SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Mathematics in Design – SMTA1103

UNIT1- MATHEMATICS IN DESIGN

Golden Ratio

The ratio of two parts of a line such that the longer part divided by the smaller part is also equal to the whole length divided by the longer part.

Results: i) The ratio of the circumference of a circle to its diameter is π . ii) The ratio of the width of a rectangular picture frame to its height is not necessarily equal to he golden ratio, though some artists and architects believe a frame made using the golden ratio makes the most pleasing and beautiful shape.iii) The ratio of two successive numbers of the Fibonacci sequence give an approximate value of the golden ratio,the bigger the pair of Fibonacci Numbers, the closer the approximation.

Example1:

The Golden Ratio = 1.61803398874989484820... = 1.618 correct to 3 decimal places.

If x_n are terms of the Fibonacci sequence, then what is the least value of n for which

($x_{n+1} / x_n = 1.618$ correct to 3 decimal places.

Example2:

A rectangle divided into a square and a smaller rectangle. If (x + y) /y = x / y ,find the value of golden ratio.

Example3:

The Golden Ratio = 1.61803398874989484820... = 1.6180 correct to 4 decimal places. If x_n are terms of the Fibonacci sequence, then what is the least value of n for which $(x_{n+1}) / x_n = 1.6180$ correct to 4 decimal places.

Example4:
Obtain golden ratio in a line segment.

Proportion

Proportion is an equation which defines that the two given ratios are equivalent to each other. For example, the time taken by train to cover 100km per hour is equal to the time taken by it to cover the distance of 500km for 5 hours. Such as100km/hr = 500km/5hrs.

Proportion is an equation which defines that the two given ratios are equivalent to each other. In other words, the proportion states the equality of the two fractions or the ratios. In proportion, if two sets of given numbers are increasing or decreasing in the same ratio, then the ratios are said to be directly proportional to each other.

Ratio and proportions are said to be faces of the same coin. When two ratios are equal in value, then they are said to be in proportion**.** In simple words, it compares two ratios.

Ratio Formula

Assume that, we have two quantities (or two numbers or two entities) and we have to find the ratio of these two, then the formula for ratio is defined as;

a: b ⇒ a/b

where a and b could be any two quantities.

Here, “a” is called the first term or **antecedent**, and “b” is called the second term or **consequent**.

Example: In ratio 4:9, is represented by 4/9, where 4 is antecedent and 9 is consequent.

If we multiply and divide each term of ratio by the same number (non-zero), it doesn’t affect the ratio.

Example: 4:9 = 8:18 = 12:27

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Golden ratio and Beauty

The Golden Ratio (or "Golden Section") is based on **Fibonacci Numbers**, where every number in the sequence (after the second) is the sum of the previous 2 numbers:

1, 1, 2, 3, 5, 8, 13, 21, ...

Two quantities *a* and *b* are said to be in the *golden ratio* φ if

(a + b) / b = a / b

Definition: A ***golden rectangle*** is a rectangle that can be cut up into a square and a rectangle similar to the original one. More precisely,

Let ABCD be a rectangle, with width AB < length BC. Then there is a point E on segment AD and a point F on segment BC so that BFEA is a square. i.e., AE = BF = AB (and consequently EF = AB also). Then FCDE is a rectangle. We say that ABCD is a ***golden rectangle*** if FCDE is similar to ABCD.

Spiraling Golden Rectangles

Since the sub-rectangle FDCE can also be divided into a square and a still smaller golden rectangle, one can continue this process to get an infinite number of smaller and smaller squares spiraling inside the original golden rectangle.

Golden triangle, pentagon and pentagram

Golden triangle

The golden triangle can be characterized as an isosceles triangle ABC with the property that bisecting the angle C produces a new triangle CXB which is a similar triangle to the original.

If angle BCX = α, then XCA = α because of the bisection, and CAB = α because of the similar triangles; ABC = 2α from the original isosceles symmetry, and BXC = 2α by similarity. The angles in a triangle add up to 180°, so 5α = 180, giving α = 36°. So the angles of the golden triangle are thus 36°-72°-72°. The angles of the remaining obtuse isosceles triangle AXC (sometimes called the golden gnomon) are 36°-36°-108°.

Suppose XB has length 1, and we call BC length φ. Because of the isosceles triangles XC=XA and BC=XC, so these are also length φ. Length AC = AB, therefore equals $\varphi + 1$. But triangle ABC is similar to triangle CXB, so AC/BC = BC/BX, $AC/\varphi = \varphi/1$, and so AC also equals φ^2. Thus $\varphi^2 = \varphi + 1$, confirming that φ is indeed the golden ratio.

Similarly, the ratio of the area of the larger triangle AXC to the smaller CXB is equal to φ, while theinverse ratio is $\varphi - 1$.

Pentagon

In a regular pentagon the ratio of a diagonal to a side is the golden ratio, while intersecting diagonals section each other in the golden ratio.

Relationship to Fibonacci sequence

The mathematics of the golden ratio and of the Fibonacci sequence are intimately interconnected. The Fibonacci sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

In geometry, the **golden angle** is the smaller of the two angles created by sectioning the circumference of a circle according to the golden ratio; that is, into two arcs such that the ratio of the length of the smaller arc to the length of the larger arc is the same as the ratio of the length of the larger arc to the full circumference of the circle.

Algebraically, let *a+b* be the circumference of a circle, divided into a longer arc of length *a* and a smaller arc of length *b* such that (a + b) /b = a/b.

Golden Ratio in Nature

The Golden Ratio has made an appearance in many notable and obvious items in nature, including trees, pine cones, and the seeds on a strawberry.

However, it's also seen in largely abstract places, like the point in a blak hole where the heat changes from positive to negative. Its consistent presence could signify the Golden Ratio as a fundamental constant of nature -- which might explain why our brains seem hard-wired to respond better to visuals that follow the Golden Ratio.

Symmetries

The golden ratio and inverse golden ratio have a set of symmetries that preserve and interrelate them. They are both preserved by the fractional linear transformations x , $1/(1-x)$, $(1-x)/x$ this fact corresponds to the identity and the definition quadratic equation. Further, they are interchanged by the three maps 1/x, 1-x, x/(1-x) they are reciprocals, symmetric about 1/2, and symmetric about 2.

Algebraic relations and its application in Egyptian pyramids:

Euclidean geometry

The study of plane and solid figures on the basis of axioms and theorems employed by the Greek mathematician Euclid . In its rough outline, Euclidean geometry is the plane and solid geometry commonly taught in secondary schools. Indeed, until the second half of the 19th century, when non-Euclidean geometries attracted the attention of mathematicians, *geometry* meant Euclidean geometry. It is the most typical expression of general mathematical thinking. Rather than the memorization of simple algorithms to solve equations by rote, it demands true insight into the subject, clever ideas for applying theorems in special situations, an ability to generalize from known facts, and an insistence on the importance of proof. In Euclid's great work, the elements, the only tools employed for geometrical constructions were the ruler and the compass—a restriction retained in elementary Euclidean geometry to this day.

Similarity of triangles

As indicated above, congruent figures have the same shape and size. Similar figures, on the other hand, have the same shape but may differ in size. Shape is intimately related to the notion of proportion, as ancient Egyptian artisans observed long ago. Segments of lengths a, b, c, and d are said to be proportional if $a:b = c:d$ (read, a is to b as c is to d; in older notation $a:b::c:d$). The fundamental theorem of similarity states that a line segment splits two sides of a triangle into proportional segments if and only if the segment is parallel to the triangle's third side.

Fundamental theorem of similarity

The formula in the figure reads k is to l as m is to n if and only if line DE is parallel to line AB. This theorem then enables one to show that the small and large triangles are similar. The similarity theorem may be reformulated as the AAA (angle-angle-angle) similarity theorem: two triangles have their corresponding angles equal if and only if their corresponding sides are proportional. Two similar triangles are related by a scaling (or similarity) factor s: if the first triangle has sides a, b, and c, then the second one will have sides sa, sb, and sc. In addition to the ubiquitous use of scaling factors on construction plans and geographic maps, similarity is fundamental to trigonometry.

Areas

Just as a segment can be measured by comparing it with a unit segment, the area of a polygon or other plane figure can be measured by comparing it with a unit square. The common formulas for calculating areas reduce this kind of measurement to the measurement of certain suitable lengths. The simplest case is a rectangle with sides a and b, which has area ab. By putting a triangle into an appropriate rectangle, one can show that the area of the triangle is half the product of the length of one of its bases and its corresponding height—$bh/2$. One can then compute the area of a general polygon by dissecting it into triangular regions. If a triangle (or more general figure) has area A, a similar triangle (or figure) with a scaling factor of s will have an area of s^2A.**area of a triangle**

Pythagorean theorem

For a triangle $\triangle ABC$ the Pythagorean theorem has two parts: (1) if $\angle ACB$ is a right angle, then $a^2 + b^2 = c^2$; (2) if $a^2 + b^2 = c^2$, then $\angle ACB$ is a right angle. For an arbitrary triangle, the Pythagorean theorem is generalized to the law of cosines: $a^2 + b^2 = c^2 - 2ab \cos (\angle ACB)$. When $\angle ACB$ is 90 degrees, this reduces to the Pythagorean theorem because $\cos (90°) = 0$.

Since Euclid, a host of professional and amateur mathematicians (even U.S. President James garfield have found more than 300 distinct proofs of the Pythagorean theorem. Despite its antiquity, it remains one of the most important theorems in Mathematics. It enables one to calculate distances or, more important, to define distances in situations far more general than elementary geometry. For example, it has been generalized to multidimensional vector space.

Fibonacci Sequence:

There is a special relationship between the Golden Ratio and the Fibonacci sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

(The next number is found by adding up the two numbers before it.)

And here is a surprise: when we take any two successive *(one after the other)* Fibonacci Numbers, **their ratio is very close to the Golden Ratio**.

In fact, the bigger the pair of Fibonacci Numbers, the closer the approximation. Let us try a few:

A	B	B/A

2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
...	...	...
144	233	1.618055556...
233	377	1.618025751...
...	...	...

We don't have to start with **2 and 3**, here I randomly chose **192 and 16** (and got the sequence *192, 16, 208, 224, 432, 656, 1088, 1744, 2832, 4576, 7408, 11984, 19392, 31376, ...*):

A	**B**	**B / A**
192	**16**	0.08333333...
16	208	13
208	224	1.07692308...
224	432	1.92857143...
...	...	...
7408	11984	1.61771058...
11984	19392	1.61815754...
...	...	...

The Fibonacci Sequence is the series of numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The next number is found by adding up the two numbers before it:

- the 2 is found by adding the two numbers before it (1+1),
- the 3 is found by adding the two numbers before it (1+2),
- the 5 is (2+3),
- and so on!

Example: the next number in the sequence above is 21+34 = **55**

Here is a longer list:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, ...

When we make squares with those widths, we get a nice spiral:

Do you see how the squares fit neatly together? For example 5 and 8 make 13, 8 and 13 make 21, and so on.

This spiral is found in nature
See: Nature, the golden ratio and Fibonacci

The Fibonacci Sequence can be written as a "Rule"

First, the terms are numbered from 0 onwards like this:

n =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*...*
x_n =	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	...

So term number 6 is called x_6 (which equals 8).

Example1:

the **8th** term is the **7th** term plus the **6th** term:

$x_8 = x_7 + x_6$

So we can write the rule $\mathbf{x_n = x_{n-1} + x_{n-2}}$

where: $\mathbf{x_n}$ is term number n , $\mathbf{x_{n-1}}$ is the previous term (n−1) and $\mathbf{x_{n-2}}$ is the term before that (n−2)

Example2:

term 9 is calculated like this:

$x_9 = x_{9-1} + x_{9-2}$

$= x_8 + x_7$

$= 21 + 13$

$= 34$

Golden ratio

And here is a surprise. When we take any two successive *(one after the other)* Fibonacci Numbers, their ratio is very close to the Golden ratio "**φ**" which is approximately **1.618034...**

In fact, the bigger the pair of Fibonacci Numbers, the closer the approximation. Let us try a few:

A	B	B / A
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
...	...	...
144	233	1.618055556...
233	377	1.618025751...
...	...	...

We don't have to start with **2 and 3**, here I randomly chose **192 and 16** (and got the sequence *192, 16, 208, 224, 432, 656, 1088, 1744, 2832, 4576, 7408, 11984, 19392, 31376, ...*):

A	B	B / A
192	**16**	0.08333333...
16	208	13
208	224	1.07692308...
224	432	1.92857143...
...	...	...
7408	11984	1.61771058...
11984	19392	1.61815754...
...	...	...

It takes longer to get good values, but it shows that not just the Fibonacci Sequence can do this!

Using The Golden Ratio to Calculate Fibonacci Numbers

And even more surprising is that we can **calculate any Fibonacci Number** using the Golden Ratio:

$$x_n = (\varphi^n - (1-\varphi)^n)/\sqrt{5}$$

The answer comes out **as a whole number**, exactly equal to the addition of the previous two terms.

Example: x_6

$x_6 = [\ (1.618034...)^6 - (1-1.618034...)^6]/\sqrt{5}$

When I used a calculator on this (only entering the Golden Ratio to 6 decimal places) I got the answer **8.00000033** , a more accurate calculation would be closer to 8.

Try n=12 and see what you get.

You can also calculate a Fibonacci Number by multiplying the previous Fibonacci Number by the Golden Ratio and then rounding (works for numbers above 1):

Example: $8 \times \varphi = 8 \times 1.618034... = 12.94427... = 13$ (rounded)

Here is the Fibonacci sequence again:

n =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*...*
x_n =	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	...

There is an interesting pattern:

- Look at the number $\mathbf{x_3 = 2}$. Every **3**rd number is a multiple of **2** (2, 8, 34, 144, 610, ...)
- Look at the number $\mathbf{x_4 = 3}$. Every **4**th number is a multiple of **3** (3, 21, 144, ...)
- Look at the number $\mathbf{x_5 = 5}$. Every **5**th number is a multiple of **5** (5, 55, 610, ...)

And so on (every **n**th number is a multiple of $\mathbf{x_n}$).

1/89 = 0.011235955056179775...

Notice the first few digits (0,1,1,2,3,5) are the Fibonacci sequence?

In a way they **all** are, except multiple digit numbers (13, 21, etc) **overlap**, like this:

0.0

0.01

0.001

0.0002

0.00003

0.000005

0.0000008

0.00000013

0.000000021

... etc ...

0.011235955056179775... = 1/89

Terms below zero

The sequence works below zero also, like this:

n =	...	−6	−5	−4	−3	−2	−1	***0***	1	2	3	4	5	6	...
x_n =	...	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	...

In fact the sequence below zero has the same numbers as the sequence above zero, except they follow a +−+− ... pattern. It can be written like this:

$$\mathbf{x}_{-n} = (-1)^{n+1}\,\mathbf{x}_n$$

Which says that term "$-n$" is equal to $(-1)^{n+1}$ times term "n", and the value $(-1)^{n+1}$ neatly makes the correct $+1, -1, +1, -1, \ldots$ pattern.

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DEPARTMENT OF MATHEMATICS

UNIT – II – Mathematics and Measurements – SMTA1103

UNIT II- **MATHEMATICS AND MEASUREMENTS**

Area of Plane shapes

1. Area of a triangle = 1/2xbxh, where b- base and h- vertical height

2. Area of a Square = side x side.

3. Area of a Rectangle = width x height

4. Area of a parallelogram = base x height

5. Area of a Trapezoid = ½ x (a + b) x h

6. Area of a circle =π x radius x radius

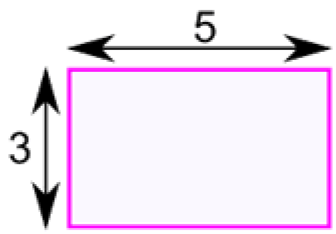
7. Circumference of a circle= 2πr

8. Area of Ellipse = πab

9. area of sector = ½ r^2 xθ

Example1:

What is the area of this rectangle?



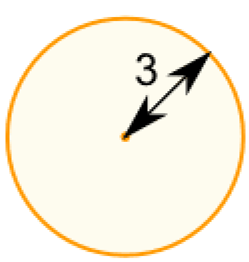
w = 5 and **h = 3**,

Area of a Rectangle = width x height

Area = 5 × 3 = **15**

Example2:

What is the area of this circle?



Radius = r = 3

Area $= \boldsymbol{\pi} \times r^2$

$= \boldsymbol{\pi} \times 3^2$

$= \boldsymbol{\pi} \times (3 \times 3)$

$= 3.14159... \times 9$

$=$ **28.27** (to 2 decimal places)

Example3:

What is the area of this triangle?

Height = h = 12

Base = b = 20

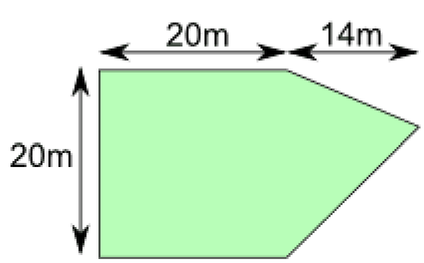
Area = ½ × b × h = ½ × 20 × 12 = **120**

A harder example:

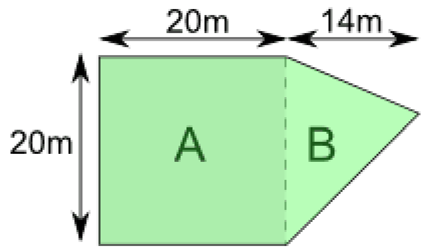
Example4:

Sam cuts grass at $0.10 per square meter

How much does Sam earn cutting this area



Let's break the area into two parts:



Part A is a square:

Area of A = a^2 = 20m × 20m = $400m^2$

Part B is a triangle. Viewed sideways it has a base of 20m and a height of 14m.

Area of B = ½b × h = ½ × 20m × 14m = $140m^2$

So the total area is:

Area = Area of A + Area of B = $400m^2$ + $140m^2$ = $540m^2$

Sam earns $0.10 per square meter

Sam earns = $0.10 × $540m^2$ = $54

Example5:

A cylindrical drum has a height of 20 cm and base radius of 14 cm. Find its curved surface area and the total surface area.

Solution

Given that, height of the cylinder h = 20 cm ; radius r =14 cm

Now, C.S.A. of the cylinder = $2\pi rh$ sq. units

C.S.A. of the cylinder = $2 \times (22/7) \times 14 \times 20 = 2 \times 22 \times 2 \times 20 = 1760$ cm^2

T.S.A. of the cylinder = $2\pi r\ (h + r)$ sq. units

$= 2 \times (22/7) \times 14 \times (20 + 14) = 2 \times 22/7 \times 14 \times 34$

$= 2992$ cm^2

Therefore, C.S.A. = 1760 cm^2 and T.S.A. = 2992 cm^2

Example6:

The curved surface area of a right circular cylinder of height 14 cm is 88 cm^2 . Find the diameter of the cylinder.

Solution

Given that, C.S.A. of the cylinder =88 sq. cm

$2\pi rh = 88$

$2 \times (22/7) \times r \times 14 = 88$ (given h=14 cm)

$2r = (88\times 7) / (22\times 14) = 2$

Therefore, diameter = 2 cm

Example6:

Find the diameter of a sphere whose surface area is 154 m^2.

Solution

Let r be the radius of the sphere.

Given that, surface area of sphere = 154 m^2

$4\pi r^2 = 154$

$$4\times\frac{22}{7}\times r^2 = 154$$

gives $r^2 = 154\times\frac{1}{4}\times\frac{7}{22}$

hence, $r^2 = \frac{49}{4}$ we get $r = \frac{7}{2}$

Therefore, diameter is 7 m

Example8:

If the base area of a hemispherical solid is 1386 sq. metres, then find its total surface area?

Solution

Let r be the radius of the hemisphere.

Given that, base area = πr^2 = 1386 sq. m

T.S.A. = $3\pi r^2$ sq.m

= $3\times1386 = 4158$

Therefore, T.S.A. of the hemispherical solid is 4158 m^2

Example9:

The volume of a solid right circular cone is 11088 cm^3. If its height is 24 cm then find the radius of the cone.

Solution

Let r and h be the radius and height of the cone respectively.

Given that, volume of the cone = 11088 cm^3

$$\frac{1}{3}\pi r^2 h = 11088$$

$$\frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 11088$$

$$r^2 = 441$$

Therefore, radius of the cone r=21 cm

Example10:

Find the volume of a cylinder whose height is 2 m and whose base area is 250 m^2.

Solution

Let r and h be the radius and height of the cylinder respectively.

Given that, height h = 2 m, base area = 250 m^2

Now, volume of a cylinder = $\pi r^2 h$ cu. Units

= *base area* × *h*

= 250×2 = 500 m^3

Therefore, volume of the cylinder = 500 m^3

Example11:

Find the volume of the iron used to make a hollow cylinder of height 9 cm and whose internal and external radii are 21 cm and 28 cm respectively.

Solution Let r, R and h be the internal radius, external radius and height of the hollow cylinder respectively.

Given that, r =21cm, R = 28 cm, h = 9 cm

Now, volume of hollow cylinder = $\pi(R^2 - r^2)h$ cu. Units

= 22/7 $(28^2 - 21^2)$×9

= 22/7 (784 − 441) ×9 = 9702

Therefore, volume of iron used = 9702 cm^3

Example7:

Perimeter of plane shapes

Square	• 4 × side
Rectangle	• 2 × (length + width)
Parallelogram	• 2 × (side1 + side2)
Triangle	• side1 + side2 + side3
Regular n-polygon	• n × side
Trapezoid	• height × (base1 + base2) / 2
Trapezoid	• base1 + base2 + height × [csc(theta1) + csc(theta2)]
Circle	• 2 ×π× radius

Area of plane shapes

Square	$side^2$
Rectangle	length × width
Parallelogram	base × height
Triangle	base × height / 2
Regular n-polygon	(1/4) × n × $side^2$ × cot(pi/n)
Trapezoid	height × (base1 + base2) / 2
Circle	pi × $radius^2$
Ellipse	pi × radius1 × radius2
Cube (surface)	6 × $side^2$
Sphere (surface)	4 × pi × r^2
Cylinder (surface of side)	perimeter of circle × height
	2 × pi × radius × height

Cylinder (whole surface)	Areas of top and bottom circles + Area of the side
	2(pi × radius^2) + 2 × pi × radius × height
Cone (surface)	pi × radius × side
Torus (surface)	pi^2 × (radius2^2 - radius1^2

Volume of plane shapes

Cube	side^3
Rectangular Prism	side1 × side2 × side3
Sphere	(4/3) × pi × radius^3
Ellipsoid	(4/3) × pi × radius1 × radius2 × radius3
Cylinder	pi × radius^2 × height
Cone	(1/3) × pi × radius^2 × height
Pyramid	(1/3) × (base area) × height
Torus	(1/4) × pi^2 × (r1 + r2) × (r1 - r2)^2

SI Metric Units of measurements for angles

Definition of Unit and its Types

The process of measurement is basically a process of comparison. To measure a quantity, we always compare it with some reference standard. For example, when we state that a rope is 10 meter long, it is to say that it is 10 times as long as an object whose length is defined as 1 metre. Such a standard is known as the unit of the quantity. Here 1 metre is the unit of the quantity 'length'.

An arbitrarily chosen standard of measurement of a quantity, which is accepted internationally is called unit of the quantity.

The units in which the fundamental quantities are measured are called *fundamental or base units* and the units of measurement of all other physical quantities, which can be obtained by a suitable multiplication or division of powers of fundamental units, are called derived *units*.

Different types of Measurement Systems

A complete set of units which is used to measure all kinds of fundamental and derived quantities is called a system of units. Here are the common system of units used in mechanics:

a. the f.p.s. system is the British Engineering system of units, which uses ***foot***, ***pound*** and ***second*** as the three basic units for measuring length, mass and time respectively.

b. The c.g.s system is the Gaussian system, which uses ***centimeter, gram*** and ***second*** as the three basic units for measuring length, mass and time respectively.

c. The m.k.s system is based on ***metre, kilogram*** and ***second*** as the three basic units for measuring length, mass and time respectively.

SI unit System

The system of units used by scientists and engineers around the world is commonly called ***the metric system*** but, since 1960, it has been known officially as the International System, or SI (the abbreviation for its French name, Systeme International). The SI with a standard scheme of symbols, units and abbreviations, were developed and recommended by the General Conference on Weights and Measures in 1971 for international usage in scientific, technical, industrial and commercial work. The advantages of the SI system are,

i. This system makes use of only one unit for one physical quantity, which means a rational system of units

ii. In this system, all the derived units can be easily obtained from basic and supplementary units, which means it is a coherent system of units.

iii. It is a metric system which means that multiples and submultiples can be expressed as powers of 10.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – Statistics and Probability – SMTA1103

Mean

The mean (also know as average), is obtained by dividing the sum of observed values by the number of observations, *n.*

$$\bar{X} = \frac{\sum_{i=1}^{i=n} X_i}{n}$$

That is Mean = $\frac{\text{Sum of Value of Entries}}{\text{Number of Entries}}$

Median

The median is the middle value of a set of data containing an odd number of values, or the average of the two middle values of a set of data with an even number of values.

Arrange all of the values from lowest to highest. If there are an odd number of entries, the median is the middle value. If there are an even number of entries, the median is the mean of the two middle entries.

Mode

The mode is the most frequently occurring value in the data set. In a data set where each value occurs exactly once, there is no mode.

Standard deviation

Standard deviation measures the variation or dispersion exists from the mean. A low standard deviation indicates that the data points tend to be very close to the mean, whereas high standard deviation indicates that the data points are spread over a large range of values.

Standard Deviation = $\sqrt{\frac{\text{Sum of (value of entry - mean of data set)}^2}{\text{Number of Entries}}}$

MEASURES OF CENTRAL TENDENCY

INDIVIDUAL SERIES	DISCRETE SERIES	CONTINUOUS SERIES
ARITHMETIC MEAN: Direct Method $\overline{X} = \frac{\sum X}{N}$ Short-cut Method $\overline{X} = A + \frac{\sum d}{N}$ Step-Deviation Method $\overline{X} = A + \frac{\sum d}{N} \times i$	Direct Method $\overline{X} = \frac{\sum f\, X}{N}$ Short-cut Method $\overline{X} = A + \frac{\sum f\, d}{N}$ Step-Deviation Method $\overline{X} = A + \frac{\sum f\, d}{N} \times i$	Direct Method $\overline{X} = \frac{\sum f\, X}{N}$ Short-cut Method $\overline{X} = A + \frac{\sum f\, d}{N}$ Step-Deviation Method $\overline{X} = A + \frac{\sum f\, d}{N} \times i$
MEDIAN: Size of $\left(\frac{N+1}{2}\right)^{th}$ term	Size of $\left(\frac{N+1}{2}\right)^{th}$ term	Size of $\left(\frac{N}{2}\right)^{th}$ term Median = $L + \frac{N/2 - c.f.}{N} \times i$
MODE: Either by inspection or the value that occurs largest number of times	Grouping Method determines that value around which most of the frequencies are concentrated.	Mode = $\mathrm{L} + \frac{\mathrm{f}_1 - \mathrm{f}_0}{2\mathrm{f}_1 - (\mathrm{f}_0 + \mathrm{f}_2)} \times \mathrm{i}$
EMPIRICAL RELATION: Mode = 3 Mean – 2 Median		

Problems

1. Find the mode, median and mean for this set of numbers: 3, 6, 9, 14, 3

Solution

First arrange the numbers from least to greatest: 3, 3, 6, 9, 14

Mode (number seen most often) = 3

3, 3, 6, 9, 14

Median (number exactly in the middle) = 6

3, 3, 6, 9, 14

Mean (add up all the numbers then divide by the amount of numbers) = 7

3 + 3 + 6 + 9 + 14 = 35 35 / 5 = 7

2. Find the mode, median and mean for this set of numbers: 1, 8, 23, 7, 2, 5

Solution

First arrange the numbers from least to greatest: 1, 2, 5, 7, 8, 23

Mode = no mode

Median = 6 (5 + 7 = 12 ; 12 / 2 = 6)

Range = 22 (23 – 1 = 22)

Mean = 7 4/6 (1 + 2 + 5 + 7 + 8 + 23 = 46 ; 46 / 6 = 7 4/6 or 7 2/3. This answer may also be stated in decimals.)

1. From the following data compute Arithmetic Mean

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 – 60
No. of students	5	10	25	30	20	10

Solution:

Marks	Midvalue X	No. of students f	f x
0 - 10	5	5	25
10 - 20	15	10	150
20 - 30	25	25	625
30 - 40	35	30	1050
40 - 50	45	20	900
50 - 60	55	10	550
		N=100	3300

Arithmetic Mean $\overline{X} = \frac{\sum f\,X}{N} = \frac{3300}{100} = 33$

2. Calculate Arithmetic Mean from the following data

Marks	0 - 10	10 - 30	30 - 60	60 - 100
No. of students	5	12	25	8

Solution:

The class intervals are unequal but still to simplify calculations we can take 5 as common factor.

Marks	Midvalue x	No. of students f	d (x - 45) / 5	f d
0 – 10	5	5	- 8	- 40
10 – 30	15	12	- 5	- 60
30 – 60	25	25	0	0
60 - 100	35	8	7	56
		N= 50		- 44

Arithmetic Mean $\bar{X} = A + \frac{\sum f\, d}{N} \times i$

A = 45, $\sum f\, d$ = - 44, N = 50, I = 5

$$\bar{X} = 45 - \frac{44}{50} \times 5 = 45 - 4.4 = 40.6$$

3. Find the missing frequency from the following data

Marks	0 – 10	10 – 20	20 - 30	30 - 40	40 -50	50 – 60
No. of Students	5	15	20	-	20	10

The arithmetic mean is 34 marks.

Solution:

Let the missing frequency be denoted by X

Marks	Midvalue x	f`	f x
0 – 10	5	5	25
10 – 20	15	15	225
20 – 30	25	20	500
30 – 40	35	X	35X
40 -50	45	20	900
50 – 60	55	10	550
		N = 70 + X	2200+35X

$$\overline{X} = \frac{\Sigma f x}{N} \qquad 34 = \frac{2200 + 35X}{70 + X}$$

34 (70 + X) = 2200 + 35X

2380 + 34X = 2200 + 35X

35X – 34X = 2380 – 2200

X = 180

4. Calculate the Median and Mode from the following data

Central size	15	25	35	45	55	65	75	85
Frequencies	5	9	13	21	20	15	8	3

Solution:

Since we are given central values first we determine the lower and upper limits of the classes. The class interval is 10 and hence the first class would be 10 – 20.

Class	Midvalue x	f	d (x – 55) / 10	f d	c.f.
10 - 20	15	5	- 4	- 20	5
20 - 30	25	9	- 3	- 27	14
30 - 40	35	13	- 2	- 26	27
40 -50	45	21	- 1	- 21	48
50 - 60	55	20	0	0	68
60 - 70	65	15	1	15	83
70 - 80	75	8	2	16	91
80 - 90	85	3	3	9	94
				Σfd = - 54	

Calculation of Median:

Med = size of $\frac{N}{2}$ th term = $\frac{94}{2} = 47$

Median lies in the class 40 – 50

Median = $L + \frac{N/2 - c.f.}{f} \times i$

$$M = 40 + \frac{47 - 27}{21} \times 10 = 40 + 9.524 = 49.524$$

5. Calculate the median and mode of the data given below. Using then find arithmetic mean

Marks	0 - 10	10 – 20	20 - 30	30 - 40	40 -50	50 – 60
No. of Students	8	23	45	65	75	80

Solution:

Marks	f`	c.f.
0 – 10	8	8
10 – 20	15	23
20 – 30	22	45
30 – 40	20	65
40 – 50	10	75
50 – 60	5	80
	N = 80	

Calculation of Median: Med = size of $\frac{N}{2}$ th term = $\frac{80}{2} = 40$ th item

Median lies in the class 20 – 30

Median = $L + \frac{N/2 - c.f.}{f} \times i$

$M = 20 + \frac{40-23}{22} \times 10 = 20 + 7.73$ = 27.73

Mode lies in the class is 20 – 30

Mode $= \mathrm{L} + \frac{\mathrm{f}_1 - \mathrm{f}_0}{2\mathrm{f}_1 - (\mathrm{f}_0 + \mathrm{f}_2)} \times \mathrm{i} = 20 + \left(\frac{22-15}{44-(15+20)}\right) \times 10 = 27.78$

MEASURES OF DISPERSION

INDIVIDUAL OBERSERVATIONS	DISCRETE& CONTINUOUS SERIES
STANDARD DEVIATION:	
Actual Mean Method: $\sigma = \sqrt{\frac{\sum f(X-\overline{X})^2}{N}}$	Actual Mean Method: $\sigma = \sqrt{\frac{\sum f(X-\overline{X})^2}{N}}$
Assumed Mean Method: $\sigma = \sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2}$	Assumed Mean Method: $\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$
Step Deviation Method $\sigma = \sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2} \times i$	Step Deviation Method $\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \times i$
$C.V. = \frac{\sigma}{\overline{\overline{X}}} \times 100$	$C.V. = \frac{\sigma}{\overline{\overline{X}}} \times 100$

1. Find the Mean and standard deviation from the following distribution

Mid value	12.0	12.5	13.0	13.5	14	14.5	15	15.5	16
No. of Students	2	16	36	60	76	37	18	3	2

Solution:

Midvalue x	No. of Students f	d (x – 14) / 0.5	f d	f d^2
12.0	2	-4	- 8	32
12.5	16	-3	- 48	144
13	36	-2	- 72	144
13.5	60	-1	- 60	60
14	76	0	0	0
14.5	37	1	37	37

15	18	2	36	72
15.5	3	3	9	27
16.0	2	4	8	32
	N= 250		Σfd = - 98	$\Sigma fd^2 = 548$

Mean $\bar{X} = A + \frac{\sum f\,d}{N} \times i = 14 - \frac{98}{250} \times 0.5 = 13.8$

Standard deviation $\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \times i$

$$\sigma = \sqrt{\frac{548}{250} - \left(\frac{-98}{250}\right)^2} \times .05 = 0.715$$

2. Find the Standard deviation and Coefficient of Variation from the following data

Marks	No. of students
Up to 10	12
Up to 20	30
Up to 30	65
Up to 40	107
Up to 50	157
Up to 60	202
Up to 70	222
Up to 80	230

Solution:

Class	Midvalue x	No. of Students f	d (x – 35) / 10	f d	f d^2
0 – 10	5	12	-3	- 36	108
10 - 20	15	18	-2	- 36	72
20 - 30	25	35	-1	- 35	35
30 - 40	35	42	0	0	0
40 -50	45	50	1	50	50
50 - 60	55	45	2	90	180
60 - 70	65	20	3	60	180
70 - 80	75	8	4	32	128
		N= 230		Σfd =125	$\Sigma fd^2 = 753$

Mean $\bar{X} = A + \frac{\sum f\,d}{N} \times i = 35 + \frac{125}{230} \times 10 = 40.43$

Standard deviation $\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \times i$

$$\sigma = \sqrt{\frac{753}{230} - \left(\frac{125}{230}\right)^2} \times 10 \;=\; 17.26$$

$$C.V. = \frac{\sigma}{\overline{X}} \times 100 \;=\; \frac{17.26}{40.43} \times 100 = 42.69$$

3. The scores of two batsmen A and B in ten innings during a certain season are:

A	32	28	47	63	71	39	10	60	96	14
B	19	31	48	53	67	90	10	62	40	80

Find which of the two batsmen more consistent in scoring

Solution:

X	$X - \overline{X}$	$\left(X - \overline{X}\right)^2$	Y	$Y - \overline{Y}$	$\left(Y - \overline{Y}\right)^2$
32	-14	196	19	- 31	961
28	-18	324	31	- 19	361
47	1	1	48	- 2	4
63	17	289	53	3	9
71	25	625	67	17	289
39	-7	49	90	40	1600
10	- 36	1296	10	- 40	1600
60	14	196	62	12	144
96	50	2500	40	-10	100
14	- 32	1024	80	30	900
$\sum X = 460$		$\sum\left(X - \overline{X}\right)^2 = 6500$	$\sum Y = 500$		$\sum\left(Y - \overline{Y}\right)^2 = 5968$

Batsman A:

Mean $\overline{X} = \frac{\sum X}{N} = \frac{460}{10} = 46$

$$\sigma = \sqrt{\frac{\sum(X - \overline{X})^2}{N}} = \sqrt{\frac{6500}{10}} = 25.495$$

$$C.V. = \frac{\sigma}{\overline{X}} \times 100 \;=\; \frac{25.495}{46} \times 100 = \;\; 55.42$$

Batsman B:

Mean $\bar{Y} = \frac{\sum Y}{N} = \frac{500}{10} = 50$

$$\sigma = \sqrt{\frac{\sum\left(Y-\bar{Y}\right)^2}{N}} = \sqrt{\frac{5968}{10}} = \quad 24.43$$

$$C.V. = \frac{\sigma}{\bar{Y}} \times 100 \quad = \quad \frac{24.43}{50} \times 100 = \quad 48.86$$

Since Coefficient of Variation is less in the case of Batsman B, we conclude that the Batsman B is more consistent.

Correlation and Regression

Correlation and Regression analyses are based on the relationship, or association, between two(or more) variables.

Karl Pearson Coefficient of Correlation

As a measure of intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called Correlation coefficient(also called as product moment correlation coefficient).

Correlation coefficient between two random variables X and Y usually denoted by r(X,Y) or simply r, is a numerical measure of linear relationship between them and is defined as

(i.e.) $$r(X,Y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 . \sum_{i=1}^{n}(y_i-\bar{y})^2\right]^{\frac{1}{2}}} \qquad \text{.................................(1)}$$

(i.e) $$r(X,Y) = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{\left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)\left(n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right)}} \qquad \text{........................(2)}$$

The correlation coefficient is a dimensionless number; it has no units of measurement. The maximum value r can achieve is 1, and its minimum value is –1. Therefore, for any given set of observations, $-1 \le r \le 1$.

Problems:

1. Calculate the correlation coefficient between X and Y from the following data:

$$\sum_{i=1}^{15}\left(X_i - \bar{X}\right)^2 = 136 \qquad \sum_{i=1}^{15}\left(Y_i - \bar{Y}\right)^2 = 138 \qquad \sum_{i=1}^{15}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) = 122$$

Solution:

We have $r(X,Y) = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 . \sum_{i=1}^{n}(y_i - \bar{y})^2\right]^{\frac{1}{2}}} = \dfrac{122}{\sqrt{136}\sqrt{138}}$ r(X,Y) = 0.89

2. Some health researchers have reported an inverse relationship between central nervous system malformations and the hardness of the related water supplies. Suppose the data were collected on a sample of 9 geographic areas with the following results:

C.N.S. malformation rate (per 1000 births)	9	8	5	1	4	2	3	6	7
Water hardness(ppm)	120	130	90	150	160	100	140	80	200

Calculate the Correlation Coefficient between the C.N.S. malformation rate and Water hardness.

Solution:

Let us denote the C.N.S. malformation rate by x and water hardness by y. The mean of the x series $\bar{x}$ = 5 and the mean of the y series $\bar{y}$ =130, hence we can use the formula (2.1)

Calculation of correlation coefficient

x	**y**	**(x – $\bar{x}$)= x – 5**	**(y – $\bar{y}$) = y – 130**	**(x – $\bar{x}$)^2**	**(y – $\bar{y}$)^2**	**(x – $\bar{x}$) (y – $\bar{y}$)**
9	120	4	– 10	16	100	– 40
8	130	3	0	9	0	0
5	90	0	– 40	0	1600	0
1	150	– 4	20	16	400	– 80
4	160	– 1	30	1	900	– 30
2	100	– 3	– 30	9	900	90

3	140	– 2	10	4	100	– 20
6	80	1	– 50	1	2500	– 50
7	200	2	70	4	4900	140
				$\Sigma(x - \bar{x})^2$= 60	**$\Sigma(y - \bar{y})^2$= 11400**	**$\Sigma(x - \bar{x})(y - \bar{y})$= 10**

$$r(X,Y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 . \sum_{i=1}^{n}(y_i - \bar{y})^2\right]^{\frac{1}{2}}} = \frac{10}{[60x11400]^{\frac{1}{2}}}$$

r(X,Y) = 0.012

Therefore, the correlation coefficient between the C.N.S. malformation rate and water hardness is 0.012.

3. Find the product moment correlation for the following data

X	57	62	60	57	65	60	58	62	56
Y	71	70	66	70	69	67	69	63	70

Solution:

X	**Y**	**XY**	**X^2**	**Y^2**
57	71	4047	3249	5041
62	70	4340	3844	4900
60	66	3960	3600	4356
57	70	3990	3249	4900
65	69	4485	4225	4761
60	67	4020	3600	4489
58	69	4002	3364	4761
62	63	3906	3844	3969
56	70	3920	3136	4900

537	**615**	**36670**	**32111**	**42077**

Thus we have, n = 9, $\sum X = 537$, $\sum Y = 615$, $\sum XY = 36670$, $\sum X^2 = 32111$, $\sum Y^2 = 42077$

$$r(X,Y) = \frac{n\sum XY - \left(\sum X\right)\left(\sum Y\right)}{\sqrt{n\sum X^2 - \left(\sum X\right)^2}\sqrt{n\sum Y^2 - \left(\sum Y\right)^2}} = \frac{9\times 36670 - 537\times 615}{\sqrt{9\times 32111 - 537^2}\sqrt{9\times 42077 - 615^2}} = -0.414$$

4. A computer operator while calculating the coefficient of correlation between two variables X and Y for 25 pairs of observations obtained the following constants: $\sum X = 125$, $\sum Y = 100$, $\sum XY = 508$, $\sum X^2 = 650$, $\sum Y^2 = 460$. However it was later discovered at the time of checking that he had copied two pairs as (6,14) and (8,6) while the correct pairs were (8,12) and (6,8). Obtain the correct correlation coefficient.

Solution:

The formula involved with the given data is, $r(X,Y) = \frac{n\sum XY - \left(\sum X\right)\left(\sum Y\right)}{\sqrt{n\sum X^2 - \left(\sum X\right)^2}\sqrt{n\sum Y^2 - \left(\sum Y\right)^2}}$

The Corrected $\sum X$ = Incorrect $\sum X$ − (6+8) + (8+6) = 125

Corrected $\sum Y$ = Incorrect $\sum Y$ − (14+6) + (12+8) = 100

Corrected $\sum X^2$ = Incorrect $\sum X^2$ − $(6^2+8^2) + (8^2+6^2) = 650$

Corrected $\sum Y^2$ = Incorrect $\sum Y^2$ − $(14^2+6^2) + (12^2+8^2) = 436$

Corrected $\sum XY$ = Incorrect $\sum XY$ − (84+48) + (96+48) = 520

Now the correct value of correlation coefficient is,

$$r(X,Y) = \frac{25\times 520 - 125\times 100}{\sqrt{25\times 650 - 125^2}\sqrt{25\times 436 - 100^2}} = 0.67$$

Spearman's Rank Correlation Coefficient

If X and Y are qualitative variables then Karl Pearson's coefficient of correlation will be meaningless. In this case, we use Spearman's rank correlation coefficient which is defined as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^{\,2}}{n\left(n^2 - 1\right)}$$ where d is the difference in ranks.

Problems:

1. The ranks of same 16 students in Mathematics and Physics are as follows. The numbers within brackets denote the ranks of the students in Mathematics and Physics. (1,1), (2,10), (3,3), (4,4), (5,5), (6,7), (7,2), (8,6), (9,8), (10,11) (11. 15), (12,9), (13,14), (14,12), (15,16), (16,13). Calculate the rank correlation coefficient for the proficiencies of this group in Mathematics and Physics.

Solution:

Ranks in Maths(X)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
Ranks in Physics(Y)	1	10	3	4	5	7	2	6	8	11	15	9	14	12	16	13	
d = X – Y	0	–8	0	0	0	–1	5	2	1	–1	–4	3	–1	2	–1	3	0
d^2	0	64	0	0	0	1	25	4	1	1	16	9	1	4	1	9	136

Spearman's Rank Correlation Coefficient is given by, $\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^{\,2}}{n\left(n^2 - 1\right)} = 1 - \frac{6 \times 136}{16\left(16^2 - 1\right)}$ = 0.8

2. The coefficient of rank correlation between the marks in Statistics and Mathematics obtained by a certain group of students is 2/3 and the sum of the squares of the differences in ranks is 55. Find the number of students in the group.

Solution:

Spearman's rank correlation coefficient is given by $\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^{\,2}}{n\left(n^2 - 1\right)}$

Here ρ = 2/3, $\sum d^2$ = 55, N = ? Therefore $\frac{2}{3} = 1 - \frac{6 \times 55}{N(N^2 - 1)}$ Solving the above equation we get N = 10.

Repeated Ranks:

If any two or more individuals are equal in the series then Spearman's formula for calculating the rank correlation coefficients breaks down. In this case, common ranks are given to the repeated ranks. This common rank is the average of the ranks which these items would have assumed if they are slightly different from each other and the next item will get the rank next the ranks already assumed. As a result of this, following adjustment is made in the formula: add the factor $\frac{m(m^2-1)}{12}$ to $\sum d^2$ where m is the number of items an item is repeated. This correction factor is to be added for each repeated value.

3. Obtain the rank correlation coefficient for the following data:

X	68	64	75	50	64	80	75	40	55	64
Y	62	58	68	45	81	60	68	48	50	70

Solution:

X	**Y**	**Rank X**	**Rank Y**	**D = X – Y**	**D^2**
68	62	4	5	– 1	1
64	58	6	7	– 1	1
75	68	2.5	3.5	– 1	1
50	45	9	10	– 1	1
64	81	6	1	5	25
80	60	1	6	– 5	25
75	68	2.5	3.5	– 1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16
					72

In X series 75 is repeated twice which are in the positions 2nd and 3rd ranks. Therefore common ranks 2.5 (which is the average of 2 and 3) is given for each 75. The corresponding correction factor is $C.F = \frac{2(2^2-1)}{12} = \frac{1}{2}$. Also in the X series 64 is repeated thrice which are in the position 5th, 6th and 7th ranks. Therefore, common ranks 6(which is the average of 5, 6 and 7) is given for each 64. The corresponding correction factor is $C.F = \frac{3(3^2-1)}{12} = 2$. Similarly, in the Y series, 68 is repeated twice which are in the positions 3rd and 4th ranks. Therefore, common ranks(which is the average of 3 and 4) is given for each 68. The corresponding correction factor is $C.F = \frac{2(2^2-1)}{12} = \frac{1}{2}$. Rank correlation coefficient is $\rho = 1 - \frac{6\left(\Sigma d^2 + TotalCorrectionFacto\,r\right)}{n\left(n^2-1\right)}$

$= 1 - \frac{6\left(72 + \frac{1}{2} + 2 + \frac{1}{2}\right)}{10(10-1)}$ = 0.5454.

Regression Analysis

Regression analysis helps us to estimate or predict the value of one variable from the given value of another. The known variable(or variables) is called independent variable(s). The variable we are trying to predict is the dependent variable.

Regression equations

Prediction or estimation of most likely values of one variable for specified values of the other is done by using suitable equations involving the two variables. Such equations are known as Regression Equations

Regression equation of y on x:

y – $\bar{y}$ = b_{yx} (x – $\bar{x}$) where y is the dependent variable and x is the independent variable and b_{yx} is given by

$$b_{yx} = \frac{\sum_{i=1}^{n}(x-\bar{x})(y-\bar{y})}{\sum_{i=1}^{n}(x-\bar{x})^2} \quad \text{or} \quad b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{n\sum_{i=1}^{n}xy - \sum_{i=1}^{n}x\sum_{i=1}^{n}y}{n\sum_{i=1}^{n}x^2 - \left(\sum_{i=1}^{n}x\right)^2}$$

Regression equation of x on y:

x – $\bar{x}$ = b_{xy} (y – $\bar{y}$) where y is the dependent variable and x is the independent variable and b_{yx} is given by

$$b_{xy} = \frac{\sum_{i=1}^{n}(x-\bar{x})(y-\bar{y})}{\sum_{i=1}^{n}(y-\bar{y})^2} \quad \text{or} \quad b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{n\sum_{i=1}^{n}xy - \sum_{i=1}^{n}x\sum_{i=1}^{n}y}{n\sum_{i=1}^{n}y^2 - \left(\sum_{i=1}^{n}y\right)^2}$$

b_{yx} and b_{xy} are called as regression coefficients of y on x and x on y respectively.

Relation between correlation and regression coefficients:

$b_{yx} = r\frac{\sigma_y}{\sigma_x}$ and $b_{xy} = r\frac{\sigma_x}{\sigma_y}$ $\qquad b_{yx} \,.\, b_{xy} = r\frac{\sigma_y}{\sigma_x}\; r\frac{\sigma_x}{\sigma_y} = r^2$ Hence

$r = \pm\sqrt{b_{yx}b_{xy}}$

Note: In the above expression the components inside the square root is valid only when b_{yx} and b_{xy} have the same sign. Therefore the regression coefficients will have the same sign.

Problems:

1. In trying to evaluate the effectiveness of its advertising campaign a company compiled the following information. Calculate the regression line of sales on advertising.

Year	1980	1981	1982	1983	1984	1985	1986	1987
Advertisement in 1000 rupees	12	15	15	23	24	38	42	48
Sales in lakhs of rupees	5	5.6	5.8	7.0	7.2	88	9.2	9.5

Solution :

Let x be advertising amount and y be the sales amount.

Here, n = 8, $\bar{x} = \frac{217}{8} = 27.1$, $\bar{y} = \frac{58.1}{8} = 7.26$

We know that, Regression equation of y on x is given by $y - \bar{y} = b_{yx}(x - \bar{x})$ where

$$b_{yx} = \frac{n\sum_{i=1}^{n}xy - \sum_{i=1}^{n}x\sum_{i=1}^{n}y}{n\sum_{i=1}^{n}x^2 - \left(\sum_{i=1}^{n}x\right)^2}$$

X	Y	X^2	XY
12	5	144	60
15	5.6	225	84
15	5.8	225	87
23	7.0	529	161
24	7.2	576	172.8
38	8.8	1444	334.4
42	9.2	1764	386.4
48	9.5	2304	456
217	**58.1**	**7211**	**1741.6**

Therefore $b_{yx} = \frac{8 \times 1741.6 - 217 \times 58.1}{8 \times 7211 - 217^2} = 0.125$ Substituting this value in the y on x equation, we get,

y – 7.26 = 0.125(x – 27.1) Therefore the required equation of Sales on Advertisement is y = 3.87 + 0.125 x

2. In a study of the effect of a dietary component on plasma lipid composition, the following ratios were obtained on a sample of experimental anumals

Measure of dietary component (X)	1	5	3	2	1	1	7	3
Measure of plasma lipid level (Y)	6	1	0	0	1	2	1	5

(i) obtain the two regression lines and hence predict the ratio of plasma lipid level with 4 dietary component.

(ii) find the correlation coefficient between X and Y

Solution:

(i)

X	Y	XY	X^2	Y^2
1	6	6	1	36
5	1	5	25	1
3	0	0	9	0
2	0	0	4	0
1	1	1	1	1
1	2	2	1	4
7	1	7	49	1
3	5	15	9	25
23	**16**	**36**	**99**	**68**

Here n = 8 $\bar{x}$ = 2.875 $\bar{y}$ = 2 The Regression equation of y on x is given by y – $\bar{y}$ = b_{yx} (x – $\bar{x}$)

Where $$b_{yx} = \frac{n\sum_{i=1}^{n} xy - \sum_{i=1}^{n} x \sum_{i=1}^{n} y}{n\sum_{i=1}^{n} x^2 - \left(\sum_{i=1}^{n} x\right)^2} \qquad b_{yx} = \frac{8\times 36 - 23\times 16}{8\times 99 - 23^2} = -0.304$$

Hence the regression equation of y on x is

y – 2 = – 0.304(x – 2.875)

(i.e) y = 2.874 – 0.304 x

when x = 4 (measure of dietary component) the plasmid lipid level is

y = 2.874 – 0.304 (4)

y = 1.658

The Regression equation of x on y is given by x – $\bar{x}$ = b_{xy} (y – $\bar{y}$)

Where $$b_{xy} = \frac{n\sum_{i=1}^{n} xy - \sum_{i=1}^{n} x \sum_{i=1}^{n} y}{n\sum_{i=1}^{n} y^2 - \left(\sum_{i=1}^{n} y\right)^2} \qquad b_{xy} = \frac{8\times 36 - 23\times 16}{8\times 68 - 16^2} = -0.278$$

Hence the regression equation of x on y is x – 2.875= – 0.278(y – 2)

(i.e) x = 3.431 – 0.278 y

(ii) The correlation coefficient between x and y is given by

$$r = \pm\sqrt{b_{yx} b_{xy}}$$

$$r = \pm\sqrt{-0.304 \times -0.278} = \pm 0.291$$

3. From the data given below find (i) two regression lines (ii) coefficient of correlation between marks in Physics and marks in Chemistry (iii) most likely marks in Chemistry when marks in Physics is 78 (iv) most likely marks in Physics when marks in Chemistry is 92

Marks in Physics (X)	72	85	91	85	91	89	84	87	75	77
Marks in Chemistry (Y)	76	92	93	91	93	95	88	91	80	81

Solution:

(i)

X	**Y**	**X^2**	**Y^2**	**XY**
72	76	5184	5776	5472
85	92	7225	8464	7820
91	93	8281	8649	8463
85	91	7225	8281	7735
91	93	8281	8649	8463
89	95	7921	9025	8455
84	88	7056	7744	7395
87	91	7569	8281	7917

75	80	5625	6400	6000
77	81	5929	6561	6237
836	**880**	**70296**	**77830**	**73957**

Here n = 10 $\bar{x}$ = 83.6 $\bar{y}$ = 88

The Regression equation of y on x is given by $y - \bar{y} = b_{yx}(x - \bar{x})$

Theory of Probability

Introduction

If an experiment is repeated under essential homogeneous and similar conditions we generally come across two types of situations:

(i) The result or what is usually known as the 'outcome' is unique or certain.

(ii) The result is not unique but may be one of the several possible outcomes.

The phenomena covered by (i) are known as deterministic. For example, for a perfect gas, PV = constant.

The phenomena covered by (ii) are known as probabilistic. For example, in tossing a coin we are not sure if a head or tail will be obtained.

In the study of statistics we are concerned basically with the presentation and interpretation of chance outcomes that occur in a planned study or scientific investigation.

Definition of various terms

Trial and event: Consider an experiment which, though repeated under essentially identical conditions, does not give unique results but may result in any one of the several possible outcomes. The experiment is known as a trial and outcomes are known as events or cases. For example, throwing of a die is a trial and getting 1(or 2 or … 6) is an event.

Exhaustive events: The total number of possible outcomes in any trial is known as exhaustive events or exhaustive cases. For example, in tossing of a coin there are two exhaustive case, viz.: Head and Tail(the possibility of the coin standing on an edge being ignored)

Favourable events or cases: The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example, in throwing of two dice, the number of cases favourable to getting the sum 3 is: (1,2) and (2,1)

Mutually exclusive events: Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the happening of all the others, that is if no two or more of them can happen simultaneously in the same trial. For example, in tossing a coin the events head and tail are mutually exclusive.

Equally likely events: Outcomes of a trial are said to be equally likely, if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example, in throwing an unbiased die, all the six faces are equally likely to come.

Sample Space: Consider an experiment whose outcome is not predictable with certainty. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the **sample space** of the experiment and is denoted by S.

Some examples follow.

1. If the outcome of an experiment consists in the determination of the sex of a newborn child, then

$$S = \{ g,b\}$$

where the outcome g means that the child is a girl and b that it is a boy.

2. If the experiment consists of flipping two coins, then the sample space consists of the following four points:

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

The outcome will be (H,H) if both coins are heads, (H,T) if the first coin is heads and the second tails, (T,H) if the first is tails and the second heads, and (T,T) if both coins are tails.

3. If the experiment consists of tossing two dice, then the sample space consists if the 36 points

$$S = \{ (i,j):i,j = 1, 2, 3, 4, 5,$$

$$= \{ (1,1)\text{------}(1,6)\text{-----}(6,1)\text{-----}(6,6) \}$$

where the outcome (i,j) is said to occur if i appears on the leftmost die and j on the other die.

Definitions of Probability

1. Mathematical or Classical or a priori probability:

If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E, then the probability ‘p’ of happening of E is given by,

$$p = P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

2. Axiomatic Definition:

Consider an experiment whose sample space is S. For each event E of the sample space S, we assume that a number P(E) is defined and satisfies the following three axioms.

Axiom 1: $0 \le P(E) \le 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events, $E_1, E_2, \ldots$ (that is, events for which $E_i E_j = \Phi$, when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Some Important Formulae

1. If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule is known as additive rule on probability.

For three events A, B and C, we have,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

2. If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

In general, if $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

3. If A and A^c are complementary events, then

$$P(A) + P(A^c) = 1$$

4. $P(S) = 1$

5. $P(\Phi) = 0$

6. If A and B are any two events, then

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

7. If A and B are independent events, then

$$P(A \cap B) = P(A) \times P(B)$$

Glossary of Probability terms:

Statement	**Meaning in terms of Set theory**
1. At least one of the events A or B occurs	$\omega \in A \cup B$
2. Both the events A and B occur	$\omega \in A \cap B$
3. Neither A nor B occurs	$\omega \in \overline{A} \cap \overline{B}$
4. Event A occurs and B does not occur	$\omega \in A \cap \overline{B}$
5. Exactly one of the events A or B occurs	$\omega \in A \Delta B$
6. If event A occurs, so does B	$A \subset B$
7. Events A and B are mutually exclusive	$A \cap B = \Phi$

8. Complementary event of A	$\overline{A}$
9. Sample space	**Universal set S**

Problems

1. Find the probability of getting a head in tossing a coin.

Solution:

When a coin is tossed, we have the sample space $\{$Head, Tail$\}$

Therefore, the total number of possible outcomes is 2

The favourable number of outcomes is 1, that is the head.

$\therefore$ The required probability is ½.

2. Find the probability of getting two tails in two tosses of a coin.

Solution:

When two coins are tossed, we have the sample space $\{$HH, HT, TH, TT$\}$

Where H represents the outcome Head and T represents the outcome Tail.

The total number of possible outcomes is 4.

The favourable number of outcomes is 1, that is TT

$\therefore$ The required probability is ¼.

3. Find the probability of getting an even number when a die is thrown

Solution:

When a die is thrown the sample space is $\{1, 2, 3, 4, 5, 6\}$

The total number of possible outcomes is 6

The favourable number of outcomes is 3, that is 2, 4 and 6

$\therefore$ The required probability is= $\frac{3}{6} = ½$.

4. What is the chance that a leap year selected at random will contain 53 Sundays?

Solution:

In a leap year(which consists of 366 days) there are 52 complete weeks and 2 days over. The following are the possible combinations for these two over days:

(i) Sunday and Monday (ii)Monday and Tuesday (iii)Tuesday and Wednesday (iv)Wednesday and Thursday (v)Thursday and Friday (vi)Friday and Saturday (vii)Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two over days must be Sunday. Since out of the above 7 possibilities, 2 viz. (i) and (ii)are favourable to this event,

Required probability $= \frac{2}{7}$

5. If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution:

We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomes – namely (1,6), (2,5), (3,4), (4,3), (5,2,), (6,1) – that result in the sum of the dice being equal to 7, the desired probability is $\frac{6}{36}$ = $\frac{1}{6}$.

6. A bag contains 3 Red, 6 White and 7 Blue balls. What is the probability that two balls drawn are white and blue?

Solution:

Total number of balls = 3 + 6 + 7 = 16.

Out of 16 balls, 2 can be drawn in $16C_2$ ways.

Therefore exhaustive number of cases is 120.

Out of 6 white balls 1 ball can be drawn in $6C_1$ ways and out of 7 blue balls 1 ball can be drawn in $7C_1$ ways. Since each of the former cases can be associated with each of the latter cases, total number of favourable cases is $6C_1$ x $7C_1$ = 6 x 7 = 42.

$\therefore$ The required probability is = $\frac{42}{120} = \frac{7}{20}$

7. A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v)exactly 1 is good, (vi) neither has major defects and (vii) neither is good.

Solution:

Although the articles may be drawn one after the other, we can consider that both articles are drawn simultaneously, as they are drawn without replacement.

(i) $P(both \text{ are good}) = \frac{\text{No. of ways drawing 2 good articles}}{\text{Total no. of ways of drawing 2 articles}}$

$$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

(ii) $P(b\text{oth have major defects}) = \frac{\text{No. of ways of drawing 2 articles with major defects}}{\text{Total no. of ways}}$

$$= \frac{2C_2}{16C_2} = \frac{1}{120}$$

(iii) P(at least 1 is good) = P(exactly 1 is good or both are good)

=P(exactly 1 is good and 1 is bad or both are good)

$$= \frac{10C_1 x 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) P(atmost 1 is good) =P(none is good or 1 is good and 1is bad)

$$=\frac{10C_0 x 6C_2 + 10C_1 x 6C_1}{16C_2} = \frac{5}{8}$$

(v) P(exactly 1is good) =P(1 is good and 1 is bad)

$$=\frac{10C_1 x 6C_1}{16C_2} = \frac{1}{2}$$

(vi) P(neither has major defects) = P(both are non-major defective articles)

$$=\frac{14C_2}{16C_2} = \frac{91}{120}$$

(vii) P(neither is good) = P(both are defective)

$$=\frac{6C_2}{16C_2} = \frac{1}{8}$$

8. From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive?

Solution:

If the product is to be positive, all the 4 numbers must be positive or all the 4 must be negative or 2 of them must be positive and the other 2 must be negative.

No. of ways of choosing 4 positive numbers= $6C_4$ =15.

No. of ways of choosing 4 negative numbers= $8C_4$ =70.

No.of ways of choosing 2 positive and 2 negative numbers

$$= 6C_2 x 8C_2 = 420.$$

Total no. of ways of choosing 4 numbers from all the 14 numbers

$= \quad 14C_4 = 1001.$

P(the product is positive)

$$= \frac{No.\text{ of ways by which the product is positive}}{\text{Total no. of ways}}$$

$$= \frac{15+70+420}{1001} = \frac{505}{1001}$$

9. If 3 balls are “randomly drawn” from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

Solution:

If we regard the order in which the balls are selected as being relevant, then the sample space consists of 11· 10 · 9 = 990 outcomes. Furthermore, there are 6· 5· 4 = 120 outcomes in which the first ball selected is white and the other two black; 5 · 6· 4 = 120 outcomes in which the first is black, the second white and the third black; and 5· 4 · 6 = 120 in which the first two are black and the third white. Hence, assuming that “randomly drawn” means that each outcome in the sample space is equally likely to occur, we see that the desired probability is $\frac{120+120+120}{990} = \frac{4}{11}$

10. In a large genetics study utilizing guinea pigs, *Cavia sp.,* 30% of the offspring produced had white fur and 40% had pink eyes. Two-thirds of the guinea pigs with white fur had pink eyes. What is the probability of a randomly selected offspring having both white fur and pink eyes?

Solution:

P(W) = 0.30, P(Pi) = 0.40, and P(Pi   W) = 0.67. Utilizing Formula 2.9,

P(Pi ∩ W) = P(Pi   W). P(W) = 0.67. 0.30 = 0.20.

Twenty percent of all offspring are expected to have both white fur and pink eyes.

11. Consider three gene loci in tomato, the first locus affects fruit shape with the oo genopyte causing oblate or flattened fruit and OO or Oo normal round fruit. The second locus affects fruit color with yy having yellow fruit and YY or Yy red fruit. The final locus affects leaf shape with pp having potato or smooth leaves and PP or Pp having the more typical cut leaves.

Each of these loci is located on a different pair of chromosomes and, therefore, acts independently of the other loci. In the following cross OoYyPp × OoYypp, what is the probability that an offspring will have the dominant phenotype for each trait? What is the probability that it will be heterozygous for all three genes? What is the probability that it will have round, yellow fruit and potato leaves?

Solution:

Genotypic array:

$$\left(\frac{1}{4}\text{OO} + \frac{2}{4}\text{Oo} + \frac{1}{4}\text{oo}\right)\left(\frac{1}{4}\text{YY} + \frac{2}{4}\text{Yy} + \frac{1}{4}\text{yy}\right)\left(\frac{1}{2}\text{pp}\right)$$

Phenotypic array:

$$\left(\frac{3}{4}\text{O-} + \frac{1}{4}\text{oo}\right)\left(\frac{3}{4}\text{Y-} + \frac{1}{4}\text{yy}\right)\left(\frac{1}{2}\text{P} + \frac{1}{2}\text{pp}\right)$$

The probabiltity of dominant phenotype for each trait from the phenotypic array above is

$$P(\text{O-Y-P-}) = P(\text{O-}) \times P(\text{Y-}) \times P(\text{P-}) = \frac{3}{4} \times \frac{3}{4} \times \frac{1}{2} = \frac{9}{32}.$$

The probability of heterozygous for all three genes from the genotypic array above is

$$P(\text{OoYyPp}) = P(\text{Oo}) \times P(\text{Yy}) \times P(\text{Pp}) = \frac{2}{4} \times \frac{2}{4} \times \frac{1}{2} = \frac{4}{32} = \frac{1}{8}.$$

The probability of a round, yellow-fruited plant with potato leaves from the phenotypic array above is

$$P(\text{O-yypp}) = P(\text{O-}) \times P(\text{yy}) \times P(\text{pp}) = \frac{3}{4} \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{32}.$$

Each answer applies the probability rules for independent events to the separate gene loci.

12. (a) Two cards are drawn at random from a well shuffled pack of 52 playing cards. Find the chance of drawing two aces.

(b) From a pack of 52 cards, three are drawn at random. Find the chance that they are a king, a queen and a knave.

(c) Four cards are drawn from a pack of cards. Find the probability that (i) all are diamond (ii) there is one card of each suit (iii) there are two spades and two hearts.

Solution:

(a) From a pack of 52 cards 2 can be drawn in $52C_2$ ways, all being equally likely. $\therefore$ Exhaustive number of cases is $52C_2$.

In a pack there are 4 aces and therefore 2 aces can be drawn in $4C_2$ ways.

$\therefore$ Required probability = $\frac{4C_2}{52C_2} = \frac{1}{221}$

(b) Exhaustive number of cases = $52C_3$

A pack of cards contains 4 kings, 4 queens and 4 knaves. A king, a queen and a knave can each be drawn in $4C_1$ ways and since each way of drawing a king can be associated with each of the ways of drawing a queen and a knave, the total number of favrourable cases = $4C_1 \times 4C_1 \times 4C_1$.

$\therefore$ Required probability = $\frac{4C_1 \times 4C_1 \times 4C_1}{52C_3} = \frac{16}{5525}$

(c) Exhaustive number of cases $52C_4$

(i) Required probability = $\frac{13C_4}{52C_4}$

(ii) Required probability = $\frac{13C_1 \times 13C_1 \times 13C_1 \times 13C_1}{52C_4}$

(iv) Required probability = $\frac{13C_2 \times 13C_2}{52C_4}$

13. What is the probability of getting 9 cards of the same suit in one hand at a game of bridge?

Solution:

One hand in a game of bridge consists of 13 cards.

$\therefore$ Exhaustive number of cases $52C_{13}$

Number of ways in which, in one hand, a particular player gets 9 cards of one suit are $13C_9$ and the number of ways in which the remaining 4 cards are of some other suit are $39C_4$. Since there are 4 suits in a pack of cards, total number of favourable cases is $4 \times 13C_9 \times 39C_4$.

$\therefore$ Required probability = $\dfrac{4 \times 13C_9 \times 39C_4}{52C_{13}}$

14. A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, two officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner:

(i) There must be one from each category
(ii) It should have at least one from the purchase department
(iii) The chartered accountant must be in the committee.

Solution:

There are 3 + 4 + 2 + 1 = 10 persons in all and a committee of 4 people can be formed out of them in $10C_4$ ways. Hence exhaustive number of cases is $10C_4$ = 210

(i) Favourable number of cases for the committee to consist of 4 members, one from each category is $4C_1 \times 3C_1 \times 2C_1 \times 1 = 24$

$\therefore$ Required probability = $\dfrac{24}{120}$

(ii) P(Committee has at least one purchase officer) = 1 – P(Committee has no purchase Officer)

In order that the committee has no purchase officer, all the four members are to be selected amongst officers of production department, sales department and chartered accountant, that is out of 3 + 2 + 1 = 6 members and this can be done in $5C_4$ = 15 ways. Hence, P(Committee has no purchase officer) = $\frac{15}{210}=\frac{1}{14}$

$\therefore$ P(Committee has at least one purchase officer) = 1 – $\frac{1}{14}$ = $\frac{13}{14}$

(iii) Favourable number of cases that the committee consists of a chartered accountant as a member and three others are:

$1\times {}^{9}C_{3}$ = 84 ways.

Since a chartered accountant can be selected out of one chartered accountant in only 1 way and the remaining 3 members can be selected out of the remaining 10 – 1 persons in ${}^{9}C_{3}$ ways. Hence the required probability = $\frac{84}{210}=\frac{2}{5}$.

15. A box contains 6 red, 4 white and 5 black balls. A persons draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.

Solution: The required event E that in a draw of 4 balls from the box at random there is at least one ball of each colour can materialize in the following mutually disjoint ways:

(i) 1 Red, 1 White and 2 Black balls
(ii) 2 Red, 1 White and 1 Black balls
(iii) 1 Red, 2 White and 1 Black balls

Hence by addition rule of probability, the required probability is given by,

P(E) = P(i) + P(ii) + P(iii)

$$=\frac{6C_1\times 4C_1\times 5C_2}{15C_4}+\frac{6C_2\times 4C_1\times 5C_1}{15C_4}+\frac{6C_1\times 4C_2\times 5C_1}{15C_4}$$

= 0.5275

16. A problem in Statistics is given to the three students A, B and C whose chances of solving it are 1/2, 3/4 and 1/4 respectively. What is the probability that the problem will be solved if all of them try independently?

Solution:

Let A, B and C denote the events that the problem is solved by the students A, B and C respectively. Then

P(A) = 1/2 P(B) = 3/4 P(C) = 1/4

P($\overline{A}$) = 1 – 1/2 = 1/2 P($\overline{B}$) = 1 – 3/4 = 1/4 P($\overline{C}$) = 1 – 1/4 = 3/4

P(Problem solved) = P(At least one of them solves the problem)

= 1 – P(None of them solve the problem)

= 1 – P($\overline{A \cup B \cup C}$)

= 1 – P($\overline{A} \cap \overline{B} \cap \overline{C}$)

= 1 – P($\overline{A}$) P($\overline{B}$) P($\overline{C}$)

$$= 1 - \frac{1}{2} \times \frac{1}{4} \times \frac{3}{4}$$

$$= \frac{29}{32}$$

17. Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random from each group. Find the probability that the three selected consist of 1 girl and 2 boys.

Solution: The required event of getting 1 girl and 2 boys among the three selected children can materialize in the following three mutually exclusive cases:

Group No. →	**I**	**II**	**III**
(i)	**Girl**	**Boy**	**Boy**
(ii)	**Boy**	**Girl**	**Boy**

(iii)	Boy	Boy	Girl

By addition rule of probability,

Required probability = P(i) + P(ii) + P(iii)

Since the probability of selecting a girl from the first group is 3/4, of selecting a boy from the second is 2/4, and of selecting a boy from the third group is ¾, and since these three events of selecting children from the three groups are independent of each other, we have,

P(i) = $\frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}$

P(ii) = $\frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}$

P(iii) = $\frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$

Hence the required probability = $\frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$

Conditional Probability

Conditional Probability and Multiplication Law

For two events A and B

P(A∩B) = P(A) . P(B/A), P(A) > 0

= P(B) . P(A/B), P(B) > 0

where P(B/A) represents the conditional probability of occurrence of B when the event A has already happened and P(A/B) is the conditional probability of occurrence of A when the event B has already happened.

Theorem of Total Probability:

If B_1, B_2, … , B_n be a set of exhaustive and mutually exclusive events, and A is another event associated with (or caused by) B_i, then

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

Problems

18. A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution:

Let A = one of the tubes drawn is good and B = the other tube is good.

$P(A \cap B)$ = P(both tubes drawn are good)

$$= \frac{6C_2}{10C_2} = \frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., P(B/A) is required.

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$$

19. A bolt is manufactured by 3 machines A, B and C. A turns out twice as many items as B, and machines B and C produce equal number of items. 2% of bolts produced by A and B are defective and 4% of bolts produced by C are defective. All bolts are put into 1 stock pile and chosen from this pile. What is the probability that it is defective?

Solution:

Let A = the event in which the item has been produced by machine A, and so on.

Let D = the event of the item being defective.

$P(A) = \frac{1}{2}$, $P(B) = P(C) = \frac{1}{4}$

P(D/A) = P(an item is defective, given that A has produced it)

$$= \frac{2}{100} = P(D/B)$$

$$P(D/C) = \frac{4}{100}$$

By theorem of total probability,

P(D) = P(A)× P(D/A) + P(B) × P(D/B) + P(C) ×P(D/c)

$$= \frac{1}{2} \times \frac{2}{100} + \frac{1}{4} \times \frac{2}{100} + \frac{1}{4} \times \frac{4}{100}$$

$$= \frac{1}{40}$$

20. In a coin tossing experiment, if the coin shows head, one die is thrown and the result is recorded. But if the coin shows tail, 2 dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2?

Solution:

When a single die is thrown, P(2) = 1/6

When 2 dice are thrown, the sum will be 2 only if each dice shows 1.

$\therefore$ P(getting 2 as sum with 2 dice) = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (since independence)

By theorem of total probability,

P(2) = P(H) × P(2/H) + P(T) × P(2/T)

$$= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{36} = \frac{7}{72}$$

21. An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and place in the second urn and then one ball is taken at random from the latter. What is the probability that it is a white ball?

Solution:

The two balls transferred may be both white or both black or one white and one black.

Let B_1 = event of drawing 2 white balls from the first urn, B_2 = event of drawing 2 black balls from it and B_3 = event of drawing one white and one black ball from it.

Clearly B_1, B_2 and B_3 are exhaustive and mutually exclusive events.

Let A = event of drawing a white ball from the second urn after transfer.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26}$$

$$P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26}$$

$$P(B_3) = \frac{10 \times 3}{13C_2} = \frac{10}{26}$$

$P(A/B_1)$ = P(drawing a white ball / 2 white balls have been transferred)

= P(drawing a white ball / urn II contains 5 white and 5 black balls)

$$= \frac{5}{10}$$

Similarly, $P(A/B_2) = \frac{3}{10}$ and $P(A/B_3) = \frac{4}{10}$

By theorem of total probability,

$$P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)$$

$$= \frac{15}{26} \times \frac{5}{10} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{4}{10} = \frac{59}{130}$$

22. In 1989 there were three candidates for the position of principal – Mr.Chatterji, Mr. Ayangar and Mr. Singh – whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0.3. The probabilities of Mr. Ayangar and Mr.Singh doing the same are respectively 0.5 and 0.8. What is the proabability that there will be co-education in the college?

Solution: Let the events and probabilities be defined as follows:

A: Introduction of co-education

E_1: Mr.Chatterji is selected as principal

E_2: Mr.Ayangar is selected as principal

E_3: Mr.Singh is selected as principal

Then,

P(E_1) = $\frac{4}{9}$ P(E_2) = $\frac{2}{9}$ P(E_3) = $\frac{3}{9}$

P(A/E_1) = 0.3 P(A/E_2) = 0.5 P(A/E_3) = 0.8

P(A) = $P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3)]$

= $P[(A \cap E_1) + (A \cap E_2) + (A \cap E_3)]$

= P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + P(E_3) P(A/E_3)

= $\frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{5}{10} + \frac{3}{9} \times \frac{8}{10} = \frac{23}{45}$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Differential and Integral Calculus – SMTA1103

Curvature: The rate of bending of a curve in any interval is called the curvature of the curve in that interval.
Curvature of a circle: The curvature of a circle at any point on it equals the reciprocal of its radius.
Radius of curvature: The radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature

Centre of curvature: The circle which touches the curve at P and whose radius is equal to the radius of curvature and its centre is known as centre of curvature.
Equation of circle of curvature:

Equation of circle of curvature: $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

Centre of curvaturee: $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$ $\quad \bar{y} = y + \frac{1}{y_2}(1+y_1^2)$

Evolute: The locus of the centre of curvature is called an evolute
Involute: If a curve C1 is the evolute of C2 , then C2 is said to be an involute of a curve C1.

Parametric equation of some standard curves

Curve	Parametric form
$Y^2 = 4ax$ (parabola)	$X = at^2$, $y = 2at$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse)	$X = a\cos\theta$, $y = b\sin\theta$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (hyperbola)	$X = a\sec\theta$, $y = b\tan\theta$
$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$	$X = a\cos^3\theta$, $y = a\sin^3\theta$
$Xy = c^2$ (rectangular hyperbola)	$X = ct$, $y = \frac{c}{t}$

Envelope: A curve which touches each member of a family of curves is called envelope of that family curves.

Envelope of a family of curves: The locus of the ultimate points of intersection of consecutive members of a family of curve is called the envelope of the family of curves.

1. **Find the radius of curvature of y=e^x at x=0**

   Solution: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

   y=e^x

   $y_1=e^x$ at x= 0 y_1=1

   $y_2=e^x$ at x= 0 y_2=1

   $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}$

2. **Find the radius of curvature of at x = $\frac{\pi}{2}$ on the curve y = 4 sin x – sin 2x**

   Solution: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

   y_1=4 cosx – 2 cos 2x at x=$\frac{\pi}{2}$ y_1=2

   y_2=4 sin x + sin 2x at x = $\frac{\pi}{2}$ y_2=-4

   $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+4)^{3/2}}{-4} = \frac{5\sqrt{5}}{2}$

3. **Given the coordinates of the centre of curvature of the curve is given as $\overline{x}$ = 2a +3at²**

$\bar{y}$ = -2at^3 Determine the evolute of the curve

Solution: $\bar{x} = 2a + 3at^2$ $\quad t^2 = (\bar{x} - 2a/_{3a})$------------ 1

$\bar{y} = -2at^3$ $\quad t^3 = \bar{y}/-2a$ ------------ 2

$(\bar{x} - 2a/_{3a})^3 = (\bar{y}/-2a)^2$

$4(\bar{x}-2a)^3=27a\bar{y}^2$

The locus of the centre of curvature (evolute) is $4(x-2a)^3=27ay^2$

4. Write the envelope of $Am^2+Bm+C=0$, where m is the parameter and A, B and C are functions of x and y.

Solution: Given $Am^2+Bm+C=0$……………………(1)

Differentiate (1) partially w.r.t. 'm'

$2Am+B=0 \quad m=-B/2A$…………(2)

Substitute (2) in (1) we get

$A(-B/2A)^2+B(-B/2A)+C=0$

$AB^2/4A^2-B^2/2A+C=0$

$AB^2-2AB^2+4A^2C=0$

$- AB^2+4A^2C=0$

Therefore $B^2-4AC=0$ which is the required envelope.

5. Find the radius of curvature at any point of the curve $y=x^2$. (NOV-07)

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $y=x^2$ $y_1=\frac{dy}{dx}=2x$ and $Y_2=\frac{d^2y}{dx^2}=2$

$$\rho = \frac{(1+(2x)^2)^{3/2}}{2} = \frac{(1+4x^2)^{3/2}}{2}$$

6. Find the envelope of the family of $x \sin \alpha + y \cos \alpha = p$, α being the parameter.

Solution: **Given** $x \sin \alpha + y \cos \alpha = p$……………. (1)

Differentiate (1) partially w.r.t. 'α'

$X \cos \alpha - y \sin \alpha = o$…………………..(2)

Eliminate α between (1) and (2)

$X \cos \alpha = y \sin \alpha \Rightarrow \frac{\sin \alpha}{\cos \alpha} = \frac{x}{y}$ $\Rightarrow$ **Tan** $\alpha = \frac{x}{y}$

Sin $\alpha = \frac{x}{\sqrt{x^2+y^2}}$ $\cos \alpha = \frac{y}{\sqrt{x^2+y^2}}$

Substitute in (1)

$x. \frac{x}{\sqrt{x^2+y^2}} + y. \frac{y}{\sqrt{x^2+y^2}} = p$

$\sqrt{x^2+y^2} = p$

Squaring on both sides, $x^2+y^2=p^2$ which is the required envelope

7. What is the curvature of $x^2+y^2-4x-6y+10=0$ at any point on it .

Solution: Given $x^2+y^2-4x-6y+10=0$

The given equation is of the form $x^2+y^2+2gx+2fy+c=0$

Here $2g=-4$ $g=-2$

$2f=-6$ $f=-3$

Centre $C(2,3)$, radius $r=\sqrt{g^2+f^2-c}$ $=\sqrt{4+9-10}=\sqrt{3}$

Curvature of the circle $=\frac{1}{r}$

Therefore Curvature of $x^2+y^2-4x-6y+10=0$ is $\frac{1}{\sqrt{3}}$

8. Find the envelope of the family of straight lines $y=mx\pm\sqrt{m^2-1}$, where m is the parameter

Solution: Given $y=mx\pm\sqrt{m^2-1}$

$(y-mx)^2=m^2-1$

$Y^2+m^2x^2-2mxy-m^2+1=0$

$m^2(x^2-1)-2mxy+y^2+1=0$ which is quadratic in 'm'

Here, $A=x^2-1$ $B=-2xy$ $C=y^2+1$

The condition is $B^2-4AC=0$

$4x^2y^2-4(x^2-1)(y^2+1)=0$

$4x^2y^2-4x^2y^2-4x^2+4y^2+4=0$

$X^2-y^2=4$ which is the required envelope

9. Find the curvature of the curve $2x^2 + 2y^2 + 5x - 2y + 1 = 0$

Solution: Given $2x^2 + 2y^2 + 5x - 2y + 1 = 0$

$\div 2$

$x^2 + y^2 + 5/2x - y + 1/2 = 0$

Here $2g = 5/2$ $g = 5/4$

$2f = -1$ $f = -1/2$ centre $C(-5/4, 1/2)$ radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{25}{16} + \frac{1}{4} - \frac{1}{2}} = \sqrt{\frac{21}{16}} = \frac{\sqrt{21}}{4}$

Therefore Curvature of the circle $2x^2 + 2y^2 + 5x - 2y + 1 = 0$ is $\frac{1}{r} = \frac{4}{\sqrt{21}}$

10. State any two properties of evolute .

Solution: (i) The normal at any point of a curve is a tangent to its evolute touching at the corresponding contre of curvature. (ii) The difference between the radii of curvature at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

11. Define the curvature of a plane curve and what the curvature of a straight line.

Solution: The rate at which the plane curve has turned at a point (rate of bending of a curve is called the curvature of a curve. The curvature of a straight line is zero.

12. Define evolute and involute .

Solution: The locus of centre of curvature of a curve ($B_1, B_2, B_3, \ldots$) is called evolute of the given curve.

If a curve C_2 is the evolute of a curve C_1 ,then C_1 is said to be an involute of a curve C_2.

13. Find the radius of curvature of the curve $x^2+y^2-6x+4y+6=0$

Solution: Given $X^2+y^2-6x+4y+6=0$

The given equation is of the form $x^2+y^2+2gx+2fy+c=0$

Here $2g=-6$ $g=-3$

$2f=4$ $f=2$

Centre C(3,-2), radius $r=\sqrt{g^2+f^2-c}=\sqrt{4+9-6}=\sqrt{7}$

Radius of Curvature of the circle = radius of the circle$=\sqrt{7}$

14. Find the envelope of the family of circles $(x-\alpha)^2+y^2=4\alpha$,where α is the parameter.(

Solution: Given $(x-\alpha)^2+y^2=4\alpha$

$X^2-2\alpha x+\alpha^2-4\alpha+y^2=0$

$\alpha^2-2\alpha(x+2)+x^2+y^2=0$ which is quadratic in $\propto$

The condition is $B^2-4AC=0$

Here $A=1$ $B=-2(x+2)$ $C=x^2+y^2$

$4(x+2)^2-4(x^2+y^2)=0$

$x^2-4x+4-x^2-y^2=0$

$y^2+4x=4$ which is the required envelope.

15. Define evolute .

Solution: The locus of centre of curvature $(\bar{x},\overline{y})$ is called an evolute .

16. Find the envelope of the family of straight lines y=mx+$\frac{a}{m}$ for different values of 'm'.

Solution: Given $y=mx+\frac{a}{m}$

$m^2x-my+a=0$ which is quadratic in 'm'

The condition is $B^2-4AC=0$

Here A=x B=-y C=a

$Y^2-4ax = 0$

There fore $y^2= 4ax$ which is the required envelope.

17. Find the envelope of the line $\frac{x}{t}$+yt=2c, where't' is the parameter.

Solution: Given $\frac{x}{t}+yt=2c$

$Yt^2-2ct+x=0$ which is quadratic in 't'

The condition is $B^2-4AC=0$

Here A=y B=-2c C=x

$C^2-xy=0$

Therefore $xy=c^2$ which is the required envelope.

18. Find the radius of curvature of the curve y=c cosh(x/c)at the point where it crosses the y-axis.

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given y=c cosh(x/c) and the curve crosses the y-axis. (i.e.)x=0 implies y=c.

Therefore the point of intersection is (0,c)

$\frac{dy}{dx}$**=c sin h(x/c)(1/c)=sin h (x/c)**

$\frac{dy}{dx}$**(0,c)=sinh 0= 0**

$\frac{d^2y}{dx^2}$**=cos h(x/c)(1/c)**

$\frac{d^2y}{dx^2}$**(0,c)= cos h(0) (1/c) = 1/c**

$$\rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{1}{c}} = c$$

19. **Find the radius of curvature of the curve xy=c²at (c,c).**

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $xy=c^2$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$ implies $\frac{dy}{dx}$ (c,c)=-1

$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y.1}{x^2}\right]$

$\frac{d^2y}{dx^2}(c,c) = -\left[\frac{c(-1)-c}{c^2}\right] = \frac{2c}{c^2} = \frac{2}{c}$

$$\rho = \frac{(1+(-1)^2)^{3/2}}{2/c} = \frac{c2\sqrt{2}}{2}$$

$$\rho = c\sqrt{2}$$

20. **Find the envelope of the family of straight lines y= mx$\pm\sqrt{a^2m^2+b^2}$, where m is the parameter**

Solution: Given y= mx$\pm\sqrt{a^2m^2+b^2}$

$(y-mx)^2=(a^2m^2+b^2)$

$Y^2+m^2x^2-2mxy-a^2m^2-b^2=0$

$m^2(x^2-a^2)-2mxy+y^2-b^2=0$ which is quadratic in 'm'

Here, $A=x^2-a^2$ $B=-2xy$ $C=y^2-b^2$

The condition is $B^2-4AC=0$

$4x^2y^2-4(x^2-a^2)(y^2-b^2)=0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is the required envelope

21. **Write down the formula for radius of curvature in terms of parametric coordinate system.**

Solution: Radius of curvature $\rho=\frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$.

22. **Define the circle of curvature at a point $P(x_1,y_2)$ on the curve $y=f(x)$.**

Solution: The circle of curvature is the circle whose centre is the centre of curvature and radius is the radius of curvature. Therefore the equation of circle of curvature is

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$$

PART-B

1. Find the radius of curvature at the point $(acos^3\theta, asin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: Given x= $acos^3\theta$……………………(1)

Y= $asin^3\theta$…………………..(2)

Differentiate (1) and (2) w.r.t θ

$\frac{dx}{d\theta} = 3acos^2\theta(-sin\theta) = -3a\, sin\theta cos^2\theta$

$$\frac{dy}{d\theta} = 3asin^2\theta(cos\theta) = 3acos\theta sin^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3acos\theta sin^2\theta}{-3asin\theta cos^2\theta} = -tan\theta$$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}(-tan\theta).\frac{d\theta}{dx}$

$= -sec^2\theta.\frac{1}{-3a\, sin\theta\, cos^2\theta}$

$\frac{d^2y}{dx^2} = \frac{1}{3a\, sin\,\theta\, cos^4\theta}$

Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+tan^2\theta)^{3/2}}{\frac{1}{3a\sin\theta cos^4\theta}} = 3a\sin\theta cos^4\theta\,(sec^2\theta)^{3/2}$

$=3a\, sin\theta cos^4\theta sec^3\theta = 3asin\theta cos\theta$

$$\rho = 3asin\theta cos\theta$$

2. Find the radius of curvature of the curve $y^2 = x^2\frac{(a+x)}{(a-x)}$ at the point (-a, 0).

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given $y^2 = x^2\frac{(a+x)}{(a-x)} = \frac{ax^2+x^3}{a-x}$

Differentiate w.r.t. 'x'

$2y\frac{dy}{dx} = \frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{(a-x)^2}$

$$\frac{dy}{dx} = \frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{2y(a-x)^2}$$

$$\frac{dy}{dx}(-a,0) = \frac{2a(-2a^2+3a^2)+(a^3-a^3)}{0} = \infty$$

$$\therefore \rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{\mathbf{d^2x}}{\mathbf{dy^2}}}$$

$$\frac{dx}{dy} = \frac{y(a-x)^2}{(a^2x+ax^2-x^3)} = \frac{0}{(a^2x+ax^2-x^3)} = 0$$

$$\frac{\mathbf{d^2x}}{\mathbf{dy^2}} = \frac{(a^2x+ax^2-x^3)\left[y.2(a-x)\left(-\frac{dx}{dy}\right)+(a-x)^2.1\right]-y(a-x)^2\left[a^2\frac{dx}{dy}+2ax\frac{dx}{dy}-3x^2\frac{dx}{dy}\right]}{(a^2x+ax^2-x^3)^2}$$

$$\frac{\mathbf{d^2x}}{\mathbf{dy^2}}(-\boldsymbol{a},\boldsymbol{o}) = \frac{(-\boldsymbol{a}^3+\boldsymbol{a}^3+\boldsymbol{a}^3)(4\boldsymbol{a}^2)}{(-\boldsymbol{a}^3+\boldsymbol{a}^3+\boldsymbol{a}^3)^2} = \frac{\mathbf{4}\boldsymbol{a}^5}{\boldsymbol{a}^6} = \frac{\mathbf{4}}{\boldsymbol{a}}$$

$$\therefore \rho = \frac{\{1+0\}^{3/2}}{\frac{\mathbf{4}}{\boldsymbol{a}}} = \frac{a}{4}$$

3. Find the radius of curvature at the point (a,0)on the curve $xy^2 = a^3 - x^3$.

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given $xy^2 = a^3 - x^3$

Differentiate w.r.t.'x'

$2xy\frac{dy}{dx}+y^2.1=-3x^2 \qquad \frac{dy}{dx} = \frac{-3x^2-y^2}{2xy}$ ……………….(1)

$$\frac{dy}{dx}(a,0) = \frac{-3a^2-0}{2a.0} = \infty$$

Therefore $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

$$\frac{dx}{dy} = \frac{2xy}{-3x^2-y^2} \dots \dots \dots (1)$$

$$\frac{dx}{dy}(a,0) = \frac{2a.0}{-3a^2-0} = 0$$

Differentiate (2) w.r.t.'y'.

$$\frac{d^2x}{dy^2} = \frac{2\left[(-3x^2-y^2)\left(x.1+y.\frac{dx}{dy}\right)-xy\left(-6x\frac{dx}{dy}-2y\right)\right]}{(-3x^2-y^2)^2}$$

$$\frac{d^2x}{dy^2}(a,0) = \frac{2[(-3a^2-0)(a+0)-0]}{(-3a^2-0)^2} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

Therefore radius of curvature $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = \frac{\{1+0\}^{3/2}}{-2/3a} = \frac{-3}{2}a$

$\rho = \frac{3}{2}a$ (since the radius of curvature is non-negative)

4. Find the curvature of the parabola y^2=4x at the vertex.

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given; y^2=4x

Differentiate w.r.t.'x'

$2y\,\frac{dy}{dx} = 4$

$\frac{dy}{dx}$ =2/y

$$\frac{dy}{dx}(0,0) = \frac{2}{0} = \infty$$

Therefore $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

$\frac{dx}{dy} = \frac{y}{2} \dots\dots\dots (1)$

$$\frac{dx}{dy}(0,0) = 0$$

Differentiate (1) w.r.t.'y'.

$$\frac{d^2x}{dy^2} = \frac{1}{2}$$

Therefore $\rho = \frac{\{1+0\}^{3/2}}{1/2}$=2

Curvature K=1/ ρ =1/2

5. Find the radius of curvature of the curve $27ay^2 = 4x^3$ at the point where the tangent of the curve makes an angle 45^0 with the X- axis.

Solution; Let (x_1, y_1) be the point on the curve at which the tangent makes an angle 45^0 with the X- axis.

$\frac{dy}{dx}(x_1,y_1) = \text{Tan } 45^o = 1$ ---------------------------------- (1)

Given $27ay^2 = 4x^3$

Differentiate w.r.t.'x'

$54ay\frac{dy}{dx} = 12x^2 \quad \frac{dy}{dx} = \frac{2x^2}{9ay}$

$\frac{dy}{dx}(x_1,y_1) = \frac{dy}{dx} = \frac{2x_1^2}{9ay_1}$ ------------------------------------(2)

$\frac{dy}{dx}(x_1,y_1) = \text{Tan } 45^o = 1 = \frac{2x_1^2}{9ay_1}$

Gives $y_1 = \frac{2x_1^2}{9a}$ ------------------------------(3)

As (x_1, y_1) lies on the curve $27ay_1^2 = 4x_1^3$ ------------------------------------(4)

Using $y_1 = \frac{2x_1^2}{9a}$ gives $x_1 = 3a$

And using (3) gives $y_1 = 2a$

Y_1 at $(3a, 2a) = 1$

$Y_2 = \frac{2}{9a}\left[\frac{y.2x - x^2.y_1}{y^2}\right]$

$Y_2 = \frac{2}{9a}\left[\frac{2.3a.2a - 9a^2.1}{4a^2}\right] = 1/6a$

Therefore radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1/6a}$

$$\rho = 12a\sqrt{2}$$

7. Find the radius of curvature for the curve r=a(1+cosθ)at $\theta = \frac{\pi}{2}$ *and prove that* $\frac{\rho^2}{r}$is a constant.

Solution: Given r=a(1+cos θ)

r'= -a sin θ and r'' = cosθ

The radius of curvature in polar form is $\rho = \frac{(r^2+r'^2)^{3/2}}{(r^2+2r'^2-rr'')}$

$$= \frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{[a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta]}$$

$$= \frac{a^3(2+2\cos\theta)^{3/2}}{a^2(1+2\cos^2\theta+2\sin^2\theta+3\cos\theta)} = \frac{a^3 2^{2/3}(1+\cos\theta)^{2/3}}{3a^2(1+\cos\theta)} = \frac{2\sqrt{2}a(1+\cos\theta)^{1/2}}{3}$$

ρ at $\theta = \frac{\pi}{2}$ is $\rho = \frac{2\sqrt{2}a}{3}$

Also, $\rho^2 = \frac{8a^2}{9}(1+\cos\theta) = \frac{8a^2}{9}r$

Therefore, $\frac{\rho^2}{r} = \frac{8}{9}a = constant.$

11. Find the equation of circle of curvature of the parabola y^2=12x at the point (3,6).

Solution: The equation of circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

Where, $\bar{x}=x-\frac{y_1}{y_2}(1+y_1^2)$

$\bar{y}=y+\frac{1}{y_2}(1+y_1^2)$

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}$$

Given y^2=12x

Differentiate w.r.t.'x' we get

$2y\frac{dy}{dx}$=12 implies $\frac{dy}{dx}=\frac{6}{y}$

$Y_1=\frac{dy}{dx}(3,6)=1$ $\quad \frac{d^2y}{dx^2}=\frac{-6}{y^2}\frac{dy}{dx}$

$Y_2=\frac{d^2y}{dx^2}(3,6)=-1/6$

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{(1+1)^{3/2}}{-1/6}=-12\sqrt{2}$$

$\rho=12\sqrt{2}$ (ρ can not be negative)

$$\bar{x}=x-\frac{y_1}{y_2}(1+y_1^2)$$

$=3-\frac{1}{-1/6}(1+1)=15$

$$\bar{y}=y+\frac{1}{y_2}(1+y_1^2)=6+\frac{1}{-1/6}(1+1)=-6$$

Therefore, the equation of circle of curvature is $(x-15)^2+(y+6)^2=288$

12. Find the radius of curvature at 't' on x=e^{t}cost,y=e^{t}sint.

Solution: Radius of curvature $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

Given $x = e^t cost\,, y = e^t sint$

X'=$\frac{dx}{dt} = e^t cost - e^t sint = e^t(cost - sint)$

Y'=$\frac{dy}{dt} = e^t cost + e^t sint = e^t(cost + sint)$

X''=$\frac{d^2x}{dt^2} = e^t(-sint - cost) + e^t(cost - sint) = -2e^t sint$

Y''=$\frac{d^2y}{dt^2} = e^t(-sint + cost) + e^t(cost + sint) = 2e^t cost$

⇒∴ The radius of curvature is $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

$$\rho = \frac{([e^t(cost - sint)]^2 + [e^t(cost + sint)]^2)^{3/2}}{e^t(cost - sint).2e^t cost - e^t(cost + sint).(-2e^t sint)}$$

$$=\frac{(e^{2t}[cos^2t+sin^2t-2sintcost+cos^2t+sin^2t+2sintcost])^{3/2}}{2e^{2t}[cos^2t-sintcost+sintcost+sin^2t]} = \frac{(2e^{2t})^{3/2}}{2e^{2t}} = \sqrt{2}e^t$$

INTRODUCTION: When a unction f(x) is integrated with respect to x between the limits a and b, get the definite integral

$$\int_a^b f(x)dx$$

If the integrand is a function f(x,y) and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (or x) between the limits y_0 and y_1 (or y).

we get a double integral that is denoted by the symbol

$$\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x,y)\, dx\, dy$$

Extending the concept of double integral one step further, we get the tripe integral

$$\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x,y,z)\, dx\, dy\, dz$$

EVALUATION OF DOUBLE INTGRALS

Before starting on double integrals let’s do a quick review of the definition of a definite integrals for functions of single variables. First, when working with the

Integral, $\int_a^b f(x)\, dx$

We think of x's as comimg from the interval a<x<b. For these integrals we can say that we are integrating over the interval a<x<b. Note that this does assume that a<b, however, if we have b<a then we can just use interval b<x<a.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval a<x<b into n subintervals of width Del(x) Δx and choose a point, x1 from each as shown below.

Each of the rectangles has height of $f(x_1)$ and we could then use the area of each of these rectangles to approximate the area as follows.

PROBLEMS BASED ON DOUBLE INTEGRATION IN CARTESIAN COORDINATES

PART A

1. **Evaluate** $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} dydx$

Solution: Let I = $\int_0^2 \left(xe^{\frac{y}{x}}\right)_0^{x^2} dx = \int_0^2 x(e^x - 1)dx$

$$= \left[(x)(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^2$$

$$= 2e^2 - 4 - e^2 + 2 + 1$$

$$= (2)(e^2 - 2) - [(e^2 - 2) - (1)]$$

$$= e^2 - 1$$

2. **Evaluate:** $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$

Solution:Let I = $\int_0^a y \int_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2 - x^2}dx$

$$= \left[\frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$$

$$\frac{= a^2}{2} sin^{-1}(1)$$

$$= \frac{\pi a^2}{4}$$

PART B

1. **Evaluate: $\iint xy(x+y)\,dxdy$ over the area between $y = x^2$ and $y = x$.**

Solution: The limits are: x varies from 0 to 1 and y varies from x^2 to x.

$$I = \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dydx$$

$$= \int_0^1 \left(x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x} \right) dx$$

$$= \int_0^1 x\left\{\left(\frac{x^3}{2} + \frac{x^3}{3}\right) - \left(\frac{x^5}{2} + \frac{x^6}{3}\right)\right\} dx$$

$$= \left(\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right)_0^1$$

$$= \frac{3}{56}$$

2. Evaluate: $\iint x^2y^2dxdy$ over the region in the first quadrant of the circle $x^2+y^2=1$.

Solution: In the given region, y varies from 0 to $\sqrt{1-x^2}$ and x varies from 0 to 1.

$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2y^2\, dydx$

$= \int_0^1 x^2 \left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}} dx = \frac{1}{3}\int_0^1 x^2(1-x^2)^{3/2}\, dx$

Put $x = \sin\theta$. Then $dx = \cos\theta d\theta$. θ varies from 0 to $\pi/2$.

$$\therefore I = \frac{1}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{1}{3}\int_0^{\pi/2} (1-\cos^2\theta)\cos^4\theta d\theta = \frac{1}{3}\left[\int_0^{\pi/2} \cos^4\theta d\theta - \int_0^{\pi/2} \cos^6\theta d\theta\right]$$

$$= \frac{1}{3}\left[\frac{3}{4}\frac{1}{2}\frac{\pi}{2} - \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right] = \frac{\pi}{96}$$

3. Evaluate: $\int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right]$

Solution : Let $I = \int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right] = \int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right] = \int_0^1\left(\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_x^{\sqrt{x}} dx$

$$= \int_0^1\left[\left(\frac{x^3}{2}+\frac{x.xy^{3/2}}{3}\right)-\left(\frac{x^4}{2}+\frac{x^4}{3}\right)\right]dx = \left[\frac{x^4}{8}+\frac{x^{7/2}}{(7/2)(3)}-\frac{5}{6}\left(\frac{x5}{5}\right)\right]_0^1$$

$$= \left(\frac{1}{8}+\frac{2}{21}+\frac{1}{6}\right)-(0) = \left(\frac{21+16-28}{168}\right) = \frac{9}{168} = \frac{3}{56}$$

DOUBLE INTEGRATION IN POLAR COORDINATES:

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)dr\, d\theta$, we first integrate w.r.to r between the limits r1 and r2. Keeping θ_1 is fixed and the resulting expression is integrated w.r.to θ from θ_1 to θ_2.

In this integral this r_1 and r_2 are functions of θ and θ_1, θ_2 are constants.

PROBLEMS BASED ON POLAR FORMS USING DOUBLE INTEGRATION:

PART A

1. **Evaluate:** $\int_0^{\pi/2}\int_0^{\infty}\frac{rdrd\theta}{(r^2+a^2)^2}$

   Solution: $Let\ I=\int_0^{\pi/2}\left[\int_0^{\infty}\frac{rdr}{(r^2+a^2)^2}\right]d\theta$

$$=\frac{1}{2}\int_0^{\pi/2}\left(\frac{-1}{(r^2+a^2)^2}\right)_0^{\infty}d\theta$$

$$=\frac{1}{2a^2}(\theta)_0^{\pi/2}$$

$$=\frac{\pi}{4a^2}$$

2. **Evaluate:** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{2cos\theta}r^2drd\theta$

   Solution: Let I = $\int_{-\pi/2}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2cos\theta}d\theta=\frac{1}{3}\int_{-\pi/2}^{\pi/2}(8cos^3\theta-0)\,d\theta$

$$=\frac{8}{3}\times 2\int_0^{\pi/2}cos^3\theta\,d\theta$$

$$=\frac{16}{3}\frac{2}{3}=\frac{32}{9}$$

PART B

1. **Evaluate: $\iint r^2 sin\theta dr d\theta$ over the cardioids r = a (1+cosθ).**

Solution: The limits of r: 0 to a (1+cosθ) and The limits of θ: 0 to π.

$$I = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 sin\theta\, dr d\theta = \int_0^{\pi} \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} sin\theta d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} sin\theta(1 + cos\theta)^3 d\theta$$

Put 1+cosθ = t then –sinθ dθ = dt

When θ = 0, t = 2

When θ = π, t = 0.

$$\therefore I = \frac{a^3}{3} \int_0^{2} t^3\, dt = \frac{a^3}{3}\left(\frac{a^4}{4}\right)_0^2 = \frac{4a^3}{3}$$

2. **Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\ dxdy$ *using polar cordinates***

Solution: $x = rcos\theta, y = rsin\theta, dxdy = rdrd\theta$ are the polar coordinates for the above integral

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\ \frac{1}{2}\, d(r^2)d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[-e^{-r^2}\right]_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[e^{-r^2}\right]_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}}[0-1]d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

CHANGE THE ORDER OF INTEGRATION:

The double integral

The double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$

when the order of integration is changed. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called change of order of integration. To effect the change of order of integration, the region of integration is identified first, a rough sketch of the region is drawn and then the new limits are fixed.

PART A

1. **Find the limits of integration in the double integral** $\iint_R f(x,y)dxdy, where\ R\ is\ in\ the\ first\ quadrant\ and\ bounded\ by: x = 0, y = 0, x + y = 1.$

   Solution: The limits are: y varies from 0 to 1 and x varies from 0 to 1-y.

2. **Change the order of integration** $\int_0^a \int_y^a f(x,y)dxdy$

   Solution: The given region of integration is bounded by y=0, y=a, x=y & x=a.

   After changing the order, we have, I = $\int_0^a \int_0^x f(x,y)dydx$

3. **Change the order of integration for the double integral** $\int_0^1 \int_0^x f(x,y)dxdy$

   Solution: $\int_0^1 \int_y^1 f(x,y)dxdy$

PART B

1. **Change the order of integration in I =** $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ **and hence evaluate it.**

   Solution: Let I = $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$

   The given region of integration is bounded by x=0, x=1, $y=x^2$ and x+y=2.

   In the given integration x is fixed and y is varying.

   So, after changing the order we have to keep y fixed and x should vary.

   After changing the order we've two regions R_1 & R_2

   $I = I_1+I_2$

   $I = \int_0^1 \int_0^{\sqrt{y}} xy\, dxdy + \int_1^2 \int_1^{2-x} xy\, dxdy$

$$= \frac{1}{2}\left[\int_0^1 (x^2y)_0^{\sqrt{y}}\, dy \int_1^2 (x^2y)_0^{2-y}\, dy\right]$$

$= \frac{1}{2}\left[\int_0^1 y^2\, dy + \int_1^2 y(2-y)^2\, dy\right]$

$$= \frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1 + \left(2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right)_1^2\right] = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ by changing the order of integration.

Solution: The given region is bounded by x=0, x=1, y=x and $x^2+y^2=2$.

$I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$

After changing the order we've,

The region R is splinted into two regions R_1& R_2.

In R_1: limits of x: 0 to y & limits of y: 0 to 1

In R_2: limits of x: 0 to $\sqrt{2-y^2}$ & limits of y: 1 to $\sqrt{2}$

$I = I_1+I_2$

$I_1 = \int_0^1 \int_0^{y} \frac{x}{\sqrt{x^2+y^2}} dxdy$

$$= \int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy = (\sqrt{2}-1)\int_0^1 y\,dy = (\sqrt{2}-1)\left(\frac{1}{2}\right)$$

$I_2 = \int_0^{\sqrt{2}} \int_x^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy$

$$= \int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy = \int_1^{\sqrt{2}} (\sqrt{2}-y)dy = (\sqrt{2}-1)\left(\frac{y2}{2}\right)_0^1 + \sqrt{2}(y)_1^{\sqrt{2}} - \left(\frac{y2}{2}\right)_1^{\sqrt{2}}$$

$$= (2-\sqrt{2}) - \frac{1}{2}$$

$I = (\sqrt{2}-1)\left(\frac{1}{2}\right) + (2-\sqrt{2}) - \frac{1}{2}$

$$= 1 - \frac{1}{\sqrt{2}}$$

3. Evaluate by changing the order of integration in $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

Solution: Let I = $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

The given region of integration is bounded by x=0, x=4, y = $x^2/4$, $y^2 = 4x$

After changing the order we've

Limits of x: $y^2/4$ to $2\sqrt{y}$

Limits of y: 0 to 4

I = $\int_0^4 \int_{y^2/4}^{2\sqrt{y}} dxdy$= 16/3.

4. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)dydx$

Solution: Given $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)dydx$

The given region of integration is bounded by x=0, x=1, $y=x^2$ and x+y=2

In the given integration x is fixed and y is varying

So, after changing the order we have to keep y fixed and x should vary.

After changing the order we have two regions R_1 & R_2

$I = I_1 + I_2$

$$I = \int_0^1 \int_0^{\sqrt{y}} f(x,y)dxdy + \int_1^2 \int_1^{2-x} f(x,y)dxdy$$

PROBLEMS BASED ON AREA AS A DOUBLE INTEGRAL:

Area of the region R in Cartesian form is given by

$$\iint_R dxdy \ \ or \ \iint_R dydx$$

Area of the region R in polar form is given by

$$\iint_R rdrd\theta$$

PROBLEMS BASED ON AREA AS A DOUBLE INTEGRAL:

- Area of the region R in Cartesian form is given by $\iint_R dxdy \ \ or \ \iint_R dydx$
- Area of the region R in polar form is given by $\iint_R rdrd\theta$

PART A

1. **Find the smaller area bounded by y = 2-x and $x^2+y^2=4$.**

Solution: Required area = $\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dydx \int_0^2 [\sqrt{4-x^2} - (2-x)]\, dx$

$$= \left[\frac{x}{2}\sqrt{4^2 - x^2} + 2sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2 = \pi - 2$$

PART B

1. **Find the area of the region outside the inner circle r=2cosθ and inside the outer circle r=4 cosθ by double integration.**

Solution: Required Area = $\iint rdrd\theta$

$$=2\int_0^{\pi/2} \int_{2cos\theta}^{4cos\theta} rdrd\theta$$

$$= \int_0^{\pi/2} (r^2)_{2\cos\theta}^{4\cos\theta}\, d\theta = 12\int_0^{\pi/2} \cos^2\theta = 12 \times \frac{1}{2} \times \frac{\pi}{2} = 3\pi sq.units$$

2. Find the area of the circle of radius 'a' by double integration.

Solution: Transforming Cartesian in Polar coordinates

(i.e.) x=rcosθ & y=rsinθ. Then dxdy = rdrdθ

limits of θ: 0 to$\frac{\pi}{2}$ and limits of r: o to 2$a$cosθ

Required Area = 2xupper area

$$=2 \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r dr d\theta$$

$$=\int_0^{\frac{\pi}{2}} (r^2)_0^{2a\cos\theta}\, d\theta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 4a^2 \frac{1}{2} . \frac{\pi}{2} = \pi a^2 sq.units$$

3. Find $\iint r^3\, dr d\theta$ over the area bounded between the circles r = 2sinθ & r= 4sinθ.

Solution: In the region of integration, r varies from r=2sinθ& r= 4sinθ and θ varies from0to π.

$$I = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 dr d\theta$$

$$= \frac{1}{4}\int_0^{\pi} (r^4)_{2\sin\theta}^{4\sin\theta}\, d\theta = \frac{24}{4}\int_0^{\pi} \sin^4\theta\, d\theta = 60 \times 2 \int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$$

$$= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{45\pi}{2} sq.units.$$

4. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: Area of the ellipse = 4 x area of the first quadrant $=4\int_0^{a} \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy dx$

$$= 4\int_0^{a} (y)_0^{\frac{b}{a}\sqrt{a^2-x^2}}\, dx = \frac{4b}{a}\int_0^{a} \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{a} = \frac{4b}{a}\left(\frac{a^2}{2}\right)\frac{\pi}{2} = \pi ab sq.units$$

5. Find the area inside the circle r=asinθ but lying outside the cardiod r=a(1-cosθ)
Solution: Given curves are r=asinθ and r =a(1-cosθ)
The curves intersect where a sin θ = a (1-cosθ)

5. Find the area inside the circle r=asinθ but lying outside the cardiod r=a(1-cosθ)

Solution: Given curves are r=asinθ and r =a(1-cosθ)

The curves intersect where a sin θ = a (1-cosθ)

$\Rightarrow$ a sin θ = a –a cosθ $\qquad$ $\Rightarrow$ a sin θ + a cosθ = a $\qquad$ $\Rightarrow$ sin θ + cosθ =1

$$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{1}{\sqrt{2}} \qquad \Rightarrow \sin\theta\cos\frac{\pi}{4}+\cos\theta\frac{\pi}{4}\cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin(\theta+\frac{\pi}{4})=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4} \qquad \Rightarrow \theta+\frac{\pi}{4}=\frac{\pi}{4}(or)\pi-\frac{\pi}{4}$$

$$\Rightarrow \theta=0(or)\theta+\frac{\pi}{4}=\pi-\frac{\pi}{4} \qquad \Rightarrow \theta=\pi-\frac{2\pi}{4}=\pi-\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow \theta=0(or)\theta=\frac{\pi}{2}$$

$$\therefore \text{The required area} = \int_0^{\pi/4}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-(1+\cos^2\theta-2\cos\theta\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta = \frac{a^2}{2}\left[\int_0^{\pi/2}\left(1-\cos^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta\right]$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(2\cos\theta-2\cos^2\theta\right)d\theta \qquad = \frac{a^2}{2}.2\int_0^{\pi/2}\left(\cos\theta-\cos^2\theta\right)d\theta$$

$$= a^2\left[\left(\sin\theta\right)_0^{\pi/2}-\int_0^{\pi/2}\cos^2\theta d\theta\right] \qquad = a^2\left[1-\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\right]$$

$$= a^2\left[1-\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right] \qquad = a^2\left[1-\frac{1}{2}\left(\frac{\pi}{2}+0\right)-0\right]$$

$$= a^2\left[1-\frac{\pi}{4}\right]=\frac{a^2(4-\pi)}{4}$$

6.Find by double integration, the area enclosed by the curves $y^2 = 4ax$ and $x^2 = 4ay$

Ans: $y^2 = 4ax \ldots\ldots\ldots\ldots(1)$ $\quad x^2 = 4ay \ldots\ldots\ldots\ldots(2)$

Sub (1) in (2) we get $\left[\frac{y^2}{4a}\right]^2 = 4ay$

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y$$

$$y^4 - 64a^3y = 0 \Longrightarrow y(y^3 - 64a^3) = 0$$

$$y = 0 \ or \ y^3 - 64a^3 = 0 \Longrightarrow y^3 = 64a^3 \Rightarrow y = \sqrt[3]{64a^3} \Rightarrow y = 4a$$

$$If\ y = 0 \Rightarrow x = 0; \quad y = 4a \Rightarrow x = 4a$$

Therefore the point of intersection of (1)&(2) is (0,0) and (4a,4a)

x Varies from 0 to 4a and **y** varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$

The required Area $= \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx = \int_0^{4a}[y]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \int_0^{4a}\left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$

$$= \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a} = \frac{16}{3}$$

The equation of the tangent at (x_1, y_1) is

$Y - y_1 = m(x - x_1)$

The equation of the normal at (x_1, y_1) is

$Y - y_1 = -1/m(x - x_1)$

Example1: Obtain the equations of tangent and normal to the curve $y = 3x^2 - 4x$ at the point (1 , - 1).

Example2: Obtain the equations of the tangent and normal to the curve $xy = 16$ at the point (2, 8)

Example3: Obtain the maxima and minima of the function $2x^3 - 3x^2 - 36x + 10$

Reduction Formulae:

Reduction formulae are integrals involving some variable n\displaystyle{n}n, as well as the usual x\displaystyle{x}x. They are normally obtained from using integration by parts.

We use the notation In\displaystyle{I}_{{n}}In when writing reduction formulae.

Reduction formula for $\int \cos^n x \, \mathrm{d}x,$

$$\int \cos^n x \, \mathrm{d}x,$$

can be evaluated by a reduction formula.

Start by setting:

$I_n = \int \cos^n x \, \mathrm{d}x.$ Now re-write as $I_n = \int \cos^{n-1} x \cos x \, \mathrm{d}x,$

Integrating by this substitution:

$$\cos x \, \mathrm{d}x = \mathrm{d}(\sin x),$$

$$I_n = \int \cos^{n-1} x \, \mathrm{d}(\sin x).$$

Now integrating by parts:

$$\begin{aligned}
\int \cos^n x \, \mathrm{d}x &= \cos^{n-1} x \sin x - \int \sin x \, \mathrm{d}(\cos^{n-1} x) \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, \mathrm{d}x - (n-1) \int \cos^n x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n,
\end{aligned}$$

solving for I_n:

$$I_n + (n-1)I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2},$$

$$nI_n = \cos^{n-1}(x) \sin x + (n-1)I_{n-2},$$

$$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2},$$

so the reduction formula is:

$$\int \cos^n x \, \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x.$$

To supplement the example, the above can be used to evaluate the integral for (say) $n = 5$;

$$I_5 = \int \cos^5 x \, \mathrm{d}x.$$

Calculating lower indices:

$$n = 5, \quad I_5 = \tfrac{1}{5}\cos^4 x \sin x + \tfrac{4}{5} I_3,$$

$$n = 3, \quad I_3 = \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3} I_1,$$

back-substituting:

$$\because I_1 = \int \cos x \, \mathrm{d}x = \sin x + C_1,$$

$$\therefore I_3 = \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x + C_2, \quad C_2 = \tfrac{2}{3} C_1,$$

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x\right] + C,$$

where C is a constant.

Reduction formula for $\int \sin^n x \, dx$

$$\begin{aligned}
\int \sin^n x \, dx &= \int \sin x \, \sin^{n-1} x \, dx \\
&= -\cos x \, \sin^{n-1} x - \int (-\cos x) \, . \, (n-1) \sin^{n-2} x \, \cos x \, dx \\
&\qquad \text{(integrating by parts)} \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, \cos^2 x \, dx \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx \\
&\qquad \text{(since} \quad \cos^2 x = 1 - \sin^2 x \text{)} \\
&= -\cos x \, \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \, . \, (1)
\end{aligned}$$

There is now a term in $\int \sin^n x\,dx$ on the right-hand side as well as on the left-hand side. Bringing these terms together on the left-hand side, (1) becomes

$$n \int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\,dx$$

$$\therefore \int \sin^n x\,dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} \int \sin^{n-2} x\,dx . \qquad (2)$$

We shall now establish a quick method to evaluate

$\int_0^{\pi/2} \sin^n x\,dx$ and $\int_0^{\pi/2} \cos^n x\,dx,\ n \in N$

consider $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$= \int_0^{\pi/2} \sin^{n-1}(x)\,dx \cdot \sin x\,dx$$

$$= \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx$$

Integrating by parts, we get

$$I_n = [\,(\sin x)^{n-1}(-\cos x)\,]_0^{\pi/2} - \int_0^{\pi/2} [\,(n-1)(\sin x)^{n-2} \cos x\,(-\cos x)\,]\,dx$$

$$I_n = [\,(\sin x)^{n-1}(-\cos x)\,]_0^{\pi/2} + (n-1)\int_0^{\pi/2} [\,(\sin x)^{n-2} \cdot \underline{\cos^2 x}\,]\,dx$$

$$I_n = [\,0-0\,] + (n-1)\int_0^{\pi/2} [\,(\sin x)^{n-2} \underline{(1-\sin^2 x)}]\,dx$$

$$I_n = (n-1)\int_0^{\pi/2} [\,(\sin x)^{n-2} - \sin^n x\,]\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx$$

$$\therefore I_n + (n-1) I_n = (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\,dx$$

$$\therefore I_n (1+n-1) = (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\,dx$$

$$\therefore \quad I_n = \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx$$

$$\therefore \qquad I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

$$\therefore \qquad I_{n-2} = \int_0^{\pi/2} \sin^{n-2}(x)\, dx$$

Thus the power of sin x reduces from n to (n - 2). We can continue this process till the power reduces to zero or one.

Consider 1) $\int_0^{\pi/2} \sin^6 x\, dx$

$$= \frac{6-1}{6} \int_0^{\pi/2} \sin^4 x\, dx$$

$$= \frac{5}{6}\left[\frac{4-1}{4} \int_0^{\pi/2} \sin^2 x\, dx\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4}\left[\frac{2-1}{2} \int_0^{\pi/2} \sin^0 x\, dx\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} 1\, dx$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} [\, x\,]_0^{\pi/2}$$

$$\therefore \qquad \int_0^{\pi/2} \sin^6 x\, dx$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

(2) $\int_0^{\pi/2} \sin^5 x\, dx$

$$= \frac{5-1}{5} \int_0^{\pi/2} \sin^3 x\, dx$$

$$= \frac{4}{5}\left[\frac{3-1}{3} \int_0^{\pi/2} \sin x\, dx\right]$$

$$= \frac{4}{5} \cdot \frac{2}{3} [\,-\cos x\,]_0^{\pi/2}$$

$$= \frac{4}{5} \cdot \frac{2}{3} [- 0 + 1]$$

$$= \frac{4}{5} \cdot \frac{2}{3}$$

In general if 'n 'is even.

$$(1) \quad \int_0^{\pi/2} \sin^n (x)\, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-2} \times \ldots \times \frac{1}{2} \times \pi/2$$

If ' n ' is odd

$$(2) \quad \int_0^{\pi/2} \sin^n (x)\, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-2} \times \ldots \times \frac{2}{3}$$

These formulas also holds for $\int_0^{\pi/2} \cos^n (x)\, dx$

These formulas have been given by ' Walli ' Hence they are known as Walli's reduction formulas.

Reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5) \ldots 5.3.1][(n-1)(n-3)(n-5) \ldots 5.3.1]}{(m+n)(m+n-2)(m+n-4) \ldots 5.3.1}\left(\frac{\pi}{2}\right), \text{if both } m \text{ and } n \text{ are even}$$

$$= \frac{[(m-1)(m-3)(m-5) \ldots (2 \text{ or } 1)][(n-1)(n-3)(n-5) \ldots (2 \text{ or } 1)]}{(m+n)(m+n-2)(m+n-4) \ldots (2 \text{ or } 1)} \quad \text{otherwise}$$

Problems

$$\text{Integrate } I = \int (\sec x)^n \, dx$$

Try integration by parts with

$$u = (\sec x)^{n-2} \qquad v = \tan x$$

$$du = (n-2)(\sec x)^{n-3} \sec x \tan x \, dx \qquad dv = \sec^2 x \, dx$$

We get

$$
\begin{aligned}
I = \int (\sec x)^n \, dx = \int u \, dv = uv - \int v \, du \\
&= \tan x (\sec x)^{n-2} - \int (\tan x)(n-2)(\sec x)^{n-3} \sec x \tan x \, dx \\
&= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} \tan^2 x \, dx \\
&= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^{n-2} (\sec^2 x - 1) \, dx \\
&= \tan x (\sec x)^{n-2} - (n-2) \int (\sec x)^n - (\sec x)^{n-2} \, dx \\
&= \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx - (n-2) \cdot I
\end{aligned}
$$

Solving for I:

$$(n-2)I + I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

or

$$(n-1)I = \tan x (\sec x)^{n-2} + (n-2) \int (\sec x)^{n-2} \, dx$$

Dividing by n gives the

Reduction Formula: $\displaystyle \int (\sec x)^n \, dx = \frac{1}{n-1} \tan x (\sec x)^{n-2} + \frac{n-2}{n-1} \int (\sec x)^{n-2} \, dx$

Exponential integral

Another typical example is:

$$\int x^n e^{ax} \, \mathrm{d}x..$$

Start by setting:

$$I_n = \int x^n e^{ax} \, \mathrm{d}x.$$

Integrating by substitution:

$$x^n \, \mathrm{d}x = \frac{\mathrm{d}(x^{n+1})}{n+1},$$

$$I_n = \frac{1}{n+1} \int e^{ax} \, \mathrm{d}(x^{n+1}),$$

Now integrating by parts:

$$\int e^{ax}\,\mathrm{d}(x^{n+1}) = x^{n+1}e^{ax} - \int x^{n+1}\,\mathrm{d}(e^{ax})$$

$$= x^{n+1}e^{ax} - a\int x^{n+1}e^{ax}\,\mathrm{d}x,$$

$$(n+1)I_n = x^{n+1}e^{ax} - aI_{n+1},$$

shifting indices back by 1 (so $n + 1 \to n$, $n \to n - 1$):

$$nI_{n-1} = x^n e^{ax} - aI_n,$$

solving for *In*:

$$I_n = \frac{1}{a}\left(x^n e^{ax} - nI_{n-1}\right),$$

so the reduction formula is:

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{1}{a}\left(x^n e^{ax} - n\int x^{n-1}e^{ax}\,\mathrm{d}x\right).$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Trigonometry – SMTA1103

RIGHT TRIANGLE TRIGONOMETRY

Objectives:

After completing this section, you should be able to do the following:

- Calculate the lengths of sides and angles of a right triangle using trigonometric ratios.
- Solve word problems involving right triangles and trigonometric ratios.

Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

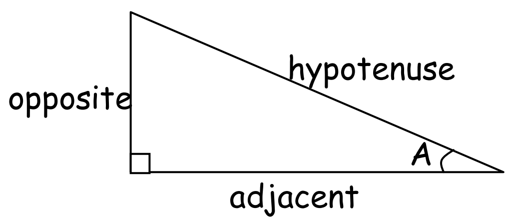
- hypotenuse
- adjacent side
- opposite side

Formulas:

You should be looking for the following formulas as you read:

- ratio for sin(*A*) where *A* is a non-right angle in right triangle
- ratio for cos(*A*) where *A* is a non-right angle in right triangle
- ratio for tan(*A*) where *A* is a non-right angle in right triangle

We will complete our study with a further study of right triangles. We will look at trigonometric value as defined by ratios of the sides of a right triangle.



In the figure above, you can see the sides of a right triangle labeled. The side labeled *hypotenuse* is always opposite the right angle of the right triangle. The names of the other two sides of the right triangle are determined by the angle that is being discusses. In our case, we will be discusing the sides in terms of the angle labeled *A*. The angle *A* is form by the hypotenuse of the right triangle and the side of the right triangle that

hypotenuse

is called *adjacent*. The *adjacent* side will always make up part of the angle that is being discussed and not be the hypotenuse. The side of the right triangle that does not form part of angle A is called the *opposite* side. The *opposite* side will never form part of the angle being discussed. adjacent opposite

The trigonometric function values of a particular value can be as the ratio of a particular pair of sides of a right triangle containing an angle of that measure. We will look at three particular trigonometric ratios.

Trigonometric Ratios

$$\sin(A) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

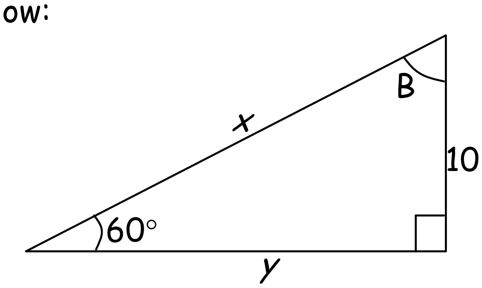
$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$

sin = shortened form of sine function
cos = shortened form of cosine function
tan = shortened form of tangent function
***A* = the angle value**

We will use these ratios to answer questions about triangles below and then we will go through a couple of application problems.

Example 1:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

We are being asked to find values for x, y, and B. We will do the angle B first.

Angle B:

We can find the measure of angle B without using any trigonometric ratios. What we need to remember to find this value is that the sum of the three angles of a triangle will always add up to 180 degrees. It does not matter the size or shape of the triangle. The sum of the three angles will always be 180 degrees. We know that one angle is a right angle. Its measure is 90 degrees. The measure of the other angle is given to be 60 degrees. Thus we just need to calculate

$$90 + 60 + B = 180$$
$$150 + B = 180$$
$$B = 30$$

Thus the measure of angle B is 30 degrees.

Side x:

We will now work to find the length of side x. We need to start by determining which angle we are going to use for our problem. As in the past, it is best to use an angle that is given. Thus we will be using the angle labeled 60 degrees. Our next step is to determine which side x is relative to the angle labeled 60 degrees. Since the side labeled x is opposite the right angle, it is the hypotenuse. There are two trigonometric ratios that include the hypotenuse. Thus, we need to determine which one to use. This will be determined by the other side of the triangle whose measure we know. This is the side labeled 10. Since this side is not one of the sides of the triangle that makes up the angle labeled 60 degrees, it is the opposite side. This means that we need to use the trigonometric ratio that has both the hypotenuse and the opposite side. That ratio is the sine ratio. We will plug into that equation and solve for x.

$$\sin(A) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(60) = \frac{10}{x}$$
$$\frac{\sin(60)}{1} = \frac{10}{x}$$
$$x\sin(60) = 1(10)$$
$$x = \frac{10}{\sin(60)}$$

Through out this solution, we have left sin(60) in this form. This saves us from needing to round until the end of the problem. sin(60) is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately x = 11.54701.

Side y:

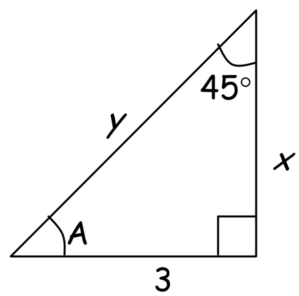
We will finish by finding the length of side y. As in the previous par of the problem, it is best to use the angle labeled 60 degrees. Our next step is to determine which side y is relative to the angle labeled 60 degrees. Since the side labeled y is forms part of the angle labeled 60 degrees, it is the adjacent side. There are two trigonometric ratios that include the adjacent side. Since the other side that is given is the side labeled 10 and this side is the opposite side (see explanation above), we will need to use the trigonometric ratio that has both the opposite side and the adjacent side. That ratio is the tangent ratio. We will plug into that equation and solve for y.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(60) = \frac{10}{y}$$
$$\frac{\tan(60)}{1} = \frac{10}{y}$$
$$y\tan(60) = 1(10)$$
$$y = \frac{10}{\tan(60)}$$

Through out this solution, we have left tan(60) in this form. This saves us from needing to round until the end of the problem. tan(60) is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately y = 5.773503.

Example 2:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

As we solve this problem, we will leave out the explanations of how we determine the names of the sides of the triangle.

Angle A:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90 + 45 + A = 180$$
$$135 + A = 180$$
$$A = 45$$

Thus the measure of angle A is 45 degrees.

Side x:

Side x forms part of the angle that is labeled to be 45 degrees, thus this is the adjacent side. We are also given the measure of the side opposite the angle to be 3. Thus we will want to use the tangent ratio.

$$\tan(45) = \frac{3}{x}$$
$$\frac{\tan(45)}{1} = \frac{3}{x}$$
$$x\tan(45) = 1(3)$$
$$x = \frac{3}{\tan(45)}$$

Our answer is $x = 3$.

We did not need to use trigonometric ratios to find x. We could have used the fact that our triangle has two angles that are equal. Such a triangle is an isosceles triangle. We should recall that the sides of an isosceles triangle opposite the equal angles are equal in length. Thus since one of the side was length 3, the side labeled x is also of length 3.

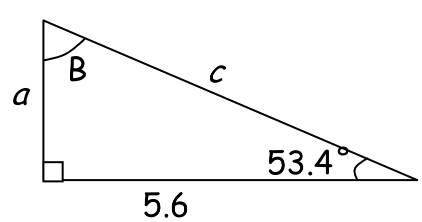
Side y:
Side y is opposite the right angle of the triangle and thus is the hypotenuse. We also have given opposite side to be 3. Thus to find y, we will need to use the sine ratio.

$$\sin(45) = \frac{3}{y}$$
$$\frac{\sin(45)}{1} = \frac{3}{y}$$
$$y\sin(45) = 1(3)$$
$$y = \frac{3}{\sin(45)}$$

Our answer is approximately $y = 4.24264$.

Example 3:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

Angle B:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90 + 53.4 + B = 180$$
$$143.4 + B = 180$$
$$B = 36.6$$

Thus the measure of angle B is 36.6 degrees.

Side c:

We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side c forms part of the triangle that is opposite the right angle. Thus it is the hypotenuse. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the cosine ratio.

$$\cos(53.4) = \frac{5.6}{c}$$
$$\frac{\cos(53.4)}{1} = \frac{5.6}{c}$$
$$c \cdot \cos(53.4) = 1(5.6)$$
$$c = \frac{5.6}{\cos(53.4)}$$

Our answer is approximately c = 9.39243.

Side a:

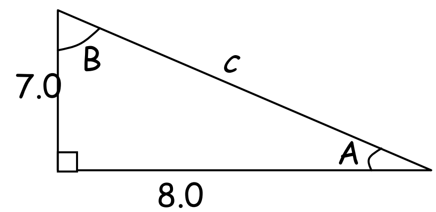
We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side a does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the tangent ratio.

$$\tan(53.4) = \frac{a}{5.6}$$
$$\frac{\tan(53.4)}{1} = \frac{a}{5.6}$$
$$5.6 \cdot \tan(53.4) = 1(a)$$
$$5.6 \cdot \tan(53.4) = a$$

Our answer is approximately a = 7.54041.

Example 4:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

In this problem, we are not given any angle to use. Instead we will need to change the labels of our sides as we solve each of the angles in turn. We will start by finding side c. Since this is a right triangle, we can use the Pythagorean theorem to find the length of c.

Side c:

The legs (a and b) are given to be 7.0 and 8.0. It does not matter which we label a and which we label b.

$$a^2 + b^2 = c^2$$
$$7.0^2 + 8.0^2 = c^2$$
$$49 + 64 = c^2$$
$$113 = c^2$$
$$\sqrt{113} = c$$

The approximate length of side c is 10.63015.

Angle A:

As we solve for angle A, we need to label the sides whose measures are given relative to angle A. The side labeled 8.0 forms part of the angle A. Thus it is the adjacent side. The side labeled 7.0 does not form any part of the angle A. Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for A.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(A) = \frac{7.0}{8.0}$$
$$\tan(A) = 0.875$$

We need to know how to solve for *A* in this equation. As in our section on exponential functions and their inverses, there is an inverse function (a functions that undoes) for the tangent function. On the calculator it is labeled $\tan^{-1}$. Thus we can finally solve for *A* by calculating

$$\tan(A) = 0.875$$
$$A = \tan^{-1}(0.875)$$

Thus the measure of angle *A* is approximately *A* = 41.18593 degrees.

Angle *B*:

As we solve for angle *B*, we need to relabel the sides whose measures are given relative to angle *B*. The side labeled 7.0 forms part of the angle *B*. Thus it is the adjacent side. The side labeled 8.0 does not form any part of the angle *B*. Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for *A*.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(A) = \frac{8.0}{7.0}$$

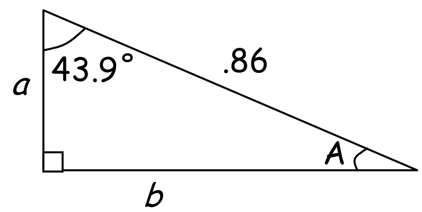
We will once again use the inverse function of the tangent function. On the calculator it is labeled $\tan^{-1}$. Thus we can finally solve for *B* by calculating

$$\tan(B) = \frac{8.0}{7.0}$$
$$B = \tan^{-1}\left(\frac{8.0}{7.0}\right)$$

Thus the measure of angle *B* is approximately *B* = 48.81407 degrees.

Example 5:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

Angle A:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90+43.9+A=180$$
$$133.9+A=180$$
$$A=46.1$$

Thus the measure of angle A is 46.1 degrees.

Side a:

We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side a forms part of the angle labeled 43.9 degrees. Thus it is the adjacent side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the cosine ratio.

$$\cos(43.9)=\frac{a}{.86}$$
$$\frac{\cos(43.9)}{1}=\frac{a}{.86}$$
$$.86\cos(43.9)=1(a)$$
$$.86\cos(43.9)=a$$

Our answer is approximately $a = 0.619674$.

Side b:

We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side b does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the sine ratio.

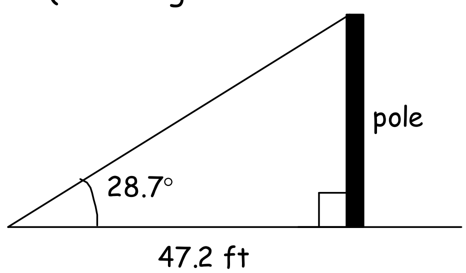
$$\sin(43.9) = \frac{b}{.86}$$
$$\frac{\sin(43.9)}{1} = \frac{b}{.86}$$
$$.86 \cdot \sin(43.9) = 1(b)$$
$$.86 \cdot \sin(43.9) = b$$

Our answer is approximately $b = 0.596326$.

We will finish by looking at some application problems for our right triangle trigonometric ratios.

Example 6:

A support cable runs from the top of the telephone pole to a point on the ground 47.2 feet from its base. If the cable makes an angle of 28.7° with the ground, find (rounding to the nearest tenth of a foot)



a. the height of the pole
b. the length of the cable

Solution:

The picture above shows that we have a right triangle situation.

Part a:

The pole is opposite the angle of 28.7 degrees that is given. The other side of the triangle that we know to be 47.2 feet forms part of the angle of 28.7 degrees. Thus it is the adjacent side. We will be able to use the tangent ratio to solve this problem since it includes both the opposite and adjacent sides.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(28.7) = \frac{\text{pole}}{47.2}$$
$$\frac{\tan(28.7)}{1} = \frac{\text{pole}}{47.2}$$
$$47.2\tan(28.7) = 1(\text{pole})$$
$$47.2\tan(28.7) = \text{pole}$$

Thus the pole is approximately 25.8 feet tall.

Part b:

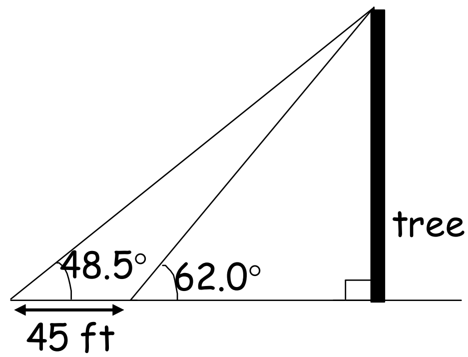
The cable is opposite the right angle of triangle and thus is the hypotenuse. We still know that the adjacent side is 47.2 feet. We will be able to use the cosine ratio to solve this problem since it includes both the hypotenuse and adjacent sides.

$$\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos(28.7) = \frac{47.2}{\text{cable}}$$
$$\frac{\cos(28.7)}{1} = \frac{47.2}{\text{cable}}$$
$$\text{cable} \cdot \cos(28.7) = 1(47.2)$$
$$\text{cable} = \frac{47.2}{\cos(28.7)}$$

Thus the cable is approximately 53.8 feet tall.

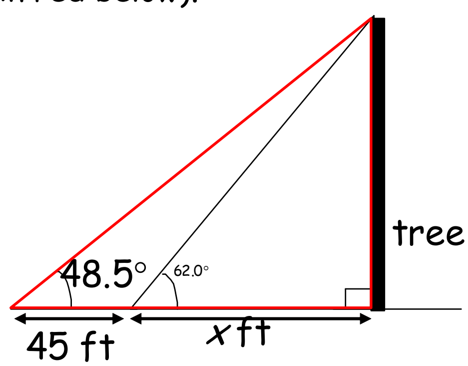
Example 7:

You are hiking along a river and see a tall tree on the opposite bank. You measure the angle of elevation of the top of the tree and find it to be 62.0°. You then walk 45 feet directly away from the tree and measure the angle of elevation. If the second measurement is 48.5°, how tall is the tree? Round your answer to the nearest foot.



Solution:

This problem will require a little more algebra than the previous problems. We will start by looking at the two different triangles that we have and writing trigonometric ratios that include the tree for each of our triangles. We will start by looking at the bigger triangles (shown in red below).

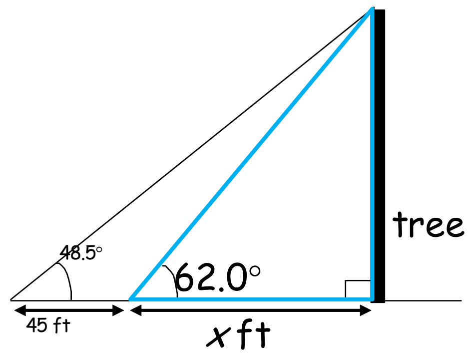


In this red triangle, the tree is opposite the angle that is given to be 48.5 degrees. We are also given the length of 45 feet as part of the side that is adjacent to the angle given to be 48.5 degrees. Since we have part of this adjacent side, we are going to label the other part of the side to be x feet. Thus the whole adjacent side is $45+x$ feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the red triangle, we have

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(48.5) = \frac{\text{tree}}{45+x}$$

We will continue by looking at the smaller triangle (shown in blue below)



In this blue triangle, the tree is opposite the angle that is given to be 62 degrees. Based on what we did for the red triangle, we know that the length of the side adjacent to the angle given to be 62 degrees is x feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the blue triangle, we have

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(62) = \frac{\text{tree}}{x}$$

Now we are ready for the algebra. We have two equations $\tan(48.5) = \frac{\text{tree}}{45 + x}$ and $\tan(62) = \frac{\text{tree}}{x}$ with two unknowns (variable). We can use the substitution method (solve one of the equations for one of the variable and then plug that in to the other equation) to determine the height of the tree. One way to go here is to solve the equation $\tan(62) = \frac{\text{tree}}{x}$ for x. This will give us

$$\tan(62) = \frac{\text{tree}}{x}$$

$$\frac{\tan(62)}{1} = \frac{\text{tree}}{x}$$

$$x\tan(62) = 1(\text{tree})$$

$$x = \frac{\text{tree}}{\tan(62)}$$

We can now plug this expression for x into the equation $\tan(48.5) = \frac{\text{tree}}{45 + x}$ and solve for the height of the tree.

$$\tan(48.5) = \frac{tree}{45 + x}$$

$$\frac{\tan(48.5)}{1} = \frac{tree}{45 + x}$$ Write both sides as fractions.

$$(45 + x)\tan(48.5) = tree$$ Cross multiply.

$$45\tan(48.5) + x\tan(48.5) = tree$$ Apply distributive property.

$$45\tan(48.5) + \left(\frac{tree}{\tan(62)}\right)\tan(48.5) = tree$$ Substitute in expression for *x*.

$$45\tan(48.5) = tree - \left(\frac{tree}{\tan(62)}\right)\tan(48.5)$$ Collect variable on the same side

$$45\tan(48.5) = tree\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)$$ Factor out the variable *tree*

$$\frac{45\tan(48.5)}{\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)} = tree$$ Divide both sides by value in parentheses.

We now just need to plug this expression into our calculator to find out the height of the tree. The tree is approximately 127 feet tall.