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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Advanced Graph Theory – SMT5207

I. Connectivity

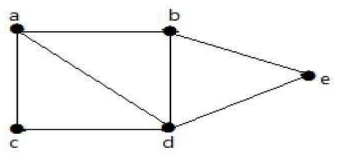
Contents - Connectivity and edge-connectivity – 2-connected graphs – Menger's theorem.

Connectivity

A graph is said the connectivity of a graph. A graph with multiple disconnected vertices and to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called edges is said to be disconnected.

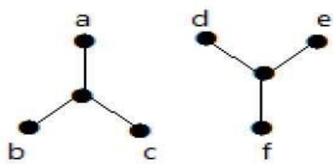
Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Example 2

In the following example, traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



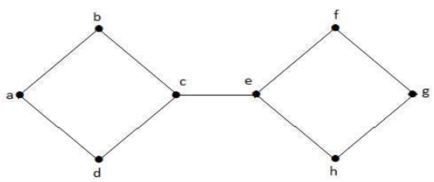
Cut Vertex

Let 'G' be a connected graph. A vertex V ∈ G is called a cut vertex of 'G', if 'G-V' (Delete 'V' from 'G') results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.

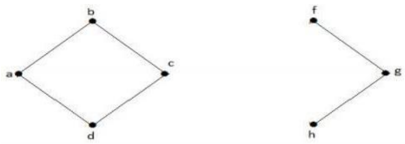
Note − Removing a cut vertex may render a graph disconnected. A connected graph 'G' may have at most (n–2) cut vertices.

Example

In the following graph, vertices 'e' and 'c' are the cut vertices.



By removing ‘e’ or ‘c’, the graph will become a disconnected graph.



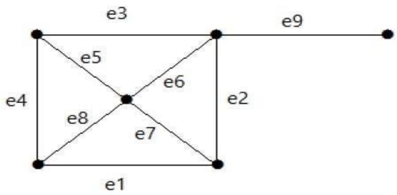
Cut Set of a Graph

Let ‘G’= (V, E) be a connected graph. A subset E’ of E is called a cut set of G if deletion of all the edges of E’ from G makes G disconnect.

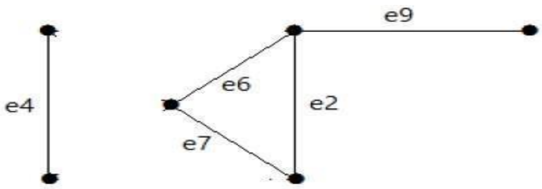
If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

Example

Take a look at the following graph. Its cut set is E1 = {e1, e3, e5, e8}.



After removing the cut set E1 from the graph, it would appear as follows −



Similarly, there are other cut sets that can disconnect the graph −

E3 = {e9} – Smallest cut set of the graph.

E4 = {e3, e4, e5}

Edge Connectivity

Let ‘G’ be a connected graph. The minimum number of edges whose removal makes ‘G’ disconnected is called edge connectivity of G.

Notation − λ(G)

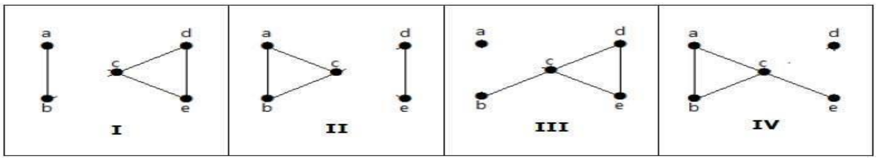
In other words, the **number of edges in a smallest cut set of G** is called the edge connectivity of G.

If ‘G’ has a cut edge, then λ(G) is 1. (edge connectivity of G.)

Example

Take a look at the following graph. By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity (λ(G)) is 2.

Here are the four ways to disconnect the graph by removing two edges −



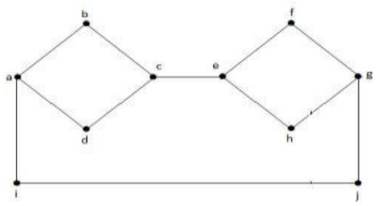
Vertex Connectivity

Let ‘G’ be a connected graph. The minimum number of vertices whose removal makes ‘G’ either disconnected or reduces ‘G’ in to a trivial graph is called its vertex connectivity.

Notation − K(G)

Example

In the above graph, removing the vertices ‘e’ and ‘i’ makes the graph disconnected.



If G has a cut vertex, then K(G) = 1.

Notation − For any connected graph G,

Vertex connectivity (K(G)), edge connectivity (λ(G)), minimum number of degrees of G(δ(G)).

Theorem (Whitney) For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof: We first prove $\lambda(G) \leq \delta(G)$.

If G has no edges, then $\lambda = 0$ and $\delta = 0$. If G has edges, then we get a disconnected graph, when all edges incident with a vertex of minimum degree are removed. Thus, in either case, $\lambda(G) \leq \delta(G)$.

We now prove $\kappa(G) \leq \lambda(G)$. For this, we consider the various cases. If G

= Kn, then κ(G) = λ (G) = n − 1. Now let G be an incomplete graph. In case G is disconnected or trivial, then obviously $\kappa = \lambda = 0$.

If G is disconnected and has a cut edge (bridge) x, then $\lambda = 1$. In this case, $\kappa = 1$, since either G has a cut vertex incident with x, or G is K2.

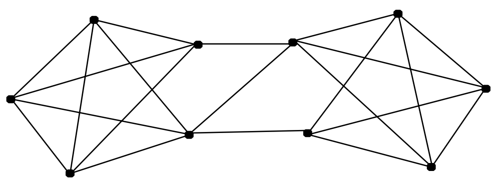
Finally, let G have $\lambda \geq 2$ edges whose removal disconnects it. Clearly, the removal of $\lambda - 1$ of these edges produces a graph with a cut edge (bridge) x

= uv. For each of these $\lambda - 1$ edges, select an incident vertex different from u or v. The removal of these vertices also removes the $\lambda - 1$ edges and quite possibly more. If the resulting graph is disconnected, then $\kappa < \lambda$. If not, x is a cut edge (bridge) and hence the removal of u or v will result in either a disconnected or a trivial graph, so that $\kappa \leq \lambda$ in every case.

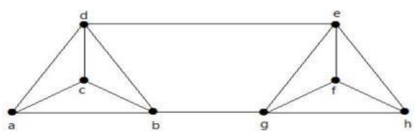
Illustration

We illustrate this by the graph shown in Figure Here $\kappa = 2$, $\lambda = 3$ and $\delta = 4$.

Example



Calculate λ(G) and K(G) for the following graph –



Solution

From the graph, $\delta(G) = 3$

$K(G) \leq \lambda(G) \leq \delta(G) = 3$ (1) $K(G) \geq 2$ (2)

Deleting the edges {d, e} and {b, h}, we can disconnect G. Therefore,

$\lambda(G) = 2$

$2 \leq \lambda(G) \leq \delta(G) = 2$ (3)

From (2) and (3), vertex connectivity $K(G) = 2$

Theorem : For any $v \in V$ and any $e \in E$ of a graph G(V, E), $\kappa(G)-1 \leq \kappa(G-v)$ and $\lambda(G)-1 < \lambda(G-e) \leq \lambda(G)$. Proof We observe that the removal of a vertex or an edge from a graph can bring down κ or λ by at most one, and that while κ may be increased by the removal of a vertex, λ cannot be increased by the removal of an edge.

Theorem : For any three integers r, s, t with $0 < r \leq s \leq t$, there is a graph G with $\kappa = r$, $\lambda = s$ and $\delta = t$. Proof Take two disjoint copies of Kt+1. Let A be a set of r vertices in one of them and B be a set of s vertices in the other. Join the vertices of A and B by s edges utilising all the vertices of B and all the vertices of A. Since A is a vertex cut and the set of these s edges is an edge cut of the resulting graph G, it is clear that κ(G) = r and λ(G) = s. Also, there is at least one vertex which is not in A∪B, and it has degree t, so that δ (G) = t.

Menger's Theorem

Harary listed eighteen variations of Menger's theorem including those for digraphs. Clearly, all these are equivalent and one can be obtained from the other.

Let u and v be two distinct vertices of a connected graph G. Two paths joining u and v are called disjoint (vertex disjoint) if they have no vertices other than u and v (and hence no edges) in common. The maximum number of such paths between u and v is denoted by p(u, v). If the graph G is to be specified, it is denoted by p(u, v|G).
The following is the vertex form of Menger's theorem. The proof is due to Nash-Williams and Tutte

Theorem (Menger-vertex form)

The minimum number of vertices separating two non-adjacent vertices s and t is equal to the maximum number of disjoint s−t paths, that is, for any pair of non-adjacent vertices s and t, the clot number equals the maximum number of disjoint s−t paths. That is, κ(s, t) = p(s, t), for every pair s, t ∈ V with st ∈/ E. Proof Let G(V, E) be a graph with |E| = m. We use induction on m, the number of edges. The result is obvious for a graph with m = 1 or m = 2. Assume that the result is true for all graphs with less than m edges. Let the result be not true for the graph G with m edges. Then we have p(s, t|G) < κ(s, t|G) = q (say), as for any graph, we obviously have p(s, t) ≤ κ(s, t). Let e = uv be an edge of G. The deletion graph G1 = G − e, and the contraction graph G2 = G|e has a smaller number of edges than G. Therefore, by induction hypothesis, we have p(s, t|G1) = κ(s, t|G1) and p(s, t|G2) = κ(s, t|G2). Let I be an (s, t) − clot in G1 and J 0 be an (s, t)− clot in G2. Then we have

|I| = κ(s, t|G1) = p(s, t|G1) ≤ p(s, t|G) < q and

|Jj| = κ(s, t|G2) = p(s, t|G2) ≤ p(s, t|G) < q, So |Jj|
< q and therefore |Jj| ≤ q − 1.

Now to Jj there corresponds an (s − t) vertex cut J of G such that |J|≤ |Jj| + 1, since, by elementary contraction, κ(s, t) can be decreased by at most one, and this decrease actually occurs when e ∈ E((J)).

Thus, |J| ≤ |Jj| + 1 ≤ q − 1 + 1
= q, that is, |J| ≤ q.
Since J is an (s, t) vertex cut in G, κ(s, t) ≤ |J|, q

≤ |J|. Thus, q ≤ |J| ≤ q, so that |J| = q.

Therefore, |I| < q and |J| = q and u, v ∈ J

Let Ht ∈ I ∪ J : there exists an s − w path in G, vertex-disjoint from I ∪ J −

= {w {w}} and

Ht = {w ∈ I ∪ J : there exists a t − w path in G, vertex-disjoint from I ∪

J − {w}}.

Clearly, Hs and Ht are (s − t) separating vertex cuts in G. Therefore,

$$|Hs| \geq q \text{ and } |Ht| \geq q.$$

Obviously, Hs ∪ Ht ⊆ I ∪ J.

We claim that Hs ∩ Ht ⊆ I ∪ J. For this, let w ∈ Hs ∩ Ht. Then there exists an s − w path P1 and w − t path P2 in G vertex disjoint from I ∪ J− {w}. So P1 ∪ P2 contains a path, say P. If e ∈ P then we have u, v ∈ V (P) ∩ J ⊆ {w}, which is impossible. Therefore e ∈/ P and so P ⊆ G − e. Since I is an (s, t) separator in G − e and J is an separator in G, P has a vertex common with I and also with J. So w ∈ I ∩ J. Thus, Hs ∩ Ht ⊆ I ∩ J.

Combining (5.17.4) and (5.17.5), and the above observation, we have

q + q ≤ |Hs| + |Ht| = |Hs ∪ Ht| + |Hs ∩ Ht | ≤ |I ∪ J| + |I ∩ J| = |I| + |J| < q + q, which is a contradiction

Thus not true, and therefore, we have κ(s, t|G) = p(s, t|G).

Definition: Two paths joining u and v are said to be edge-disjoint if they have no edges in common. The maximum number of edge- disjoint paths between u and v is denoted by l(u, v).

The following is the edge form of Menger's theorem and the proof is adopted from Wilson [196].

Theorem (Menger-edge form) For any pair of vertices s and t of a graph G, the minimum number of edges separating s and t equals the maximum number of edge-disjoint paths joining s and t, that is, λ (s, t) = l(s, t) for every pair s, t ∈ V .

Proof Let G(V, E) be a graph and let |E| = m. We use induction on the number of edges m of G. For m = 1, 2, the result is obvious. Assume the result to be true for all graphs with fewer than m edges. Let λ (s, t) = k. We have two cases to consider.

Case (i) Suppose G has an (s − t) band F such that not all edges of F are incident with s, nor all edges of F are incident with t. Then G − F consists of two non-trivial components C1 and C2 with s ∈ C1 and t ∈ C2. Let G1 be the graph obtained from G by contracting the edges of C1, and G2 be a graph obtained from G by contracting the edges of C2. Therefore,

$$G1 = G||E(C1) \text{ and } G2 = G||E(C2).$$

Since G1 and G2 have less edges than G, the induction hypothesis applies to them. Also, the edges corresponding to F provide an (s − t) band in G1 and G2, so that λ (s, t|G1) = k and λ (s, t|G2) = k. Thus, by induction hypothesis, there are k edge-disjoint paths joining s and t in G1, and there are k edge-disjoint paths joining s and t in G2. Thus l(s, t|G1) = k and l(s, t|G2) = k.

The section of the path of the k edge-disjoint paths joining s and t in G2 which are in C1 and the section of the paths of the k edge-disjoint paths joining s and t in G1 which are in C2 can now be combined to get k- edge disjoint paths between s and t in G. Hence l(s, t|G) = k.

Case (ii) Every (s − t) band of G is such that either all its edges are incident with s, or all its edges are incident with t.

If G has an edge e which is not in any (s − t) band of G, then λ (s, t|G − e) = λ (s, t|G) =k. Since the induction hypothesis is applicable to G − e, there are k edge-disjoint paths between s and t in G − e and thus in G. Hence l(s, t|G) = k.

Now, assume that every edge of G is in at least one (s − t) band of G. Then every s – t path P of G is either a single edge or a pair of edges. Any such path P can therefore contain at most one edge of any (s − t) band. Then G − E(P) = G1 is a graph with λ(s, t|G1) = κ − 1.

Appling induction hypothesis, we have l(s, t|G1) = κ − 1. Together with P, we get l(s, t|G)= k.

Theorem

A graph G with at least three vertices is 2-connected if and only if any two vertices of G are connected by at least two internally disjoint paths.

Proof Let G be 2-connected so that G contains no cut vertex. Let u and v be two distinct vertices of G. To prove the result, we induct on d(u, v).

If d(u, v) = 1, let e = uv. Since G is 2-connected and n(G) ≥ 3, therefore e cannot be a cut edge of G. For, if e is a cut edge, then at least one of u and v is a cut vertex. Now, a Theorem e belongs to a cycle C in G. Then C − e is a u − v path in G, internally disjoint from the path uv.

Assume any two vertices x and y of G, such that d(x, y) = t − 1, t ≥ 2, are joined by two internally disjoint x − y paths in G. Let d(u, v) = t and let P be a u − v path of length t, and w be the vertex before v on P. Then d(u, w) = t − 1. Therefore, by induction hypothesis, there are two internally disjoint u − w paths, say P1 and P2, in G. Since G has no cut vertex, G − w is connected and therefore there exists a u − v path Q in G − w. Clearly, Q is a u − v path in G not containing w. Suppose x is the vertex of Q such that x − v section of Q contains only the vertex x in common with P1 ∪ P2 Assume x belongs to P1. Then the union of the u − x section of P1 and x − v section of Q together with P2 ∪ {wv} are two internally disjoint u − v paths in G.

Conversely, assume any two distinct vertices of G are connected by at least two internally disjoint paths. Then G is connected. Also, G has no cut vertex. For, if v is a cut vertex of G, then there exist vertices u and w such that every u − w path contains v, contradicting the hypothesis. Thus, G is 2-connected.

QUESTION BANK

PART A

1. For a graph G with p vertices and q edges, K (G) = $\lfloor 2q/p \rfloor$` **CO2 (L2)**

2. Let G be a simple graph of order p and k be an integer with $1 \leq k \leq$ p-1. If δ(G)≥ (p+k-2)/2, then G is K – connected. **CO4 (L1)**

3. For K >0, find a k – connected graph G and a set V' of k vertices of G such that ω (G – V') >2. **CO2 (L1)**

4.Give an example to show if P is a (u,v) – path in a 2 – connected graph G, then G does not necessarily contain a (u,v) – path Q internally disjoint from P. **CO6 (L2)**

5. Prove that Connectivity of $H_{k,p}$ is k. **CO5 (L5)**

6. Show that a graph is 2 edges connected if and only if any two vertices are connected by at least two edges disjoint paths. **CO6 (L2)**

7.a) Define edge connectivity of a graph. **CO1 (L1)**

b) Show that if G is k -edge connected, then q ≥ k.p/2 **CO2 (L2)**

PART B

1. Prove that, for any graph G, k (G) ≤λ(G) ≤ δ(G). **CO2 (L5)**

2. Prove that the connectivity and edge connectivity of a simple cubic graph G are equal. **CO2 (L5)**

3. A graph with p ≥3 is 2 – connected if and only if any two vertices of G are connected by at least two internally disjoint paths - Discuss. **CO4 (L6)**

4. Prove that the minimum number of vertices separating two nonadjacent vertices u &v is equal to the maximum number of disjoint u-v paths in G. **CO2 (L2)**

5. If P (G) ≥ 3, then the following statements are equivalent.

(i) G is 2 – connected. **CO6 (L4)**

(ii) Any two vertices of G are joined by two internally disjoint paths **CO4 (L4)**

(iii) Any two vertices of G lie on a common cycle. **CO4 (L4)**

(iv) $\delta(G) \geq 1$ and only two edges of G lie on a common cycle. **CO4 (L4)**

6. a) Show that if G is simple graph with $\delta \geq p - 2$, then $k = \delta$. **CO3 (L2)**

   b) Find a simple graph G with $\delta = p-3$ and $k < \delta$. **CO2 (L1)**

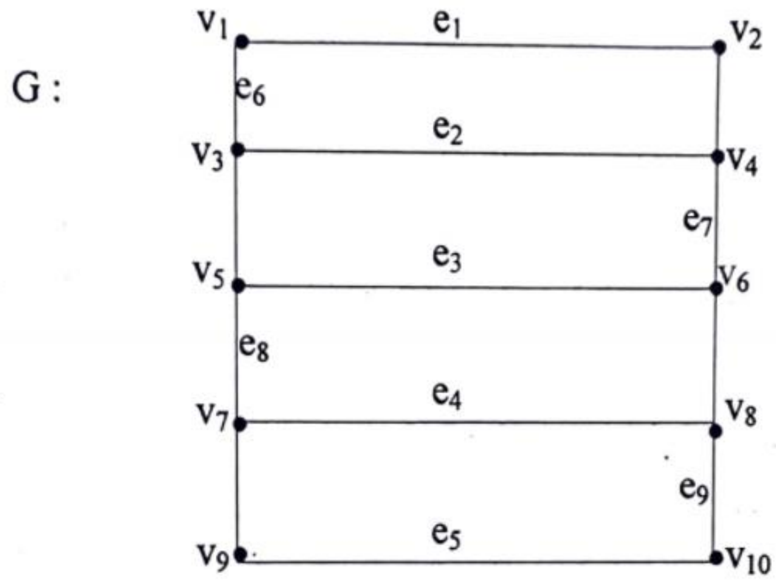
UNIT – II – Advanced Graph Theory – SMT5207

II. Matching

Content: Matching – System of Distinct Representatives and Marriage problem – Covering - 1-factor –Stable Matching.

Definition: A subset M of E is called a matching in G if no two of the edges in M are adjacent. The two ends of an edge in M are said to be matched under M.

Example: In the graph G of figure the sets $M_1= \{e_6, e_8\}$,



$M_3= \{e_6, e_7, e_8, e_9\}$ and $M_3= \{e_1, e_2, e_3, e_4, e_5\}$ are all matchings.

Definition: A matching M saturates a vèrtex v if one edge of M is incident with v. Also, we say v is M-saturated. Otherwise, v is M-unsaturated.

Example: In the graph G of figure , vi is both M_1-saturated and M_2-saturated; v_4 is M_2-saturated but M_1-unsaturated; but M_3 saturates every vertex of G**.**

Definition: If M is a matching in G such that every vertex of G is M-saturated then M is called a perfect matching.

Example: The matching M, of G of figure is a perfect matching where as M_1 and M_2 are not perfect.

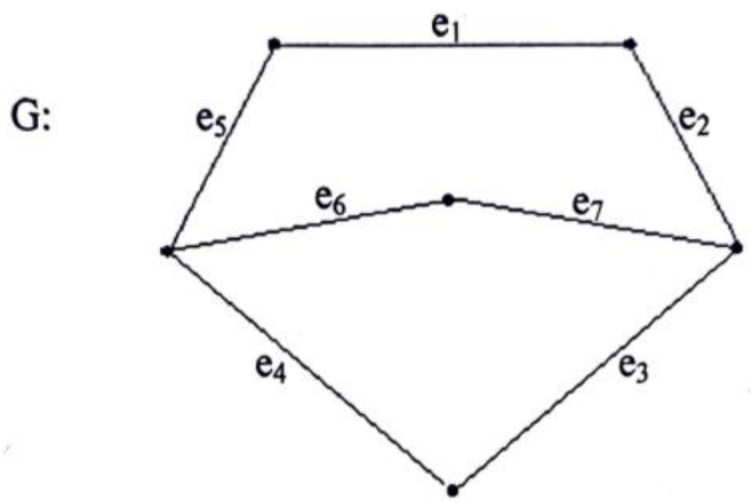
Note: If G has a perfect matching, then p is even.

Definition: A matching M is called a maximal matching of G if there is no matching M' of G such that $M' \supset M$.

Remark: Note that two maximal matchings need not have same Cardinality.

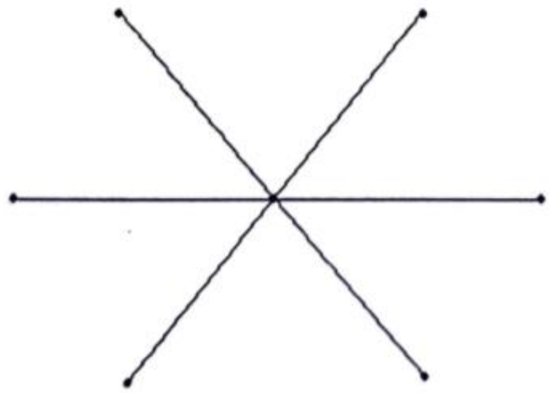
Example: In the graph G , $M_1 = (e_1, e_6, e_3\}$ and $M_2=\{e_5, e_3\}$ are maximal matchings.

Definition: A matching M of G is called a Maximum matching if G has no matching M' with |M'| > |M|. The number of edges in a maximum matching of G is called as the matching number of G.



We note that M_1= {e_1, e_6, e_3} is a maximum matching of G, but M_2= {e_5, e_3} is not a maximum matching, though it is a maximal matching of G. Clearly every perfect matching is maximum; but maximum matchings need not be perfect.

Example: Consider the star $K_{1,6}$ and in general $K_{1,p}$. Here any maximum matching contains only one edge and hence it is not perfect.



$K_{1,6}$

Definition: Let M be the matching in G. An M-alternating path in G is a path whose edges are alternately in E/M and M.

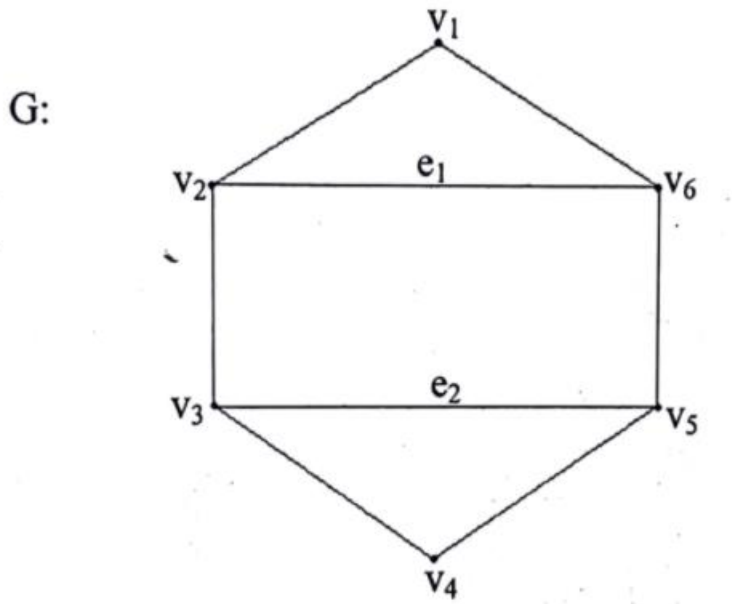
Example: In the graph G, if we consider the matching M= {e_1, e_2} then the path $v_1v_2v_6v_5v_3$ is an M- alternating path.

Definition: Let M be a matching in G. An M-augmenting path is an M-alternating path whose origin and terminus are M-unsaturated.

Example: In the graph G, if we consider the matching M={e_1 , e_2} then the path $v_1v_2v_6v_5v_3$ is an M-augmenting path.

Note: 1. In M-augmenting path initial and final edges are in E\M.

2. An M- alternating path whose initial and final edges are in E\M, need not be an M-augmenting path.



Theorem 6.1 (Berge)

A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Proof: Let M be a maximum matching in G. We prove that G has no M-augmenting path. Suppose not, let G have a M-augmenting path, $v_0e_1v_1e_2v_2...v_{2m}\ e_{2m+1}v_{2m+1}$. We note that such a path is of odd length. Now we define set M' ⊆E by,

M'= {M- {e_2, e_4,…, e_{2m}}}∪{ e_1,e_3,…,e_{2m+1} }.

Then M' is a matching in G and |M'|= |M|+1. This is a contradiction to the fact that M is maximum matching. Hence, G has no M-augmenting path.

Conversely, let G has no M-augmenting path. We prove that M is a maximum matching in G. Suppose not, let M' be a maximum matching in G.

Then, |M'|> |M| (1)

Let H G[M Δ M'] where MΔM' denotes the symmetric difference of M and M'. Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternately in M and M' or else a path with edges alternately in M and M'.

By (1), H contains more edges of M' than of M and so some path component P of H must contain more edges of M' than M and therefore must start and end with edges of M'. The origin and terminus of P being M' -saturated in H and of degree one, are M-unsaturated in G. Therefore, P is an M-augmenting path in G, which is a contradiction to our assumption. Hence, M is a maximum matching in G.

SYSTEM OF DISTINCT REPRESENTATIVES AND MARRIAGE PROBLEM

Let X be a non-empty finite set and S = {S_1,S_2,.,S_m} be a family of (not necessarily distinct) non empty subset of X. If there exists a set{X_1, X_2, X_m} of X such that x_i € S_i and $x_i \neq x_j$ if $i \neq j$ then the set {X_1, X_2, X_m} is called a system of distinct representatives (S.D.R) of the family S.

For example, consider X= {1, 2, 3, 4, 5} and S= {S_1, S_2, S_3, S_4, S_5} where S_1={1, 2}, S_2={1, 2, 3}, S_3= {1, 2, 3}, S_4={1, 4, 3} and S_5={1, 5}. Now, {1, 2, 3, 4, 5} is a system of distinct fepresentatives of the family S. Instead, if we take S_1={1, 2}, S_2={1, 2, 3}, S_3 = {1, 2, 3}, S_4={1,5} and S_5 ={2, 5} then S has no system of distinct representatives.

Naturally, we can identify S. with a bipartite graph with bipartition (S, X) in which S_i € S is joined to every x € X contained in S. A system of distinct representatives is then a set of m independent edges (thus each S_i is incident with one of these edges).

It is customary to formulate this problem of finding S.D.R in terms of marriage arrangements.

The Marriage Problem

Suppose there are n boys each of whom has several girlfriends, under what conditions can we marry off the boys in such a way that each boy marries one of his girl friends? We assume that only single life partner marriage is allowed. This is known as marriage problem.

In graph theoretical terms, the above problem, can be stated as follows. Construct a bipartite graph G with bipartition (X,Y) where X = {x_1,X_2,..,X_n} represents the. set of n boys and Y = {y_1, y_2… y_m} represents their girlfriends. An edge joins a vertex x_i to a vertex y_j if and only if y_j is a girl friend of x_i. The marriage problem is then equivalent to finding conditions for the existence of a matching in G which saturates every vertex of X.

For example, suppose there are five boys b_1, b_2, b_3, b_4 and b_5 and six girls g_1, g_2, g_3, g_4, g_5 and g_6 with their relationship as follows:

$b_1 \longrightarrow \{g_1, g_2, g_3\} = S_1$

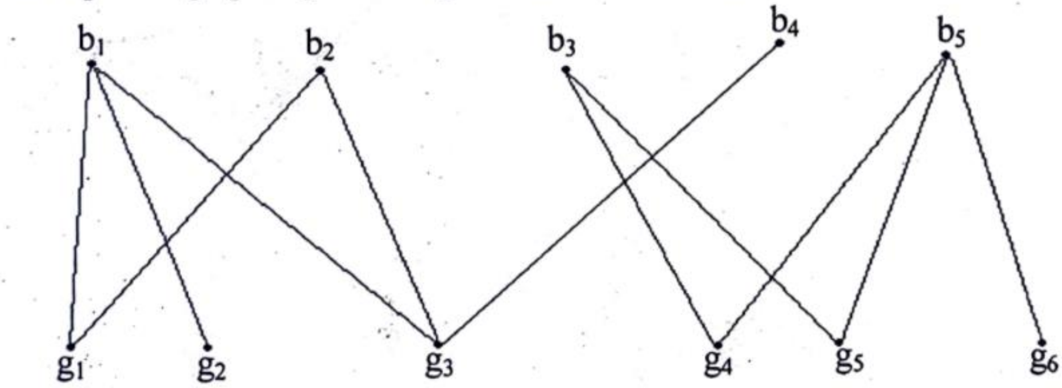
$b_2 \longrightarrow \{g_1, g_3\} = S_2$

$b_3 \longrightarrow \{g_4, g_5\} = S_3$

$b_4 \longrightarrow \{g_3\} = S_4$

$b_5 \longrightarrow \{g_4, g_5, g_6\} = S_5$

The bipartite graph representing this situation is shown



One of the solutions to this example is, b_1 to marry g_2, b_2 to marry g_1, b_3 to marry g_4, b_4 to marry g_3 and b_5 to marry g_5.

Now, we present a necessary and sufficient condition for the existence of a solution to the above marriage problem, first given by P. Hall(1935).

Theorem (Hall's Marriage theorem)

Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof: Suppose that G contains a matching M which saturates every vertex in X and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in N(S), we have $|N(S)| \geq |2S|$.

Conversely, Let G be a bipartite graph with $|N(S)| \geq |S|$ for all $S \subseteq X$. We assume that G has no matching which saturates all vertices in X. Let M* be a maximum matching in G. By our assumption, M* does not saturate all vertices in X. Let u be an M*- unsaturated vertex in X. Let Z denote the set of all vertices connected to u by M*-alternating paths. Since M* is a maximum matching in G, G has no M*-augmenting path. That is, u is the only M*-unsaturated vertex in Z. We set $S = Z \cap X$ and $T = Z \cap Y$. Clearly, the vertices in $S/\{u\}$ are matched under M* with vertices in T. So, we get $|T| = |S| - 1$ and $N(S) \supseteq T$. Since every vertex in N(S) is connected to u by an M*-alternating path, we also have $N(S) \subseteq T$ and hence $N(S) = T$. So, $|N(S)| = |T| = |S| - 1 < |S|$. This is a contradiction to the given hypothesis and hence G has a matching that saturates every vertex in X.

Now, let us reformulate the marriage theorem in terms of system of distinctive representatives.

Theorem 6.3 A family S= $\{S_1,S_2,..,S_m\}$ of sets has a system of distinctive representatives if and only if $\left|\bigcup_{i\in F} S_i\right| \geq |F|$ for every $F \subseteq \{1,2,., m\}$.

Corollary: If G is a k-regular bipartite graph with k> 0, then G has a perfect matching.

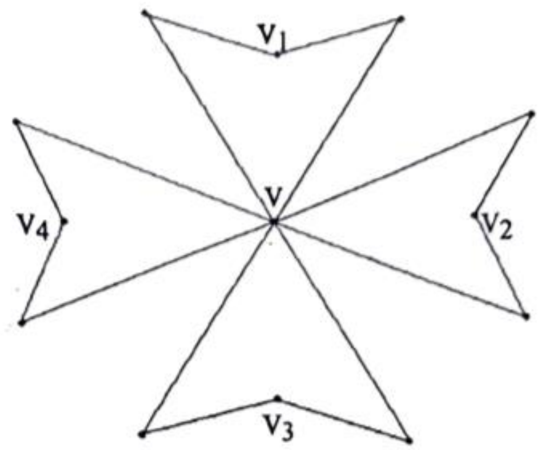
Proof: Let G be a k-regular bipartite graph with bipartition (X, Y). Since G is k-regular, |X]=|Y|.

Now, let S be a subset of X and denote E_1 and E_2 the sets of edges incident with vertices in S and N(S) respectively. By definition of N(S), $E_1 \subseteq E_2$ and therefore $k|N(S)| = |E_2| \geq |E_1| = k|S|$. Therefore, $|N(S)| \geq |S|$ and hence, by theorem, that G has a matching M that saturates every vertex in X. Since |X|=|Y|, M is a perfect matching. Hence the corollary.

COVERING

Definition: A covering of a graph G is a subset K of V such that every edge of G has at least one end in K. For example, in the graph G of figure, the set

G_1:

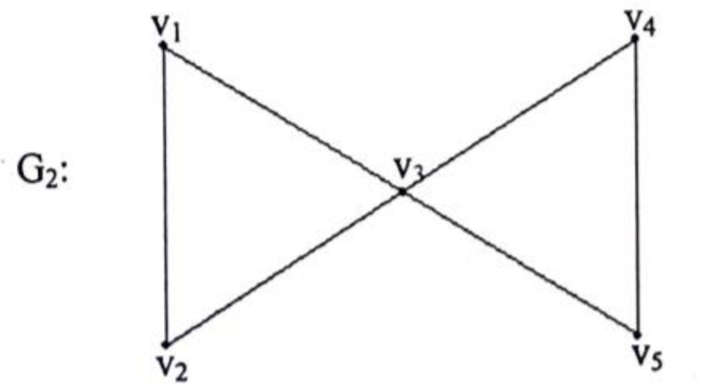


$K=\{v, v_1, v_2, v_3, v_4\}$is a covering of G1.

Definition: A covering K is called a minimal covering of G if there is no covering K' of G such that $K' \subset K$.

For example, the covering K of G_1 is a minimal covering.

Definition: A covering K is called a minimum covering of G if G has no covering K' with |K'|<|K|.



For example, {v_1, v_3, v_5} is a minimum covering of the graph G_2. Also, we note that {v_1, v_2, v_4, v_5} is a minimal covering but not minimum covering of G_2.

Remark: If K is a covering of G and M is a matching of G then K contains at least one end of each of the edges in M. Thus, for any matching M and any covering K, $|M| \leq |K|$. In particular, if M* is a maximum matching and K* is a minimum covering then,

$$|M^*| \leq |K^*|. \qquad (1)$$

In general, equality does not hold in (1). For example, consider the graph G_2. Here, $|M^*| = 2$ and $|K^*| = 3$. Under what conditions, does the equality hold? If G is bipartite then $|M^*| = |K^*|$. This result was proved by Konig and Egervary in 1931. Now, we present a lemma, which is useful in proving the Konig-Egervary theorem.

Lemma Let M be a matching and K be a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

Proof: Let M* be a maximum matching and K* be a minimum covering of G. Then, $|M| \leq |M^*| \leq |K^*| \leq |K|$. Since $|M| = |K|$, in the above, equality must hold throughout and hence the lemma.

Konig-Egervary Theorem

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof: Let G be a bipartite graph with bipartition (X, Y) and let M* be a maximum matching of G.

Suppose M* is perfect then $|X| = |Y| = |M^*|$. In this case X is a covering and the theorem holds.

So, we assume that M* is not perfect. Let U denote the set of all M*-unsaturated vertices in X and let Z be the set of all vertices connected by M*-alternating paths to vertices of U. Let $S = Z \cap X$ and $T = Z \cap Y$. Clearly every vertex in T is M*-saturated and N(S) = T (as in Hall's theorem). Define K* = (X/S) UT. Every edge of G must have at least one of its ends in K*; otherwise, there would be an edge with one end in S and one end in YT, contradicting N(S) = T. Thus, K* is a covering of G and clearly M"|= K*|. By lemma 6.5, K* is a minimum covering. Hence the theorem.

1-FACTOR

Definition: A factor of a graph G is a spanning subgraph of G.

Definition: A k-factor of a graph G is a spanning k-regular subgraph of G.

Thus, a perfect matching of a graph G induces a 1-factor of G and conversely. A 2-factor is a union of edge disjoint cycles, containing all vertices.

Definition: A component of a graph is odd or even according as it has odd or even number of vertices; the number of odd components of G is o(G).

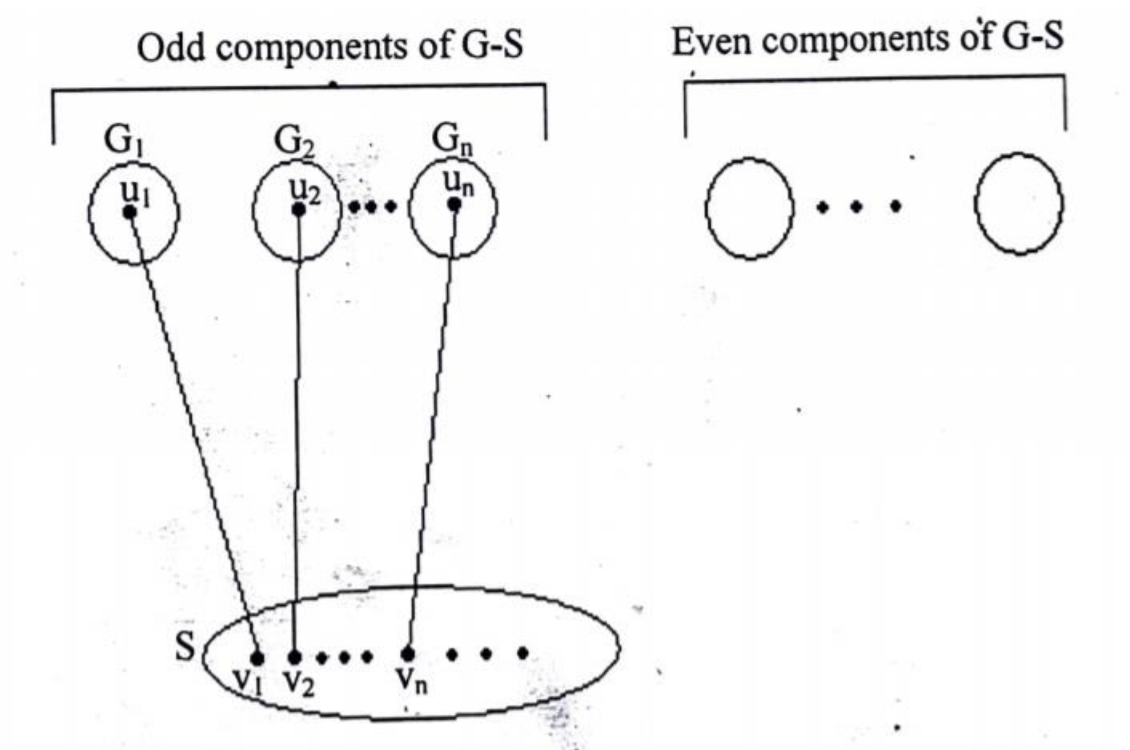
Tutte found a necessary and sufficient condition for a graph to have a 1-factor. Here, we present the proof of Lovasz (1975).

Tutte's Theorem

A graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for all $S \subseteq V$ and $S \neq V$.

Proof: It is enough if we prove the theorem for simple graphs. Let us assume that G has a 1-factor and let M be a perfect matching of G. Let S be a subset of V and $S \neq V$ and let G_1, G_2, .., G_n be the odd components of G-S. Since Gi is odd , some vertex v_i of G; must be matched under M with a vertex v_i of S. Clearly, $\{V_1,V_2,..,V_n\} \subseteq S$ and v_is are distinct and hence,

$o(G-S) = n=|\{v_1,v_2,..,v_n\} \leq |S|$.



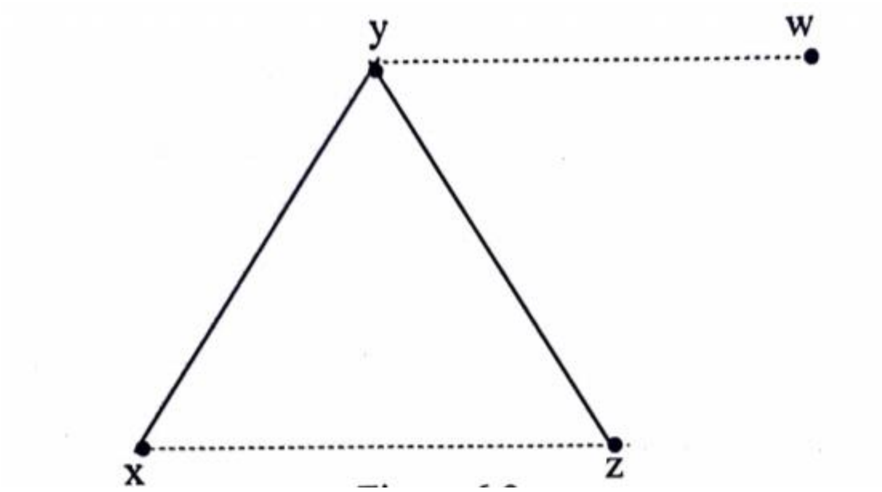
Conversely, let G satisfy the inequality $o(G-S) \leq |S|$ for all $S \subseteq V$ and $S \neq V$ and G have no perfect matching. Then G is a spanning subgraph of a maximal graph G* having no perfect matching. Since G-S is a spanning subgraph of G*-S, $o(G^*-S) \leq o(G-S)$.

Therefore, $o(G^*-S) \leq |S|$ for all $S \subseteq V(G^*)$ and $S \neq V(G^*)$. (1)

In particular, setting $S = \Phi$, we get $o(G^*) = 0$ and hence $p(G^*) = p$ is even.

Let U denote the set of all vertices of degree p-1 in G*. Since G* has a perfect matching if U = V, we may assume that $U \neq V$.

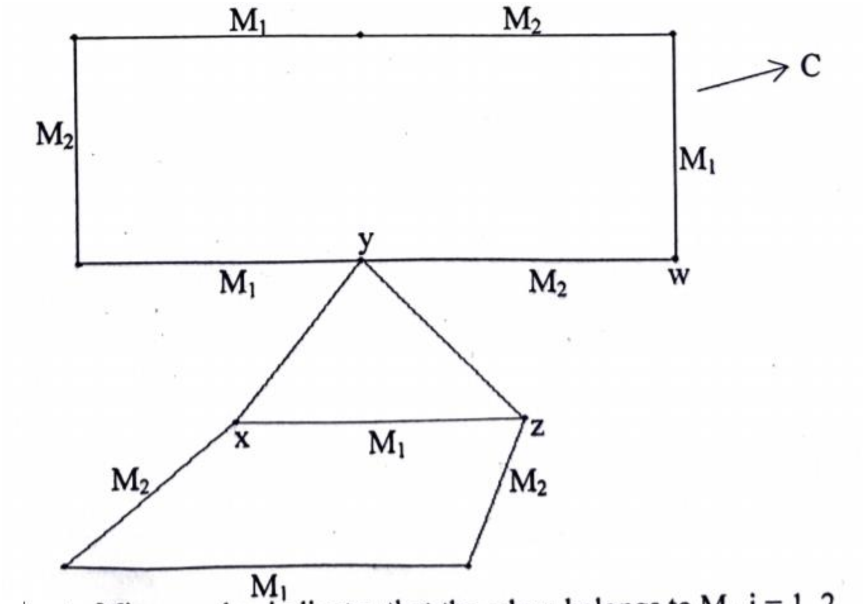
Claim: Now we prove that G*-U is a disjoint union of complete graphs. Suppose not, there is a component of G*-U which is not complete. Since this component is not complete, we can find three vertices x, y and z such that xy € E(G*), yz € E(G*) and xz $\notin$ E(G*). Also, we can find a vertex w in G-U such that yw $\notin$ E(G*), since y $\notin$ U.



By our assumption G* is a maximal graph containing no perfect matching and so G*+xz and G+yw have perfect matchings, say, M1 and M2, respectively.

Let H be the subgraph of G* U{xz, yw} induced by $M_1 \Delta M_2$ Since M_1 and M_2 are perfect matchings, each vertex of H has degree two and hence H is a disjoint union of cycles. Also, all of these cycles are even, since edges of M_1 alternate with edges of M_2 around them. We distinguish two cases.

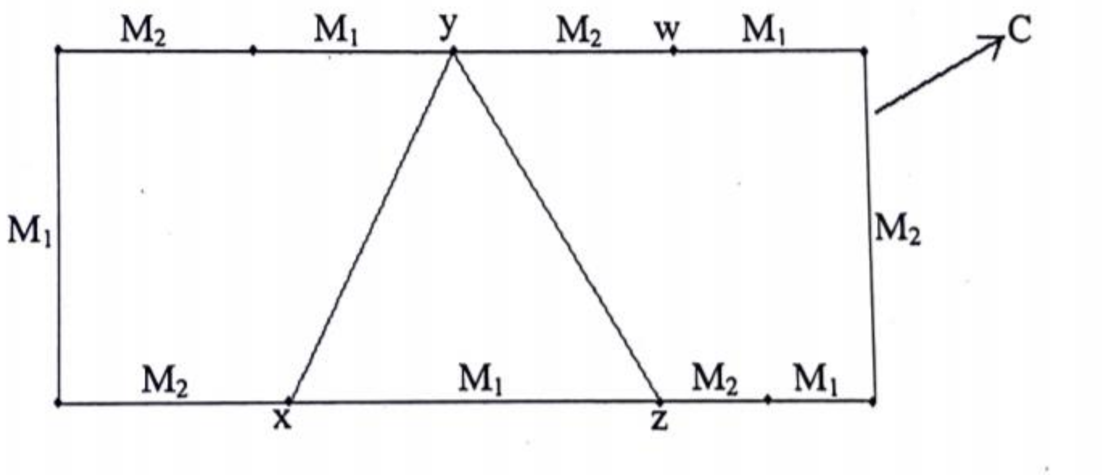
Case 1. xz and yw are in different components of H.



Mi on a edge indicates that the edges belongs to M_i, i = 1, 2.

Now if yw is in the cycle C of H, then the edges of M_1 in C together with the edges of M_2 not in C, constitute a perfect matching in G*. This is a contradiction since G* has no perfect matching.

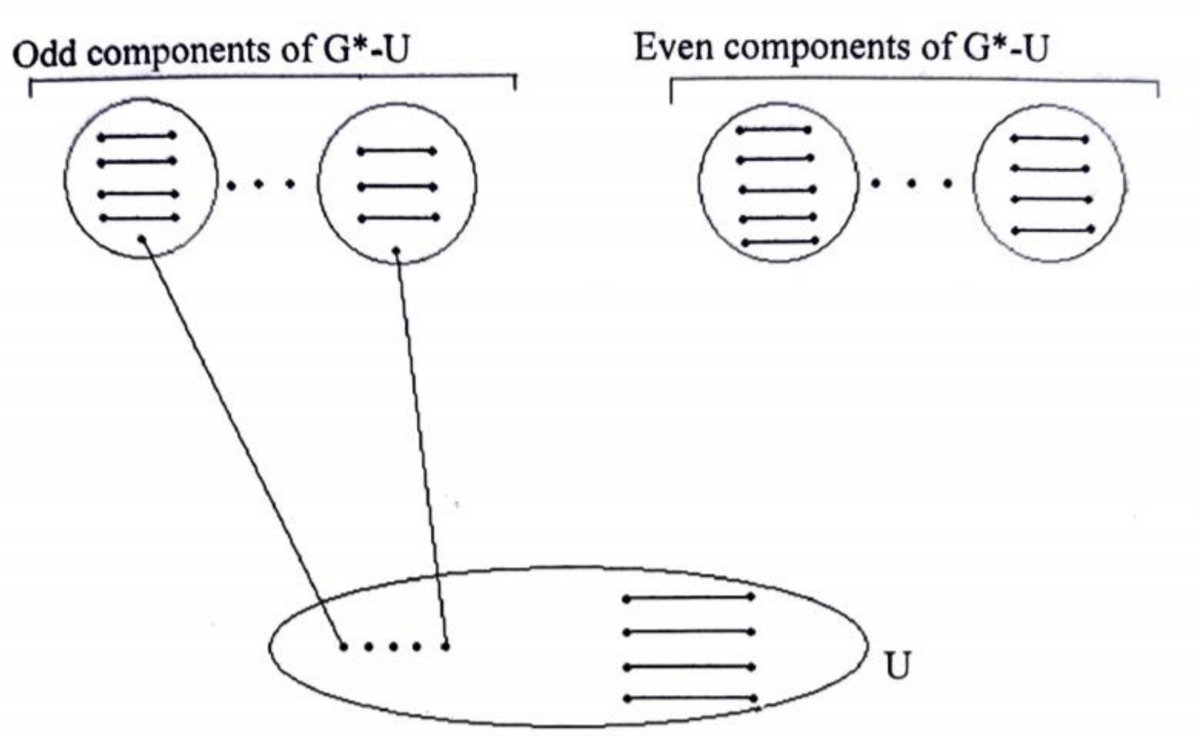
Case 2. xz and yw are in the same component C of H. By symmetry of x and z, we may assume that the vertices x, y, w and z occur in that order on C.



M_i on a edge indicates that the edge belongs to M_i, i = 1, 2.

Then the edges of M_1 in the section yw...z of C, together with the edge yz and the edges of M_2 not in the section yw...z of C, constitute a perfect matching in G*. This is a contradiction since G* has no perfect matching. Hence, G*-U is a disjoint union of complete graphs. Now by (1), $o(G^*\text{-}U) \leq |U|$. Therefore, G*-U can have almost |U| odd components. This implies that G* has a perfect matching, as below.

One vertex in each odd component of G*-U is matched with a vertex of U; the remaining vertices in U and in components of G*-U, are then matched to any vertex in the same component as illustrated. This is possible since each component is complete.



This is a contradiction to our assumption that G* has no perfect matching. Hence G has a perfect matching. That is, G has a 1-factor.

Corollary Every 3-regular graph without cut edges has a perfect matching.

Proof: Let G be a 3-regular graph without cut edges. Let S be a subset of V such that S≠V and let G_1, G_2, ..., G_n, be the odd components of G-S. Let α_1 be the number of edges with one end in G_i and the other end in S. Since G is 3-regular,

$$\sum_{v\in V(G_i)} d(v) = 3.p(G_i) \text{ for } 1 \le i \le n \text{ and } \sum_{v\in S} d(v) = 3|S|.$$

Now, $$\sum_{v\in V(G_i)} d(v) - \alpha_i = 2q(G_i)$$

$$\alpha_i = \sum_{v\in V(G_i)} d(v) - 2q(G_i)$$

We note that, since $p(G_i)$ is odd, $\sum_{v\in V(G_i)} d(v)$ is odd and hence α_i is odd

Since G has no cut edge, $\alpha_i \ne 1$ and thus $\alpha_i \ge 3$ for $1 \le i \le n$.

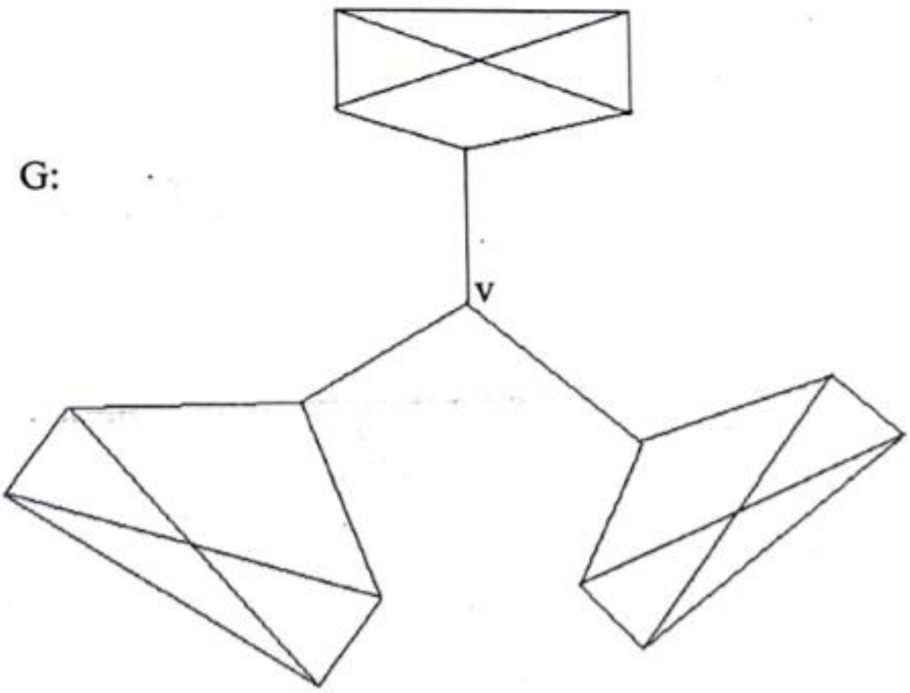
That is, $(1/3)\alpha_i \ge 1$ for $1 \le i \le n$. Hence, $(1/3)\sum_{i=1}^{n} \alpha_i \ge n$.

Now, $o(G\text{-}S) = n \le (1/3)\sum_{i=1}^{n} \alpha_i \le (1/3)\sum_{v\in S} d(v) = |S|$.

By Tutte's theorem, G has a perfect matching.

Remark: A 3-regular graph with cut edges need not have a perfect matching

Consider the graph G. Clearly G is 3-regular and has cut edge. Since o(G-v) =3, by Tutte's theorem, G has no perfect matching.



STABLE MATCHINGS

Now we turn to a special type of matchings, that is, matchings satisfying certain conditions. Matchings satisfying certain conditions are called stable matchings. In 1961, Gale and Shapley introduced stable matchings. It is customary to formulate the conditions and results in terms of marriage arrangements between boys and girls. So, naturally the corresponding graphs are simple bipartite and we consider only simple graphs, in this section. However, we have defined stable matching for a bipartite multigraph in the exercise.

Consider a bipartite graph G with bipartition (V_1, V_2) where V_1= {a, b,...} is the set of n boys and V_2= {A, B, ... } is the set of m girls. An edge a A means that the boy a knows the girl A. Suppose that each boy has an order of preferences on the set of girls he knows, and each girl has an order of preferences on the set of boys she knows. We assume that these orders are linear orders but place no other restriction on them.

Stable Matching: Given the preferences, a stable matching in G is a set M of independent edges of G such that if aB ∈ E(G) - M, then either aA∈ M for some girl A preferred to B by a, or bB ∈ M for some boy b preferred to a by B.

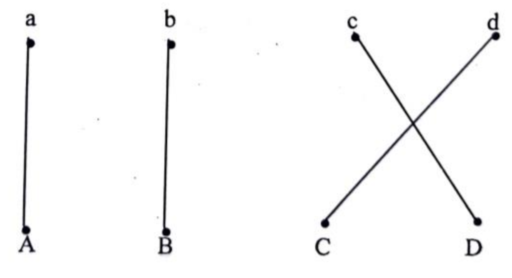
Thus, if a is not married to B, then either a is married to a girl he prefers to B, or else B is married to a boy she prefers to a. Otherwise the matching is "unstable"; a and B will leave their current partners and switch to each other.

Example. Consider a set of 4 boys {a, b, c, d}, a set of 4 girls {A, B, C, D} and their preferences as below.

Preference	1	2	3	4
a	A	B	C	D
b	A	C	B	D
c	C	D	A	B
d	C	B	A	D

	1	2	3	4
A	c	a	b	d
B	b	d	a	c
C	d	a	b	c
D	a	b	c	d

Here, the matching {aA, bB, cD, dC } is a stable matching.



Note: We have not assumed that a stable matching saturates all vertices in V_1 or V_2

Result 1. Every stable matching is a maximal matching in G.

Stable Matching Theorem For every assignment of preferences in a bipartite graph, there is a stable matching.

Proof: We consider a variant of the above algorithm, in which all boys and all girls act simultaneously, in rounds.

In every odd round, each boy proposes to his highest preference among those girls whom he knows and who have not yet refused him, and in every even round each girl refuses all but her highest suitor. The process ends when no girl refuses a suitor; then every girl marries her (only) suitor, if she has one. This process terminates after at most 2nm rounds, since at most m(n-1) proposals are refused, where n is the number of boys and m is the number of girls.

Since at every stage each boy proposes to at most one girl, and each girl rejects all but at most one boy, this algorithm results with a matching.

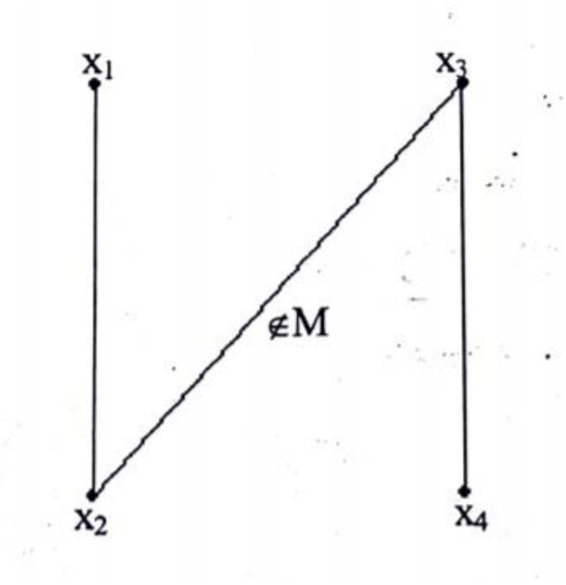
Now we prove that this matching is a stable matching. If $aB \in E(G)-M$, then either a never proposed to B, or a was refused by B during the algorithm. In the former case a marries a girl he prefers to B, as he never goes as low as B, and in the later case B refused a for a boy she prefers to a and got married. Hence this matching is a stable matching.

Definition: A cycle C is called preference-oriented cycle if it can be written in the form aAbB...zZ such that A prefers b to a, b prefers B to A..and Z prefers a to z. That is, each person prefers the next person to previous person.

Theorem Let M and M' be two stable matchings in a bipartite graph with certain preferences, and let C be a component of the subgraph H formed by the edges of MUM'. If C has at least three vertices, then it is a preference-oriented cycle. In particular, if $aA, bB \in M$ and $aB \in M$, then a prefers A to B if and only if B prefers a to b.

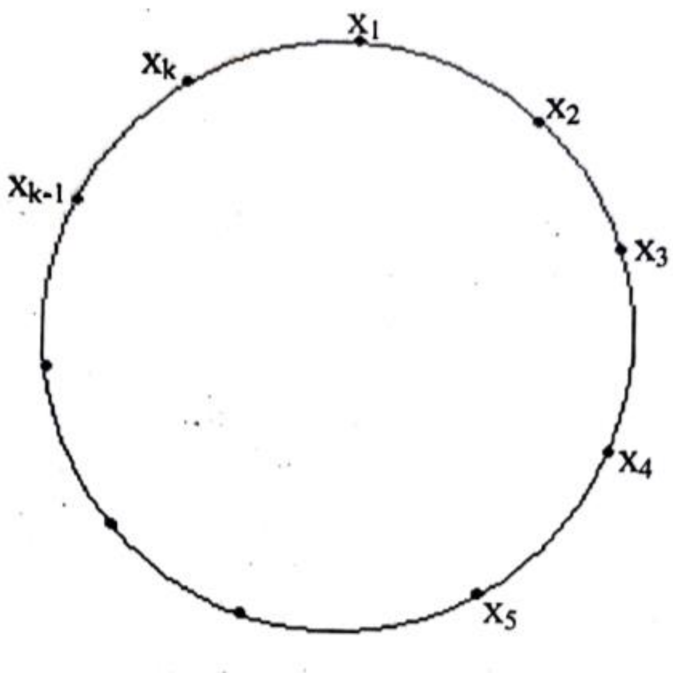
Proof: Without distinguishing between boys and girls, here, we write x_1, x_2 for either of them. Clearly, C is either a path of length at least two or a cycle of length at least four.

If C has a path $x_1x_2x_3x_4$, With x_2 preferring x_3 to x_1 and assuming $x_2x_3 \notin M$ then it is clear that x3 prefers x_4 to x_2, since M is stable. Using this fact, we prove that C is a preference-oriented cycle.



If $x_1x_2x_3...x_k$ is a cycle and x_2 prefers x_3 to x_1, then considering the path $x_1x_2x_3x_4$ we see that x_3 prefers x_4 to x_2. Next, consider the path $x_2x_3x_4x_5$ we see that x_4 prefers x_5 to x_3 .

Continuing in this way, we find that xk prefers x1 to xk-1 and x1 prefers x_2 to X_k. Thus, C is a preference-oriented cycle.



If C is a path $x_1x_2...X_l$, $l \leq 3$ and $x_1x_2 \notin M$, say, then x_2 prefers x_3 to x_1, since M is stable. Similarly, x_{l-1} prefers x_{l-2} to x_l. This is not possible, since, arguing as above, x_2 prefers x_3 to x_1, x_3 prefers x_4 to x_2, x_4 prefers x_5 to x_3, and so on, x_{l-1} prefers x_l to $x_{l-2.}$

Since the component of H containing the path AaBb is prefernce-oriented cycle, the particular case follows.

Note: It is worth to note that all stable matchings are incident with the same set of vertices.

Theorem For every assignment of preferences in a bipartite graph with bipartition (V_1, V_2), there are subsets $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ such that every stable matching saturates all vertices of U_1 and U_2.

In particular, all stable matchings have the same cardinality.

Proof: Suppose the theorem is not true. Then we can find some edge aA of M such that a is not incident with any edge of M'. Since M' is maximal, we can find some b ∈ Vi, b $\neq$a such that bA ∈ M'. But then the component of a in the subgraph formed by the edges MUM' which contains a, A and b is not a cycle. This is a contradiction. Hence the theorem.

QUESTION BANK

PART A

1. Prove that If G is a K – regular bipartite graph with k >0, then G has a perfect matching. **CO2 (L2)**

2. Prove that every 3 – regular graph without cut edges has a perfect matching. **CO2 (L2)**

3. Prove that every stable matching is a maximal matching in G. **CO2 (L2)**

4. Define stable matching. **CO1 (L1)**

5. Consider a set of boys {a, b, c, d}, a set of 4 girls {A, B, C, D} and their preferences as below. Find the stable matching.

Preferences	1	2	3	4
a	A	B	C	D
b	A	C	B	D
c	C	D	A	B
d	C	B	A	D

	1	2	3	4
A	c	a	b	d
B	b	d	a	c
C	d	a	b	c
D	a	b	c	d

CO2 (L1)

5. a) Define minimal covering and minimum covering of a graph. **CO1 (L1)**

b) Prove that if M* is a maximum matching and K* is a minimum covering then, $|M^*| \leq |K^*|$ **CO2 (L2)**

6. a) Define perfect matching. **CO1 (L1)**

b) Prove that a 3 – regular graph with cut edges need not have a perfect matching. **CO2 (L2)**

PART B

1. Matching M in G is a maximum matching if and only if G contains no M – augmenting path - Discuss **CO5 (L6)**
2. A matching M in g is a maximum matching if and only if G contains no M augmenting path. Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if ⌈ N(s) ⌈≥⌈S ⌈ for all S ⊆ X. - Discuss **CO5 (L6)**
3. Prove that in a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering. **CO4 (L2)**
4. Discuss that A graph has a 1-factor if and only if $o(G-S) \leq |S|$ for all S ⊆ V and S ≠V. **CO5 (L6)**
5. Prove that, for every assignment of preferences in a bipartite graph, there is a stable matching. **CO3 (L2)**
6. Let M and M' be two stable matching's in a bipartite graph with certain preferences, and let C be a component of the sub graph H formed by the edges of M∪M'. If C has at least three vertices, then it is a preference-oriented cycle. In particular, if aA, bB ∈M and aB∈M', then a prefers A to B if and only if B prefers a to b - Discuss **CO3 (L6)**

UNIT – III – Advanced Graph Theory – SMT5207

III. Independence and Covering

Content: Independent sets – Edge - colouring – Vizing's Theorem – Vertex Colouring – Uniquely Colourable graphs – Critical graphs.

Definition: An **independent set** or **stable set** of a graph G is a subset S of V such that no two vertices of S are adjacent in G.
For example, in the graph G of figure , the set S = $\{v_1,v_3,v_5\}$ is an independent set of G.

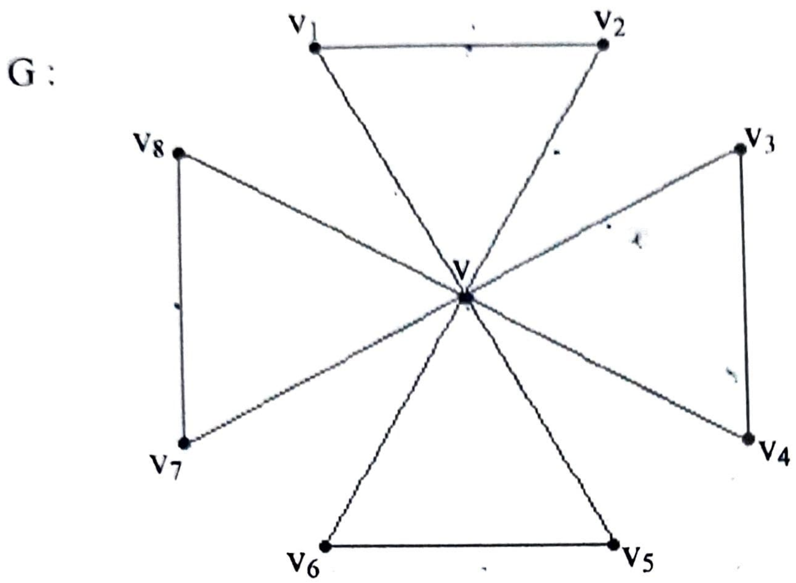


Figure . 1

Definition: An independent set S is called a **maximum independent set** of G if there is no independent set S′ of G with $|S'| > |S|$.
For example, the independent set $S_1 = \{v_1,v_3,v_5,v_7\}$ of G of figure . 1 is a maximum independent set.

Definition: The number of vertices in a maximum independent set of G is called the **independence number** or **stability number** of G and is denoted by $\alpha(G)$.
For example, the independence number of the graph G of figure . 1 is 4, since $|S_1| = 4$.

We have defined covering of a graph in the chapter 6. A **covering** of a graph G is a subset K of V such that every edge of G has at least one end in K. A covering K is called a **minimum covering** if it is a covering of least cardinality.

Definition: The number of vertices in a minimum covering of G is called the **covering number of G** and is denoted by $\beta(G)$.
For example, the covering number of the graph G of figure . 1 is 5, since $K = \{v,v_1,v_3,v_5,v_7\}$ is a minimum covering of G.

Theorem , A set $S \subseteq V$ is an independent set of G if and only if V-S is a covering of G.

Proof: By definition,
S is an independent set of G if and only if no edge of G has both ends in S.
if and only if each edge has at least one end in V-S.
if and only if V-S is a covering of G. #

Corollary In any graph G, $\alpha+\beta = p$.

Proof: Let S be a maximum independent set and K be a minimum covering of G. By theorem we get V-K is an independent set of G.. Therefore, $|V-K| \leq |S|$. This implies that $p-\beta \leq \alpha$ or $p \leq \alpha+\beta$. By theorem , since S is an independent set, V-S is a covering of G. Since K is a minimum covering, we have, $|K| \leq |V-S|$. So, $\beta \leq p-\alpha$. Thus $\alpha+\beta \leq p$. Hence the corollary.
#

Now we introduce two similar concepts with respect to edges.

Definition: An **edge independent set** or **matching** of a graph G is a subset M of E such that no two edges of M are adjacent.

Definition: The number of edges in a maximum edge-independent set of G is called the **edge independence number** of G and is denoted by $\alpha'(G)$.

For the graph G of figure . 1, $\alpha'(G) = 4$.

Definition: An **edge covering** of a graph G is a subset L of E such that each vertex of G is an end of some edge in L.

Note: A graph G has an edge covering if and only if $\delta > 0$.

Definition: The number of edges in a minimum edge covering of G is called **edge covering number** of G and is denoted by $\beta'(G)$.

For example, the edge covering number of the graph G of figure . 1 is 5, since $L = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, vv_1\}$ is a minimum edge covering of G.

Theorem In any graph G with $\delta > 0$, $\alpha'+\beta' = p$.

Proof: Consider a maximum matching M in G with $|M| = \alpha'$. Let U denotes the set of all M-unsaturated vertices in G. Since M is maximum, no two vertices of U are adjacent. Since $\delta > 0$, we can find a set E′ of |U| edges, one incident with each vertex in U. Clearly, $M \cup E'$ is an edge covering of G. This implies that,

$$\beta' \leq |M \cup E'| = |M| + |E'| = \alpha' + (p-2\alpha') = p-\alpha'.$$

Thus, $\alpha'+\beta' \leq p$.

Next, we consider a minimum edge covering L of G with $|L| = \beta'$.

Let $H = G[L]$ and let M_1 be a maximum matching in H. We denote the set of M_1-unsaturated vertices in H by U_1. Since M_1 is maximum, no two vertices of U_1 are adjacent. We can find a set F′ of $|U_1|$ edges in H one incident with each vertex in U_1 and $F' \subseteq L\text{-}M_1$. Now

$$|L|-|M_1| = |L/M_1| \geq |F'| = |U_1| = p-2|M_1|.$$

Since H is a subgraph of G, M_1 is a matching in G and $|M_1| \leq \alpha'$. Thus, $\alpha'+\beta' \geq |M_1|+|L| \geq p$. Hence, we get $\alpha'+\beta' = p$. #

Theorem In a bipartite graph with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Proof: Let G be a bipartite graph with $\delta > 0$. By corollary and theorem we have $\alpha+\beta = \alpha'+\beta'$. By Konig-Egarvary theorem, we have $\alpha' = \beta$. Hence we get $\alpha = \beta'$. #

EDGE COLOURINGS

Definition: A **k-edge colouring** of a graph G is an assignment of k colours, usually denoted by 1,2,…,k, to the edges of G (one colour per edge). Thus, a k-edge colouring of a graph G is a mapping $\pi: E(G) \to \{1,2,\ldots,k\}$.

Definition: An edge colouring is **proper** if no two adjacent edges have the same colour.

Thus, a proper k-edge colouring π of G is a mapping $\pi: E(G) \to \{1,2,\ldots,k\}$ such that $\pi(e) \neq \pi(e')$ whenever e and e′ are adjacent in G.

Remark: Clearly, if G has a k-edge colouring then the edge set E(G) has a partition $(E_1,E_2,\ldots,E_k)$ where E_i denotes all edges of E (possibly empty) which are coloured with the colour i. We note that if the colouring is proper then each E_i is a matching.

Definition: A graph G is **k-edge colourable** if G has a proper k-edge colouring.

Note:
1. Clearly, every graph G is q-edge colourable.
2. If G is k-edge colourable, then G is also k′-edge colourable for every $k' > k$.

Definition: The **edge chromatic number** $\chi_1(G)$ of a graph G is the minimum k for which G is k-edge colourable.

G is said to be **k-edge chromatic** if $\chi_1(G) = k$.

Examples:

1. Consider a path P_n ($n \geq 2$) with 2 colours. We can give a proper 2-edge colouring to P_n by alternating the 2 colours about P_n and 2 is the minimum. Hence $\chi_1(P_n) = 2$.
2. Since K_p^c has no edges, $\chi_1(K_p^c) = 0$.
3. Consider a cycle C_n. When n is even, $\chi_1(C_n) = 2$ as in the case 1. When n is odd, if we try to colour the edges of C_n with 2 colours, we must alternate the two colours about C_n. But then two adjacent edges must be assigned the same colour. So three colours are needed to give a proper edge colouring. Hence, $\chi_1(C_n) = 3$, if n is odd.
4. Consider the star $K_{1,n}$. Since any two edges are adjacent, we need at least n colours to give a proper edge colouring to the star. Hence, $\chi_1(K_{1,n}) = n$.

Observation: Since, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours, we have $\chi_1(G) \geq \Delta(G)$.

Note: We say that colour i is represented at vertex v if some edge incident with v has colour i.

Theorem . The edge chromatic number of a complete graph on n vertices is n, if n is odd ($n \neq 1$); n-1 if n is even.

Proof: If n=2 then the result is immediate. Hence we assume that $n > 2$. Let n be odd. We place the vertices of K_n in the form of a regular polygon. Colour the edges around the boundary using a different colour for each edge. Now each of the remaining 'internal edges' of G is parallel to exactly one on the boundary and we assign it the same colour as we have assigned to the edge on the boundary. Since two edges have the same colour only if they are parallel, this colouring is a proper n-edge colouring. So, $\chi_1(K_n) \leq n$.

Since the maximum number of edges with a particular colour is ½(n-1) and K_n has ½(n(n-1)) edges, we need at least $\frac{\frac{1}{2}(n(n-1))}{\frac{1}{2}(n-1)} = n$ colours for a proper edge colouring. Hence $\chi_1(K_n) = n$.

Now let us assume that n is even. For any graph G, $\chi_1(G) \geq \Delta(G)$ and so we get $\chi_1(K_n) \geq n-1$. Hence it is enough if we prove that K_n has a proper n-1 edge colouring of G. Let v be some fixed vertex of K_n. Consider K_n-{v}. This is complete with n-1 vertices. Since n-1 is odd, by the previous case, we have a proper n-1 edge colouring of K_n-{v}, as described above. With this colouring there is a colour absent at each vertex, namely the colour assigned to the edge opposite to the vertex, with different vertices having different absentees. This proper edge colouring can be extended to a proper edge colouring of K_n. Colour each edge vw where w is a vertex of K_n-{v} with the colour absent at w. This gives a proper (n-1) edge colouring of K_n and hence the theorem. #

Aliter: Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and let n be odd. For the edge joining v_i and v_j, give the colour k+1 where k = i+j (mod n). If n is even, colour K_n-v_n as above. For the edge joining v_i and v_n, give colour k+1 where k = 2i (mod n). It is easy to verify that this gives a proper edge colouring.

Theorem . Let G be a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

Proof: If G is trivial then there is nothing to prove. Hence we assume that G is non-trivial.

Case 1. G is eulerian.

If G itself is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, since G is eulerian a vertex v_0 repeats on the eulerian tour and hence G has a vertex v_0 of degree at least four. Let $v_0e_1v_1\ldots e_kv_0$ be an Euler tour of G. Now we set,

$E_1 = \{e_i / i \text{ is odd}\}$ and $E_2 = \{e_i / i \text{ is even}\}$.

Now, since each vertex of G is an internal vertex of $v_0e_1v_1\ldots e_kv_0$, the 2-edge colouring (E_1,E_2) of G satisfies the theorem.

Case 2. G is not eulerian.

Now we introduce a new vertex v_0 and join v_0 to each vertex of odd degree in G. Let the new graph be G*. Since the number of vertices of odd degree is even, the new graph G* is eulerian. Let $v_0e_1v_1\ldots e_kv_0$ be an Euler tour of G* and we define E_1 and E_2 as in the previous case. Clearly, the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of G satisfies the theorem. #

Note: Consider a k-edge colouring π of G. We denote the number of distinct colours represented at v by c(v). Clearly $c(v) \leq d(v)$ and equality holds for every vertex of G if and only if π is a proper k-edge colouring of G.

Definition: A k-edge colouring π' of G is said to be an **improvement** on π if

$$\sum_{v\in V} c'(v) > \sum_{v\in V} c(v)$$

where $c'(v)$ is the number of distinct colours represented at v in the colouring π'.

Definition: An **optimal k-edge colouring** is one, which cannot be improved.

Theorem Let $\pi = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof: Let u be a vertex of G such that colour i is not represented at u and j is represented at least twice at u. Let H be the component of $G[E_i \cup E_j]$ containing u. We prove that H is an odd cycle. If not, then by theorem 7.6, H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. Now, we recolour the edges of H with colours i and j in this way and so we get a new k-edge colouring $\pi' = (E_1', E_2', \ldots, E_k')$ of G. Let $c'(v)$ denote the number of distinct colours at v in the colouring π'. Clearly, both colours i and j are represented at u in π' and so $c'(u) = c(u) + 1$ and also $c'(v) \geq c(v)$ for $v \neq u$.

Thus, $\sum_{v\in V} c'(v) > \sum_{v\in V} c(v)$.

This is a contradiction to the fact that π is an optimal k-edge colouring. Hence H is an odd cycle. #

Theorem . If G is a bipartite graph, the $\chi_1(G) = \Delta(G)$.

Proof: Let G be a bipartite graph. We know that $\chi_1(G) \geq \Delta(G)$ for any graph. Suppose $\chi_1(G) > \Delta(G)$, let $\pi = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of G. Since π is not proper, we can find a vertex u such that $c(u) < d(u)$. That is, at u, some colour is not represented and some other colour is represented at least twice. Now, by theorem , G contains an odd cycle. This is a contradiction to the fact that G is bipartite. Hence, $\chi_1(G) = \Delta(G)$. #

Corolla $\chi_1(K_{m,n}) = \max\{m, n\}$.

APPLICATION

Now we present a simple but interesting application of edge colourings.

Latin squares are used frequently by statisticians and quality control analysts in experimental design. Here, we consider construction of Latin squares.

A Latin square is an n×n matrix having the numbers 1,2,…,n as their entries such that no single number appears more than once in any row or any column.

Here, we show that the construction of a Latin square of order n using an n-edge colouring of the complete bipartite graph $K_{n,n}$. By theorem 7.8, $K_{n,n}$ has a proper n-edge colouring but no proper edge colouring with less than n colours. Let the bipartition of $K_{n,n}$ be (X,Y) where X = $\{v_1,v_2,\ldots,v_n\}$ and Y = $\{u_1,u_2,\ldots,u_n\}$ and denote the colours of the proper n-edge colouring by 1,2,...,n. Now we define the matrix $A = (a_{ij})$ by

$a_{ij} = k$ if the edge v_iu_j is coloured with k.

Since the edges incident with the same vertex v_i have different colours, all the elements of the i^{th} row are different. Similarly for columns.

Note: Conversely any n×n Latin square can be used to give a proper edge colouring of $K_{n,n}$.

VIZING'S THEOREM

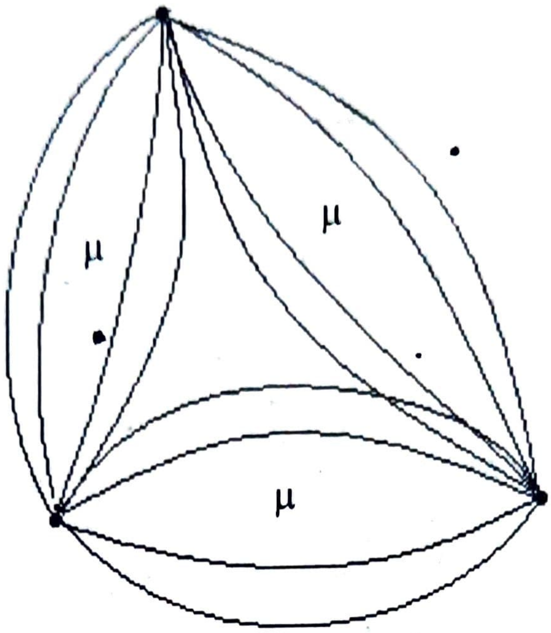
Now we find bounds for the edge chromatic number of a graph.

The maximum number of edges joining two vertices in G is called the **multiplicity** of G, and denoted by $\mu(G)$.

Theorem For any graph G, $\Delta \leq \chi_1 \leq \Delta+\mu$.

This theorem is best possible in the sense that for any μ, there exists a graph G such that $\chi_1 = \Delta+\mu$. For example, consider the graph G of figure

G:



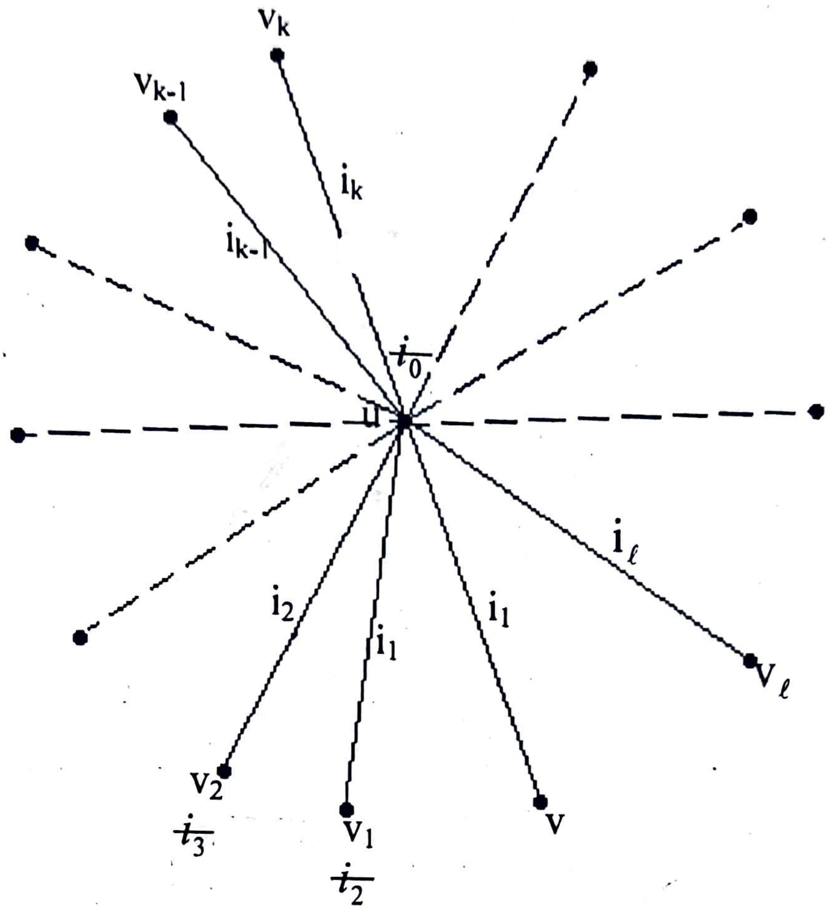
Here $\Delta = 2\mu$ and $\chi_1 = q(G) = 3\mu$ since any two edges are adjacent. Hence, $\chi_1 = \Delta+\mu$.

Now we prove this result for simple graphs and the proof is due to Fournier (1973).

Theorem If G is a simple graph, then $\Delta \le \chi_1 \le \Delta+1$.

That is, $\chi_1 = \Delta$ or $\chi_1 = \Delta+1$.

Proof: We know that, for any graph G, $\chi_1 \ge \Delta$. So, it is enough if we prove that $\chi_1 \le \Delta+1$. Suppose $\chi_1 > \Delta+1$. Let $\pi = (E_1,E_2,\ldots,E_{\Delta+1})$ be an optimal $(\Delta+1)$-edge colouring of G. Since $\chi_1 > \Delta+1$ and π is an $(\Delta+1)$-edge colouring, π is not proper and we have a vertex u such that $c(u) < d(u)$. So there exists colours i_0 and i_1 such that i_0 is not represented at u and i_1 is represented at least twice at u. Let the edges uv and uv_1 have colour i_1. Since $d(v_1) < \Delta+1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u since otherwise, recolouring uv_1 with i_2 we would obtain an improvement on π. Thus some edge uv_2 has colour i_2.



Again, since $d(v_2) < \Delta+1$, some colour i_3 is not represented at v_2 and i_3 must be represented at u since otherwise by recolouring uv_1 with i_2 ane uv_2 with i_3, we would obtain an improved (Δ+1) edge colouring. Thus some edge uv_3 has colour i_3.

Continuing this process, we construct a sequence $v_1,v_2,v_3,\ldots$ of vertices and a sequence $i_1,i_2,i_3,\ldots$ of colours such that

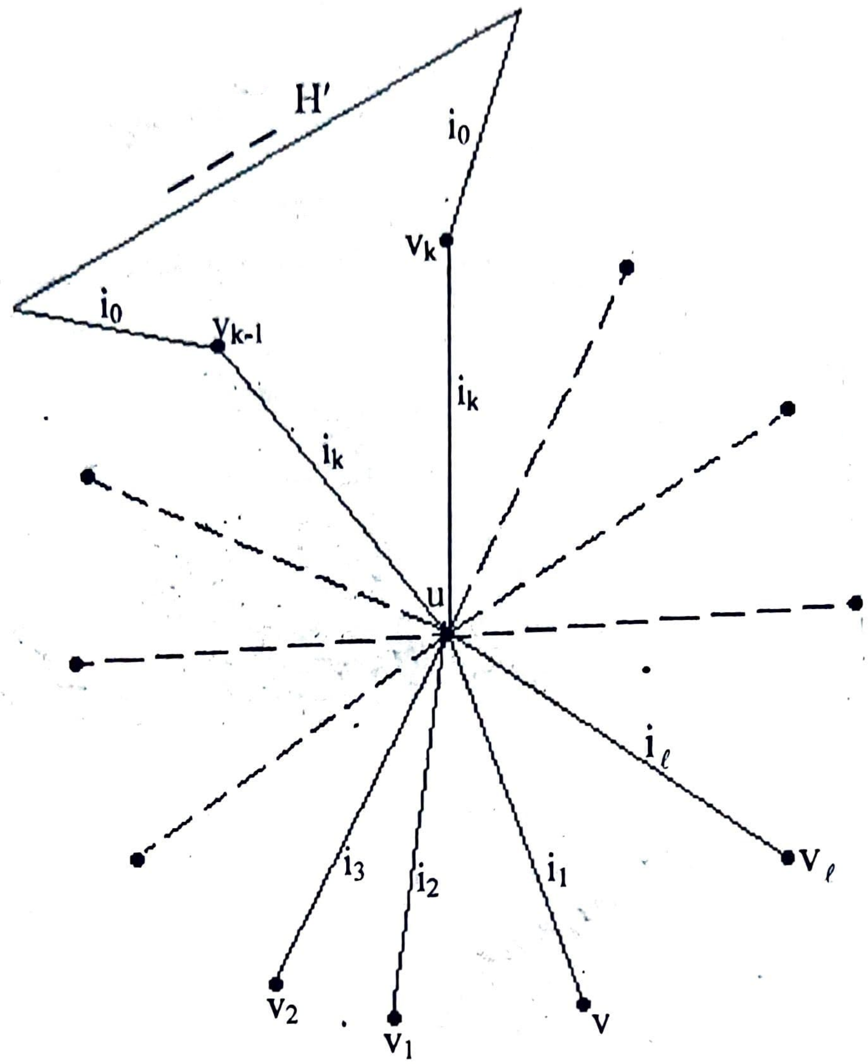
(i) uv_j has colour i_j and

(ii) i_{j+1} is not represented at v_j.

Since the degree of u is finite, there exists a smallest positive integer ℓ such that for some $k < \ell$

(iii) $i_{\ell+1} = i_k$.

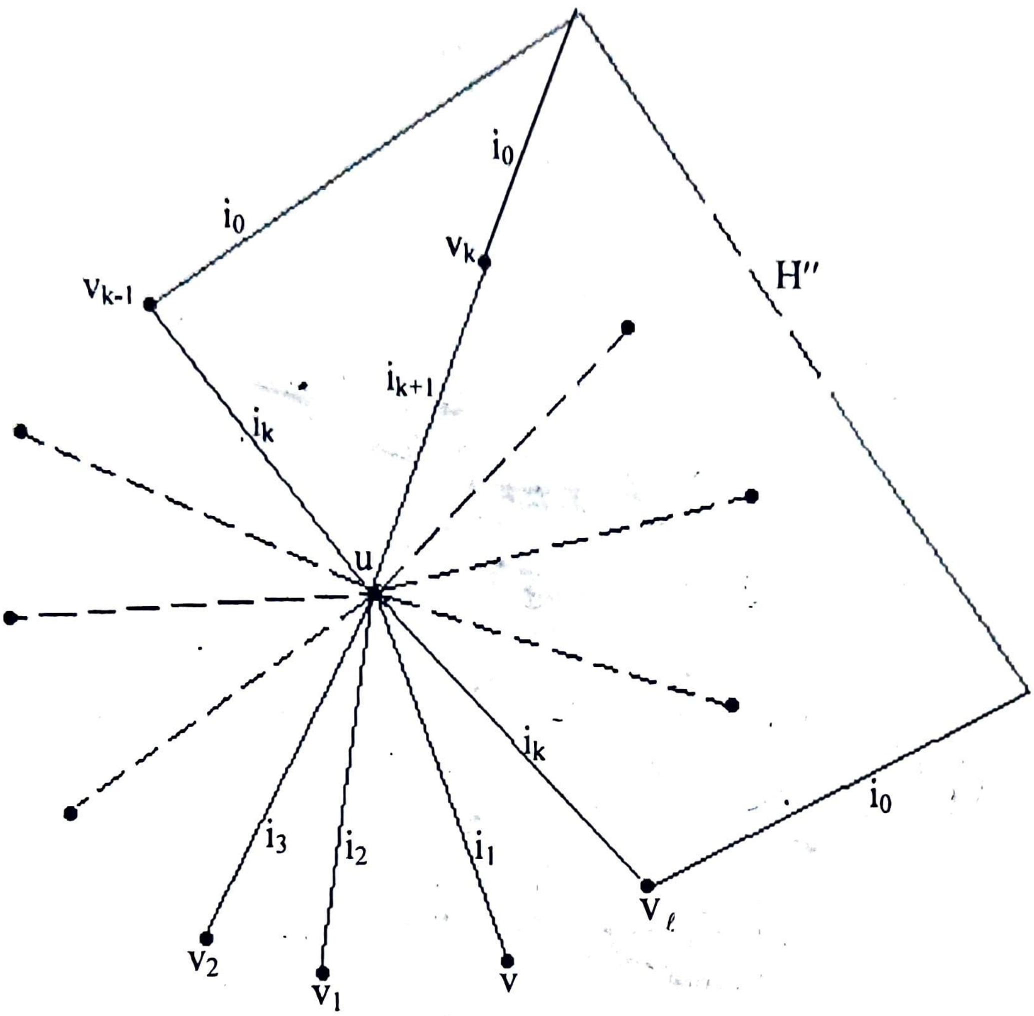
We now recolour edges of G as follows. For $1 \leq j \leq k-1$, recolour uv_j with colour i_{j+1} and leaving other edge colours unchanged we get a new (Δ+1)-edge colouring $\pi' = (E_1', E_2', \ldots, E'_{\Delta+1})$.



We note that $c'(v) \geq c(v)$ for all $v \in V$ and so π' is also an optimal $(\Delta+1)$-edge colouring of G. By theorem the component H' of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u is an odd cycle, since i_0 is not represented and i_k is represented twice at u.

Now, in addition, for $k \leq j \leq \ell$, recolour uv_j with colour i_{j+1} and this gives a new $(\Delta+1)$-edge colouring $\pi'' = (E''_1, E''_2, ..., E''_{\Delta+1})$.

Clearly, $c''(v) \geq c(v)$ for all $v \in V$ and the component H'' of $G[E''_{i_0} \cup E''_{i_k}]$ that contains u is an odd cycle.



Both H′ and H″ contain the vertex v_k. Since v_k has degree two in H′, the degree of v_k in H″ is one. This is a contradiction since H″ is an odd cycle. Hence the theorem. #

Solved Problems

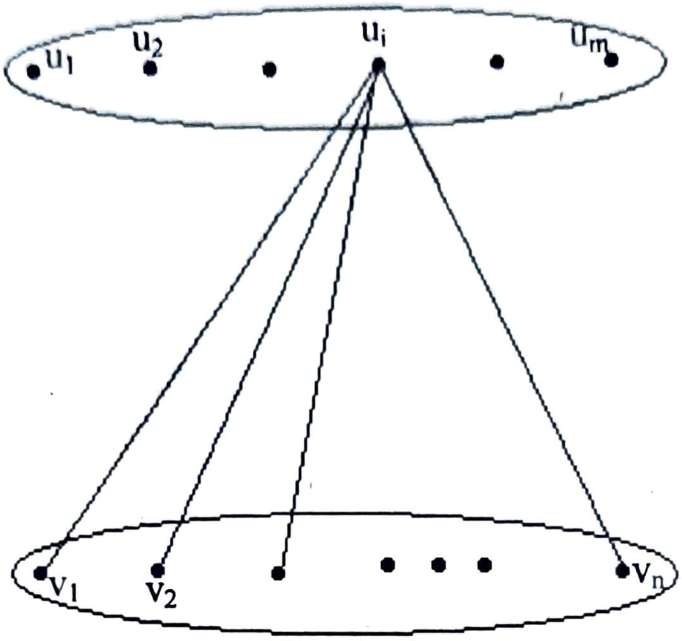
1. Show that if G is bipartite with $\delta > 0$, then G has a δ-edge colouring such that all δ colours are represented at each vertex.

Solution: Consider an optimal δ-edge colouring of the graph G. If there is some vertex u such that all the δ colours are not represented then some colour is represented at least twice at u.

Let the colour i be not represented at u and the colour j be represented twice at u. Then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle. This is a contradiction to the fact that G is bipartite. Hence the solution.

2. Show by finding an appropriate edge colouring, that $\chi_1(K_{m,n}) = \Delta(K_{m,n})$.

Solution: Consider a complete bipartite graph G with bipartition (X,Y) where X contains $u_1, u_2, \ldots, u_m$ and Y contains $v_1, v_2, \ldots, v_n$.



Figure

Let max {n, m} = n. Consider the edge u_iv_j. Divide i+j by n and let r be the remainder. Colour the edge u_iv_j by the colour r+1. This colouring uses only n colours and for any particular i, the n numbers i+1, i+2, ..., i+n gives different remainder when divided by n. It is easy to see that this is a proper n-edge colouring and so $\chi_1(K_{m,n}) \leq n$.

Since for any graph G, $\chi_1(G) \geq \Delta = n$, we conclude $\chi_1(K_{m,n}) = n = \Delta(K_{m,n})$.

VERTEX COLOURINGS

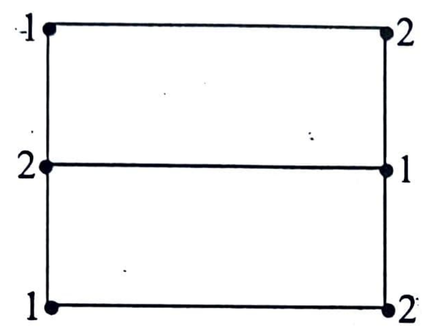
Instead of colouring edges, here, we colour vertices.

Definition: A **k-vertex colouring** of a graph G is an assignment of k colours, usually denoted by 1,2,...,k, to the vertices of G. Thus, a k-vertex colouring of a graph G is a mapping $\pi: V(G) \to \{1,2,\ldots,k\}$.

Definition: A vertex colouring is **proper** if no two distinct adjacent vertices have the same colour.
Thus, a proper k-vertex colouring π of G is a mapping $\pi: V(G) \to \{1,2,\ldots,k\}$ such that $\pi(v_1) \neq \pi(v_2)$ whenever v_1 and v_2 are adjacent in G.

Remark: Clearly, if G has a proper k-vertex colouring then the vertex set V(G) has a partition $(V_1, V_2, \ldots, V_k)$, where V_i denotes all vertices of V (possibly empty) which are coloured with the colour i; each V_i is an independent set.

Example: Consider the graph G of figure A proper 2-vertex colouring is illustrated in the figure



Figure

Definition: A graph G is **k-vertex colourable** if G has a proper k-vertex colouring.

Notation: It is customary to abbreviate a proper vertex colouring as a colouring, a proper k-vertex colouring as a k-colouring and k-vertex colourable as k-colourable.

Definition: The **chromatic number** $\chi(G)$, of a graph G, is the minimum k for which G is k-colourable.

G is said to be **k-chromatic** if $\chi(G) = k$.

Remark: Since presence of multiple edges **do not** change the chromatic number, when we consider vertex colourings, we consider only simple graphs. **Therefore, in the rest of this chapter, graph means simple graph.**

Examples:

1. Consider a path P_n $(n \geq 2)$. We can give a 2-colouring to the vertices of P_n by alternating the 2 colours about P_n and 2 is the minimum. Hence $\chi(P_n) = 2$.
2. Since K_p^c has no edges, $\chi(K_p^c) = 1$.
3. Consider a cycle C_n. When n is even, $\chi(C_n) = 2$ as in the example 1. When n is odd, if we try to colour the vertices of C_n with 2 colours, we must alternate the two colours about C_n. But then two adjacent

vertices must be assigned the same colour. So, three colours are needed to give a colouring. Hence $\chi(C_n) = 3$, if n is odd.

4. Consider the star $K_{1,n}$. Colouring all the end vertices with one colour and the other vertex by another colour, we have $\chi(K_{1,n}) = 2$.

The following results are easy to prove.

Results

1. If G is a (p,q) graph, then $\chi(G) \leq p$.
2. If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.
3. $\chi(K_p) = p$.
4. If K_p is a subgraph of G, then $\chi(G) \geq p$.
5. If $G_1, G_2, ..., G_k$ are the components of G, then $\chi(G) = \underset{1 \leq i \leq k}{\text{Max}} \chi(G_i)$.

Theorem A non-empty graph G is 2-colourable if and only if G is bipartite.

Proof: Let G be 2-colourable. Let X denotes the set of all vertices of colour 1 and Y denote the set of all vertices of colour 2. Since no two adjacent vertices can have the same colour, there is no edge between any two vertices of X and no edge between any two vertices of Y. So, (X,Y) is a bipartition of G. Hence G is bipartite.

Conversely, let G be a bipartite graph with bipartition (X,Y). Now we assign colour 1 to the vertices of X and colour 2 to the vertices of Y. Since G is non-empty, $\chi(G) = 2$. Hence, G is 2-colourable. #

UNIQUELY COLOURABLE GRAPHS

Now, we present a few results on uniquely colourable graphs. Harary, Hedetniemi and Robinson (1969) proved many results on the construction of uniquely colourable graphs. Chartrand and Geller (1969) and Aksionov (1977) obtained good results on uniquely colourable planar graphs.

Definition: A graph G is said to be **uniquely k-colourable** if all k-colourings of G, with no colour class empty, induce the same partition of V.

Definition: Two k-colourings of a graph G are different if they induce different partitions. That is, two colourings $(V_1, V_2, \ldots, V_k)$ and $(V_1', V_2', \ldots, V_k')$ are different if $\{V_1, V_2, \ldots, V_k\} \neq \{V_1', V_2', \ldots, V_k'\}$. Each V_i is called a colour class.

Examples

1. The only uniquely 1-colourable graphs are empty graphs.
2. The only uniquely 2-colourable graphs are connected bipartite graphs.
3. Clearly, K_3 is uniquely 3-colourable and K_n is uniquely n-colourable.

Theorem If G is uniquely k-colourable then $\delta(G) \geq k-1$.

Proof: Consider a vertex v of G, which is uniquely k-colourable. In any k-colouring, v must be adjacent with at least one vertex of every colour different from that assigned to v. Otherwise, by recolouring v with a colour which is not represented at any adjacent vertex of v, we get a different k-colouring and hence a different partition of V. Therefore, $d(v) \geq k-1$ and hence $\delta(G) \geq k-1$. #

Theorem If G is uniquely k-colourable then the subgraph induced by the union of any two colour classes of a k-colouring of G is connected.

Proof: Let G be a uniquely k-colourable graph and let C_1, C_2 be two colour classes in a k-colouring of G. Let $C_{1,2}$ be the subgraph induced by $C_1 \cup C_2$. Suppose $C_{1,2}$ is not connected, let H be a component of $C_1 \cup C_2$. Clearly, no vertex of H is adjacent to a vertex in V(G)/V(H) that is coloured with 1 or 2. Now, interchanging the colours of the vertices of H and retaining the original colours for all other vertices, we get a different k-colouring of G. This is not possible, since G is uniquely k-colourable. Hence $C_{1,2}$ is connected. #

Theorem If G is uniquely k-colourable then G is (k-1)-connected.

Proof: If G is a complete graph on k vertices then it is (k-1)-connected.

Suppose G is an incomplete uniquely k-colourable graph which is not (k-1)-connected. Let S be a vertex cut of G with at most (k-2) vertices. Then, at least two colours of any k-colouring of G will not be present in S. Let the colours be 1 and 2. Since the subgraph induced by the colours 1 and 2 is connected, it is contained in some component of G – S. Now, if we recolour any vertex in some other component of G – S with colour 1 or 2, we get a different k-colouring of G. This is a contradiction, since G is uniquely k-colourable. Hence the theorem. #

7.7 CRITICAL GRAPHS

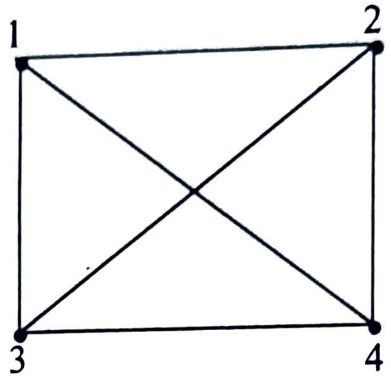
Critical graphs play a vital role in the study of colourings. Dirac (1952) was the first person to make an extensive study of critical graphs. A survey by Toft , is an interesting and useful survey which contains many results on critical graphs.

Definition: A graph G is **critical** if $\chi(H) < \chi(G)$ for every proper subgraph H of G.

Definition: A graph which is critical and k-chromatic is called a **k-critical graph.**

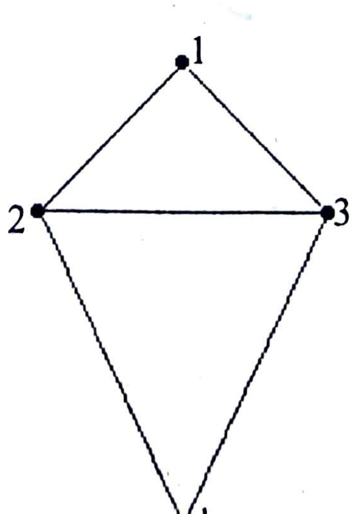
Example

Consider the complete graph on 4 vertices, K_4. We show that K_4 is 4-critical. We know that $\chi(K_4) = 4$.



If we remove 1, 2 or 3 vertices then we get K_3, K_2, K_1 respectively. In these cases, the chromatic number is 3,2,1 respectively. Therefore, the chromatic number decreases strictly.

If we remove any single edge from K_4, the resulting graph is two triangles with a common edge, which is 3 chromatic. In this case also, the chromatic number decreases strictly. All other cases can be disposed similarly.



Hence, K_4 is 4-critical.

Results

1. Every k-chromatic graph has a k-critical subgraph.

Proof: Let G be a k-chromatic graph. If G is critical, then G is the required subgraph. Otherwise, G has a subgraph H such that $\chi(H) = \chi(G)$. If H is critical, then H is the required subgraph. Otherwise, we repeat this process and we get a k-critical subgraph.

2. Every critical graph is connected.

Proof: Consider a critical graph G and let $\chi(G) = k$. Suppose G is not connected, let $G_1, G_2, \ldots, G_n$ be its components.

$$\text{Let } \alpha = \underset{1 \le i \le n}{\text{Max}}\, \chi(G_i).$$

By 7.12(5), $k = \alpha$. But since G is critical, $\alpha < k$, which is a contradiction. Hence G is connected.

Theorem If G is k-critical, then $\delta \ge k-1$.

Proof: Suppose G is a k-critical graph with $\delta < k-1$. Let v be a vertex of degree δ in G. Since G is k-critical, G-v is (k-1)-colourable. Since degree of v is δ, we can find a colour in this (k-1)-colouring which is not represented at any adjacent vertex of v. Now we can assign this colour to v. Hence G is (k-1)-colourable. This is a contradiction to the fact that G is k-chromatic. Hence $\delta \ge k-1$. #

Corollary Every k-chromatic graph has at least k vertices of degree at least k-1.

Proof: Let G be a k-chromatic graph. By result , it has a k-critical subgraph, say H. By theorem , each vertex of H has degree at least k-1 in H and hence also in G. Since H is k-chromatic, it has at least k-vertices. Hence the result. #

Corollary For any graph G, $\chi(G) \le \Delta(G)+1$.

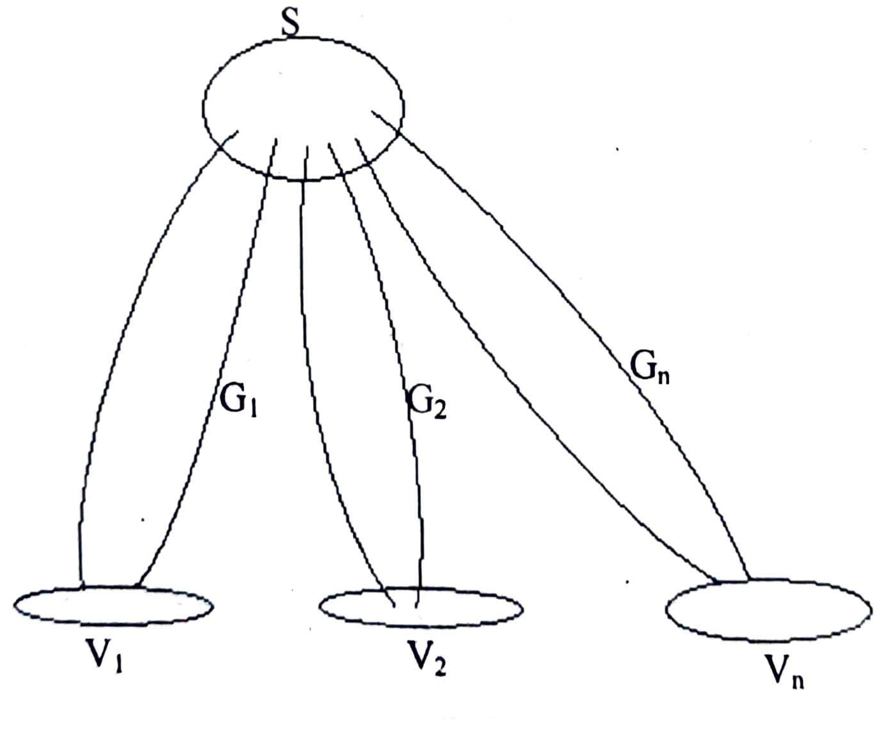
Proof: Let G be k-chromatic. By corollary , G has at least k vertices of degree at least k-1. Therefore, $\Delta(G) \ge k-1$. So, $k \le \Delta(G)+1$. Hence $\chi(G) \le \Delta(G)+1$.

Definition: Let S be a vertex cut of a connected graph G. Let the components of G–S have vertex sets $V_1, V_2, \ldots, V_n$. The subgraphs $G_i = G[V_i \cup S]$ are called the **S-components** of G.

Definition: The colourings of the S-components $G_1, G_2, \ldots, G_n$ of G are said to agree on S if, for every $v \in S$, each colouring assigns the same colour to v.

Theorem ′ In a critical graph, no vertex cut is a clique.

Proof: Let G be a k-critical graph. We prove that no vertex cut of G is a clique. Suppose not, let us assume that G has a vertex cut S that is a clique. Let $G_1, G_2, \ldots, G_n$ be the S-components of G. Since G is k-critical, each G_i is (k-1)-colourable. Furthermore, because S is a clique, the vertices in S must receive distinct colours in any (k-1)-colouring of G_i. It follows that there are



(k-1)-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. This gives a (k-1)-colouring of G. This is a contradiction to the fact that G is k-critical. Hence the theorem.

#

Corollary Every critical graph is a block.

Proof: This is immediate by theorem ′

Note: If G is k-critical with a 2-vertex cut {u,v} then u and v cannot be adjacent.

QUESTION BANK

PART A

1. Prove that In any graph G, $\alpha + \beta = p$ **CO2 (L2)**
2. Prove that If G is a bipartite graph, then $\chi_1(G) = \Delta(G)$. **CO2 (L2)**
3. Show that if G is bipartite with $\delta > 0$, then G has a δ - edge coloring such that all δ colors are represented at each vertex. **CO2 (L2)**
4. Show by finding an appropriate edge coloring, that $\chi_1(K_{m,n}) = \Delta(K_{m,n})$ **CO2 (L1)**
5. Prove that A non-empty graph G is 2 – colorable if and only if G is bipartite. **CO2 (L2)**
6. Prove that, every critical graph is connected. **CO2 (L2)**

7 .If G is k – critical then show that $\delta(G) \geq k-1$. **CO2 (L1)**

8. a) Define k- critical graph. **CO1 (L1)**

b) show that In a critical graph, no vertex cut is a clique. **CO2 (L2)**

PART B

1. In any graph G with $\delta>0$, prove that $\alpha' + \beta' = p$ **CO5 (L5)**

2. Examine that, The edge chromatic number of a complete graph on n vertices is n, if n is odd ($n\neq1$): n-1 if n is even. **CO5 (L4)**

3. Let G be a connected graph that is not an odd cycle. Then prove G has a 2 – edge coloring in which both colors are represented at each vertex of degree at least two.

CO4 (L5)

4. If G is a simple graph, then $\Delta \leq \chi_1 \leq \Delta + 1$ **CO4 (L2)**

5. a) If G is uniquely K – colorable then. $\delta(G) \geq k-1$. **CO4 (L2)**

b) If G is uniquely k – colorable then G is (k-1) connected. **CO3 (L2)**

6. In a bipartite graph with $\delta > 0$, prove that $\alpha = \beta'$ **CO4 (L5)**

UNIT – IV – Advanced Graph Theory – SMT5207

IV. LABELINGS

Content: Predecessor and Successor – Algorithm – Graceful Labeling – Sequential functions - Magic graphs – Conservative graphs.

Labeling of a graph is an assignment of labels (numbers) to its vertices or/and edges or faces, which satisfy some conditions. These are different from colouring problems since some properties and structures of numbers such as ordering, addition and subtraction are used here which are not properties of colours.

In 13th century, Chinese mathematician Yang Hui and others (1275) have studied labeling of geometric figures which are later called plane graphs. Later Chang Chhao (1670), Pao Chhi-Shou (1880), Li Nien also contributed to this area. But in 1983, the notions of magic and consecutive labelings of plane graphs were defined by Ko-Wei Lih.

In 1963, Ringel conjectured that if T is any tree with n edges, then the complete graph K_{2n+1} can be decomposed into 2n+1 subgraphs isomorphic to T.

S.W.Golomb was instrumental in coining the phraseology "Graceful Graphs" and it was popularised by the articles of S.W.Golomb and M. Gardener. Interest in this field was aroused by the conjecture of Ringel and an article of Rosa. The same concept has been used in additive number theory but under the name of restricted difference basis. Graceful graphs have several applications in coding theory, X-ray crystallography, radar, communication networks and astronomy. The conjecture, "All trees are graceful" by Ringel-Kotzig-Rosa is well-known and still open.

PREDECESSOR AND SUCCESSOR

Consider the sequence of integers $\pi : (a_1, a_2, \ldots, a_n)$, $a_i \geq 0$, not all zero. Let $b_1 = |a_1 - a_2|$, $b_2 = |a_2 - a_3|$, $b_3 = |a_3 - a_4|$, …, $b_n = |a_n - a_1|$. Let π' be $(b_1, b_2, \ldots, b_n)$. π is called a **predecessor** and π' is called its **successor.**

Clearly, every predecessor has a unique successor. But a sequence of non-negative integers, not all zero (successor), need not have a predecessor. For example, (1, 2, 5, 9) has no predecessor.

Result If $(b_1, b_2, \ldots, b_n)$ is the successor of a predecessor $(a_1, a_2, \ldots, a_n)$, then $\sum_{i=1}^{n} b_i$ is even.

Proof: Let $b_1' = a_1 - a_2$, $b_2' = a_2 - a_3$, $b_3' = a_3 - a_4$, …, $b_n' = a_n - a_1$. We know that $|b_i'| = b_i$.

Obviously, $b_1' + b_2' + \ldots + b_n' = 0$. So all b_i's cannot be positive. Some of them must be negative such that the sum of the positive terms is equal to sum of the negative terms with different sign.

Let $b_1', b_2', \ldots, b_k'$ be the positive terms and $b_{k+1}', b_{k+2}', \ldots, b_n'$ be the negative terms. Therefore, sum of $b_1, b_2, \ldots, b_k$ is equal to the sum of $b_{k+1}, b_{k+2}, \ldots, b_n$. That is, $b_1 + b_2 + \ldots + b_k + b_{k+1} + \ldots + b_n = 2(b_1, + b_2 + \ldots + b_k)$. Hence, $b_1, + b_2 + \ldots + b_n$ is even. #

Remark: From above, it is clear that if $(b_1, b_2, \ldots, b_n)$ is the successor of a predecessor $(a_1, a_2, \ldots, a_n)$ then sum of some b_is is equal to the sum of the other b_is. Now, we prove that this necessary condition is also sufficient for a successor (sequence of non-negative integers, not all zero) to have a predecessor.

Theorem A sequence of non-negative integers $(b_1, b_2, \ldots, b_n)$, not all zero, has a predecessor if and only if the sum of some b_is is equal to the sum of other b_is.

Proof: Let the sequence of non-negative integers $(b_1, b_2, \ldots, b_n)$, not all zero, has a predecessor. Then by the above remark, the sum of some b_is is equal to the sum of other b_is is clear.

Conversely, let us assume that the sum of some b_is is equal to the sum of other b_is for the sequence of non-negative integers $(b_1, b_2, \ldots, b_n)$, not all zero. We show that this has a predecessor. Suppose $\pi : (a_1, a_2, \ldots, a_n)$ be its predecessor, then we find a_is in terms of b_is. Clearly,

$b_1 = |a_1-a_2|,\ b_2 = |a_2-a_3|,\ b_3 = |a_3-a_4|, \ldots, b_n = |a_n-a_1|.$

So, $b_1 = \pm(a_1-a_2),\ b_2 = \pm(a_2-a_3),\ b_3 = \pm(a_3-a_4), \ldots, b_n = \pm(a_n-a_1).$

That is, $a_1-a_2 = \pm b_1,\ a_2-a_3 = \pm b_2,\ a_3-a_4 = \pm b_3, \ldots, a_n-a_1 = \pm b_n.$

So we get,

$$a_1 = a_1 \pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_n.$$
$$a_2 = a_1 \pm b_2 \pm b_3 \pm \ldots \pm b_n.$$
$$a_3 = a_1 \pm b_3 \pm b_4 \pm \ldots \pm b_n.$$
$$\vdots \qquad \vdots$$
$$a_{n-1} = a_1 \pm b_{n-1} \pm b_n \text{ and}$$
$$a_n = a_1 \pm b_n.$$

From the above, we have, $\pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_n = 0$. By our assumption, the sum of the some b_is is equal to the sum of other b_is. Now let us partition the set $\{b_1, b_2, \ldots, b_n\}$ into two sets namely, A and B such that the sum of A is equal to the sum of B. With out loss of generality, we can assume that b_i takes positive sign if $b_i \in A$ or negative sign if it belongs to B. Now the above set of equations becomes,

$a_1 = a_1+0,\ a_2 = a_1 \pm b_1,\ a_3 = a_1 \pm b_1 \pm b_2,\ a_4 = a_1 \pm b_1 \pm b_2 \pm b_3, \ldots,$
$a_n = a_1 \pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_{n-1}.$

At present, we assume that $a_1=0$. Hence, a_is are known in terms of b_is. Here, some a_is may be negative. To get all a_is to be non-negative, choose a_1 to be the absolute value of the most negative term among $\pm b_1$, $\pm b_1 \pm b_2$, $\pm b_1 \pm b_2 \pm b_3$, ..., $\pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_{n-1}$. Therefore, all a_is are non-negative. Hence, $(b_1, b_2, \ldots, b_n)$ has a predecessor. #

Now we present a necessary and sufficient condition on n for a sequence (1,2,3, …,n) to have a predecessor.

Theorem The sequence (1,2,3, …,n) has a predecessor if and only if $n \equiv 0,3 \pmod 4$.

Proof: First we prove that if $n \equiv 0,3 \pmod 4$, then the sequence (1,2, …,n) has a predecessor.

Case 1: Let $n \equiv 0 \pmod 4$.

It is enough if we prove that the sum of some terms of the sequence (1,2,3, …,n) is equal to the sum of the remaining terms. Let $k = n/4$ and $k_1 = n/2$. Now, the sum of the terms starting with k+1 and ending with $k+k_1$ is,

$$= (k+k_1)(k+k_1+1)/2 - k(k+1)/2$$
$$= \tfrac{1}{2}\,[k^2+kk_1+k+kk_1+k_1^2+k_1-k^2-k]$$
$$= \tfrac{1}{2}\,[2kk_1+k_1+k_1^2]$$

$= \frac{1}{2} [2(n/4)(n/2)+n/2+n^2/4]$
$= \frac{1}{2} [(n^2+n)/2]$
$= n(n+1)/4.$

The sum of the remaining terms of the sequence $= n(n+1)/2 - n(n+1)/4 = n(n+1)/4$. Hence the sequence (1,2,3, ...,n) has a predecessor.

Case 2: Let $n \equiv 3 \pmod 4$. Now taking $k = (n-3)/4$ and $k_1 = (n+1)/2$, we can prove that, as in the previous case, the sum of the terms starting with k+1 and ending with $k+k_1$ is equal to the sum of the remaining terms of the sequence. Hence the sequence (1,2,3, ...,n) has a predecessor.

Conversely, let us assume that the sequence (1,2,3, ...,n) has a predecessor. We know that by result 10.1, $n(n+1)/2 = 2k$, for some k. That is, $n(n+1) = 4k$. Hence we conclude that $n \equiv 0,3 \pmod 4$. #

ALGORITHM

Given a successor $(b_1,b_2, \ldots,b_n)$ such that the sum of some b_is is equal to the sum of other b_is, this algorithm finds its predecessor $(a_1,a_2, \ldots,a_n)$.

Step 1. Partition the set $\{b_1,b_2, \ldots,b_n\}$ into 2 subsets namely, A and B such that the sum of elements of A is equal to the sum of elements of B.

Step 2. Assume that b_i takes positive sign if b_i belongs to A or negative sign if it belongs to B.

Step 3. Assume $a_1 = 0$.

Step 4. Find $a_1, a_2, \ldots, a_n$ using the following equations.

$a_1 = a_1+0$
$a_2 = a_1 \pm b_1$
$a_3 = a_1 \pm b_1 \pm b_2$
$a_4 = a_1 \pm b_1 \pm b_2 \pm b_3$
$\vdots \qquad \vdots$
$a_n = a_1 \pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_{n-1}.$

Step 5. If all a_is are non-negative then stop.
Otherwise, choose a_1 to be the absolute value of the most negative term among $\pm b_1$, $\pm b_1 \pm b_2$, $\pm b_1 \pm b_2 \pm b_3$, ..., $\pm b_1 \pm b_2 \pm b_3 \pm \ldots \pm b_{n-1}$.

Step 6. Repeat step 4 and stop.

Illustration

Consider the successor $(4,2,7,9,3,3) = (b_1,b_2, \ldots,b_6)$.

Iteration 1.

Step 1. Let $A = \{b_1 = 4, b_3 = 7, b_6 = 3\}$ and $B = \{b_2 = 2, b_4 = 9, b_5 = 3\}$.

Step 2. We assume that b_1, b_3 and b_6 take positive sign and b_2, b_4 and b_5 take negative sign.

Step 3. Let $a_1 = 0$.

Step 4. $a_1 = 0+0 = 0$.
$a_2 = 0+4 = 4$.
$a_3 = 0+4-2 = 2$.
$a_4 = 0+4-2+7 = 9$.
$a_5 = 0+4-2+7-9 = 0$
$a_6 = 0+4-2+7-9-3 = -3$.

Step 5. We have $a_6 = -3$ and choose $a_1 = |-3| = 3$.

Iteration 2.

Step 4. $a_1 = 3$, $a_2 = 7$, $a_3 = 5$, $a_4 = 12$, $a_5 = 3$ and $a_6 = 0$.

Hence (3,7,5,12,3,0) is a predecessor of the given successor.

GRACEFUL GRAPHS

Definition: Let G be a simple graph with order p and size q. A function $f: V(G) \to \{0,1, \ldots, q\}$ is called a **graceful labeling** (numbering) if

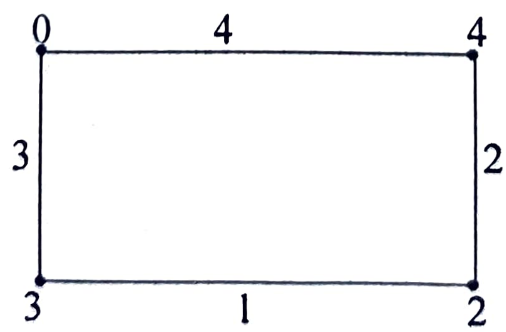
(i) f is one-to-one.

(ii) the edges receive all the labels (numbers) from 1 to q, where the label of an edge is computed to be the absolute value of the difference between the vertex labels at its ends (that is, if $e = (x,y)$ then the label of e is $|f(x)-f(y)|$).

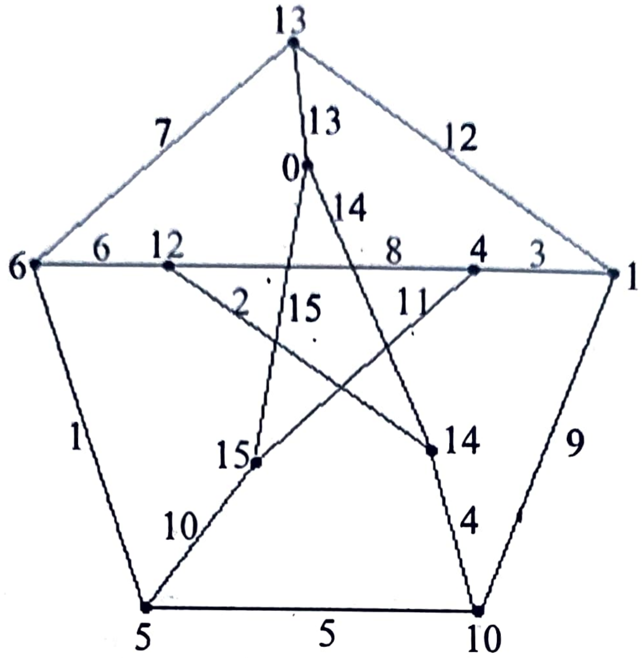
Definition: A graph that has a graceful labeling is called a **graceful graph.**

Examples:

1. The graph C_4 is a graceful graph since it has a graceful labeling as shown in figure .1.



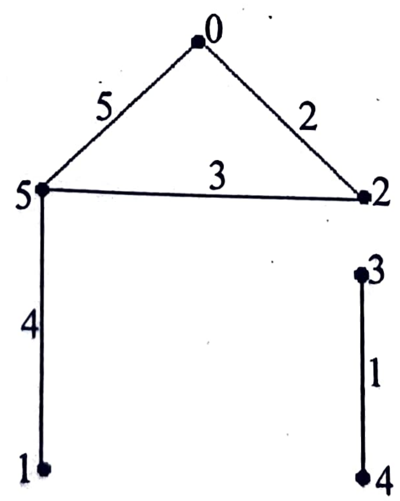
2. Petersen graph is graceful and its graceful labeling is shown in figure 2.



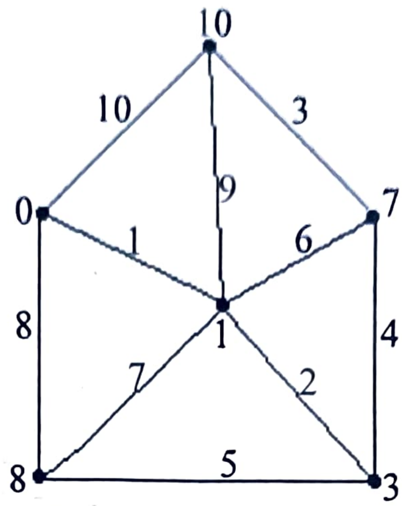
3. The graph C_5 is not graceful

Remarks:

1. A graceful graph may be a disconnected graph. The graph G of figure is a disconnected graceful graph.

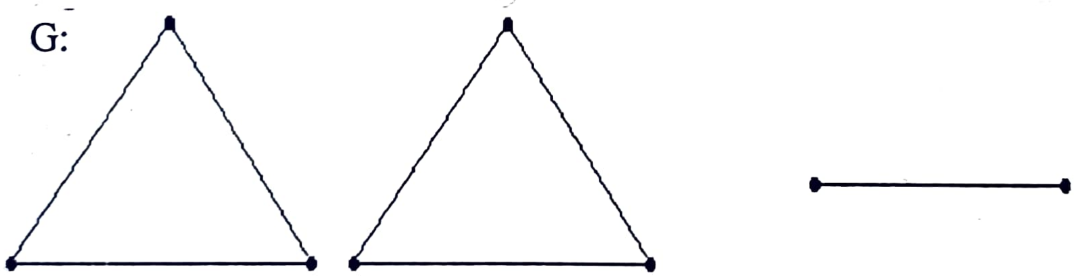


2. Subgraph of a graceful graph need not be graceful. Consider the wheel W_6, which is graceful but the subgraph C_5 is not graceful.

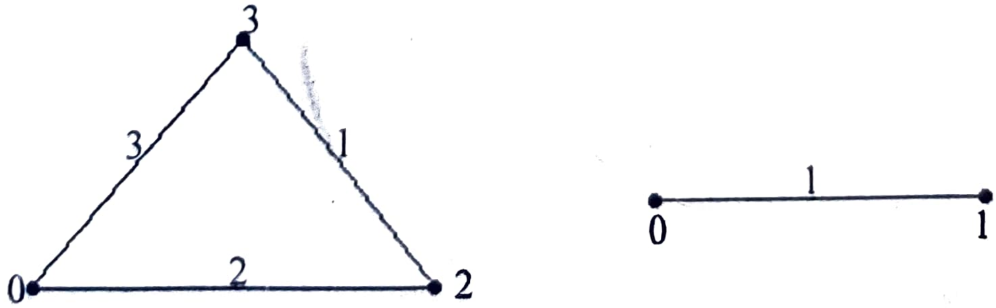


3. If G is a disconnected graph with k-components such that each component is graceful, even then G need not be graceful. Now, consider the graph G as shown in figure .5.

G:



Here, G has 3-components and a graceful labeling of each component is given below.



But, G is not graceful. If G is graceful, then it is possible to label the vertices of G with different labels from {0,1, ...,7} so that the labels from 1 to 7 are realized by the edges. Since the label 7 should be realized by an edge, the vertex labels 0 and 7 should be given to adjacent vertices say v_1 and v_2, and since the label 6 should be realized by an edge, the vertex adjacent to v_1 and v_2 should get either 6 or 1. In either case, the label 5 cannot be realized by any edge of G. Hence, G is not graceful.

4. If G is graceful, then it is connected or contains a cycle.
Proof: Let G be a graceful graph of order p and size q. Since G is graceful, it is possible to label the vertices of G with different labels from {0,1,2, ...,q} in such a way that the edges receive all the labels from 1 to q. Thus, $q+1 \geq p$. That is, $q \geq p-1$. This implies that G is connected or contains a cycle. #

Theorem If G is graceful eulerian graph with size q, then $q \equiv 0, 3 \pmod 4$.

Proof: Since G is Eulerian, by theorem and by result , the sum of the edge labels is an even number. Since G is graceful this number is equal to the sum 1+2+...+q. Thus,

$q(q+1)/2 = 2k$, for some k, or

$q(q+1) = 4k$.

Hence $q \equiv 0, 3 \pmod 4$. #

SEQUENTIAL FUNCTIONS

Simply sequential and sequential graphs were first introduced by D.W.Bange, A.E.Barkauskas and peter J.Slater in their paper, "Simply Sequential and Graceful Graphs" .

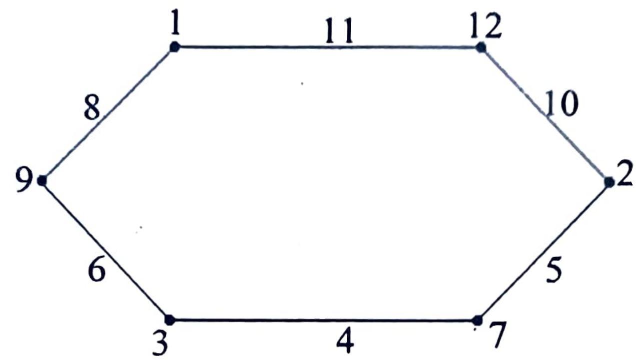
Definition: Let G be a simple graph with order p and size q. A function $f : V(G) \cup E(G) \rightarrow \{k, k+1, k+2, \ldots, p+q+k-1\}$ is called a **k-sequential function** if f is one-to-one and for any edge e, f(e) equals the absolute value of the difference between the tex labels at its end vertices. (that is, if $e = (x,y)$ then $f(e) = |f(x)-f(y)|$ }

Definition: A **simply sequential** function is a 1-sequential function.

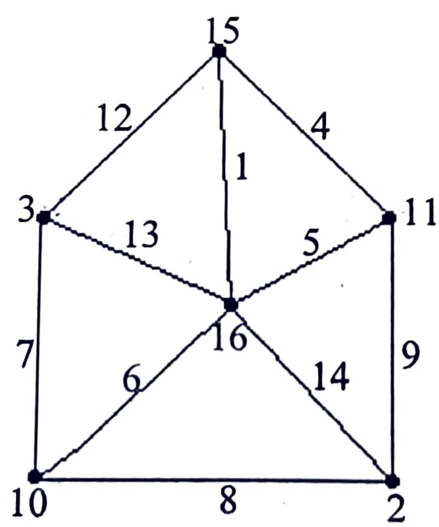
Definition: If a graph G admits a k-sequential function then it is called a **k-sequential graph.**

Examples:

1. The graph C_6 is 1-sequential graph since it has a 1-sequential labeling as shown in figure .6.



2. The wheel W_6 is 1-sequential and a 1-sequential labeling is shown in figure .7.



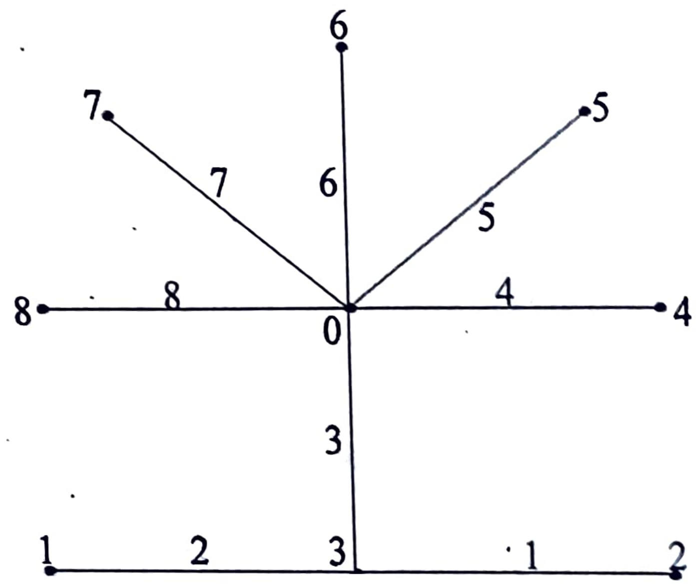
Theorem A graph G is 1-sequential if and only if G+v is graceful by a labeling f with f(v) = 0.

Proof: Let us assume that G is 1-sequential.
Clearly, $|E(G+v)| = |V(G)|+|E(G)|$. Now, we extend a 1-sequential function of G to the vertex v also by assigning zero to v. In G+v, each edge of G retains the original value of f and for all edges e = (u,v) receives the label f(u). Hence, we have a graceful labeling of G+v.

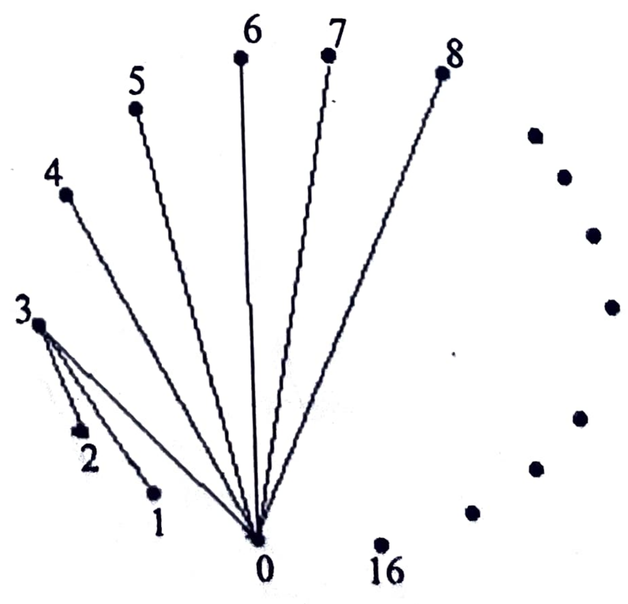
Conversely, we assume that f is a graceful labeling of G+v with f(v) = 0. For each vertex $u \in V(G)$ receives the same label of the edge e = (u,v). Thus, we get a 1-sequential function to G by restricting f to G. #

APPLICATION

Now we give a mathematical application using graceful labeling and the turning trick. Consider the following tree T with 9 vertices and 8 edges with a graceful labeling.



We would like to decompose K_{17} into seventeen copies of T. We do this in the following way. First we place seventeen vertices around a circle and number nine consecutive vertices 0,1,2, ...,8.



Then we introduce an edge between two of the numbered vertices if the vertices labeled by those two numbers are adjacent in T. Since the labeling of T is graceful, every edge has a distinct label, and thus every edge has distinct geometric length. Thus we got a copy of T.

Consequently the turning trick will give us a decomposition of K_{17} into seventeen copies of T.

This kind of construction is possible for every graceful tree and leads us to the following theorem.

Theorem If a tree T with q edges is graceful, then the complete graph K_{2q+1} is decomposable into 2q+1 trees, each isomorphic to the given tree T.

MAGIC GRAPHS

Definition: Let G be a graph with q edges. G is said to be **magic** if the edges of G can be labeled by the numbers 1,2,3, ...,q so that the sum of the labels of all the edges incident with any vertex is the same.

Examples:

1. The graph $K_{3,3}$ is magic since it has a magic labelling as shown in figure 10.

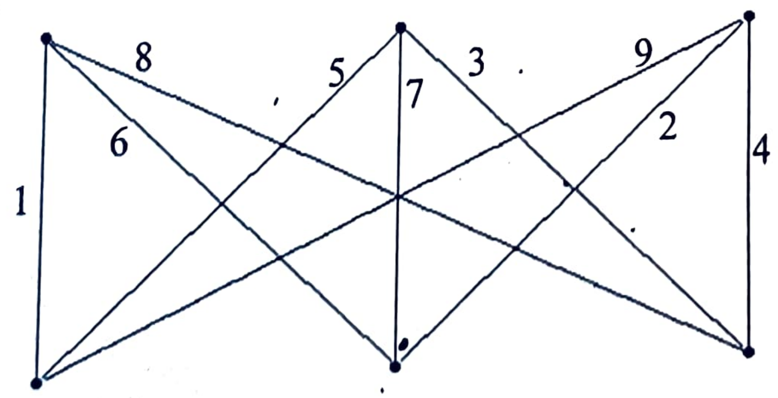
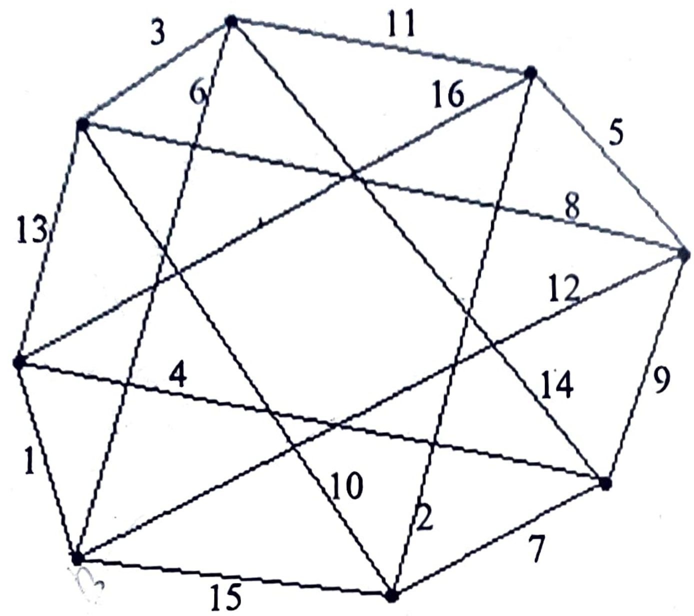


Figure 10.10

2. The graph $K_{4,4}$ is magic and a magic labeling is shown in figure 11. Notice that $K_{4,4}$ is decomposable into two Hamilton cycles. The labels on the edges of one of the Hamilton cycles are 1, 13, 3, 11, 5, 9, 7, 15. The edges of the other Hamilton cycle are 2, 16, 4, 14, 6, 12, 8, 10.

3. The complete graph K_3 is not magic since we cannot find a magic labeling of K_3 with numbers 1,2 and 3.



Theorem $K_{n,n}$ is magic for $n \neq 2$

Proof: The proof is left as an exercise to the reader. #

Theorem If a bipartite graph G is decomposable into two Hamilton cycles, then G is magic.

Proof: Since G is bipartite, the length of the Hamilton cycle is even, say 2n. Thus the number of edges in G is $q = 4n$. We label one Hamilton cycle with even numbers and the other with odd numbers.

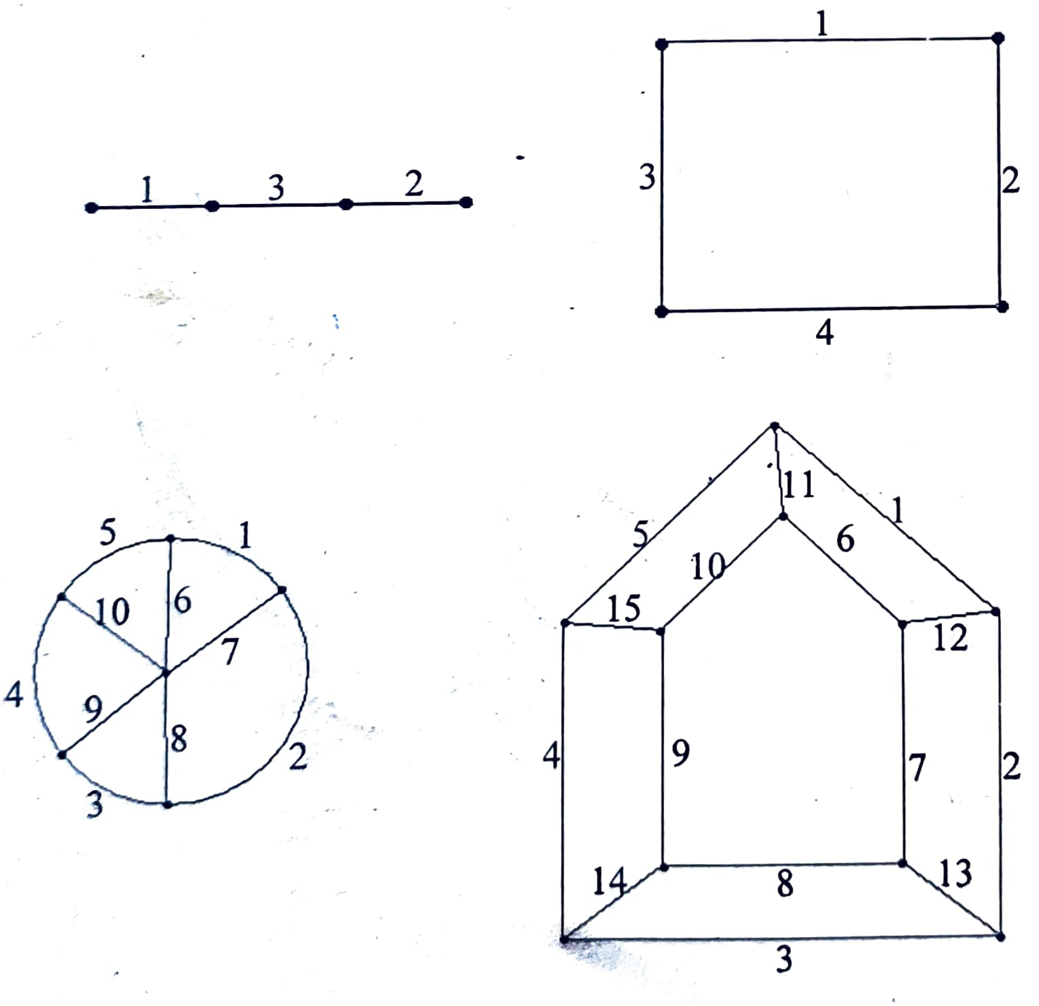
We choose a vertex a and label the edges of the first Hamilton cycle starting at a by 4n-1, 1, 4n-3, 3, …, 2n+1, 2n-1.
Then we label the edges of the second Hamilton cycle starting at a, by 2, 4n, 4, 4n-2, …, 2n, 2n+2.
Since G is bipartite, the vertices can be coloured red and blue with no two adjacent vertices of the same colour.

If a is blue, then the sum of the odd-numbered edges at all blue vertices except a is 4n-2. The sum of even-numbered edges at all blue vertices except a is 4n+4. We note that the sum of all edges at a is 8n+2. Thus the sum of all edges at any blue vertex is 8n+2. The sum of the odd-numbered edges at each red vertex is 4n. The sum of the even-numbered edges at each red vertex is 4n+2. Hence the sum of all edges at any red vertex is 8n+2. Hence G is magic. #

Theorem If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is magic.

Proof: Let q_1 and q_2 denote the number of edges of G_1 and G_2, respectively. Consider a magic labeling of G_1 and a magic labeling of G_2. To each label of G_2, we add q_1. Since G_2 is regular, we have added the same amount at each vertex. We now have the edges of G labeled with the integers $1,2,3, \ldots, q_1, q_1+1, \ldots, q_1+q_2$, and the sum of the labels at each vertex is the same. Hence G is magic. #

Now we introduce antimagic graphs.



The above figure shows some examples of graphs whose edges are labeled with the integers $1,2, \ldots, q$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum. We shall call a graph that can be so labeled as antimagic.

Conjecture Every connected graph $\neq K_2$ is antimagic.

Conjecture Every tree $\neq K_2$ is antimagic.

Definition: Let G be a graph with p vertices and q edges. G is said to be **magic total** if the vertices and edges of G can be labelled by the numbers 1,2,3,...,p+q so that for each edge $e = (u,v) \in E(G)$ we have $f(u) + f(e) + f(v)$ is the same.

Definition: Let G be a graph with q edges. G is said to be **bi-magic** if the edges of G can be labelled by the numbers 1,2,3,...,q so that the sum of the labels of all the edges incident with any vertex is either k_1 or k_2, where k_1 and k_2 are two constants.

Definition: Let G be a graph with p vertices and q edges. G is said to be **bi-magic total** if the vertices and edges of G can be labelled by the numbers 1,2,3,...,p+q so that for each edge $e = (u,v) \in E(G)$ we have $f(u) + f(e) + f(v) = k_1$ or k_2, where k_1 and k_2 are two constants.

L(2,1)-LABELING

Definition: A **L(2,1) - labeling** of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$.

A k-L(2,1)-labeling is an L(2,1)-labeling $f: V(G) \rightarrow \{0,\ldots, k\}$.

The minimum k among all such possible assignments is known as the **L(2,1)-labeling number** or **λ-number** and is denoted by λ(G).

Note: A L(2,1)-labeling is also called as Distance two labelling.

Definition: An injective L(2,1)-labeling is called an **L′(2,1)-labeling**.

A k-L′(2,1)-labeling is an L′(2,1)-labeling $f: V(G) \rightarrow \{0,\ldots, k\}$.

The minimum k among all such possible assignments is known as the **L′(2,1)-labeling number** or **λ′-number** and is denoted by λ′(G).

Definition: Let f be a labeling of a graph G. The number of occurrence of a label less one is called the multiplicity of the label in f and the sum of the multiplicity of labels of f is called the **multiplicity of f.**

Here, we find an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs such that G_2 has an injective L(2,1)-labeling and also we prove that the bound is attainable when G_1 and G_2 are complete. Also we present an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs.

Theorem . For any two graphs G_1 and G_2, $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda'(G_2) + 2$ and the bound is attainable when G_1 and G_2 are complete.

Proof: Let f_1 be the L(2,1)-labeling of G_1 corresponding to $\lambda(G_1)$ and f_2 be the injective L(2,1)-labeling of G_2 corresponding to $\lambda'(G_2)$.

Set $V(G_1) = \{u_1, u_2,\ldots, u_m\}$ and $V(G_2) = \{v_1,\ldots, v_n\}$ and define a labeling f on V $(G_1 \circ G_2)$:

$f(u_i) = f_1(u_i)$

$f(v_i) = f_2(v_i) + \lambda(G_1) + 2$, for all v_i in all copies.

Clearly f is a L(2,1) - labeling for $G_1 \circ G_2$.

Hence $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda'(G_2) + 2$.

Now let us assume that G_1 and G_2 are complete. Since G_1 is complete on m vertices, any L(2,1)-labeling of $G_1 \circ G_2$ needs 2m distinct labels for the vertices of G_1 and a different set of 2n labels for the vertices of G_2. Since we can use the label zero also,

$\lambda(G_1 \circ G_2) \geq 2m + 2n - 2 = 2m - 2 + 2n - 2 + 2 = \lambda(G_1) + \lambda'(G_2) + 2$.

That is, $\lambda(G_1 \circ G_2) = \lambda(G_1) + \lambda'(G_2) + 2$. #

Theorem For any two graphs G_1 and G_2, $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda(G_2)+m+2$, where m is the multiplicity of the L(2,1)-labeling corresponding to $\lambda(G_2)$.

Proof: Let f_1 be the L(2,1)-labeling of G_1 corresponding to $\lambda(G_1)$, f_2 be the L(2,1)-labeling of G_2 corresponding to $\lambda(G_2)$ and m be the multiplicity of f_2.

Let $V(G_1) = \{u_1, v_2,\ldots, u_m\}$ and $V(G_2) = \{v_1,\ldots, v_n\}$.

If f_2 is injective then by the above theorem, $\lambda(G_1 \circ G_2) \leq \lambda(G_1) + \lambda(G_2) + 2$ and since $m = 0$ in this case, the theorem is true. Otherwise, we rename the vertices of G_2 as below.

Let $k = \lambda(G_2)$ and let n_i denotes the multiplicity of the label i of f_2. For $i = 0,1, 2, \ldots, k$ and $j = 0,1, 2,\ldots, n_i$ let $\{v_{i,j}\}$ denote the set of all vertices of G_2 which receive the colour i in f_2 and these sets form a partition of $V(G_2)$. We note that for some i, this set may be empty. Hence the multiplicity of f_2 is $n_0 + n_1 + \ldots + n_k$.

QUESTION BANK

PART A

1. If (b_1, b_2, b_3, b_n) is the successor of a predecessor (a_1, a_2, a_3, a_n), then prove $\sum_{i=1}^{n} bi$ is even. **CO2 (L2)**

2. a) Define graceful labeling of a graph G. **CO1 (L1)**

   b) Analyse the graceful labeling of Petersen graph. **CO2 (L4)**

3. If G is graceful eulerian graph withsize q, then identify q ≡ 0,3(mod4) **CO2 (L2)**

4. Conclude that A graph G is 1- sequential if and only if G +V is graceful by a labeling f with f (v) = 0. **CO5 (L5)**

5 If G is decomposable into two Hamilton cycles, then examine that G is conservative. **CO2 (L4)**

6. If G is decomposable into two Hamilton cycles, then show that G is strongly conservative. **CO2 (L2)**

PART B

1. Prove that, the sequence (1, 2, 3, n) has a predecessor if and only if n ≡ 0, 3 (mod4). **CO2 (L5)**

2. Prove that, if a bipartite graph G is decomposable into two Hamilton cycles then G is magic. **CO2 (L5)**

3. For any two graphs G_1 and G_2, $\lambda(G_1°G_2) \leq\lambda(G_1) + \lambda'$ (G2) +2 and the bound is attainable when G_1 and G_2 are complete - Discuss **CO5 (L6)**

4. For any two graphs G_1 and G_2 , $\lambda(G_1°G_2) \leq\lambda(G_1) + \lambda(G_2)$ +m +2, where m is the multiplicity of the L (2,1)–labeling corresponding to $\lambda(G_2)$.-Discuss **CO5 (L6)**

5 If G is decomposable into two sub graphs H_1 and H_2 and if H_1 is conservative, and H_2 is strongly conservative then prove that G is strongly conservative.

CO5 (L5)

6. If G is a labeled directed graph such that Kirchhoff's current law holds at every vertex of G except a particular vertex a, then Kirchhoff's law also holds at the Vertex a. - Discuss **CO5 (L6)**

UNIT – V – Advanced Graph Theory – SMT5207

V. PERFECT GRAPHS

Content: Perfect Graphs – The Perfect Graph Theorem – Chordal Graphs – Interval Graphs – Comparability Graphs.

The chromatic number of a graph G is always greater than or equal to the clique number of the graph. For what type of graphs, equality holds?

Also, the clique cover number of G is always greater than or equal to the independence number of G. For what type of graphs, equality holds?

In 1961, Claude Berge conjectured that α-perfect and χ-perfect are equivalent. This was proved by László Lovász in 1971, at the age of 22. The above equivalence was almost proved earlier by Fulkerson. On hearing the success of Lovász from Berge, he completed his own proof, with in a few hours. No doubt it was a moment of sorrow for Fulkerson. But in this process, Fulkerson invented the notion of antiblocking pairs of polyhedra, an idea which has become an important topic in the field of polyhedral combinatorics.

In this chapter, mostly, we deal with vertex colouring and clique and so **we restrict our attention to simple graphs.**

PERFECT GRAPHS

For a graph G, we know that $\chi(G)$ denotes the chromatic number of G;

The minimum number of colours needed to properly colour the vertices of G; equivalently, the minimum number of independent sets needed to partition the vertices of G.

$\varpi(G)$ denotes the clique number of G; the cardinality of the largest clique of G.

$\alpha(G)$ denotes the independence number of G; the cardinality of the largest independent set of G.

$\theta(G)$ denotes the clique cover number of G; the minimum number of cliques needed to partition (or cover) the vertices of G.

Remark: The intersection of a clique and an independent set of a graph G can be at most one vertex. So, for any graph G,

$$\alpha(G) \le \theta(G) \text{ and } \varpi(G) \le \chi(G).$$

Notation: In this chapter, G_A denotes the subgraph induced by A, that is G[A].

Definition: A graph G is defined to be **χ-perfect** if $\chi(G_A) = \varpi(G_A)$, for all $A \subseteq V$.

Definition: A graph G is defined to be **α-perfect** if $\alpha(G_A) = \theta(G_A)$, for all $A \subseteq V$.

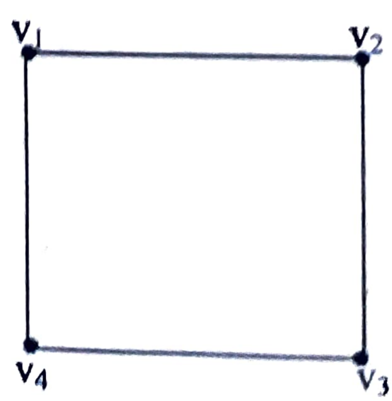
Note: 1. $\alpha(G_A) \le \alpha(G)$, $\varpi(G_A) \le \varpi(G)$
$\chi(G_A) \le \chi(G)$, $\theta(G_A) \le \theta(G)$.

2. A graph need not be χ-perfect (α-perfect) even if every proper induced subgraph is χ-perfect (α-perfect). This can be seen by considering the cycle C_5.

Examples:

1. Consider the graph C_4.

C_4:



Here $\chi(C_4) = 2 = \varpi(C_4)$.
Also $\chi(G_A) = \varpi(G_A)$, where $G = C_4$ and $A \subseteq V(C_4)$.
Hence C_4 is χ-perfect.
Also $\alpha(C_4) = 2 = \theta(C_4)$ and $\alpha(G_A) = \theta(G_A)$, where $G = C_4$ and $A \subseteq V(C_4)$
Therefore, C_4 is α-perfect.

2. If G is a bipartite graph, then we know that, $\chi(G) = 2 = \varpi(G)$, if G has an edge; Otherwise, $\chi(G) = 1 = \varpi(G)$.
   Hence G is χ-perfect
   Also, it is easy to see that, for bipartite graphs, $\alpha(G) = \theta(G)$.
   Hence it is also α-perfect.

3. Consider C_{2k+1} , $k > 1$.
   This is not χ-perfect since $\chi(C_{2k+1}) = 3$ and $\varpi(C_{2k+1}) = 2$.
   Also this is not α-perfect because $\alpha(C_{2k+1}) = k$ and $\theta(C_{2k+1}) = k+1$, a minimum partition consists of k, 2 cliques and one 1-clique.

Theorem A graph G is χ-perfect if and only if its complementary graph G^c is α-perfect.

Proof: Clearly $\alpha(G_A) = \varpi(G_A^c)$

$$\theta(G_A) = \chi(G_A^c)$$

Thus, $\alpha(G_A) = \theta(G_A)$ is equivalent to $\varpi(G_A^c) = \chi(G_A^c)$ and

$\alpha(G_A^c) = \theta(G_A^c)$ is equivalent to $\varpi(G_A) = \chi(G_A)$.

Hence the theorem. #

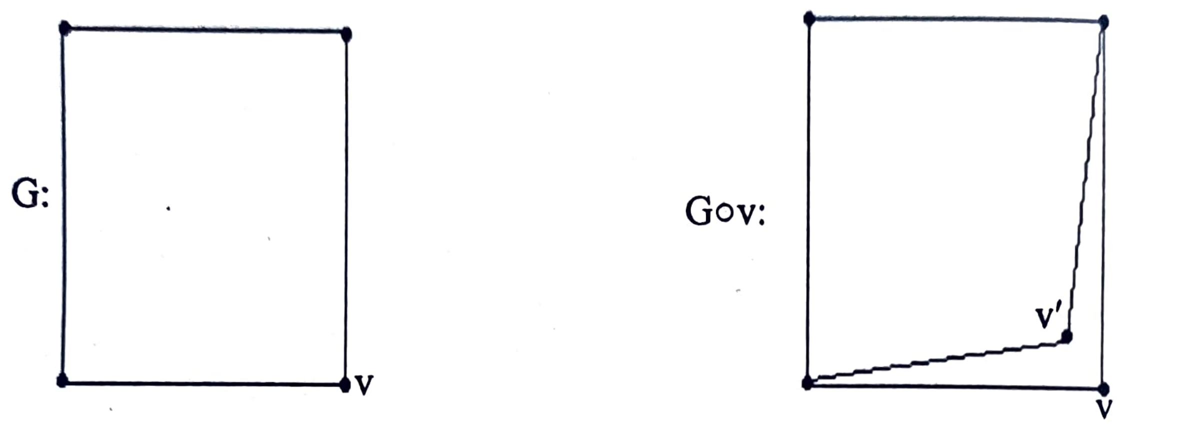
Corollary If either a graph G or its complementary graph G^c contains an odd cycle of length greater than 3 without chords, then G is neither χ-perfect nor α-perfect .

Proof: Let A be the vertex set of such a cycle of G.
Then $\chi(G_A) \neq \varpi(G_A)$, $\alpha(G_A) \neq \theta(G_A)$.
Thus G is neither χ-perfect nor α-perfect.

If the complementary graph G^c contains such a cycle, then it is neither χ-perfect nor α-perfect and by the previous theorem, G is neither χ-perfect nor α-perfect. #

Now we introduce the concept of multiplication of the vertices of a graph.

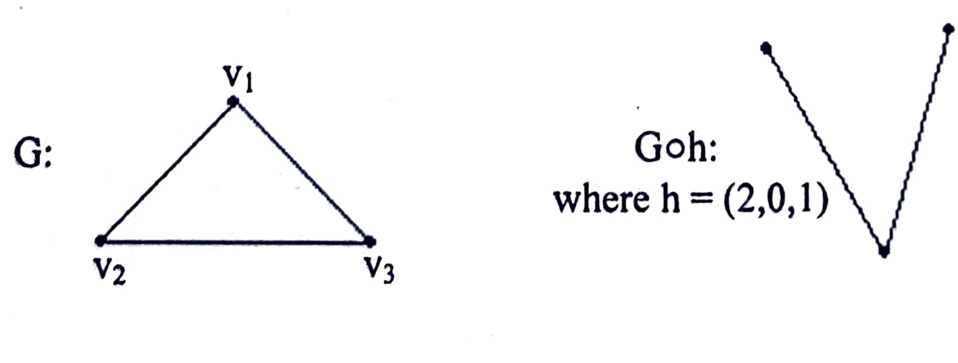
Let G be a graph with vertex v. The graph Gov is obtained from G by adding a new vertex v' which is connected to all the neighbours of v.



It is easy to see that, (Gov)-u = (G-u)ov for distinct vertices v and u.

More generally, if $v_1, v_2, \ldots, v_p$ are the vertices of G and $h = (h_1, h_2, \ldots, h_p)$ is a vector of non-negative integers, then H = Goh is constructed by substituting for each v_i an independent set of h_i vertices $v_i^1, \ldots, v_i^{h_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G. We say that H is obtained from G by multiplication of vertices.

Example



Note:

1. If we take $h_l = 0$ then H does not contain v_i.
2. Every induced subgraph of G can be obtained by multiplication of an appropriate vector, in which h_i is zero or one.
3. If each $h_i = 1$ then Goh = G.

Theorem (Berge) Let H be obtained from G by multiplication of vertices.

(i) If G is χ-perfect then H is χ-perfect.
(ii) If G is α-perfect then H is α-perfect.

Proof: The theorem is true if G has only one vertex. We shall assume that (i) and (ii) are true for all graphs with fewer vertices than G. Let H=Goh. If one of the co-ordinates of h equals zero, say $h_i = 0$, then H can be obtained from G-v_i by multiplication of vertices. But if G is χ-perfect then G-v_i also χ-perfect. Also, if G is α-perfect then G-v_i also α-perfect. Now by induction hypothesis, the theorem is true.

Thus, we may assume that each co-ordinate $h_i \geq 1$ and some $h_i \geq 2$. Since H can be built up from a sequence of smaller multiplications (Hint: refer exercise 12.4), it is sufficient to prove the result for H=Gov. Let v′ denote the added copy of v.
Let us assume that G is χ-perfect.
So, $\varpi(G_A) = \chi(G_A)$, for all $A \subseteq V$.
Since v and v′ are non-adjacent, $\varpi(Gov) = \varpi(G)$.
Let G be coloured using $\varpi(G)$ colours. Colour v′ the same colour as v. This will be a colouring of Gov in $\varpi(Gov)$ colours.
So, $\chi(Gov) = \varpi(G) = \varpi(Gov)$.
Similarly we can prove that $\chi(H_A) = \varpi(H_A)$ where H = Gov and for all $A \subseteq V(H)$. Hence, Gov is χ-perfect.
Now we assume that G is α-perfect. So, $\alpha(G_A) = \theta(G_A)$, for all $A \subseteq V$.
It is enough to prove that $\alpha(Gov) = \theta(Gov)$.
Let $\tilde{N}$ be a clique cover of G with $|\tilde{N}| = \theta(G) = \alpha(G)$ and let K_v be the clique of $\tilde{N}$ containing v. Now we consider two cases.

Case 1: v is contained in a maximum independent set S of G such that $|S| = \alpha(G)$.
Now $S \cup \{v'\}$ is a independent set of Gov and $\alpha(Gov) = \alpha(G)+1$.
Since $\tilde{N} \cup \{v'\}$ is a clique cover of Gov, we have that
$\theta(Gov) \leq \theta(G)+1 = \alpha(G)+1 = \alpha(Gov) \leq \theta(Gov)$.
Hence, $\alpha(Gov) = \theta(Gov)$.

Case 2: No maximum independent set of G contains v. So $\alpha(Gov) = \alpha(G)$.
Since each clique of $\tilde{N}$ intersects a maximum independent set exactly once, this is true in particular for K_v. But v is not a member of any maximum

independent set. Therefore, $D = K_v - \{v\}$ intersects each maximum independent set of G exactly once, so $\alpha(G_{V-D}) = \alpha(G)-1$.
Now $\theta(G_{V-D}) = \alpha(G_{V-D}) = \alpha(G)-1 = \alpha(G\circ v)-1$.
Taking a clique cover of G_{V-D} of cardinality $\alpha(G\circ v)-1$ along with extra clique $D \cup \{v'\}$, we obtain a clique cover of $G\circ v$.
Hence $\theta(G\circ v) = \alpha(G\circ v)$. #

Remark: In this chapter, our main aim is to prove the Perfect Graph theorem, which states that a graph is χ-perfect if and only if it is α-perfect. This was proved by Lovász along with a third equivalent condition,

$$\varpi(G_A).\alpha(G_A) \geq |A|, \text{ for all } A \subseteq V.$$

Theorem . (Fulkerson[1971], Lovász[1972]) Let G be a graph each of whose proper induced subgraphs are α-perfect, and let H be obtained from G by multiplication of vertices. If G satisfies the condition $\varpi(G_A).\alpha(G_A) \geq |A|$, for all $A \subseteq V$ then H also satisfies this condition.

Proof: Let G satisfy the condition,

$$\varpi(G_A).\alpha(G_A) \geq |A|, \text{ for all } A \subseteq V \qquad \text{(P)}$$

Choose H to be a graph having the smallest possible number of vertices which can be obtained from G by multiplication of vertices but which fails to satisfy (P) itself. So,

$\varpi(H).\alpha(H) < |X|$, where X denotes the vertex set of H, yet (P) does hold for each proper induced subgraph of H.

As in the proof of the above theorem, we may assume that each vertex of G was multiplied at least once and that some vertex u was multiplied at least twice (that is, $h \geq 2$). Let $U = \{u^1, u^2, \ldots, u^h\}$ be the vertices of H corresponding to u. The vertex u^1 plays an important role in this proof.
By the minimality of H, (P) is satisfied by H_{X-U^1}, which gives,

$$|X|-1 = |X-u^1| \leq \varpi(H_{X-U^1}).\alpha(H_{X-U^1}) \quad \text{by (P)}$$
$$\leq \varpi(H).\alpha(H)$$
$$\leq |X|-1.$$

Thus, equality must hold throughout, and we define,

$$p_1 = \varpi(H_{X-U^1}) = \varpi(H)$$

$$q_1 = \alpha(H_{X-U^1}) = \alpha(H)$$

and $p_1 q_1 = |X|-1$.
Since H_{X-U} is obtained from G-u by multiplication of vertices, by the previous theorem H_{X-U} is α-perfect. Thus, H_{X-U} can be covered by a set of q

cliques of H, say $K_1, K_2, \ldots, K_{q_1}$. We can assume that K_i's are pairwise disjoint and $|K_1| \geq |K_2| \geq \ldots \geq |K_{q_1}|$.

Now $\sum_{i=1}^{q_1} |K_i| = |X-U| = |X| - h = p_1q_1 + 1 - h = p_1q_1 - (h-1)$.

Since $|K_i| \leq p_1$, at most h-1 of the K_i cannot contribute p_1 to the sum.
Hence $|K_1| = |K_2| = \ldots = |K_{q_1-h+1}| = p$.

Let H′ be the subgraph of H induced by

$$X' = K_1 \cup K_2 \cup \ldots \cup K_{q_1-h+1} \cup \{u^1\}.$$

$|X'| = p_1 (q_1 - h + 1) + 1 < p_1q_1 + 1 = |X|$
So by the minimality of H, we have,

$$\varpi(H')\, \alpha(H') \geq |X'|$$

But $p_1 = \varpi(H) \geq \varpi(H')$,
So, $\alpha(H') \geq |X'| / p_1 > q_1 - h + 1$.
Let S′ be an independent set of H′ of cardinality q_1-h+2. Since S′ cannot have two vertices of a clique, $u^1 \in S'$. Hence, $S = S' \cup U$ is an independent set of H with q_1+1 vertices, which is a contradiction, to the definition of q_1. #

THE PERFECT GRAPH THEOREM

The Perfect Graph Theorem (Lovász) For a graph G, the following statements are equivalent.

1. G is χ-perfect.
2. G is α-perfect.
3. $\varpi(G_A).\alpha(G_A) \geq |A|$, for all $A \subseteq V$.

Proof: We may assume that the theorem is true for all graphs with fewer vertices than G.
(1) ⇒ (3). Let us assume that G is χ-perfect.
So we have, $\varpi(G_A) = \chi(G_A)$, for all $A \subseteq V$.
It means, we can colour G_A in $\varpi(G_A)$ colours.
Since there are at most $\alpha(G_A)$ vertices of a given colour, we get $\varpi(G_A).\alpha(G_A) \geq |A|$.
(3) ⇒ (1). Let us assume that G satisfies the condition

$$\varpi(G_A).\alpha(G_A) \geq |A|, \text{ for all } A \subseteq V.$$

Each proper induced subgraph of G satisfies (3) and by induction assumption, satisfies all the above three conditions.
So it is enough if we prove that $\varpi(G) = \chi(G)$.

If we have an independent set S of G such that $\varpi(G_{V-S}) < \varpi(G)$, then we can colour the elements of S by a new colour and G_{V-S} in $\varpi(G)$-1 other colours. This is a colouring of G and so $\chi(G) \leq \varpi(G)-1+1 = \varpi(G)$.
But we know, $\varpi(G) \leq \chi(G)$ for any graph G. Hence, $\varpi(G) = \chi(G)$.

Suppose G_{V-S} has an $\varpi(G)$-clique K(S) for every independent set S of G.
Let ξ be the collection of all independent sets of G.
Also $S \cap K(S) = \phi$.
For each $v_i \in V$, let h_i denote the number of cliques K(S) which contain v_i.
Let H be obtained from G by multiplying each v_i by h_i and let V(H) = X.
By theorem 12.4, $\varpi(H).\alpha(H) \geq |X|$.
Now we compute,

$|X| = \sum_{v_i \in V} h_i = \sum_{S \in \xi} |K(S)|$, since h_i equals the number of non-zeros in row i,

and |K(S)| equals the number of non-zeros in its corresponding column in the incidence matrix whose rows are indexed by the vertices $v_1, v_2, \ldots, v_p$ and whose columns correspond to the cliques K(S) for $S \in \xi$.
Therefore, the above equation becomes,

$$|X| = \sum_{v_i \in V} h_i = \sum_{S \in \xi} |K(S)| = \varpi(G)|\xi|.$$

Now we consider any clique in H. Since, at most one 'copy' of any vertex of G could be in a clique of H, we have $\varpi(H) \leq \varpi(G)$.
We note that, if a maximum independent set of H contains some of the 'copies' of v_i, then it will contain all of the 'copies' and hence,

$$\alpha(H) = \underset{T \in \xi}{\text{Max}} \sum_{v_i \in T} h_i$$

$$= \underset{T \in \xi}{\text{Max}} \left[\sum_{S \in \xi} |T \cap K(S)| \right],$$ by considering the entries of the rows corresponding to T in the above matrix.

$\leq |\xi|-1$ since $|T \cap K(T)| = 0$ and $|T \cap K(S)| \leq 1$, since T is an independent set and K(S) is a clique.

Now consider,
$\varpi(H).\alpha(H) \leq \varpi(G).(|\xi| -1) < |X|$,
which is a contradiction.
Hence the result.

(2) ⇔ (3)

G is α-perfect if and only if G^c is χ-perfect.

if and only if $\varpi(G^c_A).\alpha(G^c_A) \geq |A|$, for all $A \subseteq V$

if and only if $\alpha(G_A).\varpi(G_A) \geq |A|$, for all $A \subseteq V$.

Hence the theorem. #

Corollary A graph G is perfect if and only if its complement G^c is perfect.

Proof: By theorem , the corollary is immediate. #

Corollary A graph G is perfect if and only if every graph H obtained from G by multiplication of vertices is perfect.

Note: Sine χ-perfect and α-perfect are equivalent for any graph G, here after we call the graph which satisfy any one of them as perfect graph.

However, the above equivalence fails for uncountable graphs.

Now we present another characterization for perfect graphs.

Theorem . A necessary and sufficient condition for a non-empty graph G to be perfect is that for every induced subgraph $H \subseteq G$ there is an independent set of vertices I, such that $\varpi(H\text{-}I) < \varpi(H)$.

Proof: Let G be a perfect graph and H be an induced subgraph of G, and so H is also perfect. Let $k = \chi(H) = \varpi(H)$.

In this k-colouring of H, let I be a colour class. Now,

$\varpi(H\text{-}I) \leq \chi(H\text{-}I) = \chi(H)\text{-}1 < \varpi(H)$. Hence this part.

Conversely, let us assume that for each induced subgraph $H \subseteq G$, there is an independent set of vertices I, such that $\varpi(H\text{-}I) < \varpi(H)$.

We prove this result by induction on $\varpi(G)$.

If $\varpi(G) = 2$, then the result is obvious.

So, let $\varpi(G) > 2$ and we assume the result for smaller values of the clique number.

By induction hypothesis, we can colour H-I with $\varpi(H\text{-}I)$ colours and colouring the vertices of I with a new colour, we obtain a colouring of H with $\varpi(H\text{-}I)+1 \leq \varpi(H)$ colours. So $\chi(H) \leq \varpi(H)$. Hence G is perfect. #

Conjecture: When we talk about perfect graphs, it is natural to think about graphs, which are not perfect. Consider odd cycles of length at least five. It's chromatic number is three, but it's clique number is two. So it is not perfect. But we note that every subgraph of this is perfect. In 1960, Berge raised the question of the existence of other minimal imperfect graphs other than odd cycles of length at least five and their complements. He conjectured that there are none other than these. This has come to be known as **strong perfect graph conjecture**. It may be stated as below. A graph G is perfect if and only if G has no induced subgraph that is an odd cycle of length at least five or its complement.

CHORDAL GRAPHS

The concept of chordal graphs is due to Hajnal and Suranyi. In 1958, they showed that chordal graphs are α-perfect, and Berge, in 1960, proved that chordal graphs are χ-perfect. So, chordal graphs are very good examples for perfect graphs.

Definition: A graph G is called **chordal** if every cycle of length strictly greater than 3 possesses a chord, that is, an edge joining two non-consecutive vertices of the cycle.

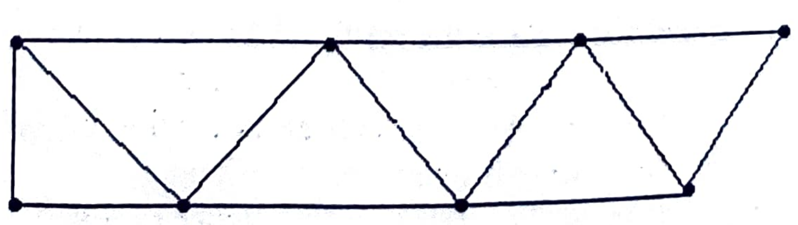
Remark: (i) By definition, G does not contain an induced subgraph isomorphic to C_n for $n > 3$.

(ii) A subgraph of a chordal graph is also chordal, since, if the subgraph has a cycle of length greater than three without chords then G would also have a cycle of length greater than three without chords.

(iii) Chordal graphs are also called as **triangulated** graphs.

Examples

1. All complete graphs are chordal graphs.
2. All trees are chordal graphs
3. The following graph is chordal.



4. Wheel W_n is not a chordal graph if $n \geq 5$.
5. $K_{m,n}$, $m, n \geq 2$ is not chordal.

Definition: A vertex v of G is called **simplicial** if its adjacency set Adj(v) is a clique of G (not necessarily maximal).

Definition: Let G be a graph and let $\sigma = [v_1, v_2, \ldots, v_p]$ be an ordering of the vertices. σ is called a **perfect vertex elimination scheme** (or perfect scheme) if each v_i is a simplicial vertex of the induced subgraph $G_{\{v_i, \ldots, v_p\}}$.

Equivalently, $X_i = \{v_j \in Adj(v_i) \,/\, j > i\}$ is a clique.

Examples:

1. In a tree, successive deletion of leaves induces a perfect vertex elimination scheme.
2. If G is a cycle of length greater than three, then G cannot have a perfect vertex elimination scheme, since a cycle has no simplicial vertex to start the elimination.

Definition: Let a and b be two non-adjacent vertices in a connected graph G. A subset $S \subseteq V$ is a vertex separator for a and b (or an a-b separator) if the removal of S from the graph separates a and b into distinct components.

If no proper subset of S is an a-b separator, then S is a **minimal vertex separator** for a and b.

Theorem For a graph G, the following statements are equivalent.

(i) G is chordal.

(ii) Every minimal vertex separator is a clique.

Proof:

(ii) $\Rightarrow$ (i) Let us assume that every minimal vertex separator is a clique.
Consider a cycle of length strictly greater than three of G, say,
$[a, x, b, y_1, y_2, \ldots, y_k, a]$, $k \geq 1$.
Any minimal a-b separator must contain vertices x and y_i for some i; So $xy_i \in E$, which is a chord of the cycle.
Hence G is a chordal.

(i) $\Rightarrow$ (ii) Let us assume that G be a chordal graph.
Let S be a minimal a-b separator with G_A and G_B being the components of G_{V-S} containing a and b respectively.
Since S is minimal, each $x \in S$ is adjacent to some vertex in A and some vertex in B.

Let $x,y \in S$. Since G_A is connected, there exists a path between x and y with internal vertices from G_A.
Now we choose such a path of smallest length say, $[x,a_1,\ldots,a_r,y]$, where $a_i \in A$.
Similarly, we choose a path of smallest length between x and y, say $[y,b_1,\ldots,b_t,x]$, where $b_i \in B$.
Now combining these two we get a cycle $[x,a_1,\ldots,a_r,y,b_1,\ldots,b_t,x]$, whose length is at least 4.
By our assumption, this cycle must have a chord. Since S is vertex separator, $a_ib_j \notin E$. Also $a_ia_j \notin E$, $b_ib_j \notin E$, $xa_i \notin E$ for $i > 1$, $ya_i \notin E$ for $i < r$, $xb_j \notin E$ for $j < t$ and $yb_j \notin E$ for $j > 1$ by the minimality of the length of the paths.
Therefore, the only possible chord is $xy \in E$.
Hence the theorem. #

Theorem Every chordal graph has a simplicial vertex. Moreover, if G is not a clique, then it has two non-adjacent simplicial vertices.

Proof: Let G be a chordal graph. The theorem is true trivially if G is complete. Now we assume that G has two non-adjacent vertices a and b and that the theorem is true for all graphs with fewer vertices than G.
Let S be a minimal vertex separator for a and b with G_A and G_B being the components of G_{V-S} containing a and b, respectively.
By induction, either the subgraph $G_{A\cup S}$ has two non-adjacent simplicial vertices one of which must be in A, since S is a clique or $G_{A\cup S}$ is itself complete and any vertex of A is simplicial in $G_{A\cup S}$. Since no vertex of A is adjacent with a vertex in B, a simplicial vertex of $G_{A\cup S}$ in A is simplicial in G. Similarly, B contains a simplicial vertex of G.
Hence the theorem. #

Now we present an equivalent condition for chordal graphs by Dirac(1961).

Theorem For a graph G, the following statements are equivalent.

(i) G is chordal.
(ii) G has a perfect vertex elimination scheme. Moreover, any simplicial vertex can start a perfect scheme.
(iii) Every minimal vertex separator is a clique.

Proof:

(i) ⇒ (ii) Let G be chordal.

Let us assume the result for all graphs with fewer vertices than G.
Since G is chordal, it has a simplicial vertex, say v.
Since any subgraph of a chordal graph is chordal, $G_{V-\{v\}}$ is chordal and smaller than G. So, by induction, $G_{V-\{v\}}$ has a perfect scheme which, when adjoined as a suffix of v, gives a perfect scheme for G.
(ii) ⇒ (i) Let G has a perfect vertex elimination scheme and let C be a cycle of G with length greater than three. Let v be the vertex of C with the smallest index in the perfect scheme of G. Since C is a cycle, $|Adj(v) \cap C| \geq 2$; and the eventual simpliciality of v gives a chord in C.
Hence G is chordal.
(i) ⇔ (iii) already proved.
Hence the theorem.

Definition: A subset $S \subseteq V$ of a connected graph G is said to be a vertex separator if G-S is disconnected.

Theorem Let S be a vertex separator of a connected graph G and let $G_{A_1}, G_{A_2}, \ldots, G_{A_t}$ be the components of G_{V-S}. If S is a clique (not necessarily maximal), then

$$\chi(G) = \operatorname{Max}_i \chi(G_{S \cup A_i})$$

and

$$\varpi(G) = \operatorname{Max}_i \varpi(G_{S \cup A_i})$$

Proof: Obviously $\chi(G) \geq \chi(G_{S \cup A_i})$ for each i.

So $\chi(G) \geq \operatorname{Max}_i \chi(G_{S \cup A_i}) = k.$

Now we show that G can be coloured using exactly k colours. First colour G_S, then independently extend the colouring to each piece of $G_{S \cup A_i}$. This will be a colouring of G with k colours and this is possible because S is a clique. Hence $\chi(G) = k$.
Now we prove the other equality.
We know, $\varpi(G) \geq \varpi(G_{S \cup A_i})$ for each i.

So, $\varpi(G) \geq \operatorname{Max}_i \varpi(G_{S \cup A_i}) = m$

Let X be a maximum clique of G with $\varpi(G)$ elements. Then X must lie wholly in one of the $G_{S\cup A_r}$, since any two vertices of X are connected and so they cannot belongs to G_{A_i} and G_{A_j}, for $i \neq j$.

So, $m \geq \varpi(G_{S\cup A_r}) \geq |X| = \varpi(G)$.

Hence $\varpi(G) = m$. #

Corollary Let S be a vertex separator of a connected graph G and let $G_{A_1}, G_{A_2}, \ldots, G_{A_t}$ be the components of G_{V-S}. If S is a clique, and if each subgraph $G_{S\cup A_i}$ is perfect, then G is perfect.

Proof: We assume that the result is true for all graphs with fewer vertices than G. So it is enough if we show that $\chi(G) = \varpi(G)$. Let each graph $G_{S\cup A_i}$ be perfect.

Now by the previous theorem,

$$\chi(G) = \underset{i}{\text{Max}}\, \chi(G_{S\cup A_i}) = \underset{i}{\text{Max}}\, \varpi(G_{S\cup A_i}) = \varpi(G).$$

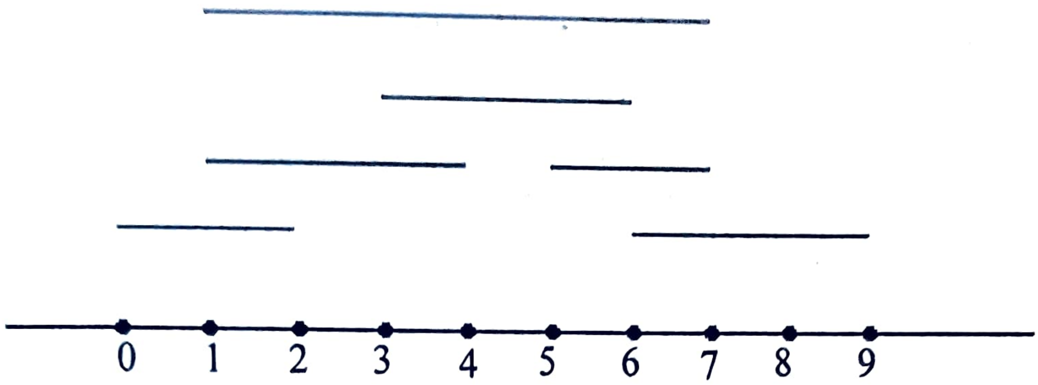
Hence G is perfect. #

Theorem Chordal graphs are perfect.

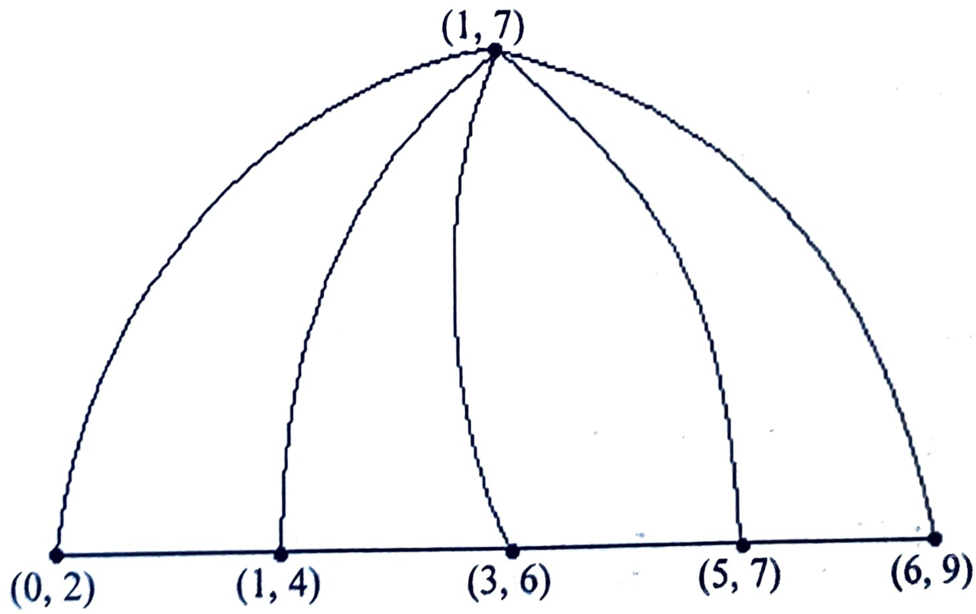
Proof: Let G be a chordal graph.
We assume that the theorem is true for all graphs having fewer vertices than G. Also, we may assume that G is connected, for otherwise we consider each component individually.
If G is complete, then G is perfect.
If G is not complete, then let S be a minimal vertex separator for some pair of non-adjacent vertices. Since G is chordal, S is a clique.
Also, by induction hypothesis, since each of the subgraphs $G_{S\cup A_i}$ (as defined in the corollary) are chordal, they are perfect.
Hence, by the previous corollary, G is perfect. #

INTERVAL GRAPHS

Consider the following open intervals on the real line,
(0, 2), (1, 4), (3, 6), (5, 7), (1, 7) and (6, 9)



Now we construct a graph from these intervals by introducing a vertex for each of these intervals and joining two such vertices by an edge whenever the corresponding intervals overlap. The graph arising from the intervals is shown in figure



Any graph, which arises in this way from a set of intervals, is called an **interval graph**. For example, the above Fan is an interval graph since we obtained this from the above intervals.

G. Hajo's and N. Wiener (1957) were the first to study interval graphs.

Definition: Consider a family $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ of intervals on the real line. The representative graph of $\mathcal{A}$ is defined to be a simple graph G in which each vertex a_i corresponds to an interval A_i, and with two vertices joined together if, and only if, the corresponding intervals overlap. Such a graph is called an **interval graph**.

Theorem Every interval graph is chordal.

Proof: Let G be an interval graph.
Suppose, G has a chordless cycle $[v_0, v_1, \ldots, v_{\ell-1}, v_0]$ with $\ell > 3$.
Let I_k denote the interval corresponding to v_k.
For $i = 1, 2, \ldots, \ell - 1$, choose a point $\alpha_i \in I_{i-1} \cap I_i$. Since I_{i-1} and I_{i+1} do not overlap, α_i constitute a strictly increasing or strictly decreasing sequence. So I_0 and $I_{\ell-1}$ cannot intersect, which is a contradiction, to the fact that $v_0 v_{\ell-1}$ is an edge of G. Hence G is chordal. #

Theorem Interval graphs are perfect.

Proof: Since every interval graph is chordal and since chordal graphs are perfect, the theorem is true. #

Solved Problems:

1. Show that the complement of a bipartite graph is perfect without using perfect graph theorem.

Solution: Since an induced subgraph of the complement of a bipartite graph is also the complement of a bipartite graph, it is enough if we prove that, if G is a bipartite graph then $\chi(G^c) = \varpi(G^c)$.
Now, in a colouring of G^c, every colour class is either a vertex or a pair of adjacent vertices in G. Thus $\chi(G^c)$ is the minimal number of vertices and edges of G, covering all vertices of G. Let H be the subgraph of G obtained by deleting all isolated vertices of G. In H the minimum number of edges covering all the vertices of H equal to maximum number of independent vertices of H, $\alpha = \beta'$. Now adding isolated vertices on both sides, we get the minimum number of vertices and edges of G covering all vertices of G is equal to the maximum number of independent vertices of G. Hence $\chi(G^c) = \varpi(G^c)$.

2. Let G be a bipartite graph with the line graph H = L(G). Show that H and H^c are perfect.
Solution: Since an induced subgraph of the line graph of a bipartite graph is also the line graph of a bipartite graph, it is enough if we prove that $\chi(H) = \varpi(H)$. Also, it is enough if we prove that $\chi(H^c) = \varpi(H^c)$.
A set of edges in G are adjacent if and only if all the edges pass through the same vertex.

Hence, $\varpi(H) = \Delta(G)$

$\chi(H) = \chi_1(G)$

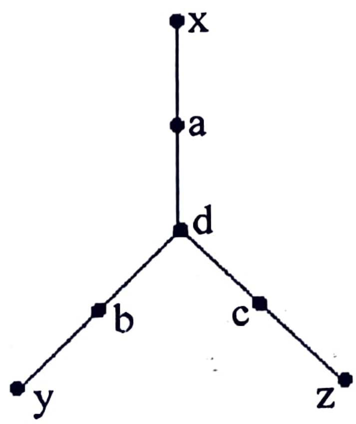
But for bipartite graphs, $\chi_1(G) = \Delta(G)$, hence H is perfect.

Now, we note that $\chi(H^c)$ is the minimal number of vertices of G covering all the edges and $\varpi(H^c)$ is the maximal number of independent edges of G. Since both of them are equal for bipartite graphs ($\alpha' = \beta$), H^c is perfect.

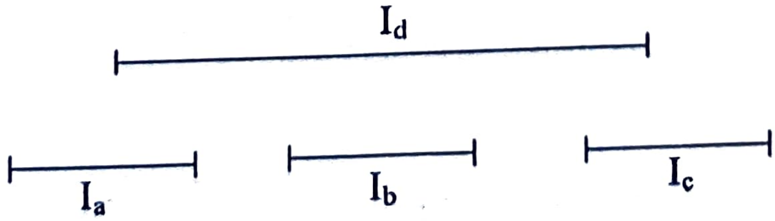
Note: Problem 1, is the first result on perfect graphs, proved by Gallai and Konig in 1932, although the concept of perfect graph was explicitly defined by Berge in 1960.

3. Give an example of a chordal graph, which is not an interval graph.

Solution:



Obviously, the above graph is chordal. In the interval representation of the above graph, the intervals I_a, I_b and I_c are disjoint.



Without loss of generality, we assume I_b be in between I_a and I_c. Since d is adjacent to a, b and c, I_d should properly include I_b. But it is impossible to have an interval I_y which has non-empty intersection with I_b, but disjoint with I_d. Hence this graph is not an interval graph.

COMPARABILITY GRAPHS

Definition: A graph G is called a **comparability graph** if it is possible to direct its edges so that the resulting graph with arc set F satisfies:

(i) $(x, y) \in F,\ (y, z) \in F \Rightarrow (x, z) \in F$ (transitivity)

(ii) $(x, y) \in F \Rightarrow (y, x) \notin F$ (anti-symmetry)

Note: A comparability graph may have more than one orientation of edges satisfying the two conditions.

Examples:

1. Every bipartite graph is a comparability graph, since if (A, B) is the bipartition then, we direct all the edges from A to B.
2. Clearly subgraph of comparability graph is a comparability graph.

Theorem Comparability graphs are perfect.

Proof: Let G be a comparability graph. Consider G with its direction.
For each vertex v, let t(v) denote the length of the longest path from v plus one. If max $t(v) = k$, there exists a k-clique containing all the vertices in the longest path from v. But in G there cannot be a (k+1)-clique; otherwise we can find a path with k+1 vertices. Thus $\varpi(G) = k$.
Consider k colours say, 1,2,...,k.. Colour each vertex v with colour t(v).
Two adjacent vertices cannot have the same colour, because if there is an arc directed from v_1 to v_2 then $t(v_1) > t(v_2)$. Thus $\chi(G) \leq K$.
But $\chi(G) \geq \varpi(G) = k$ and hence $\chi(G) = k$.
Therefore, $\chi(G) = \varpi(G)$.
Now by perfect graph theorem, G is perfect. #

QUESTION BANK

UNIT V

PART A

1. Show thatA graph G is χ - perfect if and only if its complementary graph G^c is $\boldsymbol{\alpha}$ – perfect. **CO2 (L2)**

2. Let S be a vertex separator of a connected graph G and let $G_{A1}, G_{A2}, \ldots, G_{At}$ be the components of $G_{v\text{-}s\ I}$ If S is a clique and if each sub graph $G_{s \cup Aj}$ is perfect, then G is perfect. **CO2 (L2)**

4.Explain Every chordal graph is perfect. **CO2 (L2)**

5. Explain Every interval graph is Chordal. **CO2 (L2)**

6. Explain Comparability graphs are perfect. **CO2 (L2)**

7. Give an example of a chordal graph, which is not an interval graph. **CO1 (L2)**

PART B

1. Prove that if either a graph G or its complementary graph G^c contains an odd cycle of length greater than 3 without chords, then G is neither χ - perfect nor $\boldsymbol{\alpha}$ – perfect. **CO3 (L5)**

2. Let H be obtained from G by multiplication of vertices the prove that

   i) If G is χ - perfect then H is χ - perfect

   ii) If g is $\boldsymbol{\alpha}$ – perfect then H is $\boldsymbol{\alpha}$ – perfect **CO3 (L5)**

3. Let G be a graph each of whose proper induced sub graphs are $\boldsymbol{\alpha}$ – perfect, and let H be obtained from G by multiplication of vertices. If G contains the condition $\omega(G_A)$. $\alpha(G_A) \geq |A|$ for all A $\subseteq$V then H also satisfies this condition. If g is $\boldsymbol{\alpha}$ – perfect then H is $\boldsymbol{\alpha}$ – perfect – Discuss. **CO3 (L6)**

4. For a graph G, the following Analyse that statements are equivalent.

   1) G is χ - perfect

   2) G is $\boldsymbol{\alpha}$ – perfect

3) $\omega(G_A)$. $\alpha(G_A) \geq |A|$, for all $A \subseteq V$. **CO6 (L4)**

5.A necessary and sufficient for a non-empty graph G to be perfect is that for every induced sub graph $H \subseteq G$ there is an independent set of vertices I, such that $\omega(H-I) < \omega(H)$. – Discuss **CO6 (L6)**

6. For a graph G, Prove that the following statements are equivalent.

   i) G is Chordal

   ii) Every minimal vertex separator is a clique. **CO6 (L5)**

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