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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 1 – MATRICES – SMTA1101

UNIT I

MATRICES

CHARACTERISTIC EQUATION:

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A

Working rule to find characteristic equation:

<u>For a 3 x 3 matrix:</u>

<u>Method 1:</u>

The characteristic equation is $|A - \lambda I| = 0$

<u>Method 2:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$,

$S_3 = Determinant of A = |A|$

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of A = |A|$

Problems:

1. **Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 =$ $sum of the main diagonal elements = 1 + 2 = 3$,

$S_2 = Determinant of A = |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the characteristic equation of $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$**

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 = sum of the main diagonal elements$ = 8 + 7 + 3 = 18,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 +$ $20 + 20 = 45$, $S_3 = Determinant of A = |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

3. **Find the characteristic polynomial of $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic polynomial of A is $\lambda^2 - S_1\lambda + S_2$ where $S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic polynomial is $\lambda^2 - 5\lambda + 7$

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ = $\begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

2. **If A = $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ $\boldsymbol{write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem}$**

Solution:Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 6$

$S_2 = |A| = 5$

Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

i.e., $A^2 = 6A - 5I$

3. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

<u>**Solution**</u>: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| \ = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$
$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

(1) $\Rightarrow$ $A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ---------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - \ 8A^2 + 3A$

Therefore, $A^4 \ = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by A^{-1}, $A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4. **Verify that A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4**

Solution:Given A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 =$ Sum of the main diagonal elements $= 0$

$S_2 = |A| = -1 - 4 = -5$

Therefore, the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$--------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find A^4:

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

5. **Find A^{-1} if A = $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem**

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 - 1 = 2$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (-2+1) + (-1-8) + (2+3)$
$= -1 - 9 + 5 = -5$

$S_3 = |A| = 1(-2+1) + 1(-3+2) + 4(3-4) = -1 - 1 - 4 = -6$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ---------- (1)

To find A^{-1}:

Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I)$ ----------- (2)

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

6. **If A = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $find\ A^n\ in\ terms\ of\ A$**

Solution: The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 = 3$

$S_2 = |A| = 2 - 0 = 2$

The characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$ i.e., $\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm 1}{2} = 2,1$

To find A^n:

When λ^n is divided by $\lambda^2 - 3\lambda + 2$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + b$

$\lambda^n = (\lambda^2 - 3\lambda + 2)Q(\lambda) + a\lambda + b$ --------- (1)

When $\lambda = 1, 1^n = a + b$

When $\lambda = 2, 2^n = 2a + b$

$2a + b = 2^n$ ---------- (2)

$a + b = 1^n$ -------- (3)

Solving (2) and (3), we get, (2) - (3)⇒ $a = 2^n - 1^n$

(2) – 2 x (3) ⇒ $b = -2^n + 2(1)^n$

i.e., $a = 2^n - 1^n$

$b = 2(1)^n - 2^n$

Since $A^2 - 3A + 2I = 0$ by Cayley-Hamilton theorem, (1) ⇒ $A^n = aA + bI$

$$A^n = (2^n - 1^n)\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} + [2(1)^n - 2^n]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

7. **Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ to express as a linear polynomial in A (i) $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ (ii) $A^4 - 4A^3 - 5A^2 + A + 2I$**

Solution: Given A = $\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

By Cayley-Hamilton theorem, we get, $A^2 - 4A - 5I = 0$ -------- (1)

$\lambda^3 - 2\lambda + 3$

$\lambda^2 - 4\lambda - 5 \,|\, \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

$\lambda^5 - 4\lambda^4 - 5\lambda^3$

$-2\lambda^3 + 11\lambda^2 - \lambda$

(-)$-2\lambda^3 + 8\lambda^2 + 10\lambda$

$3\lambda^2 - 11\lambda - 10$

$(-)\ 3\lambda^2 - 12\lambda - 15$

$\lambda + 5$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I = 0 + A + 5I$$

$= A + 5I$ (by (1)) which is a linear polynomial in A

(i)

λ^2

$\lambda^2 - 4\lambda - 5 \,|\, \lambda^4 - 4\lambda^3 - 5\lambda^2 + \lambda + 2$

$$\lambda^4 - 4\lambda^3 - 5\lambda^2$$

(-) $\lambda + 2$

$A^4 - 4A^3 - 5A^2 + A + 2I = A^2(A^2 - 4A - 5I) + A + 2I$ = 0 + A + 2I = A +2I (by (1)) which is a linear polynomial in A

8. Using Cayley-Hamilton theorem, find A^{-1} when A = $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 + 1 = 3$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (1-1) + (1-3) + (1-0)$
$= 0 - 2 + 1 = -1$

$S_3 = |A| = 1(1-1) + 0(2+1) + 3(-2-1) = 1(0) + 0 - 9 = -9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

Pre-multiplying by A^{-1}, we get, $A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-A^2 = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix};\ 3A = \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix};\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

9. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Solution: Given A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements = 1+2+1 = 4

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (2-6) + (1-7) + (2-12)$
$= -4 - 6 - 10 = -20$

$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3 + 42 = 35$

The characteristic equation is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

To prove that: $A^3 - 4A^2 - 20A - 35I = 0$

$$A^2 = \begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix} = \begin{bmatrix}1+12+7 & 3+6+14 & 7+9+7\\ 4+8+3 & 12+4+6 & 28+6+3\\ 1+8+1 & 3+4+2 & 7+6+1\end{bmatrix} = \begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix} = \begin{bmatrix}20+92+23 & 60+46+46 & 140+69+23\\ 15+88+37 & 45+44+74 & 105+66+37\\ 10+36+14 & 30+18+28 & 70+27+14\end{bmatrix}$$

$$= \begin{bmatrix}135 & 152 & 232\\ 140 & 163 & 208\\ 60 & 76 & 111\end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix}135 & 152 & 232\\ 140 & 163 & 208\\ 60 & 76 & 111\end{bmatrix} - 4\begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix} - 20\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix} - 35\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}135 & 152 & 232\\ 140 & 163 & 208\\ 60 & 76 & 111\end{bmatrix} - \begin{bmatrix}80 & 92 & 92\\ 60 & 88 & 148\\ 40 & 36 & 56\end{bmatrix} - \begin{bmatrix}20 & 60 & 140\\ 80 & 40 & 60\\ 20 & 40 & 20\end{bmatrix} - \begin{bmatrix}35 & 0 & 0\\ 0 & 35 & 0\\ 0 & 0 & 35\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

Therefore, Cayley-Hamilton theorem is verified.

10. Verify Cayley-Hamilton theorem for the matrix (i) A = $\begin{bmatrix}3 & -1\\ -1 & 5\end{bmatrix}$ $(ii) A = \begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$

<u>Solution</u>:(i) Given A = $\begin{bmatrix}3 & -1\\ -1 & 5\end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 3 + 5 = 8$

$S_2 = |A| = 15 - 1 = 14$

The characteristic equation is $\lambda^2 - 8\lambda + 14 = 0$

To prove that: $A^2 - 8A + 14I = 0$

$$A^2 = \begin{bmatrix}3 & -1\\ -1 & 5\end{bmatrix}\begin{bmatrix}3 & -1\\ -1 & 5\end{bmatrix} = \begin{bmatrix}9+1 & -3-5\\ -3-5 & 1+25\end{bmatrix} = \begin{bmatrix}10 & -8\\ -8 & 26\end{bmatrix}$$

$$8A = 8\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix}$$

$$14I = 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$$

$$A^2 - 8A + 14I = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} - \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Cayley-Hamilton theorem is verified.

(ii) Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

To prove that: $\boldsymbol{A^2 - 4A - 5I = 0}$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix};\ 5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Cayley-Hamilton theorem is verified.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors
2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.
   If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

Symmetric matrix:

If a square matrix A is symmetric, then A = A^T

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors
2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors
   If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

Problems:

1. **Find the eigen values and eigen vectors of the matrix** $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = sum of the main diagonal elements = 1 - 1 = 0,$

$S_2 = Determinant of A =$ $|A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

A is a non-symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$----------- **(1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$

$$3x_1 + x_2 = 0$$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$$

i.e., we get only one equation $x_1 - x_2 = 0$

$$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

<u>Solution</u>: Let A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ which is a non-symmetric matrix

<u>To find the characteristic equation:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 3 + 2 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 =$ $11,$

$S_3 = Determinant of A =$ $|A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \multicolumn{1}{|r}{0} \end{array}$$

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

Therefore, the eigen values are 1, 1, and 5

A is a non-symmetric matrix with repeated eigen values

<u>To find the eigen vectors:</u>

$[A-\lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ -------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------ (2)

$x_1 + 2x_2 - 3x_3 = 0$ ----------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccccccc} 2 & \times & 1 & \times & -3 & \times & 2 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \; \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 \, , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 . Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 1 = 2,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$

$-4 - 4 + 4 = \; -4,$

$S_3 = Determinant of A = \; |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$2 \;\; | \;\; 1 \quad -2 \quad -4 \quad 8$

$$\begin{array}{cccc} 0 & 2 & 0 & -8 \\ 1 & 0 & -4 & 0 \end{array}$$

$$(\lambda-2)(\lambda^2-4)=0 \Rightarrow \lambda=2, \qquad \lambda = 2,-2$$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

<u>To find the eigen vectors:</u>

$$[A-\lambda I]X=0$$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda=-2$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1-2x_2+2x_3=0$ -------- (1)

$x_1+3x_2+x_3=0$ ------------ (2)

$x_1+3x_2+x_3=0$ ---------- (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} -1 & 1 & 2 & -1 \\ 3 & 1 & 1 & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{-4}=\frac{x_2}{-1}=\frac{x_3}{7} \Rightarrow \frac{x_1}{4}=\frac{x_2}{1}=\frac{x_3}{-7}$$

Therefore, $X_1=\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$-------- (1)

$x_1 - x_2 + x_3 = 0$-------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$---------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} -2 & 2 & 0 & -2 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

4. **Find the eigen values and eigen vectors of** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0,$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = - 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

A is a symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ ------- (1)

$x_1 + 7x_2 + x_3 = 0$ ----------- (2)

$3x_1 + x_2 + 3x_3 = 0$ --------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{cccccccc} & & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 & \\ 7 & & 1 & & 1 & & 7 & \end{array}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{4} = \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 3$**,** $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ———— (1)

$x_1 + 2x_2 + x_3 = 0$ ——————— (2)

$3x_1 + x_2 - 2x_3 = 0$ —————— (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{cccccccc} & & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 & \\ 2 & & 1 & & 1 & & 2 & \end{array}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6$**,** $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 - x_2 + x_3 = 0$ -------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ---------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$x_1 x_2 x_3$

$$\begin{matrix} 1 & & 3 & & -5 & & 1 \\ & \times & & \times & & \times & \\ -1 & & 1 & & 1 & & -1 \end{matrix}$$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

5. **Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Determine the algebraic and geometric multiplicity**

<u>Solution</u>: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a symmetric matrix

<u>To find the characteristic equation:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where
$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0$,
$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$
$-1 - 1 - 1 = -3$,
$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$$
\begin{array}{r|rrrr}
-1 & 1 & 0 & -3 & -2 \\
 & 0 & -1 & 1 & 2 \\
\hline
 & 1 & -1 & -2 & \boxed{0}
\end{array}
$$

$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

A is a symmetric matrix with repeated eigen values. The algebraic multiplicity of $\lambda = -1$ is 2

<u>**To find the eigen vectors:**</u>

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 2$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ -------- (1)

$x_1 - 2x_2 + x_3 = 0$ ----------- (2)

$x_1 + x_2 - 2x_3 = 0$ --------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

$$\begin{array}{ccccccc} 1 & & 1 & & -2 & & 1 \\ & \times & & \times & & \times & \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = -1$, $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ------------------ (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 is orthogonal to X_1 *and* X_2 .

$[1 \quad 1 \quad 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$[0 \quad 1 \quad -1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$--------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & l & & m & & n & \\ 1 & & 1 & & 1 & & 1 \\ 1 & & -1 & & 0 & & 1 \end{array}$$

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 *and* X_3. So, the geometric multiplicity of eigen value $\lambda = -1$ is 2

Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$**

Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

Product of the eigen values = $|A|$ = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Product of two eigen values of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigen value**

Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

$$There\ fore, \lambda_1 \lambda_2 \lambda_3 = 32 \Rightarrow 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

3. **Find the sum and product of the eigen values of the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ without finding the roots of the characteristic equation**

Solution: We know that the sum of the eigen values = Trace of A = a + d

Product of the eigen values = $|A|$ = ad – bc

4. **If 3 and 15 are the two eigen values of A =** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$**, find | A| , without expanding the determinant**

Solution: Given $\lambda_1 = 3 \; and \; \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$$

$We \; know \; that \; the \; product \; of \; the \; eigen \; values = \; |A|$

$\Rightarrow (3)(15)(0) = \; |A|$

$\Rightarrow |A| = 0$

5. **If 2, 2, 3 are the eigen values of A =** $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$**, find the eigen values of** A^T

Solution: By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of $A^T \; are \; 2,2,3$

6. **Find the eigen values of A =** $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Solution: Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

7. **Two of the eigen values of A =** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **are 3 and 6. Find the eigen values of** A^{-1}

Solution: Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, $3 + 6 + k = 11 \Rightarrow k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. Find the eigen values of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ *and* -1

<u>Solution</u>: Let A = $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =$ 5 and $S_2 = |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ *and* -1 is A^{-1}

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$|A|$ = 4 - 10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

9. Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

<u>Solution</u>: We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} *are* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. Find the eigen values of A^3 given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of A^2 and A^{-1}?

Solution:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 and 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ and $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ and $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

$= 3(2)^3 + 5(2)^2 - 6(2) + 2 = 24 + 20 - 12 + 2 = 34$

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

$= 3(5)^3 + 5(5)^2 - 6(5) + 2$

$= 375 + 125 - 30 + 2 = 472$

14. Find the eigen values of adj A if A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: Given A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{1}{|A|} adj\ A$

Adj A = |A| A^{-1}

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

|A| =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

Note: A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times the third. Find them

Solution: Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1, 1, 5

16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a*

Solution: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 $\Rightarrow k = 1$

We know that product of the eigen values = |A|

$\Rightarrow$2(3)(k) = |A|

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

17. Prove that the eigen vectors of the real symmetric matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

<u>Solution</u>:The characteristic equation of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$where $S_1 = sum\ of\ the\ main\ diagonal\ elements = 7;$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements\ = 4 + (-8) + 4 = 0$

$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4) - 1(-2) + 3(-14) = -36$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

3	1-7	0	36	
	0	3	-12	-36
	1	-4	-12	0

Therefore, $\lambda = 3, \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda = 3, \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} = \frac{4 \pm 8}{2} = 6, -2$

Therefore, the eigen values of A are -2, 3, 6

<u>To find the eigen vectors:</u>

$$(A - \lambda I)X = 0$$

<u>Case 1:</u> When $\lambda = -2, \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 7x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 + 3x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccccc} 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{array}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\text{Case 2:}}$When $\lambda = 3$, $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 2x_2 + x_3 = 0$ -------- (2)

$3x_1 + x_2 - 2x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\text{Case 3:}}$When $\lambda = 6$, $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 - x_2 + x_3 = 0$ -------- (2)

$3x_1 + x_2 - 5x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To prove that: $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0$

$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$$X_2^T X_3 = [1 \; -1 \;\; 1]\begin{bmatrix}1\\2\\1\end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = [1 \;\; 2 \;\; 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal in pairs

18. Find the sum and product of all the eigen values of the matrix A = $\begin{bmatrix}1 & 2 & 3\\2 & 2 & 4\\1 & 2 & 7\end{bmatrix}$**.Is the matrix singular?**

<u>Solution</u>:Sum of the eigen values = Sum of the main diagonal elements = Trace of the matrix

Therefore, the sum of the eigen values = 1+2+7=10

Product of the eigen values = $|A|$ = 1(14 - 8) -2(14 - 4) + 3(4 - 2) = 6-20+ 6= - 8

$|A| \neq 0$. Hence the matrix is non-singular.

19. Find the product of the eigen values of A = $\begin{bmatrix}1 & 2 & -2\\1 & 0 & 3\\-2 & -1 & -3\end{bmatrix}$

<u>Solution</u>:Product of the eigen values of A = $|A| = \begin{vmatrix}1 & 2 & -2\\1 & 0 & 3\\-2 & -1 & -3\end{vmatrix} = 1(3) - 2(3) - 2(-1) =$

$3 - 6 + 2 = -1$

ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TODIAGONAL FORM:

Orthogonal matrices:

A square matrix A (with real elements) is said to be orthogonal if $AA^T = A^TA = I$ or $A^T = A^{-1}$

Problems:

1. **Check whether the matrix B is orthogonal. Justify. B =** $\begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$

<u>Solution</u>: Condition for orthogonality is $AA^T = A^TA = I$

To prove that: $BB^T = B^TB = I$

B = $\begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$; $B^T = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$

$$BB^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta + 0 & \sin^2\theta + \cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$B^TB = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, B is an orthogonal matrix

2. **Show that the matrix P = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal**

__Solution__:To prove that: $PP^T = P^TP = I$

$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$; $P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$PP^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Similarly, P^TP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, P is an orthogonal matrix

WORKING RULE FOR DIAGONALIZATION

[ORTHOGONAL TRANSFORMATION]:

Step 1: To find the characteristic equation

Step 2: To solve the characteristic equation

Step 3:To find the eigen vectors

Step 4: If the eigen vectors are orthogonal, then form a normalized matrix N

Step 5: Find N^T

Step 6: Calculate AN

Step 7: Calculate D = N^TAN

Problems:

1. **Diagonalize the matrix** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

<u>**Solution**</u>: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 3 + 5 + 3 = 11$

$$S_2 = \textit{Sum of the minors of the main diagonalelements} = (15 - 1) + (9 - 1) + (15 - 1)$$
$$= 14 + 8 + 14 = 36$$

$S_3 = |A| = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5) = 3(14) - 2 - 4 = 42 - 6 = 36$

Therefore, the characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 2, \lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2} = 6,3$

Hence, the eigen values of A are 2, 3, 6

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 2$, $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 3x_2 - x_3 = 0$ ---------- (2)

$x_1 - x_2 + x_3 = 0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{array}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$ -------- (1)

$-x_1 + 2x_2 - x_3 = 0$ ------- (2)

$x_1 - x_2 + 0x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: When $\lambda = 6$, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$ --------- (1)

$-x_1 - x_2 - x_3 = 0$ ------- (2)

$x_1 - x_2 - 3x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = [-1 \quad 0 \quad 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = [1 \quad 1 \quad 1]\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = [1 \quad -2 \quad 1]\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{6}} & \frac{9}{3} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{12}} & \frac{0}{\sqrt{18}} & \frac{36}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

The diagonal elements are the eigen values of A

2. **Diagonalize the matrix** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

<u>**Solution**</u>: Let A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 8 + 7 + 3 = 18$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (21 - 16) + (24 - 4) + (56 - 36)$
$= 5 + 20 + 20 = 45$

$S_3 = |A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 8(5) - 60 + 20 = 0$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda = 0, \lambda = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(45)}}{2(1)} = \frac{18 \pm \sqrt{324 - 180}}{2} = \frac{18 \pm 12}{2}$
$= 15, 3$

Hence, the eigen values of A are 0, 3, 15

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 0$, $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$ --------- (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ ------- (2)

$2x_1 - 4x_2 + 3x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14}=\frac{x_2}{-12+32}=\frac{x_3}{56-36}\Rightarrow\frac{x_1}{10}=\frac{x_2}{20}=\frac{x_3}{20}\Rightarrow\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

$$X_1=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Case 2: When $\lambda=3$, $\begin{bmatrix}5 & -6 & 2\\-6 & 4 & -4\\2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$5x_1-6x_2+2x_3=0$ -------- (1)

$-6x_1+4x_2-4x_3=0$ -------- (2)

$2x_1-4x_2+0x_3=0$ -------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{24-8}=\frac{x_2}{-12+20}=\frac{x_3}{20-36}\Rightarrow\frac{x_1}{16}=\frac{x_2}{8}=\frac{x_3}{-16}\Rightarrow\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$X_2=\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Case 3: When $\lambda=15$, $\begin{bmatrix}-7 & -6 & 2\\-6 & -8 & -4\\2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-7x_1-6x_2+2x_3=0$ ---------- (1)

$-6x_1-8x_2-4x_3=0$ ---------- (2)

$2x_1-4x_2-12x_3=0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -6 & 2 & -7 & -6 \\ -8 & -4 & -6 & -8 \end{array}$$

$$\frac{x_1}{24+16}=\frac{x_2}{-12-28}=\frac{x_3}{56-36}\Rightarrow\frac{x_1}{40}=\frac{x_2}{-40}=\frac{x_3}{20}\Rightarrow\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = [1 \quad 2 \quad 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 2 - 4 = 0$$

$$X_2^T X_3 = [2 \quad 1 \quad -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4-2-2 = 0$$

$$X_3^T X_1 = [2 \quad -2 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 - 4 + 2 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

AN =

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$\frac{1}{3} \begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$N^T AN = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0+0+0 & 2+2-4 & 10-20+10 \\ 0+0+0 & 4+1+4 & 20-10-10 \\ 0+0+0 & 4-2-2 & 20+20+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

The diagonal elements are the eigen values of A

QUADRATIC FORM- REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION:

Quadratic form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form

Example: $2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$ is a quadratic form in three variables

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$$

Problems:

1. **Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$**

Solution: $Q = \begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

Here $x_2x_1 = x_1x_2$; $x_3x_1 = x_1x_3$; $x_2x_3 = x_3x_2$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

2. **Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$**

Solution: $Q = \begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

3. **Write down the quadratic form corresponding to the following symmetric matrix** $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

Solution: Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

The required quadratic form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$$

$= 0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$

NATURE OF THE QUADRATIC FORM:

Rank of the quadratic form: The number of square terms in the canonical form is the rank (r) of the quadratic form

Index of the quadratic form: The number of positive square terms in the canonical form is called the index (s) of the quadratic form

Signature of the quadratic form: The difference between the number of positive and negative square terms = s – (r-s) = 2s-r, is called the signature of the quadratic form

The quadratic form is said to be

(1) **Positive definite** if all the eigen values are positive numbers
(2) **Negative definite** if all the eigen values are negative numbers
(3) **Positive Semi-definite** if all the eigen values are greater than or equal to zero and at least one eigen value is zero
(4) **Negative Semi-definite** if all the eigen values are less than or equal to zero and at least one eigen value is zero
(5) **Indefinite** if A has both positive and negative eigen values

Problems:

1. **Determine the nature of the following quadratic form f(x_1, x_2, x_3) $= x_1^2 + 2x_2^2$**

<u>**Solution**</u>: The matrix of the quadratic form is Q = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The eigen values of the matrix are 1, 2, 0

Therefore, the quadratic form is Positive Semi-definite

2. **Discuss the nature of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$ without reducing it to canonical form**

<u>**Solution**</u>: The matrix of the quadratic form is Q = $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 2(+ve)$

$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5(+ve)$

$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0 = 12 - 2 = 10(+ve)$

Therefore, the quadratic form is positive definite

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM THROUGH ORTHOGONAL TRANSFORMATION [OR SUM OF SQUARES FORM OR PRINCIPAL AXES FORM]

Working rule:

Step 1: Write the matrix of the given quadratic form

Step 2: To find the characteristic equation

Step 3: To solve the characteristic equation

Step 4: To find the eigen vectors orthogonal to each other

Step 5: Form the Normalized matrix N

Step 6: Find N^T

Step 7: Find AN

Step 8: Find D = N^TAN

Step 9: The canonical form is $[y_1 y_2 y_3][D]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Problems:

1. **Reduce the given quadratic form Q to its canonical form using orthogonal transformation Q = $x^2 + 3y^2 + 3z^2 - 2yz$**

<u>**Solution**</u>: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 + 3 = 7$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elemeents = (9-1)+(3-0)+(3-0)$
$= 8+3+3 = 14$

$S_3 = |A| = 1(9-1)+0+0 = 8$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

1	1	-7	14	-8
	0	1	-6	8
	1	-6	8	0

$$\lambda = 1, \lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 1, \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(8)}}{2(1)} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2} = 4, 2$$

The eigen values are 1, 2, 4

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + 0x_2 + 0x_3 = 0$ -------- (1)

$0x_1 + 2x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 + 2x_3 = 0$ -------- (3)

Solving (2) and (3) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

$$\begin{matrix} 2 & & -1 & & 0 & & 2 \\ & \times & & \times & & \times & \\ -1 & & 2 & & 0 & & -1 \end{matrix}$$

$$\frac{x_1}{4-1} = \frac{x_2}{0-0} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case 2: When $\lambda = 2$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + 0x_2 + 0x_3 = 0$ -------- (1)

$0x_1 + x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 + x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

$$\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case 3: When $\lambda = 4$, $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 + 0x_2 + 0x_3 = 0$ -------- (1)

$0x_1 - x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 - x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

$$\begin{array}{cccc} 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & -1 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

The Normalized matrix N = $\begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$; $N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

AN = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

The canonical form is $[y_1 y_2 y_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2$

canonical form is $[y_1 \quad y_2 \quad y_3]\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$

1. **Reduce the quadratic form to a canonical form by an orthogonal reduction $2x_1x_2 + 2x_1x_3 - 2x_2x_3$. Also find its nature.**

Solution: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 0$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = -1 - 1 - 1 = -3$

$S_3 = |A| = 0(0-1) - 1(0+1) + 1(-1-0) = 0 - 1 - 1 = -2$

The characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda^3 - 3\lambda + 2 = 0$

$$\begin{array}{c|cccc}
1 & 1 & 0 & -3 & 2 \\
 & 0 & 1 & 1 & -2 \\
\hline
 & 1 & 1 & -2 & \multicolumn{1}{|c}{0}
\end{array}$$

$$\lambda = 1, \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1, \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

The eigen values are 1, 1, -2

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = -2$, $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + x_2 + x_3 = 0$ ------ (1)

$x_1 + 2x_2 - x_3 = 0$ ------ (2)

$x_1 - x_2 + 2x_3 = 0$ ------ (3)

$$\begin{array}{ccccccc}
 & x_1 & & x_2 & & x_3 & \\
1 & \times & 1 & \times & 2 & \times & 1 \\
2 & & -1 & & 1 & & 2
\end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} \Rightarrow \frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case 2: When $\lambda = 1$, $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 + x_3 = 0$ ------- (1)

$x_1 - x_2 - x_3 = 0$ ------ (2)

$x_1 - x_2 - x_3 = 0$ ------ (3)

All three equations are one and the same.

Put $x_1 = 0$, $x_2 = -x_3$. Let $x_3 = 1$. Then $x_2 = -1$

$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. Since X_3 is orthogonal to X_1 *and* X_2, $X_1^T X_3 = 0$ *and* $X_2^T X_3 = 0$

$$[-1 \quad 1 \quad 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \text{ and } [0 \quad -1 \quad 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$-l + m + n = 0$ -------- (1)

$0l - m + n = 0$ -------- (2)

$$\begin{array}{cccccc} & l & & m & & n \\ 1 & & 1 & & -1 & & 1 \\ -1 & & 1 & & 0 & & -1 \end{array}$$

$$\frac{l}{1+1} = \frac{m}{0+1} = \frac{n}{1-0} \Rightarrow \frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N^TAN = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-6}{3} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{6}} & \frac{2}{2} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{18}} & \frac{0}{\sqrt{12}} & \frac{6}{6} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $D = N^TAN = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $[y_1 \quad y_2 \quad y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -2y_1^2 + y_2^2 + y_3^2$

Nature: The eigen values are -2, 1, 1. Therefore, it is indefinite in nature.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 2 – GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS – SMTA1101

UNIT – II

GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS

Curvature:

At each point on a curve, with equation y=f(x), the tangent line turns at a certain rate. A measure of this rate of turning is the curvature

$$K = \frac{f''(x)}{(1 + [f'(x)])^{3/2}}$$

Radius of curvature in Cartesian form:

If the curve is given in Cartesian coordinates as *y(x)*, then the radius of curvature is

$\rho = (1 + [y']^2)^{(3/2)}/y''$ where $y' = \frac{dy}{dx}$, $y'' = (d^2 y)/(dx^2)$.

Radius of curvature in Parametric form:

If the curve is given parametrically by functions *x(t)* and *y(t)*, then the radius of curvature is

$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y - ..}$, $x' = \frac{dx}{dt}$, $x = \frac{d^2x}{dt^2}$, $y' = \frac{dy}{dt}$, $y = \frac{d^2y}{dt^2}$

Examples:

1. Find the radius of the curvature at the point $\left(\frac{1}{4},\frac{1}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = 1$.

   Solution: $\sqrt{x} + \sqrt{y} = 1$

   Differentiating w. r. t x ,we get

   $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0$. $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

   At $\left(\frac{1}{4},\frac{1}{4}\right)$, $y' = -1$.

   $y'' = -[(\sqrt{x}\ 1/(2\sqrt{y})\ y' - \sqrt{y}\ 1/(2\sqrt{x}))/x]$

At $\left(\frac{1}{\frac{4,1}{4}}\right)$, $y'' = -[(1/2\ \ 1/(2\ 1/2)\ (-1) - 1/2\ \ 1/(2\ 1/2))/(1/4)]$ = 4.

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{1}{\sqrt{2}}$$

2. Show that the radius of the curvature at any point of the curve $y = c\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$.

   Solution: $y = c\cosh\left(\frac{x}{c}\right)$

   Differentiating y w. r. t x we get

   $y' = \sinh\left(\frac{x}{c}\right)$

   $y'' = 1/c\ \ \cosh(x/c)$

   $$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = c\cosh^2\left(\frac{x}{c}\right) = \frac{y^2}{c}.$$

3. Find the radius of the curvature of the curve y = x^2(x-3) at the points where the tangent is parallel to the x – axis.

   Solution: y = x^2(x-3)

   Differentiating y w. r. t x we get

   $y' = 3x^2 - 6x$

   $y'' = 6x - 6$

   The points at which the tangent parallel to the x – axis can be found by equating y' to

   zero.

   i.e., $3x^2 - 6x = 0 \Rightarrow x = 0, x = 2.$

   At x = 0,y" = -6. At x = 2, y" = 6.

   Therefore at x = 0 and x = 2, $\rho = \frac{1}{6}$.

4. Prove that the radius of the curvature of the curve at any point of the cycloid

   $x = a(t + \sin t), y = a(1 + \cos t)$ is $\frac{4a\cos t}{2}$.

Solution: We have $x = a(t + sint), y = a(1 + cost)$.

Therefore $\frac{dx}{dt} = a(1 + cost)$, $\frac{dy}{dt} = asint.$

Now $\frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{asint}{a(1 + cost)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2cos^2\frac{t}{2}} = \tan\frac{t}{2}.$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{t}{2}\right) = \left\{\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right\}\frac{dt}{dx} = \frac{1}{2}sec^2\frac{t}{2}\frac{1}{a(1 + cost)} = \frac{1}{4a}sec^4\frac{t}{2}.$

Hence $\rho = \frac{\left(1 + tan^2\frac{t}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}sec^4\frac{t}{2}} = 4a\cos\frac{t}{2}.$

Centre and Circle of curvature:

Let the equation of the curve be y = f(x). let P be the given point (x,y) on this curve and Q the point (x+Δx,y+Δy) in the neighborhood of P. let N be the point of intersection of the normals at P and Q. As Q→P, suppose N→C. Then C is the centre of curvature of P. The circle whose centre C and radius ρ is called the circle of curvature. The co-ordinates of the centre of curvature is denoted as $(\bar{x}, \bar{y})$.

where $\bar{x} = x - \frac{y'(1 + y'^2)}{y''}$, $\bar{y} = y + \frac{(1 + y'^2)}{y''}$.

Equation of the circle of curvature:

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature and ρ be the radius of curvature at any point (x,y) on a curve, then the equation of the circle of curvature at that point is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

Examples:

1. Find the centre of curvature of the curve $a^2y = x^3$.

   Solution: $a^2y = x^3$

   $\frac{dy}{dx} = \frac{3x^2}{a^2}$ and $\frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$$\overline{x} = x - \frac{x}{2}\left(1 + \frac{9x^4}{a^4}\right) = \frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right]$$

$$\overline{y} = \frac{x^3}{a^2} + \frac{\left[1 + \frac{9x^4}{a^4}\right]}{\frac{6x}{a^2}} = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$

Therefore the required centre of curvature is $\left(\frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right], \frac{5x^3}{2a^2} + \frac{a^2}{6x}\right)$.

2. Find the centre of curvature of y = x^2 at $\left(\frac{1}{\frac{2,1}{4}}\right)$.

Solution: y' = 2x, y" = 2.

At $\left(\frac{1}{\frac{2,1}{4}}\right)$, y' = 1, y" = 2.

Therefore $\overline{x} = \frac{1}{2} - \frac{(1+1)}{2} = -\frac{1}{2}, \overline{y} = \frac{1}{4} + 1 = \frac{5}{4}$.

Therefore the required centre of curvature is $\left(-\frac{1}{\frac{2,5}{4}}\right)$.

3. Find the centre of curvature of the curve xy = a^2 at (a,a).

Solution: $y^{\uparrow\prime} = -a^{\uparrow}2/x^{\uparrow}2$, $y'' = 2a^{\uparrow}2\, x^{\uparrow}(-3)$. At (a,a) y' = -1, y" = $\frac{2}{a}$

Therefore $\overline{x} = a + \frac{2}{2/a} = 2a, \overline{y} = a + \frac{2}{2/a} = 2a.$

The required centre of curvature is (2a, 2a).

4. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{\frac{2,3a}{2}}\right)$.

Solution: $x^3 + y^3 = 3axy$

$$3x^2 + 3y^2y' = 3a(xy' + y)$$

$$y' = \frac{ay - x^2}{y^2 - ax}$$

y' at $\left(\frac{3a}{\frac{2,3a}{2}}\right)$ is -1.

$y'' = ((y^{\uparrow}2 - ax)(ay^{\uparrow\prime} - 2x) - (ay - x^{\uparrow}2\ )(2yy^{\uparrow\prime} - a))/(y^{\uparrow}2 - ax)^{\uparrow}2$

$y''at(3a/2,3a/2) = (-32)/3a$

$$\rho = \frac{2\sqrt{2(3a)}}{32}$$

$$\overline{x} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}$$

$$\overline{y} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}.$$

The circle of curvature is $\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \frac{9a^2}{128}$

5. Find the circle of curvature at the point (2,3) on $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

Solution: $\frac{2x}{4} + \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{-9x}{4y} \Rightarrow y'(2,3) = \frac{-3}{2}$

$y'' = (-9(y - xy^{\uparrow\prime}))/(4y^{\uparrow}2\ )$, $y''\ at\ (2,3) = (-3)/2$

$\rho = \frac{13^{\frac{3}{2}}}{12}$, $\quad \overline{x} = 2 - \frac{\left(-3/_2\right)\left(1 + 9/_4\right)}{\frac{-3}{2}} = \frac{-5}{4}$,

$$\overline{y} = 3 + \frac{\left(1 + 9/_4\right)}{\frac{-3}{2}} = \frac{5}{6}$$

The circle of curvature is $\left(x + \frac{5}{4}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{13^3}{12^2}$

Evolute and Involute

Evolute: Evolute of the curve is defined as the locus of the centre of curvature for that curve.

Involute : If C' is the evolute of the curve C then C is called the involute of the curve C'.

Procedure to find the evolute:

Let the given curve be f(x,y,a,b) = 0. (1)

Find y' and y" at the point P.

Find the centre of curvature $(\overline{x},\overline{y})$. Using $\overline{x} = x - (y'(1+{y'}^2))/y''$,
$\overline{y} = y + ((1+{y'}^2))/y''$.   (2)

Eliminate x, y from (1), (2) we get f($\overline{x},\overline{y},a,b) = 0$. (3)

Equation (3) is the required evolute.

Examples:

1. Show that the evolute of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is another cycloid given by $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta)$.

Solution: $\frac{dx}{d\theta} = a(1 + cos\theta), \frac{dy}{d\theta} = asin\theta$

$$\frac{dy}{dx} = \frac{dy/_{d\theta}}{dx/_{d\theta}} = \frac{asin\theta}{a(1 + cos\theta)} = \frac{\tan\theta}{2}$$

$y'' = d/d\theta\,(tan\ \theta/2)\ (d\theta)/dx = (sec^4\ \theta/2)/4a$

$$\overline{x} = a(\theta + sin\theta) - \frac{\frac{\tan\theta}{2\left(1 + tan^2\frac{\theta}{2}\right)}}{sec^4\frac{\theta}{2}/_{4a}} = a(\theta + sin\theta) - 2asin\theta = a(\theta - sin\theta)$$

,

$$\overline{y} = a(1 - cos\theta) + \frac{\left(1 + tan^2\frac{\theta}{2}\right)}{sec^4\frac{\theta}{2}/_{4a}} = a(1 - cos\theta) + 4acos^2\frac{\theta}{2} = a(1 + cos\theta) + 2a.$$

$\overline{x} = a(\theta - sin\theta), \overline{y} - 2a = a(1 + cos\theta)$.

The locus of $\overline{x}$ **and** $\overline{y}$ is $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta)$.

2. Prove that the evolute of the curve $x = a(cos\theta + \theta sin\theta), y = a(sin\theta - \theta cos\theta)$ is a circle $x^2 + y^2 = a^2$.

Solution: $\frac{dx}{d\theta} = a(-sin\theta + sin\theta + \theta cos\theta) = a\theta cos\theta, \frac{dy}{d\theta} = a\theta sin\theta.$

$$\frac{dy}{dx} = \frac{dy/_{d\theta}}{dx/_{d\theta}} = \frac{a\theta cos\theta}{a\theta sin\theta} = tan\theta$$

.

$y'' = 1/(a\theta\, cos^3\,\theta)$,

$$\overline{x} = a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta\left(1+\tan^2\theta\right)}{1/_{a\theta\cos^3\theta}} = a\cos\theta,$$

$$\overline{y} = a(\sin\theta - \theta\cos\theta) + \frac{(1+\tan^2\theta)}{1/_{a\theta\cos^3\theta}} = a\sin\theta.$$

Eliminating , $\overline{x}$ ***and*** $\overline{y}$ we get $\overline{x}^2 + \overline{y}^2 = a^2$.

The evolute of the given curve is $x^2 + y^2 = a^2$.

ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter family of curves

Let us consider y = f(x,α) to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter

Step 2: By Substituting the value of parameter α in the given family of curves, we get the required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter,i.e. $A\alpha^2+B\alpha+c=0$, then envelope is given by **discriminant = 0** i.e. $B^2-4AC=0$

Case 2: Envelope of two parameter family of curves.

Let us consider y = f(x,α, β) to be the given family of curves, and a relation connecting the two parameters α and β, g(α, β) = 0

Step 1: Consider α as independent variable and β depends α . Differentiate y = f(x,α, β) and g(α, β) = 0, w.r. to the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and g(α, β) = 0, we get the required envelope.

Problems on envelope of one parameter family of curves :

1. Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants

Solution : Differentiate $y = mx + am^p$ (1)

with respect to the parameter m, we get,

$$0 = x + pam^{p-1}$$

$$\Rightarrow \quad m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} \qquad (2)$$

Using (2) eliminate m from (1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{p-1}}$$

$$\Rightarrow \quad y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^{p}$$

i.e. $ap^{p} y^{p-1} = -x^{p} p^{p-1} + (-x)^{p}$

which is the required equation of envelope of (1)

2. Determine the envelope of $x\sin\theta - y\cos\theta = a\theta$, where θ being the parameter.

Solution : Differentiate ,

$$x\sin\theta - y\cos\theta = a\theta \qquad (1)$$

with respect to θ , we get,

$$x\cos\theta + y\sin\theta = a \qquad (2)$$

As θ cannot be eliminated between (1) and (2) ,we solve (1) and (2) for x and y in terms of θ.

For this, multiply (2) by sinθ and (1) by cosθ and then subtracting, we get,

$y = a(\sin\theta - \theta\cos\theta)$. Using similar simplification, we get, $x = a(\theta\sin\theta + \cos\theta)$.

3. (Leibnitz's problem) Calculate the envelope of family of circles whose centres lie on the x-axis

and radii are proportional to the abscissa of the centre.

Solution : Let (a,0) be the centre of any one of the member of family of curves with a as the parameter. Then the equation of family of circles with centres on x-axis and radius proportional to the abscissa of the centre is

$(x-a)^2 + y^2 = ka^2$ (1)

where k is the proportionality constant. Differentiating (1) with respect to a, we get,

$-2(x-a) = 2ka$

i.e. $a = \dfrac{x}{1-k}$.

From (1), $\left(x - \dfrac{x}{1-k}\right)^2 + y^2 = \dfrac{k}{(1-k)^2}x^2$

i.e. $\left(k^2 - k\right)x^2 + (1-k)^2 y^2 = 0, \qquad k \neq 1$

4. Find the envelope of $x\sec^2\theta + y\operatorname{cosec}^2\theta = a$, where θ is the parameter.

Solution : The given equation is rewritten as , $x\left(1+\tan^2\theta\right) + y\left(1+\cot^2\theta\right) = a$

i.e. $x\tan^4\theta + (x+y-a)\tan^2\theta + y = 0$,

which is a quadratic equation in $t = \tan^2\theta$. Therefore the required envelope is given by the discriminant equation : $B^2-4AC = 0$

i.e. $(x+y-a)^2 - 4xy = 0$

i.e. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$.

Envelope of Two parameter family of curves :

1. Find the envelope of family of straight lines ax+by=1, where a and b are parametersconnected by the relation ab = 1

Solution :

$ax + by = 1$ (1)

$$ab = 1 \qquad (2)$$

Differentiating (1) with respect to a (considering 'a' as independent variable and 'b' depends on a).

$$x + \frac{db}{da} y = 0$$

i.e. $$\frac{db}{da} = \frac{-x}{y} \qquad (3)$$

Differentiating (2) with respect to a

$$b + a\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b}{a} \qquad (4)$$

From (3) and (4), we have

$$\frac{x}{y} = \frac{b}{a}$$

i.e. $$\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

$\therefore$ $a = \dfrac{1}{2x}$ and $b = \dfrac{1}{2y}$ (5)

Using (5) in (2), we get the envelope as 4xy = 1

2. Find the envelope of family of straight lines $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a and b are parameters connected by the relation $\sqrt{a} + \sqrt{b} = 1$

Solution :

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \tag{1}$$

$$\sqrt{a} + \sqrt{b} = 1 \tag{2}$$

Differentiating (1) with respect to a

$$\frac{\sqrt{x}}{-2a^{3/2}} + \frac{\sqrt{y}}{-2b^{3/2}} \frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{x}}{\sqrt{y}} \frac{b^{3/2}}{a^{3/2}} \tag{3}$$

Differentiating (2) with respect to a

$$\frac{1}{2\sqrt{a}} + \frac{1}{2\sqrt{b}} \frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{b}}{\sqrt{a}} \tag{4}$$

From (3) and (4), we have

$$\frac{\sqrt{x}}{\sqrt{y}} \frac{b}{a} = 1$$

i.e. $$\frac{\sqrt{\frac{x}{a}}}{\sqrt{a}} = \frac{\sqrt{\frac{y}{b}}}{\sqrt{b}} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{\sqrt{a} + \sqrt{b}} = \frac{1}{1}$$

$\therefore \quad a = \sqrt{x}$ and $b = \sqrt{y}$ (5)

Using (5) in (2), we get the envelope as $x^{1/4} + y^{1/4} = 1$

3. Find the envelope of family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are parameters connected by the relation $a^2b^3 = c^5$

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

$$a^2b^3 = c^5 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-b^2x}{a^2y} \qquad (3)$$

Differentiating (2) with respect to a

$$2ab^3+3a^2b^2\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-2b}{3a} \qquad (4)$$

From (3) and (4), we have

$$\frac{3x}{a}=\frac{2y}{b}$$

i.e. $$\frac{\frac{x}{a}}{3}=\frac{\frac{y}{b}}{2}=\frac{\frac{x}{a}+\frac{y}{b}}{5}=\frac{1}{5}$$

$\therefore \quad a=\frac{5x}{3}$ and $\quad b=\frac{5y}{2} \qquad (5)$

Using (5) in (2), we get the envelope as $\quad x^2y^3=\frac{72}{3125}c^5$

4. Find the envelope of the family of circles whose centres lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and which pass through its centre.

Solution: Let (α,β) be the centre of arbitrary member of family of circles which lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is (0,0). Therefore, equation of the circles passing through origin and having centreat (α,β) is

$$x^2 + y^2 - 2\alpha x - 2\beta y = 0 \tag{1}$$

with

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \tag{2}$$

Differentiating (1) with respect to α ('α' as independent variable and 'β' depends on α),

$$x+\frac{d\beta}{d\alpha}y=0$$

i.e. $$\frac{d\beta}{d\alpha}=\frac{-x}{y} \tag{3}$$

Differentiating (2) with respect to α

$$\frac{2\alpha}{a^2}+\frac{2\beta}{b^2}\frac{d\beta}{d\alpha}=0$$

i.e. $$\frac{d\beta}{d\alpha}=\frac{-b^2\alpha}{a^2\beta} \tag{4}$$

From (3) and (4), we have

$$\frac{x}{y}=\frac{b^2\alpha}{a^2\beta}$$

i.e. $$\frac{\alpha x}{\frac{\alpha^2}{a^2}}=\frac{\beta y}{\frac{\beta^2}{b^2}}=\frac{\alpha x+\beta y}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}}=\frac{k}{1}$$, where k = αx+βy

$\therefore \quad \alpha = \dfrac{a^2 x}{k}$ and $\beta = \dfrac{b^2 y}{k}$ (5)

From (1), we have , $x^2 + y^2 = 2k$ (6)

Using (5) and (6) in (2), we get the envelope as $\left(x^2 + y^2\right)^2 = 4\left(a^2 x^2 + b^2 y^2\right)$

5. Determine the equation of the envelope of family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where the parameters a and b are connected by the relation $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = 1$, *l* and m are non-zero constants.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b^3 x^2}{a^3 y^2} \qquad (3)$$

Differentiating (2) with respect to a

$$\frac{2a}{l^2} + \frac{2b}{m^2}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-m^2 a}{l^2 b} \qquad (4)$$

From (3) and (4), we have

$$\frac{b^4x^2}{a^4y^2}=\frac{m^2}{l^2}$$

i.e. $$\frac{\frac{x^2}{a^2}}{\frac{a^2}{l^2}}=\frac{\frac{y^2}{b^2}}{\frac{b^2}{m^2}}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{\frac{a^2}{l^2}+\frac{b^2}{m^2}}=\frac{1}{1}$$

$\Rightarrow$ $a^4=l^2x^2$ and $b^4=m^2y^2$

i.e. $a^2=lx$ and $b^2=my$ (5)

Using (5) in (2), we get the envelope as $\frac{x}{l}+\frac{y}{m}=1$

Problems on Evolute as envelope of its normals :

1. Determine the evolute of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ by considering it as an envelope of its normal

Solution : Let P (a cosht, b sinht) be any point on the given hyperbola. Then

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b\cosh t}{a\sinh t}=\frac{b}{a}\coth t$$

Equation of normal line to the hyperbola is

$$(y-b\sinh t)=\frac{-a}{b\cosh t}(x-a\cosh t) \qquad (1)$$

$$\Rightarrow \quad \frac{by}{\sinh t}+\frac{ax}{\cosh t}=a^2+b^2 \qquad (2)$$

Differentiating (2) partially with respect to t, we have,

$$\frac{-by}{(\sinh t)^2}\cosh t - \frac{ax}{(\cosh t)^2}\sinh t = 0$$

$$\Rightarrow \quad \tanh t = -\left(\frac{by}{ax}\right)^{1/3}$$

$$\Rightarrow \quad \sinh t = \mp\left(\frac{by}{h}\right)^{1/3} \text{ and } \cosh t = \pm\left(\frac{ax}{h}\right)^{1/3} \qquad (3)$$

Where $\qquad h = \sqrt{(ax)^{2/3} - (by)^{2/3}}$

Using (3) in (2) , we get,

$$\frac{by}{-(by)^{1/3}}h + \frac{ax}{(ax)^{1/3}}h = a^2 + b^2$$

i.e. $\left((ax)^{2/3} - (by)^{2/3}\right)\left((ax)^{2/3} - (by)^{2/3}\right)^{1/2} = a^2 + b^2$

i.e. $(ax)^{2/3} - (by)^{2/3} = \left(a^2 + b^2\right)^{2/3}$

2. By considering the evolute of a curve as the envelope of its normal, find the evolute of

$x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

Solution :

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta \, .$$

Equation of normal line to the hyperbola is

$$(y - (\sin\theta - \theta\cos\theta)) = \frac{-1}{\tan\theta}(x - (\cos\theta + \theta\sin\theta))$$

$$\Rightarrow \quad y\sin\theta - \sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$$

i.e. $$y\sin\theta + x\cos\theta = 1 \qquad (1)$$

Differentiating (1) with respect to the parameter θ, we have

$$y\cos\theta - x\sin\theta = 0 \qquad (2)$$

Multiplying (1) by cosθ and (2) by sinθ and then subtracting, we have,

$$x = \cos\theta \qquad (3)$$

Similarly we get,

$$y = \sin\theta \qquad (4)$$

Eliminating θ between (3) and (4) we get the required evolute as $x^2 + y^2 = 1$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 3 - FUNCTIONS OF SEVERAL VARIABLES – SMTA1101

Jacobians

Changing variable is something we come across very often in Integration. There are many reasons for changing variables but the main reason for changing variables is to convert the integrand into something simpler and also to transform the region into another region which is easy to work with. When we convert into a new set of variables it is not always easy to find the limits. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables. In order to change variables in an integration we will need the **Jacobian** of the transformation.

If $f_1, f_2, \ldots, f_n$ are n differentiable functions of n variables $x_1, x_2, \ldots, x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is defined as the Jacobian of $f_1, f_2, \ldots, f_n$ with respect to the n variables $x_1, x_2, \ldots, x_n$ and is denoted by $\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$.

If u and v are functions of x and y, then $J(u,v) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$

If u, v and w are functions of x, y and z, then J(u,v,w)= $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of the Jacobian

1. Chain Rule for Jacobians: If u and v are functions of independent variables r and s and each of r and s are functions of the variables x and y, then u and v are functions of x and y. Further the jacobians satisfy the chain rule $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$
2. If u and v are functions of x and y, then x and y can be solved in terms of u and v. Then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.
3. If u, v and w are functions of x, y and z and if u, v, w are functionally related or dependent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Problems

1. If u = xyz, v = $x^2 + y^2 + z^2$, w = x + y + z, find J (x, y, z)

   Solution J (x, y, z) = $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$

   $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

   Therefore $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$

2. If u = x + y + z, u^2v = y + z, u^3 w = z, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}$

   Solution: $u_x = u_y = u_z = 1$

$v_x = \dfrac{-2(y+z)}{u^3}$, $v_y = \dfrac{u-2(y+z)}{u^3}$, $v_z = \dfrac{u-2(y+z)}{u^3}$

$w_x = \dfrac{-3z}{u^4}$, $w_y = \dfrac{-3z}{u^4}$, $w_z = \dfrac{1}{u^3} - \dfrac{3z}{u^4}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} \\ \frac{-3z}{u^4} & \frac{-3z}{u^4} & \frac{u-3z}{u^4} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-2(y+z)}{u^3} & \frac{1}{u^2} & \frac{1}{u^2} \\ \frac{-3z}{u^4} & 0 & \frac{1}{u^3} \end{vmatrix} \quad c_2 \to c_2 - c_1, c_3 \to c_3 - c_1$$

$$= \frac{1}{u^5}$$

3. If $u = \dfrac{x+y}{1-xy}$ *and* $v = \tan^{-1} x + \tan^{-1} y$, show that $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

Solution: Let x = tanθ and y = tanφ,

then $u = \dfrac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = \tan(\theta+\varphi)$

And v = θ + φ

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\theta,\varphi)} \cdot \frac{\partial(\theta,\varphi)}{\partial(x,y)}$$

$u_\theta = \sec^2(\theta+\varphi)$, $u_\varphi = \sec^2(\theta+\varphi)$, $v_\theta = v_\varphi = 1$

$$\frac{\partial(u,v)}{\partial(\theta,\varphi)} = \begin{vmatrix} \sec^2(\theta+\varphi) & \sec^2(\theta+\varphi) \\ 1 & 1 \end{vmatrix} = 0$$

$$\textit{Thus } \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

4. Show that $u = x^3 - y^3z^3$, $v = x^2 + y^2z^2 + xyz$ and $w = x - yz$ are functionally dependent and also find the relation.

Solution: If u, v, w are functionally dependent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3x^2 & -3y^2z^3 & -3y^3z^2 \\ 2x+yz & 2yz^2+xz & 2y^2z+xy \\ 1 & -z & -y \end{vmatrix}$$

$$= yz\begin{vmatrix} 3x^2 & -3y^2z^2 & -3y^2z^2 \\ 2x+yz & 2yz \ \ +x & 2yz+x \\ 1 & -1 & -1 \end{vmatrix} \text{ taking } z \text{ common from } c_2 \text{ and } y \text{ from } c_3$$

$$= 0 \ \text{ (Two columns are identical)}$$

Since $x^3 - y^3z^3 = (x - yz)(x^2 + xyz + y^2z^2)$

u = v w is the relation between the three variables.

5. If $u = e^x\cos y$, $v = e^x\sin y$, where $x = lr + sm$, $y = mr - sl$, verify if

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

6. $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x}$$

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} l & m \\ m & -l \end{vmatrix} = -l^2 - m^2$$

$u = e^{lr+sm}\cos(mr-sl)$, $v = e^{lr+sm}\sin(mr-sl)$

$u_r = le^{lr+sm}\cos(mr-sl) - m\, e^{lr+sm}\sin(mr-sl)$

$u_s = me^{lr+sm}\cos(mr-sl) + l\, e^{lr+sm}\sin(mr-sl)$

$v_r = le^{lr+sm}\sin(mr-sl) + m\, e^{lr+sm}\cos(mr-sl)$

$v_s = me^{lr+sm}\sin(mr-sl) - l\, e^{lr+sm}\cos(mr-sl)$

$$\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} = -e^{2x}(l^2+m^2) = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

Useful Links for this topic

1. http://mathwiki.ucdavis.edu/Calculus/Vector_Calculus/Multiple_Integrals/Jacobians
2. http://www-astro.physics.ox.ac.uk/~sr/lectures/multiples/Lecture5reallynew.pdf
3. http://math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/jacpol/jacpol.html
4. http://www.google.co.in/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CCMQFjAB&url=http%3A%2F%2Fwww.tcc.edu%2FVML%2FMth163%2Fdocuments%2FJacobians.pptx&ei=1DKSVZfbG867uAS_t4CABg&usg=AFQjCNHQDmFpTK-pU16sC61WTkwouEvUFA&bvm=bv.96783405,d.c2E
5. http://math.etsu.edu/multicalc/prealpha/Chap3/Chap3-3/printversion.pdf

Taylor's Series

Statement : Let $f(x,y)$ be a function of two variables x, y which possess continuous partial derivatives at all points (x,y). Then

$$f(x+h,y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f$$
$$+ \frac{1}{4!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^4 f + \cdots$$

Another form of Taylor series :

$$f(x,y) = f(a,b) + \frac{1}{1!}[\,(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+ \frac{1}{2!}[\,(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$
$$+ \frac{1}{3!}[\,(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)$$
$$+ (y-b)^3 f_{yy}(a,b)] + \ldots$$

Maclaurin's series :

The Taylor series expansion of $f(x,y)$ about the point (0, 0) is called Maclaurin's series.

$f(x,y) = f(0,0) + \frac{1}{1!}\,[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}\,[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] +$
$\frac{1}{3!}\,[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \frac{1}{4!}\,[x^4 f_{xxxx}(0,0) +$
$4x^3 y f_{xxxy}(0,0) + 6x^2y^2 f_{xxyy}(0,0) + 4xy^3 f_{xyyy}(0,0) + y^4 f_{yyyy}(0,0)]$
$+ \ldots$

Problems

1. Find the Taylor series expansion of $\cos(x-y)$upto second degree terms.

Solution:

Taylor series expansion of $f(x,y)$ upto second degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}\,[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

Let $f(x,y) = \cos(x-y)$	$f(0,0) = \cos 0 = 1$
$f_x(x,y) = -\sin(x-y)$	$f_x(0,0) = -\sin 0 = 0$
$f_y(x,y) = \sin(x-y)$	$f_y(0,0) = \sin 0 = 0$
$f_{xx}(x,y) = -\cos(x-y)$	$f_{xx}(0,0) = -\cos 0 = -1$
$f_{xy}(x,y) = \cos(x-y)$	$f_{xy}(0,0) = \cos 0 = 1$
$f_{yy}(x,y) = -\cos(x-y)$	$f_{yy}(0,0) = -\cos 0 = -1$

Taylor series expansion of $f(x,y) = \cos(x-y)$upto second degree terms

$$\cos(x-y) = 1 + \frac{1}{1!}[x(0) + y(0)] + \frac{1}{2!}\,[x^2(-1) + 2xy(1) + y^2(-1)]$$

i.e., $\cos(x-y) = 1 - \frac{1}{2!}\,[x^2 - 2xy + y^2]$

2. Expand e^{x+y} in powers of $(x-1)$ and $(y+1)$ upto and including second degree term.

Solution:

Taylor series expansion of $f(x,y)$ about the point (a,b) i.e., in powers of $(x-a)$ and $(y-b)$ upto second degree term is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

Let $f(x,y) = e^{x+y}$ $\quad f(1,-1) = e^{1-1} = e^0 = 1$

$f_x(x,y) = e^{x+y}$ $\quad f_x(1,-1) = e^{1-1} = e^0 = 1$

$f_y(x,y) = e^{x+y}$ $\quad f_y(1,-1) = e^{1-1} = e^0 = 1$

$f_{xx}(x,y) = e^{x+y}$ $\quad f_{xx}(1,-1) = e^{1-1} = e^0 = 1$

$f_{xy}(x,y) = e^{x+y}$ $\quad f_{xy}(1,-1) = e^{1-1} = e^0 = 1$

$f_{yy}(x,y) = e^{x+y}$ $\quad f_{yy}(1,-1) = e^{1-1} = e^0 = 1$

Taylor series expansion of $f(x,y) = e^{x+y}$ in powers of $(x-1)$ and $(y+1)$ upto second degree term is given by

$$f(x,y) = f(1,-1) + \frac{1}{1!}[(x-1)f_x(1,-1) + (y+1)f_y(1,-1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,-1) + 2(x-1)(y+1)f_{xy}(1,-1) + (y+1)^2 f_{yy}(1,-1)]$$

i.e.,

$$e^{x+y} = 1 + \frac{1}{1!}[(x-1)(1) + (y+1)(1)] + \frac{1}{2!}[(x-1)^2(1) + 2(x-1)(y+1)(1) + (y-1)^2(1)]$$

i.e., $e^{x+y} = 1 + \frac{1}{1!}[(x-1) + (y+1)] + \frac{1}{2!}[(x-1)^2 + 2(x-1)(y+1) + (y-1)^2]$

3. Expand $tan^{-1}\frac{y}{x}$ as a Taylor series in the neighbourhood of (1, 1) upto second degree term

Solution:

Taylor series expansion of $f(x,y)$ in the neighbourhood of (a,b) upto second degree term is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

Let $f(x,y) = tan^{-1}\frac{y}{x}$ $\quad f(1,\ 1) = tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

$f_x(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$ $\quad f_x(1,1) = -\frac{1}{2}$

$f_y(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$ $\quad f_y(1,1) = \frac{1}{2}$

$f_{xx}(x,y) = -\frac{(x^2+y^2)(0)-y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$ $\qquad f_{xx}(1,1) = \frac{1}{2}$

$f_{xy}(x,y) = -\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad f_{xy}(1,1) = 0$

$f_{yy}(x,y) = \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$ $\qquad f_{yy}(1,1) = -\frac{1}{2}$

Taylor series expansion of $f(x,y) = tan^{-1}\frac{y}{x}$ in the neighbourhood of (1, 1) upto second degree term is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}[(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)]$$

i.e, $tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \frac{1}{1!}\left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + \right.$

$\left. 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$

i.e, $tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2]$

4. Using Taylor series show that $\log(1+x+y) = \frac{(x+y)}{1} - \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3} - \ldots$

Solution:

Taylor series expansion of $f(x,y)$ upto third degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}[xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$$

Let $f(x,y) = \log(1+x+y)$ $\quad f(0,\ 0) = \log 1 = 0$

$f_x(x,y) = \frac{1}{1+x+y}$ $\qquad f_x(0,0) = 1$

$f_y(x,y) = \frac{1}{1+x+y}$ $\qquad f_y(0,0) = 1$

$f_{xx}(x,y) = -\frac{1}{(1+x+y)^2}$ $\qquad f_{xx}(0,0) = -1$

$f_{xy}(x,y) = -\frac{1}{(1+x+y)^2}$ $\qquad f_{xy}(0,0) = -1$

$f_{yy}(x,y) = -\frac{1}{(1+x+y)^2}$ $\qquad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = \frac{2}{(1+x+y)^3}$ $\qquad f_{xxx}(0,0) = 2$

$f_{xxy}(x,y) = \frac{2}{(1+x+y)^3}$ $\qquad f_{xxy}(0,0) = 2$

$f_{xyy}(x,y) = \frac{2}{(1+x+y)^3}$ $\qquad f_{xyy}(0,0) = 2$

$f_{yyy}(x,y) = \frac{2}{(1+x+y)^3}$ $\qquad f_{yyy}(0,0) = 2$

Taylor series expansion of $f(x,y) = \log(1+x+y)$ upto third degree term is given by

$$\log(1+x+y) = 0 + \frac{1}{1!}[x(1) + y(1)] + \frac{1}{2!}[x^2(-1) + 2xy(-1) + y^2(-1)] + \frac{1}{3!}[x^3(2) + 3x^2y(2) + 3xy^2(2) + y^3(2)]$$

i.e., $\log(1+x+y) = (x+y) - \frac{1}{2!}[x^2 + 2xy + y^2] + \frac{1}{3}[x^3 + 3x^2y + 3xy^2 + y^3]$

i.e., $\log(1+x+y) = \frac{(x+y)}{1} - \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3} - \ldots$

5. Expand $\cos x \ \cos y$ in powers of $x, \ y$ upto fourth degree terms.

Solution:

Taylor series expansion of $f(x,y)$ upto third degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}[xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)] + \frac{1}{3!}[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)] + \frac{1}{4!}[x^4f_{xxxx}(0,0) + 4x^3yf_{xxxy}(0,0) + 6x^2y^2f_{xxyy}(0,0) + 4xy^3f_{xyyy}(0,0) + y^4f_{yyyy}(0,0)]$$

Let $f(x,y) = \cos x \cos y$ $\qquad f(0,\ 0) = 1$

$f_x(x,y) = -\sin x \cos y$ $\qquad f_x(0,0) = 0$

$f_y(x,y) = -\cos x \sin y$ $\qquad f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos x \cos y$ $\qquad f_{xx}(0,0) = -1$

$f_{xy}(x,y) = sinx \sin y$ $\qquad f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\cos x \cos y \qquad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = \sin x \cos y \qquad f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \cos x \sin y \qquad f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = \sin x \cos y \qquad f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = \cos x \sin y \qquad f_{yyy}(0,0) = 0$

$f_{xxxx}(x,y) = \cos x \cos y \qquad f_{xxxx}(0,0) = 1$

$f_{xxxy}(x,y) = -\sin x \sin y \qquad f_{xxxy}(0,0) = 0$

$f_{xxyy}(x,y) = \cos x \cos y \qquad f_{xxyy}(0,0) = 1$

$f_{xyyy}(x,y) = -\sin x \sin y \qquad f_{xyyy}(0,0) = 0$

$f_{yyyy}(x,y) = \cos x \cos y \qquad f_{yyyy}(0,0) = 1$

Taylor series expansion of $f(x,y) = \cos x \cos y$ in powers of $x,\ y$ upto fourth degree terms

$$\begin{aligned}\cos x \cos y = {} & 1 + \frac{1}{1!}[\,x(0) + y(0)] + \frac{1}{2!}[\,x^2(-1) + 2xy(0) + y^2(-1)] \\ & + \frac{1}{3!}[\,x^3(0) + 3x^2y(0) + 3xy^2(0) + y^3(0)] \\ & + \frac{1}{4!}[\,x^4(1) + 4x^3y(0) + 6x^2y^2(1) + 4xy^3(0) + y^4(1)]\end{aligned}$$

i.e., $\cos x \cos y = 1 - \frac{1}{2}(x^2 + y^2) + \frac{1}{24}(x^4 + 6x^2y^2 + y^4)$

Maxima and Minima of functions of two variables

The problem of determining the maximum or minimum of a function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century.

A function of two variables can be written in the form z = f(x, y). A critical point is a point (a, b) such that the two partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero at the point (a, b). A relative maximum or a relative minimum occurs at a critical point.

A critical point is a maximum if the value of f at that point is greater than its value at all its sufficiently close neighboring points.

A critical point is a minimum if the value of f at that point is less than its value at all its sufficiently close neighboring points.

A critical point is a saddle point if the value of f at that point is greater than its value at some neighboring point and if the value of f at that point is less than its value at some other neighboring point. Saddle point is a point which is neither a maximum nor a minimum.

Working rule for identifying critical points of the function z = f(x,y) and to classify them

Step 1: Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Solving $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ gives the critical points (a,b) at which a maxima or minima may exist.

Step 2: Find the value of $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$ at all the points (a,b) got in step 1.

Step 3:

i. If $r < 0$ and $rt - s^2 > 0$ the f(x,y) has a maximum point at (a,b) and the corresponding maximum value is f(a,b).

ii. If $r > 0$ and $rt - s^2 > 0$ the f(x,y) has a minimum point at (a,b) and the corresponding minimum value is f(a,b).

iii. If $rt - s^2 < 0$ the f(x,y) has neither a maximum nor a minimum point at (a,b) and the point is called a saddle point.

iv. If $rt - s^2 = 0$ the further investigation is required to classify the point

Problems

1. Find the maxima and minima of the function, if any, for the function $f(x,y) = y^2 + 4xy + 3x^2 + x^3$

   Solution: $f_x = 4y + 6x + 3x^2$, $f_y = 2y + 4x$. Equate f_x and f_y to zero

   $4y + 6x + 3x^2 = 0$ ......(1)
   $2y + 4x = 0$ ......(2)
   Solving equations (1) and (2) we get the critical points (0,0) and (2/3,-4/3)
   $r = f_{xx} = 6 + 6x$, $t = f_{yy} = 2$, $s = f_{xy} = 4$

Critical Point	r	$rt - s^2$	Classification
(0,0)	6	-4	Saddle point
(2/3,-4/3)	10	4	Minimum point

The point (2/3, -4/3) is a minimum point of the function and the minimum value F(2/3,-4/3) = -4/27

2. Find the maxima and minima of the function $f(x,y) = xy\,(a - x - y)$

Solution: $f_x = ay - 2xy - y^2$, $f_y = ax - x^2 - 2xy$. Equate f_x and f_y to zero

$y(a - 2x - y) = 0$ ……(1)
$x(a - x - 2y) = 0$ ……(2)
Solving equations (1) and (2) we get the critical points (0,0), (a,0), (0,a) and (a/3, a/3).

$r = f_{xx} = -2y$, $t = f_{yy} = -2x$, $s = f_{xy} = a - 2x - 2y$

Critical Point	r	$rt - s^2$	Classification
(0,0)	0	$-a^2$	Saddle point
(a,0)	0	$-a^2$	Saddle point
(0,a)	-2a	$-a^2$	Saddle point
(a/3, a/3)	-2a/3	$a^2/3$	Maximum or minimum point

(a/3, a/3) is the only point which could be either be a maximum or a minimum.
r depends on the value of ‘a’.
$r = -2a/3 < 0$ if ‘a’ is positive
$r = -2a/3 > 0$ if ‘a’ is negative
Hence f(x,y) has a maximum at (a/3, a/3) if ‘a’ is positive and has a minimum at (a/3, a/3) if ‘a’ is negative.

The value is $f(a/3, a/3) = \frac{a^3}{27}$

3. Examine the function $f(x,y) = x^3 + 3xy^2 - 15x^2 + 72x - 15y^2$ for extreme values

Solution: $f_x = 3x^2 + 3y^2 - 30x + 72$, $f_y = 6xy - 30y$. Equate f_x and f_y to zero

$3x^2 + 3y^2 - 30x + 72 = 0$ ......(1)
$6xy - 30y = 0$ ......(2)
Solving equations (1) and (2) we get the critical points (5,1), (5,-1), (4,0) and (6,0).

$r = f_{xx} = 6x-30$, $t = f_{yy} = 6x-30$, $s = f_{xy} = 6y$

Critical Point	r	$rt - s^2$	Classification
(5,1)	0	-36	Saddle point
(5,-1)	0	-36	Saddle point
(4,0)	-6	36	Maximum point
(6, 0)	6	36	Minimum point

(4, 0) is a maximum point and the maximum value is f(4,0) = 112
(6, 0) is a minimum point and the minimum value is f(6,0) = 108

4. Find the extreme values of the function $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

Solution: $u_x = 2xy^2 - 10x - 8y$, $u_y = 2x^2y - 8x - 10y$. Equate f_x and f_y to zero

$2xy^2 - 10x - 8y = 0$ ......(1)
$2x^2y - 8x - 10y = 0$ ......(2)
Since (0,0) satisfies both (1) and (2), (0, 0) is a critical point. To get the other points rewrite equation (1)

From (1) we get $x = \frac{8y}{2y^2 - 10}$ ....(3).
Substitute this in (2)

$$2\left(\frac{8y}{2y^2-10}\right)^2 y - 8\left(\frac{8y}{2y^2-10}\right) - 10y = 0 \text{ ....(4)}$$

Solving (4) we get y = 3, -3, 1, -1 Substitute these values in (3) we get the critical points (0, 0), (1,-1), (-1,1), (3,3), (-3,-3).

$r = f_{xx} = 2y^2 - 10$, $t = f_{yy} = 2x^2 - 10$, $s = f_{xy} = 4xy - 8$

Critical Point	r	rt – s^2	Classification
(0,0)	-10	36	Maximum point
(1,-1)	-8	-80	Saddle point
(-1,1)	-8	-80	Saddle point
(3, 3)	8	-720	Saddle point
(-3, -3)	8	-720	Saddle point

(0,0) is a maximum point and the maximum value is f(0,0) = 0.

5. Show that x = a/2, y = a/3 makes the function $u = ax^3 y^2 - x^4 y^2 - x^3y^3$ a maximum.

Solution:

$u_x = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3$........(1)

$u_y = 2ax^3y - 2x^4y - 3x^3y^2$............(2)

Put x = a/2 and y =a/3 in both equations (1) and (2)

Since both u_x and u_y are zero at (a/2,a/3), it is a critical point.

$$u_{xx} = 6axy^2 - 12x^2y^2 - 6xy^3$$
$$u_{yy} = 2ax^3 - 2x^4 - 6x^3y$$
$$u_{xy} = 6ax^2y - 8x^3y - 9x^2y^2$$

r at (a/2, a/3) = $-\dfrac{a^4}{9}$ which is negative for any value of ‘a’.

t at (a/2, a/3) = $-\dfrac{a^4}{8}$

s at (a/2, a/3) = $-\dfrac{a^4}{12}$

rt – s^2 at (a/2, a/3) = $\dfrac{a^8}{144}$ which is positive for any value of ‘a’.

Since r is negative and rt – s^2 is positive the point (a/2, a/3) is a maximum point

Useful Links for this topic

1. http://personal.maths.surrey.ac.uk/st/S.Zelik/teach/calculus/max_min_2var.pdf
2. http://www.maths.manchester.ac.uk/~mheil/Lectures/2M1/Material/Chapter2.pdf
3. http://tutorial.math.lamar.edu/Classes/CalcIII/RelativeExtrema.aspx
4. http://www.ccs.neu.edu/home/lieber/courses/algorithms/cs4800/f10/lectures/11.4.Maximizing.pdf
5. http://www.maths.manchester.ac.uk/~ngray/MATH19662/Section%204%20-%20Functions%20of%20Two%20Variables.pdf

Constrained Maxima and Minima

Sometimes we may require to find the extreme values of a function of three (or more) variables say $f(x,y,z)$ which are not independent, but are connected by a relation say $g(x,y,z) = 0$. The extreme values of a function in such a situation is called constrained extreme values.

In such situations, we use $g(x,y,z) = 0$ to eliminate one of the variables, say z from the given function, thus converting the function of three variables as a function of only two variables. Then we find the unconstrained maxima and minima of the converted function.

When this procedure cannot be used, we use Lagrange's method.

Lagrange's Multiplier Method

Sometimes we may require to find the maximum and minimum values of a function $f(x,y,z)$ where x, y, z subject to the constraint $g(x,y,z) = 0$.

(1)

We define a function $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

where λ is the Lagrange's multiplier independent of x, y, z.

The neccessary condition for a maximum or minimum are

$$\frac{\partial F}{\partial x} = 0$$

(2)

$$\frac{\partial F}{\partial y} = 0$$

(3)

$$\frac{\partial F}{\partial z} = 0$$

(4)

Solving the four equations (1), (2), (3) and (4) we get the values of x, y, z, λ which give the extreme values of $f(x,y,z)$

Problems

1. Prove that the stationary values of $a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ occur at $x = \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$

Solution:

Let $f = a^3x^2 + b^3y^2 + c^3z^2$

$g = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$

and $F = f + \lambda g = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$

$\frac{\partial F}{\partial x} = 0$ implies $2a^3x - \frac{\lambda}{x^2} = 0 \Rightarrow a^3x^3 = \frac{\lambda}{2}$ (1)

$\frac{\partial F}{\partial y} = 0$ implies $2b^3y - \frac{\lambda}{y^2} = 0 \Rightarrow b^3y^3 = \frac{\lambda}{2}$ (2)

$\frac{\partial F}{\partial z} = 0$ implies $2c^3x - \frac{\lambda}{z^2} = 0 \Rightarrow c^3z^3 = \frac{\lambda}{2}$ (3)

From (1), (2) and (3) we get $a^3x^3 = b^3y^3 = c^3z^3$

i.e., $ax = by = cz$

i.e., $\frac{a}{\frac{1}{x}} = \frac{b}{\frac{1}{y}} = \frac{c}{\frac{1}{z}} = \frac{a+b+c}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = \frac{a+b+c}{1}$

consider $\frac{a}{\frac{1}{x}} = \frac{a+b+c}{1} \Rightarrow x = \frac{a+b+c}{a}$

consider $\frac{b}{\frac{1}{y}} = \frac{a+b+c}{1} \Rightarrow y = \frac{a+b+c}{b}$

consider $\frac{c}{\frac{1}{z}} = \frac{a+b+c}{1} \Rightarrow z = \frac{a+b+c}{c}$

Thus f is stationary at this point $= \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$.

2. Find three positive constants such that their sum is a constant and their product is maximum.

Solution:

Let the three positive constants be x, y, z such that $x + y + z = a$.

Let $f = xyz$

and $g = x + y + z - a$ (1)

We have to maximize $f = xyz$ subject to the constraint $g = x + y + z - a$

Let $F = f + \lambda g = xyz + \lambda(x + y + z - a)$

$\frac{\partial F}{\partial x} = 0$ implies $yz + \lambda(1) = 0 \Rightarrow yz = -\lambda$

(2)

$\frac{\partial F}{\partial y} = 0$ implies $xz + \lambda(1) = 0 \Rightarrow xz = -\lambda$

(3)

$\frac{\partial F}{\partial z} = 0$ implies $xy + \lambda(1) = 0 \Rightarrow xy = -\lambda$

(4)

From (2), (3) and (4) we get $yz = xz = xy$

Consider $yz = xz \Rightarrow y = x$

Consider $xz = xy \Rightarrow z = y$

Therefore $x = y = z$

Substituting in (1), we get $x + x + x = a \Rightarrow 3x = a \Rightarrow x = \frac{a}{3}$

Therefore $= \frac{a}{3}$, $z = \frac{a}{3}$.

Hence the three numbers are $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}$.

3. Split 24 into three parts such that continued product of the first , square of the second and cube of the third may be minimum.

Solution:

Let the three parts be x, y, z such that $x + y + z = 24$.

Let $f = xy^2z^3$ (1)

We have to minimize $f = xy^2z^3$ subject to the constraint $g = x + y + z - 24$

Let $F = f + \lambda g = xy^2z^3 + \lambda(x + y + z - 24)$

$\frac{\partial F}{\partial x} = 0$ implies $y^2z^3 + \lambda(1) = 0 \Rightarrow y^2z^3 = -\lambda$

(2)

$\frac{\partial F}{\partial y} = 0$ implies $2xyz^3 + \lambda(1) = 0 \Rightarrow 2xyz^3 = -\lambda$

(3)

$\frac{\partial F}{\partial z} = 0$ implies $3xy^2z^2 + \lambda(1) = 0 \Rightarrow 3xy^2z^2 = -\lambda$

(4)

From (2), (3) and (4) we get $y^2z^3 = 2xyz^3 = 3xy^2z^2$

Consider $y^2z^3 = 2xyz^3 \Rightarrow y = 2x$

Consider $y^2z^3 = 3xy^2z^2 \Rightarrow z = 3x$

Substituting in (1), we get $x + 2x + 3x = 24 \Rightarrow 6x = 24 \Rightarrow x = 4$

Therefore $= 8$, $z = 12$.

Hence the three parts of 24 are $4, 8, 12$.

4. Find the shortest distance of the point (2, 1, -3) from the plane 2x + y = 2z + 4

Solution:

Let the foot of the perpendicular from the point (2, 1, -3) to the plane be $2x + y = 2z + 4$ be $P(x, y, z)$.

Shortest distance from (2, 1, -3) to the point $P(x, y, z)$ on the plane is the perpendicular distance $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+3)^2}$

$\therefore d^2 = (x-2)^2 + (y-1)^2 + (z+3)^2$

We have to find the minimum distance d equivalently d^2 subject to the constraint $2x + y = 2z + 4$.

Let $f = (x-2)^2 + (y-1)^2 + (z+3)^2$

and $g = 2x + y - 2z - 4$ (1)

Let $F = f + \lambda g = (x-2)^2 + (y-1)^2 + (z+3)^2 + \lambda(2x + y - 2z - 4)$

$\frac{\partial F}{\partial x} = 0$ implies $2(x-2) + \lambda(2) = 0 \Rightarrow x - 2 = -\lambda$ (2)

$\frac{\partial F}{\partial y} = 0$ implies $2(y-1) + \lambda(1) = 0 \Rightarrow 2(y-1) = -\lambda$ (3)

$\frac{\partial F}{\partial z} = 0$ implies $2(z+3) + \lambda(-2) = 0 \Rightarrow -(z+3) = -\lambda$

(4)

From (2), (3) and (4) we get $x - 2 = 2(y-1) = -(z+3)$

Consider $x - 2 = 2(y-1) \Rightarrow x = 2y \Rightarrow y = \frac{x}{2}$

Consider $x - 2 = -z - 3 \Rightarrow x = -z - 1 \Rightarrow z = -1 - x$

Substituting this in (1) we get $2x + \frac{x}{2} + 2x + 2 = 4 \Rightarrow x = \frac{4}{9}$

$\Rightarrow y = \frac{4}{18}$ i.e., $y = \frac{2}{9}$ and $z = -\frac{13}{9}$

Shortest distance from (2, 1, -3) to the point $P\left(\frac{4}{9}, \frac{2}{9}, -\frac{13}{9}\right)$ on the plane is given by

$\sqrt{(\frac{4}{9} - 2)^2 + (\frac{2}{9} - 1)^2 + (-\frac{13}{9} + 3)^2} = \frac{7}{3}$

5. Find the points on the surface $z^2 = xy + 1$ nearest to the origin

Solution:

Let the point on the surface $z^2 = xy + 1$, which is nearest to the origin be $P(x, y, z)$.

Distance from this point $P(x, y, z)$ to the origin is $d = \sqrt{x^2 + y^2 + z^2}$

$\therefore f(x, y, z) = x^2 + y^2 + z^2$ (1)

But $z^2 = xy + 1$ (2)

Using (2) in (1), we get $f(x, y, z) = x^2 + y^2 + xy + 1$

Now $\frac{\partial f}{\partial x} = 2x + y$ $\qquad \frac{\partial f}{\partial y} = 2y + x$

$r = \frac{\partial^2 f}{\partial x^2} = 2$ $\quad$ $t = \frac{\partial^2 f}{\partial y^2} = 2$ $\quad$ $s = \frac{\partial^2 f}{\partial x \partial y} = 1$

To find the point at which maximum and minimum occurs we equate

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + y = 0$ (3)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + x = 0$ (4)

Solving (3) and (4) we get $\Rightarrow x = 0, y = 0$

Substituting for $x = 0, y = 0$ in (2) we get $z^2 = 1 \Rightarrow z = \pm 1$

Therefore stationary points are (0, 0 , 1) and 0, 0, -1).

At the stationary point (0, 0, 1) $rt - s^2 = 3 > 0 \Rightarrow$ the function has a minimum at (0, 0, 1)

At the stationary point (0, 0, -1) $rt - s^2 = 3 > 0 \Rightarrow$ the function has a minimum at (0, 0,-1)

Hence the points on the surface nearest to the origin are (0, 0 , 1) and 0, 0, -1).

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 4 - INTEGRAL CALCULUS I– SMTA1101

- Definite integrals
- Properties of definite integrals and problems
- Beta and Gamma integrals
- Relation between them
- Properties of Beta and Gamma integrals with proofs
- Evaluation of definite integrals in terms of Beta and Gamma function.

Definite Integrals

Property 1: $$\int_a^b f(x)dx = \int_a^b f(z)dz$$

Property 2: $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property 3: $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Property 4: $$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 5: $$\int_{-a}^a f(x)dx = \begin{cases} 0 & if\ f(x)\, is\ odd \\ 2\int_0^a f(x)dx & if\ f(x)\, is\ even \end{cases}$$

Problems based on definite Integrals

PROBLEM (1) Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x)dx \qquad (1)$$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)dx \qquad (2)$$

Adding (1) & (2)

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x dx + \int_0^{\frac{\pi}{2}} \log \cos x dx \qquad (\text{Since } \because \log a + \log b = \log ab)$$

$$= \int_0^{\frac{\pi}{2}} \log [\sin x \cos x] dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{\sin 2x}{2} \right) dx \qquad \left(\because \sin x \cos x = \frac{\sin 2x}{2} \right)$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx \qquad (3)$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log \sin y dy$$

$$= \frac{1}{2}(2)\int_0^{\pi/2} \log \sin y dy$$

$$= \int_0^{\pi/2} \log \sin y dy$$

$$= \int_0^{\pi/2} \log \sin x dx \quad \text{\_} \qquad (4)$$

To evaluate $\int_0^{\frac{\pi}{2}} \log(\sin 2x)dx$

Put $\quad 2x = y, 2dx = dy$

when $\quad x = 0, y = 0$

$$x = \frac{\pi}{2}, y = \pi$$

sub (4) in (3)

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \frac{-\pi}{2}\log 2$$

PROBLEM (2) Evaluate $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$

$$\text{let } I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta \qquad (1)$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan\theta}\right]d\theta \qquad (2)$$

$(1) + (2) \Rightarrow$

$$2I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta + \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log\left[(1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right)\right]d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log 2 d\theta = \log 2 \int_0^{\frac{\pi}{4}} d\theta$$

$$2I = \log 2[\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$$

$$\therefore 2I = \frac{\pi}{4}\log 2$$

$$\therefore I = \frac{\pi}{8}\log 2$$

BETA AND GAMMA FUNCTIONS

Gamma Functions:

Gamma function is defined as $\int_0^\infty e^{-x} x^{n-1} dx; n > 0$ and it is denoted by Γn

(i.e) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx,\ n > 0$

Beta function:

Beta function is defined as $\int_0^1 x^{m-1}.(1-x)^{n-1} dx, m > 0, n > 0$ and it in denoted by β(m, n)

(i.e) $\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}.dx; m > 0, n > 0$

Result : 1 Recurrence formula for Γn

$$\Gamma(n+1) = n\Gamma n$$

Result : 2 $\Gamma 1 = 1$

Result 3: when 'n' is a positive integer, then $\Gamma n+1 = n!$

Properties of Beta function:

1) **Symmertric Property:** $\beta(m, n) = \beta(n, m)$

2) Transformation of Beta function:

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{n+m}} . dy$$

3) **Trigonometric form of Beta function:**

$$\beta(m,n) = 2\int_0^{\pi/2} \sin 2^{m-1}\theta . \cos^{2n-1}\theta d\theta$$

Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Proof: W.K.T $\Gamma n = \int_0^{\infty} e^{-x}\cdot x^{n-1}dx$

Put $x = y^2$

$$dx = 2y\,dy$$

$$\Gamma n = \int^{\infty} e^{-y2}\cdot(y^2)^{n-1}2y\cdot dy$$

$$= 2\int_0^{\infty} e^{-y2}.y^{2x-2}.y^1 dy$$

$$\Gamma n = 2\int_0^{\infty} e^{-y2}\cdot y^{2x-1}dy$$

Similarly $\Gamma(m) = 2\int_0^{\infty} e^{-x^2}\cdot x^{2m-1}\cdot dx$

$$\therefore \overline{|(m)} \cdot \overline{|(n)} = 2\int_0^\infty e^{-x^2} \cdot x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2} y^{2x-1} \cdot dy$$

$$= 4.\int_0^\infty \int_0^\infty e^{-(x^2-y^2)} x^{2m-1}.y^{2n-1}.dx.dy$$

Put $\quad x = r\cos\theta; \quad y = r\sin\theta$

Hence |J| = r, by change of variables (jacobian)

$$dxdy = r.dr.d\theta, where\, r = |J|\, (ie) r^2 = x^2 + y^2$$

The region of integration is the complete first quadrant.

In which r varies from 0 to ∞

θ varies from 0 to $\dfrac{\pi}{2}$.

$$\therefore \overline{|(m)} \cdot \overline{|(n)} = 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-2} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} \cdot |r| \cdot dr \cdot d\theta$$

$$= 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-1} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} .dr.d\theta$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2m+2n-1} dr. \int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1} .d\theta$$

$$= 4\int_0^{\infty} e^{-r^2} [r^2]^{m+n-1} \frac{1}{2} d(r)^2 . \int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1} .(\sin\theta)^{2n-1} d\theta$$

$$\therefore \Gamma(m)\cdot\Gamma(n) = 4\left[\frac{1}{2}\Gamma(m+n)\right]\cdot\left[\frac{1}{2}\cdot\beta(m,n)\right]$$

Using Beta & Gamma Properties.

$$= \frac{4}{4}\left[\Gamma(m+n)\right]\cdot\beta(m,n)$$

$$= \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Result : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: W.K.T $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2n-1}\, d\theta$

Put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2\times\frac{1}{2}-1}.(\cos\theta)^{2\times\frac{1}{2}-1}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1.d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{2}} = 2\times\frac{\pi}{2} = \pi \qquad (1)$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

$$\text{W.K.T } \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \quad \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\text{By (1)} \quad \pi = \frac{\left[\Gamma\frac{1}{2}\right]^2}{\Gamma(1)} = \frac{\left[\Gamma\frac{1}{2}\right]^2}{1}$$

$$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

Hence proved

PROBLEM (3)

Evaluate $\int_0^\infty e^{-x^2}\,dx$

Solution

Put $x^2 = t;\;\; 2x\,dx = dt$

$$\therefore \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sqrt{\pi}$$

PROBLEM (4) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x dx$ using Gamma functions

Solution:

we know that

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

Here p = 6, q = 7

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x dx = \frac{1}{2} \frac{\Gamma\frac{7}{2} \times \Gamma\left(\frac{8}{2}\right)}{\Gamma\frac{15}{2}} = \frac{\frac{1}{2}\Gamma\left(\frac{7}{2}\right) \times 3!}{\frac{13}{2} \times \frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{1}{2} \times \frac{6 \times 2^4}{13 \times 11 \times 9 \times 7}$$

$$= \frac{16}{3003}$$

Property 1: $\int_a^b f(x)dx = \int_a^b f(z)dz$

$$\text{Proof : L.H.S } = \int_a^b f(x)dx = [F(x)]_a^b$$

$$= F[b] - F[a]$$

$$R.H.S = \int_a^b f(z)dz = [F(Z)]_a^b$$

$$= F[b] - F[a]$$

$$\text{L.H.S } = \text{R.H.S}$$

Property 2: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$\text{Proof : L.H.S} = \int_a^b f(x)dx = [F(x)]_a^b = F[b] - F[a]$$

$$\text{R.H.S} = -\int_b^a f(x)dx = -[F(x)]_b^a$$

$$= -\left[F(a) - F(b)\right]$$

$$= \left[F(b) - F(a)\right]$$

$$\text{L.H.S} = \text{R.H.S}$$

Property 3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_a^b f(x)dx$

$$\text{Proof : L.H.S} = \int_a^b f(x)dx$$

$$= [F(x)]_a^b = F(b) - F(a)$$

$$\text{R.H.S} = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

Hence L.H.S = R.H.S

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Consider, LHS

Put $x = a - z$

$dx = -dz$

If $x = 0 \Rightarrow z = a$

$x = a \Rightarrow z = 0$

$$\int_0^a f(x)dx = \int_a^0 f(a-z)(-dz)$$

$$= -\int_a^0 f(a-z)dz$$

$$= \int_0^a f(a-z)dz \qquad \text{[by property 2]}$$

$$= \int_0^a f(a-x)dx \qquad \text{[by property 1]}$$

$$= \text{R.H.S}$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 5: $\Gamma 1 = 1$

we know that $\Gamma n = \int_0^\infty e^{-x} x^{n-1} . dx$

Put n = 1 $\qquad \Gamma 1 = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1} \right]_0^\infty . dx$

$$= \left(\frac{e^{-\infty}}{-1} \right) - \left(\frac{e^{-0}}{-1} \right) = 0 + 1 = 1$$

$$\Gamma 1 = 1$$

Property 6: $\beta(m, n) = \beta(n, m)$

Proof : W.K.T $\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}.dx$

W.K.T $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \beta(m,n) = \int_0^1 (1-x)^{m-1}.[1-(1-x)]^{n-1}.dx$$

$$= \int_0^1 (1-x)^{m-1}.x^{n-1}.dx$$

$$= \int_0^1 x^{n-1}.(1-x)^{m-1}.dx$$

$\beta(m,n) = \beta(n,m)$, by definition of Beta function.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – 5 - INTEGRAL CALCULUS II – SMTA1101

- Double integrals in Cartesian and polar co-ordinates
- Change the order of integration
- Change of variables from Cartesian to polar co-ordinates
- Area of plane curves using double integrals
- Triple integrals
- Volume using triple integrals in Cartesian co-ordinates

Problem (1) Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

$$\int_2^a \int_2^b \frac{dxdy}{xy} = \int_2^a \frac{1}{y} [\log x]_{x=2}^{x=b} \, dy$$

$$= \int_2^a \frac{1}{y} [\log b - \log 2] \, dy$$

$$= \int_2^a \frac{1}{y} [\log\left(\frac{b}{2}\right)] \, dy$$

$$= \log\left(\frac{b}{2}\right) \int_2^a \frac{1}{y} \, dy$$

$$= \log\left(\frac{b}{2}\right) [\log y]_{y=2}^{y=a}$$

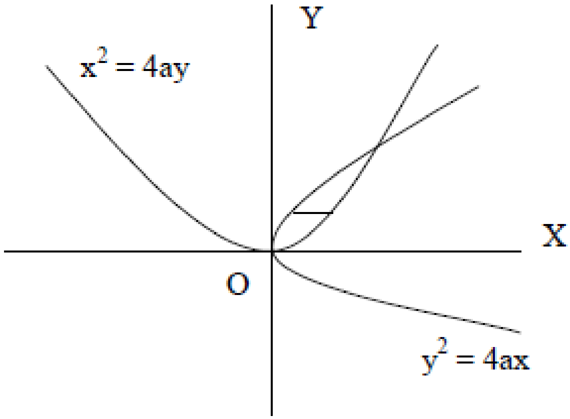
$$= \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$$

Problem (2)

Change the order of integration and hence evaluate $\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$

$$\text{Let } I = \int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$$

The limits for y varies from $y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and the limits for x varies from x = 0 to x = 4a. The region of integration is enclosed between the curves (parabolas) $x^2 = 4ay$ and $y^2 = 4ax$ and the lines x = 0 and x = 4a. The two parabolas intersect at (0, 0) and (4a, 4a).



To change the order of integration, first integrate w.r.t x and then w.r.t y. Since first integration is w.r.t x, we consider a horizontal strip. The limits for x varies from x = $y^2/4a$ to $x = 2\sqrt{ay}$ and then y varies from y = 0 to y = 4a.

Hence,

$$I = \int_o^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy = \int_o^{4a}\left[\frac{x^2}{2}.y\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_o^{4a}\left[\frac{4ay}{2}.y - \frac{y^4}{32a^2}.y\right]dy$$

$$= \left[2a.\frac{y^3}{3} - \frac{1}{32a^2}.\frac{y^6}{6}\right]_0^{4a}$$

$$= \left[2a.\frac{64a^3}{3} - \frac{1}{32a^2}\frac{(4a)^6}{6}\right]$$

$$= \frac{128a^4}{3} - \frac{64a^4}{3}$$

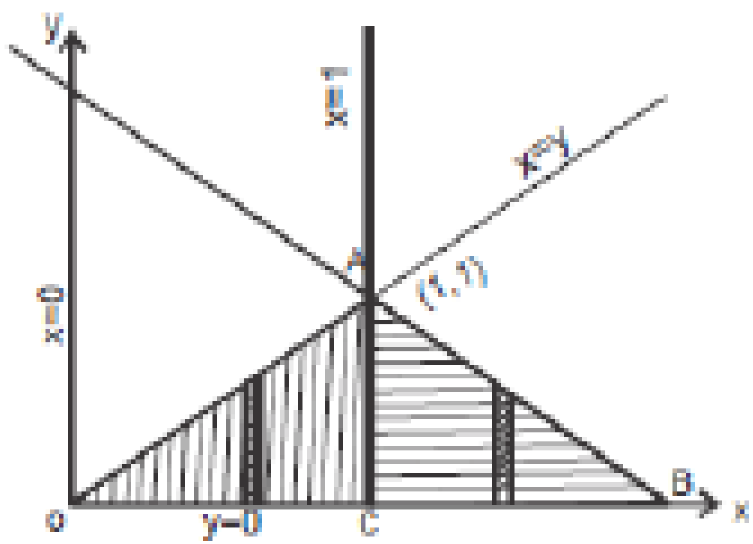
$$= \frac{64a^4}{3}$$

Problem (3)

Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ and hence evaluate it.

Solution:

The region of integration is bounded by $x = y$, $x = 2 - y$, $y = 0$ and $y = 1$ which is shown in the figure.



When we change the order of integration, we first integrate with respect to y keeping x as constant. When the region of integration is covered by vertical strip, it does not intersect the region of integration in the same fashion. Hence the region ΔOAB is splitted into two subregions ΔOAC and ΔCAB. Hence

$$\iint_{OAB} xydxy = \iint_{OAC} xydydx + + \iint_{CAB} xydydx$$

$$\int_0^1 \int_y^{2-y} xydxdy = \int_0^1 \int_0^x xydydx + \int_1^2 \int_0^{2-x} xydydx$$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)_0^x dx + \int_1^2 \left(\frac{xy^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3 dx + \int_1^2 \frac{x(2-x)^2}{2} dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1 + \int_1^2 \frac{x(4+x^2-4x)}{2} dx$$

$$=\frac{1}{2}\left(\frac{1}{4}-0\right)+\frac{1}{2}\left(4\frac{x^2}{2}+\frac{x^4}{4}-4\frac{x^3}{3}\right)_1^2$$

$$=\frac{1}{2}\left[\frac{1}{4}+\left(8+\frac{16}{4}-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\left(12-\frac{32}{3}\right)-\left(\frac{24+3-16}{12}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{5}{12}\right]$$

$$=\frac{8}{12\times 2}$$

$$=\frac{1}{3}$$

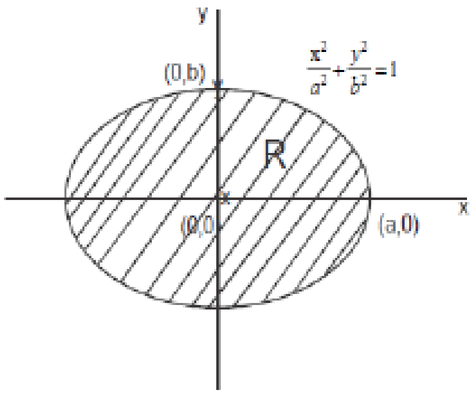
Area Using Double Integral

Problem (4) Find the Area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Solution:

From the equation of the ellipse, we have

$$\frac{y}{b}=\pm\sqrt{1-\frac{x^2}{a^2}}$$



So, the region of integration R can be considered as the area bounded by

$$x=-a\ and\ x=a,\ y\frac{-b}{a}\sqrt{a^2-x^2}\ and\ y=\frac{b}{a}\sqrt{a^2-x^2}$$

Area $=\iint\limits_R dydx = 4\times$ Area in first quadrant

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[(0-0)+\frac{a^2}{2}\left(\frac{\pi}{2}-0\right)\right]$$

$= \pi ab$ square units.

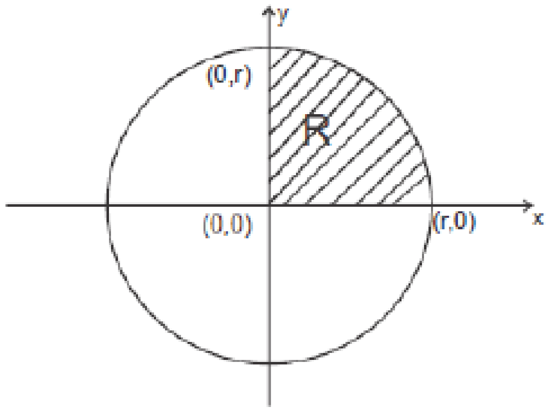
Problem (5)

Find the area of the circle $x^2 + y^2 = r^2$ lies in the positive quadrant

Solution

The circle lies in the first quadrant is bounded by $x = 0, y = 0, x^2 + y^2 = r^2$

Therefore, the region of integration R can be considered as the area bounded by $x = 0, x = r, y = 0 \, and \; y = \sqrt{r^2 - x^2}$



$$\text{Area} = \iint_R dxdy$$

$$= \int_{x=0}^{r} \left[\int_{y=0}^{\sqrt{r^2-x^2}} dy \right] dx$$

$$\text{Area} = \int_{x=0}^{r} [y]_0^{\sqrt{r^2-x^2}}\, dx$$

$$= \int_0^r \sqrt{r^2 - x^2}\, dx$$

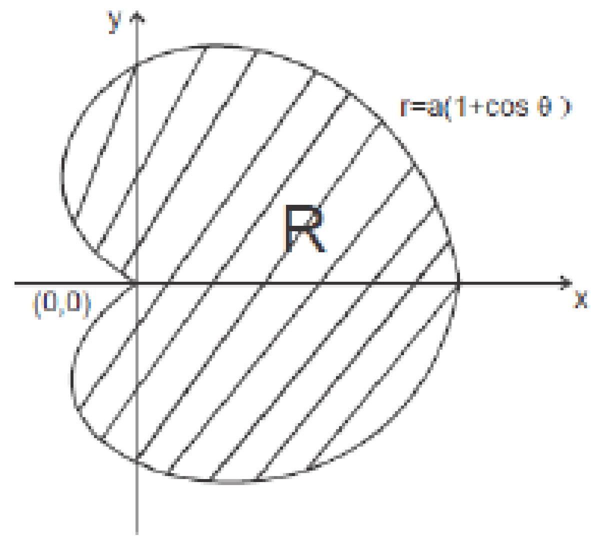
$$= \left[\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r$$

$$= 0 + \frac{r^2}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi r^2}{4} \text{ square units.}$$

Problem (6) Find the area of the cardioids $r = a(1+\cos\theta)$

Solution



Area $= \iint\limits_R dxdy$

$= \iint r dr d\theta$

Given $r = a(1+\cos\theta)$

Limits

$r : 0 \to a(1+\cos\theta)$

$\theta . 0 \to 2\pi$

$$\text{Area} = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r dr d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^2}{2} 2\int_0^{\pi} (1+\cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2 \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= a^2 \left[(\pi - 0) + 2(0-0) + \frac{1}{2}\left[(\pi - 0) + \frac{1}{2}(0-0)\right]\right]$$

$$= a^2\left(\pi + \frac{1}{2}\pi\right) = \frac{3\pi}{2}a^2 \qquad \text{Square units.}$$

Volume using Triple Integrals

Problem (7)

Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ without transformation.

Solution

V = 8 x volume in the first octant

z varies from $z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$

$y$varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$

$x$varies from $x = 0$ to $x =$ a

$$\therefore V = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\, dy\,dx$$

$$= 8\int_0^a \int^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right) + \frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-(a^2-x^2)}\right] dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}(1) + 0\right] dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\cdot\frac{\pi}{2}\right) dx$$

$$= 8 \times \frac{\pi}{4} \int_0^a \left(a^2 - x^2\right) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^2 \times a - \frac{a^3}{3} - 0 \right]$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\left[\frac{3a^3 - a^3}{3} \right] = \frac{4\pi a^3}{3} \, cubic\ units$$

Problem (8)

Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant using triple integration.

Solution

Given $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

$$\text{Volume} = \iiint dz\,dy\,dx$$

To find x limit put $y=0$ *and* $z=0$ we get (line integral)

$$(1) \Rightarrow \frac{x^2}{a^2}=1 \Rightarrow x^2=a^2 \Rightarrow x=\pm a$$

ie, $x=0$ to $x=a$ ($\because$ first octant area)

To find y limit put $z=0$ we get (surface integral)

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow y = \pm b\sqrt{1 - x^2/a^2}$$

$$\Rightarrow y = b\sqrt{1 - x^2/a^2}$$

ie, $y = 0, y = b\sqrt{1 - \frac{x^2}{a^2}}$ ($\because$ first octant area)

To find z limit [volume integral]

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\Rightarrow z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\Rightarrow z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$z = 0 \; to \; z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\text{volume} = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= \int_{b}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\, dydx$$

$$= \int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} - 0 \right] dydx$$

$$= c\int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dydx$$

$$= c\int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left[b^2\left(1-\frac{x^2}{a^2}\right)\right]-y^2}\,dydx$$

$$= \frac{c}{b}\int_0^a \left[\frac{y\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}}{2}+\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}}\right)\right]_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx$$

$$= \frac{c}{b}\int_0^a \left[0+\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\left(\frac{\pi}{2}\right)\right]dx$$

$$= \frac{\pi c b^2}{4b} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{\pi b^2 c}{4b} \left(x - \frac{x^3}{a^2 \times 3}\right)_0^a$$

$$= \frac{\pi bc}{4} \left(a - \frac{a^3}{3a^2}\right)$$

$$= \frac{\pi bc}{4} \left(a - \frac{a}{3}\right)$$

$$= \frac{\pi bc}{4} \frac{2a}{3} = \frac{\pi abc}{6}$$

Hence the volume of the ellipsoid

$$V = 8 \times \frac{\pi abc}{6} = \frac{4}{3} \pi abc \text{ cubic units.}$$