

$$= 2a^2 - 2a^2\cos\theta = 2a^2(1-\cos\theta) = 4a^2\sin^2\theta/2$$

$$\therefore \left(\frac{ds}{d\theta}\right) = 2a\sin\theta/2.$$

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2. S. Arumugam, A.T. Issac, Calculus, New Gamma Publications, Revised Edition, 2011.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – Differential Calculus (Continued)– SMTA1106

UNIT 2

DIFFERENTIAL CALCULUS

INTRODUCTION

In the late seventeenth century the invention of Calculus by Newton and Leibnitz has turned the house of math into a metropolis. Differential geometry and curvature were natural applications for the calculus because they provided words to its music. More specifically, Calculus methods of infinitesimals and limits were the perfect tools for the problem of curvature because most curves have a different degree of bending at every point. The use of infinitesimals to study rates of change can be found in Indian Mathematics, Perhaps as early as 500 AD, when the astronomer and Mathematician Aryabhata (476 – 550) used infinitesimals to study the motion of the moon. The motivation of optics in Differential geometry yielded concept of involutes and evolutes (Huygens in 1673) and later envelope, a representative of family of curves.

Curvature

Nature is too beautiful for words. Many curves in the plane and in space are simply beautiful. Mathematicians have developed several ways of describing them. One of the elegant method of describing a curve is to say that how much the curve "bends" at each point. This measure of bending is known by the technical word "Curvature". In many practical problems, we are concerned with comparision of bending of two curves or bending of a curve at its different points, for example, is laying the rail tracks and calculating the maximum speed that a train can have when it turns or designing high ways or constructing curved focal planes of telescope.

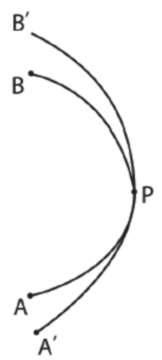


Fig 2.1

For getting the idea of Curvature let us consider two curves APB and $A'PB'$ as shown in the figure. The curve APB is bending more rapidly than $A'PB'$ in the neighborhood of P.

In otherwords it may be said that the curvature of APB is greater than that of $A'PB'$. Here if we consider the curves of arcs of circles then it is very clear that radius of APB is lesser than that of $A'PB'$.

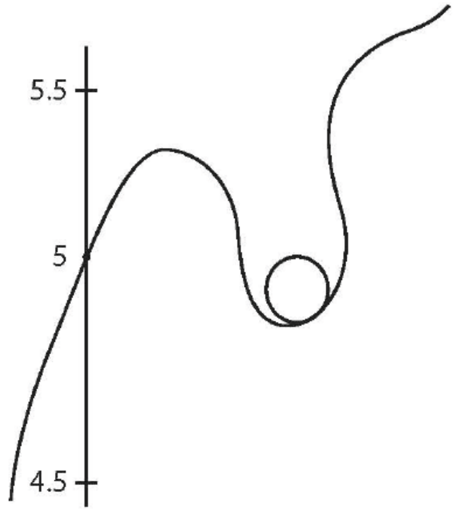


Fig 2.2

The curvature of a given curve at a particular point is the curvature of the approximating circle at the point. The radius of curvature of the curve is defined as the radius of the approximating circle. This radius changes as we move along the curve. The approximating circle is said to be circle of curvature. The formal definitions of above terms can be given as follows.

Definition of Curvature

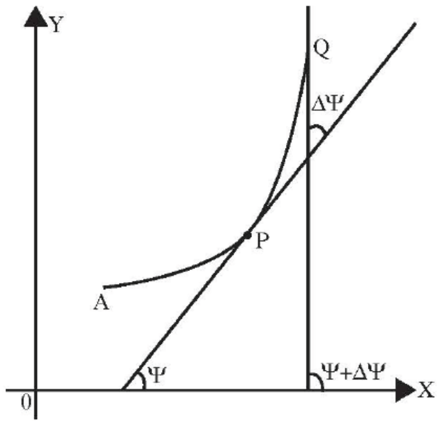


Fig 2.3

Let P and Q be any two close points on a plane curve. Let the arcual distances of P and Q measured from a fixed point A on the given curve be s and $s + \Delta s$, so that PQ (The arcual length of PQ) is Δs.

Let the tangent at P and Q to the curve make angles Ψ and $\Psi + \Delta\Psi$ with a fixed line in the plane of the curve, say the x-axis.

Then the angle between the tangents at P and Q is $\Delta\Psi$.

Thus for a change of Δs in the arcual length of the curve, the direction of the tangent to the curve changes by $\Delta\Psi$.

Hence $\frac{\Delta\Psi}{\Delta s}$ is the average rate of bending of the curve (or average rate of change of direction of the tangent to the curve in the arcual interval PQ) or average curvature of the arc PQ.

$\therefore \underset{\Delta s \to 0}{\text{limit}} \left(\frac{\Delta\Psi}{\Delta S}\right) = \frac{d\Psi}{ds}$ is the rate of bending of the curve with respect to arcual distance at P called the curvature of the curve at the point P and is denoted by K.

For example, Let us find the curvature of a circle of radius at any point on it.

Let the arcual distances of points on the circle be measured from A, the lower point of the circle and let the tangent at A be chosen as the x-axis. Let $AP = s$ and let the tangent at P makes an angle Ψ with the x-axis

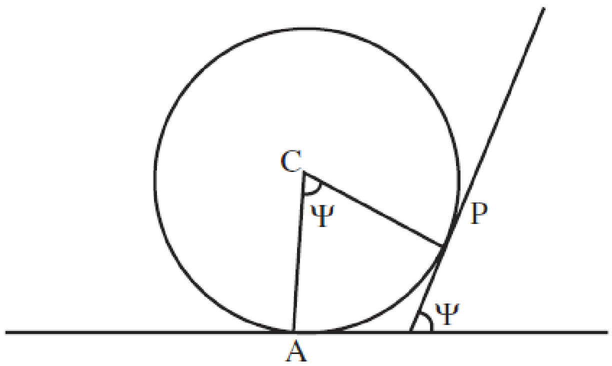


Fig 2.4

Then s = a $\hat{ACP}$

$= a\Psi$

or $\Psi = \frac{1}{a}s$ [$\because$ The angle between CA and CP equals the angle between the respective perpendicular AT and PT.]

$$\frac{d\Psi}{ds} = \frac{1}{a}$$

Thus the Curvature of a circle at any point on it equals the reciprocal of its radius. Equivalently, the radius of a circle equals the reciprocal of the curvature at any point on it.

Radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature of the curve at that point and denoted by ρ Thus $\rho = \frac{1}{K} = \frac{ds}{d\Psi}$.

To find K or ρ of a curve at any point on it, we should know the relation between s and Ψ for that curve, which is not easily derivable in most cases.

Some Basic Results

Let P(x, y) and Q $\left(x+\Delta x,\ y+\Delta y\right)$ be any two close points on a curve $y = f\left(x\right)$. Let $AP = s$ and $AQ = s + \Delta s$ where A is a fixed point on the curve. Let a chord PQ make an angle θ with the x-axis.

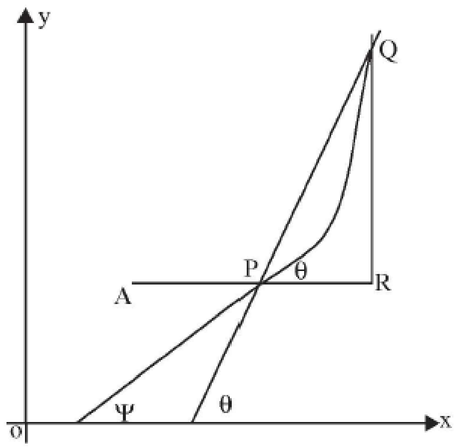


Fig 2.5

From $\Delta PQR,\ sin\theta = \frac{RQ}{\text{Chord PQ}} = \frac{RQ}{\Delta s}.\frac{\Delta s}{\text{Chord PQ}}$ (1)

where $PQ = \Delta s$

$$= \frac{\Delta y}{\Delta s}.\frac{\Delta s}{\text{Chord PQ}}$$

and

$$\cos\theta = \frac{PR}{\text{Chord PQ}} = \frac{\text{Chord PR}}{\Delta s}.\frac{\Delta s}{\text{Chord PQ}}$$

$$= \frac{\Delta x}{\Delta s}.\frac{\Delta s}{PQ} \qquad (2)$$

when Q approaches P chord PQ $\to$ tangent at P and hence $\theta \to \Psi$. Also $\frac{\Delta s}{PQ} \to 1$.

Thus in the limiting case when $Q \to P$, (1) and (2) becomes

$$\sin\Psi = \frac{dy}{ds} \; and \; \cos\Psi = \frac{dx}{ds}$$

$$\therefore \tan\Psi = \frac{dy}{dx}$$

I. Formula for Radius of Curvature in Cartesian co-ordinates

Let Ψ be the angle made by the tangent at any point (x, y) on the curve $y = f(x)$.

Then $\tan\Psi = \frac{dy}{dx}$ ______(1)

Differentiating both sides of (1) w.r.t x, we get

$$\sec^2\Psi \frac{d\Psi}{dx} = \frac{d^2y}{dx^2}$$

i.e, $$\sec^2\Psi \frac{d\Psi}{ds} . \frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\Psi \frac{1}{\rho} .\sec\Psi = \frac{d^2y}{dx^2} \qquad \left[\because \cos\Psi = \frac{dx}{ds}\right] \left[\because \frac{1}{\cos\Psi} = \sec\Psi\right]$$

$$\therefore \quad \rho = \frac{\sec^3\Psi}{\frac{d^2y}{dx^2}} = \frac{\left(\sec^2\Psi\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\tan^2\Psi\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \left(\because \sec^2\psi = 1+\tan^2\psi\right)$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \text{by eqn (1)}$$

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

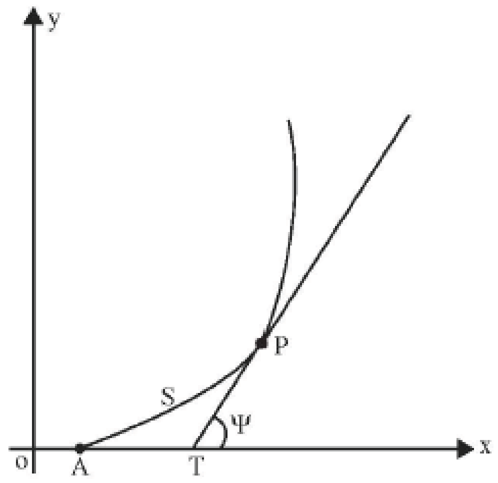


Fig 2.6

Note :

This formula does not hold good where the tangent at the point (x, y) is parallel to y axis. In that case $\frac{dy}{dx}$ is not defined. Since the value of ρ is independent of the choice of axis of co-ordinates, in this case we take the formula for ρ as

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

II. Formula for Radius of Curvature in Parametric co-ordinates

Let the parametric equation of the curve be $x = f(t) \text{ and } y = \phi(t)$.

Then $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Let $$x' = \frac{dx}{dt} \; and \; y' = \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right)$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{y'}{x'}\right)}{x'}$$

$$= \frac{1}{x'}\left(\frac{x'\frac{d}{dt}(y') - y'\frac{d}{dt}(x')}{x'^2}\right) \quad \text{Using Quotient Rule of differention}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x'^3}\left(x'y'' - y'x''\right)$$

$$\therefore \quad \rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\left(\frac{y'}{x'}\right)^2\right)^{\frac{3}{2}}}{\frac{1}{x'^3}\left(x'y'' - y'x''\right)}$$

$$= \frac{x'^3 \left(\frac{x'^2 + y'^2}{x'^2} \right)^{\frac{3}{2}}}{x'y'' - y'x''}$$

$$= x'^3 \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{\left(x'^2\right)^{\frac{3}{2}} \left(x'y'' - y'x''\right)}$$

$$= \frac{x'^3 \left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'^3 \left(x'y'' - y'x''\right)}$$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{\left(x'y'' - y'x''\right)}$$

Examples

1. Find the Curvature and radius of curvature of $x^2 + y^2 = 25$.

Solution:

$x^2 + y^2 = 25$ represents a circle of radius 5. We know that the curvature of a circle of radius r is $\frac{1}{r}$

Hence curvature of $x^2 + y^2 = 25$ is $\frac{1}{5}$

Also the radius of curvature is the reciprocal of the curvature

$\therefore$ Radius of curvature of the circle

$x^2 + y^2 = 25$ is 5.

2. Find the radius of curvature at any point (x, y) on the curve $y = c \log \sec\left(x / c\right)$.

Solution:

$$y = c \log \sec\left(x / c\right)$$

Differentiating y w.r.t x we get

$$y' = c\frac{1}{\sec(x/c)}.\sec(x/c)\tan(x/c)1/c$$

$$= \tan\left(\frac{x}{c}\right)$$

$$y'' = \sec^2\left(\frac{x}{c}\right).\frac{1}{c}$$

Radius of curvature $= \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{|y''|}$

$$= \frac{\left(1+\tan^2\left(\frac{x}{c}\right)\right)^{\frac{3}{2}}}{\frac{1}{c}\sec^2\left(\frac{x}{c}\right)}$$

$$= \frac{\left(\sec^2\left(\frac{x}{c}\right)\right)^{\frac{3}{2}}}{\frac{1}{c}\sec^2\left(\frac{x}{c}\right)}$$

$$= \frac{c\ \sec^3\left(\frac{x}{c}\right)}{\sec^2\left(\frac{x}{c}\right)} = c\sec\left(\frac{x}{c}\right).$$

3. Find the radius of curvature at (a, 0) of the curve $xy^2 = a^3 - x^3$.

Solution:

Differentiating $xy^2 = a^3 - x^3$ with respect to x we get,

$$2xyy' + y^2 = -3x^2$$

$$y^1 = \frac{-\left(3x^2+y^2\right)}{2xy}$$

$$At(\text{a},0), \quad y' = \infty.$$

Hence we find $\frac{dx}{dy} = \frac{-2xy}{3x^2+y^2}$

$$\therefore \left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

Now $$\frac{d^2x}{dy^2} = \frac{-\left[2\left(3x^2+y^2\right)\left(x+y\frac{dx}{dy}\right)-2xy\left(6x\frac{dx}{dy}+2y\right)\right]}{\left(3x^2+y^2\right)^2}$$

$$(\text{a},0), \frac{d^2x}{dy^2} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

$$= \frac{1}{\left|\frac{-2}{3a}\right|} = \frac{3a}{2}.$$

4. Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy.$

Solution:

Differentiating $x^3 + y^3 = 3axy$ with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx}\left(y^2 - ax\right) = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$At\left(\frac{3a}{2},\frac{3a}{2}\right), \frac{dy}{dx}=\frac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=\frac{\frac{-3}{4}a^2}{\frac{3}{4}a^2}=-1$$

$$\frac{d^2y}{dx^2}=\frac{\left(y^2-ax\right)\left(a\frac{dy}{dx}-2x\right)-\left(ay-x^2\right)\left(2y\frac{dy}{dx}-a\right)}{\left(y^2-ax\right)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}=\frac{\left[\left[\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)\right]\left(-a-2\left(\frac{3a}{2}\right)\right)-\left(a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2\right)\left(-2\left(\frac{3a}{2}\right)-a\right)\right]}{\left(\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)\right)^2}$$

$$=\frac{\frac{3}{4}a^2\left(-a-3a\right)-\left(\frac{-3a^2}{4}\right)\left(-3a-a\right)}{\frac{9a^4}{16}}$$

$$=\frac{\left(-3a^3-3a^3\right)}{9a^4}\times 16=\frac{-32}{3a}$$

$$\rho=\frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''}=\frac{\left(1+1\right)^{\frac{3}{2}}}{\frac{32}{3a}}=\left(\frac{2\sqrt{2}}{32}\right)3a$$

$$|\rho|=\frac{3\sqrt{2}}{16}a.$$

5. Show that for the curve $y=\frac{ax}{a+x}$, the radius of curvature ρ at (x, y) related as $\left(\frac{2\rho}{a}\right)^{\frac{2}{3}}=\frac{x^2}{y^2}+\frac{y^2}{x^2}$.

Solution:

The given equation is $y=\frac{ax}{a+x}$;

Differentiating y w.r.t to x, we get

$$y' = \frac{(a+x).a - ax(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2} = \frac{y^2}{x^2}\left[\because \frac{y}{x} = \frac{a}{a+x}\right]$$

$$y'' = \frac{(a+x)^2(0) - a^2(2(a+x))}{(a+x)^4} = \frac{-2a^2}{(a+x)^3}$$

$$= \frac{-2}{a}\frac{y^3}{x^3}$$

$$\rho = \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{|y''|}$$

$$\rho = \frac{\left(1+\frac{y^4}{x^4}\right)^{\frac{3}{2}}}{\left|\frac{-2}{a}\frac{y^3}{x^3}\right|} = \frac{\left(1+\frac{y^4}{x^4}\right)^{\frac{3}{2}} \cdot \frac{x^3}{y^3}}{\frac{2}{a}}$$

$$\rho = \frac{a}{2}\left[\left(1+\frac{y^4}{x^4}\right)^{\frac{3}{2}}\frac{x^3}{y^3}\right]$$

$$\frac{2\rho}{a} = \left(1+\frac{y^4}{x^4}\right)^{\frac{3}{2}}\frac{x^3}{y^3}$$

$$\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left[\left(1+\frac{y^4}{x^4}\right)^{\frac{3}{2}}\frac{x^3}{y^3}\right]^{\frac{2}{3}} = \left(1+\frac{y^4}{x^4}\right)\left(\frac{x^2}{y^2}\right)$$

$$\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \frac{x^2}{y^2} + \frac{y^2}{x^2}.$$

6. Find the radius of curvature at the point $\left(a\cos^3\theta, a\sin^3\theta\right)$ on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution:

The parametric equations of the given curve are $x = acos^3\theta$ and $y = a\sin^3\theta$

Differentiating twice with respect to θ,

$$\dot{x} = \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\dot{y} = \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -3a\left(\cos^3\theta + 2\cos\theta\left(-\sin\theta\right)\sin\theta\right)$$

$$= -3a\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)$$

$$\ddot{y} = \frac{d^2y}{d\theta^2} = 3a\left(\sin^2\theta(-\sin\theta) + 2\sin\theta\cos^2\theta\right)$$

$$= 3a\left(-\sin^3\theta + 2\sin\theta\cos^2\theta\right)$$

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \frac{\left(9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta\right)^{\frac{3}{2}}}{\left(-3a\cos^2\theta\sin\theta\right)\left(3a\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)\right) + \left(3a\sin^2\theta(\cos\theta)\right)\left(-3a\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)\right)}$$

$$= \frac{\left(9a^2\right)^{\frac{3}{2}}\left(\sin^2\theta\cos^2\theta\right)^{\frac{3}{2}}\left(\cos^2\theta + \sin^2\theta\right)^{\frac{3}{2}}}{9a^2\cos^2\theta\sin^2\theta\left(-\left(\cos^2\theta + \sin^2\theta\right)\right)}$$

$$= 3a\sin\theta\cos\theta.$$

7. Show that the radius of curvature at the point ‘θ’ on the curve $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.

Solution:

Differentiating with respect to ‘θ’, we get

$$\dot{x} = 3a\left(-\sin\theta\right) - a\left(-3\sin 3\theta\right)$$
$$= -3a\sin\theta + 3a\sin 3\theta$$

$$\dot{y} = 3a\cos\theta - a\left(3\cos 3\theta\right)$$
$$= 3a\cos\theta - 3a\cos 3\theta$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3a\left(\cos\theta - \cos 3\theta\right)}{3a\left(\sin 3\theta - \sin\theta\right)}$$

$$=\frac{2\sin 2\theta \sin\theta}{2\cos 2\theta \sin\theta}=\tan 2\theta$$

$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}(\tan 2\theta).\frac{d\theta}{dx}$$

$$=2\sec^2 2\theta.\frac{1}{3a(\sin 3\theta-\sin\theta)}$$

$$=\frac{2\sec^2 2\theta}{3a(2\cos 2\theta \sin\theta)}$$

$$=\frac{2\sec^2 2\theta}{6a\cos 2\theta \sin\theta}$$

$$=\frac{\sec^2 2\theta}{3a\sin\theta}$$

$$\rho=\frac{(1+y'^2)^{\frac{3}{2}}}{|y''|}=\frac{(1+\tan^2 2\theta)^{\frac{3}{2}}}{\sec^3 2\theta}3a\sin\theta$$

$$=3a\sin\theta.$$

EXERCISES

Part – A

1. Find the radius of curvature at the point $\left(\frac{1}{4},\frac{1}{4}\right)$ on the curve $\sqrt{x}+\sqrt{y}=1.$
2. Find the radius of curvature at $x=\frac{\pi}{2}$ on the curve $y = 4\sin x.$
3. Find the radius of curvature of $y = \log\sin x.$
4. Show that the radius of curvature at any point of the catenary $y=c\cosh(x/c)$ is $\frac{y^2}{c}.$
5. Find the radius of curvature at any point of the curve $x=a\cos\theta, y=a\sin\theta.$
6. Find the radius of curvature for the curve $x = at^2, y = 2at.$
7. Find the radius of curvature on $y=e^x$ at the point where the curve cuts the y-axis

Part – B

1. Find the radius of curvature for $y=\frac{\log x}{x}$ at $x=1$

2. Find the radius of curvature at any point of the parabola $y^2 = 4ax$.

3. Show that the measure of curvature of the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ at any point (x, y) on it is $\frac{ab}{2(ax+by)^{\frac{3}{2}}}$.

4. Find the radius of curvature of the curve $y = x^2(x-3)$ at the points where the tangent is parallel to the x – axis

5. Find the radius of curvature at θ on the curve $x = a(\theta - \sin\theta), y = a(1-\cos\theta)$.

6. For the curve x = $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ prove that the radius of curvature is $a\theta$

7. Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$ is $4a\cos\frac{\theta}{2}$

ANSWER

Part A

1. $\frac{1}{\sqrt{2}}$

2. $\frac{1}{4}$

3. $\text{cosec}\, x$

5. a

6. $2a\left(1+t^2\right)^{\frac{3}{2}}$

7. $2\sqrt{2}$

Part B

1. $\frac{2\sqrt{2}}{3}$

2. $\frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}}$

4. $\frac{1}{b}$

5. $4a\sin\frac{\theta}{2}$

8. $\sqrt{2}\,e^{t}$

CENTRE AND CIRCLE OF CURVATURE

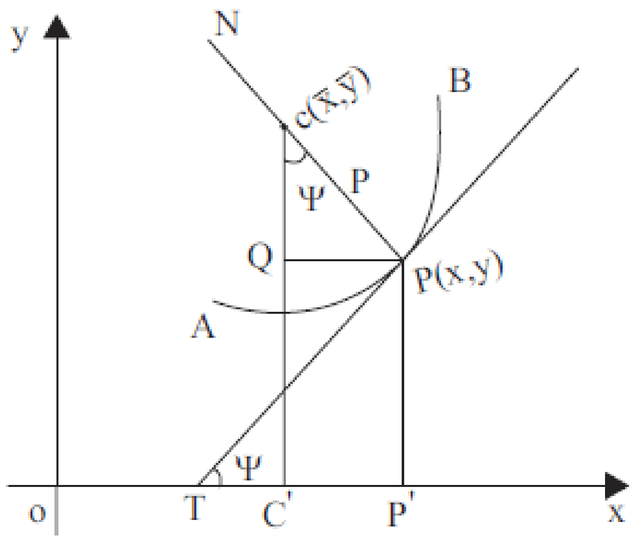


Fig 2.7

Let APB be a curve $y=f(x)$ and P be a point (x, y) on the curve $y=f(x)$. Draw the tangent TP and the normal PN at P(x, y). Along PN, cut off a length PC = ρ, such that C and the curve lie on the same side of the tangent TP. Note that ρ is the radius of curvature of the curve at P. The point C is called the centre of curvature at P for the curve.

The circle whose centre is at C and radius ρ is called the circle of curvature.

Let $C(\overline{x}, \overline{y})$ be the co-ordinates of the centre of curvature of the curve at point P(x, y).

$$\begin{aligned} Then \quad \bar{x} &= OC' \\ &= OP' - C'P' \\ &= OP' - PQ \end{aligned}$$

$$From\, \Delta CPQ, \sin\Psi = \frac{PQ}{CP}$$

$$= \frac{PQ}{\rho} \qquad (\because CP = \rho)$$

$$i.e, \qquad PQ = \rho \sin\Psi$$

$$\therefore \bar{x} = x - \rho\sin\Psi$$

$$= x - \frac{\left(1 + y_1^{\,2}\right)^{\frac{3}{2}}}{y_2} . \sin\Psi$$

We know that $\tan\Psi = \frac{dy}{dx} = y_1$

Hence $\sin\Psi = \frac{\sin\Psi}{\cos\Psi} . \cos\Psi$

$$= \frac{\tan\Psi}{\sec\Psi} = \frac{\tan\Psi}{\sqrt{1 + \tan^2\Psi}}$$

$$i.e, \sin\Psi = \frac{y_1}{\sqrt{1 + y_1^{\,2}}}$$

$$\cos\Psi = \frac{1}{\sec\Psi} = \frac{1}{\sqrt{1 + \tan^2\Psi}}$$

$$= \frac{1}{\sqrt{1 + y_1^{\,2}}}$$

$$\therefore \bar{x} = x - \frac{\left(1 + y_1^{\,2}\right)^{\frac{3}{2}}}{y_2} \frac{y_1}{\sqrt{1 + y_1^{\,2}}}$$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^{\,2}\right)$$

$$\begin{aligned} Also \quad \bar{y} &= C'C \\ &= P'P + QC \\ &= y + QC \end{aligned}$$

$$from \quad \Delta CPQ$$

$$cos\Psi = \frac{QC}{CP} = \frac{QC}{\rho}\left(\therefore CP = \rho\right)$$

$$i.e, \qquad QC = \rho\cos\Psi$$

$$\therefore \bar{y} = y + \rho\cos\Psi$$

$$= y + \frac{\rho}{\sec\Psi}$$

$$= y + \frac{\rho}{\sqrt{1+\tan^2\Psi}}$$

We know that $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}, \quad y_1 = \frac{dy}{dx} \ and \ y_2 = \frac{d^2y}{dx^2}$

$$\bar{y} = y + \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}}$$

$$= y + \frac{\left(1+y_1^2\right)}{y_2}$$

Therefore the equation of circle curvature is given by

$$\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2$$

$$where \quad \bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$$

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

Example 1

Find the centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at the point $(1,-1)$.

Solution:

Given $\quad y = x^3 - 6x^2 + 3x + 1$

$$y_1 = 3x^2 - 12x + 3$$

$$y_{1(1,-1)} = 3 - 12 + 3$$

$$= -6$$

$$y_2 = 6x - 12$$

$$y_{2(1,-1)} = 6 - 12 = -6$$

$$\therefore \quad \bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^{\,2}\right)$$

$$\bar{x}_{(1,-1)} = 1 - \frac{6}{6}\left[1 + (-6)^2\right]$$

$$= -36$$

$$\bar{y} = y + \frac{1}{y_2}\left(1 + y_1^{\,2}\right)$$

$$\bar{y}_{(1,-1)} = -1 - \frac{1}{6}\left[1 + (-6)^2\right]$$

$$= \frac{-6 - 37}{6} = \frac{-43}{6}$$

$$\therefore \text{ Centre of curvature } \left(\bar{x}, \bar{y}\right) = \left(-36, \frac{-43}{6}\right)$$

Example 2

Find the equation of the circle of curvature of the rectangular hyperbola xy = 12 at the point (3, 4).

Solution:

Given xy = 12

Differentiating with respect to x

$$xy_1 + y.1 = 0$$

$$y_1 = \frac{-y}{x}$$

$$y_{1(3,4)} = \frac{-4}{3}$$

$$y_2 = -\left[\frac{x.y_1 - y.1}{x^2}\right]$$

$$y_{2(3,4)} = -\left[\frac{3.\left(\frac{-4}{3}\right)-4}{9}\right]$$

$$= \frac{8}{9}$$

$$\therefore \bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$$

$$= 3 + \frac{4/3}{8/9}\left(1+\frac{16}{9}\right)$$

$$= 3 + \frac{25}{6}$$

$$= \frac{43}{6}$$

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

$$= 4 + \frac{1}{8/9}\left(\frac{25}{9}\right)$$

$$= 4 + \frac{9}{8}\left(\frac{25}{9}\right)$$

$$= \frac{57}{8}$$

Radius of curvature $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$

$$= \frac{\left(1+\frac{16}{9}\right)^{\frac{3}{2}}}{8/9}$$

$$= \frac{\left(25/9\right)^{\frac{3}{2}}}{8/9}$$

$$= \frac{125}{24}$$

Circle of curvature is

$$\left(x-\bar{x}\right)^2+\left(y-\bar{y}\right)^2=\rho^2$$

$$\left(x-\frac{43}{6}\right)^2+\left(y-\frac{57}{8}\right)^2=\left(\frac{125}{24}\right)^2$$

Example 3

Find the centre and circle of curvature for the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}\,at\left(\frac{a}{4},\frac{a}{4}\right)$

Solution:

Given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiating with respect to x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1=0$$

$$(or)\qquad y_1=-\sqrt{\frac{y}{x}}$$

Hence $y_{1\left(\frac{a}{4},\frac{a}{4}\right)}=-\sqrt{\frac{a/4}{a/4}}=-1$

Differentiating y_1 with respect to x, we get

$$y_2=-\left[\frac{\sqrt{x}.\frac{1}{2\sqrt{y}}y_1-\sqrt{y}.\frac{1}{2\sqrt{x}}}{x}\right]$$

$$y_{2\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{-\left(\frac{-1}{2}-\frac{1}{2}\right)}{\frac{a}{4}}=\frac{4}{a}$$

$$\rho=\frac{\left(1+y_1{}^2\right)^{\frac{3}{2}}}{y_2}=\frac{\left(1+(-1)^2\right)^{\frac{3}{2}}}{4/a}$$

$$= \frac{a}{4} 2\sqrt{2} = \frac{a}{\sqrt{2}}$$

Centre of curvature at $\left(\frac{a}{4}, \frac{a}{4} \right)$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^{\ 2}\right)$$
$$= \frac{a}{4} + \frac{a}{4}(1+1)$$
$$= \frac{3a}{4}$$
$$\bar{y} = y + \frac{1}{y_2}\left(1 + y_1^{\ 2}\right)$$
$$= \frac{a}{4} + \frac{a}{4}(1+1)$$
$$= \frac{3a}{4}$$

Hence the centre of curvature is $\left(\bar{x}, \bar{y}\right) = \left(\frac{3a}{4}, \frac{3a}{4} \right)$

Equation of circle of curvature is

$$\left(x - \bar{x}\right)^2 + \left(y - \bar{y}\right)^2 = \rho^2$$
$$\Rightarrow \left(x - \frac{3a}{4} \right)^2 + \left(y - \frac{3a}{4} \right)^2 = \left(\frac{a}{\sqrt{2}} \right)^2 = \frac{a^2}{2}$$

Example 4

Find the radius and centre of curvature of any point on $x = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right), \ y = a\sec\theta$

Solution:

$$Given \quad x = a\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right), \qquad y = a\sec\theta$$
$$\frac{dx}{d\theta} = a.\frac{1}{\tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right)} \sec^2\left(\frac{\pi}{4} + \frac{\theta}{2} \right).\frac{1}{2}$$

$$= \frac{a\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} \cdot \frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \frac{a}{\sin\left(\frac{\pi}{2}+\theta\right)} \qquad [\because 2\sin A\cos A = \sin 2A]$$

$$= \frac{a}{\cos\theta}$$

$$\frac{dy}{d\theta} = a\sec\theta.\tan\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{a\sec\theta\tan\theta}{\frac{a}{\cos\theta}}$$

$$= a\sec\theta\tan\theta \times \frac{\cos\theta}{a}$$

$$= \tan\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \sec^2\theta\frac{d\theta}{dx}$$

$$= \sec^2\theta\frac{\cos\theta}{a} = \frac{\sec\theta}{a}$$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1{}^2\right)$$

$$= a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \frac{\tan\theta}{\frac{\sec\theta}{a}}\left(1+\tan^2\theta\right)$$

$$= a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \frac{a\sin\theta}{\cos\theta.\sec\theta}\sec^2\theta$$

$$= a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - \frac{a\sin\theta}{\cos\theta}.\sec\theta$$

$$= a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - a\tan\theta\sec\theta$$

$$\bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right)$$

$$= a\sec\theta + \frac{1}{\left(\frac{\sec\theta}{a}\right)}\left(1 + \tan^2\theta\right)$$

$$= a\sec\theta + \frac{a}{\sec\theta}\sec^2\theta$$

$$= a\sec\theta + a\sec\theta = 2a\sec\theta$$

Centre of curvature = $\left(a\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - a\tan\theta\sec\theta,\ 2a\sec\theta\right)$

Radius of curvature = $\frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2}$

$$= \frac{\left(1 + \tan^2\theta\right)^{\frac{3}{2}}}{\left(\frac{\sec\theta}{a}\right)}$$

$$= \left(\sec^2\theta\right)^{\frac{3}{2}} \times \frac{a}{\sec\theta} = a\sec^2\theta$$

Example 5

Find the equation of the circle of curvature of the parabola $y^2 = 12x$, at the point (3, 6)

Solution:

Given $y^2 = 12x$

Differentiating with respect to x, we get

$$2yy_1 = 12$$

$$(or) \qquad y_1 = \frac{6}{y}$$

$$y_{1(3,6)} = \frac{6}{6} = 1$$

Differentiating again with respect to x, we get

$$y_2 = \frac{-6}{y^2} y_1$$

$$y_{2(3,6)} = \frac{-1}{6}$$

$$\therefore \rho = \frac{\left(1+y_1{}^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left(1+(-1)^2\right)^{\frac{3}{2}}}{\frac{-1}{6}}$$

$$= -6(2)^{\frac{3}{2}}$$

$$= -6 \times 2\sqrt{2}$$

$$= -12\sqrt{2}$$

So radius of curvature = $|\rho| = 12\sqrt{2}$

Centre of curvature

$$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$$

$$\bar{x} \;\; at \;\; (3,6) = 3 - \frac{1}{\left(\frac{-1}{6}\right)}(1+1)$$

$$= 3 + 6 \times 2$$

$$= 15$$

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

$$= 6 + \frac{1}{\left(\frac{-1}{6}\right)}(1+1) = -6$$

Circle of curvature

$$\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2$$

$$\Rightarrow (x-15)^2 + (y+6)^2 = \left(12\sqrt{2}\right)^2$$

$$\Rightarrow x^2 - 30x + 15^2 + y^2 + 12y + 6^2 = 144 \times 2$$

$$\Rightarrow x^2 + y^2 - 30x + 12y - 27 = 0$$

EXERCISES

Part – A

1. Find the x-coordinate of the centre of curvature of the curve $y = x^2$ at the origin.

2. Find the y-coordinate of the centre of curvature of the curve $xy = 1$ at (1, 1).

3. State the formula for finding the centre of curvature at any point (x, y) on a given curve.

4. Find the centre of curvature of $y = x^2$ at the origin.

PART – B

1. Find the equation of the circle of curvature at the point (2, 3) on $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

2. Find the co-ordinates of the centre of curvature at the point (a, 2a) on the parabola $y^2 = 4ax$.

3. Find the equation of the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

4. Show that the circle of curvature of the parabola $y = mx + \frac{x^2}{a}$ at (0, 0) is $x^2 + y^2 = a\left(1+m^2\right)\left(y - mx\right)$.

5. Find the centre of curvature of $y = x^2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

6. Find the centre of curvature of $xy = c^2$ at $\left(c, c\right)$.

7. Find the centre of curvature of $x = a\left(\cos t + t\sin t\right),\ y = a\left(\sin t - t\cos t\right) at\ 't'$.

8. Find the centre of curvature of $y = x\log x$ at the point where $y' = 0$.

ANSWERS

Part A:

1. zero

2. 2

3. $\left(x-\frac{y_1}{y_2}\left(1+y_1^2\right), y+\frac{1+y_1^{2}}{y_2}\right)$

4. $\left(0,\frac{1}{2}\right)$

Part B:

1. $\left(x+\frac{5}{4}\right)^2+\left(y-\frac{5}{6}\right)^2=\frac{13^2}{12^2}$

2. $(5a,-2a)$

3. $\left(x-\frac{21a}{16}\right)^2+\left(y-\frac{21a}{16}\right)^2=\frac{9a^2}{128}$

5. $\left(\frac{-1}{2},\frac{3}{4}\right)$

6. $(2c,2c)$

7. $(a\cos t, a\sin t)$

8. $\left(\frac{1}{e},0\right)$

Evolute

Let $C\left(\overline{x},\overline{y}\right)$ be the centre of curvature of the given curve C_1 at the point $P(x,y)$. When P moves on the curve C_1, centre of curvature will also take different position and move on another curve C_2 which is called as evolute of the given curve C_1. Hence evolute is defined as the locus of centres of curvature of a curve.

Involute

If C_2 is evolute of the given curve C_1 then the given curve C_1 is called the involute of C_2.

Procedure

Let the given curve be

$$f(x,y,a,b)=0 \qquad (1)$$

Find $y_1 = \frac{dy}{dx}$ at point P and $y_2 = \frac{d^2y}{dx^2}$ at point P

Find the centre of curvature $\left(\bar{x}, \bar{y}\right)$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right) \qquad (2)$$

$$\bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right) \qquad (3)$$

Eliminate x & y from (1) , (2) & (3) we get

$$f\left(\bar{x}, \bar{y}, a, b\right) = 0 \qquad (4)$$

Locus of $\left(\bar{x}, \bar{y}\right)$ is the required evolute.

PARAMETRIC REPRESENTATION OF SOME STANDARD CURVES

Curve	Cartesian Form	Parametric Equations
Parabola (Horizontal)	$y^2 = 4ax$	$x = at^2;\ y = 2at$
Parabola (Vertical)	$x^2 = 4ay$	$x = 2at;\ y = at^2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = acos\theta;\ \ y = b\sin\theta$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta;\ y = b\tan\theta$
Rectangular Hyperbola	$xy = c^2$	$x = ct;\ \ y = \frac{c}{t}$
Circle	$\left(x-a\right)^2 + \left(y-b\right)^2 = r^2$	$x = a + r\cos\theta;\ \ y = b + r\sin\theta$
Astroid	$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$	$x = a\cos^3\theta;\ \ y = a\sin^3\theta$
Cycloid	-	$x = a\left(\theta - \sin\theta\right);\ y = a\left(1 - \cos\theta\right)$ (or) $x = a\left(\theta + \sin\theta\right);\ \ y = -a\left(1 - \cos\theta\right)$
Tractrix	-	$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right);\ y = a\sin\theta$

PROPERTIES OF EVOLUTE

I. The involute of a curve is orthogonal to all the tangents of that curve.

II. The evolute of the curve is independent of parametrization of any differentiable function

III. The evolute of the curve is the envelope of the normal to the given curve.

Problems

1. Find the evolute of the parabola $y^2 = 4ax$.

Solution:

The parametric form of the parabola $y^2 = 4ax$ is $x = at^2,\ y = 2at$

$$\Rightarrow \frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at}$$

$$= \frac{1}{t} \qquad (1)$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{1}{dx/dt}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{1}{2at}$$

$$= \left(-\frac{1}{t^2}\right)\frac{1}{2at}$$

$$y_2 = -\frac{1}{2at^3} \qquad (2)$$

The co-ordinates of centre of curvature $\left(\bar{x}, \bar{y}\right)$ is given by

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right), \quad \bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right).$$

$$\bar{x} = 2at - \frac{\left(1/t\right)}{\left(-1/2at^3\right)}\left(1+\frac{1}{t^2}\right) \quad \left(from\,(1)\, and\ (2)\right) \pm$$

$$= at^2 + 2at^3\left(\frac{t^2+1}{t^2}\right)$$

$$\bar{x} = at^2 + 2at^2 + 2a$$

$$\Rightarrow \quad \bar{x} = 2a + 3at^2$$

$$\Rightarrow 3at^2 = \bar{x} - 2a$$

$$\Rightarrow t^2 = \frac{\bar{x} - 2a}{3a} \qquad (3)$$

$$\bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right)$$

$$= 2at + \frac{1}{-1/2at^3}\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3\left(\frac{t^2+1}{t^2}\right)$$

$$= 2at - 2at^2 - 2at$$

$$\bar{y} = -2at^3$$

$$\Rightarrow \left(\bar{y}\right)^2 = \left(-2at^3\right)^2 = 4a^2\left(t^2\right)^3$$

$$= 4a^2\left(\frac{\bar{x}-2a}{3a}\right)^3 \qquad \left[from\left(3\right)\right]$$

$$\Rightarrow \left(\bar{y}\right)^2 = 4a^2\frac{\left(\bar{x}-2a\right)^3}{27a^3}$$

$$\Rightarrow 27a\left(\bar{y}\right)^2 = 4\left(\bar{x}-2a\right)^3$$

The Locus of $\left(\bar{x}, \bar{y}\right)$ is $27ay^2 = 4\left(x-2a\right)^2$ is the required evolute of $y^2 = 4ax$.

2. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$.

Solution:

The parametric form of the ellipse $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$ is $x = a\cos\theta,\ \ y = b\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta, \;\; \frac{dy}{d\theta} = b\cos\theta$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$y_1 = \frac{-b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta \qquad (1)$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{1}{dx/d\theta}$$

$$= \frac{d}{d\theta}\left(\frac{-b\cos\theta}{a\sin\theta}\right)\frac{1}{-a\sin\theta}$$

$$= \frac{b}{a}\cos ec^2\theta\left(\frac{1}{-a\sin\theta}\right)$$

$$y_2 = -\frac{b}{a^2}\cos ec^3\theta = -\frac{b}{a^2\sin^3\theta} \qquad (2)$$

The co-ordinates of centre of curvature $\left(\bar{x},\bar{y}\right)$ is given by $\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$,

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

$$\bar{x} = a\cos\theta - \frac{\left(-b\cos\theta/a\sin\theta\right)}{\left(-b/a^2\sin^3\theta\right)}\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right) \qquad \left[\mathit{from}(1)\,\mathit{and}\,(2)\right]$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\sin^2\theta\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\left[a^2\left(1-\cos^2\theta\right) + b^2\cos^2\theta\right]$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\left[a^2 - a^2\cos^2\theta + b^2\cos^2\theta\right]$$

$$= a\cos\theta - a\cos\theta + a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \cos^3\theta\left(\frac{a^2-b^2}{a}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{\frac{1}{3}} \qquad (3)$$

$$\bar{y} = b\sin\theta + \left(\frac{1}{-b\big/a^2\sin^3\theta}\right)\left[\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right] \qquad \left[\mathit{from}(1)\,\mathit{and}\,(2)\right]$$

$$= \frac{b\sin\theta - a^2\sin^3\theta}{b}\left[\frac{a^2\sin^2\theta + b^2\left(1-\sin^2\theta\right)}{a^2\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{\sin\theta}{b}\left(a^2\sin^2\theta + b^2 - b^2\sin^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta + b\sin^3\theta$$

$$\bar{y} = \sin^3\theta\left(\frac{b^2-a^2}{b}\right)$$

$$\Rightarrow \sin\theta = \left[\frac{b\bar{y}}{-\left(a^2-b^2\right)}\right]^{\frac{1}{3}} \qquad (4)$$

$$(3)^2 + (4)^2$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = \left[\frac{a\bar{x}}{a^2-b^2}\right]^{\frac{2}{3}} + \left[\frac{-b\bar{y}}{a^2-b^2}\right]^{\frac{2}{3}}$$

$$\Rightarrow 1 = \left(\frac{1}{a^2-b^2}\right)^{\frac{2}{3}}\left[\left[a\bar{x}\right]^{\frac{2}{3}} + \left(b\bar{y}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow \left(a^2-b^2\right)^{\frac{2}{3}} = \left(a\bar{x}\right)^{\frac{2}{3}} + \left(b\bar{y}\right)^{\frac{2}{3}} \qquad (5)$$

The locus of $\left(\bar{x},\bar{y}\right)$ is $(ax)^{2/3} + (by)^{2/3} = \left(a^2-b^2\right)^{2/3}$ which is the required evolute of the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solution:

The parametric form of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x=a\sec\theta,\ y=b\tan\theta$

$$\Rightarrow \frac{dx}{d\theta}=a\sec\theta\tan\theta, \quad \frac{dy}{d\theta}=b\sec^2\theta$$

$$y_1=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\sec\theta\tan\theta}{b\sec^2\theta}$$

$$\Rightarrow y_1=\frac{b\sec\theta}{a\tan\theta}=\frac{b}{a}\cos ec\theta \qquad (1)$$

$$y_2=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{1}{\left(dx/d\theta\right)}$$

$$=\frac{d}{d\theta}\left(\frac{b}{a}\cos ec\theta\right).\frac{1}{a\sec\theta.\tan\theta}$$

$$=-\frac{b}{a}\cos ec\theta\cot\theta\left(\frac{1}{a\sec\theta\tan\theta}\right)$$

$$y_2=-\frac{b}{a^2}\cot^3\theta \qquad (2)$$

The co-ordinates of the centre of curvature $\left(\bar{x},\bar{y}\right)$

$$\bar{x}=x-\frac{y_1}{y_2}\left(1+y_1^2\right),\ \bar{y}=y+\frac{1}{y_2}\left(1+y_1^2\right)$$

using (1) & (2)

$$\bar{x}=a\sec\theta-\left(\frac{\frac{b\sec\theta}{a\tan\theta}}{\frac{-b}{a^2\tan^3\theta}}\right)\left(1+\frac{b^2\sec^2\theta}{a^2\tan^2\theta}\right)$$

$$=a\sec\theta+a\sec\theta\tan^2\theta\left(\frac{a^2\tan^2\theta+b^2\sec^2\theta}{a^2\tan^2\theta}\right)$$

$$= a\sec\theta + \frac{\sec\theta}{a}\left[a^2\left(\sec^2\theta - 1\right) + b^2\sec^2\theta\right]$$

$$= a\sec\theta + a\sec^3\theta - a\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$\bar{x} = \sec^3\theta\left(a + \frac{b^2}{a}\right)$$

$$\Rightarrow \sec\theta = \left(\frac{a\bar{x}}{a^2 + b^2}\right)^{\frac{1}{3}} \qquad (3)$$

$$\bar{y} = b\tan\theta + \frac{1}{\left(-b/a^2\tan^3\theta\right)}\left[\frac{a^2\tan^2\theta + b^2\sec^2\theta}{a^2 tan^2\theta}\right]$$

$$\Rightarrow \bar{y} = b\tan\theta - \frac{a^2\tan^3\theta}{b}\left[\frac{a^2\tan^2\theta + b^2\left(1 + \tan^2\theta\right)}{a^2 tan^2\theta}\right]$$

$$= b\tan\theta - \frac{1}{b}\tan\theta\left(a^2\tan^2\theta + b^2 + b^2\tan^2\theta\right)$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\tan\theta - b\tan^3\theta$$

$$= \tan^3\theta\left(-\frac{a^2}{b} - b\right)$$

$$= -\tan^3\theta\left(\frac{a^2 + b^2}{b}\right)$$

$$\Rightarrow \tan\theta = \left(\frac{-b\bar{y}}{a^2 + b^2}\right)^{\frac{2}{3}} \qquad (4)$$

$$(3)^2 - (4)^2$$

$$\Rightarrow \sec^2\theta - \tan^2\theta = \left[\frac{a\bar{x}}{a^2 + b^2}\right]^{\frac{2}{3}} - \left[\frac{-b\bar{y}}{a^2 + b^2}\right]^{\frac{2}{3}}$$

$$\Rightarrow 1 = \frac{1}{\left(a^2 + b^2\right)^{\frac{2}{3}}}\left[\left(a\bar{x}\right)^{\frac{2}{3}} - \left(b\bar{y}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow \left(a\bar{x}\right)^{\frac{2}{3}} - \left(b\bar{y}\right)^{\frac{2}{3}} = \left(a^2 + b^2\right)^{\frac{2}{3}} \qquad (5)$$

The locus of $\left(\bar{x}, \bar{y}\right)$ is $\left(ax\right)^{\frac{2}{3}} - \left(by\right)^{\frac{2}{3}} = \left(a^2 + b^2\right)^{\frac{2}{3}}$ which is the required evolute of the given by hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

4. Find the evolute of the rectangular hyperbola $xy = c^2$.

Solution:

The parametric form of the rectangular hyperbola $xy = c^2$ is $x = ct \quad y = \dfrac{c}{t}$

$$\Rightarrow \frac{dx}{dt} = c \;\; \& \;\; \frac{dy}{dt} = \frac{-c}{t^2}$$

$$y_1 = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c} = -\frac{1}{t^2} \qquad (1)$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{1}{\left(dx/dt\right)}$$

$$= \frac{d}{dt}\left(-\frac{1}{t^2}\right) \cdot \frac{1}{c}$$

$$= \frac{2}{ct^3} \qquad (2)$$

The co-ordinates of centre of curvature $\left(\bar{x}, \bar{y}\right)$ is given by

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right), \quad \bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right)$$

$$\bar{x} = ct - \frac{\left(-1/t^2\right)}{\left(2/ct^3\right)}\left(1 + \frac{1}{t^4}\right) \qquad \left[from(1) and (2) \right]$$

$$= ct + \frac{ct}{2}\left(1 + \frac{1}{t^4}\right)$$

$$= ct + \frac{ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2}\left(3t + \frac{1}{t^3}\right) \qquad (3)$$

$$\bar{y} = \frac{c}{t} + \frac{1}{\left(2/ct^3\right)}\left(1 + \frac{1}{t^4}\right) \qquad \left[from(1)\,and\,(2)\right]$$

$$= \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

$$\bar{y} = \frac{c}{2}\left(\frac{3}{t} + t^3\right) \qquad (4)$$

$$(3) + (4) \Rightarrow \bar{x} + \bar{y} = \frac{c}{2}\left[t^3 + 3t + \frac{3}{t} + \frac{1}{t^3}\right]$$

$$\Rightarrow \bar{x} + \bar{y} = \frac{c}{2}\left(t + \frac{1}{t}\right)^3 \Rightarrow t + \frac{1}{t} = \left[\frac{2}{c}\left(\bar{x} + \bar{y}\right)\right]^{\frac{1}{3}} \qquad (5)$$

Similarly

$$\bar{x} - \bar{y} = \frac{-c}{2}\left(t - \frac{1}{t}\right)^3 \Rightarrow t - \frac{1}{t} = \left[-\frac{2}{c}\left(\bar{x} - \bar{y}\right)\right]^{\frac{1}{3}} \qquad (6)$$

$$(5)^2 - (6)^2$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 = \left(\tfrac{2}{c}\right)^{2/3}\left[\left(\bar{x} + \bar{y}\right)^{\frac{2}{3}} - \left(\bar{x} - \bar{y}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow \left[\left(t + \tfrac{1}{t}\right) + \left(t - \tfrac{1}{t}\right)\right]\left[\left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right)\right] = \left(\frac{2}{c}\right)^{\frac{2}{3}}\left[\left(\bar{x} + \bar{y}\right)^{\frac{2}{3}} - \left(\bar{x} - \bar{y}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow (2t)\left(\frac{2}{t}\right) = \frac{2^{2/3}}{c^{2/3}}\left[\left(\bar{x} + \bar{y}\right)^{\frac{2}{3}} - \left(\bar{x} - \bar{y}\right)^{\frac{2}{3}}\right]$$

$$\Rightarrow \frac{4c^{2/3}}{2^{2/3}} = \left(\bar{x} + \bar{y}\right)^{\frac{2}{3}} - \left(\bar{x} - \bar{y}\right)^{\frac{2}{3}} \qquad (7)$$

The locus of $\left(\bar{x}, \bar{y}\right)$ is $\left(x + y\right)^{\frac{2}{3}} - \left(x - y\right)^{\frac{2}{3}} = \left(4c\right)^{\frac{2}{3}}$

which is the required evolute of $xy = c^2$.

5. Find the evolute of the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution:

The parametric equation of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $x = a\cos^3\theta, y = a\sin^3\theta$

$$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$y_1 = \frac{-\sin\theta}{\cos\theta} = -\tan\theta \qquad (1)$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{1}{dx/d\theta}$$

$$y_2 = \frac{d}{d\theta}(-\tan\theta)\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= -\sec^2\theta\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$y_2 = \frac{1}{3a\cos^4\theta\sin\theta} \qquad (2)$$

The co-ordinates of centre of curvature $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2), \bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$\bar{x} = a\cos^3\theta - \frac{\left(\frac{-\sin\theta}{\cos\theta}\right)}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)}\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right) \qquad [from(1)\,and\,(2)]$$

$$= a\cos^3\theta + 3a\cos^3\theta\sin^2\theta\left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right)$$

$$= a\left[\cos^3\theta + 3\cos\theta\sin^2\theta\right] \qquad (3)$$

$$\bar{y} = a\sin^3\theta + \frac{1}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)}\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) \qquad \left[\mathit{from}(1)\,\mathit{and}\,(2)\right]$$

$$= a\sin^3\theta + 3a\cos^4\theta\sin\theta\left(\frac{1}{\cos^2\theta}\right)$$

$$= a\left[\sin^3\theta + 3\cos^2\theta\sin\theta\right] \qquad (4)$$

(3) + (4)

$$\Rightarrow \bar{x} + \bar{y} = a\left[\cos^3\theta + 3\cos\theta\sin^2\theta + 3\cos^2\theta\sin\theta + \sin^3\theta\right]$$

$$= a(\cos\theta + \sin\theta)^3$$

$$\Rightarrow \cos\theta + \sin\theta = \left(\frac{\bar{x} + \bar{y}}{a}\right)^{\frac{1}{3}} \qquad (5)$$

(3) - (4)

$$\Rightarrow \bar{x} - \bar{y} = a\left[\cos^3\theta + 3\cos\theta\sin^2\theta - 3\cos^2\theta\sin\theta - \sin^3\theta\right]$$

$$= a(\cos\theta - \sin\theta)^3$$

$$\Rightarrow \cos\theta - \sin\theta = \left(\frac{\bar{x} - \bar{y}}{a}\right)^{\frac{1}{3}} \qquad (6)$$

$(5)^2 + (6)^2$

$$\Rightarrow (\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = \left(\frac{\bar{x} + \bar{y}}{a}\right)^{\frac{2}{3}} + \left(\frac{\bar{x} - \bar{y}}{a}\right)^{\frac{2}{3}}$$

$$\Rightarrow \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta$$

$$= \frac{(\bar{x} + \bar{y})^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{(\bar{x} - \bar{y})^{\frac{2}{3}}}{a^{\frac{2}{3}}}$$

$$\Rightarrow 2(\cos^2\theta + \sin^2\theta) = \frac{1}{a^{2/3}}\left[(\bar{x} + \bar{y})^{\frac{2}{3}} + (\bar{x} - \bar{y})^{\frac{2}{3}}\right]$$

$$\Rightarrow (\bar{x} + \bar{y})^{\frac{2}{3}} + (\bar{x} - \bar{y})^{\frac{2}{3}} = 2a^{\frac{2}{3}} \qquad (7)$$

The locus of $\left(\overline{x}, \overline{y}\right)$ is $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$, which the required evolute of the asteroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

6. Prove that the evolute of the cycloid $x=a\left(\theta-sin\ \theta\right),\ \ y=a\left(1-cos\ \theta\right)$ is another cycloid.

Solution:

$$\text{Given} \quad x=a(\theta-\sin\theta), \qquad y=a(1-\cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta}=a\left(1-\cos\theta\right), \qquad \frac{dy}{d\theta}=a\sin\theta$$

$$=2a\sin^2\frac{\theta}{2} \qquad\qquad =2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore y_1=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}$$

$$y_1=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2} \qquad (1)$$

$$y_2=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{1}{\frac{dx}{d\theta}}$$

$$=\frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right).\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$=-\operatorname{cosec}^2\frac{\theta}{2}\left(\frac{1}{2}\right)\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$y_2=-\frac{1}{4a\sin^4\frac{\theta}{2}} \qquad (2)$$

The co-ordinates of centre of curvature $\left(\overline{x}, \overline{y}\right)$ is given by

$$\overline{x}=x-\frac{y_1}{y_2}\left(1+y_1^2\right),\ \ \overline{y}=y+\frac{1}{y_2}\left(1+y_1^2\right)$$

$$\bar{x} = a(\theta - \sin\theta) - \frac{\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)}{\left(\frac{-1}{4a\sin^4\frac{\theta}{2}}\right)}\left(1 + \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) \qquad [\,from\,(1)\,and\,(2)]$$

$$= a\theta - a\sin\theta + 4a\cos\frac{\theta}{2}\sin^3\frac{\theta}{2}\left(\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right)$$

$$= a\theta - a\sin\theta + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a\theta - a\sin\theta + 2a\sin\theta$$

$$\bar{x} = a(\theta + \sin\theta) \qquad (3)$$

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$$

$$= a(1 - \cos\theta) + \frac{1}{\left(\frac{-1}{4a\sin^4\frac{\theta}{2}}\right)}\left(\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) \qquad [\,from\,(1)\,and\,(2)]$$

$$= a\left(2\sin^2\frac{\theta}{2}\right) - 4a\sin^2\frac{\theta}{2}$$

$$= -2a\sin^2\frac{\theta}{2}$$

$$\bar{y} = -2a(1 - \cos\theta) \qquad (4)$$

The locus of $(\bar{x}, \bar{y})$ is $x = a(\theta + \sin\theta),\ y = -2a(1 - \cos\theta)$ which is another cycloid.

7. Find the evolute of the tractrix $x = a(\cos t + \log\tan\frac{t}{2}),\ y = a\sin t$.

Solution:

Given $x = a(\cos t + \log\tan\frac{t}{2}),\ y = a\sin t$

$$\Rightarrow \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}.\left(\frac{1}{2}\right)\right]$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)}$$

$$= \frac{\sin t}{\cos t} = \tan t \qquad (1)$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{1}{\frac{dx}{dt}}$$

$$= \frac{d}{dt}(\tan t).\frac{1}{\left(\frac{a\cos^2 t}{\sin t}\right)}$$

$$= \text{s}ec^2 t\,\frac{\sin t}{a\cos^2 t}$$

$$y_2 = \frac{\sin t}{a\cos^4 t} \qquad (2)$$

The co-ordinates of centre of curvature $\left(\bar{x}, \bar{y}\right)$ is given by

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right), \quad \bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right)$$

$$\bar{x} = a\left(\cos t + \log\tan\frac{t}{2}\right) - \frac{\left(\frac{\sin t}{\cos t}\right)}{\left(\frac{\sin t}{a\cos^4 t}\right)}\left(1 + \frac{\sin^2 t}{\cos^2 t}\right) \qquad \left[from\left(1\right) and \left(2\right)\right]$$

$$= a\cos t + a\log\tan\frac{t}{2} - a\cos^3 t\left(\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right)$$

$$= a\cos t + a\log\tan\frac{t}{2} - a\cos t$$

$$\bar{x} = a\log\tan\frac{t}{2}$$

$$\log\tan\frac{t}{2} = \frac{\bar{x}}{a}$$

$$\Rightarrow \tan\frac{t}{2} = e^{\frac{\bar{x}}{a}} \qquad (3)$$

$$\bar{y} = a\sin t + \frac{1}{\left(\frac{\sin t}{a\cos^4 t}\right)}\left(\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right) \qquad \left[from\left(1\right) and \left(2\right)\right]$$

$$= a\sin t + \frac{a\cos^2 t}{\sin t}$$

$$= \frac{a\sin^2 t + a\cos^2 t}{\sin t}$$

$$\bar{y} = \frac{a}{\sin t}$$

$$= \frac{a}{\left(\frac{2\tan\frac{t}{2}}{1 + \tan^2\frac{t}{2}}\right)}$$

$$\Rightarrow \bar{y} = \frac{a\left[1+\tan^2\frac{t}{2}\right]}{2\tan\frac{t}{2}}$$

$$= \frac{a}{2}\left(\frac{1+e^{2\frac{\bar{x}}{a}}}{e^{\frac{\bar{x}}{a}}}\right) \qquad \text{using (3)}$$

$$\bar{y} = \frac{a}{2}\left(e^{-\frac{\bar{x}}{a}} + e^{\frac{\bar{x}}{a}}\right)$$

$$\Rightarrow \bar{y} = \frac{a}{2}2\cosh\frac{\bar{x}}{a} \qquad \left[\because \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)\right] \qquad (4)$$

The locus of $\left(\bar{x},\bar{y}\right)$ is $y = a\cosh\frac{x}{a}$ which is a catenary.

EXERCISES

1. Show that the evolute of the cycloid $x = a(\theta+\sin\theta),\ y = a(1-\cos\theta)$ is another cycloid given by $x = a(\theta-\sin\theta),\ y-2a = a(1+\cos\theta)$.

2. Prove that the evolute of the curve $x = a(\cos\theta+\theta\sin\theta),\ y = a(\sin\theta-\theta\cos\theta)$ is a circle $x^2+y^2 = a^2$.

ENVELOPES

Introduction

In the plane, an envelope is a curve that is a tangent at some point to each member of the family of curves. Hence envelope can be viewed as a curve that touches every member of the family at some point. Classically, a point on the envelope can be imagined as the limit of intersection of nearby curves. This idea can be generalized to an envelope of surfaces in space and also extended to higher dimensions. A family of curves may have no envelope or unique envelope or several envelopes.

> **Note:** In the study of ordinary differential equations, envelopes are considered as singular solutions of ODEs.

Consider the equation $f(x,y,\alpha)=0$ where α is an arbitrary constant. Assigning different values for α results in number of equations representing a family of curves. Hence the quantity α is called parameter of the family of curves.

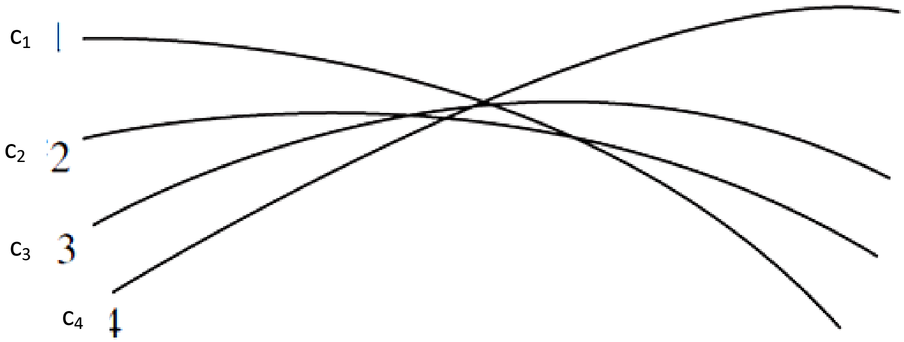


Fig 2.9

Definition

The locus of the limiting positions of the points of intersection of consecutive members of a family of curves is called the envelope of the family.

Theorem

The envelope of a family of curves touches every member of the family of curves.

Proof:

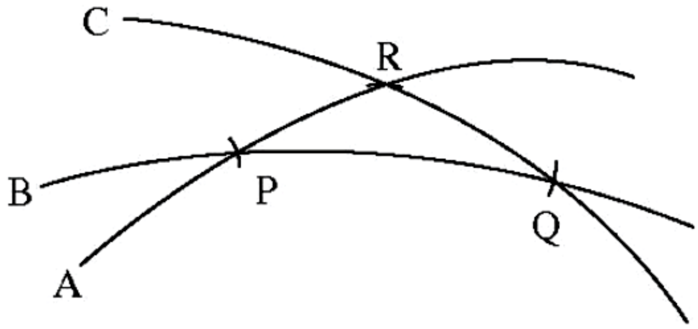


Fig 2.10

Consider three consecutive intersecting member of the family, given by $f\left(x, y, \alpha\right) = 0.$ Let curves A and B intersect at the point P, curves B and C intersect at the point Q. The points P and Q lie on the envelope and on the curve B. Now there exist a common tangent for the curve B and the envelope. In a similar manner, it can be proved that the envelope touches every point of the curve of the family.

Note: The envelope of a family of curves is the curve which touches every member of the family of curves.

Method of finding the equation of the envelope of single parameter family of curves

Let $f\left(x, y, \alpha\right) = 0$ be the equation of the given family of curves, where α is the parameter. The two consecutive members of the family corresponding to two close values of α are given by

$$f\left(x, y, \alpha\right) = 0 \qquad (1)$$

and $$f\left(x, y, \alpha + \Delta\alpha\right) = 0 \qquad (2)$$

The co-ordinates of the points of intersection of (1) and (2) will satisfy (1) and (2) and hence satisfy $\dfrac{f\left(x, y, \alpha + \Delta\alpha\right) - f\left(x, y, \alpha\right)}{\Delta\alpha} = 0$

Hence the co-ordinates of the limiting positions of the points of intersection of (1) and (2) will satisfy the equation $\lim\limits_{\Delta\alpha \to 0}\left[\dfrac{f\left(x, y, \alpha + \Delta\alpha\right) - f\left(x, y, \alpha\right)}{\Delta\alpha}\right] = 0$

$$i.e.\ \frac{\partial}{\partial\alpha} f\left(x, y, \alpha\right) = 0 \Rightarrow \frac{\partial f}{\partial\alpha} = 0 \qquad (3)$$

These limiting points will continue to lie on (1) and satisfy $f\left(x, y, \alpha\right) = 0$ Eliminating α between $f\left(x, y, \alpha\right) = 0$ and $\dfrac{\partial f}{\partial\alpha} = 0$, the required envelope of the family of curves is obtained.

Equation of the envelope of the family $\mathbf{A}\alpha^2 + \mathbf{B}\alpha + \mathbf{C} = 0$, where α is the parameter and A, B, C are functions of x and y

Let the family of curves be quadratic in the parameter α given by

$$A\alpha^2 + B\alpha + C = 0 \qquad (1)$$

Differentiating (1) partially w.r.t α,

$$2A\alpha + B = 0 \Rightarrow \alpha = \frac{-B}{2A} \qquad (2)$$

Substituting (2) in (1)

$$A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C = 0$$
$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$
$$\Rightarrow B^2 - 4AC = 0,$$

which is the equation of the required envelope of the given family.

Note: If the family of curves is a quadratic in the parameter α then the required envelope is given by $B^2 - 4AC = 0$.

Examples

1. Find the envelope of the curve $y = mx + \frac{a}{m}$, where m is the parameter.

Solution:

$$\text{Given } y = mx + \frac{a}{m}$$
$$ym = m^2x + a$$
$$m^2x - ym + a = 0$$

This is a quadratic equation in m with

$$A = x, \qquad B = -y, \qquad C = a$$

Hence the required envelope is given by

$$B^2 - 4AC = 0$$
$$\Rightarrow (-y)^2 - 4(x)(a) = 0$$
$$\Rightarrow y^2 - 4ax = 0$$
$$\Rightarrow y^2 = 4ax$$

The envelope is a parabola.

2. Find the envelope of the family of curves $y = mx + am^2, m$ being the parameter.

Solution:

Given $y = mx + am^2$ (1)

Differentiating (1) with respect to m,

$$0 = x + 2am$$

$$\Rightarrow m = \frac{-x}{2a} \qquad (2)$$

Substituting (2) in (1)

$$y = \left(\frac{-x}{2a}\right)x + a\left(\frac{-x}{2a}\right)^2$$
$$y = \frac{-x^2}{2a} + a\frac{x^2}{4a^2}$$
$$y = \frac{-x^2}{4a}$$

$x^2 = -4ay$ is the required envelope. Envelope is a parabola.

3. Find the envelope of family of straight lines given by $y = mx \pm \sqrt{a^2m^2 + b^2}$, m being the parameter.

Solution:

$$\text{Given} \quad y = mx \pm \sqrt{a^2m^2 + b^2}$$
$$(y - mx) = \pm\sqrt{a^2m^2 + b^2}$$
$$(y - mx)^2 = a^2m^2 + b^2$$
$$y^2 + m^2x^2 - 2xym = a^2m^2 + b^2$$
$$(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$$

which is quadratic in m, with

$$A = x^2 - a^2;\ B = -2xy;\ C = y^2 - b^2$$

Envelope is given by $B^2 - 4AC = 0$

$$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$
$$\cancel{x^2y^2} = \cancel{x^2y^2} - x^2b^2 - a^2y^2 + a^2b^2$$
$$b^2x^2 + a^2y^2 = a^2b^2$$

Dividing by a^2b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The required envelope is an ellipse.

4. Find the envelope of the family of straight lines $y = mx - 2am - am^3$, where m is a parameter.

Solution:

Given $y = mx - 2am - am^3$ (1)

Differentiating partially w.r.t m,

$$0 = x - 2a - 3am^2 \quad (2)$$

$$\Rightarrow \qquad m^2 = \frac{x-2a}{3a} \quad (3)$$

From (1) $\Rightarrow$ $y = m\left(x - 2a - am^2\right)$

$$y = m\left[x - 2a - a\left(\frac{x-2a}{3a}\right)\right] \qquad [from(3)]$$

$$= m\left[(x-2a) - \frac{1}{3}(x-2a)\right]$$

$$y = m\frac{2}{3}(x-2a)$$

$$m = \frac{3y}{2(x-2a)} \quad (4)$$

From (3) and (4), we get

$$\frac{(x-2a)}{3a} = \left[\frac{3y}{2x-4a}\right]^2$$

$$\Rightarrow \frac{x-2a}{3a} = \frac{9y^2}{4(x-2a)^2}$$

$$\Rightarrow (x-2a)^3 = \frac{27}{4}a\,y^2$$

$$\Rightarrow 4(x-2a)^3 = 27a\,y^2$$

is the required envelope.

5. Find the envelope of the family of curves $x = my + \frac{1}{m}, m$ being the parameter.

Solution:

Given $x = my + \frac{1}{m}$

$$xm = m^2 y + 1$$

$$m^2 y - xm + 1 = 0$$

This is a quadratic equation in 'm' $A = y; B = -x;\ C = 1$

Required envelope: $B^2 - 4AC = 0$

$$(-x)^2 - 4(y)(1) = 0$$
$$x^2 - 4y = 0$$
$$x^2 = 4y$$

Envelope is a parabola.

6. Find the envelope of $\frac{x}{t} + yt = 2c,\ t$ being the parameter.

Solution:

Given $\frac{x}{t} + yt = 2c$

$$\Rightarrow yt^2 - 2ct + x = 0$$

$$\Rightarrow yt^2 + x = 2ct$$

This is a quadratic equation in't' with $A = y;\ B = -2c;\ C = x$

Required Envelope $B^2 - 4AC = 0$

$$(-2c)^2 - 4(y)\,(x) = 0$$

$$4c^2 - 4yx = 0$$

$$c^2 - yx = 0$$

$$xy = c^2$$

The envelope is rectangular hyperbola.

7. Find the envelope of family of circles $(x-a)^2 + y^2 = 2a$ where 'a' is a parameter.

Solution:

Given $(x-a)^2 + y^2 = 2a$ (1)

Differentiating w.r.t 'a' we get

$$2(x-a)(-1)=2$$

$$a=x+1 \qquad (2)$$

Substituting (2) in (1)

$$[x-(x+1)]^2+y^2=2(x+1)$$
$$1+y^2=2x+2$$
$$y^2=2x+1$$

The required envelope is a parabola.

8. Find the envelope to the family of circles $x^2+(y-b)^2=a^2$ with centres on y-axis & of given radius ‘a’ with ‘b’ as the parameter.

Solution:

Consider $x^2+(y-b)^2=a^2$ (1)

Differentiate (1) w.r.t ‘b’

$$-2(y-b)=0 \Rightarrow y-b=0$$

$$\Rightarrow b=y \qquad (2)$$

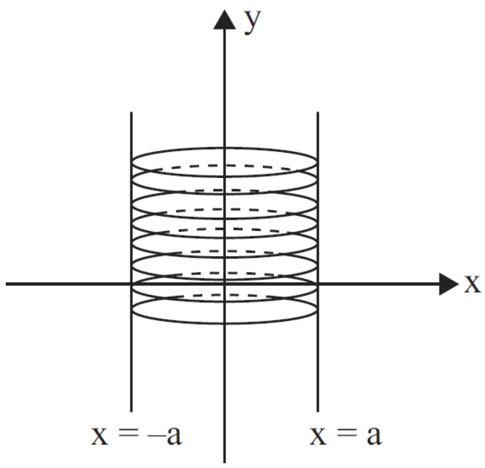


Fig 2.11

Substitute (2) in (1) we get

$$x^2=a^2$$
$$\Rightarrow x=\pm a$$

$\therefore$ The two lines $x = a$ *and* $x = -a$ are the two envelopes to this family of circles

9. Show that the x-axis $y = 0$ is the envelope of the family of semicubical parabola $y^2 - (x+b)^3 = 0$ with b as the parameter.

Solution:

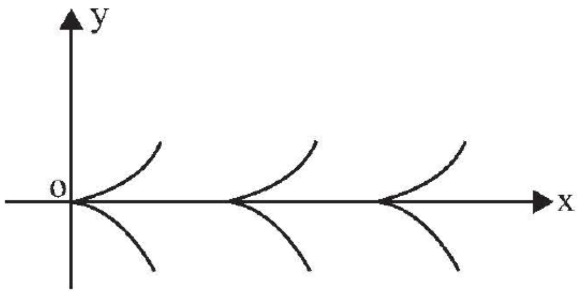


Fig 2.12

Consider $y^2 - (x+b)^3 = 0$ (1)

Differentiate (1) w.r.t 'b'

$$-3(x+b)^2 = 0$$
$$\Rightarrow x+b = 0 \qquad (2)$$

Substituting (2) in (1), we get

$$y^2 = 0$$

$\Rightarrow y = 0$, is the envelope

10. Show that the family of straight lines $2y - 4x + \alpha = 0$ has no envelope, where α is the parameter.

Solution:

Differentiating $2y - 4x + \alpha = 0$, with respect to α

we get $0+0+1=0$ which is a contradiction. We observe that the given family of straight lines $y = 2x - \frac{\alpha}{2}$ are all parallel with common slope m = 2. Hence no curve (envelope) exists which touches each member of this parallel straight lines.

11. Show that the family of circles $x^2 + (y-b)^2 = b^2$ with centres lying on the y-axis has no envelope.

Solution:

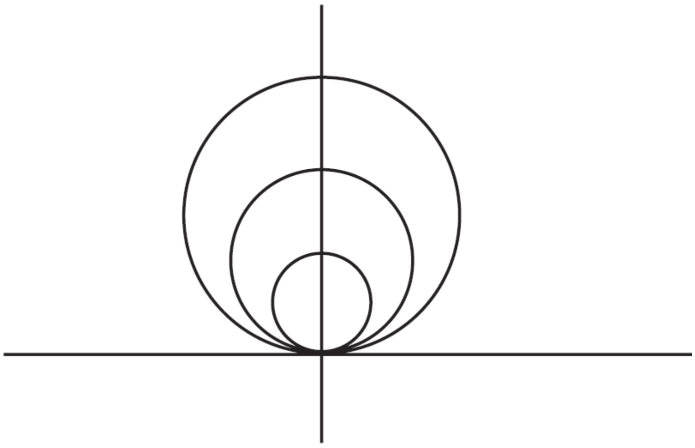


Fig 2.13

Differentiating $x^2+(y-b)^2=b^2$ with respect to b we get,

$$0-2(y-b)=2b$$
$$\Rightarrow y=0$$

We observe that the given family of circles meets at the origin and that we could not find a curve which covers or touches every member.

Hence the envelope does not exist for this family of circles.

12. Find the envelope of the family of lines $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$ θ being the parameter.

Solution:

Given $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ (1)

Differentiating (1) partially with respect to θ

$$\frac{-x}{a}\sin\theta+\frac{y}{b}\cos\theta=0 \quad (2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2+\left(\frac{-x}{a}\sin\theta+\frac{y}{b}\cos\theta\right)^2=1^2+0^2$$

$$\left(\frac{x^2}{a^2}\cos^2\theta+\frac{y^2}{b^2}\sin^2\theta+\cancel{\frac{2xy}{ab}\sin\theta\cos\theta}\right)+\left(\frac{x^2\sin^2\theta}{a^2}+\frac{y^2\cos^2\theta}{b^2}-\cancel{\frac{2xy}{ab}\sin\theta\cos\theta}\right)=1$$

$$\frac{x^2}{a^2}\left(\cos^2\theta+\sin^2\theta\right)+\frac{y^2}{b^2}\left(\cos^2\theta+\sin^2\theta\right)=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The required envelope is an ellipse.

13. Find the envelope of the family of curves $x\cos\theta+y\sin\theta=\alpha,\theta$ being the parameter.

Solution:

Given $x\cos\theta+y\sin\theta=\alpha$ (1)

Differentiating (1) with respect to θ

$-x\sin\theta+y\cos\theta=0$ (2)

Squaring and adding (1) and (2)

$$\left(x\cos\theta+y\sin\theta\right)^2+\left(-x\sin\theta+y\cos\theta\right)^2=\alpha^2+0^2$$

$$x^2\left(\cos^2\theta+\sin^2\theta\right)+y^2\left(\sin^2\theta+\cos^2\theta\right)=\alpha^2$$

$$x^2+y^2=\alpha^2$$

The required envelope is a circle.

14. Find the envelope of the family of curves $\left(\frac{a^2}{x}\right)\cos\theta-\left(\frac{b^2}{y}\right)\sin\theta=c,\theta$ being the parameter.

Solution:

Given $\left(\frac{a^2}{x}\right)\cos\theta-\left(\frac{b^2}{y}\right)\sin\theta=c$ (1)

Differentiating with respect to θ

$$\left(\frac{a^2}{x}\right)(-\sin\theta)-\left(\frac{b^2}{y}\right)\cos\theta=0$$

$$\left(\frac{a^2}{x}\right)\sin\theta+\left(\frac{b^2}{y}\right)\cos\theta=0 \qquad (2)$$

The equation of the envelope is obtained by eliminating θ between (1) and (2)

Now squaring and adding (1) and (2)

$$\left(\frac{a^2}{x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{b^2}{y}\right)\left(\sin^2\theta+\cos^2\theta\right)=c^2$$

$$\frac{a^4}{x^2}+\frac{b^4}{y^2}=c^2$$

$$a^4y^2+b^4x^2=c^2x^2y^2$$

is the required envelope.

15. Find the envelope of the family of straight lines $x\cos\alpha+y\sin\alpha=c\sin\alpha\cos\alpha,\alpha$ being the parameter.

Solution:

Given by $x\cos\alpha+y\sin\alpha=c\sin\alpha\cos\alpha$

Dividing by $\sin\alpha\cos\alpha,$ we get

$$\frac{x}{\sin\alpha}+\frac{y}{cos\alpha}=c \qquad (1)$$

Differentiating (1) with respect to α, we get

$$\frac{-x}{\sin^2\alpha}\cos\alpha+\frac{y}{\cos^2\alpha}\sin\alpha=0 \qquad (2)$$

$$\frac{x\cos\alpha}{\sin^2\alpha}=\frac{y\sin\alpha}{\cos^2\alpha}$$

$$\Rightarrow\frac{x}{\sin^3\alpha}=\frac{y}{\cos^3\alpha}=k\left(say\right) \qquad (3)$$

$$\sin^3\alpha=\frac{x}{k} \qquad \cos^3\alpha=\frac{y}{k}$$

We know $\sin^2\alpha+\cos^2\alpha=1$

$$\left(\frac{x}{k}\right)^{\frac{2}{3}}+\left(\frac{y}{k}\right)^{\frac{2}{3}}=1$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}} \qquad (4)$$

From (3) we have $\sin\alpha=\dfrac{x^{\frac{1}{3}}}{k^{\frac{1}{3}}};\ \cos\alpha=\dfrac{y^{\frac{1}{3}}}{k^{\frac{1}{3}}}$ (5)

Using (5) in (1)

$$k^{\frac{1}{3}}\frac{x}{x^{\frac{1}{3}}}+k^{\frac{1}{3}}\frac{y}{y^{\frac{1}{3}}}=c$$

$$\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)k^{\frac{1}{3}}=0$$

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^{\frac{1}{2}}=c \qquad \left[from(4)\right]$$

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=c \Rightarrow x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$$

is the required envelope.

16. Find the envelope of $y=x\tan\alpha+2\sec\alpha,\ \alpha$ being the parameter.

Solution:

Given $y=x\tan\alpha+2\sec\alpha$ (1)

Differentiating with respect to α

$$0=x\sec^2\alpha+2\sec\alpha\tan\alpha$$

Dividing by $\sec\alpha$

$$x\sec\alpha+2\tan\alpha=0 \qquad (2)$$

$(1)^2-(2)^2$ gives

$$\left(x\tan\alpha+2\sec\alpha\right)^2-\left(x\sec\alpha+2\tan\alpha\right)^2=y^2$$

$$\left(x^2\tan^2\alpha+4\sec^2\alpha+2x\tan\alpha\sec\alpha\right)-\left(x^2\sec^2\alpha+4\tan^2\alpha+2x\sec\alpha\tan\alpha\right)=y^2$$

$$x^2\left(\tan^2\alpha-\sec^2\alpha\right)+4\left(\sec^2\alpha-\tan^2\alpha\right)=y^2$$

$$-x^2+4=y^2$$

$$x^2+y^2=4$$

The required envelope is a circle.

17. Find the envelope of $x\cos ec\theta-y\cot\theta=a,\theta$ being the parameter.

Solution:

Given $x\cos ec\theta-y\cot\theta=a$ (1)

Differentiating with respect to θ,

$$-x\cos ec\theta\cot\theta+y\cos ec^2\theta=0$$

$$x\cos ec\theta\cot\theta=y\cos ec^2\theta$$

$$\frac{x}{y}=\frac{\cos ec^2\theta}{\cos ec\theta\cot\theta}$$

$$\frac{x}{y}=\frac{1}{\cos\theta}$$

$$\cos\theta=\frac{y}{x};\quad \sin\theta=\frac{\sqrt{x^2-y^2}}{x} \qquad (2)$$

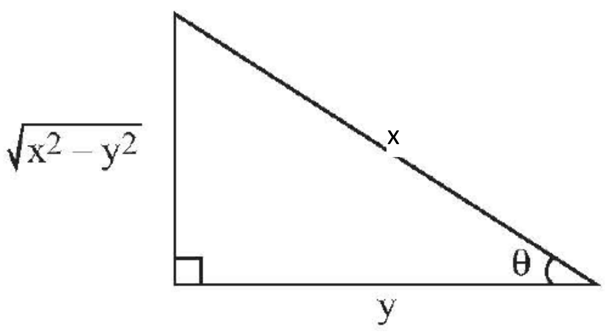


Fig 2.14

Substituting (2) in (1)

$$x\cos ec\theta - y\cot\theta = a$$

$$x\left(\frac{x}{\sqrt{x^2-y^2}}\right) - y\left(\frac{y}{\sqrt{x^2-y^2}}\right) = a$$

$$\Rightarrow \frac{x^2-y^2}{\sqrt{x^2-y^2}} = a$$

$$\Rightarrow \sqrt{x^2-y^2} = a$$

$\Rightarrow x^2 - y^2 = a^2$ is the required envelope.

EXERCISE

Part – A

1. Define envelope of a family of curves.
2. Show that the envelope of the curve $y = mx + \frac{1}{m}, m$ being the parameter is given by $y^2 = 4x$.
3. Show that the envelope of the family of lines $y = mx + 4m^2, m$ being the parameter is given by $x^2 = -16y$.
4. Show that the envelope of $y = mx + \sqrt{1+m^2}, m$ being the parameter is given by $x^2 + y^2 = 1$.
5. Show that the envelope of $y = mx + \sqrt{m^2-1}, m$ being the parameter is given by $x^2 - y^2 = 1$.
6. Show that the envelope of the family of circles $(x-\alpha)^2 + y^2 = 4\alpha, \alpha$ being the parameter is given by $y^2 = 4(x+1)$.
7. Show that the envelope of the lines $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1, \theta$ being the parameter is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
8. Show that the envelope of the lines $x\sec\theta - y\tan\theta = a, \theta$ being the parameter is given by $x^2 - y^2 = a^2$.
9. Show that the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = a\sec\alpha, \alpha$ being the parameter is given by $y^2 = -4a(x-a)$.
10. Show that the envelope of the family of straight lines $y = mx - \sqrt{a^2m^2 - b^2}, m$ being the parameter is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
11. Show that the envelope of the family of straight lines $y = mx + am^p, m$ being the parameter is given by $ap^p y^{p-1} + (p-1)^{p-1} x^p = 0$.

12. Show that the envelope of family of curves $y = mx + am^3, m$ being the parameter is given by $27ay^2 + 4x^3 = 0$.

13. Show that the envelope of $1 - x^2 + (y-k)^2 = 0$, k being the parameter is given by $x^2 = 1$.

14. Show that the family of circles $(x-a)^2 + y^2 = a^2$ has no envelope, 'a' being the parameter.

15. Show that the envelope of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{k^2 - a^2} = 1$, 'a' being the parameter is given by $x \pm y = \pm k$.

Part B

16. Show that the envelope of the lines $x\sec\theta + y\cos ec\theta = c, \theta$ being the parameter is given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

17. Show that the envelope of the family of straight lines $y\cos\theta - \sin\theta = a\cos 2\theta, \theta$ being the parameter is given by $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

18. Show that the envelope of the family of circles $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = c^2$, θ being the parameter is given by $4a^2x^2 + 4a^2y^2 = (x^2 + y^2 - c^2)^2$

19. Show that the envelope of the family of straight lines $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2, \theta$ being the parameter is given by $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$

20. Show that the envelope of the lines $x\cos^3\alpha + y\sin^3\alpha = a,$ were α the parameter is given by $x^2y^2 = a^2(x^2 + y^2)$.

21. Show that the envelope of the family of curves $x^2(x-a) + (x+a)(y-m)^2 = 0, m$ being the parameter is given by $(x-y)^{\frac{2}{3}} + (x+y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

22. Find the envelope of y = mx + $\sqrt{a^2m^2 - b^2}$, m is the parameter.

23. Find the envelope of the family of curves y = mx + a $\sqrt{1+m^2}$, m being the parameter.

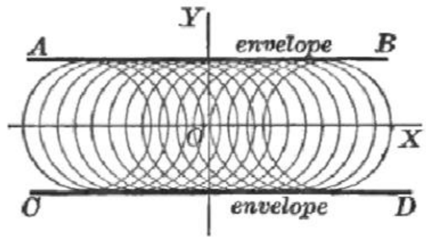
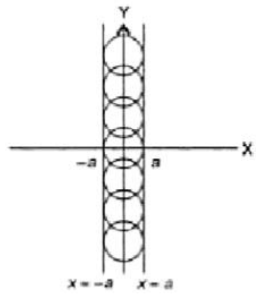
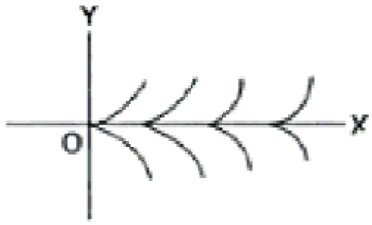
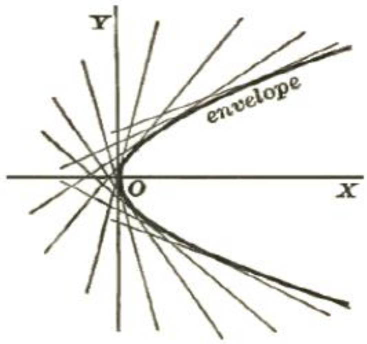
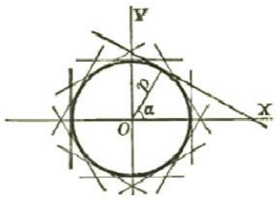
24. Find the envelope of the family of curves y = x tan $\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$ where α is the parameter.

25. Find the envelope of the circles described on the radii vectors of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameters.

26. Find the envelope of the circles passing through the origin and with their centres lying on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

27. Find the envelope of the circle whose centre lies on $xy = c^2$ and passes through the origin.

Envelopes of families of curves with single parameters

Equation of the Curve	Envelope of family with respect to given conditions
$(x - \alpha)^2 + y^2 = r^2$ α is the parameter	
$x^2 + (y - b)^2 = a^2$ b is the parameter	
$y^2 - (x + b)^3 = 0$ (semicubical parabolas) b is the parameter	
$y = \text{mx} + \dfrac{c}{m}$ m is the parameter	
x cos α + y sin α = p α is the parameter	

ENVELOPE OF FAMILY OF CURVES WITH TWO PARAMETERS

In this section we find envelopes of families of curves with two parameters, the parameters being connected by a relation.

For example, to find the envelope of a line segment of constant length c sliding on two fixed perpendicular lines leads to the problem of finding the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$ where $a^2+b^2=c^2$

In some problems of finding the envelope of two parameter family of curves, it may be possible to express one of the parameters explicitly interms of the other. This reduces the problem to a single parameter problem. When this is not possible we proceed as below.

Consider the equations $f\left(x,y,a,b\right)=0$ (1)

$$\varphi\left(a,b\right)=0 \quad (2)$$

Differentiating (1) and (2) with respect to 'a' (Treating b as a function of a)

We get, $\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}.\frac{db}{da}=0$ (3)

and $\frac{\partial \varphi}{\partial a}+\frac{\partial \varphi}{\partial b}.\frac{db}{da}=0$ (4)

Substitute for $\frac{db}{da}$ from (4) in (3) and eliminating a and b from the resulting equation and the relations (1) and (2) we get the envelope.

Examples

1. Find the envelope of the straight line $\frac{x}{a}+\frac{y}{b}=1$ where a and b are connected by the relation $a+b=c$ where c is a constant, a and b are parameters.

Solution:

Given $\frac{x}{a}+\frac{y}{b}=1$ (1)

and $a+b=c$ (2)

Differentiating (1) with respect to 'a'

$$\frac{-x}{a^2} - \frac{y}{b^2}\,\frac{db}{da} = 0 \qquad (3)$$

Differentiating (2) with respect to 'a'

$$1 + \frac{db}{da} = 0$$

$$\Rightarrow \frac{db}{da} = -1 \qquad (4)$$

Substituting (4) in (3)

$$\frac{-x}{a^2} - \frac{y}{b^2}(-1) = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{b^2}$$

(i.e.) $$\frac{\left(\frac{x}{a}\right)}{a} = \frac{\left(\frac{y}{b}\right)}{b} = \frac{\left(\frac{x}{a} + \frac{y}{b}\right)}{a+b} \quad \left[Since\, \frac{a}{b} = \frac{c}{d} = \frac{e}{f} .... = \frac{a+c+e+.........}{b+d+f+.......}\right]$$

$$= \frac{1}{a+b} = \frac{1}{c} \quad \left[From\ (1)\ and\ (2)\right]$$

$$\frac{x}{a^2} = \frac{1}{c}$$

$$\Rightarrow cx = a^2$$

$$\Rightarrow a = (cx)^{\frac{1}{2}}$$

Similarly $b = (cy)^{\frac{1}{2}}$

Substitute a and b in (2)

$$(cx)^{\frac{1}{2}} + (cy)^{\frac{1}{2}} = c$$

$$\Rightarrow c^{\frac{1}{2}}\left[x^{\frac{1}{2}} + y^{\frac{1}{2}}\right] = c$$

$$\Rightarrow x^{\frac{1}{2}} + y^{\frac{1}{2}} = \frac{c}{c^{\frac{1}{2}}}$$

$$\Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{c}$$

is the envelope of the given family of curves.

2. Find the envelope of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters are related by the equation $a^2+b^2=c^2$ where c is constant.

Solution:

$$Given \quad \frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

$$Also \quad a^2+b^2=c^2 \qquad (2)$$

Differentiating (1) with respect to 'a'

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \qquad (3)$$

Differentiating (2) with respect to 'a'

$$2a+2b\frac{db}{da}=0$$

$$\Rightarrow \frac{db}{da}=\frac{-a}{b} \qquad (4)$$

Substituting (4) in (3)

$$\frac{-x}{a^2}-\frac{y}{b^2}\left(\frac{-a}{b}\right)=0$$

$$\Rightarrow \frac{x}{a^2}=\frac{ay}{b^3}$$

$$\Rightarrow \frac{\left(\frac{x}{a}\right)}{a}=\frac{\left(\frac{y}{b}\right)}{\frac{b^2}{a}}=\frac{\left(\frac{x}{a}+\frac{y}{b}\right)}{\left(a+\frac{b^2}{a}\right)}=\frac{1}{\frac{a^2+b^2}{a}}=\frac{a}{a^2+b^2}=\frac{a}{c^2} \qquad [\text{From } (1)\&(2)]$$

$$\Rightarrow \frac{x}{a^2}=\frac{a}{c^2}\Rightarrow a^3=c^2x\Rightarrow a=\left(c^2x\right)^{\frac{1}{3}}$$

$$Similarly \ \frac{ay}{b^3}=\frac{a}{c^2}\Rightarrow b^3=c^2y\Rightarrow b=\left(c^2y\right)^{\frac{1}{3}}$$

Substituting a and b in (2)

$$\left(c^2x\right)^{\frac{2}{3}}+\left(c^2y\right)^{\frac{2}{3}}=c^2$$

$$\Rightarrow c^{\frac{4}{3}}x^{\frac{2}{3}}+c^{\frac{4}{3}}y^{\frac{2}{3}}=c^2$$

$$\Rightarrow c^{\frac{4}{3}}\left[x^{\frac{2}{3}}+y^{\frac{2}{3}}\right]=c^2$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}},$$

is the required envelope of the family of straight lines.

3. Find the envelope of the straight line $\frac{x}{a}+\frac{y}{b}=1$ where the parameters a and b are related by the equation $a^n+b^n=c^n$, c being a constant.

Solution: $\frac{x}{a}+\frac{y}{b}=1$ (1)

$a^n+b^n=c^n$ (2)

Differentiating (1) w.r.t 'a'

$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$ (3)

Differentiating (2) w.r.t 'a'

$$na^{n-1}+nb^{n-1}\frac{db}{da}=0$$

$\Rightarrow\frac{db}{da}=\frac{-a^{n-1}}{b^{n-1}}$ (4)

Substituting (4) in (3)

$$\frac{-x}{a^2}-\frac{y}{b^2}\left[\frac{-a^{n-1}}{b^{n-1}}\right]=0$$

$$\Rightarrow\frac{-x}{a^2}+\frac{ya^{n-1}}{b^{n+1}}=0$$

$$\Rightarrow\frac{x}{a^2}=\frac{ya^{n-1}}{b^{n+1}}\Rightarrow\frac{x}{a^{n+1}}=\frac{y}{b^{n+1}}$$

$$\Rightarrow \frac{x}{a.a^n}=\frac{y}{b.b^n}=\frac{\left(\frac{x}{a}\right)}{a^n}=\frac{\left(\frac{y}{b}\right)}{b^n}=\frac{\left(\frac{x}{a}+\frac{y}{b}\right)}{a^n+b^n}=\frac{1}{c^n}$$

$$\frac{x}{a^{n+1}}=\frac{1}{c^n}\Rightarrow a^{n+1}=x.c^n\Rightarrow a=\left[x.c^n\right]^{\frac{1}{n+1}}$$

$$\frac{y}{b^{n+1}}=\frac{1}{c^n}\Rightarrow b^{n+1}=y.c^n\Rightarrow b=\left[y.c^n\right]^{\frac{1}{n+1}}$$

Substituting a and b in (2)

$$(xc^n)^{\frac{n}{n+1}}+(yc^n)^{\frac{n}{n+1}}=c^n$$

$$\Rightarrow c^{\frac{n^2}{n+1}}\left[x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right]=c^n$$

$$\Rightarrow x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=\frac{c^n}{c^{\frac{n^2}{n+1}}}$$

$$=c^{n-\frac{n^2}{n+1}}$$

$$x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=c^{\frac{n}{n+1}},$$

which is the required envelope.

4. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ where a and b are connected by the relation $ab = c^2$.

Solution:

Given $\frac{x}{a}+\frac{y}{b}=1$ (1)

Also $ab = c^2$ (2)

Differentiating (1) w.r.t 'a'

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \qquad (3)$$

Differentiating (2) w.r.t 'a'

$$a\frac{db}{da}+b=0$$

$$\frac{db}{da} = \frac{-b}{a} \qquad (4)$$

Substitute (4) in (3)

$$\frac{-x}{a^2} - \frac{y}{b^2}\left(\frac{-b}{a}\right) = 0$$

$$\frac{x}{a^2} = \frac{y}{ab}$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} \Rightarrow \frac{\left(\frac{x}{a}\right)}{1} = \frac{\left(\frac{y}{b}\right)}{1} \Rightarrow \frac{\left(\frac{x}{a} + \frac{y}{b}\right)}{1+1} = \frac{1}{2} \qquad [\mathit{from}\ (1)]$$

$$\Rightarrow \frac{x}{a} = \frac{1}{2} \Rightarrow a = 2x$$

$$and\ \frac{y}{b} = \frac{1}{2} \Rightarrow b = 2y$$

Substitute a, b in (2)

$$(2x)(2y) = c^2$$
$$\Rightarrow 4xy = c^2$$

which is the required envelope

5. Determine the envelope of the two parameter family of parabolas $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ where the two parameters a and b are connected by the relation $a + b = c$ where c is a given constant.

Solution:

Using the given relation $a + b = c$,

Eliminate $b = c - a$ (1)

From the given family,

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \qquad (2)$$

Substitute (1) in (2)

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{c-a}}=1 \qquad (3)$$

which is now a one - parameter family of parabolas with 'a' as the parameter

Differentiating (3) w.r.t 'a'

$$\sqrt{x}\left[\frac{-1}{2}\right]\times\frac{1}{a^{\frac{3}{2}}}+\sqrt{y}\left[\frac{-1}{2}\right]\times\frac{1}{(c-a)^{\frac{3}{2}}}(-1)=0$$

$$\left(\frac{c-a}{a}\right)^{\frac{3}{2}}=\left(\frac{y}{x}\right)^{\frac{1}{2}}$$

$$\frac{c-a}{a}=\left(\frac{y}{x}\right)^{\frac{1}{2}\times\frac{2}{3}}$$

$$=\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$\frac{c}{a}-\frac{a}{a}=\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\frac{c}{a}=\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}+1$$

$$\frac{c}{a}=\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\Rightarrow a=\frac{cx^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}} \qquad (4)$$

Substitute (4) in (3) we get the required envelope as

$$\left[x^{\frac{2}{3}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\right]^{\frac{1}{2}}+\left[y^{\frac{2}{3}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\right]^{\frac{1}{2}}=c^{\frac{1}{2}}$$

$$\Rightarrow\left[x^{\frac{1}{3}}+y^{\frac{1}{3}}\right]\left[\left(x^{\frac{2}{3}}\right)^{\frac{1}{2}}+\left(y^{\frac{2}{3}}\right)^{\frac{1}{2}}\right]=c^{\frac{1}{2}}$$

$$\Rightarrow\left[x^{\frac{1}{3}}+y^{\frac{1}{3}}\right]^{\frac{3}{2}}=c^{\frac{1}{2}}$$

Thus the envelope is the asteroid given by $x^{\frac{1}{3}}+y^{\frac{1}{3}}=c^{\frac{1}{3}}$.

6. Show that the envelope of a family of parabolas $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ under the condition $ab=c^2$ is a hyperbola having its asymptotes coinciding with the axes.

Solution:

Equation of hyperbolas is $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$ (1)

where $ab = c^2$ (2)

Differentiating (1) with respect to 'a' regarding b as a function of a

$$x^{\frac{1}{2}}\left(\frac{-1}{2}\right)a^{\frac{-3}{2}}+y^{\frac{1}{2}}\left(\frac{-1}{2}\right)b^{\frac{-3}{2}}\frac{db}{da}=0 \quad (3)$$

Differentiating (2) w.r.t 'a' regarding b as a function of a

$$a\frac{db}{da}+b.1=0$$

$$\Rightarrow \frac{db}{da}=\frac{-b}{a} \quad (4)$$

Substitute (4) in (3)

$$\left(\frac{-1}{2}\right)x^{\frac{1}{2}}a^{\frac{-3}{2}}+\left(\frac{-1}{2}\right)y^{\frac{1}{2}}b^{\frac{-3}{2}}\left(\frac{-b}{a}\right)=0$$

Dividing by $\left(\frac{-1}{2}\right)$ and multiplying by 'a'

$$x^{\frac{1}{2}}a^{\frac{-1}{2}}-y^{\frac{1}{2}}b^{\frac{-1}{2}}=0$$

$$\Rightarrow \sqrt{\frac{x}{a}}-\sqrt{\frac{y}{b}}=0$$

$$\Rightarrow \sqrt{\frac{x}{a}}=\sqrt{\frac{y}{b}}$$

$$\Rightarrow \frac{\left(\sqrt{\frac{x}{a}}\right)}{1}=\frac{\left(\sqrt{\frac{y}{b}}\right)}{1}=\frac{\left(\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}\right)}{1+1}=\frac{1}{2} \qquad \text{From}\,(1)$$

$$\Rightarrow \sqrt{\frac{x}{a}} = \frac{1}{2} \; and \; \sqrt{\frac{y}{b}} = \frac{1}{2}$$

Squaring,

$$\Rightarrow \frac{x}{a} = \frac{1}{4} \; and \; \frac{y}{b} = \frac{1}{4}$$
$$\Rightarrow a = 4x \; and \; b = 4y$$

Substitute a and b in (2), we get

$$(4x)(4y) = c^2$$
$$\Rightarrow 16xy = c^2$$

which we know is a rectangular hyperbola with asymptotes as axes.

7. Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are connected by $\sqrt{a} + \sqrt{b} = \sqrt{c}$ and c is a constant.

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1)

Also $\sqrt{a} + \sqrt{b} = \sqrt{c}$ (2)

Differentiating (1) w.r.t 'a'

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3} \frac{db}{da} = 0 \quad (3)$$

Differentiating (2) w.r.t 'a'

$$\frac{1}{2\sqrt{a}} + \frac{1}{2\sqrt{b}} \frac{db}{da} = 0$$

$$\Rightarrow \frac{db}{da} = \frac{-\sqrt{b}}{\sqrt{a}} \quad (4)$$

Substitute (4) in (3)

$$\Rightarrow \frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\left(\frac{-\sqrt{b}}{\sqrt{a}}\right) = 0$$

$$\Rightarrow \frac{x^2}{a^3} = \frac{y^2\sqrt{b}}{b^3\sqrt{a}}$$

$$\Rightarrow \frac{\left(\frac{x^2}{a^2}\right)}{a} = \frac{\left(\frac{y^2}{b^2}\right)}{\left(\frac{b\sqrt{a}}{\sqrt{b}}\right)}$$

$$\Rightarrow \frac{\left(\frac{x^2}{a^2}\right)}{a} = \frac{\left(\frac{y^2}{b^2}\right)}{\sqrt{ab}}$$

$$\Rightarrow \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a + \sqrt{ab}} = \frac{1}{a + \sqrt{ab}} = \frac{1}{\sqrt{a}\left[\sqrt{a} + \sqrt{b}\right]} = \frac{1}{\sqrt{a}\sqrt{c}} \qquad \text{(From (1)}$$

$$\therefore \frac{x^2}{a^3} = \frac{1}{\sqrt{a}\sqrt{c}} \Rightarrow x^2\sqrt{c} = \frac{a^3}{\sqrt{a}}$$

$$\Rightarrow a^{\frac{5}{2}} = x^2 c^{\frac{1}{2}} \Rightarrow a = \left(x^2 c^{\frac{1}{2}}\right)^{\frac{2}{5}}$$

Similarly

$$\frac{y^2\sqrt{b}}{b^3\sqrt{a}} = \frac{1}{\sqrt{a}\sqrt{c}}$$

$$\Rightarrow b^{\frac{5}{2}} = y^2\sqrt{c} \Rightarrow b = \left(y^2 c^{\frac{1}{2}}\right)^{\frac{2}{5}}$$

Substitute a and b in (2)

$$\left[x^2 c^{\frac{1}{2}}\right]^{\frac{1}{5}} + \left[y^2 c^{\frac{1}{2}}\right]^{\frac{1}{5}} = c^{\frac{1}{2}}$$

$$\Rightarrow c^{\frac{1}{10}}\left[x^{\frac{2}{5}} + y^{\frac{2}{5}}\right] = c^{\frac{1}{2}}$$

$$\Rightarrow x^{\frac{2}{5}} + y^{\frac{2}{5}} = c^{\frac{2}{5}},$$

which is the required envelope of the given family.

8. Find the envelope of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ where a and b are connected by the relation $a^2+b^2=c^2$, c being a constant.

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

Also $a^2+b^2=c^2$ (2)

Eliminating b from (2) we get

$b^2=c^2-a^2$ (3)

Substitute (3) in (1), we get $\frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$

$\Rightarrow\left(c^2-a^2\right)x^2+a^2y^2=a^2\left(c^2-a^2\right)$

$\Rightarrow a^4-a^2\left(c^2+x^2-y^2\right)+c^2x^2=0$ (4)

(4) is a quadratic equation in a^2.

The envelope is given by $B^2-4AC=0$

$$\Rightarrow \quad \left(c^2+x^2-y^2\right)^2-4c^2x^2=0$$
$$\Rightarrow \quad \left[\left(c^2+x^2-y^2\right)+2cx\right]\left[c^2+x^2-y^2-2cx\right]=0$$
$$\Rightarrow \quad (x+c)^2-y^2=0,\ (x-c)^2-y^2=0$$
$$\Rightarrow \quad x+c=\pm y \text{ and } x-c=\pm y\Rightarrow \quad x=-c\pm y \text{ and } x=c\pm y$$
$$\Rightarrow \quad x\pm y=\pm c.$$

9. If $a^2+b^2=c,$ show that the envelopes of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with a, b as parameters are the straight lines $\pm x\pm y=\sqrt{c}$.

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

Treating b as a function of a and differentiating equation (1) w.r.t 'a', we get

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0 \qquad (2)$$

Also given $a^2 + b^2 = c$ (3)

Differentiating (3) w.r.t 'a'

$$2a + 2b\frac{db}{da} = 0$$

$$\frac{db}{da} = \frac{-a}{b} \qquad (4)$$

Substitute (4) in (2)

$$\frac{x^2}{a^4} = \frac{y^2}{b^4}$$

$$\frac{\left(\frac{x^2}{a^2}\right)}{a^2} = \frac{\left(\frac{y^2}{b^2}\right)}{b^2} = \frac{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}{a^2 + b^2} = \frac{1}{c} \qquad \left[\mathit{from}\ (1)\,\text{and}\,(3)\right]$$

$$\Rightarrow \frac{x^2}{a^4} = \frac{1}{c}\,\text{and}\,\frac{y^2}{b^4} = \frac{1}{c}$$

$\Rightarrow a^2 = \pm\sqrt{c}$ x and b^2 = $\pm\sqrt{c}y$

Substitute a^2 *and* b^2 in (3)

$$(3) \Rightarrow c = a^2 + b^2$$

$$c = \pm\sqrt{c}x \pm \sqrt{c}y$$

So the required envelopes are $\pm x \pm y = \sqrt{c}$.

10. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a^n + b^n = c^n$, a and b being the parameters.

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1)

Also $a^n + b^n = c^n$ (2)

Differentiating (1) w.r.t 'a'

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0 \qquad (3)$$

Differentiating (2) w.r.t 'a'

$$na^{n-1} + nb^{n-1}\frac{db}{da} = 0$$

$$\Rightarrow \frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \qquad (4)$$

Substitute (4) in (3)

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\left(\frac{-a^{n-1}}{b^{n-1}}\right) = 0$$

$$\Rightarrow \frac{x^2}{a^3} = \frac{y^2 a^{n-1}}{b^3 b^{n-1}} \Rightarrow \frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}}$$

$$\frac{\left(\frac{x^2}{a^2}\right)}{a^n} = \frac{\left(\frac{y^2}{b^2}\right)}{b^n} = \frac{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}{a^n + b^n} = \frac{1}{c^n} \qquad \left[from\ (1)\ and\ (2)\right]$$

$$\therefore \frac{x^2}{a^{n+2}} = \frac{1}{c^n}\ and\ \frac{y^2}{b^{n+2}} = \frac{1}{c^n}$$

$$\Rightarrow a^{n+2} = x^2c^n\ and\ b^{n+2} = y^2c^n$$

$$\Rightarrow a = \left[x^2c^n\right]^{\frac{1}{n+2}}\ and\ b = \left[y^2c^n\right]^{\frac{1}{n+2}}$$

Substitute a and b in (2)

$$\left[x^2\, c^n\right]^{\frac{n}{n+2}} + \left[y^2c^n\right]^{\frac{n}{n+2}} = c^n$$

$$\Rightarrow c^{\frac{n^2}{n+2}}\left[\left(x^2\right)^{\frac{n}{n+2}} + \left(y^2\right)^{\frac{n}{n+2}}\right] = c^n$$

$$\Rightarrow \left(x^2\right)^{\frac{n}{n+2}} + \left(y^2\right)^{\frac{n}{n+2}} = \frac{c^n}{c^{\frac{n^2}{n+2}}}$$

$$= c^n . c^{\frac{-n^2}{n+2}}$$

$$\Rightarrow \left[x^2\right]^{\frac{n}{n+2}} + \left[y^2\right]^{\frac{n}{n+2}} = \left[c^2\right]^{\frac{n}{n+2}}$$

which is the required envelope.

11. Find the envelope of the family of ellipses whose axes coincide and whose area is constant.

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

is the equation of the ellipse where a and b are the variables

Parameters connected by the equation $\pi ab = k$ (2)

πab being the area of an ellipse whose semi - axes are a and b.

Differentiating (1) and (2) regarding a and b as variables, we get

$$\frac{x^2}{a^3}+\frac{y^2 db}{b^3 da}=0 \Rightarrow \frac{db}{da}=\frac{-x^2b^3}{a^3y^2}$$

$$a\frac{db}{da}+b.1=0 \Rightarrow \frac{db}{da}=\frac{-b}{a}$$

From the above equations, we get $\frac{x^2}{a^2}=\frac{y^2}{b^2}$

From (1) $\frac{x^2}{a^2}=\frac{1}{2}\, and\, \frac{y^2}{b^2}=\frac{1}{2}$

$$\Rightarrow a=\pm x\sqrt{2} \quad and \quad b=\pm y\sqrt{2}$$

Substitute a and b in (2),

we get the envelope $xy=\pm\frac{k}{2\pi}$ a pair of conjugate rectangular hyperbolas.

12. Find the envelopes of the family of curves $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ where the parameters a and b are connected by the relation $a^p+b^p=c^p$.

Solution:

Equation of the given family of curves is

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1 \quad (1)$$

where the parameters a and b are connected by the relation

$$a^p+b^p=c^p \quad (2)$$

Now we shall differentiate (1) and (2) w.r.t 'a' regarding b as a function of a.

From (1) we get, $\frac{-mx^m}{a^{m+1}} - \frac{my^m}{b^{m+1}}\frac{db}{da} = 0$

$$\frac{db}{da} = \frac{\left(\frac{-x^m}{a^{m+1}}\right)}{\left(\frac{y^m}{b^{m+1}}\right)} \qquad (3)$$

From (2) we get,

$$pa^{p-1} + pb^{p-1}\frac{db}{da} = 0$$

$$\Rightarrow \frac{db}{da} = \frac{-a^{p-1}}{b^{p-1}} \qquad (4)$$

Equating the two values of $\frac{db}{da}$, we get

$$\frac{\left(\frac{x^m}{a^{m+1}}\right)}{\left(\frac{y^m}{b^{m+1}}\right)} = \frac{a^{p-1}}{b^{p-1}}$$

$$\Rightarrow \frac{\left(\frac{x^m}{a^m}\right)}{\left(\frac{y^m}{b^m}\right)} = \frac{a^p}{b^p} \qquad (5)$$

Eliminating a and b between (1), (2) and (5) we get the required envelope

From (5) we have

$$\frac{\left(\frac{x^m}{a^m}\right)}{a^p} = \frac{\left(\frac{y^m}{b^m}\right)}{b^p} = \frac{\left(\frac{x^m}{a^m} + \frac{y^m}{b^m}\right)}{a^p + b^p} = \frac{1}{c^p} \qquad \left[\, from \;\; (1)\, and\, (2)\right]$$

$$\therefore \frac{x^m}{a^{p+m}} = \frac{1}{c^p}$$

$$\Rightarrow a^{p+m} = x^m c^p \Rightarrow a = \left[x^m c^p\right]^{\frac{1}{p+m}}$$

$$\Rightarrow a^p = \left[x^m c^p\right]^{\frac{p}{p+m}} = x^{\frac{mp}{p+m}}.c^{\frac{p^2}{p+m}}$$

Similarly $b^p = y^{\frac{mp}{p+m}}.C^{\frac{p^2}{p+m}}$

Substitute $a^p and\, b^p$ in (2) we get

$$c^{\frac{p^2}{p+m}}\left\{x^{\frac{mp}{p+m}} + y^{\frac{mp}{p+m}}\right\} = c^p$$

$$\Rightarrow x^{\frac{mp}{p+m}} + y^{\frac{mp}{p+m}} = c^{p-\frac{p^2}{p+m}}$$

$$\Rightarrow x^{\frac{mp}{p+m}} + y^{\frac{mp}{p+m}} = C^{\frac{mp}{p+m}}$$

which is the required envelope.

Exercise:

1. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ where $a^2 + b^2 = 4$.

2. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where parameters a and b are connected by the relation $a^3 + b^3 = c^3, c$ being a constant.

3. Find the envelope of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ where ab = 4.

4. Find the envelope of the family of curves $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ where a and b are connected by the relation (i) $a^n + b^n = c^n$.

   (ii) a + b = c, c being a constant.

5. Find the envelope of curves $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

   when (i) $a + b = c$.

   (ii) $ab = c^2, c$ being a constant.

6. Show that the envelope of the straight line of given length l which slides with extremities on two fixed straight lines at right angles is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$.

Hint : Let us take the two fixed straight lines at right angles as axes. Let the equation of the line with its ends on the given perpendicular lines as axes be $\frac{x}{a}+\frac{y}{b}=1$ where $a^2+b^2=l^2$.

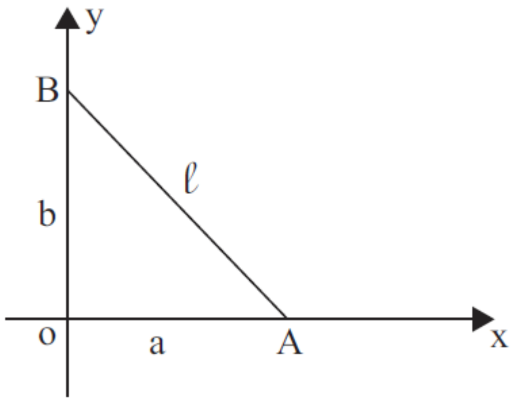


Fig.2.15

7. Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the parameters are connected by a + b = c, c is a constant.

8. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ where the parameters a & b are connected by the relation $a^m b^m = c^{m+n}$, c is a constant.

Answers

1. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

2. $x^{\frac{3}{4}}+y^{\frac{3}{4}}=c^{\frac{3}{4}}$.

3. 4xy = 1.

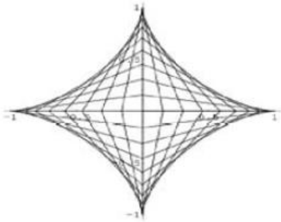
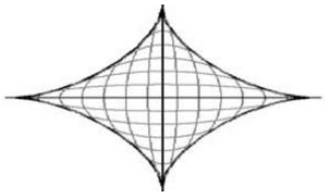
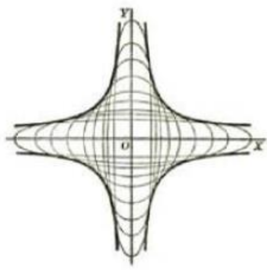
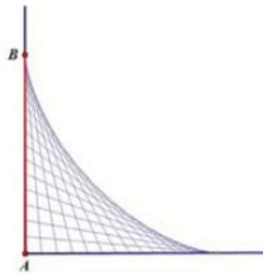
4. i) $x^{\frac{n}{2n+1}}+y^{\frac{n}{2n+1}}=c^{\frac{n}{2n+1}}$. ii) $x^{\frac{1}{3}}+y^{\frac{1}{3}}=c^{\frac{1}{3}}$.

5. i) $x^{\frac{m}{m+1}}+y^{\frac{m}{m+1}}=c^{\frac{m}{m+1}}$ ii) $4xy=c^2$.

7. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$

8. $x^m y^n=\left(\frac{c}{m+n}\right)^{m+n} m^m n^n$

Envelope of Family of Curves with Two Parameters

Equation of the Curve	Condition for Parameters	Envelope of Family with Respect to Given Condition
$\frac{x}{a}+\frac{y}{b}=1$	*$a^2 + b^2 = c^2$*	
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	*$a + b = c$*	
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	*$ab = c$*	
$\frac{x}{a}+\frac{y}{b}=1$	*$a + b = c$*	

EVOLUTE AS ENVELOPE OF NORMALS

Let P_1, P_2, P_3 be consecutive points on a curve and the normal at P_1, P_2 cut at Q_1 and the normal at P_2, P_3 cut at Q_2. In the limiting process as P_3 *and* P_1 moves towards P_2, Q_2 moves towards Q_1. Q_2 *and* Q_1 are the centres of curvature at the points P_2 *and* P_1. As both Q_2 *and* Q_1 lie on the evolute and also on the normals, it is clear that the normals to the original curve are tangents to the evolute. Hence the evolute can also be thought of as the envelope of the normals to the original curve.

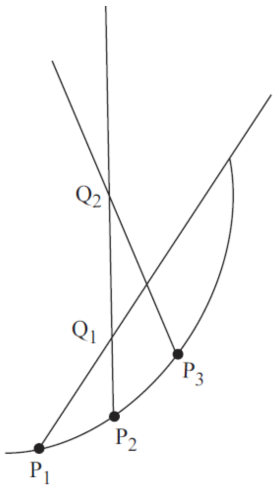


Fig 2.16

In the differential geometry of curves, the evolute of a curve is the locus of all its centres of curvature, Equivalently it is the envelope of the normals to a curve. The original curve is an involute of its evolute.

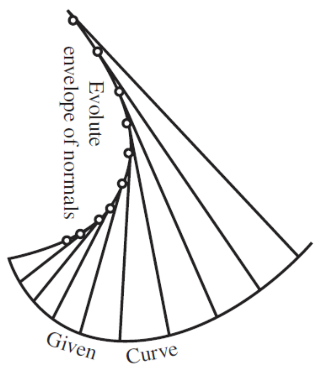


Fig 2.17

The normals to the curve form a family of straight lines. Thus the envelope of the normals is the locus of the ultimate points of intersection of consecutive normals. But we know that the centres of curvature is the point of intersection of consecutive normals. So the envelope of the normals must be the locus of the centres of curvatures, which is the evolute of the given curve.

Hence, the evolute of a given curve can also be considered as the envelope of the normals to the curve.

Problems:

1. Find the evolute of $y^2 = 4ax$ considering it as the envelope of the normals.

Solution:

Any point on $y^2 = 4ax$ is $\left(at^2, 2at\right)$

$$x = at^2, \qquad y = 2at$$
$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$
$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

The equation of the normal to $\left(x_1, y_1\right)$ with m as the slope of the tangent to a curve is given by

$$\left(y - y_1\right) = \frac{-1}{m}\left(x - x_1\right)$$
$$y - 2at = \frac{-1}{\left(\frac{1}{t}\right)}\left(x - at^2\right)$$

$$y + xt = at^3 + 2at \qquad (1)$$

Differentiate equation (1) partially with respect to t we have

$$x = 3at^2 + 2a$$

$$t^2 = \frac{x - 2a}{3a} \Rightarrow t = \left(\frac{x - 2a}{3a}\right)^{\frac{1}{2}}$$

Substituting the value of t in equation (1)

$$y+x\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}=a\left(\frac{x-2a}{3a}\right)^{\frac{3}{2}}+2a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}$$

$$=a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left[\frac{x-2a}{3a}+2\right]$$

$$=a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left[\frac{x-2a+6a}{3a}\right]$$

$$=a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left[\frac{x+4a}{3a}\right]$$

$$y=a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left(\frac{x+4a}{3a}\right)-x\left[\frac{x-2a}{3a}\right]^{\frac{1}{2}}$$

$$=\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left[a\left(\frac{x+4a}{3a}\right)-x\right]$$

$$=\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left[\frac{x+4a-3x}{3}\right]$$

$$=\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left(\frac{4a-2x}{3}\right)$$

$$=-2a\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}\left(\frac{x-2a}{3a}\right) \qquad \left[\begin{array}{l}\text{Multiply the numerator}\\ \text{and denominator by a}\end{array}\right]$$

$$y=-2a\left(\frac{x-2a}{3a}\right)^{\frac{3}{2}}$$

$$y=\frac{-2}{3^{\frac{3}{2}}a^{\frac{1}{2}}}(x-2a)^{\frac{3}{2}}$$

$$3^{\frac{3}{2}}.a^{\frac{1}{2}}.y=-2(x-2a)^{\frac{3}{2}}$$

Squaring on both side,

$$27ay^{2}=4(x-2a)^{3}.$$

2. Regarding the evolute as the envelope of the normals, show that the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

Solution:

Any point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $(a\cos\theta, b\sin\theta)$

$x=a\cos\theta,\qquad y=b\sin\theta$

$\frac{dx}{d\theta}=-\sin\theta\qquad \frac{dy}{d\theta}=b\cos\theta$

$m=\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\cos\theta}{-a\sin\theta}=\frac{-b}{a}\cot\theta$

Equation of the normal at (x_1, y_1) is

$$(y-y_1)=-\frac{1}{m}(x-x_1)$$

$$y-b\sin\theta=\frac{a}{b\cot\theta}(x-a\cos\theta)$$

$$b\cot\theta(y-b\sin\theta)=a(x-a\cos\theta)$$

$$b\frac{\cos\theta}{\sin\theta}(y-bsin\theta)=a(x-a\cos\theta)$$

by $\cos\theta-b^2\cos\theta\sin\theta=ax\sin\theta-a^2\cos\theta\sin\theta$

Divide by $\sin\theta\cos\theta$

$$\frac{by}{\sin\theta}-b^2=\frac{ax}{\cos\theta}-a^2$$

$$\frac{ax}{\cos\theta}\frac{-by}{\sin\theta}=a^2-b^2 \qquad (1)$$

Now partially differentiate equation (1) with respect to θ

$$ax\times\frac{1}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta}\right)=-by.\frac{1}{\sin\theta}.\frac{\cos\theta}{\sin\theta}$$

$$\frac{ax}{\cos\theta}\tan\theta=\frac{-by\cot\theta}{\sin\theta}$$

$$ax(\tan\theta)^2 = -by\cot\theta$$

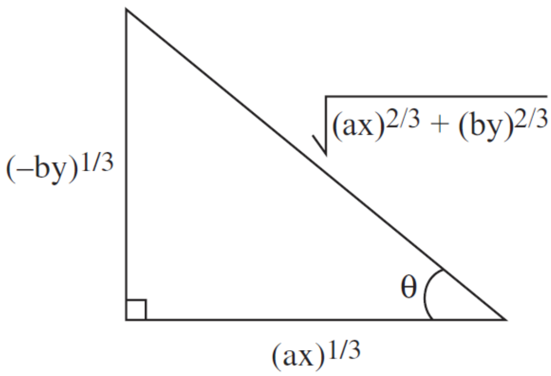


Fig 2.17

$$ax(\tan\theta)^3 = -by$$

$$(\tan\theta)^3 = \frac{-by}{ax}$$

$$(\tan\theta) = \left(\frac{-by}{ax}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(-by)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}$$

$$\cos\theta = \frac{(ax)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}$$

Equation (1) becomes

$$\Rightarrow \frac{ax\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}{(ax)^{\frac{1}{3}}} + \frac{by\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}{(by)^{\frac{1}{3}}} = a^2 - b^2$$

$$\Rightarrow \sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}\left[(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right] = a^2 - b^2$$

$$\Rightarrow (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$$

which is the required evolute.

3) Considering the evolute as the envelope of the normals, find the evolute of the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution:

Any point on the asteroid

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is given by $\left(a\cos^3\theta, a\sin^3\theta\right)$

$$i.e.\ x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta\left(-\sin\theta\right) \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\left(\cos\theta\right)$$

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$m = -\tan\theta$$

Equation of the normal is

$$y - a\sin^3\theta = \frac{1}{\tan\theta}\left(x - a\cos^3\theta\right)$$
$$\sin\theta\left(y - a\sin^3\theta\right) = \cos\theta\left(x - a\cos^3\theta\right)$$
$$y\sin\theta - a\sin^4\theta = x\cos\theta - a\cos^4\theta$$
$$y\sin\theta - x\cos\theta = -a(\cos^4\theta - \sin^4\theta)$$
$$= -a\left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) \qquad \left[\because a^2 - b^2 = (a+b)(\mathrm{a}-\mathrm{b})\right]$$
$$= -a\cos 2\theta \qquad \left[\because \cos 2A = \cos^2 A - \sin^2 A\right]$$

$$i.e.\quad y\sin\theta - x\cos\theta = -a\cos 2\theta \qquad (1)$$

Differentiate equation (1) with respect to θ

$$y\cos\theta + x\sin\theta = 2a\sin 2\theta \qquad (2)$$

Eqn $(1)\times\cos\theta \Rightarrow y\cos\theta\sin\theta - x\cos^2\theta = -a\cos 2\theta\cos\theta$

Eqn $(2)\times\sin\theta \Rightarrow y\sin\theta\cos\theta + x\sin^2\theta = 2a\sin 2\theta\sin\theta$

Subtracting we have,

$$-x(\cos^2\theta+\sin^2\theta)=-a\cos 2\theta\cos\theta-2a\sin 2\theta\sin\theta$$
$$x=a\cos 2\theta\cos\theta+2a\sin 2\theta\sin\theta$$
$$=a(\cos^2\theta-\sin^2\theta)\cos\theta+2a\left(2\sin\theta\cos\theta\right)\sin\theta$$
$$=a\left[\cos^3\theta-\sin^2\theta\cos\theta\right]+4a\sin^2\theta\cos\theta$$

$$x=a\cos^3\theta+3a\sin^2\theta\cos\theta \qquad (3)$$

Now, Eqn $\left(1\right)\times\sin\theta \Rightarrow y\sin^2\theta-x\cos\theta\sin\theta=-a\cos 2\theta\sin\theta$

Eqn $\left(2\right)\times\cos\theta \Rightarrow y\cos^2\theta+x\sin\theta\cos\theta=2a\sin 2\theta\cos\theta$

Adding we have,

$$y(\sin^2\theta+\cos^2\theta)=2a\sin 2\theta\cos\theta-a\cos 2\theta\sin\theta$$

$$y=2a(2\sin\theta\cos\theta.\cos\theta)-a(\cos^2\theta-\sin^2\theta)\sin\theta$$

$$y=4a\sin\theta\cos^2\theta-a\cos^2\theta\sin\theta+a\sin^3\theta$$

$$y=3a\sin\theta\cos^2\theta+a\sin^3\theta \qquad (4)$$

Adding (3) & (4) we get

$$x+y=a\left(\cos^3\theta+\sin^3\theta\right)+3a\sin\theta\cos\theta(\sin\theta+\cos\theta)$$
$$=a\cos^3\theta+3a\sin^2\theta\cos\theta+3a\cos^2\theta\sin\theta+a\sin^3\theta$$

$$x+y=a\left(\cos\theta+\sin\theta\right)^3$$
$$\left(x+y\right)^{\frac{2}{3}}=a^{\frac{2}{3}}\left(\cos\theta+\sin\theta\right)^2$$

Similarly,

$$x-y=a\cos^3\theta-3a\cos^2\theta\sin\theta+3a\sin^2\theta\cos\theta-a\sin^3\theta$$
$$x-y=a\left(\cos\theta-\sin\theta\right)^3$$
$$\left(x-y\right)^{\frac{2}{3}}=a^{\frac{2}{3}}\left(\cos\theta-\sin\theta\right)^2$$

Thus,

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}\left[(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2\right]$$

$$=a^{\frac{2}{3}}\left[\cos^2\theta+\sin^2\theta+\cancel{2\cos\theta\sin\theta}+\cos^2\theta+\sin^2\theta-\cancel{2\cos\theta\sin\theta}\right]$$

$$=2a^{\frac{2}{3}}$$

$$\therefore (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$$

which is the required evolute.

4. Find the evolute of cycloid $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$ treating it as the envelope of its normals.

Solution:

$$x=a(\theta-\sin\theta)$$

$$y=a(1-\cos\theta)$$

$$\frac{dx}{d\theta}=a(1-\cos\theta);\qquad \frac{dy}{d\theta}=a\sin\theta$$

$$\frac{dy}{dx}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{\sin\theta}{1-\cos\theta}$$

Equation of the normal,

$$y-y_1=-\frac{1}{m}(x-x_1)$$

$$m=\frac{dy}{dx}$$

$$y-a(1-\cos\theta)=\frac{-1}{\left(\frac{\sin\theta}{1-\cos\theta}\right)}\left(x-a(\theta-\sin\theta)\right)$$

$$y-a+a\cos\theta=\frac{-1(1-\cos\theta)}{\sin\theta}(x-a\theta+a\sin\theta)$$

$$y\sin\theta\cancel{-a\sin\theta}+\cancel{a\sin\theta\cos\theta}=-x+a\theta-\cancel{a\sin\theta}+x\cos\theta-a\theta\cos\theta+\cancel{a\sin\theta\cos\theta}$$

$$x+y\sin\theta=a\theta-a\theta\cos\theta+x\cos\theta$$

$$x + y\sin\theta = a\theta(1-\cos\theta) + x\cos\theta$$

$$\Rightarrow x - x\cos\theta + y\sin\theta = a\theta(1-\cos\theta)$$

$$\Rightarrow x(1-\cos\theta) + y\sin\theta = a\theta(1-\cos\theta)$$

$$\Rightarrow x + \frac{y\sin\theta}{1-\cos\theta} = a\theta \qquad (1)$$

Since $\displaystyle \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$

From (1)

$$x + y\cot\frac{\theta}{2} = a\theta \qquad (2)$$

Differentiate (2) with respect to θ

$$y\left(-\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{2}\right) = a$$

$$y = \frac{-2a}{\operatorname{cosec}^2\frac{\theta}{2}} \qquad \left(\because 2\sin^2\frac{\theta}{2} = 1-\cos\theta\right)$$

$$y = -2a\sin^2\frac{\theta}{2}$$

$$y = -a(1-\cos\theta)$$

Substitute in equation (1)

$$x + \left[-a(1-\cos\theta)\right]\frac{\sin\theta}{1-\cos\theta} = a\theta$$

$$x - a\sin\theta = a\theta$$

$$x = a\theta + a\sin\theta$$

$$x = a(\theta + \sin\theta)$$

From (3) & (4) envelope of the normal is a cycloid.

Hence evolute of the given cycloid is another cycloid.

Exercise

1. Define Evolute of a curve as an envelope of its normals.

2. Regarding the evolute of a curve as the envelope of its normals, find the evolute of $x^2 = 4ay$.

3. Find the evolute of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ considering it as an envelope of its normals.

4. Considering the evolute of a curve as the envelope of its normals, find the evolute of the rectangular hyperbola $xy = c^2$.

5. Show that the evolute of the cycloid $x = a(\theta + \sin\theta), y = -a(1 - \cos\theta)$, treating it as the envelope of its normals is another cyclord $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

Answers

2. $27ax^2 = 4(y - 2a)^3$

3. $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$

4. $(x + y)^{\frac{2}{3}} - (x - y)^{\frac{2}{3}} = (4c)^{\frac{2}{3}}$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –III – Multivariable Calculus – SMTA1106

UNIT 3

MULTI VARIABLE CALCULUS

Introduction

A function of two variables maps points (x, y) in the XY plane to numbers z on the Z axis. These functions are generally denoted by f (x, y). It can also be considered as an assignment of a real number to a point (x, y) in the XY plane.

The equation in two variables geometrically represents a curve which indicates the dependency between the variable quantities. This idea is originated from Descartes during 1596-1650.

Later it was Leibnitz (1646-1716) who first used the term function in 1673. He also introduced the terms constants, variables and parameters.

A function f of two variables is a relation, which maps every point of a set D in the XY plane to at most one real number z. The set D is called the domain of the function f. This representation of two variables as a function is identified by Euler during the period 1707-1783, in practice.

Examples of Functions of Two variables

Consider the functions $z = xy$, $z = cosx\,siny$. In these examples *x, y* are called the **independent variables** and z is called the **dependent variable**.

The graph of the functions of two variables is a surface z = f (x, y) where z is the height of the surface at (x, y).

For example $z = 2x^2 + 2y^2 - 4$ is an elliptic paraboloid.

Applications of functions of two variables

The functions of many variables are useful in every field of engineering applications. For example when a violin is placed in the XY plane with strings of length *l* coincides on the x axis, then u (x, t) is defined as the displacement of the string above or below a point x on the x axis at a time t, then $y = u\ (x,\ t)$ is the shape of the string at a fixed time t. Likewise u (x, t) might represent the evolution of the temperature distribution of a thin rod, where u represents the temperature at time t at a distance x from one end. The distribution of temperature on a thin metal plate with surfaces insulated is also a function of two variables $u(x,\ y)$ in the XY plane. The Ideal gas law is also an example of function of two variables. The Ideal gas law given by $P = \rho \mathrm{RT}$. Here *P* is a function of both density ρ, temperature T and R is the gas constant.

Limits

The function $f(x, y)$ is said to tend to the limit *l* as x$\rightarrow$a and y$\rightarrow$b if and only if the limit '*l*' is independent of the path. In this case we write

$$\lim_{\substack{x\to a \\ y\to b}} f(x, y) = l$$

The limit can also be defined in terms of a circular neighbourhood as follows. The function f (x, y) defined in a region R is said to tend to the limit '*l*' as x$\rightarrow$a and y$\rightarrow$b if and only if corresponding to a positive number ε there exists another positive number δ such that $\left| f(x, y) - l \right| < \varepsilon$ for $0 < (\mathrm{x}-\mathrm{a})^2 + (\mathrm{y}-\mathrm{b})^2 < \sigma = \delta^2$ for every point (x, y) in R

Continuity:

A function f (x, y) is said to be continuous at the point (a, b) if

$\lim_{\substack{x\to a \\ y\to b}} f(x, y)$ exists and $\lim_{\substack{x\to a \\ y\to b}} f(x, y) = f(a, b)$.

Partial derivatives

Functions of multiple variables can be differentiated with respect to either of their independent variables, the other variable being treated as constant during the differentiation. Such derivatives are known as Partial derivatives.

Let $z = \mathrm{f}(x, y)$ be a function of two variables x & y. The partial derivative of z with respect to x keeping y as a constant is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$\text{Similarly } \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

The partial derivative exists only when the above limit exists.

First Order Derivatives:

For a function of two variables there are two first order derivatives, For example if z is a function of x and y then the first order derivatives are $\frac{\partial z}{\partial x} = z_x = F_x\left(x, y\right)$ and

$\frac{\partial z}{\partial y} = z_y = F_y\left(x, y\right)$. These derivatives are also functions of x and y.

Second order and Higher order derivatives

For functions of two variables there are 3 second order partial derivatives which are defined as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(z_x) = z_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(z_y) = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(z_y) = z_{xy} \qquad (1)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(z_x) = z_{yx} \qquad (2)$$

Generally $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

The Higher order derivatives are recursively defined as

$$\frac{\partial^5 z}{\partial x^3 \partial y^2} = \frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 z}{\partial y^2}\right) = z_{xxxyy}$$

Problems

1) Find the first and the second order partial derivatives of $z = x^3 + 3y - y^3 - 3x$.

Solution:

$$\frac{\partial z}{\partial x} = 3x^2 - 3, \quad \frac{\partial z}{\partial y} = 3 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3) = 6x.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3-3y^2) = -6y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3-3y^2) = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2-3) = 0$$

2) If $z = xcos\ y - ycos\ x$, then prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Solution:

Let $\frac{\partial z}{\partial x} = cos\ y + y\sin x$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial y} = -x\sin y - \cos x.$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(-x\sin y - \cos x)$$

$$= -\ sin\ y\ +\ sin\ x \qquad (1)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\cos y + y\sin x\right)$$

$$= -\ sin\ y\ +\ sin\ x \qquad (2)$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

3) Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ for $z = e^x \cos y$

Solution:

$$\frac{\partial z}{\partial x} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y \qquad (1)$$

$$\frac{\partial z}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-e^x \sin y) = -e^x \cos y \qquad (2)$$

(1)+(2) $\Rightarrow$ $\quad \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \cos y - e^x \cos y = 0.$

4) If $u = log\left(ax+by\right)$ find $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial x \partial y}$

Solution:

$$\frac{\partial u}{\partial x} = \frac{a}{ax+by}$$

$$\frac{\partial u}{\partial y} = \frac{b}{ax+by}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[a(ax+by)^{-1}\right]$$

$$= a(-1)(ax+by)^{-2}(a) = \frac{-a^2}{(ax+by)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[b(ax+by)^{-1}\right]$$

$$= b(-1)(ax+by)^{-2}\left(b\right) = \frac{-b^2}{(ax+by)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{b}{ax+by}\right)$$

$$= \frac{\partial}{\partial x}\left(b[ax+by]^{-1}\right)$$

$$= b(-1)(ax+by)^{-2}\left(a\right) = \frac{-ab}{(ax+by)^2}$$

5) Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = \sin^{-1}\left(y/x\right)$

Solution:

$$\frac{\partial u}{\partial x} = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(1/x\right)$$

$$= \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\left(x^2-y^2\right)^{-1/2}\right)$$

$$= \left(\frac{-1}{2}\right)(x^2-y^2)^{\frac{-3}{2}}(2x)$$

$$= \frac{-x}{(x^2-y^2)^{3/2}} \qquad (1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-1}{x}y(x^2-y^2)^{\frac{-1}{2}}\right)$$

$$= \frac{-1}{x}\left[y\left(\frac{-1}{2}\right)(x^2-y^2)^{\frac{-3}{2}}(-2y) + (x^2-y^2)^{-1/2}\right]$$

$$= \frac{-1}{x}\left[\frac{y^2+(x^2-y^2)}{(x^2-y^2)^{\frac{3}{2}}}\right]$$

$$= \frac{-x}{(x^2-y^2)^{\frac{3}{2}}} \qquad (2)$$

Hence $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

6) If $u = x^y$, Show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Solution:

Given $u = x^y$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= yx^{y-1} \log x + x^y . \frac{1}{x}$$

$$= x^{y-1}\left(y \log x + 1\right)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^{y-1}\left(y \log x + 1\right)\right] \qquad (1)$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= x^{y-1} + yx^{y-1} \log x$$

$$= x^{y-1}(\text{ylogx+1})$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^{y-1}\left(y \log x + 1\right)\right] \qquad (2)$$

Hence $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

7. If $z = e^{ax+by} f(ax-by)$, show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution:

Differentiating z partially with respect to x

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)a + f(ax-by)a\, e^{ax+by}$$

$$b\frac{\partial z}{\partial x} = ab\, e^{ax+by} \left[f'(ax-by) + f(ax-by) \right] \qquad (1)$$

Differentiating z partially with respect to y

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)b\, e^{ax+by}$$

$$a\frac{\partial z}{\partial y} = ab\, e^{ax+by} \left[-f'(ax-by) + f(ax-by) \right] \qquad (2)$$

Adding (1) & (2)

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abe^{ax+by} f(ax-by)$$

= 2abz

Exercise problems

1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z = log $(x^2 + y^2)$

2. If $z = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

3. If $z = sin3x\, cos4y$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$

4. If $u = x^4 + y^4 + 3x^2y^2$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u$

5. If $z = sin(y + ax)$, then prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

6. If $x = r\cos\theta$, $y = r\sin\theta$, then prove that $\frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial \theta} = r\sin\theta$

7. If $x = r\cos\theta$, $y = r\sin\theta$, then prove that $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

8. If $u = x^2 y + y^2 z + z^2 x$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$

9. If $u = x^3 + y^3 - 3axy$, then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

10. If $f = \frac{x - 2y}{x + y}$, then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Answers:

1. $\frac{\partial z}{\partial x} = \frac{2y}{x^2 + y^2}$; $\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

2. $\frac{\partial z}{\partial x} = \frac{x^2 + 2xy - y^2}{(x+y)^2 + (\mathrm{x}^2 + \mathrm{y}^2)^2}$, $\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2 + (\mathrm{x}^2 + \mathrm{y}^2)^2}$

3. $\frac{\partial z}{\partial x} = 3\cos 3x \cos 4y$, $\frac{\partial z}{\partial y} = -4\sin 3x \sin 4y$, $\frac{\partial^2 z}{\partial x^2} = -9\sin 3x \cos 4y$

10. $\frac{\partial f}{\partial x} = \frac{3y}{(x+y)^2}$; $\frac{\partial f}{\partial y} = \frac{-3x}{(x+y)^2}$

HOMOGENEOUS FUNCTIONS

A function in which every term is of the same degree, is known as a homogeneous function of that degree.

Consider,

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + ... + a_{n-2} x^2 y^{n-2} + a_{n-1} x y^{n-1} + a_n y^n \quad (1)$$

Since every term of this function has the same degree, it is a homogeneous function of degree n in x and y.

Examples:

1) $f(x,y)=ax^2+by^2+cxy$ is a homogeneous function of degree 2

2) $f(x,y)=x+y$ is a homogeneous function of degree 1

Now the expression (1) can be written as

$$f(x,y)=x^n\left\{a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+....+a_n\left(\frac{y}{x}\right)^n\right\} \text{ which is of the form } x^nF\left(\frac{y}{x}\right)$$

[or]

$$f(x,y)=y^n\left\{a_n+a_{n-1}\left(\frac{x}{y}\right)+a_{n-2}\left(\frac{x}{y}\right)^2+.....a_0\left(\frac{x}{y}\right)^n\right\} \text{ which is of the form } y^nG\left(\frac{x}{y}\right)$$

Thus, every homogeneous function of degree n in x and y can be expressed in the form $x^nF\left(\frac{y}{x}\right)$ or $y^nG\left(\frac{x}{y}\right)$.

Examples:

1) $x^3\cos\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree 3

2) $\tan^{-1}\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree 0.

TEST FOR HOMOGENEITY OF A FUNCTION OF TWO VARIABLES

If $f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is called a homogeneous function of degree n where n is any real number.

The above equation is called Euler's equation

Example:

$f(x,y)=\dfrac{x^3+y^3}{x+y}$ is a homogeneous function of degree 2

Since $f(tx,ty)=\dfrac{(tx)^3+(ty)^3}{(tx)+(ty)}=\dfrac{t^3x^3+t^3y^3}{tx+ty}=\dfrac{t^3(x^3+y^3)}{t(x+y)}$

$$= t^2\left(\frac{x^3+y^3}{x+y}\right) = t^2 f(x,y)$$

NOTE:

If the *numerator* of a function is homogeneous of degree *p* and the *denominator* of the same function is homogeneous of degree *q*, then the *degree of the function* is given by,

Degree of a function = degree of numerator - degree of denominator = p - q

Example:

Let $f(x,y) = \left[\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}\right]^{\frac{1}{2}}$

Here numerator $= \left[x^{\frac{1}{3}}+y^{\frac{1}{3}}\right]^{\frac{1}{2}} = f_1(x,y)(say)$

$$f_1(tx,ty) = \left[(tx)^{\frac{1}{3}}+(ty)^{\frac{1}{3}}\right]^{\frac{1}{2}} = \left\{t^{\frac{1}{3}}\left[x^{\frac{1}{3}}+y^{\frac{1}{3}}\right]\right\}^{\frac{1}{2}}$$

$$f_1(tx,ty) = t^{\frac{1}{6}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)^{\frac{1}{2}} = t^{\frac{1}{6}} f_1(x,y)$$

$\Rightarrow$ Numerator is a homogeneous function of degree $\dfrac{1}{6}$

Denominator $= \left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)^{\frac{1}{2}} = f_2(x,y)$ (say)

$$f_2(tx,ty) = \left[(tx)^{\frac{1}{4}}-(ty)^{\frac{1}{4}}\right]^{\frac{1}{2}} = \left[t^{\frac{1}{4}}\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)\right]^{\frac{1}{2}}$$

$$f_2(tx,ty) = t^{\frac{1}{8}}\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)^{\frac{1}{2}} = t^{\frac{1}{8}} f_2(x,y)$$

$\Rightarrow$ Denominator is a homogeneous function of degree $\dfrac{1}{8}$

$\therefore f\,(x, y)$ is also a homogeneous function of degree $\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

APPLICATIONS:

Homogeneous functions have enormous applications in various fields. It is applicable in projective geometry, differential equations, special function, calculus of variations, analytical mechanics, dimensional analysis, economics, thermodynamics and so on…

LEONHARD EULER [1707-1783] was a pioneering Swiss Mathematician and Physicist. He made important discoveries in various fields like calculus, graph theory, number theory, physics and so on. His contributions were so numerous that terms like Euler's formula or Euler's theorem can mean many different things depending on the context.

EULER'S Theorem on Homogeneous Functions

If u(x, y) is homogeneous function of degree n in x and y then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$

Proof:

Since u is a homogeneous function of degree n in x and y, can be expressed as

$$u=x^nF\left(\frac{y}{x}\right) \qquad (1)$$

Differentiating (1) partially with respect to x, we have

$$\frac{\partial u}{\partial x}=x^nF'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)+F\left(\frac{y}{x}\right).nx^{n-1}$$

$$\frac{\partial u}{\partial x}=-yx^{n-2}F'\left(\frac{y}{x}\right)+nx^{n-1}F\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x}=-yx^{n-1}F'\left(\frac{y}{x}\right)+nx^nF\left(\frac{y}{x}\right) \qquad (2)$$

Differentiating (1) partially with respect to y, we have

$$\frac{\partial u}{\partial y}=x^nF'\left(\frac{y}{x}\right).\frac{1}{x}=x^{n-1}F'\left(\frac{y}{x}\right)$$

$$\therefore y\frac{\partial u}{\partial y} = x^{n-1} yF\left(\frac{y}{x}\right) \qquad (3)$$

$$(2)+(3) \text{ gives, } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1}F'\left(\frac{y}{x}\right) + nx^n F\left(\frac{y}{x}\right) + x^{n-1}yF'\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n F\left(\frac{y}{x}\right) = nu$$

Hence the theorem is proved.

The above theorem can be generalized to homogeneous functions of any number of variables. Thus if $u = f(x_1, x_2, ...x_m)$ is a homogeneous function of degree n in variables $x_1, x_2, ...x_m$ then

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + ... + x_m\frac{\partial u}{\partial x_m} = nu$$

COROLLARY 1:

If u is a homogeneous function of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 y}{\partial y^2} = n(n-1)u$$

Proof:

Since u is a homogeneous function of degree n, by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad (1)$$

Differentiating (1) partially with respect to x, we get,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n.\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x} \qquad (2)$$

Differentiating (1) partially with respect to y on both sides, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n.\frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \qquad (3)$$

$(2)\times x+(3)\times y$ gives,

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u \qquad \left[\text{since}\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

COROLLARY 2:

If v is a homogeneous function of degree n in x and y and if $v=f(u)$ then

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$ where $g(u)=n\frac{f(u)}{f'(u)}$

Proof:

Since v is a homogeneous function, by Euler's theorem, $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv$

Again since $v=f(u)$, $\frac{\partial v}{\partial x}=f'(u)\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}=f'(u)\frac{\partial u}{\partial y}$

$$\therefore\ xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}=nf(u)$$

(or) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}$

Taking $\frac{nf(u)}{f'(u)}=g(u)$

$$\text{we get } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g(u) \qquad (1)$$

Differentiating (1) partially with respect to x,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left[g'(u)-1\right]\frac{\partial u}{\partial x} \qquad (2)$$

Differentiating (1) partially with respect to y,

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\left[g'(u)-1\right]\frac{\partial u}{\partial y} \qquad (3)$$

$$(2)\times x+(3)\times y\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=\left[g'(u)-1\right]\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left[g'(u)-1\right]g(u)$$

$$\left[\text{since } \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

APPLICATIONS:

Euler's theorem on homogeneous function is applicable in Lagrangian Dynamics, useful in developing thermodynamic distinction between extensive and intensive variables of state and deriving Gibb's-Duhem relation [energy form of Euler's equation] Also useful in production economics theory.

Examples

1. Verify Euler's theorem for the following functions

(i) $3x^2yz+5xy^2z+4z^4$ (ii) $\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$ (iii) $x^3\log\left(\dfrac{y}{x}\right)$

(iv) $\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$

Solution

(i) Let $u=3x^2yz+5xy^2z+4z^4$

Then, $\dfrac{\partial u}{\partial x}=6xyz+5y^2z,\qquad \dfrac{\partial u}{\partial y}=3x^2z+10xyz,\qquad \dfrac{\partial u}{\partial z}=3x^2y+5xy^2+16z^3$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x(6xyz+5y^2z)+y(3x^2z+10xyz)+z(3x^2y+5xy^2+16z^3)$$

$$=6x^2yz+5xy^2z+3x^2yz+10xy^2z+3x^2yz+5xy^2z+16z^4$$
$$=12x^2yz+20xy^2z+16z^4=4\left[3x^2yz+5xy^2z+4z^4\right]$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4u$$

$\therefore$ Euler's theorem is verified.

(ii) $u=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$

Solution:

$$u(tx,ty)=\frac{(tx)^{\frac{1}{4}}+(ty)^{\frac{1}{4}}}{(tx)^{\frac{1}{5}}+(ty)^{\frac{1}{5}}}=\frac{t^{\frac{1}{4}}\left\{x^{\frac{1}{4}}+y^{\frac{1}{4}}\right\}}{t^{\frac{1}{5}}\left\{x^{\frac{1}{5}}+y^{\frac{1}{5}}\right\}}=t^{\frac{1}{4}-\frac{1}{5}}\frac{(x^{\frac{1}{4}}+y^{\frac{1}{4}})}{(x^{\frac{1}{5}}+y^{\frac{1}{5}})}$$

$u(tx,ty)=t^{\frac{1}{20}}u(x,y)\Rightarrow$ u is a homogeneous function of degree $\dfrac{1}{20}$

Differentiating u partially with respect to x, we get

$$\frac{\partial u}{\partial x}=\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\frac{1}{4}x^{\frac{-3}{4}}-\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\frac{1}{5}x^{\frac{-4}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$\therefore\ x\frac{\partial u}{\partial x}=\frac{\frac{1}{4}x^{\frac{1}{4}}\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)-\frac{1}{5}x^{\frac{1}{5}}\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

Differentiating u partially with respect to y, we get

$$\frac{\partial u}{\partial y}=\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\frac{1}{4}y^{\frac{-3}{4}}-\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\frac{1}{5}y^{\frac{-4}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$\therefore y\frac{\partial u}{\partial y}=\frac{\frac{1}{4}y^{\frac{1}{4}}\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)-\frac{1}{5}y^{\frac{1}{5}}\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\frac{1}{4}\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)-\frac{1}{5}\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$=\frac{\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(\frac{1}{4}-\frac{1}{5}\right)}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}=\frac{1}{20}\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}=\frac{1}{20}u$$

$\therefore$ Euler’s theorem is verified.

(iii) $u=x^3\log\left(\frac{y}{x}\right)$

The given function is of the form $x^nF\left(\frac{y}{x}\right)\Rightarrow u$ is a homogeneous function of degree 3

Differentiating u partially with respect to x, we get

$$\frac{\partial u}{\partial x}=x^3\frac{1}{\left(\frac{y}{x}\right)}.\left(\frac{-y}{x^2}\right)+\log\left(\frac{y}{x}\right)\left(3x^2\right)=\left(\frac{x^4}{y}\right)\left(\frac{-y}{x^2}\right)+3x^2\log\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x}=-x^3+3x^3\log\left(\frac{y}{x}\right) \qquad (1)$$

Differentiating *u* partially with respect to *y*, we get,

$$\frac{\partial u}{\partial y}=x^3.\frac{1}{\left(\frac{y}{x}\right)}.\left(\frac{1}{x}\right)=\frac{x^3}{y}.$$

$$\therefore y\frac{\partial u}{\partial y}=x^3 \qquad (2)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-x^3+3x^3\log\left(\frac{y}{x}\right)+x^3$$

$$x\frac{\partial u}{\partial u}+y\frac{\partial u}{\partial y}=3x^3\log\left(\frac{y}{x}\right)=3u$$

Hence Euler's theorem is verified.

(iv) $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

$$u(tx,ty)=\sin^{-1}\left(\frac{tx}{ty}\right)+\tan^{-1}\left(\frac{ty}{tx}\right)=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=u(x,\ y)$$

$\Rightarrow u$ is a homogeneous function of degree 0

Differentiating u partially with respect to x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

Differentiating *u* partially with respect to *y*, we get

$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{-x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}\ \text{ and } y\frac{\partial u}{\partial y}=\frac{-x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=0=0.\,u$$

$\therefore$ Euler's theorem is verified.

2. If $u=f\left(\frac{y}{x}\right)$ evaluate $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}$

Solution:

Since $u=f\left(\frac{y}{x}\right)$, u is a homogeneous function of degree ‘0’.

Hence by Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ becomes,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

3. If $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

Solution:

$$u(tx,ty,tz)=\frac{tx}{ty}+\frac{ty}{tz}+\frac{tz}{tx}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=u(x,y,z)$$

$\Rightarrow u$ is a homogeneous function of degree 0

Hence by Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

4. If $z=xy\,f\left(\frac{y}{x}\right)$, Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

Solution:

$$Given\ \ z(x,y)=xy\,f\left(\frac{y}{x}\right)\qquad \therefore z(tx.\,ty)=(tx)(ty)\,f\left(\frac{ty}{tx}\right)=t^2xy\,f\left(\frac{y}{x}\right)$$

$$z(tx,ty)=t^2\ z(x,\ y)$$

$\Rightarrow z$ is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

5. If $\frac{1}{u}=\sqrt{x^2+y^2+z^2}$, *then* prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u$

Given $\frac{1}{u}=\sqrt{x^2+y^2+z^2}\Rightarrow u(x,\ y,\ z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$

$$u(tx,ty,tz)=\frac{1}{\sqrt{t^2x^2+t^2y^2+t^2z^2}}=\frac{1}{\sqrt{t^2(x^2+y^2+z^2)}}=\frac{1}{t\sqrt{x^2+y^2+z^2}}$$

Solution: $u(tx,ty,tz)=t^{-1}u(x,y,z)$

$\Rightarrow u$ is a homogeneous function of degree -1

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=(-1)u=-u$

6. If $u = \cos^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\cot u$

Solution:

Given $u = \cos^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right) \Rightarrow \cos u = \frac{x^3+y^3}{x^2+y^2} = v(\text{say})$

Then v is a homogeneous function of degree1

$$\text{since, v}(tx, ty) = \frac{(tx)^3 + (ty)^3}{(tx)^2 + (ty)^2} = \frac{t^3x^3 + t^3y^3}{t^2x^2 + t^2y^2} = \frac{t^3(x^3+y^3)}{t^2(x^2+y^2)}$$

$$v(tx, ty) = tv(x, y)$$

(ie) *u* is not homogeneous, but $v = \cos u$ is homogeneous

$\therefore$ By corollary 2 of Euler's theorem on homogeneous function,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\, f(u)}{f'(u)} \text{ where } f(u) = v = \cos u$$

$$(ie) \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\ (\cos u)}{(-\sin u)} = -\cot u$$

$$\therefore \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\cot u$$

Hence proved.

7. If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\frac{\partial u}{\partial x} = \frac{-y}{x}\frac{\partial u}{\partial y}$

Solution:

$\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = v$ is a homogeneous function, since $v(tx, ty)$

$$= \frac{\sqrt{tx}-\sqrt{ty}}{\sqrt{tx}+\sqrt{ty}} = \frac{\sqrt{t}(\sqrt{x}-\sqrt{y})}{\sqrt{t}\,(\sqrt{x}+\sqrt{y})} = v(x, y)$$

$\Rightarrow v = \sin u$ is a homogeneous function of degree 0

$\therefore$ By corollary (2) of Euler's theorem on homogeneous function

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}=0.\ \frac{\sin u}{\cos u}=0.\ \text{ where } f(u)=\sin u=v.$$

$$\Rightarrow x\frac{\partial u}{\partial x}=-y\frac{\partial u}{\partial y}\ \text{(or)}\ \frac{\partial u}{\partial x}=\frac{-y}{x}\frac{\partial u}{\partial x}.$$

Hence it is proved.

8. If $u=e^{x^3+y^3}$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u\log u.$

Solution:

u is not homogeneous as $u(tx,ty)\neq t^n u(x,y)$ but $\log u=x^3+y^3=\upsilon$ is homogeneous, since,

$$v(tx,ty)=(tx)^3+(ty)^3=t^3x^3+t^3y^3=t^3(x^3+y^3)=t^3v(x,y)$$

$\Rightarrow v$ is homogeneous of degree 3.

$\therefore$ By corollary 2 of Euler's theorem on homogeneous function,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}\ \text{ where } f(u)=v=\log u$$

$$=\frac{3.\log u}{\frac{1}{u}}=3u\log u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u\log u$$

9. If $u=x\sin^{-1}\left(\frac{y}{x}\right)$ prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 y}{\partial y^2}=0$

Solution:

$$u(x,y)=x\sin^{-1}\left(\frac{y}{x}\right)\Rightarrow u\left(tx,ty\right)=tx\sin^{-1}\left(\frac{ty}{tx}\right)$$

$$u(tx,ty)=tx\sin^{-1}\left(\frac{y}{x}\right)=tu(x,y)$$

Given $\Rightarrow u$ is a homogeneous function of degree 1

$\therefore$ By corollary 1 of Euler's theorem on homogeneous function,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

10. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u.$

Solution:

Here $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ is not homogeneous, but

$\tan u=\upsilon=\frac{x^3+y^3}{x-y}$ is homogeneous.

Also $v(tx,ty)=\frac{t^3x^3+t^3y^3}{tx-ty}=\frac{t^3\left(x^3+y^3\right)}{t(x-y)}=t^2\upsilon(x,y)$

$\Rightarrow \upsilon$ is a homogeneous function of degree 2.

$\therefore$ By corollary 2 of Euler's theorem on homogeneous function,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f\,'(u)} \qquad (1)$$

Where $f(u)$ = tan u

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g\,'(u)-1\right] \qquad (2)$$

Where $g(u)=\frac{nf(u)}{f\,'(u)}$

(1) becomes, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.\frac{\tan u}{\sec^2 u}=\frac{2.\sin u}{\cos u\sec^2 u}$

$$=2\sin u\cos u=\sin 2u$$

$(ie)\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$. Hence (i) is proved

(2) becomes, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(\sin 2u)\left[2\cos 2u-1\right]$

$$=2\sin 2u\cos 2u-\sin 2u=\sin 4u-\sin 2u$$

(using $2\sin\theta\cos\theta=\sin 2\theta$)

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

Hence (ii) is proved.

11. If $u=x\phi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$, then show that

(i) $x\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=x\phi\left(\frac{y}{x}\right)$ (ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$

Solution:

(i) Let $v=x\phi\left(\frac{y}{x}\right),\ \omega=\psi\left(\frac{y}{x}\right)=x^0\psi\left(\frac{y}{x}\right)$

Then *v* is a homogeneous function of degree 1 and ω is a homogeneous function of degree 0

$$\Rightarrow x\frac{\partial \upsilon}{\partial x}+y\frac{\partial \upsilon}{\partial y}=\upsilon \qquad \text{(1) (By Euler's theorem)}$$

$$x\frac{\partial \omega}{\partial x}+y\frac{\partial \omega}{\partial y}=0 \qquad \text{(2) (By Euler's theorem)}$$

But $u=\upsilon+\omega$

$$\therefore (1)+(2)\Rightarrow x\frac{\partial \upsilon}{\partial x}+y\frac{\partial \upsilon}{\partial y}+x\frac{\partial \omega}{\partial x}+y\frac{\partial \omega}{\partial y}=\upsilon+0$$

$$\Rightarrow x\frac{\partial}{\partial x}(\upsilon+\omega)+y\frac{\partial}{\partial y}(\upsilon+\omega)=\upsilon=x\phi\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\phi\left(\frac{y}{x}\right)$$

Hence (i) is proved.

(ii) Similarly since υ and ω are homogeneous functions, by corollary 1 of Euler's theorem on homogeneous functions,

$$x^2\frac{\partial^2\upsilon}{\partial x^2}+2xy\frac{\partial^2\upsilon}{\partial x\partial y}+y^2\frac{\partial^2\upsilon}{\partial y^2}=n(n-1)\upsilon=0 \qquad (3)$$

$$x^2\frac{\partial^2\omega}{\partial x^2}+2xy\frac{\partial^2\omega}{\partial x\partial y}+y^2\frac{\partial^2\upsilon}{\partial y^2}=n(n-1)\omega=0 \qquad (4)$$

$$(3)+(4)\Rightarrow x^2\frac{\partial^2\upsilon}{\partial x^2}+2xy\frac{\partial^2\upsilon}{\partial x\partial y}+y^2\frac{\partial^2\upsilon}{\partial y^2}+\frac{x^2\partial^2\omega}{\partial x\partial y}+y^2\frac{\partial^2\upsilon}{\partial y^2}$$

= 0 + 0 = 0

$$\Rightarrow x^2\frac{\partial^2}{\partial x^2}(v+\omega)+2xy\frac{\partial^2}{\partial x\partial y}(v+\omega)+y^2\frac{\partial^2}{\partial y^2}(v+\omega)=0$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}=0$$

Hence (ii) is proved.

12. If $p+iq=(x-iy)^2$ and $u=\frac{p}{q}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Solution

Given $p+iq=(x-iy)^2$

$$p+iq=x^2-2ixy-y^2=(x^2-y^2)-2ixy$$

Equating real and imaginary parts,

$$p=x^2-y^2,\ q=-2xy$$

$$\therefore u=\frac{p}{q}=\frac{x^2-y^2}{-2xy}=\frac{y^2-x^2}{2xy}$$

Now $u(x,y)=\frac{y^2-x^2}{2xy}$ $\qquad \therefore u(tx,ty)=\frac{(ty)^2-(tx)^2}{2(tx)(ty)}$

$$\Rightarrow u(tx,ty)=\frac{t^2y^2-t^2x^2}{2t^2xy}=\frac{t^2(y^2-x^2)}{2t^2xy}=\frac{y^2-x^2}{2xy}=u(x,y)$$

$$\Rightarrow u(tx,ty)=u(x,y)$$

$\therefore u$ is a homogeneous function of degree 0

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0.u=0$

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Hence proved

Exercise Problems

PART A

1. State which of the following functions are homogeneous. If so, find the degree.

(i) x^2y+xy^2 (ii) $\frac{x^3+y^3}{x+y}$ (iii) $xy+xy^2$

(iv) $\tan\left(\frac{x^3+y^3}{x-y}\right)$ (v) $x^2\sin\left(\frac{y}{x}\right)$

2. Give an example of a homogeneous function of degree "–1".

3. If $u=e^{x/y}$, find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

4. If $u=\sin\left(\frac{x^2+y^2-z^2}{xy+yz+zx}\right)$ find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

5. If $u=x^4y^2\sin^{-1}\left(\frac{y}{x}\right)$ find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

6. If $z=\tan^{-1}\left(\frac{y}{x}\right)+\cot^{-1}\left(\frac{x}{y}\right)+\sin\left(\frac{x}{y}\right)+\sqrt{\frac{y}{x}}$ then prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$

PART B

7. Verify Euler's theorem for the following functions:

(i) $ax^2+2hxy+by^2$ (ii) $\frac{x^2y^2}{x+y}$ (iii) $\frac{1}{y^2}+\frac{\log x-\log y}{x^2}$

(iv) $x^3\cos\left(\frac{y}{x}\right)$

8. If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

9. If $u = \log\left(\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$

10. If $u = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$

11. It $t = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$ prove that $x\dfrac{\partial t}{\partial x} + y\dfrac{\partial t}{\partial y} + \dfrac{1}{2}\cot t = 0$

12. If $u = \dfrac{x^2 y^2}{x^2 + y^2}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

13. If $u = x\sin^{-1}\left(\dfrac{y}{x}\right)$ prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

14. If $u = \dfrac{xy}{x + y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

15. If $u = f\left(\dfrac{y}{x}\right) + \sqrt{x^2 + y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$

16. If $u = \sin^{-1}\left(\dfrac{y}{x}\right)$ evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$

17. If *u* is a homogeneous function of degree *n*, prove that

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$

18. If $\upsilon = \log_e \sin\left\{\dfrac{\pi\left(2x^2 + y^2 + xz\right)^{\frac{1}{2}}}{2\left(x^2 + xy + 2yz + z^2\right)^{\frac{1}{3}}}\right\}$, find the value of $x\dfrac{\partial \upsilon}{\partial x} + y\dfrac{\partial \upsilon}{\partial y} + z\dfrac{\partial \upsilon}{\partial z}$

when $x = 0,\ y = 1,\ z = 2$

19. If $u + i\upsilon = \left(ax + iby\right)^3$, prove the following:

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$ (ii) $x\dfrac{\partial \upsilon}{\partial x} + y\dfrac{\partial \upsilon}{\partial y} = 3\upsilon.$

20. *If* $u=\tan^{-1}\left(\frac{y^2}{x}\right)$ show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u\sin^2 u$

21. *Given* $u=\sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ Show that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{12}\tan u$

22. (ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}\left(13+\tan^2 u\right)$

Answers:

1. (i)Homogeneous, 3 (ii) Homogeneous, 2 (iii) Non-homogeneous

(iv)Non-homogeneous (v) Homogeneous, 2

3. 0

4. 0

5. 6*u*

16. 0

18. $\frac{\pi}{12}$

TOTAL DIFFERENTIATION

Introduction

The total derivative of a function *f* is the best linear approximation of the value of the function with respect to its arguments. Unlike partial derivatives, the total derivative approximates the function with respect to all of its arguments. The term “total derivative” is used only when *f* is a function of several variables. Where *f* is a function of single variable, the total derivative is the same as the derivative of the function.

1) If $z=f(x,y)$ then the total differential of *z* is given by $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$.

2) If $z = f(x,y)$, where x and y are continuous functions of another variable t, then the total differential coefficient is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

3) If $z = f(u,v)$; where u and v are functions of other variables x and y, then the partial derivative of z with respect to x and y are given by

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

4) If $f(x, y) = 0$ is an implicit function of x then the derivative $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

The second order derivative

$$\frac{d^2 y}{dx^2} = -\left(\frac{p^2 t - 2pqs + q^2 r}{q^3}\right)$$

where $p = \frac{\partial f}{\partial x},\ q = \frac{\partial f}{\partial y},\ r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$

Examples

1) If $u = \sin(xy^2)$, express the total differential of u in terms of those of x and y.

Solution:

Given $u = \sin(xy^2)$

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$$

$$\text{Now, } \frac{\partial u}{\partial x} = \cos(xy^2) \times y^2$$

$$= y^2 \cos(xy^2)$$

$$\frac{\partial u}{\partial y} = \cos(xy^2) \times 2xy$$

$$= 2xy\cos(xy^2)$$

$$\therefore du = y^2 \cos(xy^2)dx + 2xy\cos(xy^2)dy$$

2) Find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$.

Solution:

Given $u = x^3y^2 + x^2y^3$ and $x = at^2$, $y = 2at$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Now, $\frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3$

$\frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$

$\frac{dx}{dt} = 2at$; $\frac{dy}{dt} = 2a$

$\therefore \frac{du}{dt} = (3x^2y^2 + 2xy^3)2at + (2x^3y + 3x^2y^2)2a$

$= \left(12a^4t^6 + 16a^4t^5\right)2at + \left(4a^4t^7 + 12a^4t^6\right)2a$

$= 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$

$= 32a^5t^7 + 56a^5t^6$

$= 8a^5t^6\left(4t + 7\right)$

3) Find $\frac{du}{dt}$, if $u = \log(x + y + z)$, where $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

Solution:

Given $u = \log(x + y + z)$ and $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}}$$

Now, $\frac{\partial u}{\partial x} = \frac{1}{x + y + z}$

$\frac{\partial u}{\partial y} = \frac{1}{x + y + z}$

$$\frac{\partial u}{\partial z} = \frac{1}{x+y+z}$$

$$\frac{dx}{dt} = -e^{-t}; \quad \frac{dy}{dt} = \cos t; \quad \frac{dz}{dt} = -\sin t$$

$$\therefore \frac{du}{dt} = \frac{-e^{-t}}{x+y+z} + \frac{\cos t}{x+y+z} - \frac{\sin t}{x+y+z}$$

$$= \frac{-e^{-t} + \cos t - \sin t}{x+y+z}$$

$$= \frac{-e^{-t} + \cos t - \sin t}{e^{-t} + \sin t + \cos t}$$

4) Find $\frac{dy}{dx}$, using partial derivatives when $x^3 + 3x^2y + 6xy^2 + y^3 = 1$

Solution:

Given $x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$

$$\boxed{\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}}$$

$$f(x,y) = x^3 + 3x^2y + 6xy^2 + y^3 - 1$$

$$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$$

$$\frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$$

$$\therefore \frac{dy}{dx} = -\left[\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}\right]$$

$$= -\frac{3\left[x^2 + 2yx + 2y^2\right]}{3\left[x^2 + 4xy + y^2\right]}$$

$$= -\frac{\left[x^2 + 2xy + 2y^2\right]}{\left[x^2 + 4xy + y^2\right]}$$

5) Find $\frac{du}{dx}$, when $u = \sin\left(x^2 + y^2\right)$, where $x^2 + 4y^2 = 9$.

Solution:

Given $u = \sin(x^2 + y^2)$ and $x^2 + 4y^2 = 9$,

$$\boxed{\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}}$$

Now, $\frac{\partial u}{\partial x}=\cos(x^2+y^2)\times 2x$

$$=2x\cos(x^2+y^2)$$

$$\frac{\partial u}{\partial y}=\cos(x^2+y^2)\times 2y$$

$$=2y\cos(x^2+y^2)$$

Differentiating $x^2+4y^2=9$ with respect to *x*,

$$2x+4\times 2y\frac{dy}{dx}=0$$

$$8y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-x}{4y}$$

$$\therefore \frac{du}{dx}=2x\cos(x^2+y^2)+2y\cos(x^2+y^2)\times\frac{-x}{4y}$$

$$=2x\cos(x^2+y^2)-\frac{x}{2}\cos(x^2+y^2)$$

$$=\frac{3x}{2}\cos(x^2+y^2).$$

6) If $u=f(x-y,\ y-z,\ z-x)$ Prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

Solution:

Given $u=f(x-y,\ y-z,\ z-x)$

Let $A=x-y; B=y-z;\ \ C=z-x$

$\therefore u=f(A,\ B,\ C)$, where $A,\ B,\ C,$ are functions of $x,$ y, z as assumed

Now $\boxed{\frac{\partial u}{\partial x}=\frac{\partial u}{\partial A}\frac{\partial A}{\partial x}+\frac{\partial u}{\partial c}\frac{\partial c}{\partial x}}$

$$= \frac{\partial u}{\partial A} \times 1 + \frac{\partial u}{\partial c} \times -1$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial c} \qquad (1)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial y} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial y}}$$

$$= \frac{\partial u}{\partial A} \times -1 + \frac{\partial u}{\partial B} \times 1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial A} + \frac{\partial u}{\partial B} \qquad (2)$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial B}\frac{\partial B}{\partial Z} + \frac{\partial u}{\partial c}\frac{\partial c}{\partial z}}$$

$$= \frac{\partial u}{\partial B} \times (-1) + \frac{\partial u}{\partial c} \times 1$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial B} + \frac{\partial u}{\partial c} \qquad (3)$$

From (1), (2) and (3) we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial c} - \frac{\partial u}{\partial A} + \frac{\partial u}{\partial B} - \frac{\partial u}{\partial B} + \frac{\partial u}{\partial c} = 0$$

7) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ Prove that $\sum x \frac{\partial u}{\partial z} = 0.$

Solution:

Given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let $A = \frac{x}{y}; \quad B = \frac{y}{z}; \quad C = \frac{z}{x}$

$\therefore u = f(A, B, C),$ where A, B, C are functions of x, y, z as assumed.

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} . \frac{\partial A}{\partial x} + \frac{\partial u}{\partial c}\frac{\partial c}{\partial x}}$$

$$= \frac{\partial u}{\partial A} \times \frac{1}{y} + \frac{\partial u}{\partial c} \times \frac{-z}{x^2}$$

$$= \frac{1}{y}\frac{\partial u}{\partial A} - \frac{z}{x^2}\frac{\partial u}{\partial c} \qquad (1)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial y} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial y}}$$

$$= \frac{\partial u}{\partial A} \times \frac{-x}{y^2} + \frac{\partial u}{\partial B} \times \frac{1}{z}$$

$$= \frac{-x}{y^2}\frac{\partial u}{\partial A} + \frac{1}{z}\frac{\partial u}{\partial B} \qquad (2)$$

$$= \frac{\partial u}{\partial B} \times \frac{-y}{z^2} + \frac{\partial u}{\partial c} \times \frac{1}{x}$$

$$= \frac{-y}{z^2}\frac{\partial u}{\partial B} + \frac{1}{x}\frac{\partial u}{\partial c} \qquad (3)$$

$\therefore$ From (1), (2) and (3), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left[\frac{x}{y}\frac{\partial u}{\partial A} - \frac{z}{x}\frac{\partial u}{\partial c}\right] + \left[\frac{-x}{y}\frac{\partial u}{\partial A} + \frac{y}{z}\frac{\partial u}{\partial B}\right] + \left[\frac{-y}{z}\frac{\partial u}{\partial B} + \frac{z}{x}\frac{\partial u}{\partial c}\right]$$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

8) If $z = f(x, y),$ where $x = u + v,\ y = uv,$ prove that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

Solution:

Given $z = f(x, y),$ where $x = u + v,\ y = uv.$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

Now, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times 1 + \frac{\partial z}{\partial y} v$

$$= \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \qquad (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times 1 + \frac{\partial z}{\partial y} u$$

$$= \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \qquad (2)$$

$\therefore$ From (1) and (2), we have

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y}$$

$$= (u+v)\frac{\partial z}{\partial x} + 2uv\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$$

9) If $z = f(u,v)$, where $u = x^2 - y^2$ and $v = 2xy$,

Prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(x^2+y^2\right)\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}$

Solution:

Given $z = f(u,v)$, where $u = x^2 - y^2$ and $v = 2xy$.

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}}$$

Now, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}2x + \frac{\partial z}{\partial v}2y$

$$\left(\frac{\partial z}{\partial x}\right)^2 = 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \qquad (1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}2x$$

$$\Rightarrow \left(\frac{\partial z}{\partial y}\right)^2 = 4y^2\left(\frac{\partial z}{\partial u}\right)^2 + 4x^2\left(\frac{\partial z}{\partial v}\right)^2 - 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \qquad (2)$$

$\therefore$ From (1) and (2) we have

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4\left(x^2+y^2\right)\left(\frac{\partial z}{\partial u}\right)^2+4\left(x^2+y^2\right)\left(\frac{\partial z}{\partial v}\right)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4\left(x^2+y^2\right)\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$$

10) If $z = f\left(x,\, y\right)$ where $x=X\cos\alpha-Y\sin\alpha$ and $y=X\sin\alpha+Y\cos\alpha$ show that $z_{xx}+z_{yy}=z_{xx}+z_{yy}$

Solution:

Given $z=f(x,y)$ and $x=x\cos\alpha-y\sin\alpha$

$$y=x\sin\alpha+y\cos\alpha$$

Now $\boxed{\frac{\partial z}{\partial X}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}}$

$$=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\sin\alpha$$

$$\therefore \frac{\partial}{\partial X}=\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}$$

$$=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)$$

$$=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad (1)$$

Similarly $\boxed{\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial y}}$

$$=\frac{\partial z}{\partial x}\times(-\sin\alpha)+\frac{\partial z}{\partial y}\times(\cos\alpha)$$

$$\therefore \frac{\partial}{\partial y}=-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha \frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 z}{\partial y^2} \qquad (2)$$

$\therefore$ From (1) and (2) we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\cos^2\alpha + \sin^2\alpha\right)\frac{\partial^2 z}{\partial x^2} + \left(\cos^2\alpha + \sin^2\alpha\right)\frac{\partial^2 z}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

11) If $z = f(u,v)$, where $u = lx + my$ and $v = ly - mx$,

Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(l^2 + m^2\right)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

Solution:

Given $z = f(u,v)$ and $u = lx + my,\ v = ly - mx$

Now $\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}}$

$$= \frac{\partial z}{\partial u} \times l + \frac{\partial z}{\partial v} \times -m$$

$$= l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}$$

$$= \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u \partial v} + m^2\frac{\partial^2 z}{\partial v^2} \qquad (1)$$

Similarly

$$\boxed{\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}}$$

$$=\frac{\partial z}{\partial u}m+\frac{\partial z}{\partial v}l$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)}$$

$$=\left(m\frac{\partial}{\partial u}+l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u}+l\frac{\partial z}{\partial v}\right)$$

$$=m^2\frac{\partial^2 z}{\partial u^2}+2lm\frac{\partial^2 z}{\partial u\partial v}+l^2\frac{\partial^2 z}{\partial v^2} \qquad (2)$$

$\therefore$ From (1) and (2) we have

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(l^2+m^2\right)\frac{\partial^2 z}{\partial u^2}+\left(l^2+m^2\right)\frac{\partial^2 z}{\partial v^2}$$

$$=\left(l^2+m^2\right)\left[\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial v^2}\right]$$

EXERCISE:

1) If $f(x,y)=x^2+xy+y^2,$ where $x=r\cos\theta,\ y=r\sin\theta$, then find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

2) If $u=x^3+y^3,$ where $x=a\ \cos t,\ y=b\sin t,$ *then* find $\frac{du}{dt}$.

3) If $u=\sin\frac{x}{y},\ x=e^t,\ y=t^2,$ then find $\frac{du}{dt}$.

4) If $u=xyz,\ x=t^2,\ y=e^t,\ z=e^{-t},$ then find $\frac{du}{dt}$.

5) If $u=e^x\sin y,\ x=st^2,\ y=s^2t,$ *then* find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

6) Find $\frac{dy}{dx}$, if $x^3+y^3=3axy$

7) Find $\frac{dy}{dx}$, if $3x^2+xy-y^2+4x-2y+1=0$

8) If *z* is a function of *x* and *y*, where

$x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$, then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$

9) If $x = u + v + w;$

$y = vw + uw + uv;$

$z = uvw$ and

$F = f(x, y, z),$ *then* show that $uF_u + vF_v + wF_w = xF_x + 2yF_y + 3zF_z$

10) If $u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

11) If $u = f\left[x^2 - y^2, y^2 - z^2, z^2 - x^2\right]$, then prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$

12) If $z = f(u, v)$, where $u = x + y$ and $v = x - y$, then show that $2z_u = z_x + z_y$

13) If $f = f\left[\frac{y-x}{xy}, \frac{z-x}{zx}\right]$, then show that $x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + z^2\frac{\partial f}{\partial z} = 0$

14) If $z = f(x, y);\ x = u + v,\ y = uv,$ *then* prove that $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$

15) If $u = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, then prove that

$$(u_x)^2 + (u_y)^2 = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2$$

16) If $f = f(u, v)$ and $u = e^x \cos y;\ v = e^x \sin y$, then prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2 + v^2)\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right].$$

17) *If* $x = u\cos\alpha - v\sin\alpha,\ y = u\sin\alpha + v\cos\alpha,$ then prove that

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

18) If $u = u(x, y)$ and $x = e^r \cos\theta;\ y = e^r \sin\theta,$

then show that $(u_x)^2 + (u_y)^2 = e^{-2r}\left[(u_r)^2 + (u_\theta)^2\right]$

19) If $u = u(x, y)$ and $x = e^r \cos\theta;\ y = e^r \sin\theta,$ then show that $u_{xx} + u_{yy} = e^{-2r}\left[u_{rr} + u_{\theta\theta}\right]$

20) $z = f(u,v)$, where $u = \cosh x \ \cos y$ and

$v = \sinh x \sin y$ then prove that $z_{xx} + z_{yy} = \left[\sinh^2 x + \sin^2 y\right] \ [z_{uu} + z_{vv}]$.

21) If $z = f(u, v)$ where $u = x^2 - y^2$, $v = 2xy$, then show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$$

ANSWERS:

1) $r(2 + \sin 2\theta);\ r^2 \cos 2\theta$

2) $3\cos t \sin t\left[b^3 \sin t - a^3 \cos t\right]$

3) $\dfrac{e^t(t-2)}{t^3}\cos\left(\dfrac{e^t}{t^2}\right)$

4) 2t

5) $t^2 e^{st^2} \sin s^2 t + 2st\ e^{st^2} \cos s^2 t;\ 2ste^{st^2} \sin s^2 t + s^2 e^{st^2} \cos s^2 t.$

6) $\dfrac{ay - x^2}{y^2 - ax}$

7) $-\left[\dfrac{6x + y + 4}{x - 2y - 2}\right]$

TAYLOR SERIES

Introduction:

Classically, algebraic functions are defined by an algebraic equation, and transcendental functions are defined by some property that holds for them, such as differential equations. One may equally well, define an analytic function by its Taylor Series.

Taylor Series is used to define functions and “operators” in diverse areas of Mathematics. In particular, this is true in areas where the classical definitions of functions break down. Using Taylor Series one may define functions of matrices, such as the matrix exponential or matrix logarithm.

In other areas, such as formal analysis, it is more convenient to work directly with the power series themselves. Thus one may define a solution of a differential equation as a power series which is the Taylor Series of the desired solution.

Definition:

Taylor Series expansion of a function $f(x, y)$ in powers of x and y at (a, b) defined as

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$

$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

$$+\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2$$

$$f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + ...$$

Example 1: Expand $e^x \cos y$ in powers of x & y as far as the terms of the third degree.

Solution:

Given $f(x,y) = e^x \cos y$ $\qquad f(a,b) = f(0,0) = 1$

$f_x(x,y) = e^x \cos y$ $\qquad f_x(0,0) = 1$

$f_{xx}(x,y) = e^x \cos y$ $\qquad f_{xx}(0,0) = 1$

$f_{xxx}(x,y) = e^x \cos y$ $\qquad f_{xxx}(0,0) = 1$

$f_y(x,y) = -e^x \sin y$ $\qquad f_y(0,0) = 0$

$f_{yy}(x,y) = -e^x \cos y$ $\qquad f_{yy}(0,0) = -1$

$f_{yyy}(x,y) = e^x \sin y$ $\qquad f_{yyy}(0,0) = 0$

$f_{xy}(x,y) = -e^x \sin y$ $\qquad f_{xy}(0,0) = 0$

$f_{xxy}(x,y) = -e^{-x} \sin y$ $\qquad f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = -e^x \cos y$ $\qquad f_{xyy}(0,0) = -1$

Taylor series of $f(x,y)$ in powers of x and y is

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[xf_x(0,0) + yf_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + ...$$

$$e^x \cos y = 1 + \frac{1}{1!}\left[x.1 + y.0\right] + \frac{1}{2!}\left[x^2.1 + 2xy.0 + y^2.(-1)\right]$$
$$+\frac{1}{3!}\left[x^3.1 + 3x^2y.0 + 3\text{x}\text{y}^2.(-1) + \text{y}^3.0\right]$$
$$= 1 + \frac{x}{1!} + \frac{1}{2!}\left(x^2 - y^2\right) + \frac{1}{3!}\left(x^3 - 3xy^2\right) + \ldots$$

Example 2: Expand x^y in powers of x and y near the point (1,1) up to the second degree terms.

Solution:

Given $f(x,y) = x^y$ $\quad\quad (a,b) = (1,1)$

$$f(x,y) = x^y \qquad f(1,1) = 1$$
$$f_x(x,y) = yx^{y-1} \qquad f_x(1,1) = 1$$
$$f_y(x,y) = x^y \log x \left[\because \frac{\partial a^x}{\partial x} = a^x \log a\right] \qquad f_y(1,1) = 0$$

$$f_{xx}(x,y) = y(y-1)x^{y-2} \qquad f_{xx}(1,1) = 0$$
$$f_{yy}(x,y) = x^y(\log x)^2 \qquad f_{yy}(1,1) = 0$$
$$f_{xy}(x,y) = x^{y-1} + \text{y}x^{y-1}\log x \qquad f_{xy}(1,1) = 1$$

Taylor's series expansion of $f(x,y)$ at $(1,1)$ is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}\left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ldots\ldots$$

$$x^y = 1 + \left[(x-1).1 + (y-1).0\right] + \frac{1}{2}\left[(x-1)^2.0 + 2(x-1)(y-1).1 + (y-1)^2.0\right]$$
$$= 1 + (x-1) + (x-1)(y-1) + \ldots..$$

Example 3: Find the Taylor Series expansion of $e^x \sin y$ near the point $\left(-1, \frac{\pi}{4}\right)$.

Solution:

Given $f(x,y) = e^x \sin y$ $\quad\quad (a,b) = \left(-1, \frac{\pi}{4}\right)$

$$f(x,y)=e^x \sin y \qquad f\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_x(x,y)=e^x \sin y \qquad f_x\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{xx}(x,y)=e^x \sin y \qquad f_{xx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{xxx}(x,y)=e^x \sin y \qquad f_{xxx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_y(x,y)=e^x \cos y \qquad f_y\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{yy}(x,y)=-e^x \sin y \qquad f_{yy}\left(-1,\frac{\pi}{4}\right)=\frac{-1}{e\sqrt{2}}$$

$$f_{yyy}(x,y)=-e^x \cos y \qquad f_{yyy}\left(-1,\frac{\pi}{4}\right)=\frac{-1}{e\sqrt{2}}$$

$$f_{xy}(x,y)=e^x \cos y \qquad f_{xy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{xxy}(x,y)=e^x \cos y \qquad f_{xxy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{xyy}(x,y)=-e^x \cos y \qquad f_{xyy}\left(-1,\frac{\pi}{4}\right)=\frac{-1}{e\sqrt{2}}$$

Taylor's expansion of $f(x,y)$ at $\left(-1,\frac{\pi}{4}\right)$ is

$$f(x,y)=f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left[(x+1)\,f_x\left(-1,\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right]$$

$$+\frac{1}{2!}\left[(x+1)^2\,f_{xx}\left(-1,\frac{\pi}{4}\right)+2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)^2\,f_{yy}\left(-1,\frac{\pi}{4}\right)\right]$$

$$+\frac{1}{3!}[(x+1)^3\,f_{xxx}(-1,\frac{\pi}{4})+3(x+1)^2(y-\frac{\pi}{4})\,f_{xxy}(-1,\frac{\pi}{4})+3(x+1)(y-\frac{\pi}{4})^2$$

$$f_{xyy}(-1,\frac{\pi}{4})+(y-\frac{\pi}{4})^3\,f_{yyy}(-1,\frac{\pi}{4})+]\ldots$$

$$e^y \sin y=\frac{1}{e\sqrt{2}}+\frac{1}{1!}[(x+1)\frac{1}{e\sqrt{2}}+(y-\frac{\pi}{4})\frac{1}{e\sqrt{2}}]$$

$$\frac{1}{2!}[(x+1^{2}.\frac{1}{e\sqrt{2}}+2(x+1)(y-\frac{\pi}{4})\frac{1}{e\sqrt{2}}+(y-\frac{\pi}{4})^2.\frac{-1}{e\sqrt{2}}]$$

$$+\frac{1}{3!}[(x+1)^3\frac{1}{e\sqrt{2}}+3(x+1)^2(y-\frac{\pi}{4})\frac{1}{e\sqrt{2}}+3(x+1)(y-\frac{\pi}{4})^2\frac{-1}{e\sqrt{2}}+(y-\frac{\pi}{4})^3\frac{-1}{e\sqrt{2}}+]...$$

$$=\frac{1}{e\sqrt{2}}[1+\frac{1}{1!}[(x+1)+(y-\frac{\pi}{4})]+\frac{1}{2!}[(x+1)^2+2(x+1)(y-\frac{\pi}{4})-(y-\frac{\pi}{4})^2]]$$

$$+\frac{1}{3!}[(x+1)^3+3(x+1)^2(y-\frac{\pi}{4})-3(x+1)(y-\frac{\pi}{4})^2-(y-\frac{\pi}{4})^3]+...$$

Example 4: Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in a series of powers of h and k up to the second degree terms.

Solution:

Let $f(x+h,y+k)=\frac{(x+h)(y+k)}{x+h+y+k}$

$$\therefore f(x,y)=\frac{xy}{x+y}$$

Taylor's series of $(x+h)(y+k)$ in powers of h and k is

$$f(x+h,y+k)=f(x,y)+\frac{1}{1!}\left[h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right]+$$

$$\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right]+..... \qquad (1)$$

Now, $f_x(x,y)=\frac{(x+y).y-xy.1}{(x+y)^2}=\frac{y^2}{(x+y)^2}$

$$f_y(x,y)=\frac{(x+y).x-xy.1}{(x+y)^2}=\frac{x^2}{(x+y)^2}$$

$$f_{xx}(x,y)=\frac{(x+y)^2.0-y^2.2.(x+y)}{(x+y)^4}=\frac{-2y^2}{(x+y)^3}$$

$$f_{xy}(x,y)=\frac{(x+y)^2.2y-y^2.2.(x+y)}{(x+y)^4}$$

$$=\frac{2xy}{(x+y)^3}$$

$$f_{yy}(x,y)=\frac{(x+y)^2.0-y^2.2.(x+y)}{(x+y)^4}=\frac{-2x^2}{(x+y)^4}.$$

Using these values in (1), we have

$$\frac{(x+h)(y+k)}{x+h+y+k}=\frac{xy}{x+y}+\frac{hy^2}{(x+y)^2}+\frac{kx^2}{(x+y)^2}-\frac{h^2y^2}{(x+y)^3}$$
$$+\frac{2hkxy}{(x+y)^3}-\frac{k^2x^2}{(x+y)^3}+.....$$

Example 5: Find the Taylor's series expansion of $x^2y^2+2x^2y+3xy^2$ in powers of $(x+2)$ and $(y-1)$ up to the third degree.

Solution:

Taylor's series of $f(x,y)$ in powers of $(x+2)$ and $(y-1)$ is

$$f(x,y)=f(-2,1)+\frac{1}{1!}[(x+2)f_x(-2,1)+(y-1)f_y(-2,1)]$$

$$\frac{1}{2!}[(x+2)^2f_{xx}(-2,1)+2(x+2)(y-1)f_{xy}(-2,1)+(y-1)^2f_{yy}(-2,1)]+$$

$$\frac{1}{3!}[(x+2)^3f_{xxx}(-2,1)+3(x+2)^2(y-1)f_{xxy}(-2,1)+3(x+2)(y-1)^2f_{xyy}(-2,1)$$

$$+(y-1)^3f_{yyy}(-2,1)]+... \qquad (1)$$

$f(x,y)=x^2y^2+2x^2y+3xy^2$ $\qquad f(-2,1)=6$

$f_x(x,y)=2xy^2+4xy+3y^2$ $\qquad f_x(-2,1)=-9$

$f_y(x,y)=2x^2y+2x^2+6xy$ $\qquad f_y(-2,1)=4$

$f_{xx}(x,y)=2y^2+4y$ $\qquad f_{xx}(-2,1)=6$

$f_{yy}(x,y)=2x^2+6x$ $\qquad f_{yy}(-2,1)=-4$

$f_{xy}(x,y)=4xy+4x+6y$ $\qquad f_{xy}(x,y)=-10$

$f_{xxx}(x,y)=0$ $\qquad$ $f_{xxx}(-2,1)=0$

$f_{yyy}(x,y)=0$ $\qquad$ $f_{yyy}(-2,1)=0$

$f_{xyy}(x,y)=4x+6$ $\qquad$ $f_{xyy}(-2,1)=-2$

$f_{xxy}(x,y)=4y+4$ $\qquad$ $f_{xxy}(-2,1)=8$

Using these values in (1), we have

$$x^2y^2+2x^2y+3xy^2=6+\frac{1}{1!}\left[-9(x+2)+4(y-1)\right]$$
$$+\frac{1}{2!}\left[6(x+2)^2-20(x+2)(y-1)-4(y-1)^2\right]$$
$$+\frac{1}{3!}\left[24(x+2)^2(y-1)-6(x+2)(y-1)^2\right]+.....$$

Example 6: Find the Taylor's Series expansion of the function $e^x\log(1+y)$ near the point (0, 0).

Solution:

Given: $f(x,y)=e^x\log(1+y);\quad (a,b)=(0,0).$

$f(x,y)=e^x\log(1+y)$ $\qquad$ $f(0,0)=0$

$f_x(x,y)=e^x\log(1+y)$ $\qquad$ $f_x(0,0)=0$

$f_{xx}(x,y)=e^x\log(1+y)$ $\qquad$ $f_{xx}(0,0)=0$

$f_y(x,y)=\dfrac{e^x}{1+y}$ $\qquad$ $f_y(0,0)=1$

$f_{yy}(x,y)=\dfrac{-e^x}{1+y}$ $\qquad$ $f_{yy}(0,0)=-1$

$f_{xy}(x,y)=\dfrac{e^x}{1+y}$ $\qquad$ $f_{xy}(0,0)=1$

Taylor's series expansion of $e^x\log(1+y)$ at (0, 0) is

$$e^x\log(1+y)=f(0,0)+\frac{1}{1!}\left[x.f_x(0,0)+yf_y(0,0)\right]$$
$$+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]+....$$
$$=0+\frac{1}{1!}\left[x.0+y.1\right]+\frac{1}{2!}\left[0+2xy-1\right]+....$$
$$=y+xy-\frac{y^2}{2}+....$$

Example 8: Expand e^{xy} at (1, 1) as the Taylor's Series

Solution: $f(x,y)=e^{xy}$ and $(a,b)=(1,1)$

$$\begin{array}{ll} f(x,y)=e^{xy} & f(1,1)=e \\ f_x(x,y)=y.e^{xy} & f_x(1,1)=e \\ f_{xx}(x,y)=y^2e^{xy} & f_{xx}(1,1)=e \\ f_{xy}(x,y)=y.e^{xy}(x)+1.e^{xy} & f_{xy}(1,1)=2e \\ f_y(x,y)=x.e^{xy} & f_y(1,1)=e \\ f_{yy}(x,y)=x^2e^{xy} & f_{yy}(1,1)=e \end{array}$$

The Taylor's Series expansion of $f(x,y)$ at $(1,1)$

is $f(x,y)= f(1,1)+\frac{1}{1!}\left[(x-1)\,f_x(1,1)+(y-1)\,f_y(1,1)\right]$

$$+\frac{1}{2!}\left[(x-1)^2\,f_{xx}(1,1)+2(x-1)(y-1)\,f_{xy}(1,1)+(y-1)^2\,f_{yy}(1,1)\right]+....$$

$$e^{xy}=e+\frac{1}{1!}\left[(x-1).e+(y-1).e\right]+\frac{1}{2!}\left[(x-1)^2.e+2(x-1)(y-1).2e+(y-1)^2.e\right]+....$$

$$=e\left[1+(x-1)+(y-1)+\frac{(x-1)^2}{2}+2(x-1)(y-1)+\frac{(y-1)^2}{2}+.....\right]$$

Example 9: Expand sin(*xy*)at $\left(1,\frac{\pi}{2}\right)$ as the Taylor's Series.

Solution: $f(x,y)=\sin\ (xy)$ $\qquad f\left(1,\frac{\pi}{2}\right)=1$

$$\begin{array}{ll} f_x(x,y)=y\cos(xy) & f_x\left(1,\frac{\pi}{2}\right)=0 \\ f_{xx}(x,y)=-y^2\sin(xy) & f_{xx}\left(1,\frac{\pi}{2}\right)=-\frac{\pi^2}{4} \\ f_{xy}(x,y)=-xy\sin(xy)+\cos(xy) & f_{xy}\left(1,\frac{\pi}{2}\right)=-\frac{\pi}{2} \\ f_y(x,y)=x\cos(xy) & f_y\left(1,\frac{\pi}{2}\right)=0 \\ f_{yy}(x,y)=-x^2\sin(xy) & f_{yy}(x,y)=-1 \end{array}$$

Taylor's Series is

$$f(x,y)=f\left(1,\frac{\pi}{2}\right)+\frac{1}{1!}\left[(x-1)\,f_x\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2\,f_{xx}\left(1,\frac{\pi}{2}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(1,\frac{\pi}{2}\right)\right]+\dots$$

$$\sin xy=1+\frac{1}{1!}[(x-1).0+(y-\frac{\pi}{2}).0]+\frac{1}{2!}[(x-1)^2(\frac{\pi^2}{4})+2(x-1)$$
$$(y-\frac{\pi}{2}).(-\frac{\pi}{2})+(y-\frac{\pi}{2})^2(-1)]+\dots$$

$$=1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi(x-1)(y-\frac{\pi}{2})}{2}-\frac{(y-\frac{\pi}{2})^2}{2}+\dots..$$

EXERCISE

1. Expand $e^{ax}\sin by$ at (0,0) as Taylor's Series.

2. Expand $e^{x}\cos y$ in the neighborhood of $\left(0,\frac{\pi}{4}\right)$

3. Expand $e^{2x}\cos 2y$ in the neighborhood of $\left(0,\frac{\pi}{2}\right)$.

4. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ at $(a,b)=(1,1)$

5. Expand $\cos(x-y)$ up to second degree terms.

6. Find the Taylor's Expansion for $e^{x}\sin y$ at $\left(0,\frac{\pi}{2}\right)$ up to third-degree terms.

7. Expand $2x^2y+x+y$ using Taylor's theorem about $(1,-2)$ up to terms of third order

8. Expand x^y near (1,1) up to second term.

9. Expand $e^{ax}\sin by$ in powers of *x* and *y* in terms of third degree.

10. Find the expansion for $\cos x\cos y$ in powers of x,y up to second degree.

11. Expand $f(x,y)=2x^2-xy+y^2+3x-4y+1$ about the point $(-1,1)$.

12. Expand $\sin(x+h)(y+k)$ by Taylor's Series Expansion.

13. Expand $e^x \cos y$ in the neighborhood of $(1, \frac{\pi}{4})$

14. Expand $x^2y + 3y - 2$ in power of (x – 1) and (y + 2) up to third degree terms.

ANSWERS

1. $by + abxy + \frac{1}{2!}\left[3a^2bx^2y - b^3y^3\right] + ...$

2. $y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + ...$

3. $-1 - x + (y - \frac{\pi}{2}) + 2[-x^2 + (y - \frac{\pi}{2})^2 + ...$

4. $\frac{\pi}{4} + \frac{1}{2}[y - x] + \frac{1}{4}[(x-1)^2 - (y-1)^2] + ...$

5. $1 - \frac{1}{2!}(x - y)^2 + ...$

6. $1 + x + \frac{1}{2}x^2 - \frac{1}{2}(y - \frac{\pi}{2})^2 + \frac{1}{6}x^3 - \frac{1}{2}(y - \frac{\pi}{2})^3 + ...$

7. $-5 - 7(x-1) + 3(y+2) - 4(x-1)^2 + 4(x-1)(y+2)2(x-1)^2(y+2) + ...$

8. $1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + ...$

9. $by + abxy + \frac{1}{3!}(3a^2bx^2y - b^3y^3) + ...$

10. $1 - \frac{x^2}{2} - \frac{y^2}{2} + ..$

11. $-2 - 2(x+1) - (y-1) + 2(x+1)^2 - (x+1)(y-1) + (y-1)^2 + ...$

12. $\sin xy + (hy + kx)\cos xy + hk\cos xy - \frac{1}{2}(hy + kx)^2 \sin xy + ...$

13. $\frac{e}{v2}[1 + (x-1) - (y - \frac{\pi}{4}) + \frac{(x-1)^2}{21} - (x-1)(y - \frac{\pi}{4}) - (y - \frac{\pi}{4})^2 + ...]$

14. $-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + ...$

MAXIMA AND MINIMA OF A FUNCTION OF TWO VARIABLES

Introduction

The problem of finding the maximum or minimum of a function is encountered in geometry, mechanics, physics and other fields which were the motivating factors in the development of the calculus in the 17^{th} century. The calculation of the optimum value of a function of two variables is a common requirement in many areas of engineering, for example in thermodynamics. Unlike the case of a function of one variable we have to use more complicated criteria to distinguish between the various types of stationary points.

To optimize something means to maximize (or) minimize some aspects of it. An important application of multivariate differential calculus is finding the maximum and minimum values of functions of several variables and determining where they occur. In the study of stability of the equilibrium state of mechanical and physical systems, determination of extrema is of greatest importance. Lagrange multipliers method developed by Lagrange in 1755 is a powerful method for finding extreme values of constrained functions. Numerous cases present, themselves, both in engineering theory and practice, in which the value of one quantity which depends on the former, has a maximum (or) minimum value when the former has the determined value.

Definition

Let f (x, y) be a function of two variables. Let (a, b) be a point such that $f(a,\ b) > f(a+h,\ b+k)$ in some neighborhood of (a, b), then we say that f(x, y) attains its maximum value at (a, b) and f (a, b) is called the maximum value of f (x, y)

If f(a, b) is such that $f(a,\ b) < f(a+h,\ b+k)$ in some neighborhood of (a, b) we say that f(x, y) attains its minimum value at (a, b) and f(a, b) is called a minimum value of f(x, y)

f(a, b) is said to be an extremum of f(x, y) if f(a, b) is either a maximum or a minimum value and (a, b) is called an extreme point.

The necessary and sufficient conditions for (a,b) to be an extreme point:

By Taylors theorem,

$$\Delta f = f(a+h,\ b+k) - f(a,b)$$

$$= hf_x(a,b) + kf_y(a,b) + \frac{1}{2!}\left(h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right) + \dots \quad (1)$$

For small values of h and k, the second and higher order terms are still smaller and may be neglected. Thus sign of $\Delta f = sign\ of \left[hf_x(a,b) + kf_y(a,b)\right]$

Taking h = 0, the sign of Δf changes with the sign of k.

Similarly taking k= 0, the sign if Δf changes with the sign if h. Since Δf changes sign with h and k, f(x, y) cannot have a maximum (or) minimum value at (a, b) unless $f_x(a,b)=0=f_y(a,b)$.

Hence the necessary conditions for (a,b) to a extreme point are

$$f_x(a,b)=0, \quad f_y(a,b)=0$$

If these conditions are satisfied, then for small values of h and k

$$\Delta f=\frac{1}{2!}\left(h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)\right)+....$$

$$=\frac{1}{2!}\left(h^2r+2hks+k^2t\right)+...$$

Where $r=f_{xx}(a,b), \quad s=f_{xy}(a,b), \quad t=f_{yy}(a,b)$

$$\Delta f=\frac{1}{2r}\left(h^2r^2+2hkrs+k^2rt\right)+....$$

$$=\frac{1}{2r}\left(h^2r^2+2hkrs+k^2s^2+k^2rt-k^2s^2\right)+....$$

$$=\frac{1}{2r}\left[(hr+ks)^2+k^2\left(rt-s^2\right)+.......\right] \qquad (2)$$

Now $(hr+ks)^2$ is always positive and $k^2\left(rt-s^2\right)$ will be positive if $rt-s^2>0$. In this case Δf will have the same sign as that of r for all values of h & k.

Hence if $rt-s^2>0$, then f(x, y) has a maximum/minimum value at (a, b) according as $r<0/r>0$.

If $rt-s^2<0$, then Δf changes sign with h & k. Hence there is neither a maximum nor a minimum value at (a, b).

If $rt-s^2=0$, no conclusion can be drawn about a maximum (or) a minimum value at (a, b) and hence further investigation is required, (i.e. higher partial derivatives must be considered).

Stationary values (or) stationary points (or) **critical points**

A function f(x, y) is said to be stationary at (a, b) if $\frac{\partial f}{\partial x}=0,\frac{\partial f}{\partial y}=0$ at (a, b). The stationary points are the points at which $\frac{\partial f}{\partial x}=0,\frac{\partial f}{\partial y}=0$.

Saddle points:

If at a point $rt-s^2<0$, then f(x, y) has neither maxima nor minima for the function. Such points are called saddle points.

Extreme value (or) turning value

The value f(a, b) is said to be an extreme value of the function f(x, y) at the point (a, b) if it is either maximum nor minimum.

Note:

Every extreme value is a stationary value but a stationary value need not be an extreme value.

Working procedure to find the maximum or minimum value of f(x,y)

Let f(x,y) be a given function.

Step 1:

Find $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$

Solve the equations $f_x = 0$ and $f_y = 0$ to find the values of x and y. These values of x and y gives the points at which maxima or minima exists.

Let the points be (a_1,b_1), (a_2,b_2) etc.

Step 2:

Find $\frac{\partial^2 f}{\partial x^2} = f_{xx} = r$, $\frac{\partial^2 f}{\partial y^2} = f_{yy} = t$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = s$.

Step 3:

Calculate the values of r, s and t at each of the points found in step 1.

Step 4:

i) If $r < 0$ and $rt - s^2 > 0$, then f(x,y) has a maximum and the corresponding value of f(x,y) is called the maximum value.

ii) If $r > 0$ and $rt - s^2 > 0$,then f(x,y) has a minimum and the corresponding value of f(x,y) is called the minimum value;

iii) If $rt - s^2 < 0$, then f(x,y) has neither a maximum nor a minimum.

iv) If $rt - s^2 = 0$, further considerations are required.

Geometrical interpretation of Maxima and Minima

Geometrically, z = f(x,y) represents a surface. The maximum is a point or the surface (hill top) from which the surface descends (comes down) in every direction towards the XY plane (refer fig.1). The minimum is the bottom of depression from which the surface ascends (climbs up) in every direction (see fig.2). In either case, the tangent plane to the surface is horizontal (parallel to XY plane) and perpendicular to z axis).

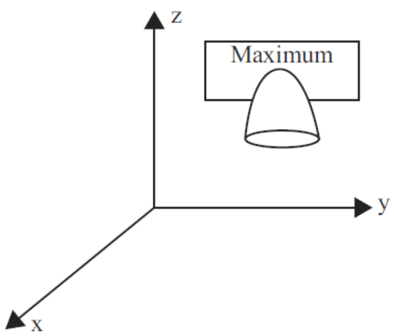

Fig.1

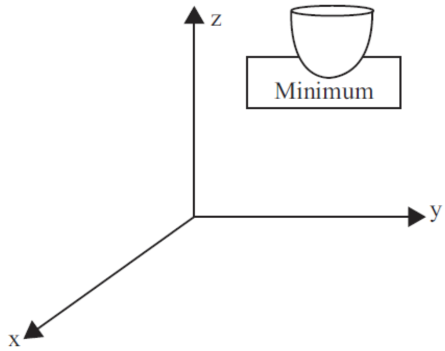

Fig.2

At saddle point f(x,y) is maximum in one direction while minimum in another direction. Geometrically such a surface (looks like the leather seat on back of a horse). Fig.3 forms a ridge rising in one direction and falling in another direction.

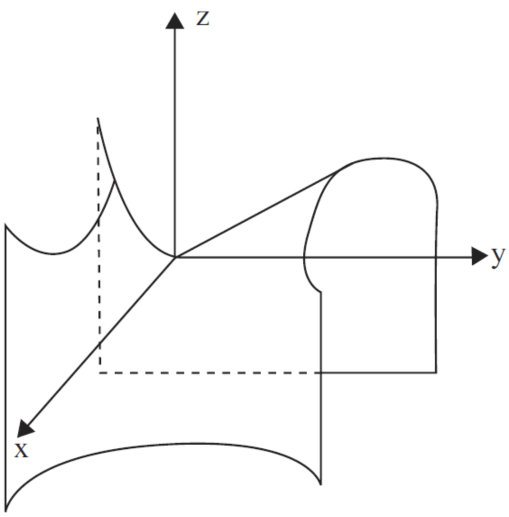

Fig.3

Example:

$z = xy$, hyperbolic paraboloid has a saddle point at the origin.

Solved problems:

1. Find the stationary points of $f(x,y)=\frac{1}{2}x^2-xy$

Solution:

Given $f(x,y)=\frac{1}{2}x^2-xy$

$$f_x = x-y;\ f_y = -x;$$
$$f_x = 0;\ f_y = 0 \Rightarrow x-y=0,\ -x=0$$

$$\Rightarrow x=0$$
$$y=0.$$

$\therefore$ The only stationary point is (0,0)

2. Find the critical points of $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

Solution:

Given, $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$f_x=4\left(x^3-x+y\right)$, $f_y=4\left(y^3+x-y\right)$

Critical points are the solutions of $f_x=0$ & $f_y=0$

(i.e.) $x^3-x+y=0$ (1)

$y^3+x-y=0$ (2)

Adding these two equations, we get

$x^3+y^3=0\ (or)\ x=-y$

Substituting in (1) gives $x\left(x^2-2\right)=0$

$$\therefore\ x=0,\pm\sqrt{2},\quad y=0,\mp\sqrt{2}$$

$\therefore$ The critical points are (0,0), ($\sqrt{2}$,- $\sqrt{2}$) & (- $\sqrt{2}$, $\sqrt{2}$)

3. Find the extreme values of the function $x^3 + y^3 - 3axy$

Solution:

Given $f(x, y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay, \quad f_y = 3y^2 - 3ax$

Now $f_x = 0$ and $f_y = 0$

$$\Rightarrow x^2 - ay = 0 \qquad (1)$$

$$y^2 - ax = 0 \qquad (2)$$

$$\Rightarrow y = \frac{x^2}{a}$$

(2) $\Rightarrow (2) \Rightarrow \quad \dfrac{x^4}{a^2} - ax = 0$

$$\Rightarrow x\left(x^3 - a^3\right) = 0$$

$$\Rightarrow x = 0, a$$

When x = 0, y = 0

y = a, x = a

$\therefore$ The two stationary points are (0,0), (a,a)

Now $r = f_{xx} = 6x$

$s = f_{xy} = -3a$

$t = f_{yy} = 6y$

At (0,0)

$$rt - s^2 = -9a^2 < 0$$

$\Rightarrow$ There is no extreme value at (0,0)

At (a,a)

$$rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$$

$\Rightarrow$ f(x,y) has extreme value at (a,a)

Now $r = 6a$

If $a > 0$ *then* $r > 0$ So that f(x,y) has a minimum value at (a,a)

Minimum value of $f(x,y) = a^3 + a^3 - 3a^3$

$= -a^3$

If $a < 0$ *then* $r < 0$ So that $f(x,\ y)$ has a maximum value at (a,a)

Maximum value of $f(x,y) = -a^3 - a^3 + 3a^3 = a^3$

4) Find the maximum and minima of $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$.

Solution:

Given $f(x,\ y) = x^3 + y^3 - 63(x+y) + 12xy$

$f(x) = 3x^2 - 63 + 12y, \quad f(y) = 3y^2 - 63 + 12x$

For extremum $f_x = 0,\ f_y = 0$

$\Rightarrow 3x^2 - 63 + 12y = 0 \ \& \ 3y^2 - 63 + 12x = 0$

$\Rightarrow \ x^2 + 4y = 21 \qquad (1)$

$\Rightarrow \ y^2 + 4x = 21 \qquad (2)$

Solving (1) & (2)

$x^2 - y^2 - 4(x-y) = 0$

$(x-y)\ (x+y-4) = 0$

$x = y \ \& \ x+y = 4 \qquad (3)$

$If \quad x = y,\ from\ (1),\ x^2 + 4x - 21 = 0$

$(x+7)\ (x-3) = 0$

$x = -7,\ 3$

$y = -7,\ 3$

Hence stationary points are (-7,-7), (3,3)

If $x + y = 4$ from (1)

$x^2 + 4(4-x) = 21$

$\Rightarrow \ x^2 - 4x - 5 = 0$

$\Rightarrow \ x = -1,\ 5$

$\therefore \ y = 5,\ -1$

Hence other stationery points are (-1,5), (5, -1)

Now $r = f_{xx} = 6x,\ s = f_{xy} = 12,\ t = f_{yy} = 6y$

At (-7,-7)

$$rt - s^2 = 36(-7)(-7) - 144$$

$$= 1620$$

$$> 0$$

$$Also,\ r = -42$$

$$< 0$$

Hence f(x,y) has a maximum at (-7,-7)

Maximum value = $(-7)^3 + (-7)^3 - 63(-14) + 12(-7)(-7)$

$= 784$

At (3,3)

$$rt - s^2 = 180 > 0$$

$$r = 18 > 0$$

Hence f(x,y) has a minimum at (3,3)

Minimum value $= -216$

At (-1,5)

$$rt - s^2 = -324 < 0$$

f(x,y) has neither maximum nor minimum at (-1,5)

$\therefore$ (-1,5) is a saddle point

At (5,-1)

$$rt - s^2 = -324 < 0$$

(5,-1) is a saddle point.

5) Find the extremum points of $f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$.

Solution:

Given, $f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$

$$f_x = 4x^3 - 4x, \quad f_y = -4y^3 + 4y$$

Solving $f_x = 0\ \&\ f_y = 0$

$$4\left(x^3 - x\right) = 0 \ \& \ 4\left(y - y^3\right) = 0$$
$$4x\left(x^2 - 1\right) = 0 \ \& \ 4y\left(1 - y^2\right) = 0$$
$$x = 0 \ (or) \pm 1 \quad \& \ y = 0 \ (or) \pm 1$$

$\therefore$ (0,0), (1,1), (1,-1), (-1,1), (-1,-1), (1,0), (-1,0), (0,1) and (0,-1) are the critical points.

Now $r = 12x^2 - 4, \ s = 0, \ t = -12y^2 + 4$

Point	rt-s^2	r	Nature
(0,0)	< 0	-	Saddle
(1,1)	< 0	-	Saddle
(1,-1)	< 0	-	Saddle
(-1,1)	< 0	-	Saddle
(-1,-1)	< 0	-	Saddle
(1,0)	> 0	> 0	Minimum
(-1,0)	> 0	> 0	Minimum
(0,1)	> 0	< 0	Maximum
(0,-1)	> 0	< 0	Maximum

$\therefore$ f(x,y) attains its minimum at (1,0) & (-1,0) and the minimum value is -1

f(x,y) attains its maximum at (0,1) & (0,-1) and the maximum value is +1.

6) Examine $f(x, y) = x^3y^2(1 - x - y)$ for extreme values.

Solution:

Given $f(x, y) = x^3y^2(1 - x - y)$

$$f_x = (1 - x - y)3x^2y^2 - x^3y^2$$
$$f_y = 2x^3y(1 - x - y) - x^3y^2$$

For Critical points, $f_x = 0, \quad f_y = 0$

$$(i.e.) \quad x^2y^2\left(3(1 - x - y) - x\right) = 0 \Rightarrow 4x + 3y = 3 \qquad (1)$$
$$x^3y\left(2(1 - x - y) - y\right) = 0 \Rightarrow 2x + 3y = 2 \qquad (2)$$

From Equation (1) x = 0 or y = 0 or 4x + 3y = 3

Equation (2) x = 0 or y = 0 or 2x + 3y = 2

$\therefore$ The critical points are $(0,0),(0,\frac{2}{3}),\ (\frac{3}{4},0),\ (\frac{1}{2},\frac{1}{3}),(0,1),(1,0)$

Now $r\ =\ 6(1-x-y)xy^2-6x^2y^2$

$s=6x^2y(1-x-y)-2x^3y-3x^2y^2$
$t=2x^3(1-x-y)-4x^3y$

At $(0,0),(0,\frac{2}{3}),\ (\frac{3}{4},0),(0,1),(1,0)$ the value of Δ = rt – s^2 = 0

At $\left(\frac{1}{2},\frac{1}{3}\right)$

$r=-\frac{1}{9}\ s=-\frac{1}{12}\ t=\frac{1}{8}$

$rt-s^2=\frac{1}{72}-\frac{1}{144}>0$

Also $r<0$

f(x,y) is maximum at ($\frac{1}{2},\frac{1}{3}$) and the maximum value of

$f(x,\ y)=(\frac{1}{2})^3\ (\frac{1}{3})^2\ (1-\frac{1}{2}-\frac{1}{3})$

$=\frac{1}{432}$

7) Test the function $f(x,\ y)\ =\ \left(x^2+y^2\right)e^{-(x^2+y^2)}$ for extremum points which do not lie on the circle $x^2+y^2\ =\ 1$

Solution:

Given $f(x,\ y)\ =\ \left(x^2+y^2\right)e^{-(x^2+y^2)}$

$f_x\ =\ \left(x^2+y^2\right)e^{-(x^2+y^2)}(-2x)+2xe^{-(x^2+y^2)}$

$=\ 2x\ \left(1-x^2-y^2\right)e^{-(x^2+y^2)}$

$f_y=\ \left(x^2+y^2\right)e^{-(x^2+y^2)}(-2y)\ +\ 2ye^{-(x^2+y^2)}$

$=\ 2y\ \left(1-x^2-y^2\right)e^{-(x^2+y^2)}$

Now, $f_x = 0 \text{ and } f_y = 0$

$$\Rightarrow 2x(1-x^2-y^2)e^{-(x^2+y^2)} = 0 \,\&\, 2y(1-x^2-y^2)e^{-(x^2+y^2)} = 0$$

$$\Rightarrow x=0, y=0 \quad \& \quad x^2+y^2=1$$

Since points lying on the circle $x^2 + y^2 = 1$should not be considered, (0, 0) is the only stationary point.

Now,

$$f_x = \left(2x - 2x^3 - 2xy^2\right)e^{-(x^2+y^2)}$$

$$r = f_{xx} = \left(2-6x^2 - 2y^2\right)e^{-(x^2+y^2)} + \left(2x - 2x^3 - 2xy^2\right)e^{-(x^2+y^2)}\left(-2x\right)$$

$$= e^{-(x^2+y^2)}\left(4x^4 - 10x^2 + 4x^2y^2 - 2y^2 + 2\right)$$

$$s = f_{xy} = -4xye^{-(x^2+y^2)} + \left(2x - 2x^3 - 2xy^2\right)e^{-(x^2+y^2)}(-2y)$$

$$=\left(-8xy + 4x^3y + 4xy^3\right)e^{-(x^2+y^2)})$$

$$t = e^{-(x^2+y^2)}\left(2 - 2x^2 - 6y^2\right) + e^{-(x^2+y^2)}\left(2y - 2y\,x^2 - 2y^3\right)\left(-2y\right)$$

$$= e^{-(x^2+y^2)}\left(2-2x^2 - 10y^2 + 4x^2y^2 + 4y^4\right)$$

At (0, 0)

r = 2, s = 0, t = 2

$rt - s^2 = 4 > 0$

& $r = 2 > 0$

∴ f(x, y) has a minimum value at (0, 0)

Minimum value $= 0$

8) Find a point within a triangle such that the sum of the square of its distances from the three vertices is a minimum.

Solution:

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the vertices of the triangle ABC. Consider a point p(x, y) inside the triangle.

Let $f\left(x, y\right) = \sum_{i=1}^{3}((x-x_i)^2 + (y-y_i)^2)$

For a maximum or minimum

$f_x = 0$, $f_y = 0$

$\Rightarrow \; 2\sum_{i=1}^{3}(\mathrm{x}-\mathrm{x}_i) = 0, \; 2\sum_{i=1}^{3}(\mathrm{y}-y_i) = 0$

$\Rightarrow \; (\mathrm{x}-\mathrm{x}_1)+(\mathrm{x}-\mathrm{x}_2)+(\mathrm{x}-\mathrm{x}_3) = 0$ and $(\mathrm{y}-y_1)+(\mathrm{y}-y_2)+(\mathrm{y}-y_3) = 0$

$\Rightarrow \; 3x = x_1 + x_2 + x_3$ and $3y = y_1 + y_2 + y_3$

$\Rightarrow \; x = \dfrac{x_1 + x_2 + x_3}{3}$ and $y = \dfrac{y_1 + y_2 + y_3}{3}$

$\Rightarrow$ The extreme may occur at the centroid

Now $r = 6$ $s = 0$ $t = 6$

Thus $rt - s^2 = 36 > 0$ and $r > 0$

Hence f(x, y) is a minimum at the centroid of the triangle.

Exercise

1. Given $f_{xx} = 6x,\; f_{xy} = 0,\; f_{yy} = 6y,$ find the nature of the stationary point (1, 2) of the function f (x, y)

2. What is the relation between a stationary point and extreme point of a function.

3. Find the stationary points of $x^2 + y^2 + 6y + 12$.

4. Find the critical points of $f(x, y) = x^3 + y^3 - 3xy$

5. Find the stationary points of $f(x, y) = x^2 - xy + y^2 - 2x + y$

6. Find the extreme values of $f(x, y) = x^4 + 2x^2y - x^2 + 3y^2$

7. Find the maxima and minima of $f(x, y) = x^3y^2(12 - 3\mathrm{x} - 4\mathrm{y})$

8. Examine the function for extreme values $F(x, y) = xy + 27\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

9. Find the saddle points and extreme points of the function $xy(3x + 2y + 1)$

10. Find the maxima and minima of $f(x, y) = Sinx + Siny + Sin(x+y); 0 \le x, y \le \frac{\pi}{2}$

11. Discuss the maximum and minimum of $(x^2 + y^2)\ e^{6x+2x^2}$

12. Find the extreme values of $x^3 + y^2 - 3x - 6y - 1$

13. Find the maxima and minima of $x^4 + y^4 - 36xy$

14. Discuss the maxima and minima of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

15. Find the maxima and minima of $x^3 + y^3 - x - 6y + 10$

16. Examine for extreme values $log(x^2 + y^2 + 2)$

17. Find the minimum value of the function $f(x, y) = x^2 + y^2 + xy + ax + by$

18. Discuss the maxima and minima of $f(x, y) = xy^2(3x + 6y - 2)$

19. Find the maxima and minima of $f(x, y) = x^4 + y^4 - 2(x-y)^2$

20. Examine the maxima and minima of $f(x, y) = x^3 - 4x^2 - xy - y^2$

21. Find the maxima and minima of $f(x, y) = (y - x^2)\ (2 - x - y)$

22. Discuss the maxima and minima of $f(x, y) = x^3 + y^3 - x - 6y + 10$

23. Examine for extreme values of $u = 2(x-y)^2 - x^4 - y^4$

24. Find the maximum value of the function $xye^{-(2x+3y)}$

25. Find the maximum and minimum value of and (x, y) = sin x sin y sin $(x + y)$, $0 < x, y < \pi$

Answers:

1. Minimum

3. (0, -3)

4. (0, 0),(1, 1)

5. (1, 0)

6. Minimum at $\left(\frac{\pm\sqrt{3}}{2}, \frac{-1}{4}\right)$

7. Maximum at(2, 1)

8. Minimum at (3, 3)

9. (0, 0), $\left(\frac{-1}{3}, 0\right)$, $\left(0, \frac{-1}{2}\right)$ are saddle points

10. The maximum point is $\left(\pi/3, \pi/2\right)$

11. Minimum at (0,0), Minimum value = 0

Minimum at (-1,0), Minimum value = e^{-4}

$\left(-1/2, 0\right)$ is a saddle point.

12. (0,0) is a saddle point.

13. (0,0) is a saddle point.

(-3, -3) & (3, 3) give minimum value.

14. Minimum at (6,0), Maximum at (4,0)

(5,1), (5,-1) is a saddle points.

15. Minimum at $\left(1/\sqrt{3}, 2\right)$, Maximum at $\left(-1/\sqrt{2}, -\sqrt{2}\right)$

$\left(-1/\sqrt{3}, \sqrt{2}\right)$ & $\left(1/\sqrt{3}, -\sqrt{2}\right)$ are saddle points.

16. Minima at (0, 0)

17. Minimum at $\left(\frac{6-2a}{3}, a-2b\right)$

Minimum value = $\frac{13b^2 + a^2 + 13ab}{9}$

18. Minimum at $\left(1/6, 1/6\right)$

$\left(0, 1/3\right)$ Saddle point.

at (0,0) no conclusion.

19. Maximum at (0,0)

Minimum at $\left(\sqrt{2},-\sqrt{2}\right)$ & $\left(-\sqrt{2},\sqrt{2}\right)$

20. Maximum at (0,0), $\left(\frac{5}{2},\frac{-5}{4}\right)$ is a saddle point.

21. (1,1) & (-2,4) are saddle points

$\left(-\frac{1}{2},\frac{11}{8}\right)$ is the maximum point. Minimum at $\left(\frac{1}{\sqrt{3}},\sqrt{2}\right)$,

22. Maximum at $\left(-\frac{1}{\sqrt{2}},-\sqrt{2}\right)$

$\left(\frac{1}{\sqrt{3}},-\sqrt{2}\right)$ *and* $\left(\frac{-1}{\sqrt{3}},\sqrt{2}\right)$ are saddle points

23. Maximum at $\left(\sqrt{2},-\sqrt{2}\right)$ & $\left(-\sqrt{2},\sqrt{2}\right)$

(0,0) is not an extreme point, Maximum value is 8.

24. (0,0) is not a extreme point, Maximum value at $\left(\frac{1}{2},\frac{1}{3}\right)$

Maximum value is $\frac{1}{6e^2}$

25. f(x,y) attains minimum value at ($\frac{2\pi}{3},\frac{2\pi}{3}$) and the minimum value is $\frac{-3\sqrt{3}}{8}$.

CONSTRAINED MAXIMA MINIMA (LAGRANGIAN MULTIPLIER METHOD)

Suppose we require to find the maximum and minimum value of $f(x,y,z)$ where x,y,z are subject to a constraint equation $g(x,y,z)=0$

We define a function

$$F(x,y,z)=f(x,y,z)+\lambda g(x,y,z) \qquad (1)$$

Where λ is the Lagrange multiplier which is independent of x, y, z

The necessary conditions for a maximum or minimum are

$$\frac{\partial F}{\partial x}=0 \qquad (2)$$

$$\frac{\partial F}{\partial y}=0 \qquad (3)$$

$$\frac{\partial F}{\partial z}=0 \qquad (4)$$

$$\frac{\partial F}{\partial \lambda}=0 \qquad (5)$$

Solving the four equations for four unknowns λ, x, y, z we obtain (x, y, z)

1. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Solution:

Let the auxiliary function F be

$$F(x,y,z)=\left(x^2+y^2+z^2\right)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right) \qquad (1)$$

$$\frac{\partial F}{\partial x}=2x+\lambda\left(\frac{-1}{x^2}\right)=2x-\frac{-\lambda}{x^2}$$

$$\frac{\partial F}{\partial y}=2y+\lambda\left(\frac{-1}{y^2}\right)=2y-\frac{\lambda}{y^2}$$

$$\frac{\partial F}{\partial z}=2z+\lambda\left(\frac{-1}{x^0}\right)=2z-\frac{-\lambda}{z^2}$$

For a minimum at (x,y,z) we have

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2x-\frac{\lambda}{x^2}=0 \Rightarrow x^3=\frac{\lambda}{2} \Rightarrow x=\left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \qquad (2)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2y-\frac{\lambda}{y^2}=0 \Rightarrow y^3=\frac{\lambda}{2} \Rightarrow y=\left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \qquad (3)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2z-\frac{\lambda}{z^2}=0 \Rightarrow z^3=\frac{\lambda}{2} \Rightarrow z=\left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \qquad (4)$$

From (2), (3) & (4) we get

$$x=y=z \qquad (5)$$

Given $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \qquad [\therefore x=y=z]$

$$3\left(\frac{1}{x}\right)=1$$

$$3 = x \qquad [i.e. \quad x = 3]$$

$\therefore (3,3,3)$ is the point where minimum value occur

The minimum value is

$$3^2+3^2+3^2 \Rightarrow 9+9+9$$
$$=27$$

2. Find the minimum value of $x^2+y^2+z^2$ subject to condition $x+y+z=3a$

Solution:

To find the stationary value

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2x+\lambda=0 \Rightarrow 2x=-\lambda \qquad (1)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2y+\lambda=0 \Rightarrow 2y=-\lambda \qquad (2)$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+\lambda=0 \Rightarrow 2z=-\lambda \qquad (3)$$

From (2), (3) & (4) we get

$$2x=2y=2z \Rightarrow x=y=z \qquad (4)$$

$$\text{Given}: x+y+z=3a$$
$$x+x+x=3a \qquad [by \;\; 4]$$
$$3x=3a$$
$$x=a \Rightarrow y=a; \quad z=a$$

(a, a, a) is where minimum value occur. The minimum value is $a^2+a^2+a^2=3a^2$

3. Find the maximum value of $x^m.y^n z^p$ when x + y + z = a

Solution:

Let $f=x^m y^n z^p$ *and* $g=x+y+z-a$, then the auxiliary function is

$$F=f+\lambda g$$

Stationary points are given by

$$\frac{\partial F}{\partial x}=0; \quad \frac{\partial F}{\partial y}=0; \frac{\partial F}{\partial z}=0 \text{ and } \frac{\partial F}{\partial \lambda}=0$$

$$mx^{m-1}y^n z^p+\lambda=0 \qquad (1)$$

$$nx^m y^{n-1}z^p+\lambda=0 \qquad (2)$$

$$px^m y^n z^{p-1}+\lambda=0 \qquad (3)$$

$$x+y+z-a=0 \qquad (4)$$

From (1), (2), (3) we get

$$-\lambda=mx^{m-1}y^n z^p=nx^m y^{n-1}z^p=px^m y^n z^{p-1}$$

$$i.e.\ \frac{m}{x}=\frac{n}{y}=\frac{p}{z}=\frac{m+n+p}{x+y+z}=\frac{m+n+p}{a}\ by\ (4)$$

f attains maximum when

$$x=\frac{am}{m+n+p}; \quad y=\frac{an}{m+n+p}; \quad z=\frac{ap}{m+n+p}$$

Maximum value of $f=\dfrac{a^{m+n+p}m^m n^n p^p}{(m+n+p)^{m+n+p}}$

4. Find the maximum volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Solution:

Let the sides of the rectangular parallelepiped be $2x, 2y, 2z$

Hence the volume $v=(2x)(2y)(2z)=8xyz$

Now, we have to maximize V subject to the condition.

$$g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

Let $F = f + \lambda g = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$ (1)

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0 \quad (2)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0 \quad (3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xy + \frac{2z\lambda}{c^2} = 0 \quad (4)$$

(2) $\Rightarrow 4yz = \frac{-x\lambda}{a^2}$

Multiply by x on both sides,

$$\Rightarrow 4xyz = \frac{-x^2}{a^2}\lambda$$

$$\Rightarrow \frac{4xyz}{-\lambda} = \frac{x^2}{a^2} \quad (5)$$

(3) $4xz = \frac{-y\lambda}{b^2}$

Multiply by 'y' on both sides,

$$\Rightarrow 4xyz = \frac{-y^2}{b^2}\lambda$$
$$\Rightarrow \frac{4xyz}{-\lambda} = \frac{y^2}{b^2} \quad (6)$$

(4) $4xy = \frac{-z\lambda}{c^2}$

Multiply by 'z' on both sides,

$$\Rightarrow 4xyz = \frac{-z^2}{c}\lambda$$
$$\Rightarrow \frac{4xyz}{-\lambda} = \frac{z^2}{c^2} \quad (7)$$

From (5) (6) & (7) we get

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=k, \text{ say} \qquad (8)$$

Given: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\Rightarrow k+k+k=1$$
$$\Rightarrow 3k=1$$
$$\Rightarrow k=\frac{1}{3}$$

$$\Rightarrow \frac{x^2}{a^2}=\frac{1}{x} \Rightarrow x^2=\frac{a^2}{3} \Rightarrow x=\frac{a}{\sqrt{3}}$$

Similarly $y=\frac{b}{\sqrt{3}}$; $\qquad z=\frac{c}{\sqrt{3}}$

The extremum point is $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$.

$$\therefore \text{ Maximum volume } = 8\left(\frac{abc}{3\sqrt{3}}\right)$$

6. Find the shortest and longest distance from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$ using Lagrange's method of constrained maxima and minima

Solution:

Let (x, y, z) be any point on the sphere. Distance of the point (x, y, z) from (1, 2, -1) is given by

$$d=\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$$

To find the maximum and minimum values of d or equivalently of d^2

$$d^2=(x-1)^2+(y-2)^2+(z+1)^2$$

Subject to constraint $x^2+y^2+z^2-24=0$

Here $f=(x-1)^2+(y-2)^2+(z+1)^2$ and
$g=x^2+y^2+z^2-24$

Auxiliary function $F = f + \lambda g$ where λ is the Lagrange multiplier. The stationary points of F are given by $\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial z} = 0; \quad \frac{\partial F}{\partial \lambda} = 0$

(i.e) $2(x-1) + 2\lambda x = 0$ (1)

$2(y-2) + 2\lambda x = 0$ (2)

$2(z+1) + 2\lambda z = 0$ (3)

$x^2 + y^2 + z^2 = 24$ (4)

From (1), (2) & (3) we get

$$x = \frac{1}{1+\lambda}, y = \frac{2}{1+\lambda}, z = \frac{1}{1+\lambda}$$

Using three values in (4) we get

$$\frac{6}{(1+\lambda)^2} = 24; \qquad i.e,\ (1+\lambda)^2 = \frac{1}{4}$$

$$\lambda = \frac{-1}{2}\ or\ \frac{-3}{2}$$

When $\lambda = \frac{-1}{2}$ the point on the sphere is $(2, 4, -2)$

When $\lambda = \frac{-3}{2}$ the point on the sphere is $(-2, -4, 2)$

When the point is $(2, 4, -2)$ we get

$$d = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$$

When the point is $(-2, -4, 2)$ we get

$$d = \sqrt{(-3)^2 + (-0)^2 + 3^2} = 3\sqrt{6}$$

Shortest and longest distances are $\sqrt{6}\ and\ 3\sqrt{6}$ respectively.

7. A rectangular box open at the top is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction

Solution:

Let x, y, z be length, breadth, height of the box

Surface area = $xy + 2yz + 2zx$ (A)

Volume = x y z = 32 (B)

Let auxiliary function F be

$$F(x, y, z) = (xy + 2yz + 2zx) + \lambda(xyz - 32) \quad (1)$$

Where λ is Lagrange multiplier

$$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda yx$$

$$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0 \Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \quad (2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda zx = 0 \Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \quad (3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda xy = 0 \Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \quad (4)$$

From (2) & (3) we get

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y \quad (5)$$

From (3) & (4) we get

$$\frac{1}{z} + \frac{2}{x} = \frac{1}{y} + \frac{2}{x}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$y = 2z \quad (6)$$

From (5) & (6) we get x = y = 2z

(B) Volume = xyz = 32

$$(2z)(2z)z = 32$$
$$4z^3 = 32$$
$$z^3 = \frac{32}{4} = 8$$
$$z = 2$$
$$\therefore x = 4; \quad y = 4; \quad z = 2$$

Cost minimum when x = 4; y = 4; z = 2. Thus dimensions of the box are [4, 4, 2]

8. A closed rectangular box is to have one edge equal to twice the other and a constant volume $72m^3$. Find the least surface area of the box.

Solution: Let x, y, 2y be the length, breadth and height of the box respectively

$$\text{Surface Area} = 2(x) + 2(y)(2y) + 2(x)(2y)$$
$$= 2xy + 4y^2 + 4xy$$
$$= 6xy + 4y^2 \qquad (A)$$

Volume xyz = 72

$$(i.e) \qquad xy(2y) = 72$$
$$2xy^2 = 72$$
$$xy^2 = 36 \qquad (B)$$

Let the auxiliary function F be

$$F(x, y, z) = \left(6xy + 4y^2\right) + \lambda\left(xy^2 - 36\right) \qquad (1)$$

$$\frac{\partial F}{\partial x} = 6y + \lambda y^2$$

$$\frac{\partial F}{\partial y} = 6x + 8y + 2\lambda xy$$

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = 0 \qquad \Rightarrow 6y + \lambda y^2 = 0$$

$$\lambda = \frac{-6y}{y^2} = \frac{-6}{y} \qquad (2)$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow 6x+8y+2xy=0$$

$$\lambda=\frac{-6x+8y}{2xy} \qquad (3)$$

From (2) & (3) we get

$$\frac{-6}{y}=\frac{-(6x+8y)}{2xy}$$
$$\frac{6}{y}=\frac{2(3x+4y)}{2xy}$$
$$6x=3x+4y$$
$$4y=3x \Rightarrow y=\frac{3}{4}x$$

(B) $\Rightarrow xy^2=36$

$$x\left(\frac{9}{16}x^2\right)=36$$
$$x^3=36\times\frac{16}{9} \Rightarrow x^3=4\times 16$$
$$x^3=4\times 4\times 4$$
$$x=4$$
$$y=\frac{3}{4}\times 4$$
$$\Rightarrow y=3$$

$\therefore f$ is minimum at (4, 3)

$$\text{The minimum surface area is} = (6)(4)(3)+4(3)^2$$
$$=72+36$$
$$=108$$

9. Show that, if the perimeter of a triangle is a constant, the triangle has maximum area when it is an equilateral

Solution:

Perimeter $S=\frac{a+b+c}{2}$

$$\text{Area of the triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$(\text{Area})^2 = s(s-a)(s-b)(s-c)$$

Consider $F = s(s-a)(s-b)(s-c) + \lambda(a+b+c-2s)$

F is extremum when,

$$\frac{\partial f}{\partial a} = -s(s-b)(s-c) + \lambda = 0$$
$$\frac{\partial f}{\partial b} = -s(s-a)(s-c) + \lambda = 0$$
$$\frac{\partial f}{\partial c} = -s(s-a)(s-b) + \lambda = 0$$
$$\therefore \lambda(s-a) = \lambda(s-b) = \lambda(s-c)$$
$$\text{or} \quad s-a = s-b = s-c$$
$$\therefore a = b = c$$

The triangle is equilateral.

10. Show that of all rectangular parallelepiped of given volume, the cube has the least surface area.

Solution:

Let x, y, z be the length, breadth, height of the rectangular parallelepiped

Volume is given as constant

$$v = xyz = k\,(say)$$

Surface Area A = $2(xy + yz + zx)$

To find minimum of A, subject to $xyz - k = 0$

Let F = $f + \lambda g$

$$= 2(xy + yz + zx) + \lambda(xyz - k)$$
$$\frac{\partial F}{\partial x} = 2(y+z) + \lambda yz = 0$$
$$\frac{\partial F}{\partial y} = 2(x+z) + \lambda xz = 0$$
$$\frac{\partial F}{\partial z} = 2(x+y) + \lambda xy = 0$$
$$\frac{2}{x} + \frac{2}{y} = -\lambda$$

$$\frac{2}{z}+\frac{2}{x}=-\lambda$$
$$\frac{2}{z}+\frac{2}{y}=-\lambda$$

Solving the three equations we get

x = y = z

Hence the parallelepiped is a **cube.**

11. Find the foot of the perpendicular from the origin on the plane $2x+3y-z-5=0$

Solution:

Let A be (0, 0, 0). We have to find a point B (x, y, z) such that the distance d is minimum

$$AB = d = \sqrt{x^2+y^2+z^2}$$
$$(i.e) \quad f = d^2 = x^2+y^2+z^2 \qquad (A)$$
$$g = 2x+3y-z-5=0 \qquad (B)$$

Let the auxiliary function F be $F = f+\lambda g$

(i.e.) $F(x, y, z) = \left(x^2+y^2+z^2-d^2\right)+\lambda(2x+3y-z-5)$

$$\frac{\partial F}{\partial x}=2x+2\lambda$$
$$\frac{\partial F}{\partial y}=2y+3\lambda$$
$$\frac{\partial F}{\partial z}=2z-\lambda$$

For extremum

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2x+2\lambda=0 \Rightarrow x=-\lambda \qquad (2)$$
$$\frac{\partial F}{\partial y}=0 \Rightarrow 2y+3\lambda=0 \Rightarrow \frac{2}{3}y=-\lambda \qquad (3)$$
$$\frac{\partial F}{\partial z}=0 \Rightarrow 2z-\lambda=0 \Rightarrow -2z=-\lambda \qquad (4)$$

From (2), (3), & (4) we get $x=\frac{2}{3}y=-2z$

(B) $=2(-2z)+3(-3z)-z-5=0$

$$-4z-9z-z-5=0$$

$$-14z \ =5$$

$$z=\frac{-5}{14}$$

$$x=-2\left(\frac{-5}{14}\right)\Rightarrow x=\frac{5}{7}$$

$$\frac{2}{3}y=-2\left(\frac{-5}{14}\right)$$

$$\therefore y=3\left(\frac{5}{14}\right)=\frac{15}{14}$$

Hence the extremum occurs at $x=\frac{5}{7};\quad y=\frac{15}{14};\quad z=\frac{-5}{14}$

The required point is $\left(\frac{5}{7},\frac{15}{14},\frac{-5}{14}\right)$.

12. The temperature $u(x,y,z)$ at any point in space is $u=400xyz^2$. Find the point on the surface of the sphere $x^2+y^2+z^2=1$ with highest temperature

Solution:

Given $u=f=400xyz^2$

$$g=x^2+y^2+z^2-1=0$$

Let the auxiliary function F be $F=f+\lambda g$

(i.e.) $F(x,y,\ z)=\left(400xyz^2\right)+\lambda\left(x^2+y^2+z^2-1\right)$

$$\frac{\partial F}{\partial x}=400yz^2+\lambda(2x)$$

$$\frac{\partial F}{\partial x}=400xz^2+\lambda(2y)$$

$$\frac{\partial F}{\partial x}=800xyz+\lambda(2z)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 400yz^2+\lambda 2x=0 \Rightarrow \frac{200yz^2}{x}=-\lambda \quad (2)$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow 400xz^2+\lambda 2y=0 \Rightarrow \frac{200xz^2}{y}=-\lambda \quad (3)$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow 800xyz+\lambda 2z=0 \Rightarrow 400xy=-\lambda \quad (4)$$

From (2) & (3) we get

$$\frac{200yz^2}{x}=\frac{200xz^2}{y}$$

$$\frac{y}{x}=\frac{x}{y}$$

$$y^2=x^2 \quad (5)$$

From (3) & (4) we get

$$\frac{200xz^2}{y}=400xy$$

$$z^2=2y^2 \quad (6)$$

From (5) & (6)

$$x^2=y^2=\frac{z^2}{2}$$

$$x^2+y^2+z^2=1$$

$$\Rightarrow \frac{z^2}{2}+\frac{z^2}{2}+z^2=1$$

$$z^0\left[\frac{1}{2}+\frac{1}{2}+1\right]=1$$

$$z^2\left[2\right]=1$$

$$z^2=\frac{1}{2}$$

$$z = \pm \frac{1}{\sqrt{2}}$$

$$x^2 = \frac{z^2}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

$$y^2 = x^2 \frac{1}{4} \Rightarrow y = \pm \frac{1}{2}$$

$$Subsitute \;\; x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{2} \; in \; u = 400xyz^2$$

$$u = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$u = 50$$

Maximum temperature is = 50

13. Assuming the dimensions of a triangle ABC varies show that the maximum value of $\cos A \cos B \cos C$ is obtained when the triangle is equilateral using Lagrange's method of multipliers.

Solution :

Let $f = \cos A \cos B \cos C$ where A + B + C = π

Suppose $g = A + B + C - \pi$

Consider F = $f + \lambda g$

$$= \cos A \cos B \cos C + \lambda\left(A + B + C - \pi\right)$$

$$\frac{\partial F}{\partial A} = \frac{\partial F}{\partial B} = \frac{\partial F}{\partial C} = 0 \; gives$$

$$\frac{\partial F}{\partial A} = -\sin A \cos B \cos C + 1 = 0$$

$$\frac{\partial F}{\partial B} = -\sin B \cos C \cos A + 1 = 0$$

$$\frac{\partial F}{\partial C} = -\sin C \cos A \cos B + 1 = 0$$

Hence $\sin A\cos B\cos C = \sin B\cos B\cos C\cos A = \sin C\cos A\cos B = \cos A\cos B\cos C$

$$\tan A = \tan B = \tan C$$
$$\Rightarrow A = B = C$$

The triangle is equilateral.

12. Find the dimensions of an open rectangular box of maximum capacity with surface area 432 m^2.

Solution:

Let x, y, z be the length, breadth, & height of box

Surface area = $xy + 2yz + 2zx = 432$ (1)

Volume = xyz

Let the auxiliary function F be,

$$F(x, y, z) = xyz + \lambda(xy + 2yz + 2zx - 432)$$

$$\frac{\partial F}{\partial x} = Yz + \lambda(y + 2z)$$

$$\frac{\partial F}{\partial y} = zx + \lambda(x + 2z)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2x + 2y)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0$$
$$\Rightarrow yz = -\lambda(y + 2z)$$
$$\Rightarrow \frac{yz}{(y + 2z)} = -\lambda \qquad (2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow zx + \lambda(x + 2z) = 0$$
$$\Rightarrow zx = -\lambda(x + 2z)$$
$$\Rightarrow \frac{zx}{(x + 2z)} = -\lambda \qquad (3)$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow xy+\lambda(2x+2y)=0$$

$$\Rightarrow xy=-\lambda(2x+2y)$$

$$\Rightarrow \frac{xy}{(2x+2y)}=-\lambda \qquad (4)$$

From (2) & (3) we get

$$\frac{yz}{-(y+2z)}=\frac{zx}{-(x+2z)}$$

$$x=y \qquad (5)$$

From (3) & (4) we get

$$\frac{zx}{x+2z}=\frac{xy}{2x+2y}$$

$$2x^2z+2xyz=x^2y+2xyz$$

$$2z=y \qquad (6)$$

$$xy+2yz+2zx=432$$

$$(2z)(2z)+2(2z)z+2z(2z)=432$$

$$4z^2+4z^2+4z^2=432$$

$$12z^2=432$$

$$z^2=36; \quad z=6$$

$$x=12; y=12; z=6$$

Thus the dimensions of the box should be 12, 12 and 6. Maximum volume = $12\times12\times16=864m^3$

Exercise

1. Find the maxima and minima of the function $f(x,y)=3x^2+4y^2-xy$ if $2x+y=21$

2. Find the maxima and minima if any of the function $f(x,y)=12xy-3y^2-x^2$

   Subject to x + y = 16.

3. Show that the maximum value of $x^2y^2z^2$ subject to the constraint $x^2+y^2+z^2=a^2 \, is \left(\frac{a^2}{3}\right)^3$

4. Find the minimum value of $x^2+y^2+z^2$, when (i) $xyz=a^3$ and (ii) $xy+yz+zx=3a^2$

5. Show that the minimum value of $\left(a^3x^2+b^3y^2+c^3z^2\right)$, when $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{k}$ is $k^2(a+b+c)^3$

6. The temperature at any point (x, y, z) is space is given by $T=k\,xyz^2$, where k is constant. Find the highest temperature on the surface of the sphere $x^2+y^2+z^2=a^2$

7. Show that, of all rectangular parallelepipeds with given surface area, the cube has the greatest volume

8. Prove that the rectangular solid of Maximum volume which can be inscribed in a sphere is a cube

9. Find the points on the surface $z^2=xy+1$ whose distance from the origin is the least.

10. Find the point on the surface $z=x^2+y^2$, that is nearest to the point $(3,-6,4)$.

11. Find the minimum distance from the point (3, 4, 15) to the cone $x^2+y^2=4z^2$

12. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$

Answers

1. $f(x,y)=\frac{987}{4}$ attains the minimum at $\left(\frac{17}{2},4\right)$

2. $f(x,y)=528$ attains the maximum at (9,7)

4. (i) $3a^2$, (ii) $3a^2$

6. $\frac{ka^4}{8}$

9. (0, 0, 1) and (0, 0, -1)

10. (1,-2,5)

11. 125

12. $\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right);\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right);$

JACOBIANS

Introduction:

Carl Gustav Jacob Jacobi [10 December 1804-18 February 1851] was a German Mathematician, who invented Jacobian determinant formed from the n^2 differential co-efficients of n given functions of n independent variables. He made fundamental contributions to elliptic functions, dynamics, differential equations and number theory. In vector calculus, the Jacobian matrix is the matrix of all first order partial derivatives of vector valued function.

Suppose $F:\mathbb{R}^n \to \mathbb{R}^m$ is a function form Euclidean n-space to Euclidean m-space. Such a function is given by m real-valued component functions $F_1(x_1, x_2 \ldots x_n)$, $F_2(x_1, x_2 \ldots x_n)$,…, $F_m(x_1, x_2 \ldots x_n)$. The partial derivatives of all these functions with respect to the variables $x_1, x_2 \ldots x_n$ (if they exist) can be organized in $m \times n$ matrix, the Jacobian matrix J of F as follows:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

If $m = n$ the Jacobian matrix will be a square matrix and its determinant, a function of $x_1, x_2 \ldots x_n$ is the Jacobian determinant of F

Geometrical Interpretations of F:

(i) Area

(ii) Orientation

Area: The Jacobian of a matrix allows us to understand the area. When the determinant of the Jacobian is not equal to zero, the area is not annihilated but may be enlarged or shrunk.

Definition:

If u and v be two continuous functions of two independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix}$ is called Jacobian of u and v with respect to x and y. It is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $\mathrm{J}\left(\frac{u,v}{x,y}\right)$.

Note:

If $u_1, u_2 \ldots u_n$ be n continuous functions of n independent variables $x_1, x_2 \ldots x_n$ such that $\frac{\partial u_i}{\partial x_j}$, $i = 1, 2, \ldots n$ are also continuous in $x_1, x_2 \ldots x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called of Jacobian $u_1, \mathrm{u}_2 \ldots, u_n$ with respect to $x_1, x_2 \ldots x_n$. It is denoted by $\frac{\partial(\mathrm{u}_1, \mathrm{u}_2 \ldots u_n)}{\partial(x_1, x_2 \ldots x_n)}$ or $J(\frac{u_1, \mathrm{u}_2 \ldots u_n}{x_1, x_2 \ldots x_n})$

Properties of Jacobians:

1. If *u* and *v* are functions of *x* and y, then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Proof:

Solve $u = f(x,y)$ and $v = g(x,y)$ for x and y. Let $x = \phi(u,v)$ and $y = \chi(u,v)$

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial u}&=\frac{\partial u}{\partial u}=1\\
\frac{\partial u}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial v}&=\frac{\partial u}{\partial v}=0\\
\frac{\partial v}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}.\frac{\partial y}{\partial u}&=\frac{\partial v}{\partial u}=0\\
\frac{\partial v}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}.\frac{\partial y}{\partial v}&=\frac{\partial v}{\partial v}=1
\end{aligned}\right\} \qquad (1)$$

Now

$$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\times\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial v}\\ \frac{\partial v}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}.\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}.\frac{\partial y}{\partial u}\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\text{[From(1)]}$$

$$=1$$

(2) If u and v are functions of r and s, where r and s are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}$$

Proof:

$$\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\times\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial x}\end{vmatrix}$$

By rewriting the second determinant, we have

$$\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\times\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

(3) If the function u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w with respect to x, y, z vanishes.

Proof:

If u, v, and w are not independent variables then there will be a relation $F(u,v,w)=0$, which will connect these dependent variables.

Differentiating this relation with respect to x, y, and z we get

$$\frac{\partial F}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}.\frac{\partial v}{\partial x}+\frac{\partial F}{\partial w}.\frac{\partial w}{\partial x}=0$$

$$\frac{\partial F}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}.\frac{\partial v}{\partial y}+\frac{\partial F}{\partial w}.\frac{\partial w}{\partial y}=0$$

$$\frac{\partial F}{\partial u}.\frac{\partial u}{\partial z}+\frac{\partial F}{\partial v}.\frac{\partial v}{\partial z}+\frac{\partial F}{\partial w}.\frac{\partial w}{\partial z}=0$$

Eliminating $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}$ *and* $\frac{\partial F}{\partial w}$ we get $\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$

Note:

If the transformations *x* = *x* (*u*, *v*), *y* = *y* (*u, v*) are made in the double integral $\iint f(x,y)dxdy$ then *f*(*x*, *y*) = *F*(*u, v*) and *dxdy* = |J| *dudv* where J = $\frac{\partial(x,y)}{\partial(u,v)}$

Worked Example

1) If $x = r\cos\theta$, $y = r\sin\theta$, then verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Solution:

Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Now $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore\ 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

Similarly $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$

Hence

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1$$

2. If we transform from three dimensional cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r,θ,ϕ), then show that the Jacobian of x, y, z with respect to r,θ,ϕ is $r^2\sin\theta$

Solution:

The transformation equations are

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2[\sin\theta\cos\phi(0+\sin^2\theta cos\phi) - \cos\theta\cos\phi(0-\sin\theta\cos\theta\cos\phi)$$
$$-\sin\theta\sin\phi(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)]$$

$$= r^2[\sin^3\theta\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin^3\theta\sin^2\phi$$
$$+\sin\theta\sin^2\phi\cos^2\theta]$$

$$= r^2\sin\theta[(\sin^2\phi+\cos^2\phi)(\sin^2\theta+\cos^2\theta)]$$

$$= r^2\sin\theta$$

3. If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$, then show that

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3$$

Solution:

Let $y_1 = f_1(x_1)$, $y_2 = f_2(x_1,x_2)$, $y_3 = f_3(x_1,x_2,x_3)$ then

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial y_2}{\partial x_2}\cdot\frac{\partial y_3}{\partial x_3}$$
$$=(-\sin x_1)(-\sin x_1 \sin x_2)(-\sin x_1 \sin x_2 \sin x_3)$$
$$=-\sin^3 x_1 \sin^2 x_2 \sin x_3$$

4. If $u = 2x$, $v = x^2 - y^2$, $x = \cos\theta$ and $y = r\sin\theta$, then compute $\frac{\partial(u,v)}{\partial(r,\theta)}$

Solution:

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= -4(x^2 + y^2) r(\cos^2\theta + \sin^2\theta)$$

$$= -4r^3$$

5. If $f(0) = 0$, $f'(x) = \frac{1}{1+x^2}$, $f'(y) = \frac{1}{1+y^2}$; then without integrating prove that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Solution:

Let $u = f(x) + f(y)$ *and* $v = \frac{x+y}{1-xy}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad (1)$$

$$\frac{\partial u}{\partial x} = f'(x) = \frac{1}{1+x^2}, \frac{\partial u}{\partial y} = f'(y) = \frac{1}{1+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1 - xy + (x+y)y}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}; \frac{\partial v}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

Substituting in (1) we have

$\frac{\partial(u,v)}{\partial(x,y)} = 0$. Thus there is a relation between u and v.

Let $u = \phi(v)$ then $f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right)$

Put $y = 0$, we get $f(x) + f(0) = \phi(x)$

i.e. $f(x) = \phi(x)$

Hence the function $\phi = f$.

$$\therefore\ f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

6) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1=\frac{x_2x_3}{x_1}$, $y_2=\frac{x_3x_1}{x_2}$, $y_3=\frac{x_1x_2}{x_3}$.

Solution:

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3}\\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1}\\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_2}\\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2}\end{vmatrix}=\frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix}-x_2x_3 & x_3x_1 & x_3x_1\\ x_2x_3 & -x_3x_1 & x_1x_2\\ x_2x_3 & x_3x_1 & -x_1x_2\end{vmatrix}$$

$$=\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4.$$

7) Examine if the following functions are functionally dependent. If they are, find also the functional relationship.

(i) $u=\sin^{-1}x+\sin^{-1}y; v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

(ii) $u=y+z, v=x+2z^2, w=x-4yz-2y^2$

Solution (i):

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-x^2}};\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-y^2}};\frac{\partial v}{\partial x}=\frac{1}{\sqrt{1-y^2}}\frac{-xy}{\sqrt{1-x^2}};\frac{\partial v}{\partial y}=\frac{-xy}{\sqrt{1-y^2}}+\sqrt{1+x^2}$$

$$\therefore\ \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}}\\ \sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}}+\sqrt{1-x^2}\end{vmatrix}=0$$

$\therefore$ u and v are functionally dependent.

Now $\sin u = \sin[\sin^{-1} x + \sin^{-1} y]$

$$= \sin(\sin^{-1} x)\cos(\sin^{-1} y) + \cos(\sin^{-1} x)\sin(\sin^{-1} y)$$
$$= x.\cos\left(\cos^{-1}\sqrt{1-y^2}\right) + y.\cos\left(\cos^{-1}\sqrt{1-x^2}\right)$$
$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$
$$= v$$

$\therefore$ The functional relationship between u and v is $\sin u = v$

(ii) $u = y + z, \quad v = x + 2z^2 \quad w = x - 4yz - 2y^2$

$$\frac{\partial u}{\partial x} = 0;\ \frac{\partial v}{\partial x} = 1\ \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1\quad \frac{\partial v}{\partial y} = 0\quad \frac{\partial w}{\partial y} = -4y - 4z$$

$$\frac{\partial u}{\partial z} = 1\quad \frac{\partial v}{\partial z} = 4z\quad \frac{\partial w}{\partial z} = -4y$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix} = 0$$

$\therefore$ u, v and w are functionally dependent.

Now $v - w = 2z^2 + 4yz + 2y^2$

$$= 2(y+z)^2 = 2u^2$$

$\therefore$ The functional relationship among u, v and w is $2u^2 = v - w$

8). Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$

Solution:

Let $u = ax^2 + 2hxy + by^2, \quad v = Ax^2 + 2Hxy + By^2$

u and *v* are not independent, if there exists a relationship between them and in that case $\frac{\partial(u,v)}{\partial(x,y)}$ should vanish identically.

i.e. $\begin{vmatrix} 2(ax+hy) & 2(hx+by) \\ 2(Ax+Hy) & 2(Hx+By) \end{vmatrix} = 0$

$$\text{i.e. } (ax+hy)(Hx+By) - (Ax+Hy)(hx+by) = 0$$
$$(aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2 = 0$$

Now the variables *x* and *y* are independent and as such the co-efficient of x^2 and y^2 should vanish simultaneously.

$\therefore\ aH - Ah = 0 \quad \text{or} \quad \frac{a}{A} = \frac{h}{H} \quad \text{and}$

$Bh - bH = 0 \quad \text{or} \quad \frac{h}{H} = \frac{b}{B}.$

Hence $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$ are the required conditions.

9. Express $\iiint \sqrt{xyz(1-x-y-z)}\,dxdydz$ in terms of u, v, w given that $x+y+z=u, y+z=uv, z=uvw$.

Solution:

The given transformations are

$$x+y+z=u \qquad (1)$$
$$y+z=uv \qquad (2)$$
$$z=uvw \qquad (3)$$

Using (3) in (2), we have $y=uv(1-w)$

Using (2) in (1), we have $x=u(1-v)$

$dxdydz = |\mathrm{J}|\,dudvdw$ where

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}1-v & -u & 0\\ v(1-w) & u(1-w) & -uv\\ vw & wu & uv\end{vmatrix}=u^2v \qquad (4)$$

Using (1), (2), (3) and (4) in the given triple integral I, we have

$$I=\iiint\sqrt{u^3v^2w(1-v)(1-w)(1-u)}\,u^2v\,dudvdw$$
$$=\iiint u^{\frac{7}{2}}v^2w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\,dudvdw.$$

EXERCISE

1. If $u=\frac{x}{y-z}, v=\frac{y}{z-x}, w=\frac{z}{x-y}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

2. If $u_1=x_1+x_2+x_3+x_4$, $u_1u_2=x_2+x_3+x_4$, $u_1u_2u_3=x_3+x_4$, $u_1u_2u_3u_4=x_4$ then show that $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)}=u_1^3u_2^2u_3$

3. If $u=x(1-y), v=xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$

4. If $x = uv$, $y = \frac{u+v}{u-v}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

5. If $u=x^2-y^2, v=2xy$ find $\frac{\partial(x,y)}{\partial(u,v)}$

6. If $x_1+x_2+\cdots+x_n=y_1$

$x_2+\cdots+x_n=y_1y_2$

$x_3+\cdots+x_n=y_1y_2y_3$

$\vdots$

$x_n=y_1y_2y_3\cdots y_n$

Then show that $\frac{\partial(x_1,x_2,\ldots x_n)}{\partial(y_1,y_2,\ldots,y_n)}= y_1^{n-1}y_2^{n-2},\cdots,y_{n-2}^2y_{n-1}$.

7. If $y_1 = 1 - x_1,\ y_2 = x_1(1 - x_2),\ y_3 = x_1 x_2 (1 - x_3), \cdots, y_n = x_1 x_2 \cdots x_{n-1}(1 - x_n)$ then show that $\dfrac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$.

8. If $x = u(1+v)$ and $y = v(1+u)$, find the Jacobian of *x, y* with respect to *u* and *v.*

9. If $x = u(1 - v),\ y = uv$ verify that $\dfrac{\partial(x, y)}{\partial(u,v)} \times \dfrac{\partial(u,v)}{\partial(x, y)} = 1$.

10. If $u = x^2,\ v = y^2$ find $\dfrac{\partial(u,v)}{\partial(x, y)}$

11. If $x = r\cos\theta\ ,\ y = r\sin\theta\ ,\ z = z$ find $J\left(\dfrac{x, y, z}{r\theta\ , z}\right)$.

12. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, Find $J\left(\dfrac{x, y, z}{u, v, w}\right)$

13. $F = xu + v - y,\ G = u^2 + vy + w,\ H = zu - v + vw$ Find $\dfrac{\partial(F, G, H)}{\partial(u,v,w)}$

ANSWERS

1. 0

2. x

4. $\dfrac{(\mathrm{u} - \mathrm{v})^2}{(u+v)^2}$

5. $\dfrac{1}{4(x^2 + y^2)}$

10. $4xy$

11. r

12. $\dfrac{-1}{2(x - y)(y - z)(z - x)}$

13. $x(vy + 1 - w) + z - 2uv$.

Part – B

1. If $x = a\cosh a \ \cos b$, $y = a\sinh a \ \sin b$ then show that

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{a^2}{2}[\cosh 2\alpha - \cos 2\beta)$$

2. If $u = x(1 - v^2)^{-1/2}$, $v = y(1 - r^2)^{-1/2}$, $w = z(1 - r^2)^{-1/2}$ where $r^2 = x^2 + y^2 + z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1 - r^2)^{\frac{-5}{2}}$

3. Examine the functional dependence of the following functions. If they are dependent find the relation between them.

i) $u = \frac{x+y}{x-y}, \ v = \frac{xy}{(x-y)^2}$

ii) $u = \frac{x+y}{1-xy}, \ v = \tan^{-1}x + \tan^{-1}y$

iii) $f_1 = x+y+z, \ f_2 = x^2+y^2+z^2, \ f_3 = xy+yz+zx$

iv) $u = \frac{x-y}{x+z}, \ v = \frac{x+z}{y+z}$

v) $u = 3x+2y-z, \ v = x-2y+z, \ w = x(x+2y-z)$

vi) $u = x+y+z, \ v = x^2+y^2+z^2, \ w = x^3+y^3+z^3-3xyz$

vii) $u = x+y-z, \ v = -x+y+z, \ w = x^2+y^2+z^2-2yz$

viii) $u = xy+yz+zx, \ v = x^2+y^2+z^2, \ w = x+y+z$

4. If $u_1 = x_1x_2 + x_3$, $u_1^2u_2 = x_2 + x_3$ and $u_1^3u_3 = x_3$ then prove that $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = -\frac{1}{u_1^5}$

5. If $u^3+v^3 = x+y$, $u^2+v^2 = x^3+y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2-x^2}{2uv(u-v)}$

6. If $u^3 = xyz,\ \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ w^2 = x^2 + y^2 + z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$= \frac{v(x-y)(y-z)(z-x)(x+y+z)}{3u^2w(xy+yz+zx)}$$

7. If $u^3 + v + w + = x + y^2 + z^2$

$u + v^3 + w = x^2 + y + z^2$

$u + v + w^3 = x^2 + y^2 + z$

Then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$

Answers:

3)

(i) $u^2 = 1 + 4v$

$(ii)\ u = \tan v$

$(iii)\ f_1^2 = f_2 + 2f_3$

$(iv)\ v = \frac{1}{1-u}$

$(vi)\ u^3 = 3uv - 2w$

$(vii)\ u^2 + v^2 = 2w$

$(viii)\ w^2 = v + 2u$

TEXT / REFERENCE BOOKS

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Integral Calculus – SMTA1106

UNIT-IV

INTEGRAL CALCULUS

INTRODUCTION: The integral of the function g(x) with respect to x is the function whose derivative with respect to x is g(x) and is written as $\int g(x)dx$.

Example: $\int \cos x\, dx = sinx$ and $\frac{d}{dx}(sinx) = cosx$.

BASIC FORMULAE:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
2. $\int \frac{1}{x} dx = logx + c$
3. $\int e^x dx = e^x + c$
4. $\int \cos x\, dx = sinx + c$
5. $\int sinx\, dx = -cosx + c$
6. $\int sec^2 xdx = tanx + c$
7. $\int cosec^2 xdx = -cotx + c$
8. $\int secx\, tanxdx = secx + c$
9. $\int cosecxcotx\, dx = -cosecx + c$
10. $\int tanxdx = \log(secx) + c$
11. $\int cotxdx = \log(sinx) + c$
12. $\int secxdx = \log(secx + tanx) + c$
13. $\int cosecxdx = \log(cosecx - cotx) + c$
14. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ if $(n \neq -1)$
15. $\int \frac{1}{1+x^2} dx = tan^{-1}x + c$
16. $\int \frac{-1}{1+x^2} dx = cot^{-1}x + c$
17. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + c$
18. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}x + c$
19. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c$
20. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} log\left(\frac{a+x}{a-x}\right) + c$
21. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} log\left(\frac{x-a}{x+a}\right) + c$
22. $\int \frac{1}{\sqrt{x^2-a^2}} dx = log\left[x + \sqrt{x^2 - a^2}\right] + c$
23. $\int \frac{1}{\sqrt{x^2+a^2}} dx = log\left[x + \sqrt{x^2 + a^2}\right] + c$
24. $\int \frac{1}{\sqrt{a^2-x^2}} dx = sin^{-1}\left(\frac{x}{a}\right) + c$

DEFINITE INTEGRALS

The definite integral of $f(x)$ between the limits $x = a \ \ and \ x = b$ is defined by $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$.

Properties of Definite Integrals:

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Proof: $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

$$\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$$

$$\therefore \int_a^b f(x)dx = \int_a^b f(t)dt$$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof: Consider R.H.S

$$-\int_b^a f(x)dx = -[F(x)]_b^a = -[F(a) - F(b)] = F(b) - F(a) = \int_a^b f(x)dx = L.H.S$$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Proof: Consider R.H.S

$$\int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) = \int_a^b f(x)dx = L.H.S$$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Consider R.H.S $\int_0^a f(a-x)dx$

Put $a - x = t \ \Rightarrow -dx = dt \ \Rightarrow$ dx = −dt

Limits: When x = 0 ⇒ t = a and x = a ⇒ t = 0

$$\int_0^a f(a-x)dx = \int_a^0 f(t)(-dt) = -\int_0^a f(t)(-dt) = \int_0^a f(t)dt$$

$= \int_0^a f(x)dx = L.H.S$ (By property (1)

5. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Proof: Consider $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

Put $x = 2a - t$ in the second integral.

⇒ dx = −dt When x = a ⇒ t = 0 and x = 2a ⇒ t = 0

$$\int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

6. (i) If $f(2a-x) = f(x), then\ \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

(ii) If $f(2a-x) = -f(x), then\ \int_0^{2a} f(x)dx = 0$

Proof: (i) By property (5) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$$= \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

(ii) By property (5) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$$= \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

7. (i) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ is an even function.

(ii) $\int_{-a}^a f(x)dx = 0$, if $f(x)$ is an odd function.

Proof: Suppose $\boldsymbol{f(x)}$ **is an even function** $\boldsymbol{f(-x) = f(x)}$

$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$

$$= \int_{-a}^0 f(-x)dx + \int_0^a f(x)dx$$

Put $-x = t$ in the first integral.

⇒ dx = −dt When x = 0 ⇒ t = 0 and x = -a ⇒ t = a

$$= \int_a^0 f(t)(-dt) + \int_0^a f(x)dx = \int_0^a f(t)dt + \int_0^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

(ii) Suppose $\boldsymbol{f(x)}$ **is an odd function** $\boldsymbol{f(-x) = -f(x)}$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx$$

Put $-x = t$ in the first integral.

$\Rightarrow$ dx = −dt When x = 0 ⇒ t = 0 and x = -a ⇒ t = a

$$= -\int_{a}^{0} f(t)(-dt) + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

8. $\boldsymbol{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx}$

Proof: Consider R.H.S $\int_a^b f(a+b-x)dx$

Put $a + b - x = t \Rightarrow -dx = dt \Rightarrow dx = -dt$

When x = b ⇒ t = a and x = a ⇒ t = b

$$= \int_{b}^{a} f(t)(-dt) = \int_{a}^{b} f(t)(dt) = \int_{a}^{b} f(x)(dx) = L.H.S$$

<u>Problemss</u>

1. Evaluate

(a) $\int_0^{\pi/4} \tan x \sec^2 t \; dx$

(b) $\int_{-\pi/4}^{0} \tan x \sec^2 x \; dx$

<u>**Solution**</u>

1. (a) $\int_0^{\pi/4} \tan x \sec^2 t \; dx$

Let $u = \tan x \Rightarrow du = \sec^2 x \; dx;\; x = 0 \Rightarrow u = 0,\; x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_0^{\pi/4} \tan x \sec^2 t \; dx = \int_0^1 u \; du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2}$$

(b) $\int_{-\pi/4}^{0} \tan x \sec^2 x \; dx$

Use the same substitution as in part (a); $x = -\frac{\pi}{4} \Rightarrow u = -1, \; x = 0 \Rightarrow u = 0$

$$\int_{-\pi/4}^{0} \tan x \sec^2 x \; dx \; = \int_{-1}^{0} u \; du \; = \left[\frac{u^2}{2}\right]_{-1}^{0} = 0 - \frac{1}{2} = -\frac{1}{2}$$

2. Evaluate

(a) $\int_{0}^{\pi} 3\cos^2 x \sin x \; dx$

(b) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \; dx$

Solution

(a) $\int_{0}^{\pi} 3\cos^2 x \sin x \; dx$

Let $u = \cos x \Rightarrow du = -\sin x \; dx \Rightarrow -du = \sin x \; dx; \; x = 0 \Rightarrow u = 1, \; x = \pi \Rightarrow u = -1$

$$\int_{0}^{\pi} 3\cos^2 x \sin x \; dx = \int_{1}^{-1} -3u^2 du = \left[-u^3\right]_{1}^{-1} = -(-1)^3 - \left(-(1)^3\right) = 2$$

(b) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \; dx$

Use the same substitution as in part (a); $x = 2\pi \Rightarrow u = 1, \; x = 3\pi \Rightarrow u = -1$

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \; dx = \int_{1}^{-1} -3u^2 du = 2$$

3. Evaluate

(a) $\int_{0}^{\pi/6} (1 - \cos 3t) \sin 3t \; dt$

(b) $\int_{\pi/6}^{\pi/3} (1 - \cos 3t) \sin 3t \; dt$

Solution

(a) $\int_{0}^{\pi/6} (1 - \cos 3t) \sin 3t \; dt$

Let $u = 1 - \cos 3t \Rightarrow du = 3\sin 3t \; dt \Rightarrow \frac{1}{3}du = \sin 3t \; dt; \; t = 0 \Rightarrow u = 0, \; t = \frac{\pi}{6} \Rightarrow u = 1 - \cos\frac{\pi}{2} = 1$

$$\int_{0}^{\pi/6} (1 - \cos 3t) \sin 3t \; dt \; = \int_{0}^{1} \frac{1}{3} u \; du \; = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_{0}^{1} = \frac{1}{6}(1)^2 - \frac{1}{6}(0)^2 = \frac{1}{6}$$

(b) $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\ dt$

Use the same substitution as in part (a); $t=\dfrac{\pi}{6} \Rightarrow u=1,\ t=\dfrac{\pi}{3} \Rightarrow$

$u = 1-\cos\pi = 2$

$$\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\ dt\ = \int_1^2 \frac{1}{3}u\ du\ = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_1^2 = \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

4. Evaluate

(a) $\displaystyle\int_{-\pi/2}^{0} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}dt$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt$

Solution

(a) $\displaystyle\int_{-\pi/2}^{0} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}dt$

Let $u = 2+\tan\dfrac{t}{2} \Rightarrow du = \dfrac{1}{2}\sec^2\dfrac{t}{2}dt \Rightarrow 2du = \sec^2\dfrac{t}{2}dt;\ t=\dfrac{-\pi}{2} \Rightarrow u = 2+\tan\left(\dfrac{-\pi}{4}\right) = 1,\ t=0 \Rightarrow u=2$

$$\int_{-\pi/2}^{0} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}dt = \int_1^2 u\,(2du) = \left[u^2\right]_1^2 = 2^2-1^2 = 3$$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt$

Use the same substitution as in part (a); $t=\dfrac{-\pi}{2} \Rightarrow u=1,\ t=\dfrac{\pi}{2} \Rightarrow u=3$

$$\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt\ = 2\int_1^3 u\ du\ = \left[u^2\right]_1^3 = 3^2-1^2 = 8$$

5. Evaluate

(a) $\displaystyle\int_{9}^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz$

(b) $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz$

Solution

(a) $\int_9^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz$

Let $u = 4 + 3\sin z \Rightarrow du = 3\cos z\ dz \Rightarrow \frac{1}{3}du = \cos z\ dz;\ z = 0 \Rightarrow u = 4,\ z = 2\pi \Rightarrow u = 4$

$\int_9^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz\ = \int_4^4 \frac{1}{\sqrt{u}}\ \left(\frac{1}{3}\ du\right) = 0$

(b) $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz$

Use the same substitution as in part (a); $z = -\pi \Rightarrow u = 4 + 3\sin(-\pi) = 4,\ z = \pi \Rightarrow u = 4$

$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz\ = \int_4^4 \frac{1}{\sqrt{u}}\ \left(\frac{1}{3}\ du\right) = 0$

6. Evaluate

$$\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx$$

<u>Solution</u>

$\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx$

Let $u = \frac{x}{6} \Rightarrow du = \frac{1}{6}dx \Rightarrow 6du = dx;\ x = \pi \Rightarrow u = \frac{\pi}{6},\ x = 3\pi \Rightarrow u = \frac{\pi}{2}$

$\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx = \int_{\pi/6}^{\pi/2} 6\cot^2 u\ du = 6\int_{\pi/6}^{\pi/2} \left(\csc^2 u - 1\right) du = \left[6\left(-\cot u - u\right)\right]_{\pi/2}^{\pi/6} =$

$= 6\left(-\cot\frac{\pi}{2} - \frac{\pi}{2}\right) - 6\left(-\cot\frac{\pi}{6} - \frac{\pi}{6}\right) = 6\sqrt{3} - 2\pi$

7. Evaluate

$$\int_{-1}^{1} 2x\sin\left(1 - x^2\right) dx$$

<u>Solution</u>

$\int_{-1}^{1} 2x\sin\left(1 - x^2\right) dx$

Let $u = 1 - x^2 \Rightarrow du = -2x\ dx \Rightarrow -du = 2xdx;\ x = -1 \Rightarrow u = 0,\ x = 1 \Rightarrow u = 0$

$\int_{-1}^{1} 2x\sin\left(1 - x^2\right) dx = \int_0^0 -\sin u\ du = 0$

8. Evaluate

$$\int_{\pi/4}^{3\pi/4} \csc z \cot z \ dz$$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc z \cot z \ dz$$

Let $\int_{\pi/4}^{3\pi/4} \csc z \cot z \ dz = [-\csc z]_{\pi/4}^{3\pi/4} = \left(-\csc \frac{3\pi}{4}\right) - \left(-\csc \frac{\pi}{4}\right) = -\sqrt{2} + \sqrt{2} = 0$

9. Evaluate

$$\int_0^{\pi/2} 5 (\sin x)^{3/2} \cos x \ dx$$

Solution

$$\int_0^{\pi/2} 5 (\sin x)^{3/2} \cos x \ dx$$

Let $u = \sin x \Rightarrow du = \cos x \ dx; \ x = 0 \Rightarrow u = 0, \ x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_0^{\pi/2} 5 (\sin x)^{3/2} \cos x \ dx = \int_0^1 5u^{3/2} du = \left[5\left(\frac{2}{5}\right) u^{5/2}\right]_0^1 = [2u^{5/2}]_0^1 =$$

$$= 2(1)^{5/2} - 2(0)^{5/2} = 2$$

10. Evaluate

$$\int_0^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$$

Solution

$$\int_0^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$$

Let $u = \cos\left(\frac{x}{2}\right) \Rightarrow du = -\frac{1}{2}\sin\left(\frac{x}{2}\right) dx \Rightarrow -2du = \sin\left(\frac{x}{2}\right) dx;$

$x = 0 \Rightarrow u = \cos\left(\frac{0}{2}\right) = 1, \ x = \frac{2\pi}{3} \Rightarrow u = \cos\left(\frac{\frac{2\pi}{3}}{2}\right) = \frac{1}{2}$

$$\int_0^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx = \int_1^{1/2} u^{-4}\,(-2du) = \left[-2\left(\frac{u^{-3}}{-3}\right)\right]_1^{1/2}$$
$$= \frac{2}{3}\left(\frac{1}{2}\right)^{-3} - \frac{2}{3}(1)^{-3} = \frac{2}{3}(8-1) = \frac{14}{3}$$

Reduction Formulae:

Reduction formula for $\displaystyle\int \cos^n x\,\mathrm{d}x,$

$$\int \cos^n x\,\mathrm{d}x,$$

can be evaluated by a reduction formula.

Start by setting:

$I_n = \displaystyle\int \cos^n x\,\mathrm{d}x.$ Now re-write as $I_n = \displaystyle\int \cos^{n-1} x \cos x\,\mathrm{d}x,$

Integrating by this substitution:

$$\cos x\,\mathrm{d}x = \mathrm{d}(\sin x),$$
$$I_n = \int \cos^{n-1} x\,\mathrm{d}(\sin x).$$

Now integrating by parts:

$$\begin{aligned}
\int \cos^n x\,\mathrm{d}x &= \cos^{n-1} x \sin x - \int \sin x\,\mathrm{d}(\cos^{n-1} x) \\
&= \cos^{n-1} x \sin x + (n-1)\int \sin x \cos^{n-2} x \sin x\,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,\mathrm{d}x - (n-1)\int \cos^n x\,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}$$

solving for I_n:

$$I_n + (n-1)I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2},$$

$$nI_n = \cos^{n-1}(x) \sin x + (n-1)I_{n-2},$$

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2},$$

so the reduction formula is:

$$\int \cos^n x \, \mathrm{d}x = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, \mathrm{d}x.$$

To supplement the example, the above can be used to evaluate the integral for (say) $n = 5$;

$$I_5 = \int \cos^5 x \, \mathrm{d}x.$$

Calculating lower indices:

$$n = 5, \quad I_5 = \tfrac{1}{5} \cos^4 x \sin x + \tfrac{4}{5} I_3,$$

$$n = 3, \quad I_3 = \tfrac{1}{3} \cos^2 x \sin x + \tfrac{2}{3} I_1,$$

back-substituting:

$$\because I_1 = \int \cos x \, \mathrm{d}x = \sin x + C_1,$$

$$\therefore I_3 = \tfrac{1}{3} \cos^2 x \sin x + \tfrac{2}{3} \sin x + C_2, \quad C_2 = \tfrac{2}{3} C_1,$$

$$I_5 = \frac{1}{5} \cos^4 x \sin x + \frac{4}{5} \left[\frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x \right] + C,$$

where C is a constant.

Reduction formula for $\int \sin^n x \, dx$

$$\begin{aligned}
\int \sin^n x \, dx &= \int \sin x \, \sin^{n-1} x \, dx \\
&= -\cos x \, \sin^{n-1} x - \int (-\cos x) \,.\, (n-1) \sin^{n-2} x \, \cos x \, dx \\
&\qquad \text{(integrating by parts)} \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, \cos^2 x \, dx \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx \\
&\qquad \text{(since } \cos^2 x = 1 - \sin^2 x \text{)} \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \,. \quad (1)
\end{aligned}$$

There is now a term in $\int \sin^n x\,dx$ on the right-hand side as well as on the left-hand side. Bringing these terms together on the left-hand side, (1) becomes

$$n\int \sin^n x\,dx = -\cos x\,\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n}\cos x\,\sin^{n-1}x + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx\,. \qquad (2)$$

We shall now establish a quick method to evaluate

$\int_0^{\pi/2} \sin^n x\,dx$ and $\int_0^{\pi/2} \cos^n x\,dx$, $n \in N$

consider $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$= \int_0^{\pi/2} \sin^{n-1}(x)\,dx \cdot \sin x\,dx$$

$$= \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx$$

Integrating by parts, we get

$$I_n = [(sin^{n-1}x)(-cos\,x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\,(sin^{n-2}x)(cos\,x)(-cos\,x)dx$$

$$= (0-0) + \int_0^{\frac{\pi}{2}} (n-1)\,(sin^{n-2}x)cos^2x dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} (sin^{n-2}x)(1-sin^2x)\,dx$$

$$= (n-1)\left\{\int_0^{\frac{\pi}{2}} sin^{n-2}x\;dx - \int_0^{\frac{\pi}{2}} sin^n x\;dx\right\}$$

$I_n = (n-1)\{I_{n-2} - I_n\}$

i.e., $(1+(n-1))I_n = (n-1)I_{n-2}$

i.e., $nI_n = (n-1)I_{n-2}$

i.e., $I_n = \frac{(n-1)}{n} I_{n-2}$

Thus the power of sin x reduces from n to (n - 2). We can continue this process till the power reduces to zero or one.

Consider 1) $\int_0^{\pi/2} \sin^6 x\, dx$

$$= \frac{6-1}{6} \int_0^{\pi/2} \sin^4 x\, dx$$

$$= \frac{5}{6}\left[\frac{4-1}{4} \int_0^{\pi/2} \sin^2 x\, dx\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4}\left[\frac{2-1}{2} \int_0^{\pi/2} \sin^0 x\, dx\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} 1\, dx$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} [\, x\,]_0^{\pi/2}$$

$$\therefore \int_0^{\pi/2} \sin^6 x\, dx$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

(2) $\int_0^{\pi/2} \sin^5 x\, dx$

$$= \frac{5-1}{5} \int_0^{\pi/2} \sin^3 x\, dx$$

$$= \frac{4}{5}\left[\frac{3-1}{3} \int_0^{\pi/2} \sin x\, dx\right]$$

$$= \frac{4}{5} \cdot \frac{2}{3} [-\cos x]_0^{\pi/2}$$

$$= \frac{4}{5} \cdot \frac{2}{3} [-0+1]$$

$$= \frac{4}{5} \cdot \frac{2}{3}$$

In general if 'n 'is even.

(1) $\int_0^{\pi/2} \sin^n (x)\, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-2} \times \ldots \times \frac{1}{2} \times \pi/2$

If ' n ' is odd

(2) $\int_0^{\pi/2} \sin^n (x)\, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-2} \times \ldots \times \frac{2}{3}$

These formulas also holds for $\int_0^{\pi/2} \cos^n (x)\, dx$

These formulas have been given by ' Walli ' Hence they are known as Walli's reduction formulas.

Reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5)\ \ldots 5.3.1][(n-1)(n-3)(n-5)\ \ldots 5.3.1]}{(m+n)(m+n-2)(m+n-4)\ldots 5.3.1}\left(\frac{\pi}{2}\right), \text{if both } m \text{ and } n \text{ are even}$$

$$= \frac{[(m-1)(m-3)(m-5)\ \ldots (2 \text{ or } 1)][(n-1)(n-3)(n-5)\ \ldots (2 \text{ or } 1)]}{(m+n)(m+n-2)(m+n-4)\ldots(2 \text{ or } 1)} \quad \text{otherwise}$$

Problems

Integrate $I = \int (\sec x)^n \, dx$

Try integration by parts with

$$u = (\sec x)^{n-2} \qquad v = \tan x$$
$$du = (n-2)(\sec x)^{n-3} \sec x \tan x \, dx \qquad dv = \sec^2 x \, dx$$

We get

$$I = \int (\sec x)^n \, dx = \int u \, dv = uv - \int v \, du$$
$$= \tan x (\sec x)^{n-2} - \int (\tan x)(n-2)(\sec x)^{n-3} \sec x \tan x \, dx$$
$$= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} \tan^2 x \, dx$$
$$= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} (\sec^2 x - 1) \, dx$$
$$= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^n - (\sec x)^{n-2} \, dx$$
$$= \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx - (n-2) \cdot I$$

Solving for I:

$$(n-2)I + I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

or

$$(n-1)I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

Dividing by n gives the

Reduction Formula: $\int (\sec x)^n \, dx = \frac{1}{n-1} \tan x (\sec x)^{n-2} + \frac{n-2}{n-1} \int (\sec x)^{n-2} \, dx$

Exponential integral

Another typical example is:

$$\int x^n e^{ax}\,\mathrm{d}x.$$

Start by setting:

$$I_n = \int x^n e^{ax}\,\mathrm{d}x.$$

Integrating by substitution:

$$x^n\,\mathrm{d}x = \frac{\mathrm{d}(x^{n+1})}{n+1},$$

$$I_n = \frac{1}{n+1}\int e^{ax}\,\mathrm{d}(x^{n+1}),$$

Now integrating by parts:

$$\int e^{ax}\,\mathrm{d}(x^{n+1}) = x^{n+1}e^{ax} - \int x^{n+1}\,\mathrm{d}(e^{ax})$$

$$= x^{n+1}e^{ax} - a\int x^{n+1}e^{ax}\,\mathrm{d}x,$$

$$(n+1)I_n = x^{n+1}e^{ax} - aI_{n+1},$$

shifting indices back by 1 (so $n + 1 \to n$, $n \to n - 1$):

$$nI_{n-1} = x^n e^{ax} - aI_n,$$

solving for I_n:

$$I_n = \frac{1}{a}\left(x^n e^{ax} - nI_{n-1}\right),$$

so the reduction formula is:

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{1}{a}\left(x^n e^{ax} - n\int x^{n-1}e^{ax}\,\mathrm{d}x\right).$$

TEXT / REFERENCE BOOKS

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Multiple Calculus – SMTA1106

UNIT 5

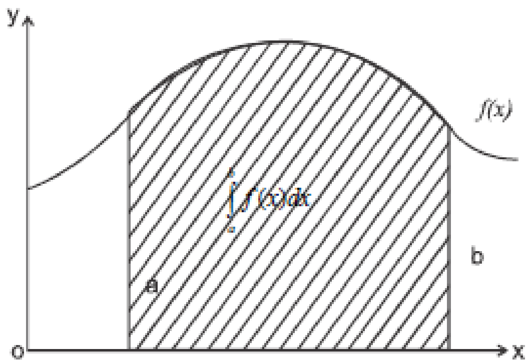
MULTIPLE INTEGRALS

Multiple Integrals

The principle of integration were formulated independently by Isaac Newton and Gollfried Leibnitz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentation.

If f is a continuous real valued function defined on a closed interval [*a,b*] then once an anti-derivative *F* of f is known, the definite integral of *f* over that interval is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$



The integral of the function *f(x)* over the range $x = a$ *to* $x = b$ gives the area under the curve between theordinates $x = a$ and $x = b$

Integrals and derivatives become the basic tools of calculus with numerous applications in Science and Engineering.

The multiple integral is a generalisation of definite integral to functions of more than one real variable, for example *f(x,y)* or *f(x,y,z)*. Integrals of a function of two variables over a region R^2 are called <u>double integrals</u>.Integrals of a function of three variables over a region R^3 are called <u>triple integrals</u>.

Double Integrals:

Let *f(x,y)* be a continuous and single valued function of *x* and *y* within some region R and on its boundary.Let the region R be subdivided into n sub-regions of areas $\delta A_1, \delta A_2 ....\delta A_n$ .Let (x_r, y_r) be any point inside the sub-region of area δA_r and consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

The limit of this sum as $n \to \infty$ and $\delta A_r \to 0 (r = 1, 2, ...n)$ is defined as the double integral of *f(x,y)*over the region R.

Thus $\iint_R f(x, y)\, dA = \underset{N \to \infty}{lt} \sum_{r=1}^{N} f(x_r, y_r)\, \delta A_r$

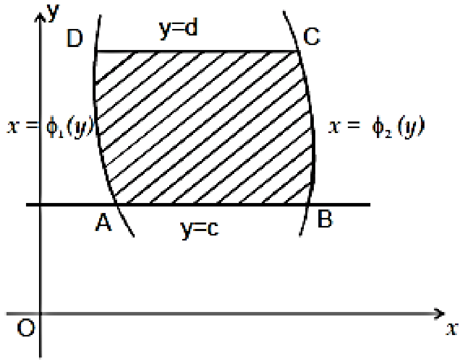
Suppose the region R is divided into rectangular partitions by a network of lines parallel to the coordinate axes, the integral $\iint f(x, y) dA$ is written as $\iint_R f(x, y) dxdy$

Consider the double integral $\int_{y=c}^{y=d} \int_{x=\varphi_1(y)}^{x=\varphi_2(y)} f(x, y)\; dxdy$

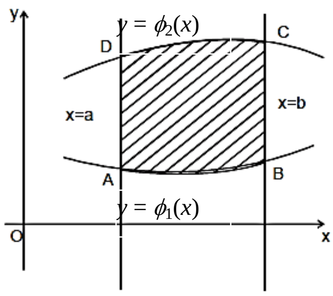
The region of integration is bounded by the lines $y = c, y = d, c < d$ and the curves $x = \phi_1(y), x = \phi_2(y), \phi_1(y) \le \phi_2(y)$ which is shown below.

The region ABCD is kno

wn as the region of integration of the given double integral



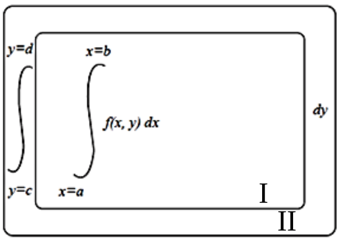
Consider the double integral $\int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\, dydx$



The region of integration is bounded by the lines $x = a, x = b, a < b$ and curves $y = \phi_1(x), y = \phi_2(x), \phi_1(x) \le \phi_2(x)$ which is shown below.

The region of integration is the region ABCD

To evaluate $\int\limits_{y=c}^{d} \int\limits_{x=a}^{b} f(x, y)\, dxdy$



The above illustration gives order in which integrations are performed.

Evaluation of double integral:

To evaluate $\iint\limits_{A} f(x, y) dxdy = \int\limits_{y_1}^{y_2} \int\limits_{x_1}^{x_2} f(x, y)\, dxdy$

(i) If x_1, x_2, y_1, y_2 are constants, then

Thus, $\int\limits_{y_1}^{y_2} \int\limits_{x_1}^{x_2} f(x, y)\, dxdy = \int\limits_{x_1}^{x_2} \int\limits_{y_1}^{y_2} f(x, y)\, dxdy$

(ii) If x_1, x_2 are functions of y, let $x_1 = \phi_1(y)$, $x_2 = \phi_2(y)$ and y_1, y_2 are constants then,

$$\iint\limits_{A} f(x, y) dxdy = \int\limits_{y_1}^{y_2} \int\limits_{\phi_1(y)}^{\phi_2(y)} f(x, y) dxdy$$

(iii) If y_1, y_2 are functions of x, let $y_1 = \phi_1(x), y_2 = \phi_2(x)$, and x_1, x_2 are constants then,

$$\iint\limits_{A} f(x, y) dxdy = \int\limits_{x_1}^{x_2} \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dydx$$

(iv) If $f(x, y) = 1$, then the double integral $\iint\limits_{A} dxdy$ gives the area of the region A.

(v) To evaluate $\int\limits_{y_1}^{y_2} \int\limits_{x_1}^{x_2} f(x, y)\, dxdy = \int\limits_{y_1}^{y_2} \left[\int\limits_{x_1}^{x_2} f(x, y) dx \right] dy.$ we integrate $f(x, y)$ with respect of x, treating y as a constant, and then the resultant function of y is integrated with respect to y.

Sketch roughly the region of integration for the following double integrals.

(1) $\int_{-b}^{b}\int_{-a}^{a} f(x,y)\,dxdy$

$$I = \int_{y=-b}^{b}\left[\int_{x=-a}^{a} f(x,y)dx\right]dy$$

The region of integration bounded by lines $x = -a$, $x = a$, $y = -b$, $y = b$ and is shown in Fig. 1

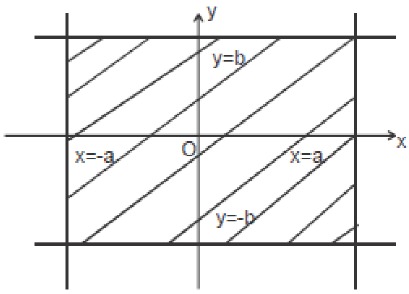


Fig.1

(2) $\int_{0}^{1}\int_{0}^{x} f(x,y)dxdy$

$$I = \int_{x=0}^{1}\left[\int_{y=0}^{x} f(x,y)dy\right]dx$$

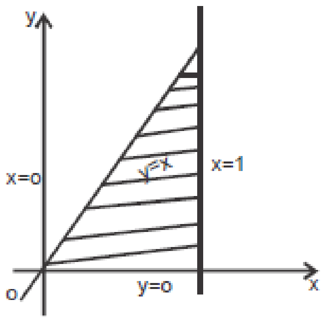


Fig. 2

The region of integration bounded by lines $x = 0$, $x = 1$, $y = 0$, $y = x$ is shown in Fig. 2.

(3) $\iint \frac{x^2y^2}{x^2+y^2}\,dxdy$

$$I = \int_{x=0}^{a} \left[\int_{y=0}^{\sqrt{a^2-x^2}} f(x,y)dy \right] dx$$

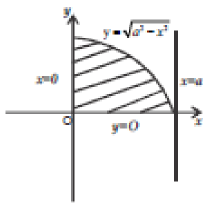


Fig. 3

The region of integration bounded by $x = 0$, $x = a$, $y = 0$, $y = \sqrt{a^2 - x^2}$ *(ie)* $x^2 + y^2 = a^2$ is shown in Fig. 3.

(4) $$\int_{0}^{b} \int_{0}^{\frac{a}{b}(b-y)} f(x,y)dxdy$$

$$I = \int_{y=0}^{b} \left[\int_{x=0}^{\frac{a}{b}(b-y)} f(x,y)dx \right] dy$$

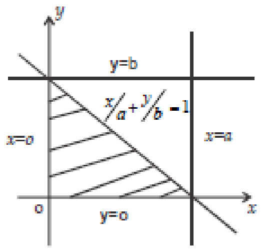


Fig. 4

The region of integration bounded by $x = 0, \frac{x}{a} + \frac{y}{b} = 1, y = 0, y = b$ is shown in Fig. 4

(5) $$\int_{0}^{a} \int_{a-x}^{\sqrt{a^2-x^2}} f(x,y)\ dydx$$

$$I = \int_{x=0}^{a} \left[\int_{y=a-x}^{\sqrt{a^2-x^2}} f(x,y)dy \right] dx$$

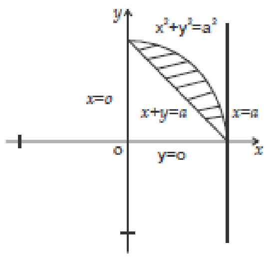


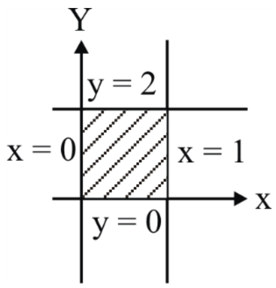
Fig. 5

The region of integration bounded by $x = 0$, $x = a$, $x + y = a$, $x^2 + y^2 = a^2$ is shown in Fig. 5

Double integrals with constant limits :

(6) Evaluate $\int_{0}^{2} \int_{0}^{1} 4xydxdy$

Solution:



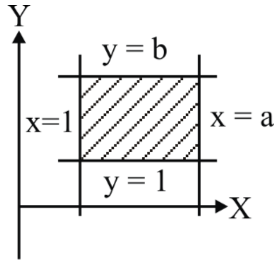
Let $$I = \int_{0}^{2} \left[\int_{0}^{1} 4xydx \right] dy$$

$$= \int_{0}^{2} 4y\left(\frac{x^2}{2}\right)_0^1 dy = \int_{0}^{2} 2y(1-0)dy$$

$$= 2\int_{0}^{2} ydy = 2\left[\frac{y^2}{2}\right]_0^2 = 4 - 0 = 4$$

(7) Evaluate $\int_1^b \int_1^a \frac{dxdy}{xy}$

Solution



Let $I = \int_1^b \left[\int_1^a \frac{dx}{xy} \right] dy$

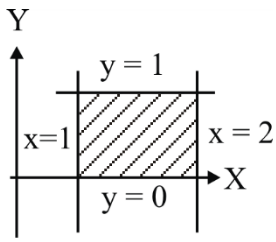
$= \int_1^b \frac{1}{y} \left(\int_1^a \frac{dx}{x} \right) dy = \int_1^b \frac{1}{y} [\log x]_1^a dy$

$= \int_1^b \frac{\log a - \log 1}{y} dy = \log a \int_1^b \frac{dy}{y}$ $\quad (\because \log 1 = 0)$

$= \log a [\log y]_1^b = \log a [\log b - \log 1]$

$= \log a \log b$ $\quad (\because \log 1 = 0)$

(8) Evaluate $\int_0^1 \int_1^2 (x^2 + y^2) dxdy$



Solution

Let $I = \int_0^1 \left[\int_1^2 (x^2 + y^2) dx \right] dy$

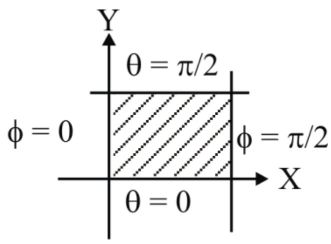
$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_1^2 dy$

$= \int_0^1 \left[\left(\frac{8}{3} + 2y^2 \right) - \left(\frac{1}{3} + y^2 \right) \right] dy = \int_0^1 \left(\frac{7}{3} + y^2 \right) dy$

$= \left[\frac{7y}{3} + \frac{y^3}{3} \right]_0^1 = \left[\frac{7}{3} + \frac{1}{3} \right] = \frac{8}{3}$

(9) Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi) d\theta d\phi$

Solution



Let $I = \int_0^{\pi/2} \left[\int_0^{\pi/2} \sin(\theta + \phi) d\theta \right] d\phi$

$$= -\int_{0}^{\pi/2} [\cos(\theta+\phi)]_0^{\pi/2} d\phi$$

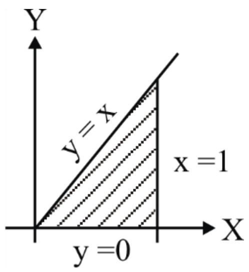
$$= -\int_{0}^{\pi/2} (\sin\phi + \cos\phi) d\phi$$

$$= [-\cos\phi + \sin\phi]_0^{\pi/2}$$

$$= (0 + 1) - (-1 + 0)$$

$$= 2$$

Double Integrals with variable limits:



(10) Evaluate $\int_{0}^{1}\int_{0}^{x} dxdy$

Solution

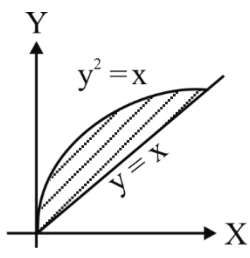
Let $I = \int_{0}^{1}\left[\int_{0}^{x} dy\right]dx$

Here, innermost limits are in terms of *x*, therefore they are limits of y and outermost limits are those of *x*.

$$I = \int_{x-0}^{1}\left[\int_{y=0}^{x} dy\right]dx = \int_{0}^{1} [y]_0^x dx$$

$$= \int_{0}^{1} xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1-0}{2} = \frac{1}{2}$$

11) Evaluate $\int_{0}^{1}\int_{x}^{\sqrt{x}} xy(x+1)dxdy$



Solution

Let $I = \int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}} \left(x^2y + xy^2\right)dy\right]dx$

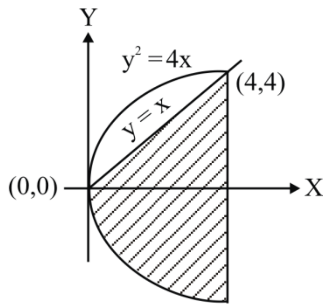
$$= \int_{0}^{1} \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_{0}^{1} \left[\left(\frac{x^2}{2}(x) + \frac{x}{3}\left(x^{3/1}\right)\right) - \left(\frac{x^4}{2} + \frac{x^4}{3}\right)\right]dx$$

$$= \int_0^1 \left[\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{5x^4}{6}\right]dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{3}\left[\frac{2x^{7/2}}{7}\right]_0^1 - \frac{5}{6}\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{8}+\frac{2}{21}-\frac{1}{6} = \frac{21+16-28}{168} = \frac{9}{168} = \frac{3}{56}$$

12) Evaluate $\int_0^4 \int_{y^2/4}^{y} \frac{ydxdy}{x^2+y^2}$



Solution:

Let $I = \int_{y=0}^{4}\left[\int_{x=y^2/4}^{y} \frac{ydx}{x^2+y^2}\right]dy$

$$= \int_0^4 \left[\tan^{-1}\left(\frac{x}{y}\right)\right]_{y^2/4}^{y} dy = \int_0^4 \left[\tan^{-1}\left(\frac{y}{y}-\tan^{-1}\right)\right]dy$$

$$= \int_0^4 \left[\tan^{-1}(1)-\tan^{-1}\left(\frac{y}{4}\right)\right]dy = \int_0^4 \frac{\pi}{4}dy - \int_0^4 \tan^{-1}\frac{y}{4}dy$$

$$= \int_0^4 \left[\tan^{-1}(1)-\tan^{-1}\left(\frac{y}{4}\right)\right]dy$$

$$= \int_0^4 \frac{\pi}{4}dy - \int_0^4 \tan^{-1}\frac{y}{4}dy$$

$$= \frac{\pi}{4}[y]_0^4 - \left\{\left[y\tan^{-1}\left(\frac{y}{4}\right)\right]_0^4 - \int_0^4 \frac{4ydy}{16+y^2}\right\}$$

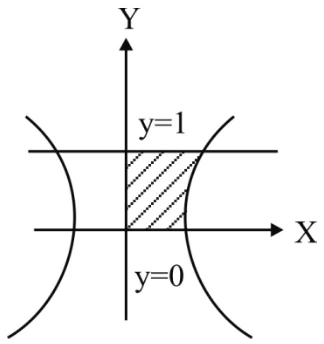
$$= 4\frac{\pi}{4} - \left\{\left[4\tan^{-1}(1)\right] - 2\log\left(16+y^2\right)\Big]_0^4\right\}$$

$$= \pi - 2\left[\log 32 - \log 16\right]$$

$$= \pi - 2\log\left(\frac{32}{16}\right)$$

$$= \pi - 2\log 2$$

13) Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dxdy}{1+x^2+y^2}$



Solution

Let $I = \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{1+y^2}} \frac{dx}{(1+y^2)+x^2} \right] dy$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+y^2}} \left(\tan^{-1} \frac{x}{\sqrt{1+y^2}} \right) \right]_0^{\sqrt{1+y^2}} dy$$

$$= \int_0^1 \frac{1}{\sqrt{1+y^2}} \left[\tan^{-1} \frac{\sqrt{1+y^2}}{\sqrt{1+y^2}} - \tan^{-1} 0 \right] dy$$

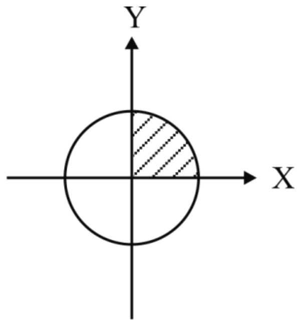
$$= \int_0^1 \frac{1}{\sqrt{1+y^2}} \left[\tan^{-1} 1 - 0 \right] dy$$

$$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}$$

$$= \frac{\pi}{4} \left[\log\left(y + \sqrt{1+y^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \left[\log\left(1+\sqrt{2} \right) - \log 1 \right]$$

$$= \frac{\pi}{4} \log\left(1+\sqrt{2} \right) \qquad (\because \log 1 = 0)$$



14) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dydx$

Solution

Let $I = \int_{x=0}^{a} \left[\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy \right] dx$

$$= \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} - \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[0 - \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{a^2-x^2}\right) - \left(0 - \frac{a^2-x^2}{2}\sin^{-1}(0)\right) dx \right]$$

$$= \int_0^a \left[-\left(\frac{a^2-x^2}{2}\right)\sin^{-1}(1) - 0 \right] dx$$

$$= \frac{-1}{2}\int_0^a \frac{\pi}{2}\left(a^2-x^2\right)dx = -\frac{\pi}{4}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \quad -\frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = -\frac{\pi}{4}\left[\frac{2a^3}{3}\right] = -\frac{\pi a^3}{6}$$

15) Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r dr d\theta$

Solution

Let $$I = \int_{\theta=0}^{\pi}\left[\int_{r=0}^{\sin\theta}\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{\sin^2\theta}{2}d\theta = \frac{1}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - 0\right] = \frac{\pi}{4} \qquad (\because \sin 2\pi = \sin 0 = 0)$$

16) Find the limits of integration in the double integral $\iint_R f(x,y)dxdy$, R lies in the first quadrant and boundedby the following curves.

i) $x = 0, y = 0, x + y = 1$

To find the Limits of x: $\quad Put\ y = 0\ in\ x + y = 1$

$$x = 1$$

Limits of $\quad x : 0 \to 1$

To find the Limits of y : $x+y=1 \Rightarrow y=1-x$

Limits of $y: 0 \rightarrow 1-x$

The region of integration in bounded by $x=0, x=1, y=0 \, and \; y=1-x$

Thus $\iint\limits_R f(x,y)dxdy$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)dydx$$

ii) $x=0, y=0, \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

To find the Limits of x : Put y = 0 in $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$x^2=a^2$

$x=\pm a$

Since R lies in the first quadrant $x=a$

Limits of $x: 0 \rightarrow a$

To find the Limits of y : From $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Since R lies in the first quadrant, $y=\dfrac{b}{a}\sqrt{a^2-x^2}$, $\therefore$ lim its of y: 0 $\rightarrow \dfrac{b}{a}\sqrt{a^2-x^2}$

Excercise

1) Evaluate $\int_0^1\int_0^2 (x^2+3xy^2)dydx$ Ans : $\dfrac{14}{3}$

2) Evaluate $\int_0^a\int_0^{\sqrt{ay}} xydxdy$ Ans : $\dfrac{a^4}{6}$

3) Evaluate $\int_0^2 \int_2^{x+2} (x+y)dydx$ Ans : 12

4) Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dxdy}{\sqrt{1-x^2-y^2}}$ Ans : $\frac{\pi}{4}$

5) Evaluate $\int_0^2 \int_0^{x^2} e^{y/x} dydx$ Ans : $e^{-2} - 1$

6) Evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta \, drd\theta$ Ans : 0

7) Evaluate $\int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta drd\theta$ Ans : $\frac{8\pi ra^3}{3}$

8) Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta drd\theta$ Ans : $\frac{5a^3\pi}{6}$

9) Evaluate $\int_0^{\pi/2} \int_0^{2\cos\theta} r \cos\theta drd\theta$ Ans: $\frac{\pi}{2}$

10) Evaluate $\int_0^{\pi/2} \int_0^{\infty} \frac{r}{r^2+a^2} drd\theta$ Ans : $\frac{\pi}{4a^2}$

CHANGE OF ORDER OF INTEGRATION

In calculus, interchange of the order of integration is a methodology that tranforms iterated integrals offunctions into other hopefully simpler integrals by changing the order in which integrations are performed.In a double integral if the limits of the integration are constants, then the order of the integration isimmaterial, provided the limits of integration are changed accordingly. Thus,

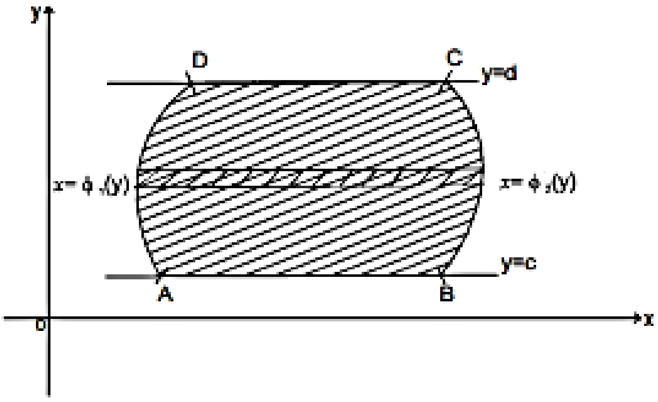
$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

But if the limits of the integration are variables the change of order of integration, change the limits ofintegration. While doing so, sometimes it is required to split up the region of integration and the given integralis expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it isalways advisable to draw a rough sketch of the region of integration.

Region of Integration and change of order:

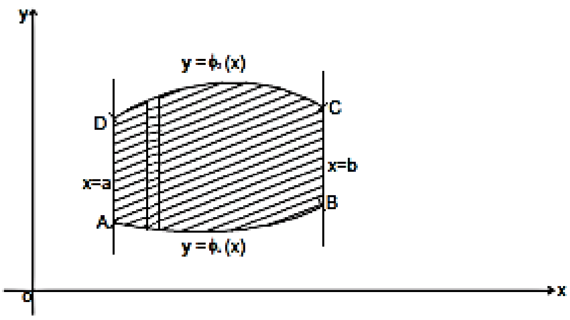
Consider the double integral $\int_{c}^{d}\int_{\phi_1(y)}^{\phi_2(y)} dxdy$. In this integral x varies from $\phi_1(y)$ to $\phi_2(y)$ and y varies fromc to d. (ie) $\phi_1(y) \le x \le \phi_2(y)$ & $c \le y \le d$

The above intequalities determine a region in the XOY plane whose boundaries are the curves $x = \phi_1(y)$ and $x = \phi_2(y)$ along with the boundaries of the lines $y = c$, $y = d$. This region of integration is shown as follows.



The region ABCD is known as the region of integration of the given double integral.

After changing the order the region ABCD has the boundaries a $y = \phi_1(x)$, $y = \phi_2(x)$ with the lines $x = a$, $x = b$ which is shown in the following figure.



Problems

1) Change the order of integration and compare the results after evaluating it with the givenorder.

$$\int_{1}^{2}\int_{0}^{1} (x^2 + y^2)dxdy$$

Solution: Let $I = \int_{1}^{2}\int_{0}^{1} (x^2 + y^2)dxdy$

$$= \int_1^2 \left(\frac{x^2}{3} + y^2 x\right)_0^1 dy$$

$$= \int_1^2 \left(\frac{1}{3} + y^2\right) dy$$

$$= \left(\frac{y}{3} + \frac{y^3}{3}\right)_1^2$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \frac{10}{3} - \frac{2}{3} = \frac{8}{3} \qquad (1)$$

Now let us change the order of integration and then evaluate it.

$$\text{I} = \int_0^1 \int_1^2 \left(x^2 + y^2) dy dx\right)$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_1^2 dx$$

$$= \int_0^1 \left(x^2 (2-1) + \frac{8}{3} - \frac{1}{3}\right) dx$$

$$= \int_0^1 \left(x^2 + \frac{7}{3}\right) dx = \left(\frac{x^3}{3} + \frac{7}{3}(x)\right)_0^1$$

$$= \frac{1}{3} + \frac{7}{3} = \frac{8}{3} \qquad (2)$$

Since (1) = (2), the change of order of the integral will not affect its solution.

Note:

If the limits of the integral are constants the order of the integration is immaterial provided the relevant limits are taken for the concerned variable and the integrand is continuous in the region of integration. This results hold for a triple integral also.

(2) Change the order of integration and then evaluate

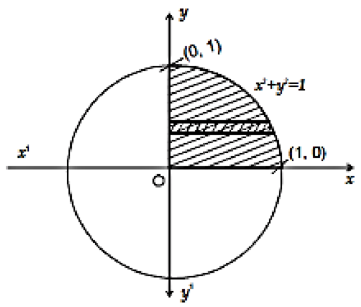
$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$$

Solution:

The region is bounded by the lines $y = 0, y = \sqrt{1-x^2}$ or ($y^2 + x^2 = 1$) (ie)
The region of integration is the unit circle centred at the origin. The other boundaries are represented by x= 0, x = 1. The line x = 0 represents the y axis. Hence the region of integration is the +ve quadrant of the unit circle which is plotted in the following figure.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$$

After changing the order, the integration is with respect to x first. The region is covered by horizontal stip. The end points of the horizontal strip gives the limits along the x - direction.



As one end of the Horizontal strip lies on the circle. We have from the circle

$x^2 = 1 - y^2$ or $x = \pm\sqrt{1-y^2}$

The limits of x are $x = 0,\ x = \sqrt{1-y^2}$

The number of Horizontal strips used to cover the region determines the limits along the y direction. Hence y varies from 0 to 1.

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dx dy$$

$$= \int_0^1 y^2 (x)_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} dy$$

Let $y = \sin\theta$, $dy = \cos\theta d\theta$ the limits becomes $\theta = 0$ to $\frac{\pi}{2}$

$$\therefore\ I = \int_0^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\ \cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta$$

$$\left(\therefore \int_0^{\pi/2} \sin^m\theta\cos^n\theta d\theta = \frac{[(n-1)(n-3)...1][(m-1)(m-3)...1]}{[(m+n)(m+n-2)...]}\times\frac{\pi}{2}\right)$$

m, n are even

$$= \left(\frac{1}{4}\times\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi}{16}$$

3) Change the order of integration and evaluate

$$\int_1^4 \int_{\sqrt{y}}^2 (x^2+y^2)dxdy$$

Solution

The region is bounded by the lines $x=\sqrt{y}$ (or $x^2 = y$) and $x = 2$. This region is plotted in the following figure.

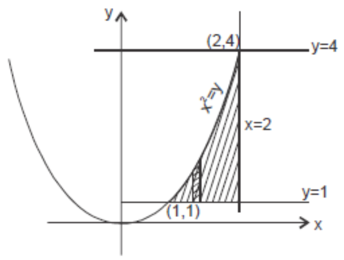
After changing the order, the integration is with respect to y first. Hence the region of integration is covered by vertical strips.

The limits of y are $y = 1$ & $y = x^2$

Hence

$$I\int_0^2 \int^{x^2} (x^2+y^2)dydx \quad \text{(After changing the order)}$$

$$= \int_1^2 \left(x^2y+\frac{y^3}{3}\right)_1^{x^2} dx$$



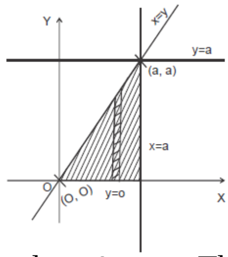
$$= \int_1^2 \left[\left(x^4+\frac{x^6}{3}\right)-\left(x^2+\frac{1}{3}\right)\right]dx$$

$$= \left(\frac{x^5}{5}+\frac{x^7}{7\times3}-\frac{x^3}{3}-\frac{x}{3}\right)_1^2$$

$$= \left(\frac{32}{5}+\frac{128}{21}-\frac{8}{3}-\frac{2}{3}\right)-\left(\frac{1}{5}+\frac{1}{21}-\frac{1}{3}-\frac{1}{3}\right)$$

$$= \frac{1026}{105}$$

4) Change the order of integration in $\int_0^a \int_y^a \frac{xdxdy}{\sqrt{x^2+y^2}}$ and then evaluate it



Solution :

Let $I = \int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}$

The region of integration is bounded by the lines $x = y, x = a$ and $y = 0, y = a.$ The sketch of the boundaries of the region is given in the figure.

After changing the order, the integration is with respect to y first.
The limits of y are y = 0 and y = x. The limits of x are x = 0 and x = a

Let $I = \int_0^a \int_0^x \frac{xdydx}{\sqrt{x^2+y^2}}$

$$= \int_0^a x\left[\log\left(y+\sqrt{y^2+x^2}\right)\right]_0^x dx \qquad \because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)$$

$$= \int_0^a x\left[\log\left(x+\sqrt{x^2+x^2}\right) - \log\left(\sqrt{x^2}\right)\right]dx$$

$$= \int_0^a x\left[\log\left(x+\sqrt{2x}\right) - \log x\right]dx$$

$$= \int_0^a x\left[\log\left(1+\sqrt{2}\right)x - \log x\right]dx$$

$$= \int_0^a x.\log\left(\frac{\left(1+\sqrt{2}\right)x}{x}\right)dx$$

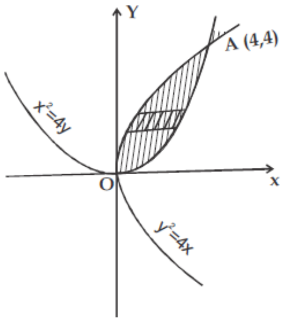
$$\therefore I = \int_0^a x\log\left(1+\sqrt{2}\right)dx$$

$$= \log\left(1+\sqrt{2}\right)\int_0^a xdx$$

$$= \log\left(1+\sqrt{2}\right)\left(\frac{x^2}{2}\right)_0^a$$

$= \frac{a^2}{2}\log\left(1+\sqrt{2}\right)$

5) Change the order of integration in $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dydx$ and then evaluate it.



Solution:

The region of integration is bounded by the curve $y = \frac{x^2}{4}$ and $y = 2\sqrt{x}$. (ie) the parabolas $x^2 = 4y$ and $y^2 = 4x$ which is shown inthe figure.

The points of intersection of the two parabolas are obtained bysolving the equations.

$x^2 = 4y \, and \, y^2 = 4x$

$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{16} = 4x \Rightarrow x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0 \, or \, x^3 = 64$

$x^3 64 \Rightarrow x = 4$

When $x = 0, y = 0$

$x = 4, y = 4$

The points of intersection are O(0,0) and A(4,4). After changing the order the given integral is

$I = \iint_R dxdy$

$= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dxdy$

$= \int_0^4 (x)_{\frac{y^2}{4}}^{2\sqrt{y}} dy$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right) dy$$

$$= \int_0^4 \left(2y^{1/2} - \frac{y^2}{4}\right) dy$$

$$= \int_0^4 2y^{1/2} dy - \int_0^4 \frac{y^2}{4} dy$$

$$= 2\left(\frac{y^{3/2}}{3/2}\right)_0^4 - \frac{1}{4}\left(\frac{y^3}{3}\right)_0^4$$

$$= \frac{4}{3}\left[4^{3/2} - 0\right] - \frac{1}{12}(64-0)$$

$$= \frac{4}{3}(2^3) - \frac{1}{12}(64)$$

$$= \frac{4}{3}(8) - \frac{16}{3}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

6) By changing the order of integration, prove that

$$\int_0^\infty \int_0^y y e^{-y^2/x} dx dy = \frac{1}{2}$$

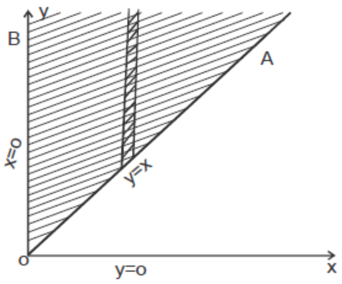
Proof:

The region of integration is bounded by the lines $x = 0, x = y, y = 0$ and $y = \infty$. The region of integrationis the infinite triangular region AOB which is shown in the figure.

After changing the order, the integration is with respect to y first. Hence the region of integration iscovered by vertical strips.

The limits of y are $y = x \, to \, y = \infty$. The limits of x are $x = 0 \, to \, x = \infty$ to

Hence



$$\int_0^\infty \int_0^y y e^{-y^2/x} dx dy = \int_0^\infty \int_x^\infty y e^{-y^2/x} dy dx$$

$$= \int_0^\infty \int_x^\infty e^{-y^2/x} d\left(\frac{y^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\infty}\int_x^{\infty} e^{-y^2/x}\, d(y^2)dx$$

$$= \frac{1}{2}\int_0^{\infty} \left(\frac{e^{-y^2/x}}{-1/x}\right)_x^{\infty} dx$$

$$= \frac{1}{2}\int_0^{\infty} -x(e^{-\infty} - e^{-x})dx$$

$$= \frac{1}{2}\int_0^{\infty} xe^{-x}dx \qquad (\because e^{-\infty} = 0)$$

$$= \frac{1}{2}\left[\left(x\frac{e^{-x}}{-1}\right)_0^{\infty} - \int_0^{\infty} -e^{-x}dx\right]$$

$$= \frac{1}{2}\left[-xe^{-x} - e^{-x}\right]_0^{\infty}$$

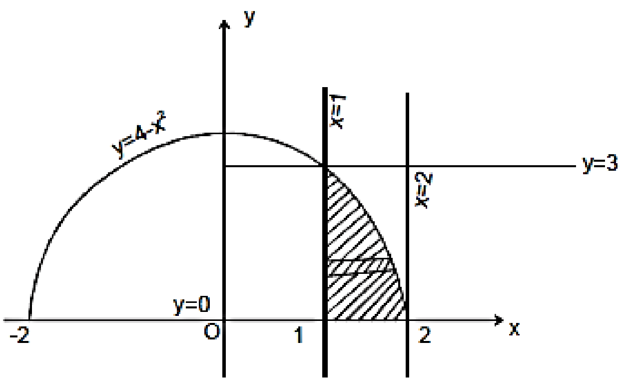
$$= \frac{1}{2}[0-(-1)]$$

$$= \frac{1}{2}$$

7) Change the order of integration in $\int_1^2 \int_0^{4-x^2} (x+y)dydx$ and hence evaluate it.

Solution:

The region of integration is bounded by the curves $x = 1$, $x = 2$, $y = 0$ and $y = 4 - x^2$. The region ofintegration is shown in the figure.



After changing the order, the integration is with respect to x first. Hence the region of integration is covered by Horizontal strips. The limits of x are $x = 1$, $x = \sqrt{4-y}$.

The limits of y are y =0, y =3.

Hence

$$\int_{1}^{2}\int_{0}^{4-x^2}(x+y)dydx=\int_{0}^{3}\int_{1}^{\sqrt{4-y}}(x+y)dxdy$$

$$=\int_{0}^{3}\left(\frac{x^2}{2}+xy\right)_{1}^{\sqrt{4-y}}dy$$

$$=\int_{0}^{3}\left(\frac{4-y}{2}+y\sqrt{4-y}-\frac{1}{2}-y\right)dy$$

$$=\int_{0}^{3}\left(\frac{3}{2}-\frac{3y}{1}+y\sqrt{4-y}\right)dy$$

$$=\left(\frac{3y}{2}-\frac{3y^2}{4}\right)_{0}^{3}+\int_{0}^{3}y\sqrt{4-ydy}$$ Put $t^2=4-y$

$y=4-t^2$

$2tdt=-dy$

When $y=0,t=2,$

$y=3.t=1$

$$=\left(\frac{3y}{2}-\frac{3y^2}{4}\right)_{0}^{3}+\int_{2}^{1}\left(4-t^2\right)t\left(-2tdt\right)$$

$$=\left(\frac{3y}{2}-\frac{3y^2}{4}\right)_{0}^{3}+\int_{2}^{1}\left(-8t^2+2t^4\right)dt$$

$$=\left(\frac{3y}{2}-\frac{3y^2}{4}\right)_{0}^{3}+\left(\frac{-8t^3}{3}+\frac{2t^5}{5}\right)_{2}^{1}$$

$$=\left(\frac{9}{2}-\frac{27}{4}\right)-(0)+\left(\frac{-8}{3}+\frac{2}{5}\right)-\left(-\frac{64}{3}+\frac{64}{5}\right)$$

$$=\frac{-9}{4}-\frac{34}{15}+\frac{128}{15}$$

$$=\frac{241}{60}$$

8) Change the order of integration in $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dxdy$ and evaluate it.

Solution

The region of integration is bounded by the curves $x = 0$, x = $\sqrt{a^2 - y^2}$, y = 0 and $y = a$. The region of integration is the +ve quadrant of the circle of radius 'a' which is plotted in the figure.

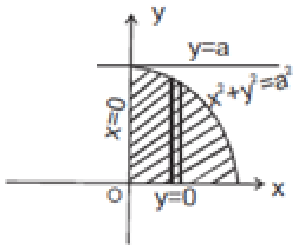
When we change the order of integration, we first integrate with respect to y, keeping x constant.Hence the region of integration is covered by vertical strips.
The limits of y are $y = 0$, $y = \sqrt{a^2 - x^2}$, thelimits of x are $x = 0$, $x = a$.

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\,dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dydx$$

$$= \int_0^a \left[\frac{y}{2}\sqrt{(a^2 - x^2)y^2} + \left(\frac{a^2 - x^2}{2}\right)\sin^{-1}\frac{y}{\sqrt{(a^2 - x^2)}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$\left(\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)$$

$$= \int_0^a \frac{a^2 - x^2}{2}\sin^{-1}(1)dx \quad \left(\because \sin^{-1} 0 = 0\right)$$



$$= \frac{\pi}{4}\int_0^q \left(a^2 - x^2\right)dx$$

$$= \frac{\pi}{4}\left(a^2 x - \frac{x^3}{3}\right)_0^a dx$$

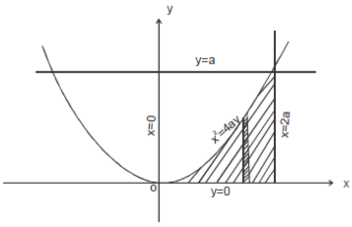
$$= \frac{\pi}{4}\left(\left(a^3 - \frac{a^3}{3}\right) - (0)\right)$$

$$= \frac{\pi a^3}{6}$$

9) Change the order of integration and hence evaluate the integral $\int_0^a \int_{2\sqrt{xy}}^{2a} xydxdy$

Solution:

The region of integration is bounded by the curves $x = 2\sqrt{ay}, x = 2a, y = 0$ and $y = a$. This region isplotted in the following figure.



After changing the order, the integration is with respect to y first keeping x as constant. Hence the regionof integration is covered by vertical strips.

The limits of y are y =0 and $y = \frac{x^2}{4a}$. The limits of x are x =0 and x =2a

$$\int_0^a \int_{2\sqrt{ay}}^{2a} xydxdy = \int_0^{2a} \int_0^{\frac{x^2}{4a}} xydydx$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\frac{x^2}{4a}} dx$$

$$= \int_0^{2a} \frac{x}{2}\left(\frac{x^4}{16a^2}\right)dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5 dx$$

$$= \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a}$$

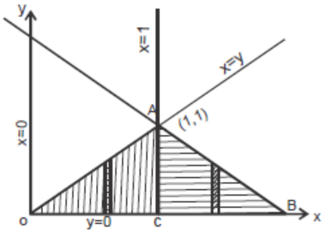
$$= \frac{1}{32a^2}\left(\frac{64a^6}{6}\right)$$

$$= \frac{a^4}{3}$$

10) Change the order of integration in $\int_0^1 \int_y^{2-y} xydxdy$ and hence evaluate it.

Solution:

The region of integration is bounded by x =y, x =2 - y, y =0 and y =1 which is shown in the figure.



When we change the order of integration, we first integrate with respect to y keeping x as constant. When the region of integration is covered by vertical strip, it does not intersect the region of integration in the same fashion. Hence the region ΔOAB is splitted into two subregions ΔOAC and ΔCAB. Hence

$$\iint\limits_{OAB} xydxy = \iint\limits_{OAC} xydydx + + \iint\limits_{CAB} xydydx$$

$$\int_0^1 \int_y^{2-y} xydxdy = \int_0^1 \int_0^x xydydx + \int_1^2 \int_0^{2-x} xydydx$$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)_0^x dx + \int_1^2 \left(\frac{xy^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3 dx + \int_1^2 \frac{x(2-x)^2}{2} dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1 + \int_1^2 \frac{x(4+x^2-4x)}{2} dx$$

$$= \frac{1}{2}\left(\frac{1}{4} - 0\right) + \frac{1}{2}\left(4\frac{x^2}{2} + \frac{x^4}{4} - 4\frac{x^3}{3}\right)_1^2$$

$$= \frac{1}{2}\left[\frac{1}{4} + \left(8 + \frac{16}{4} - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \left(12 - \frac{32}{3}\right) - \left(\frac{24+3-16}{12}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{5}{12}\right]$$

$$= \frac{8}{12 \times 2}$$

$$= \frac{1}{3}$$

11) Change the order of integration in $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ and hence evaluate it.

Solution:

The region of integration is bounded by the curves $y = x, y = \sqrt{2-x^2}, x = 0$ an$x$ = 1. The region ofintegration is a sector AOB which is shown in the figure.

When we change the order of integration, we first integrate with respect to 'x' keeping y constant. So,when the region of integration is covered by horizontal strip, it does not intersect the region of integration inthe same fashion. Hence the sector AOB is splitted into two subregions OAC and ACB.

Hence $= \iint_{OAB} \frac{x}{\sqrt{x^2+y^2}} dydx = \iint_{OAB} \frac{x}{\sqrt{x^2+y^2}} dxdy \iint_{ACB} \frac{x}{\sqrt{x^2+y^2}} dxdy$

$$= \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}} dxdy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy$$

$$= \int_0^1 \frac{1}{2}\left[\int_0^y \frac{2x}{\sqrt{x^2+y^2}} dx\right]dy + \int_1^{\sqrt{2}} \frac{1}{2}\left[\int_0^{\sqrt{2-y^2}} \frac{2x}{x^2+y^2} dx\right]dy$$

$$= \frac{1}{2}\int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy + \frac{1}{2}\int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy$$

$$\left(\because \int \frac{2xdx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} + c\right)$$

$$= \int_0^1 \left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right)dy$$

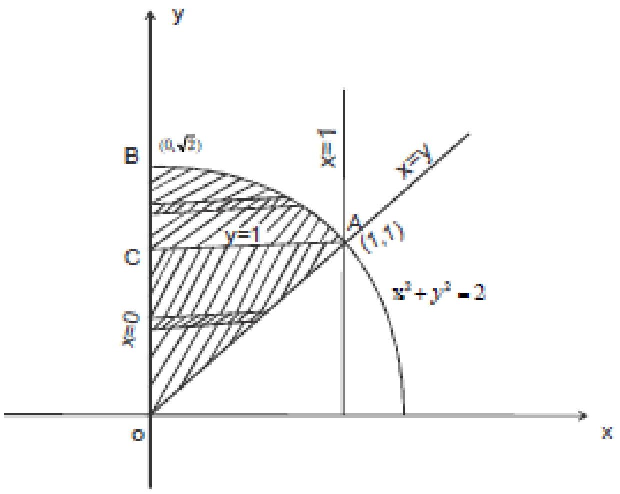
$$= \left(\sqrt{2}-1\right)\left(\frac{y^2}{2}\right)_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= \left[\left(\sqrt{2}-1\right)\left(\frac{1}{2}-0\right) + (2-1) - \left(\sqrt{2}-\frac{1}{2}\right)\right]$$

$$= \left[\frac{\sqrt{2}-1}{2} + \frac{3}{2} - \sqrt{2}\right]$$

$$= \frac{2-\sqrt{2}}{2}$$

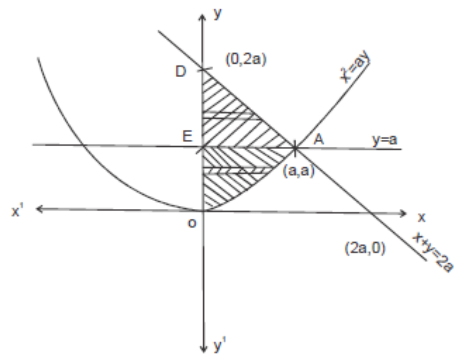
$$= 1 - \frac{1}{\sqrt{2}}$$



12) Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$ and then evaluate it.

Solution:

Let $I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$



The region of integration is bounded by the curves $y = \frac{x^2}{a} or\, x^2 = ay,$ $y = 2a - x\, or\, x + y = 2a\, x = 0,$ and $x = a$ is plotted in the following figure.

After changing the order, the integration is with respect to x first. When we draw lines parallel to the x axis to evaluate the limits of the inner integral it does not intersect the region of integration in the samefashion. Hence the region is divided into two subregions OAE and EAD.

Hence $I = \int\limits_{OAE} xydydx + \iint\limits_{EAD} xydydx$

$$= \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy$$

(The co-ordinates of A is obtained by solving the equations $x + y = 2a\, and\, x^2 = ay$)

$$\therefore I = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy + \int_a^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{1}{2}\int_0^a y(ay)dy + \frac{1}{2}\int_a^{2a} y(2a-y)^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a + \frac{1}{2}\int_a^{2a} y\left(4a^2 - 4ay + y^2\right)dy$$

$$= \frac{a}{6}(a^3) + \frac{1}{2}\int_a^{2a} \left(4a^2y - 4ay^2 + y^3\right)dy$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[4a^2\frac{y^2}{2} - 4a\frac{y^3}{3} + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2\left(4a^2 - a^2\right) - \frac{4a}{3}(8a^3 - a^3) + \frac{1}{4}(16a^4 - a^4)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 - \frac{4a}{3}(7a^3) + \frac{1}{4}15a^4\right]$$

$$= \frac{a^4}{6} + 3a^4 - \frac{14a^4}{3} + \frac{15}{8}a^4$$

$$= \frac{a^4(4+72-112+45)}{24} = \frac{9a^4}{24} = \frac{3a^4}{8}$$

Exercise:

I Change the order of Integration in the following integrals.

1) $\int_0^1 \int_0^{1-y} f(x,y)dxdy$ Ans: $\int_0^1 \int_0^{1-x} f(x,y)dydx$

2) $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)dydx$ Ans: $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)dxdy$

II Change the order of integration and hence evaluate the following integrals.

3) $\int_0^a \int_x^a \frac{e^{-y}}{y}dydx$ Ans : 1

4) $\int_0^a \int_x^a \frac{xdxdy}{x^2+y^2}$ Ans: $\frac{\pi a}{4}$

5) $\int_0^3 \int_1^{\sqrt{4-x}} (x+y)\,dydx$ Ans : $\frac{241}{60}$

6) $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} y^2\,dxdy$ Ans : $\frac{\pi a_4}{8}$

7) $\int_0^1 \int_{y^2}^{y} \frac{ydxdy}{x^2+y^2}$ Ans: $\frac{1}{2}\log 2$

8) $\int_1^2 \int_0^{4/x} xydydx$ Ans : $8\log 2$

9) $\int_0^1 \int_x^{\sqrt{2-x^2}} dydx$ Ans : $\frac{\pi}{4}$

10) $\int_0^1 \int_0^{1-x} e^{2x+y}\,dydx$ Ans : $\frac{1}{2}(e-1)^2$

11) $\int_2^3 \int_1^2 \frac{1}{xy}\,dxdy$ Ans: $\log 2.\log\frac{3}{2}$

12) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$ Ans : $\pi\, a/2$

13) $\int_0^1 \int_{\sqrt{y}}^{y} e^{y/x}\,dydx$ Ans: $3e^4-7$

14) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dxdy$ Ans : $\pi\, a^2/2$

15) $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dxdy, a>0$ Ans: $\frac{\pi a^2}{2}\left(\log a-\frac{1}{2}\right)$

16) $\int_0^a \int_{x/2}^{\sqrt{x}/a} \left(x^2+y^2\right)dxdy$ Ans: $\frac{a^2}{28}+\frac{a}{20}$

17) $\int_0^a \int_0^x x.e^{-x^2/y} dydx$ Ans : $e^2 - 1$

18) $\int_0^{a1} \int_0^{b/a\sqrt{a^2-x^2}} x^2 dydx$ Ans : $\frac{\pi}{16}a^3 b$

19) $\int_0^{a1} \int_{a-y}^{\sqrt{a^2-x^2}} y.dxdy$ Ans : $\frac{a^3}{6}$

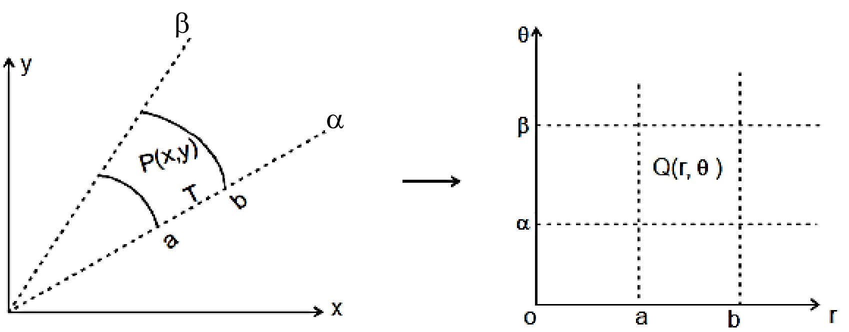
20) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3ydxdy$ Ans : 2

Change of variables from cartesian to polar co-ordinates

The evaluation of a double or a triple integral sometimes becomes easier when we transform the given variables into new variables.

In R^2 , if the domain has a circular symmetry and the function has some particular characteristics one can apply the transformation to polar co-ordinates, which means that the point P(x,y) in cartesian coordinates switch to their respective points in polar co-ordinates, that allows one to change the shape of the domain and simplify the operations.

The polar co-ordinates r and θare defined by $x = r \cos\theta$, $y = r \sin\theta$



Transformation from cartesian to polar co-ordinates

Then on substituting for x and y, the double integral $\iint_T f(x,y)dxdy$ is transformed to $\iint_U f(r\cos\theta, r\sin\theta)|J|drd\theta$ where $J(r,\theta)$ is the jacobian of (x,y) with respect to (r, θ).

$$\text{ie, } J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Therefore $dxdy = |J|\,d\theta dr = rd\theta dr$

1) By changing to polar coordinates, find the value of the integral $\int_0^{2a}\int_0^{\sqrt{2a-x^2}} (x^2+y^2)dydx$

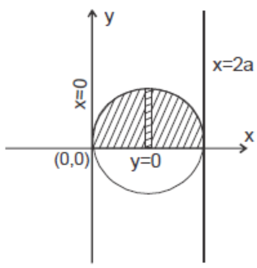
Solution

The region of integration is bounded by $x=0, x=2a, y=0$ and $y=\sqrt{2ax-x^2}$

Take $y=\sqrt{2ax-x^2}$

$y^2 = 2ax - x^2$

$x^2 + y^2 - 2ax = 0 \qquad (1)$



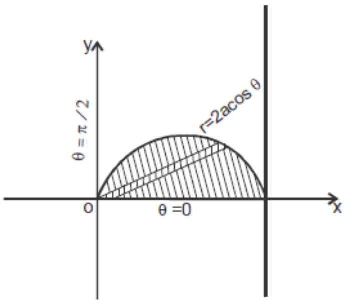
The polar coordinates are $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$

The polar equation of the circle $x^2 + y^2 - 2ax = 0$ is

$r^2\cos^2\theta + r^2\sin^2\theta - 2ar\cos\theta = 0$

$\Rightarrow r^2 - 2ar\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$



The region of integration is bouned by

$\theta = 0, \theta = \frac{\pi}{2}, \quad r = 0\, and\, r = 2a\cos\theta$

$$\therefore I = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \left[(r\cos\theta)^2 + (r\sin\theta)^2\right] rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 \left(\cos^2\theta + \sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{(2a\cos\theta)^4}{4} - 0\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2^4 a^4 \cos^4\theta}{4}\,d\theta$$

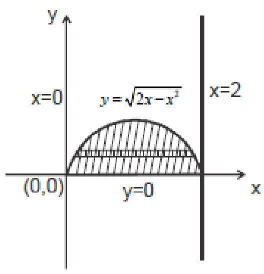
$$= \int_0^{\frac{\pi}{2}} \frac{16a^4}{4}\left(\cos^4\theta\right) d\theta$$

$$= 4a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$= 4a^4\,\frac{3}{4}\times\frac{1}{2}\,\frac{\pi}{2} = \frac{3\pi a^4}{4} \quad \because \int_0^{\pi/2} \cos^n\theta\,d\theta = \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\times\frac{\pi}{2} \text{ if n is even}$$

2) By changing into polar coordinates, evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$$



Solution :

Given $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dx\,dy$

ie, $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$ [Standard form]

The region of integration is bounded by $x=0, x=2, y=0 \, and \; y=\sqrt{2x-x^2}$

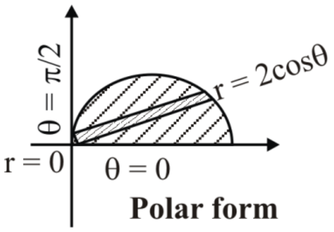
Take $y=\sqrt{2x-x^2}$

$$y^2 = 2x - x^2$$

$$\therefore x^2 + y^2 - 2x = 0$$

In this region the polar equation of the circle is

$$(r\cos\theta)^2 + (r\sin\theta)^2 - 2r\cos\theta = 0$$



$$\Rightarrow r = 2\cos\theta$$

Since $x = r\cos\theta$

$$y = r\sin\theta$$

$$dxdy = rdrd\theta$$

In polar co-ordinates the same region is bounded by the curves $r = 0, r = 2\cos\theta, \theta = 0, \theta = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{(r\cos\theta)^2 + (r\sin\theta)^2} rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r^2\cos\theta}{r^2} drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \cos\theta drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta [r]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta [2\cos\theta - 0] d\theta$$

$$= 2.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{2} \qquad \left(\because \int \cos^n\theta d\theta = \frac{(n-1)(n-3)....1}{n(n-2).....2} \times \frac{\pi}{2}\right)$$

3) Transform the integral into polar co-ordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dydx$

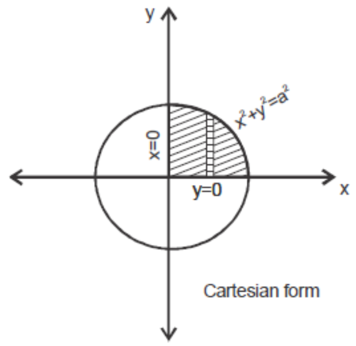
Solution

The region of integration is bounded by

$$y = 0, y = \sqrt{a^2 - x^2}, x = 0 \, and \; x = a$$

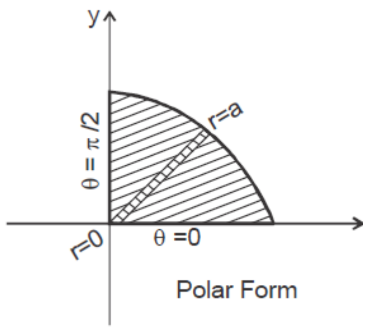
ie, $\quad y = 0, x^2 + y^2 = a^2, x = 0 \, and \; x = a$

(ie) The given region is a quadrant of the circle $x^2 + y^2 = a^2$



In this region the polar equation of the circle is

$$(r\cos\theta)^2 + (r\sin\theta)^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$$



In polar co-ordinate the same region is bounded by the curve

$$r = 0, r = a, \theta = 0 \, and \; \theta = \frac{\pi}{2} \quad \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y^2} \, dydx = \int_0^{\frac{\pi}{2}} \int_0^a rrdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}\right)_0^a d\theta = \frac{a^3}{3} [\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3} \times \frac{\pi}{2} = \frac{\pi a^3}{6}$$

4. Evaluate $\iint \frac{x^2y^2}{x^2+y^2}dxdy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2 \,(b>a)$ by transforming into polar co-ordinates.

Solution

Putting $x=r\cos\theta, y=r\sin\theta$ the given circles becomes

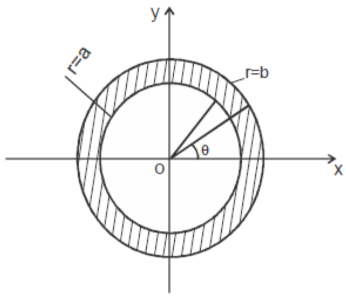
$$x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=a^2$$

$$\Rightarrow r^2=a^2 \Rightarrow r=a$$

$$x^2+y^2=b^2 \Rightarrow r^2=b^2 \Rightarrow r=b$$

andθ varies from 0 to 2π

In polar coordinates the annular region is bounded by the curves $r=a, r=b, \theta=0 \, and \, \theta=2\pi$



$$\iint \frac{x^2y^2}{x^2+y^2}dxdy=\int_0^{2\pi}\int_a^b \frac{r^2\cos^2\theta r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}.rdrd\theta \quad (\because dxdy=rdrd\theta)$$

$$=\int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta drd\theta$$

$$=\int_0^{2\pi}\cos^2\theta\sin^2\theta\left(\frac{r^4}{4}\right)_a^b d\theta$$

$$=\left(\frac{b^4-a^4}{4}\right).4\int_0^{\pi/2}\cos^2\theta\sin^2\theta d\theta$$

$$=(b^4-a^4)\frac{1.1}{4.2}.\frac{\pi}{2}$$

$$\left[\because \int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta=\frac{[(n-1)(n-3).....3.1][(m-1)(m-3)....3.1]}{(m+n)(m+n-2).....3.1}\times\frac{\pi}{2}\right],$$

where m & n are even

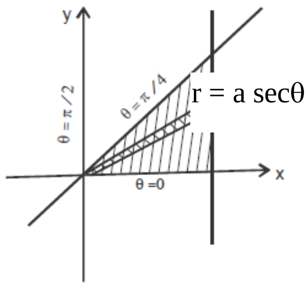
$$= \frac{\pi}{16}(b^4 - a^4)$$

5) Evaluate by changing to polars, the integral $\int_0^a \int_y^a \frac{x^2 dxdy}{\sqrt{x^2+y^2}}$

Solution

The region of integration is bounded by $y = 0, y = a, x = y \, and \, x = a$

Let us transform this integral into polar co-ordinates by taking $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$



In polar co-ordinates the given region of integration is bounded by the curves $r = 0, r = a\sec\theta, \theta = 0 \, and \, \theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}} rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{\sqrt{r^2}} drd\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{\sqrt{r^2}} drd\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} r^2\cos^2\theta drd\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\cos^2\theta \frac{r^3}{3}\right]_0^{a\sec\theta} d\theta$$

$$= a^3 \int_0^{\frac{\pi}{4}} \cos^2\theta \frac{\sec^3\theta}{3} d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$= \frac{a^3}{3} \left[\log(\sec\theta + \tan\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3} \left[\log\left(\sqrt{2}+1\right) - \log(1+0)\right]$$

$$= \frac{a^3}{3} \left[\log\left(\sqrt{2}+1\right) - 0\right]$$

$$= \frac{a^3}{3} \log\left(\sqrt{2}+1\right)$$

Exercise:

1) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dydx$ Ans : $\frac{\pi a^4}{8}$

2) Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2} dxdy$ Ans : $8\left[\frac{\pi}{2} - \frac{5}{3}\right]a^2$

3) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2 y + y^3) dxdy$ Ans : $\frac{a^5}{5}$

4) Evaluate $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{(a^2-x^2-y^2)}$ Ans : a

5) Evaluate $\int_0^a \int_y^a \frac{x^2 dxdy}{\left(x^2+y^2\right)^{3/2}}$ Ans : $\frac{a\sqrt{2}}{2}$

6) Evaluate $\int_0^\infty \int_y^\infty e^{-(x^2+y^2)} dxdy$ Ans : $\frac{\pi}{4}$

7) Evaluate $\iint (x^2+y^2)dydx$ over the circle $x^2+y^2=a^2$

Ans : $\pi\dfrac{a^4}{4}$

8) Evaluate $\iint \dfrac{dxdy}{\sqrt{x^2+y^2-a^2}}dxdy$ taken over the circle $x^2+y^2=1$

Ans : π

Area Using Double Integral

1) Find the Area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Solution:

From the equation of the ellipse, we have

$$\frac{y}{b}=\pm\sqrt{1-\frac{x^2}{a^2}}$$

So, the region of integration R can be considered as the area bounded by

$$x=-a\, and\, x=a, y\frac{-b}{a}\sqrt{a^2-x^2}\, and\, y=\frac{b}{a}\sqrt{a^2-x^2}$$

Area $= \iint\limits_R dydx = 4\times$ Area in first quadrant

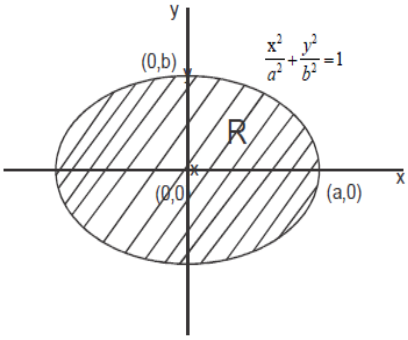
$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$



$= \frac{4b}{a}\left[(0-0)+\frac{a^2}{2}\left(\frac{\pi}{2}-0\right)\right]$

$= \pi ab$ square units.

2) Find the area enclosed by the parabola $y^2 = 4ax, x-axis$ and the latus rectum of the parabola

Solution:

Given the area is enclosed by $y^2 = 4ax$ (1)

$x-axis(y=0)$ and the

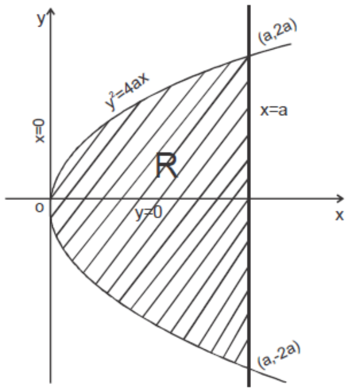
Lactus rectum of (1)

i.e, x = a

Points of intersection of (1) & (2)

is $(a,2a),(a-2a)$

Therefore the region of integration R can be considered as the area bounded by $x=0, x=a, y=-\sqrt{4ax}\ and\ y=\sqrt{4ax}$



Area $= \iint_R dxdy$

$= \int_{x=0}^{a} \int_{y=\sqrt{4ax}}^{\sqrt{4ax}}$

$= 2\int_0^a \left[\int_0^{\sqrt{4ax}} dy\right]dx$

$= 2\int_0^a \sqrt{4ax}dx$

$= 2\sqrt{4a}\int_0^a x^{\frac{1}{2}}dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$

$= \frac{8\sqrt{a}}{3}\left(a^{3/2}-0\right)$

$= \frac{8a^2}{3}$ square units.

3) Find the area in the first quadrant included between the parabola $x^2 = 16y, y - axis$ and the line y = 2.

Solution :

The area in the first quadrant is enclosed by

$x^2 = 16y$ (1)

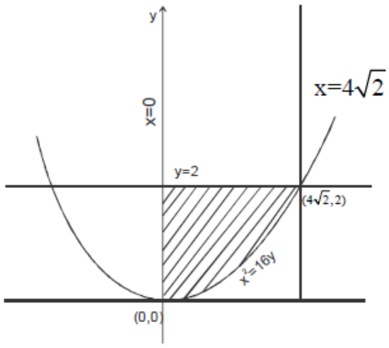
$y = 2$ (2)

and $x = 0 (y - axis)$ (3)

Points of intersection of (1) & (2) is

$(-4\sqrt{2}, 2) \& (4\sqrt{2}, 2)$

Therefore the region of integration R is considered as the area bounded by $x = 0, x = 4\sqrt{2}, y = \frac{x^2}{16}$ and $y = 2$



Area $= \iint_R dxdy$

$$= \int_{x=0}^{4\sqrt{2}} \left[\int_{y=\frac{x^2}{16}}^{2} dy \right] dx$$

$$= \int_{x=0}^{4\sqrt{2}} [y]_{\frac{x^2}{16}}^{2} dx$$

$$= \int_{x=0}^{4\sqrt{2}} \left[2 - \frac{x^2}{16} \right] dx$$

$$= \left[2x - \frac{1}{16} \frac{x^3}{3} \right]_0^{4\sqrt{2}}$$

$$= 4\sqrt{2} \left[2 - \frac{1}{48} 16(2) \right]$$

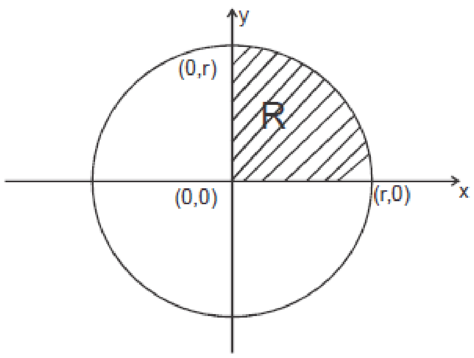
$$= 4\sqrt{2} \left[2 - \frac{2}{3} \right]$$

$$= \frac{16\sqrt{2}}{3} \text{ Square units.}$$

4) Find the area of the circle $x^2 + y^2 = r^2$ lies in the positive quadrant

Solution

The circle lies in the first quadrant is bounded by $x = 0, y = 0, x^2 + y^2 = r^2$



Therefore, the region of integration R can be considered as the area bounded by $x = 0, x = r, y = 0 \, and \; y = \sqrt{r^2 - x^2}$

$$\text{Area} = \iint_R dxdy$$

$$= \int_{x=0}^{r} \left[\int_{y=0}^{\sqrt{r^2-x^2}} dy \right] dx$$

$$\text{Area} = \int_{x=0}^{r} [y]_0^{\sqrt{r^2-x^2}} dx$$

$$= \int_0^r \sqrt{r^2 - x^2} dx$$

$$= \left[\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) \right]_0^r$$

$$= 0 + \frac{r^2}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi r^2}{4} \text{ square units.}$$

5) Find the area of the region R bounded by the parabola $y = x^2 \; and \; x = y^2$

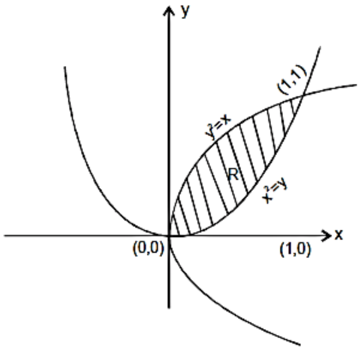
Solution:

$$\text{Area} = \iint_R dxdy$$

Region of integration is bounded by

$y = x^2$ (1) &

$x = y^2$ (2)



From (1) & (2)

$y = (y^2)^2$

$\Rightarrow y(1 - y^3) = 0$

$\Rightarrow y = 0, y = 1$

The points of intersection are (0, 0) & (1, 1) therefore, the region of integration can be considered as the area bounded by $x = 0, x = 1, y = x^2 and\ y = \sqrt{x}$

Area = $\int_{x=0}^{1} \left[\int_{y=x^2}^{\sqrt{x}} dy \right] dx$

$I = \int_{x=0}^{1} [y]_{x^2}^{\sqrt{x}} dx$

$= \int_{x=0}^{1} \left[\sqrt{x} - x^2 \right] dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1$

$= \frac{1}{3} \left[2x^{3/2} - x^3 \right]_0^1$

$= \frac{1}{3} [2(1-0) - (1-0)]$

$= \frac{1}{3} Squareunits.$

6) Find the area of the cardioids $r = a(1 + \cos\theta)$

Solution

Area $= \iint_R dxdy$

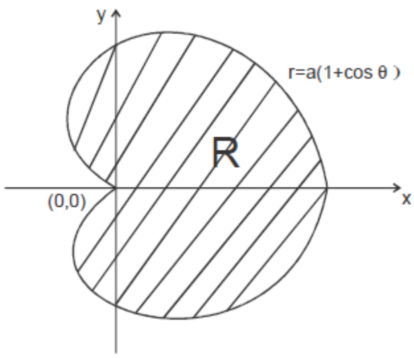
$= \iint r dr d\theta$

Given $r = a(1+\cos\theta)$

Limits

$r: 0 \to a(1+\cos\theta)$

$\theta . 0 \to 2\pi$



Area $= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} r dr d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$

$= \frac{a^2}{2} 2\int_0^{\pi} (1+\cos\theta)^2 d\theta$

$= a^2 \int_0^{\pi} (1+2\cos\theta+\cos^2\theta) d\theta$

$= a^2 \int_0^{\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$

$= a^2 \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$

$= a^2 \left[(\pi-0) + 2(0-0) + \frac{1}{2}\left[(\pi-0) + \frac{1}{2}(0-0)\right]\right]$

$= a^2 \left(\pi + \frac{1}{2}\pi\right) = \frac{3\pi}{2} a^2$ Square units.

7) Find the area inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$

Solution

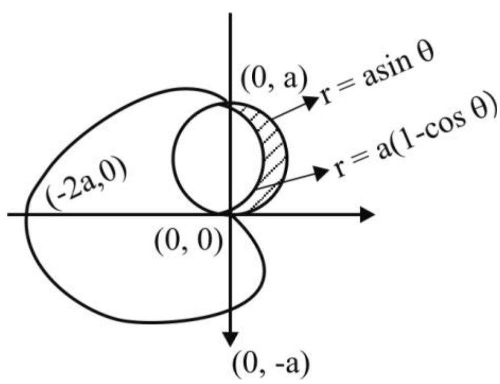
Given $r = a\sin\theta$ _______ (1)

$r = a(1-\cos\theta)$ _______ (2)

Limits

$r: a(1-\cos\theta) \to a\sin\theta$

$\theta: 0 \to \frac{\pi}{2}$



Area $= \iint_R r dr d\theta$

$$= \int_{0}^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a\sin\theta} r dr d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \left[a^2 \sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} a^2\left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_{0}^{\frac{\pi}{2}} \left[-\cos 2\theta - 1 + 2\cos\theta\right] d\theta$$

$$= \frac{a^2}{2}\left[-\frac{\sin 2\theta}{2} - \theta + 2\sin\theta\right]_{0}^{\frac{\pi}{2}}$$

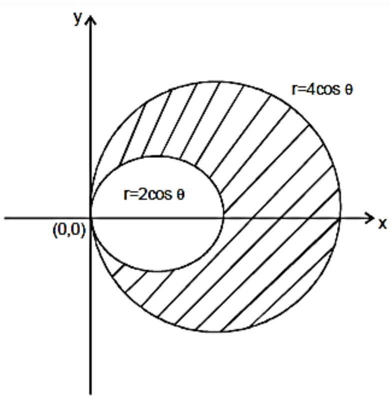
$$= \frac{a^2}{2}\left[-\frac{1}{2}(0-0) - \left(\frac{\pi}{2} - 0\right) + 2(1-0)\right]$$

$$= \frac{a^2}{2}\left(-\frac{\pi}{2} + 2\right)$$

$$= \frac{a^2}{4}(4-\pi)\, Squareunits.$$

8) Find the area of the region outside the inner circle r =2cosθand inside the outer circle r =4cosθ

Solution:



Given r =2cosθ (1)

ie, r = 4cosθ (2)

From (1) & (2)

$$r : 2\cos\theta \to 4\cos\theta$$

$$\theta : 0 \to \frac{\pi}{2}$$

Area = $2\int_0^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r dr d\theta$

$= 2\int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$

$= 1\int_0^{\frac{\pi}{2}} \left[(4\cos\theta)^2 - (2\cos\theta)^2\right] d\theta$

$= 12\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 12\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta$

$= 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= 6\left[\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(0-0)\right]$

Area = 3π square units.

Exercise:

1) Find the area bounded by the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant

Ans : $\frac{a^2}{4}(\pi - 2)$

2) Find the area which lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$

Ans : πa^2

3) Find the area enclosed by the curve $y^2 = 4ax$ and the lines $x + y = 3a$ and $x - axis$

Ans : $\frac{10a^2}{3}$

4) Find the area enclosed by the lines $x = 0, y = 0, \frac{x}{a} + \frac{y}{b} = 1$

Ans : $\frac{ab}{2}$

5) Find the area in the first quadrant bounded by the x-axis and the curves $x^2 + y^2 = 10, y^2 = 9x$

Ans : $\frac{27}{4}$

6) Find the area enclosed by the parabola $y^2 = (4 - x)$ and the x-axis.

Ans : $\frac{32}{3}$

7) Find the area bounded by the lines $x = 0, y = 0, 5y = 3$ and the curve $x^2 + y^2 = 1$

Ans : $\frac{6}{25} + \frac{1}{2}\sin^{-1}(1)$

8) Find the area bounded by the parabolas $y^2 = 4 - x \, and \, y^2 = 4 - 4x$

Ans : 8.

TRIPLE INTEGRAL

Triple integral is defined similar to that of double integral. The general form of the triple integral is $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)dxdydz$

To evaluate the triple integral, first we integrate $f(x, y, z)$ with respect to x keeping y and z as constant and substitute the limits x_1 and x_2 which will be either constants or functions of y and z . Next we integrate the resulting function of y and z with respect to y keeping z as constant and substitute the limits y_1 and y_2 which will be either constants or functions of z. Finally we integrate the resulting function of z with respect to z and substitute the limits z_1 and z_2 which will be constants.

To evaluate $\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} f(x, y, z)dxdydz$

The order in which the integrations are performed is illustrated as follows.

$$\int_{z=0}^{c}\left[\int_{y=0}^{b}\left[\int_{x=0}^{a} f(x,y,z)\,dx\right]dy\right]dz$$

Problems based on triple integration

1. Evaluate $\int_0^c \int_0^b \int_0^a xyz\,dxdydz$

Solution:

Let $I = \int_0^c \int_0^b \int_0^a xyz\,dxdydz$

$$= \left[\int_0^a x\,dx\right]\left[\int_0^b y\,dy\right]\left[\int_0^c z\,dz\right]$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \left[\frac{a^2}{2}-0\right]\left[\frac{b^2}{2}-0\right]\left[\frac{c^2}{2}-0\right]$$

$$= \frac{(abc)^2}{8}$$

2. Evaluate $\int_0^1 \int_0^2 \int_0^3 (x+y+z)\,dzdydx$

Solution:

Let $I = \int_0^1 \int_0^2 \int_0^3 (x+y+z)\,dzdydx$

$$= \int_0^1 \int_0^2 \left[(x+y)z + \frac{z^2}{2}\right]_0^3 dydx$$

$$= \int_0^1 \int_0^2 \left[3(x+y) + \frac{9}{2}\right] dydx$$

$$= \int_0^1 \left(3xy + \frac{3y^2}{2} + \frac{9}{2}y\right)_0^2 dx$$

$$= \int_0^1 (6x + 6 + 9)dx = \left(\frac{6x^2}{2} + 15x\right)_0^1 = 3 + 15 = 18$$

3. Evaluate $\int_0^1 \log\frac{(1+x)}{1+x^2}dx$

Solution:

Let $I = \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)dxdydz$

$$= \int_0^a \int_0^b \left[\frac{x^3}{3} + y^2x + z^2x\right]_{x=0}^c dydz$$

$$= \int_0^a \int_0^b \left[\frac{c^3}{3} + cy^2 + cz^2\right]dydz$$

$$= \int_0^a \left[\frac{c^3}{3}y + \frac{cy^3}{3} + cz^2y\right]_{y=0}^b dz$$

$$= \int_0^a \left[\frac{bc^3}{3} + \frac{cb^3}{3} + cbz^2\right]dz$$

$$= \left[\frac{bc^3}{3}z + \frac{cb^3}{3}z + \frac{cbz^3}{3}\right]_0^a$$

$$= \frac{abc^3}{3} + \frac{acb^3}{3} + \frac{bca^3}{3}$$

$$= \frac{abc}{3}\left[a^2 + b^2 + c^2\right]$$

4. Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} zdzdydx$

Solution

Let $I = \int_0^1 \int_0^x \int_0^{\sqrt{x+y}} zdzdydx$

$$= \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dydx$$

$$= \int_0^1 \int_0^x \left[\frac{x+y}{2} - 0\right] dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^x (x+y) dydx$$

$$= \frac{1}{2}\int_0^1 \left(xy + \frac{y^2}{2}\right)_{y=0}^{x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\left(x^2 + \frac{x^2}{2}\right) - (0+0)\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{3}{2}x^2 dx = \frac{3}{4}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{4}\left[\frac{1}{3} - 0\right] = \frac{1}{4}$$

5. Evaluate $I = \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dzdydx$

Solution:

$$I = \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^x . e^y . e^z dzdydx$$

$$= \int_0^{\log 2} \int_0^x e^x . e^y \left[e^z\right]_{z=0}^{x+\log y} dydx$$

$$= \int_0^{\log 2} \int_0^x e^x e^y \left[e^{x+\log y} - e^0\right] dydx$$

$$= \int_0^{\log 2} \int_0^x \left[e^{2x} . e^y . e^{\log y} - e^x . e^y\right] dydx$$

$$= \int_0^{\log 2} \int_0^x \left[e^{2x} e^y y - e^x e^y\right] dydx \qquad \left[\because e^{\log y} = y\right]$$

$$= \int_0^{\log 2} \left[e^{2x}\left(ye^y - e^y\right) - e^x e^y\right]_{y=0}^{x} dx \qquad \left[\int ye^y dy = ye^y - e^y\right]$$

$$= \int_0^{\log 2} \left[e^{2x}\left(xe^x - e^x\right) - e^x e^x - \left(e^{2x}(0-1) - e^x\right)\right] dx$$

$$= \int_0^{\log 2} \left[e^{3x}\left(x-1\right) - e^{2x} + e^{2x} + e^x\right] dx$$

$$= \int_0^{\log 2} \left[e^{3x}(x+1) + e^x\right] dx$$

$$= \left[\left(x-1\right)\frac{e^{3x}}{3} - (1)\frac{e^{3x}}{9} + e^x\right]_0^{\log 2} \qquad \text{(By Bernoullis formula)}$$

$$= \left[\left(x-1\right)\frac{e^{3x}}{3} - \frac{e^{3x}}{9} + e^x\right]_0^{\log 2}$$

$$= \left[\left(\left(\log 2 - 1\right)\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2}\right) - \left(\left(-1\right)\frac{1}{3} - \frac{1}{9} + 1\right)\right]$$

$$= \left[\left(\left(\log 2 - 1\right)\frac{e^{\log 2^3}}{3} - \frac{e^{\log 2^3}}{9} + e^{\log 2}\right) - \left(\frac{-1}{3} - \frac{1}{9} + 1\right)\right]$$

$$= \left[\left(\left(\log 2 - 1\right)\frac{8}{3} - \frac{8}{9} + 2\right) - \left(\frac{-3-1+9}{9}\right)\right] \qquad \left[\because e^{\log 2^3} = 2^3 = 8\right]$$

$$= \left[\frac{8}{3}\log 2 - \frac{8}{3} - \frac{8}{9} + 2\right] - \left(\frac{5}{9}\right)$$

$$= \frac{8}{3}\log 2 + \left[\frac{-2-8-18-5}{9}\right]$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

6. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dx\,dy\,dz$

Solution:

Given $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdxdydz$ (not in standard form)

$$= \int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy \quad \text{(Standard form)}$$

$$= \int_0^1 \int_{y^2}^1 (xz)_{z=0}^{1-x} dxdy$$

$$= \int_0^1 \int_{y^2}^1 x(1-x)dxdy$$

$$= \int_0^1 \int_{y^2}^1 \left(x-x^2\right)dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{x=y^2}^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy$$

$$= \int_0^1 \left(\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right)dy$$

$$= \left(\frac{1}{6}y-\frac{y^5}{2\times 5}+\frac{y^7}{3\times 7}\right)_0^1$$

$$= \frac{1}{6}-\frac{1}{10}+\frac{1}{21}$$

$$= \frac{4}{35}$$

7. Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} rdzdrd\theta$

Solution

$$\int_0^{\pi/2} \int_0^{a\sin\theta} \int^{(a^2-r^2)/a} rdzdrd\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{a\sin\theta} dr[rz]_0^{(a^2-r^2)/a}$$

$$= \int_0^{\pi/2} d\theta \int_0^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right)dr$$

$$= \int_0^{\pi/2} \frac{1}{a}\left(\frac{a^2r^2}{2}-\frac{r^4}{4}\right)_0^{a\sin\theta}$$

$$= \int_0^{\pi/2} \frac{1}{a}\left(\frac{a^4\sin^2\theta}{2}-\frac{a^4\sin^4\theta}{4}\right)d\theta$$

$$= \frac{a^3}{4}\int_0^{\pi/2} \left(2\sin^2\theta - \sin^4\theta\right)d\theta$$

$$= \frac{a^3}{4}\left[2.\frac{1}{2}.\frac{\pi}{2}-\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{5a^3\pi}{64}$$

8. **Evaluate** $\iiint_v \frac{dzdydx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $x=0, y=0, z=0, x+y+z=1$

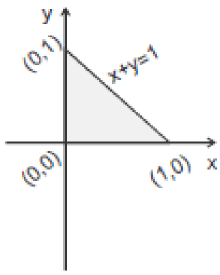
Solution

The given region is a tetrahedron. The projection of the given region in the xy plane is a triangle bounded by the line $x=0, y=0\, and\, x+y=1$ as shown in figure.

Here $x\, \text{var}\, ies\ from\, x=0\, to\, x=1$

$y\, \text{var}\, ies\ from\, y=0\, to\, y=1-x$

$z\, \text{var}\, ies\ from\, z=0\, to\, z=1-x-y$



$$\iiint_V \frac{dzdydx}{(x+y+z+1)^3} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}dzdydx$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}dzdydx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dydx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[(x+y+1-x-y+1)^{-2}-(x+y+0+1)^{-2}\right]dydx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(2)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left[(2)^{-2} y - \frac{(x+y+1)^{-1}}{(-1)}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[(2)^{-2}(1-x) + (x+1-x+1)^{-1} - 0 - (x+1)^{-1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{1}{4}\left(x - \frac{x^2}{2}\right) + \frac{1}{2}x - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{4}\left(1 - \frac{1}{2}\right) + \frac{1}{2} - \log 2 - \left(0 + 0 - \log 1\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4} \times \frac{1}{2} + \frac{1}{2} - \log 2 + 0\right]$$

$$= -\frac{1}{2}\left[\frac{1+4-8\log 2}{8}\right] = -\frac{1}{16}\left[5 - 8\log 2\right]$$

$$= \frac{1}{16}\left(8\log 2 - 5\right)$$

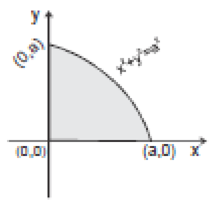
9. Evaluate $\iiint \frac{dxdydz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2$

Solution

The projection of the given region in the $x\,y$ plane $(z = 0)$ is the region of the circle $x^2 + y^2 = a^2$ lying in the first quadrant which is shown in the figure.

$\therefore$ In the region x varies from 0 to a. For a fixed x, y varies from 0 to $\sqrt{a^2 - x^2}$. For a fixed $(x, y), z$ varies from 0 to $\sqrt{a^2 - x^2 - y^2}$.

$$\iiint \frac{dxdydz}{\sqrt{a^2 - x^2 - y^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2 - x^2 - y^2 - z^2}}$$



$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{a^2 - x^2 - y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$\left[\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left[\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\right] - \sin^{-1}(0)\right] dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\frac{\pi}{2} - 0\right) dydx$$

$$= \frac{\pi}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$$

$$= \frac{\pi}{2}\int_0^a \left[y\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^a \left[\sqrt{a^2-x^2} - 0\right] dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) + \frac{a}{2}\sqrt{a^2-a^2} - \frac{a^2}{2}\sin^{-1}(0) - 0\right]$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}(1) + 0 - 0\right]$$

$$= \frac{\pi}{2} \times \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

10. Evaluate $\iiint_V xyzdxdydz$. whereV is the region of space inside the tetrahedron bounded by the planes $x = 0, y = 0, z = 0\, and\, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

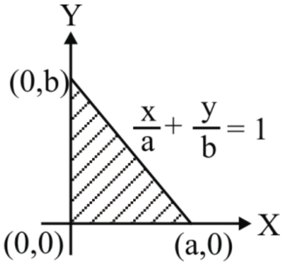
Solution

The projection of the given region in the xy-plane is the triangle bounded by the lines x = 0, y = 0 and $\frac{x}{a} + \frac{y}{b} = 1$ as shown in the figure

Here x varies from x = 0 to x = a

y varies from y = 0 to y = b $\left(1-\frac{x}{a}\right)$ z varies from z = 0 to z = c $\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$\therefore \iiint_V xyz dxdydz = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xyz dzdydx$$



$$= \int_0^a \int^{b\left(1-\frac{x}{a}\right)} xy\left(\frac{z^2}{2}\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \frac{c^2}{2}\int_0^a \int_0^b xy\left(t-\frac{y}{b}\right)^2 dydx$$

$$\text{where } t = 1-\frac{x}{a}$$

$$= \frac{c^2}{2}\int_0^a x\left(t^2.\frac{y^2}{2}-\frac{2t}{b}\frac{y^3}{3}+\frac{1}{b^2}\frac{y^4}{4}\right)_0^{bt} dx$$

$$= \frac{c^2}{2}\int_0^a \left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)b^2 x t^4 dx$$

$$= \frac{b^2c^2}{24}\int_0^a x\left(1-\frac{x}{a}\right)^4 dx$$

$$= \frac{b^2c^2}{24}\int_0^a a\left[1-\left(1-\frac{x}{a}\right)\right]\left(1-\frac{x}{a}\right)^4 dx$$

$$= \frac{ab^2c^2}{24}\int_0^a \left[\left(1-\frac{x}{a}\right)^4-\left(1-\frac{x}{a}\right)^5\right]dx$$

$$= \frac{ab^2c^2}{24}\int_0^a \left[\frac{\left(1-\frac{x}{a}\right)^5}{-5/a}-\frac{\left(1-\frac{x}{a}\right)^6}{-6/a}\right]_0^a dx$$

$$= \frac{a^2b^2c^2}{24}\left[\frac{1}{5}-\frac{1}{6}\right]$$

$$= \frac{a^2b^2c^2}{720}$$

Exercise

1. Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\,dzdydx$ Ans : $(e^a-1)(e^b-1)(e^c-1)$

2. Evaluate $\int_0^1 \int_0^1 \int_0^1 (xy+yz+zx)\,dxdydz$ Ans : $\frac{3}{4}$

3. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ Ans : $\frac{\pi^2}{8}$

4. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,dxdydz$ Ans : $\frac{1}{2}$

5. Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2z\,dzdydx$ Ans : 26

6. Evaluate $\int_2^4 \int_0^x \int_0^{\sqrt{x+y}} z\,dzdydx$ Ans : 14

7. Evaluate $\int_0^a \int_0^x \int_0^y xyz\,dzdydx$ Ans : $\frac{a^6}{48}$

8. Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2z\,dzdydx$ Ans : 26

9. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{(x+y)^2} x\,dzdydx$ Ans : $\frac{1}{10}$

10. Evaluate $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xy\,dzdydx$ Ans : $\frac{2}{25}\left[2.(3)^{5/2}-5-5\log 3\right]$

11. Evaluate $\iiint_V \frac{dxdydz}{(x+y+2z+1)^3}$ where V is the region enclosed by the planes

$x=0, y=0, z=0 \; and \; x+y+z=1$ Ans : $\frac{1}{4}[\log 3 - 1]$

12. Evaluate $\iiint_V (x+y+z)dxdydz$ where the region V is bounded by

$x+y+z=a(a>0), x=0, y=0, z=0$ Ans : $\frac{a^4}{8}$

13. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xdxdydz$ Ans : $\frac{\pi a^4}{16}$

14. Evaluate $\int_0^a \int_0^{1-x} \int_0^{x+y} e^{z+1}dxdydz$ Ans : $\frac{e}{2}$

15. Evaluate $\iiint_V \sqrt{1-x^2-y^2-z^2}\,dxdydz$ where V is the volume of the sphere

$x^2+y^2+z^2=1$ Ans : $\frac{\pi^2}{4}$

16. Evaluate $\iiint \frac{dxdydz}{x^2+y^2+z^2}$ over the sphere $x^2+y^2+z^2=a^2$

Ans : 4πa

17. Evaluate $\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{y}{b}-\frac{x}{a}\right)} x^2yzdzdydx$ Ans : $\frac{a^3b^2c^2}{2520}$

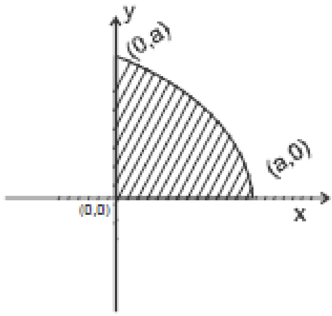
18. Find the value of $\iiint xyz\left(x^2+y^2+z^2\right)dxdydz$ taken over the positive octant for which $x^2+y^2+z^2 \le a^2$ Ans : $\frac{a^8}{64}$

19. Evaluate $\iiint xyzdxdydz$ over the region of integration bounded by $x, y, z \ge 0$ and $x^2+y^2+z^2 \le 9$

Ans : $\frac{243}{16}$

Volume using Triple Integrals

1. Find the volume of the region bounded by the surfaces $y^2 = 4ax$ and $x^2 = 4ay$ and the plane $z = 0$ and $z = 3$



Solution:

$$y^2 = 4ax \qquad (1)$$

$$x^2 = 4ay \qquad (2)$$

Solving (1) & (2)

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow x^4 - 64a^3x = 0$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$\Rightarrow x = 0, x^3 = 64a^3$$

$$x = 0, x = 4a$$

Required volume = $\displaystyle\int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}}\int_0^3 dzdydx$

$$= \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} [z]_0^3 dydx = \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} 3dydx$$

$$= 3\int_0^{4a} [y]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= 3\int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right]dx$$

$$= 3\left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{4a\times 3}\right]_0^{4a}$$

$$= 3\left[\sqrt{4a}\,\frac{4a\sqrt{4a}}{3/2} - \frac{(4a)^3}{12a} - 0\right]$$

$$= 3\left[4a \times 4a \times \frac{2}{3} - \frac{64a^3}{12a}\right]$$

$$= 3\left[\frac{32a^2}{3} - \frac{16a^2}{3}\right] = 3\left[\frac{32a^2 - 16a^2}{3}\right]$$

$$= 16a^2 \text{ Cubic units}$$

2. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ without transformation.

Solution

V = 8 x volume in the first octant

z varies from z = 0 to z = $\sqrt{a^2 - x^2 - y^2}$

y varies from y = 0 to y = $\sqrt{a^2 - x^2}$

x varies from x = 0 to x = a

$$\therefore \text{V} = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a \int^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2 - x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\frac{a^2 - x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}}\right) + \frac{\sqrt{a^2 - x^2}}{2}\sqrt{a^2 - x^2 - (a^2 - x^2)}\right]dx$$

$$= 8\int_0^a \left[\frac{a^2 - x^2}{2}\sin^{-1}(1) + 0\right]dx$$

$$= 8\int_0^a \left(\frac{a^2 - x^2}{2} \cdot \frac{\pi}{2}\right)dx$$

$$= 8 \times \frac{\pi}{4}\int_0^a \left(a^2 - x^2\right)dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[a^2 \times a - \frac{a^3}{3} - 0\right]$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right] = 2\left[\frac{3a^3 - a^3}{3}\right] = \frac{4\pi a^3}{3} \textit{cubic units}$$

3. Evaluate $\iiint_V dxdydz$, where V is the volume of the tetrahedron whose vertices are (0, 0, 0), (0, 1, 0), (1, 0, 0) and (0, 0, 1).

Solution

Now the plane through the points (0, 1, 0), (1, 0, 0) and (0, 0, 1) is
$x + y + z = 1$

If we first integrate w.r.t 'x' then its limits are 0 and $1 - (y + z)$

If the second integration is w.r.t. 'y', its limits are 0 and $1 - z$.

Finally, the limits of integration for z are 0 and 1.

$$\iiint_V dxdydz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dxdydz$$

$$= \int_0^1 \int_0^{1-z} [x]_0^{1-y-z} dydz$$

$$= \int_0^1 \int_0^{1-z} [1 - y - z]dydz$$

$$= \int_0^1 \left(y - \frac{y^2}{2} - yz\right)_0^{1-z} dz$$

$$= \int_0^1 \left((1-z) - \frac{(1-z)^2}{2} - (1-z)z - 0\right)dz$$

$$= \int_0^1 \left(1 - z - z + z^2 - \frac{(1-z)^2}{2}\right)dz$$

$$= \int_0^1 \left(1 - 2z + z^2 - \frac{(1-z)^2}{2}\right)dz$$

$$= \left(z - \frac{2x^2}{2} + \frac{z^3}{3} - \frac{(1-z)^3}{2 \times 3(-1)} \right)_0^1$$

$$= \left[1 - 1 + \frac{1}{3} - 0 \right] - \left[0 - \frac{1}{-6} \right]$$

$$= \left[1 - 1 + \frac{1}{3} \right] - \frac{1}{6} = \frac{1}{3} - \frac{1}{6}$$

$$= \frac{2-1}{6} = \frac{1}{6} \, cubic\, units$$

4. Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which lies in the first octant using triple integration.

Solution

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (1)

Volume $= \iiint dz dy dx$

To find x limit put $y = 0 \, and \, z = 0$ we get (line integral)

$$(1) \Rightarrow \frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$$

ie, $x = 0$ to $x = a$ ($\because$ first octant area)

To find y limit put $z = 0$ we get (surface integral)

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2 \left(1 - \frac{x^2}{a^2} \right)$$

$$\Rightarrow y = \pm b \sqrt{1 - x^2 / a^2}$$

$$\Rightarrow y = b \sqrt{1 - x^2 / a^2}$$

ie, $y = 0, y = b\sqrt{1-\frac{x^2}{a^2}}$ $(\because$ first octant area)

To find z limit [volume integral]

$$(1) \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\Rightarrow \frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Rightarrow z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$\Rightarrow z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

ie, $z = 0 \, to \, z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$\text{volume} = \int_0^q \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= \int_b^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-0\right] dydx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dydx$$

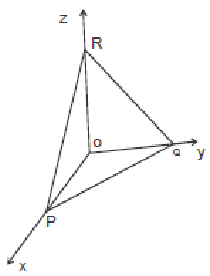
$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left[b^2\left(1-\frac{x^2}{a^2}\right)\right]-y^2}\, dydx$$

$$= \frac{c}{b}\int_0^a \left[\frac{y\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}}{2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}}\right)\right]_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx$$

$$= \frac{c}{b}\int_0^a \left[0 + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\left(\frac{\pi}{2}\right)\right]dx$$

$$= \frac{\pi c b^2}{4b}\int_0^a \left(1-\frac{x^2}{a^2}\right)dx$$



$$= \frac{\pi b^2 c}{4b}\left(x - \frac{x^3}{a^2\times 3}\right)_0^a$$

$$= \frac{\pi bc}{4}\left(a - \frac{a^3}{3a^2}\right)$$

$$= \frac{\pi bc}{4}\left(a - \frac{a}{3}\right)$$

$$= \frac{\pi bc}{4}\frac{2a}{3} = \frac{\pi abc}{6}$$

Hence the volume of the ellipsoid

$$V = 8\times\frac{\pi abc}{6} = \frac{4}{3}\pi abc \text{ cubic units.}$$

5. Find by triple integral the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solution:

The projection of the given region of the xy plane is the triangle bounded by the lines $x = 0$, $y = 0$, and $\frac{x}{a}+\frac{y}{b}=1$.

In this region x varies from 0 to a. For fixed x, y varies from 0 to $\left(1-\frac{x}{a}\right)b$. For fixed (x, y), z varies from 0 to $\left(1-\frac{x}{a}-\frac{y}{b}\right)c$.

$$\therefore V = \iiint_D dxdydz$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= \int_0^a \int^{b\left(1-\frac{x}{a}\right)} [z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int^{b\left(1-\frac{x}{a}\right)} c\left[1-\frac{x}{a}-\frac{y}{b}\right]dydx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= bc\int_0^a \left[\left(1-\frac{x}{a}\right)^2-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= bc\int_0^a \frac{1}{2}\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\times\frac{-1}{a}}\right]_0^a$$

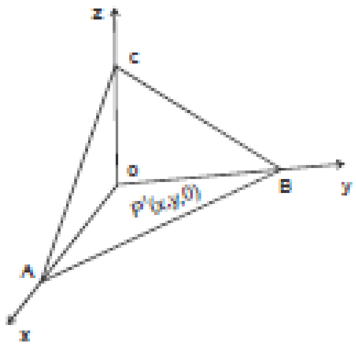
$$= \frac{bc}{2}\left[0-\frac{1}{-3/a}\right] = \frac{bc}{2}\times\frac{a}{3} = \frac{abc}{6} \quad \text{cubic units.}$$

6) Evaluate $\iiint_V dxdydz$, where V is the finite region of space (tetrahedron) formed by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 4z = 12$

Solution:

The projection of the given region on the xy plane is the triangle bounded by the lines $x=0, y=0$ and $2x+3y=12$.

In this region x varies from 0 to 6. For fixed x, y varies from 0 to $\frac{1}{3}(12-2x)$. For fixed (x, y), z varies from 0 to $\frac{1}{4}(12-2x-3y)$.



$$\therefore \iiint_V dxdydz = \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \int_0^{\frac{1}{4}(12-2x-3y)} dzdydx$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} [z]_0^{\frac{1}{4}(12-2x-3y)} dydx$$

$$= \frac{1}{4}\int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12-2x-3y)dydx$$

$$= \frac{1}{4}\int_0^6 \left[(12-2x)y - 3\frac{y^2}{2}\right]_0^{\frac{1}{3}(12-2x)} dx$$

$$= \frac{1}{4}\int_0^6 \left[(12-2x)\frac{1}{3}(12-2x) - \frac{3}{2}\times\frac{1}{9}(12-2x)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \left[\frac{1}{3}(12-2x)^2 - \frac{1}{6}(12-2x)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \frac{1}{6}(12-2x)^2\, dx$$

$$= \quad \frac{1}{4}\times\frac{1}{6}\left[\frac{(12-2x)^3}{3\times(-2)}\right]_0^6$$

$$= \quad \frac{1}{24}\left[0-\frac{12^3}{-6}\right]=\frac{1}{24}\times\frac{12^3}{6}=12 \text{ cubic units.}$$

Exercise:

1. Find the volume of the region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$, $z = 3$

Ans : 1

2. Find the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$

Ans : $\frac{1}{6}$

3. Evaluate $\iiint_v dxdydz$, where V is the region of space bounded by $x^2+y^2+z^2=9$

Ans : $\frac{108\pi}{3}$

4. Find the volume of the solid in the first octant bounded by the planes $x=0, y=0, z=0, x+2y+z-6=0$

Ans : 18

TEXT / REFERENCE BOOKS

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