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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – BASIC DIFFERENTIAL CALCULUS - SMTA1105

UNIT - I

BASIC DIFFERENTIAL CALCULUS

Introduction to Derivative of a function – Rules of Differentiation – Product Rule – Quotient Rule – Implicit Functions - Evaluating Higher order Derivatives –Maxima and minima of functions of one variable

Definition 1. Differentiation

The rate at which a function changes with respect to the independen derivative of the function.

(i.e) If y= f(x) be a function, where x and y are real variables which dependent variables respectively, then the derivative of y with respect to

Definition 2. Derivative of addition or subtraction of functions

If f(x) and g(x) are two functions of x, then $\frac{d[f(x) \pm g(x)]}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$

Definition 3. Product rule

If $y = uv$, where u and v are functions of x, then $\frac{d[uv]}{dx} = v\frac{d[u]}{dx} + u\frac{d[v]}{dx}$

Definition 4. Quotient rule

If $y = \frac{u}{v}$, where u and v are functions of x, then $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Important Derivatives Formulae

1. $\frac{d}{dx}(c) = 0$ where ‘c’ is any constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$.
3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.
4. $\frac{d}{dx}(a^x) = a^x \log a$

5. $\frac{d}{dx}\left(e^{x}\right)=e^{x}$.

6. $\frac{d}{dx}(\sin x)=\cos x$.

7. $\frac{d}{dx}(\cos x)=-\sin x$.

8. $\frac{d}{dx}(\tan x)=\sec^{2} x$.

9. $\frac{d}{dx}(\cot x)=-\cos ec^{2} x$.

10. $\frac{d}{dx}(\sec x)=\sec x \tan x$.

11. $\frac{d}{dx}(\cos ecx)=-\cos ecx \cot x$.

12. $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^{2}}}$.

13. $\frac{d}{dx}\left(\cos^{-1} x\right)=\frac{-1}{\sqrt{1-x^{2}}}$.

14. $\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^{2}}$.

15. $\frac{d}{dx}\left(\cot^{-1} x\right)=\frac{-1}{1+x^{2}}$.

16. $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{1-x^{2}}}$.

Problems

I. Ordinary Differentiation Problems

- **Differentiate** $\mathbf{x + \frac{1}{x}}$

Solution Let $y = x + \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{d\left(x + \frac{1}{x}\right)}{dx} \quad = \frac{d(x)}{dx} + \frac{d(x^{-1})}{dx} = 1 - \frac{1}{x^2}$

- **Differentiate** $\mathbf{3\tan x + 2\cos x - e^x + 5}$

Solution:

Let $y = 3\tan x + 2\cos x - e^x + 5$

Then $\frac{dy}{dx} = \frac{d(3\tan x + 2\cos x - e^x + 5)}{dx} \quad = 3\frac{d(\tan x)}{dx} + 2\frac{d(\cos x)}{dx} - \frac{d(e^x)}{dx} + \frac{d(5)}{dx}$

$= 3sec^2x - 2\sin x - e^x$

Differentiate $\mathbf{y = e^{2x}\cos 3x}$

Solution: $\frac{dy}{dx} = \frac{d(e^{2x}\cos 3x)}{dx} \quad = \cos 3x \frac{d(e^{2x})}{dx} + e^{2x}\frac{d(\cos 3x)}{dx}$

$= 2\cos 3x\, e^{2x} - 3e^{2x}\sin 3x$

Differentiate y = $\mathbf{x^3 e^{-x}\tan x}$

Solution: $\frac{dy}{dx} = \frac{d(x^3 e^{-x}\tan x)}{dx}$

$= e^{-x}\tan x\frac{d(x^3)}{dx} + x^3\tan x\frac{d(e^{-x})}{dx} + x^3 e^{-x}$

$= 3x^2e^{-x}\tan x - x^3e^{-x}\tan x + x^3e^{-x}\sec^2 x$

Differentiate y $= \mathbf{\frac{e^x}{\cos x}}$

Solution: $\frac{dy}{dx} = \frac{d\left(\frac{e^x}{\cos x}\right)}{dx} = \frac{\cos x\, e^x - e^x\,(-\sin x)}{\cos^2 x}$

$= \frac{\cos x\, e^x + e^x\,(\sin x)}{\cos^2 x}$

Differentiate $\mathbf{y = \frac{ax+b}{cx+d}}$

Solution: $\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$ (by quotient rule)

Differentiate $\mathbf{\frac{x^2+2x+3}{\sqrt{x}}}$

Solution: $\frac{dy}{dx} = \frac{\sqrt{x}(2x+2)-(x^2+2x+3)\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2} = \frac{2\sqrt{x}(x+1)-(x^2+2x+3)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$

$= \frac{2\sqrt{x}\times 2\sqrt{x}(x+1)-(x^2+2x+3)}{2\sqrt{x}(\sqrt{x})^2} = \frac{4x(x+1)-(x^2+2x+3)}{2x^{3/2}}$

$= \frac{4x^2+4x-x^2-2x-3}{2x^{3/2}} = \frac{3x^2+2x-3}{2x^{3/2}}$

Differentiate $\mathbf{y = (3x^2-1)^3}$

Solution: Given $y = (3x^2-1)^3$

Differentiating w.r.to x, we get

$\Rightarrow \frac{dy}{dx} = 3(3x^2-1)^2 6x$

$= 3(9x^4-6x^2+1) = 27x^4 - 18x^2 + 3$

Differentiate: $\mathbf{\log\left(\frac{1+\sin x}{1-\sin x}\right)}$

Solution: Let $y = \log\left(\frac{1+\sin x}{1-\sin x}\right)$

$\Rightarrow y = \log(1+\sin x) - \log(1-\sin x)$

Differentiate y w.r.to x, we get

$\frac{dy}{dx} = \frac{1}{1+\sin x}\cos x - \frac{1}{1-\sin x}(-\cos x)$

$= \frac{(1-\sin x)\cos x+\cos x(1+\sin x)}{(1+\sin x)(1-\sin x)}$

$= \frac{\cos x-\sin x\cos x+\cos x+\cos x\sin x}{1-\sin^2 x}$

$= \frac{2\cos x}{\cos^2 x} = 2\frac{1}{\cos x} = 2\sec x$

II. Differentiation Problems on Logarithmic Functions

Differentiate $\mathbf{x^{\sin x}}$

Solution: Let $y = x^{\sin x}$

Taking log on both sides, we get $\log y = \sin x \log x$

Now differentiating with respect to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x(\cos x) + \sin x\frac{1}{x} \quad \text{(Using product rule)}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\log x(\cos x) + \sin x\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x\cos x\log x+\sin x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{x\cos x\log x+\sin x}{x}\right)$$

2. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution: Given $x^y = e^{x-y}$

Taking log on both sides, we get $\log x^y = \log$

$$\Rightarrow y\log x = (x-y)\log_e e$$

$$\Rightarrow y\log x = (x-y)\ldots\ldots\ldots(1)$$

$$\Rightarrow \frac{1}{x}y + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \log x\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}(\log x + 1) = \frac{x-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\log x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\log x}{x(1+\log x)}\ldots.(2)$$

Again from (1) $y + y\log x = x$

$$\Rightarrow y(1+\log x) = x, \frac{y}{x} = \frac{1}{1+\log x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

III. Differentiation of Implicit functions

If two variables x and y are connected by the relation f(x, y) = 0 and none of the variable is directly expressed in terms of the other, then the relation is called an implicit function.

Problems

Find $\frac{dy}{dx}$, if $x^3+y^3 = 3axy$

Solution:

Differentiating w.r.to x, we get

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$\Rightarrow \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3ay-3x^2)}{3y^2-3ax} = \frac{3(ay-x^2)}{3(y^2-ax)} = \frac{(ay-x^2)}{(y^2-ax)}$$

2. **Find $\frac{dy}{dx}$, if $x^2 + y^2 = 16$**

Solution:

Given $x^2 + y^2 = 16$

$$\Rightarrow y^2 = 16 - x^2$$

$$\Rightarrow y = \sqrt{16 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(16 - x^2)^{-1/2} \times (-2x)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{16-x^2}} = -\frac{x}{y}$$

3. **Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$**

Solution: Given $x = at^2, y = 2at$

$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

Now $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$

4. Find $\frac{dy}{dx}$, if $y^2 + x^3 - xy + \cos y = 0$

Solution:

Given $y^2 + x^3 - xy + \cos y = 0$

$\Rightarrow 2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} = 0$

$\Rightarrow (2y - \sin y)\frac{dy}{dx} + 3x^2 - \left(x\frac{dy}{dx} + y \times 1\right)$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y = 0$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} = y - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{2y-\sin y-x}$

Maxima and Minima of one Variable

If you have a differentiable function $f[x]$ to extremize over a compact interval $[a, b]$

(a) Compute $f'[x]$. (Be sure it is defined on all of $[a, b]$.)

(b) Find the critical points, that is, all solutions c of $f'[c] = 0$ with $a < c < b$.

(c) Make a table of the values $f[x]$ at $x =$ endpoints and critical points.

(d) Select the largest and smallest values of the function at the candidate points.

Find the maximum and minimum of

$$f[x] = x^3 - 6x^2 + 9x + 1$$

Solution:

First, we isolate the possible candidates. The endpoints are

$$x = 0 \qquad \text{and} \qquad x = 5$$

The interior critical points are found by first computing $f'[x]$ and then finding all solutions of the equation $f'[x] = 0$.

$$\begin{aligned} \frac{df}{dx} = f'[x] &= 3x^2 - 12x + 9 \\ &= 3(x-1)(x-3) \end{aligned}$$

The derivative is always defined, so $f[x]$ is continuous and differentiable on $[0, 5]$.

The solutions of $f'[x] = 0$ are

$$3(x-1)(x-3) = 0 \qquad \Leftrightarrow \qquad x = 1 \quad \text{or} \quad x = 3$$

on the interval $[0, 5]$.

This isolates the candidates, so we compute their values:

Candidate	Value
$x =$	$f[x] =$
0	1
1	5
3	1
5	21

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –II –FUNCTIONS OF SEVERAL VARIABLES– SMTA1105

UNIT II

FUNCTIONS OF SEVERAL VARIABLES

Partial derivatives– Euler's theorem for homogeneous functions– Jacobians Maxima and Minima for functions of several variables– Method of Lagrangian multipliers

Partial Differentiation:

Consider $z = f(x, y)$, here z is a function of two independent variables x and y. z can be differentiated with respect to x or y but when we are differentiating z with respect to x (or y) we must keep the variable y (or x) as a constant.

Notations:

Let z=f(x,y)

First order partial derivatives of f(x, y) with respect to x and y.

$\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

Second order partial derivatives of f(x, y) with respect to x and y

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

Second order mixed partial derivatives of f(x, y)

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Problems:

If $u = x^3 + y^3 + 3xy$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

Solution: Given If $u = x^3 + y^3 + 3xy$

$\frac{\partial u}{\partial x} = 3x^2 + 3y$, $\frac{\partial u}{\partial y} = 3y^2 + 3x$

2. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}$

Solution: $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3x^2 - 3yz ,$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3y^2 - 3xz,$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3z^2 - 3xy$$

Now $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3yz-3xz-3xy}{x^3 + y^3 + z^3 - 3xyz}$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$$

3. **If $f(x,y) = x^2 \sin y + y^2 \cos x$, then find its all first and 2nd order partial derivatives.**

Solution: Given $f(x,y) = x^2 \sin y + y^2 \cos x$

$f_x = 2x \sin y - y^2 \sin x$; $f_y = x^2 \cos y + 2y \cos x$.

$f_{xx} = 2 \sin y - y^2 \cos x$; $f_{yy} = -x^2 \sin y + 2 \cos x$;

$f_{xy} = 2x \cos y - 2y \sin x$; $fyx = 2x \cos y - 2y \sin x$.

4. **If $f(x,y) = \frac{y}{x} \log x$, then find its all 1st and 2nd order derivatives.**

Solution: $f_x = \frac{y}{x}\frac{1}{x} + \log x\left(\frac{-y}{x^2}\right) = \frac{y}{x^2}(1-\log x), f_y = \frac{\log x}{x}$,

$f_{xx} = \frac{y}{x^2}\left(-\frac{1}{x}\right) - \frac{2y}{x^3}(1-\log x) = \frac{y}{x^3}\left(-1-2(1-\log x)\right) = \frac{y}{x^3}(\log x - 3)$;

$f_{yy} = 0$, $f_{yx} = \frac{1}{x^2}(1-\log x)$; $f_{xy} = \frac{1}{x}\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1-\log x)$.

5. **Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = \sin(ax+by+cz)$**

Solution:

$$\frac{\partial u}{\partial x} = a\cos(ax+by+cz)$$
$$\frac{\partial u}{\partial x} = b\cos(ax+by+cz)$$
$$\frac{\partial u}{\partial x} = c\cos(ax+by+cz)$$

VI. Euler's Theorem for Homogeneous Functions

A homogenous function of degree *n* of the variables *x, y, z* is a function in which each term degree *n*. For example, the function $f(x, y, z) = Ax^3 + By^3 + Cz^3 + Dxy^2 + Exz^2 + Fyz^2 + Gy$ $Hzx^2 + Izy^2 + Jxyz$, is a homogeneous function of *x, y, z*, in which all terms are of degree th

Note:

A function f(x ,y) of two independent variables x and y is said to be homogeneous in x and degree n if $f(tx,ty) = t^n f(x,y)$ for any positive quantity t.

Euler's theorem:

1). If $f(x,y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

2). If $f(x,y,z)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

Result: If z is a homogeneous function of x, y of degree n and z=f(u) then

(i). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

1. Verify Euler's theorem when $u = x^3 + y^3 + z^3 + 3xyz$

Solution:

Given $u = x^3 + y^3 + z^3 + 3xyz$

Now $tu = (tx)^3 + (ty)^3 + (tz)^3 + 3txtytz$

$= t^3(x^3 + y^3 + z^3 + 3xyz) = t^3u$

Therefore u is a homogeneous function of degree 3.

$\frac{\partial u}{\partial x} = 3x^2 + 3yz$

$\frac{\partial u}{\partial y} = 3y^2 + 3xz$

$\frac{\partial u}{\partial z} = 3z^2 + 3xy$

Therefore $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3xz) + z(3z^2 + 3xy)$

$= 3x^3 + 3y^3 + 3z^3 + 9xyz$

$= 3(x^3 + x^3 + 3xy) = 3u$

Hence Euler's theorem is verified.

2. If $u = xlog\left(\frac{y}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Solution:

Given $u = x\log\left(\frac{y}{x}\right)$

u is a homogeneous function of degree 1.

Therefore by Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \times u = u$

3. If $(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$, then prove that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0$

Solution:

$f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$

Now $f(tx,ty)\ =\frac{1}{(tx)^2}+\frac{1}{txty}+\frac{\log tx-\log ty}{(tx)^2+(ty)^2}$

$=\ \frac{1}{t^2x^2}+\frac{1}{t^2xy}+\frac{\log\frac{tx}{ty}}{t^2(x^2+y^2)}$

$=\frac{1}{t^2}\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}\right)$

$=t^{-2}\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}\right)$

Therefore f(x, y) is a homogeneous function of degree -2

By Euler's theorem , $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-2f$

$\Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0$

4. If u = $\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x\ \frac{\partial u}{\partial x}\ +\ y\ \frac{\partial u}{\partial y}\ =\ \sin 2u$

Solution: Given u = $\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\Rightarrow \tan u\ =\left(\frac{x^3+y^3}{x-y}\right)$

Let $z=\tan u\ =\left(\frac{x^3+y^3}{x-y}\right)$

And z is a homogeneous function of order 2.

We know that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Here $f(u) = \tan u$

$\Rightarrow f'(u) = \sec^2 u$

Therefore by the result,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2\sin u \times \cos u = \sin 2u$$

(Or)

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2z$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{1}{\cos^2 u}\frac{\partial u}{\partial x} + y\frac{1}{\cos^2 u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{1}{\cos u}\frac{\partial u}{\partial x} + y\frac{1}{\cos u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{1}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u = \sin 2u.$$

Jacobians

Changing variable is something we come across very often in Integration. There are many reasons for changing variables but the main reason for changing variables is to convert the integrand into something simpler and also to transform the region into another region which is easy to work with. When we convert into a new set of variables it is not always easy to find the limits. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables. In order to change variables in an integration we will need the **Jacobian** of the transformation.

If u and v are functions of x and y, then $J(u,v)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$

If $f_1, f_2, \ldots, f_n$ are n differentiable functions of n variables $x_1, x_2, \ldots, x_n$, then the determinant

$$\begin{vmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{vmatrix}$$

is defined as the Jacobian of $f_1, f_2, \ldots, f_n$ with respect to the n variables $x_1, x_2, \ldots, x_n$ and is denoted by $\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$.

Maxima and Minima of Functions two variables

Maximum Value: A function f (x;y) is said to have a maximum value at x = a; y = b
if f (a; b) > f (a+ h ;b +k); for small and independent values of h and k; positive or negative.

Minimum Value: A function f (x ;y) is said to have a maximum value at x = a; y = b
If f (a;b) < f (a +h ; b +k); for small and independent values of h and k; positive or negative.

Extreme Value: f (a ;b) is said to be an extremum value of f (x ;y) if it is either maximum or minimum.

Working rule to find extreme values (Necessary Conditions)

Step 1: Find ∂ f /∂x and ∂ f /∂y

Step 2: Solve the equations ∂ f/ ∂x = 0 and ∂ f /∂y = 0 simultaneously.

Let the solutions be (a, b), (c,d),...

Stationary Points: The point (a,b) at which ∂ f/ ∂x = 0 and ∂ f /∂y = 0 are called stationary points of the function f(x,y)

Stationary values: The values of f(x,y) at the stationary points are called stationary values of the function f(x,y).

Note: Every extremum value is a stationary value but a stationary value need not be an extremum.

Sufficient Condition for Maxima and Minima

Let (a,b) be a stationary point.

Then if $rt - s^2 > 0$ at (a, b) and $r < 0$ ($t < 0$) then f(a ,b) is maximum value.

$rt - s^2 > 0$ at (a,b) and $r > 0$ ($t > 0$) then f(a ,b) is minimum value.

$rt - s^2$ called a saddle point of the function f(x ,y).

if $rt - s^2 = 0$, then the case is doubtful and hence further investigations are required.

Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$.

Solution: Let $f(x,y) = x^2 + y^2 + 6x + 12$

Now $p = 2x + 6, q = 2y, r = 2, s = 0$ and $t = 2$

The stationary points are given by $p = 0, q = 0$

$\Rightarrow 2x + 6 = 0$ and $2y = 0 \Rightarrow x = -3$ and $y = 0$

(-3, 0) is the stationary point

Hence f(x,y) is minimum when $x = -3$ and $y = 0$.

Examine $f(x,y) = x^3 + y^3 - 3xy$ for maximum and minimum values

Solution: Let $f(x,y) = x^3 + y^3 - 3xy$

Now $p = 3x^2 - 3y$, $q = 3y^2 - 3x$, $r = 6x$, $s = -3$ and $t = 6y$

The stationary points are given by $p = 0, q = 0$

$\Rightarrow 3x^2 - 3y = 0$ and $3y^2 - 3x = 0$

$x^2 = y$ (1) and $y^2 = x$ (2)

Substituting (2) in (1), we get $x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0$

$\Rightarrow x = 0,1$ and $y = 0,1$

Examine $f(x,y) = x^3 + y^3 - 3axy$ for maxima and minima.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$

Now $p = 3x^2 - 3ay, q = 3y^2 - 3ax, r = 6x, s = -3a$ and $t = 6y$

The stationary points are obtained by equating $p = 0$ and $q = 0$

$\Rightarrow 3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0 \Rightarrow x^2 = ay$ and $y^2 = ax$

Solving these two equations, we get (0,0) and (a,a).

Therefore the stationary points are (0,0) and (a,a)

Hence the point (a,a) is a minimum if $a > 0$ and (a,a) is a maximum if $a < 0$

Lagrange's Method of Undetermined Multipliers

The conditions for f(x,y,z) to have a maximum point or a minimum point is $du = 0$.

Therefore we get $\partial f/ \partial x .dx + \partial f /\partial y .dy + \partial f/ \partial z .dz = 0$

Multiply by λ, we get

$\lambda\, \partial g/\partial x\, .dx + \lambda\, \partial g/\partial y\, .dy + \lambda\, \partial g/\partial z\, .dz = 0$

Adding we get

$\partial f/\partial x + \lambda\, \partial g/\partial x \;\; dx + \;\; \partial f/\partial y + \lambda\, \partial g/\partial y \;\; dy + \;\; \partial f/\partial z + \lambda\, \partial g\, \partial z \;\; dz = 0$

A rectangular box open at the top is to have volume of 32 cubic ft. find the dimensions in order that the total surface area is minimum.

Solution: Given $g(x,y,z) = xyz - 32 = 0$

Let x,y,z be the dimension of rectangular box open at the top.

Total surface area (S): $f(x,y,z) = xy + 2xz + 2yz$

We define the function $F(x,y,z) = xy + 2xz + 2yz + \lambda(xyz - 32)$

At the critical points, we have $\partial f/\partial x + \lambda\, \partial g/\partial x = 0 \Rightarrow y + 2z + \lambda yz = 0$

$\partial f/\partial y + \lambda\, \partial g/\partial y = 0 \Rightarrow x + 2z + \lambda xz = 0$

$\partial f/\partial z + \lambda\, \partial g/\partial z = 0 \Rightarrow 2x + 2y + \lambda xy = 0$

$,x - y = 0 \Rightarrow x = y$

$\Rightarrow y^2 - 2yz = 0$

$\Rightarrow y(y - 2z) = 0 \Rightarrow y=0$ and $y - 2z = 0$

$\Rightarrow y = 2z$

We get $x = 4, y = 4, z = 2$.

Hence the dimensions are 4cm, 4cm and 2cm

Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.cm.

Solution: Let x, y, z be the dimensions of the rectangular box, open at the top.

Given its surface area $g(x, y, z) = xy + 2yz + 2zx - 432 = 0$

The volume is (V): $f(x,y,z) = xyz$

We define the function $F(x, y, z) = xyz + \lambda(xy + 2yz + 2zx - 432)$

At the critical points, we get $yz + \lambda(y + 2z) = 0$

$xz + \lambda(x + 2z) = 0$

$xy + \lambda(2y + 2x) = 0$

$\Rightarrow x = y$ and $y = 2z$

We get $x = 12.\ y = 12, z = 6$.

Hence the dimensions of the rectangular box are 12 cm, 12 cm and 6 cm

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – BASIC INTEGRAL CALCULUS- SMTA1105

UNIT - III

BASIC INTEGRAL CALCULUS

Review of Integration and its methods – Definite Integrals – Properties of Definite Integrals – Problems on Evaluating Definite Integrals – Beta and Gamma Functions – Relation between Beta and Gamma functions (without proof)– Properties and Simple problems.

Definite Integrals

Property 1: $\int_a^b f(x)dx = \int_a^b f(z)dz$

Property 2: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property 3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Property 5:

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & if\ f(x)\ is\ odd \\ 2\int_0^a f(x)dx & if\ f(x)\ is\ even \end{cases}$$

Problems based on definite Integrals

PROBLEM (1)

Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x)dx \qquad (1)$$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)dx \qquad (2)$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x dx + \int_0^{\frac{\pi}{2}} \log\cos x dx \qquad (\text{Since } \because \log a + \log b = \log ab)$$

$$= \int_0^{\frac{\pi}{2}} \log[\sin x \cos x]dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx \qquad \left(\because \sin x \cos x = \frac{\sin 2x}{2}\right)$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log\sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx \qquad (3)$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log\sin y dy$$

$$= \frac{1}{2}(2)\int_0^{\pi/2} \log\sin y dy$$

$$= \int_0^{\pi/2} \log\sin y dy$$

$$= \int_0^{\pi/2} \log\sin x dx \qquad - \qquad (4)$$

To evaluate $\int_0^{\frac{\pi}{2}} \log(\sin 2x)dx$

Put $\quad 2x = y, 2dx = dy$

when $\quad x = 0, y = 0$

$\quad x = \frac{\pi}{2}, y = \pi$

sub (4) in (3)

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \frac{-\pi}{2}\log 2$$

PROBLEM (2)

Evaluate $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$

$$\text{let } I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta \qquad (1)$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan\theta}\right]d\theta \qquad (2)$$

$(1) + (2) \Rightarrow$

$$2I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta + \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log\left[(1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right)\right]d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log 2 d\theta = \log 2 \int_0^{\frac{\pi}{4}} d\theta$$

$$2I = \log 2[\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$$

$$\therefore 2I = \frac{\pi}{4}\log 2$$

$$\therefore I = \frac{\pi}{8}\log 2$$

Gamma Functions:

Gamma function is defined as $\int_0^{\infty} e^{-x}.x^{n-1}dx; n > 0$ and it is denoted by $\lceil n$

(i.e) $\lceil n = \int_0^{\infty} e^{-x}x^{n-1}dx,\ n > 0$

Beta function:

Beta function is defined as $\int_0^{1} x^{m-1}.(1-x)^{n-1}dx, m > 0, n > 0$ and it in denoted by β(m, n) (i.e) $\beta(m,n)=\int_0^1 x^{m-1}.(1-x)^{n-1}.dx; m>0, n>0$

Result : 1 Recurrence formula for $\overline{|n}$

$$\overline{|(n+1)}=n\overline{|n}$$

Result : 2 $\overline{|1}=1$

Result 3: when 'n' is a positive integer, then $\overline{|n+1}=n!$

Properties of Beta function:

1) **Symmertric Property:** β(m, n) = β(n, m)

2) Transformation of Beta function:

$$\beta(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{n+m}}.dy$$

3) **Trigonometric form of Beta function:**

$$\beta(m,n)=2\int_0^{\pi/2} \sin 2^{m-1}\theta.\cos^{2n-1}\theta d\theta$$

Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Proof: W.K.T $\Gamma n = \int_{0}^{\infty} e^{-x}\cdot x^{n-1}dx$

Put $x = y^2$

$dx = 2ydy$

$$\Gamma n = \int^{\infty} e^{-y2}\cdot(y^2)^{n-1}2y\cdot dy$$

$$= 2\int_{0}^{\infty} e^{-y2}.y^{2x-2}.y^{1}dy$$

$$\Gamma n = 2\int_{0}^{\infty} e^{-y2}\cdot y^{2x-1}dy$$

Similarly $\Gamma(m) = 2\int_{0}^{\infty} e^{-x^2}\cdot x^{2m-1}\cdot dx$

$$\therefore \Gamma(m)\cdot\Gamma(n) = 2\int_0^\infty e^{-x^2}\cdot x^{2m-1}dx\cdot 2\int_0^\infty e^{-y^2}y^{2x-1}\cdot dy$$

$$= 4.\int_0^\infty\int_0^\infty e^{-(x^2-y^2)}x^{2m-1}.y^{2n-1}.dx.dy$$

Put $\quad x = r\cos\theta; \quad y = r\sin\theta$

Hence |J| = r, by change of variables (jacobian)

$$dxdy = r.dr.d\theta, where\, r = |J|\,(ie) r^2 = x^2 + y^2$$

The region of integration is the complete first quadrant.

In which r varies from 0 to ∞

θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}r^{2m+2n-2}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}\cdot|r|\cdot dr\cdot d\theta$$

$$= 4\left[\frac{1}{2}\Gamma(m+n)\right]\cdot\left[\frac{1}{2}\cdot\beta(m,n)\right]$$

Using Beta & Gamma Properties.

$$= \frac{4}{4}\left[\Gamma(m+n)\right]\cdot\beta(m,n)$$

$$= \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Result : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: W.K.T $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2n-1}\,d\theta$

Put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi} (\sin\theta)^{2\times\frac{1}{2}-1}.(\cos\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1.d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{2}} = 2\times\frac{\pi}{2} = \pi \qquad (1)$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

W.K.T $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \qquad \beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

By (1) $\pi = \dfrac{\left[\Gamma\frac{1}{2}\right]^2}{\Gamma(1)} = \dfrac{\left[\Gamma\frac{1}{2}\right]^2}{1}$

$$\left[\Gamma\frac{1}{2}\right]^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

Hence proved

Evaluate $\int_0^\infty e^{-x^2}dx$

Solution

Put $x^2 = t;\ 2xdx = dt$

$$\therefore \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t}\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sqrt{\pi}$$

PROBLEM (4)

Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^7 x dx$ using Gamma functions

Property 1: $$\int_a^b f(x)dx = \int_a^b f(z)dz$$

$$\text{Proof : L.H.S } = \int_a^b f(x)dx = [F(x)]_a^b$$

$$= F[b] - F[a]$$

$$R.H.S = \int_a^b f(z)dz = [F(Z)]_a^b$$

$$= F[b] - F[a]$$

$$\text{L.H.S } = \text{R.H.S}$$

Property 2: $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\text{Proof : L.H.S } = \int_a^b f(x)dx = [F(x)]_a^b = F[b] - F[a]$$

$$\text{R.H.S } = -\int_b^a f(x)dx = -[F(x)]_b^a$$

$$= -[F(a) - F(b)]$$

$$= [F(b) - F(a)]$$

$$\text{L.H.S } = \text{R.H.S}$$

Property 3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_a^b f(x)dx$

Proof : L.H.S $= \int_a^b f(x)dx$

$$= [F(x)]_a^b = F(b) - F(a)$$

R.H.S $= \int_a^c f(x)dx + \int_c^b f(x)dx$

$$= [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

Hence L.H.S = R.H.S

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Consider, LHS

Put $x = a - z$

$dx = -dz$

If $x = 0 \Rightarrow z = a$

$x = a \Rightarrow z = 0$

$$\int_0^a f(x)dx = \int_a^0 f(a-z)(-dz)$$

$$= -\int_a^0 f(a-z)dz$$

$$= \int_0^a f(a-z)dz \qquad \text{[by property 2]}$$

$$= \int_0^a f(a-x)dx \qquad \text{[by property 1]}$$

$$= \text{R.H.S}$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 5: $\Gamma 1 = 1$

we know that $\Gamma n = \int_0^\infty e^{-x} x^{n-1}.dx$

Put n = 1 $\quad \Gamma 1 = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty .dx$

$$= \left(\frac{e^{-\infty}}{-1}\right) - \left(\frac{e^{-0}}{-1}\right) = 0 + 1 = 1$$

$\Gamma 1 = 1$

Property 6:

β(m, n) = β(n, m)

Proof : W.K.T $\beta(m,n)=\int_0^1 x^{m-1}.(1-x)^{n-1}.dx$

W.K.T $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$\therefore \beta(m,n)=\int_0^1 (1-x)^{m-1}.[1-(1-x)]^{n-1}.dx$$

$$=\int_0^1 (1-x)^{m-1}.x^{n-1}.dx$$

$$=\int_0^1 x^{n-1}.(1-x)^{m-1}.dx$$

$\beta(m,n)=\beta(n,m)$, by definition of Beta function.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – NUMERICAL METHODS FOR SOLVING EQUATIONS**-** SMTA1105

UNIT - IV

NUMERICAL METHODS FOR SOLVING EQUATIONS

Solution of algebraic equation and transcendental equation: Regula Falsi Method, Newton Raphson Method – Solution of simultaneous linear algebraic equations: Gauss Elimination Method, Gauss Jacobi & Gauss Seidel Method.

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, $a_0 \neq 0$ is called a polynomial of degree 'n'and the polynomial $f(x) = 0$ is called an algebraic equation of n^{th} degree.If $f(x)$ contains trigonometric, logarithmic or exponential functions, then $f(x) = 0$ is called a transcendental equation. For example $x^2 + 2\sin x + e^x = 0$ is a transcendental equation.

If $f(x)$is an algebraic polynomial of degree less than or equal to 4, direct methods for finding the roots of such equation are available. But if $f(x)$is of higher degree or it involves transcendental functions, direct methods do not exist and we need to apply numerical methods to find the roots of the equation $f(x) = 0$.

Most numerical methods use iterative procedures to find an approximate root of an equation $f(x) = 0$. They require an initial guess of the root as starting value and each subsequent iteration leads closer to the actual root.

Order of convergence: For any iterative numerical method, each successive iteration gives an approximation that moves progressively closer to actual solution. This is known as convergence. Any numerical method is said have order of convergence ρ, if ρ is the largest positive number such that $|\epsilon_{n+1}| \leq k|\epsilon_n|^\rho$, where ϵ_n and ϵ_{n+1} are errors in $n^{th}$and $(n+1)^{th}$ iterations, k is a finite positive constant.

Regula-Falsi method is also known as method of false position as false position of curve is taken as initial approximation. Let $y = f(x)$ be represented by the curve AB.The real root of equation $f(x) = 0$ is α as shown in adjoining figure. The false position of curve AB is taken as chord AB and initial approximation x_0 is the point of intersection of chord

AB with x-axis. Successive approximations $x_1, x_2, \ldots$are given by point of intersection of chord $A'B, A''B, \ldots$with $x -$ axis, until the root is found to be of desired accuracy.

Now equation of chord AB in two-point form is given by:

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

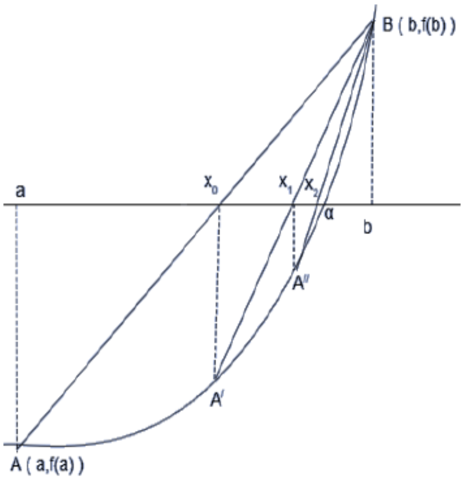
To find x_0 (point of intersection of chord AB with x-axis), put $y = 0$

$$\Rightarrow -f(a) = \frac{f(b)-f(a)}{b-a}(x_0-a)$$

$$\Rightarrow (x_0 - a) = \frac{-(b-a)f(a)}{f(b)-f(a)}$$

$$\Rightarrow x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$$

Repeat the procedure until the root is found to the desired accuracy.



Remarks:

- Rate of convergence is much faster than that of bisection method.
- Unlike bisection method, one end point will converge to the actual root α , whereas the other end point always remains fixed. As a result Regula- Falsi method has linear convergence.

Example5 Apply Regula-Falsi method to find a root of the equation $x^3 + x - 1 = 0$ correct to two decimal places.

Solution: $f(x) = x^3 + x - 1$

Here $f(0) = -1$ and $f(1) = 1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$is continuous in [0,1], $\therefore$ atleast one root exists in [0,1]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$; $a = 0, b = 1$

$$\Rightarrow x_0 = 0 - \frac{(1-0)}{f(1)-f(0)}f(0) = 0 - \frac{1}{1-(-1)}(-1) = 0.5$$

$f(0.5) = -0.375,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$$x_1 = 0.5 - \frac{(1-0.5)}{f(1)-f(0.5)}f(0.5) = 0 - \frac{0.5}{1-(-0.375)}(-0.375) = 0.636$$

$f(0.636) = -0.107,\ f(0.636).f(1) < 0$

Second approximation: $a = 0.636, b = 1$

$$x_2 = 0.636 - \frac{(1-0.636)}{f(1)-f(0.636)}f(0.636) = 0.636 - \frac{0.364}{1-(-0.107)}(-0.107) = 0.6711$$

$f(0.6711) = -0.0267,\ f(0.6711).f(1) < 0$

Third approximation: $a = 0.6711, b = 1$

$$x_3 = .6711 - \frac{(1-0.6711)}{f(1)-f(0.6711)} f(.6711) = .6711 - \frac{0.3289}{1-(-0.0267)}(-.0267) = 0.6796$$

First 2 decimal places have been stabilized; hence 0.6796 is the real root correct to two decimal places.

Example6 Use Regula-Falsi method to find a root of the equation $x \log_{10} x - 1.2 = 0$ correct to two decimal places.

Solution: $f(x) = x \log_{10} x - 1.2$

Here $f(2) = -0.5979$ and $f(3) = 0.2314 \Rightarrow f(2).f(3) < 0$

Also $f(x)$ is continuous in $[2,3]$, $\therefore$ atleast one root exists in $[2,3]$

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 2, b = 3$

$$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)} f(2) = 2 - \frac{1}{0.2314-(-0.5979)}(-0.5979) = 2.721$$

$f(2.721) = -0.0171,\ f(2.721).f(3) < 0$

First approximation: $a = 2.721, b = 3$

$$x_1 = 2.721 - \frac{(3-2.721)}{f(3)-f(2.721)} f(2.721) = 2.721 - \frac{0.279}{.2314-(-0.0171)}(-0.0171) = 2.7402$$

$f(2.7402) = -0.0004,\ f(2.7402).f(3) < 0$

Second approximation: $a = 2.7402, b = 3$

$$x_2 = 2.7402 - \frac{(3-2.7402)}{f(3)-f(2.7402)} f(2.7402) = 2.7402 - \frac{0.2598}{.2314-(-.0004)}(-.0004) = 2.7407$$

First two decimal places have been stabilized; hence 2.7407 is the real root correct to two decimal places.

Example7 Use Regula-Falsi method to find a root of the equation $\tan x + \tanh x = 0$ upto three iterations only.

Solution: $f(x) = \tan x + \tanh x$

Here $f(2) = -1.2210$ and $f(3) = 0.8525 \Rightarrow f(2).f(3) < 0$

Also $f(x)$is continuous in $[2,3]$, $\therefore$ atleast one root exists in $[2,3]$

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 2, b = 3$

$$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)} f(2) = 2 - \frac{1}{0.8525-(-1.221)}(-1.221) = 2.5889$$

$f(2.5889) = 0.3720,\ f(2).f(2.5889) < 0$

First approximation: $a = 2, b = 2.5889$

$$x_1 = 2 - \frac{(2.5889-2)}{f(2.5889)-f(2)} f(2) = 2 - \frac{0.5889}{0.3720-(-1.2210)}(-1.2210) = 2.4514$$

$f(2.4514) = 0.1596,\ f(2).f(2.4514) < 0$

Second approximation: $a = 2, b = 2.4514$

$$x_2 = 2 - \frac{(2.4514-2)}{f(2.4514)-f(2)} f(2) = 2 - \frac{0.4514}{0.1596-(-1.2210)}(-1.2210) = 2.3992$$

$f(2.3992) = 0.0662,\ f(2).f(2.3992) < 0$

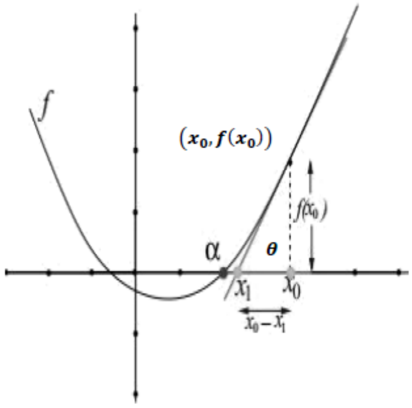
Third approximation: $a = 2, b = 2.3992$

$$x_2 = 2 - \frac{(2.3992-2)}{f(2.3992)-f(2)} f(2) = 2 - \frac{0.3992}{0.0662-(-1.2210)}(-1.2210) = 2.3787$$

∴Real root of the equationtan $x + \tanh x\ = 0$ after three iterations is 2.3787

Newton-Raphson method named after Isaac Newton and Joseph Raphson is a powerful technique for solving equations numerically. The Newton-Raphson method in one variable is implemented as follows:

Let α be an exact root and x_0 be the initial approximate root of the equation $f(x) = 0$. First approximation x_1 is taken by drawing a tangent to curve $y = f(x)$ at the point $(x_0, f(x_0))$. If θ is the angle which tangent through the point $(x_0, f(x_0))$ makes with x- axis, then slope of the tangent is given by:



$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

The required root to desired accuracy is obtained by drawing tangents to the curve at points $(x_n, f(x_n))$ successively.

$$\therefore\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 9 Use Newton-Raphson method to find a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places.

Solution: $f(x) = x^3 - 5x + 3$

$\Rightarrow f'(x) = 3x^2 - 5$

Here $f(0) = 3$ and $f(1) = -1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: Let initial approximation (x_0) in the interval $[0,1]$ be 0.8

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 0.8$, $f(0.8) = -0.488$, $f'(0.8) = -3.08$

$\Rightarrow x_1 = 0.8 - \frac{-0.488}{-3.08} = 0.6416$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 0.6415$, $f(0.6416) = 0.0561$, $f'(0.6416) = -3.7650$

$$\Rightarrow x_2 = 0.6416 - \frac{0.05611}{-3.7650} = 0.6565$$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 0.6565$, $f(0.6565) = 0.0004$, $f'(0.6565) = -3.7070$

$\Rightarrow x_3 = 0.6565 - \frac{0.0004}{-3.7070} = 0.6566$

Hence a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places is 0.6566

Example 10 Find the approximate value of $\sqrt{28}$ correct to 3 decimal places using Newton Raphson method.

Solution: $x = \sqrt{28} \quad \Rightarrow x^2 - 28 = 0$

$\therefore f(x) = x^2 - 28$

$\Rightarrow f'(x) = 2x$

Here $f(5) = -3$ and $f(6) = 8 \Rightarrow f(5).f(6) < 0$

Also $f(x)$ is continuous in $[5,6]$, $\therefore$ atleast one root exists in $[5,6]$

Initial approximation: Let initial approximation (x_0) in the interval $[5,6]$ be 5.5

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 5.5$, $f(5.5) = 2.25$, $f'(5.5) = 11$

$$\Rightarrow x_1 = 5.5 - \frac{2.25}{11} = 5.2955$$

Second approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \text{ where } x_1 = 5.2955, f(5.2955) = 0.0423, f'(5.2955) = 10.591$$

$$\Rightarrow x_2 = 5.2955 - \frac{0.0423}{10.591} = 5.2915$$

Third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \text{ where } x_2 = 5.2915, f(5.2915) = -0.00003, f'(5.2915) = 10.583$$

$$\Rightarrow x_3 = 5.2915 - \frac{-0.00003}{10.583} = 5.2915$$

Hence value of $\sqrt{28}$ correct to three decimal places is 5.2915

Consider a system of linear equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \ldots ①$$

We have been using direct methods for solving a system of linear equations. Direct methods produce exact solution after a finite number of steps whereas iterative methods give a sequence of approximate solutions until solution is obtained up to desired accuracy. Common iterative methods for solving a system of linear equations are:

1. Gauss-Jacobi’s iteration method
2. Gauss-Seidal’s iteration method

Example 14 Solve the following system of equations using Gauss Jacobi's method

$$5x - 2y + 3z = -1$$

$$-3x + 9y + z = 2$$

$$2x - y - 7z = 3$$

$$z_2 = \frac{1}{7}(-3 + 2(0.146) - 0.319) = -0.432$$

Third Approximation:

$$x_3 = \frac{1}{5}(-1 + 2y_2 - 3z_2)\ ,\ y_3 = \frac{1}{9}(2 + 3x_3 - z_2)\ ,\ z_3 = \frac{1}{7}(-3 + 2x_3 - y_3)$$

$$\Rightarrow x_3 = \frac{1}{5}(-1 + 2(0.319) - 3(-0.432)) = 0.187$$

$$y_3 = \frac{1}{9}(2 + 3(0.187) + 0.432) = 0.333$$

$$z_3 = \frac{1}{7}(-3 + 2(0.187) - 0.333) = -0.423$$

$$z_2 = \frac{1}{7}(-3 + 2(0.146) - 0.319) = -0.432$$

Third Approximation:

$$x_3 = \frac{1}{5}(-1 + 2y_2 - 3z_2)\ ,\ y_3 = \frac{1}{9}(2 + 3x_3 - z_2)\ ,\ z_3 = \frac{1}{7}(-3 + 2x_3 - y_3)$$

$$\Rightarrow x_3 = \frac{1}{5}(-1 + 2(0.319) - 3(-0.432)) = 0.187$$

$$y_3 = \frac{1}{9}(2 + 3(0.187) + 0.432) = 0.333$$

$$z_3 = \frac{1}{7}(-3 + 2(0.187) - 0.333) = -0.423$$

Solution: The given system of equations is satisfying rules of partial pivoting.

Using Gauss Seidal's approximations, system can be rewritten as

$$x_{n+1} = \frac{1}{5}(-1 + 2y_n - 3z_n)$$

$$y_{n+1} = \frac{1}{9}(2 + 3x_{n+1} - z_n)$$

$$z_{n+1} = \frac{1}{7}(-3 + 2x_{n+1} - y_{n+1})$$

Taking $x_0 = y_0 = z_0 = 0$ as initial approximation

First Approximation:

$$x_1 = -\frac{1}{5} = -0.2\ y_1 = \frac{2}{9} = 0.222\ ,\ z_1 = -\frac{3}{7} = -0.429$$

Second Approximation:

$$x_2 = \frac{1}{5}(-1 + 2y_1 - 3z_1)\ ,\ y_2 = \frac{1}{9}(2 + 3x_2 - z_1)\ ,\ z_2 = \frac{1}{7}(-3 + 2x_2 - y_2)$$

$$\Rightarrow x_2 = \frac{1}{5}(-1 + 2(0.222) - 3(-0.429)) = 0.146$$

$$y_2 = \frac{1}{9}(2 + 3(0.146) + 0.429) = 0.319$$

$$z_2 = \frac{1}{7}(-3 + 2(0.146) - 0.319) = -0.432$$

Third Approximation:

$$x_3 = \frac{1}{5}(-1 + 2y_2 - 3z_2)\ ,\ y_3 = \frac{1}{9}(2 + 3x_3 - z_2)\ ,\ z_3 = \frac{1}{7}(-3 + 2x_3 - y_3)$$

$$\Rightarrow x_3 = \frac{1}{5}(-1 + 2(0.319) - 3(-0.432)) = 0.187$$

$$y_3 = \frac{1}{9}(2 + 3(0.187) + 0.432) = 0.333$$

$$z_3 = \frac{1}{7}(-3 + 2(0.187) - 0.333) = -0.423$$

Fourth Approximation:

$$x_4 = \frac{1}{5}(-1 + 2y_3 - 3z_3)\ ,\ y_4 = \frac{1}{9}(2 + 3x_4 - z_3)\ ,\ z_4 = \frac{1}{7}(-3 + 2x_4 - y_4)$$

$$\Rightarrow x_4 = \frac{1}{5}(-1 + 2(0.333) - 3(-0.423)) = 0.187$$

$$y_4 = \frac{1}{9}(2 + 3(0.187) + 0.423) = 0.332$$

$$z_4 = \frac{1}{7}(-3 + 2(0.187) - 0.332) = -0.423$$

Values of variables have been stabilized, $\therefore$ approximate solution is given by

$x = 0.187$, $y = 0.332$ and $z = -0.423$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – INTERPOLATION NUMERICAL DIFFERENTION AND INTEGRATION - SMTA1105

UNIT - V

INTERPOLATION NUMERICAL DIFFERENTIATION AND INTEGRATION

Interpolation: Newton forward and backward interpolation formula, Lagrange’s formula for unequal intervals – Numerical differentiation: Newton’s forward and backward differences to compute first and second derivatives – Numerical integration: Trapezoidal rule, Simpson’s 1/3rd rule and Simpson’s 3/8th rule.

Differentiation and integration are basic mathematical operations with a wide range of applications in various fields of science and engineering. Simple continuous algebraic or transcendental functions can be easily differentiated or integrated directly. However at times there are complicated continuous functions which are tedious to differentiate or integrate directly or in the case of experimental data, where tabulated values of variables are given in discrete form, direct methods of calculus are not applicable.

Newton’s forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots, \quad p = \frac{x-x_0}{h} \qquad \ldots ①$$

Newton’s backward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots, \quad p = \frac{x-x_n}{h} \qquad \ldots ①$$

Example 1 Given a cubic polynomial with following data points

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{f(x)}$	5	6	3	8

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0$

Solution: Derivative has to be evaluated near the starting of the table, thereby constructing forward difference table for the function $y = f(x)$

x	y	Δ	Δ^2	Δ^3
0	5			
		1		
1	6		-4	
		-3		12
2	3		8	
		5		
3	8			

To find the derivative at $x = 0$, taking $x_0 = 0$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \quad \ldots (1)$$

From table $h = 1$, $\Delta y_0 = 1$, $\Delta^2 y_0 = -4$, $\Delta^3 y_0 = 12$, $\Delta^4 y_0 = 0$

Substituting these values in (1), we get

$$\left.\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[1 - \frac{(-4)}{2} + \frac{12}{3} + 0\right] = 7$$

Also $\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{1^2}[-4 - 12 + 0] = -16$$

Example 2 Given a polynomial with following data points:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$ and $x = 1.5$

Solution: Derivatives has to be evaluated near the starting as well as towards the end of the table, thereby constructing difference table for the function $y = f(x)$

x	$y = f(x)$	1^{st}diff	2^{nd}diff	3^{rd}diff	4^{th}diff	5^{th}diff	6^{th}diff
1.0	7.989						
		0.414					
1.1	8.403		−0.036				
		0.378		0.006			
1.2	8.781		−0.030		−0.002		
		0.348		0.004		0.001	
1.3	9.129		−0.026		−0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		−0.023		0.002		
		0.299		0.005			
1.5	9.750		−0.018				
		0.281					
1.6	10.031						

To find the derivative at $x = 1.1$, taking $x_0 = 1.1$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} - \cdots\right] \quad \ldots ①$$

From table $h = 0.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$, $\Delta^3 y_0 = 0.004$, $\Delta^4 y_0 = -0.001$, $\Delta^5 y_0 = 0.003$

Substituting these values in ①, we get

$$\left.\frac{dy}{dx}\right]_{x=1.1} = \frac{1}{0.1}\left[0.378 - \frac{(-0.03)}{2} + \frac{0.004}{3} - \frac{(-0.001)}{4} + \frac{0.003}{5}\right] = 3.952$$

Also $\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{(0.1)^2}\left[-0.03 - 0.004 + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

To find the derivative at $x = 1.5$, taking $x_n = 1.5$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \frac{\nabla^4 y_n}{4} + \cdots\right] \qquad \ldots ②$$

From table $h = 0.1$, $\nabla y_n = 0.299$, $\nabla^2 y_n = -0.023$, $\nabla^3 y_n = 0.003$, $\nabla^4 y_n = -0.001$, $\nabla^5 y_n = 0.001$

Substituting these values in ②, we get

$$\left.\frac{dy}{dx}\right]_{x=1.5} = \frac{1}{0.1}\left[0.299 + \frac{(-0.023)}{2} + \frac{0.003}{3} + \frac{(-0.001)}{4} + \frac{0.001}{5}\right] = 2.8845$$

Also $\left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=n} = \frac{1}{(0.1)^2}\left[-0.023 + 0.003 + \frac{11}{12}(-0.001) + \frac{5}{6}(0.001)\right] = -2.0083$$

Example 8 From the following table, find x for which y is maximum.

x	3	4	5	6	7	8
$f(x)$	0.205	0.240	0.259	0.262	0.250	0.224

Also find maximum value of y.

Solution: Constructing forward difference table for the function $y = f(x)$, upto third differences

x	$y = f(x)$	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		-0.016	
		0.019		0
5	0.259		-0.016	
		0.003		0.001
6	0.262		-0.015	
		-0.012		0.001
7	0.250		-0.014	
		-0.026		
8	0.224			

Newton's forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots, \quad p = \frac{x - x_0}{h} \qquad \ldots ①$$

Taking $x_0 = 3$, $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$, $\Delta^3 y_0 = 0$

Substituting these values in ①, we get

$$y \equiv (0.205) + p(0.035) + \frac{p(p-1)}{2}(-0.016) + 0 \quad \ldots ②$$

Differentiating with respect to p, we get

$$\frac{dy}{dp} = 0.035 + \frac{2p-1}{2}(-0.016) = 0.035 - (0.008)(2p-1)$$

For y to be maximum, $\frac{dy}{dp} = 0$

$\Rightarrow 0.035 - (0.008)(2p-1) = 0$

$\Rightarrow p = 2.6875$

Also $p = \frac{x-x_0}{h}$ or $x = x_0 + ph$

$\Rightarrow x = 3 + 2.6875(1) = 5.6875$

$\therefore$ y is maximum when $x = 5.6875$ or $p = 2.6875$

Substituting in ② , maximum value of y is given by

$$y \equiv (0.205) + (2.6875)(0.035) + \frac{(2.6875)(2.6875-1)}{2}(-0.016) = 0.2628$$

Numerical Integration is the process of computing the value of definite integral $\int_a^b y\,dx$, when the integrand function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0,1,2,3, \ldots, n$. As in case of numerical differentiation, here also the integrand $y = f(x)$ is first replaced with an interpolating polynomial, and then it is integrated to compute the value of the definite integral. This gives us 'quadrature formula' for numerical integration.

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

This is known as trapezoidal rule to evaluate $\int_a^b f(x)dx$, where the function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0,1,2,3, \ldots, n$.

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$
$$+ \cdots + \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1})$$
$$+2(y_3 + y_6 + \cdots + y_{n-3})]$$

This is known as Simpson's three-eighths rule to evaluate $\int_a^b f(x)dx$, where the function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0,1,2,3,\ldots.,n$.

Example9 Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using

i. Trapezoidal rule taking $h = \frac{1}{5}$
ii. Simpson's $\frac{1}{3}$ rule taking $h = \frac{1}{4}$
iii. Simpson's $\frac{3}{8}$ rule taking $h = \frac{1}{6}$

x	0	0.2	0.4	0.6	0.8	1
$y = f(x)$	1	0.96	0.86	0.74	0.61	0.5

By trapezoidal rule $\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5]$
$$= \frac{0.2}{2}[1 + 2(0.96 + 0.86 + 0.74 + 0.61) + 0.5]$$

$\therefore$ $\int_0^1 \frac{1}{1+x^2}dx = 0.784$ using trapezoidal rule.

ii. To solve $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's $\frac{1}{3}$ rule

Taking $h = \frac{1}{4} = 0.25$, $n = \frac{b-a}{h} = \frac{1-0}{0.25} = 4$

$\therefore$ Dividing the interval (0,1) into 4 equal parts for the function $f(x) = \frac{1}{1+x^2}$

x	0	0.25	0.5	0.75	1
$y = f(x)$	1	0.94	0.8	0.64	0.5

By Simpson's $\frac{1}{3}$ rule $\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$
$$= \frac{0.25}{3}[(1 + 0.5) + 4(0.94 + 0.64) + 2(0.8)]$$

Solution: i. To solve $\int_0^1 \frac{1}{1+x^2}dx$ using trapezoidal rule

Taking $h = \frac{1}{5} = 0.2$, $n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$

$\therefore$ Dividing the interval $(0,1)$ into 5 equal parts for the function $f(x) = \frac{1}{1+x^2}$

Example13 From the following table, find the area bounded by the curve and $x -$ axis, between the ordinates $x = 7.47$ to $x = 7.52$.

x	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution: As $n = 5$, Simpson's $\frac{1}{3}$ rule Simpson's $\frac{3}{8}$ rules are not applicable. Applying trapezoidal rule with $h = 0.01$

$$\int_{7.47}^{7.52} f(x)dx = \frac{.01}{2}[1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06]$$
$$= 0.005[19.93] = 0.09965 \quad \text{square units}$$

Example14 The velocity v of an airplane which starts from rest is given at fixed intervals of time t as shown:

t (minutes)	2	4	6	8	10	12	14	16	18	20
$v = f(t)$ (km/minutes)	8	17	24	28	30	20	12	6	2	0

Estimate the approximate distance covered in 20 minutes.

Solution: Since the airplane starts from rest, its initial velocity is zero. So the time/velocity relationship may be tabulated as:

t (minutes) 0 2 4 6 8 10 12 14 16 18 20