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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

FUZZY ANALYSIS

UNIT – I - From Classical Sets To Fuzzy sets – SMT5205

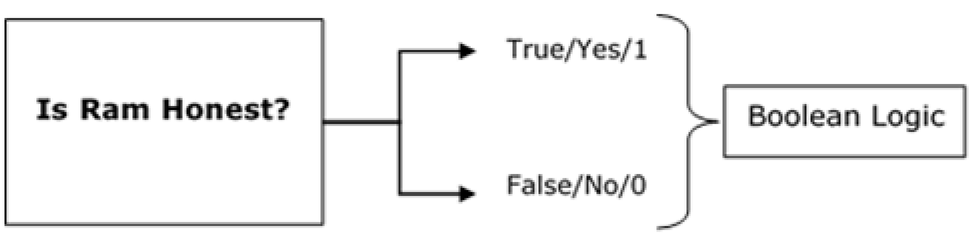
I. Fuzzy Logic Introduction

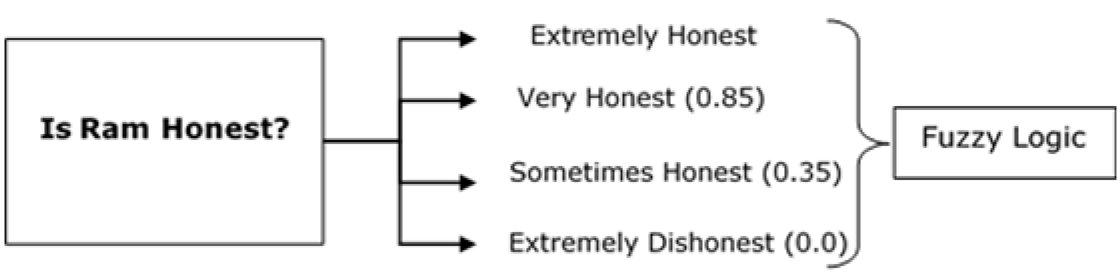
The word **fuzzy** refers to things which are not clear or are vague. Any event, process, or function that is changing continuously cannot always be defined as either true or false, which means that we need to define such activities in a Fuzzy manner.

What is Fuzzy Logic?

Fuzzy Logic resembles the human decision-making methodology. It deals with vague and imprecise information. This is gross oversimplification of the real-world problems and based on degrees of truth rather than usual true/false or 1/0 like Boolean logic.

Take a look at the following diagram. It shows that in fuzzy systems, the values are indicated by a number in the range from 0 to 1. Here 1.0 represents **absolute truth** and 0.0 represents **absolute falseness**. The number which indicates the value in fuzzy systems is called the **truth value**.



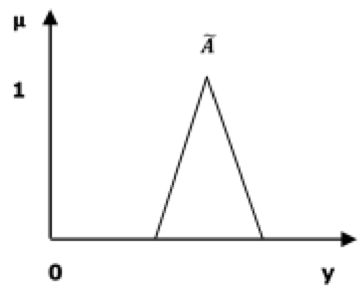


In other words, we can say that fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness. There can be numerous other examples like this with the help of which we can understand the concept of fuzzy logic.

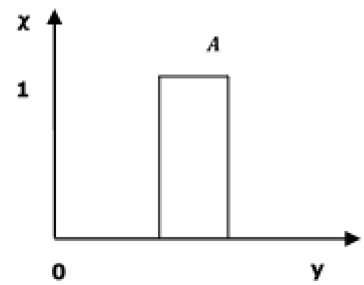
Fuzzy Logic was introduced in 1965 by Lofti A. Zadeh in his research paper “Fuzzy Sets”. He is considered as the father of Fuzzy Logic.

Fuzzy Logic – Set Theory

Fuzzy sets can be considered as an extension and gross oversimplification of classical sets. It can be best understood in the context of set membership. Basically it allows partial membership which means that it contain elements that have varying degrees of membership in the set. From this, we can understand the difference between classical set and fuzzy set. Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.



Membership Function of Fuzzy set $\widetilde{A}$



Membership Function of classical set A

Mathematical Concept

A fuzzy set $\widetilde{A}$ in the universe of information U can be defined as a set of ordered pairs and it can be represented mathematically as −

$$\widetilde{A} = \left\{ \left(y, \mu_{\widetilde{A}}(y) \right) | y \in U \right\}$$

Here $\mu_{\widetilde{A}}(y)$ = degree of membership of y in \widetilde{A}, assumes values in the range from 0 to 1, i.e., $\mu_{\widetilde{A}}(y) \in [0,1]$.

Representation of fuzzy set

Let us now consider two cases of universe of information and understand how a fuzzy set can be represented.

Case 1

When universe of information U is discrete and finite –

$$\widetilde{A} = \{\frac{\mu_{\widetilde{A}}(y_1)}{y_1} + \frac{\mu_{\widetilde{A}}(y_2)}{y_2} + \frac{\mu_{\widetilde{A}}(y_3)}{y_3} + \dots\}$$

$$= \{\sum_{i=1}^{n} \frac{\mu_{\widetilde{A}}(y_i)}{y_i}\}$$

Case 2:

When universe of information U is continuous and infinite −

$$\widetilde{A} = \{\int \frac{\mu_{\widetilde{A}}(y)}{y}\}$$

In the above representation, the summation symbol represents the collection of each element.

Operations on Fuzzy Sets

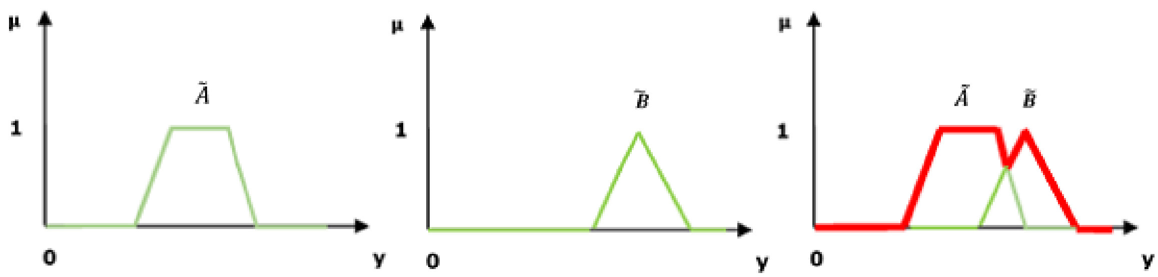
Having two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, the universe of information U and an element y of the universe, the following relations express the union, intersection and complement operation on fuzzy sets.

Union/Fuzzy 'OR'

Let us consider the following representation to understand how the **Union/Fuzzy 'OR'** relation works −

$$\mu_{\widetilde{A} \cup \widetilde{B}}(y) = \mu_{\widetilde{A}} \vee \mu_{\widetilde{B}} \quad \forall y \in U$$

Here ∨ represents the 'max' operation.



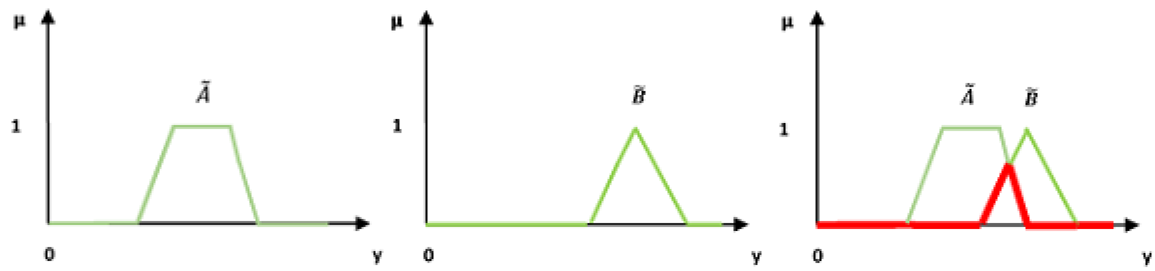
Fuzzy set $\tilde{A}$ **Fuzzy set $\tilde{B}$** **Union of two Fuzzy sets**

Intersection/Fuzzy 'AND'

Let us consider the following representation to understand how the **Intersection/Fuzzy 'AND'** relation works −

$$\mu_{\tilde{A} \cap \tilde{B}}(y) = \mu_{\tilde{A}} \wedge \mu_{\tilde{B}} \quad \forall y \in U$$

Here ∧ represents the 'min' operation.

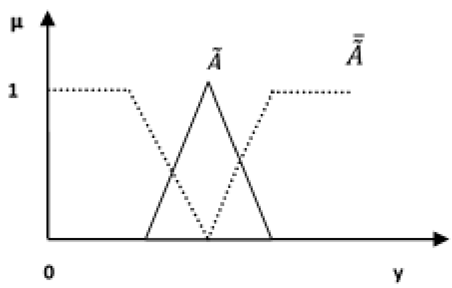


Fuzzy set $\tilde{A}$ Fuzzy set $\tilde{B}$ Intersection of two Fuzzy sets

Complement/Fuzzy 'NOT'

Let us consider the following representation to understand how the **Complement/Fuzzy 'NOT'** relation works –

$$\mu_{\tilde{A}} = 1 - \mu_{\tilde{A}}(y) \quad y \in U$$



Complement of a fuzzy set

Properties of Fuzzy Sets

Commutative Property:

Having two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, this property states −

$$\widetilde{A} \cup \widetilde{B} = \widetilde{B} \cup \widetilde{A}$$

$$\widetilde{A} \cap \widetilde{B} = \widetilde{B} \cap \widetilde{A}$$

Distributive Property

Having three fuzzy sets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, this property states −

$$\widetilde{A} \cup \left(\widetilde{B} \cap \widetilde{C}\right) = \left(\widetilde{A} \cup \widetilde{B}\right) \cap \left(\widetilde{A} \cup \widetilde{C}\right)$$

$$\widetilde{A} \cap \left(\widetilde{B} \cup \widetilde{C}\right) = \left(\widetilde{A} \cap \widetilde{B}\right) \cup \left(\widetilde{A} \cap \widetilde{C}\right)$$

Idempotency Property

For any fuzzy set $\widetilde{A}$, this property states −

$$\widetilde{A} \cup \widetilde{A} = \widetilde{A}$$

$$\widetilde{A} \cap \widetilde{A} = \widetilde{A}$$

Identity Property

For fuzzy set $\widetilde{A}$ and universal set U, this property states −

$$\widetilde{A} \cup \varphi = \widetilde{A}$$

$$\widetilde{A} \cap U = \widetilde{A}$$

$$\widetilde{A} \cap \varphi = \varphi$$

$$\widetilde{A} \cup U = U$$

Fuzzy Sets: Basic Types

○ Fuzzy sets

－Sets with vague boundaries

－Membership of *x* in *A* is a matter of degree to which *x* is in *A*

○ Utilization of fuzzy sets

(1) Representation of uncertainty
(2) Representation of conceptual entities
e.g., expensive, close, greater, sunny, tall

○ **Fuzzy Sets** ⇔ **Crisp Sets**

membership function ⇔ characteristic function

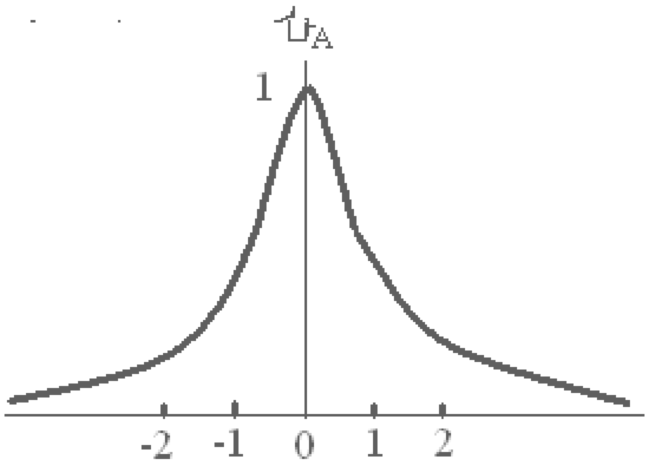
$$\mu_A : X \rightarrow [0,1] \quad \Leftrightarrow \quad m_A : X \rightarrow \{0,1\}$$

e.g.,

i) "close to 0" : $\mu_A(x) = \dfrac{1}{1+10x^2}$

ii) “very close to 0” : $\mu_A(x) = \left(\frac{1}{1+10x^2} \right)^2$

iii) “close to a” : $\mu_A(x) = \frac{1}{1+10(x-a)^2}$



○ Difference between crisp, random, and fuzzy variables:

Crisp variable: a uniform probability distribution

Random variable: a probability distribution

Fuzzy variable: a membership function

is associated with its domain.

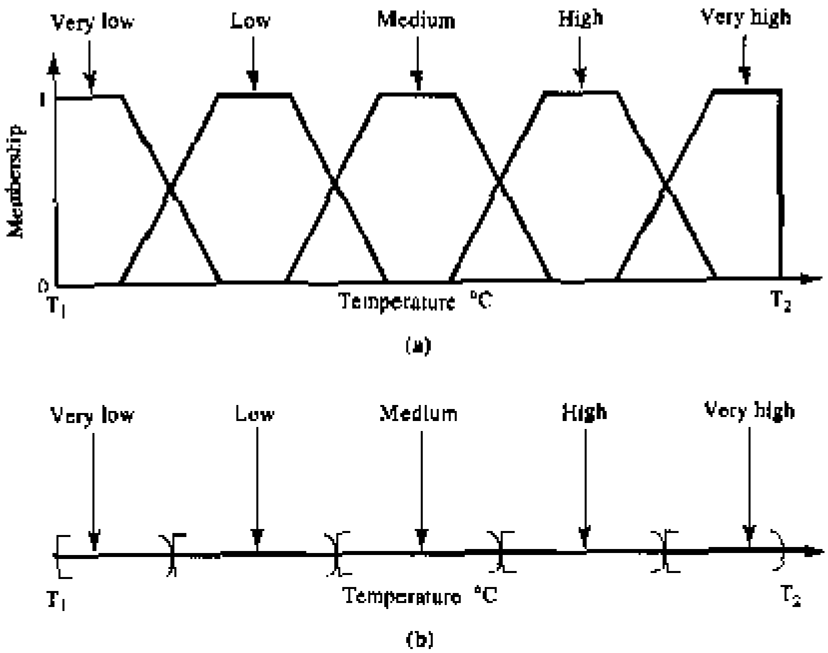


Figure 1.4 Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

○ Generalization

i) *Ordinary fuzzy sets*: $\mu_A : X \to [0,1]$

Abbreviated as $A : X \to [0,1]$.

i.e., Each element of *X* is assigned a particular real number (i.e., precise membership grades).

ii) *L-fuzzy sets*: $A : X \to L$, where *L* is a partial order set.

iii) *Interval–valued fuzzy sets*: $A : X \to \varepsilon([0,1])$,

where $\varepsilon([0,1])$ is the family of all closed interval in [0,1].

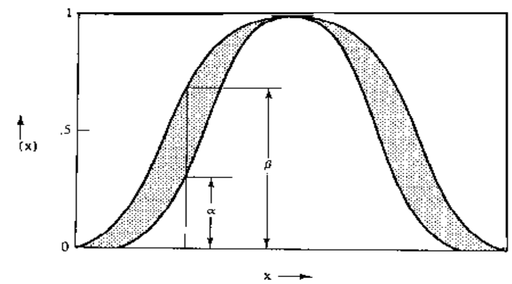


Figure 1.3. An example of an interval-valued fuzzy set ($\mu_A(a) = [\alpha, \beta]$).

iv) *Fuzzy sets of type-K*

-- Interval–valued fuzzy sets possess

fuzzy Intervals

(a) Type-2: $A : X \rightarrow \Xi([0,1])$, where

$\Xi([0,1])$: fuzzy power set of [0,1], the set of all ordinary fuzzy sets defined on [0,1].

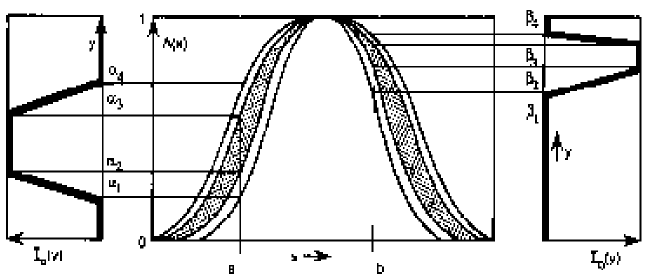
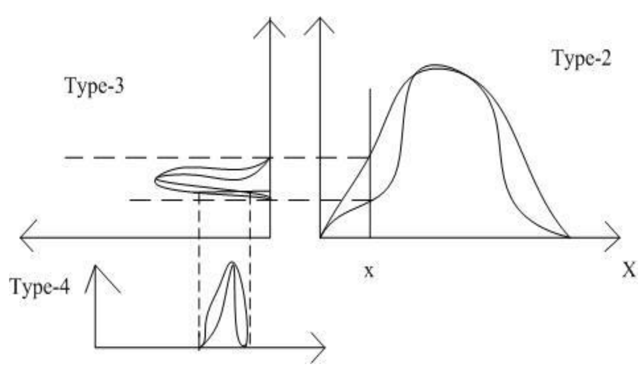


Figure 1.6 Illustration of the concept of a fuzzy set of type 2.

(b) Type-3



v) *Level-K fuzzy sets*

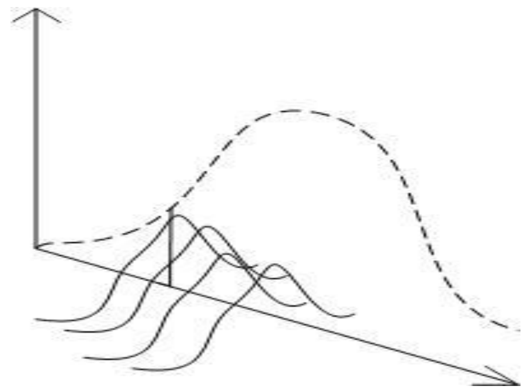
-- Elements in a universal set are themselves fuzzy sets.

(a) Level-2: $A : \Xi(X) \to [0,1]$

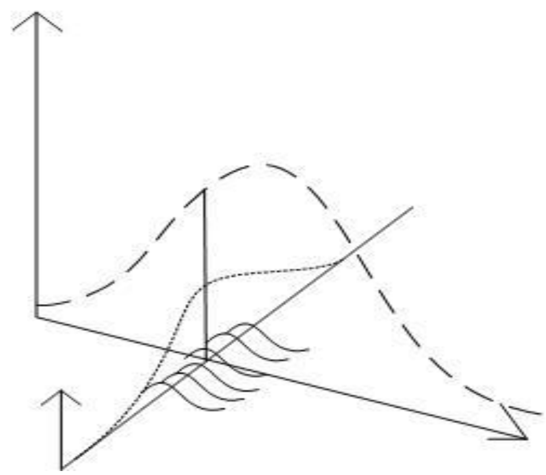
e.g., fuzzy set "*x* is close to *r*"

x : a fuzzy variable

r : a particular number , e.g., 5.



(b) Level 3:



vi) Combinations of interval-valued, *L*,

type-*K*, level-*K* fuzzy sets.

1.4 Fuzzy Sets: Basic Concept

○ Example: 3 fuzzy sets defined on age.

A_1 : "young", A_2 :"middle-aged", A_3 : "old"

Membership functions:

$$A_1(x)=\begin{cases}1 & x\le 20\\ (35-x)/15 & 20<x<35\\ 0 & x\ge 35\end{cases}$$

$$A_2(x)=\begin{cases}0 & x\le 20\, or \quad x\ge 60\\ (x-20)/15 & 20<x<35\\ (60-x)/15 & 45<x<60\\ 1 & 35\le x\le 45\end{cases}$$

$$A_3(x)=\begin{cases}0 & x\le 45\\ (x-45)/15 & 45<x<60\\ 1 & x\ge 60\end{cases}$$

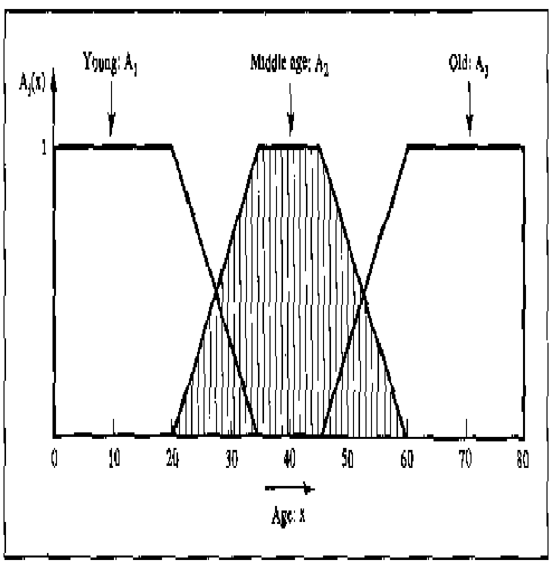
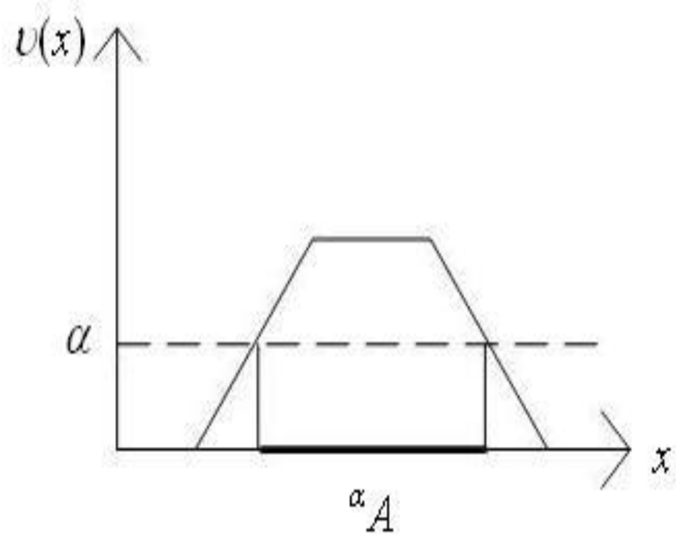


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \to [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \ldots, 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

○ α -cut ${}^{\alpha}A$: ${}^{\alpha}A=\{x \mid A(x)\ge\alpha\}$



If $\alpha_1 < \alpha_2 \Rightarrow {}^{\alpha_1}A \supseteq {}^{\alpha_2}A$

Strong α -cut ${}^{\alpha+}A$: ${}^{\alpha+}A = \{x \mid A(x) > \alpha\}$

e.g.,

$$\left.\begin{aligned} {}^{\alpha}A_1 &= [0, 35 - 15\alpha] \\ {}^{\alpha}A_2 &= [15\alpha + 20, 60 - 15\alpha] \\ {}^{\alpha}A_3 &= [15\alpha + 45, 80] \end{aligned}\right\} \forall \alpha \in (0,1]$$

$$\left.\begin{aligned} {}^{\alpha+}A_1 &= (0, 35 - 15\alpha) \\ {}^{\alpha+}A_2 &= (15\alpha + 20, 60 - 15\alpha) \\ {}^{\alpha+}A_3 &= (15\alpha + 45, 80) \end{aligned}\right\} \forall \alpha \in [0,1)$$

○ Level set $\wedge(A)$:

$$\wedge(A) = \{\alpha \mid \exists x \in X, s.t \;\; A(x) = \alpha\}$$

or $= \{\alpha \mid {}^{\alpha}A \neq \phi\}$

e.g.,

Continuous case ---

$$\wedge(A_1) = \wedge(A_2) = \wedge(A_3) = [0,1]$$

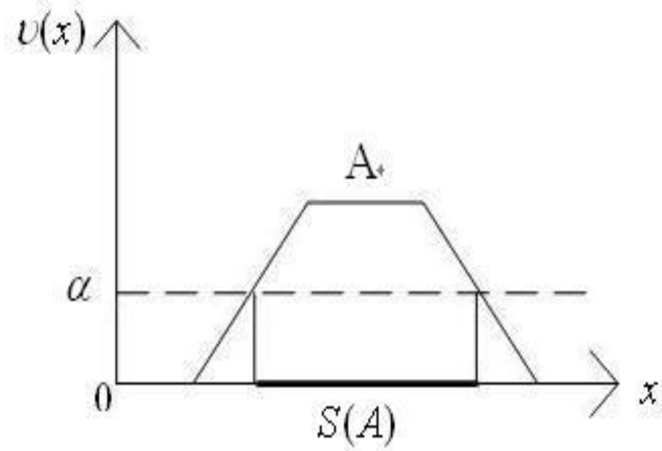
Discrete case ---

$$\wedge(D_1) = \{0 , 0.13 , 0.27 , 0.4 , 0.5 , 0.67 , 0.8 , 0.93 , 1\}$$

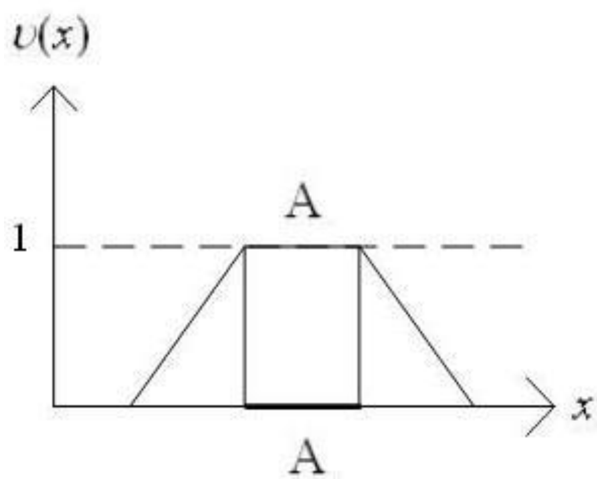
○ Support :

$$S(A) = [x \in X \mid A(x) > 0]$$

$S(A) = {}^{0+}A$, e.g., $S(D_2) = \{22, 24, \cdots, 58\}$

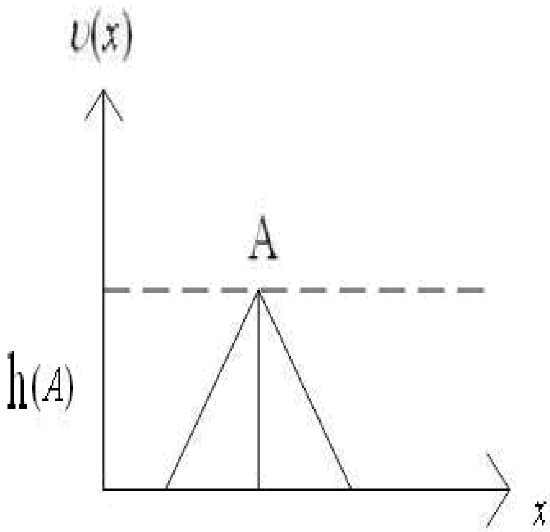


○ Core : ${}^{1}A$ (i,e, 1 - cut)



○ Hight $h(A)$：the largest membership grade

$$h(A) = \sup_{x \in X} A(x)$$

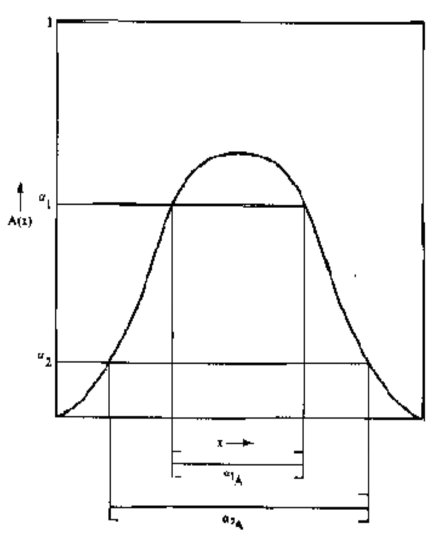


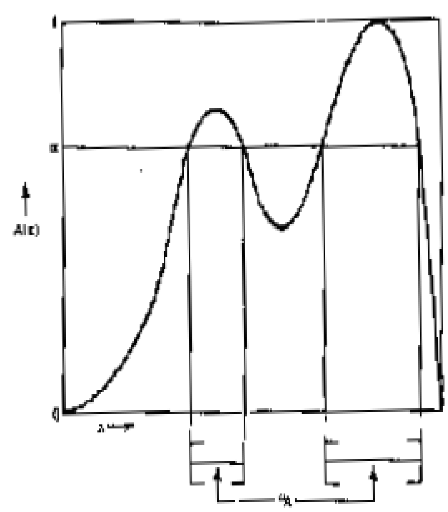
○ Normal：$h(A) = 1$

Subnormal：$h(A) < 1$

○ Convex fuzzy set：

$$\forall \alpha \in (0,1] \quad , \alpha\text{-cut is convex}$$





○ **Theorem 1.1:** A convex fuzzy set on R

iff $\forall x_1, x_2 \in R$, $\forall \lambda \in [0,1]$,

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\ [A(x_1), A(x_2)]$$

Proof :

$i,\ (\Rightarrow)$ Given A : convex ,

$\forall x_1, x_2,$ Let $a = \min[A(x_1), A(x_2)]$

$\Rightarrow\ x_1, x_2 \in {}^{a}A$

$\because A$: convex $\Rightarrow {}^{a}A$ convex

$\therefore \forall \lambda \in [0,1],\ x = \lambda x_1 + (1-\lambda)x_2 \in {}^{a}A$

(definition of convex set)

$\Rightarrow A(x) \geq a = \min[A(x_1), A(x_2)]$

$ii,\ (\Leftarrow)$

$\forall x_1, x_2,$ Given $A(\lambda x_1 + (1-\lambda)x_2)$

$\geq \min[A(x_1), A(x_2)]$

(Show that $\forall \alpha \in (0,1]$, ${}^{\alpha}A$: convex)

$\forall x_1, x_2$, $\exists \alpha$, s,t.

$A(x_1) \geq \alpha,\ A(x_2) \geq \alpha$ (i,e., $x_1, x_2 \in {}^{\alpha}A$) $-(1)$

$$\because \forall \lambda \in [0,1]$$

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)]$$

$$\geq \min(\alpha, \alpha) = \alpha,$$

$$\text{i.e., } \lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A \qquad -(2)$$

$$(1),(2) \Rightarrow {}^{\alpha}A : \text{convex} \Rightarrow A : \text{convex}$$

◎ Fuzzy Set Operations

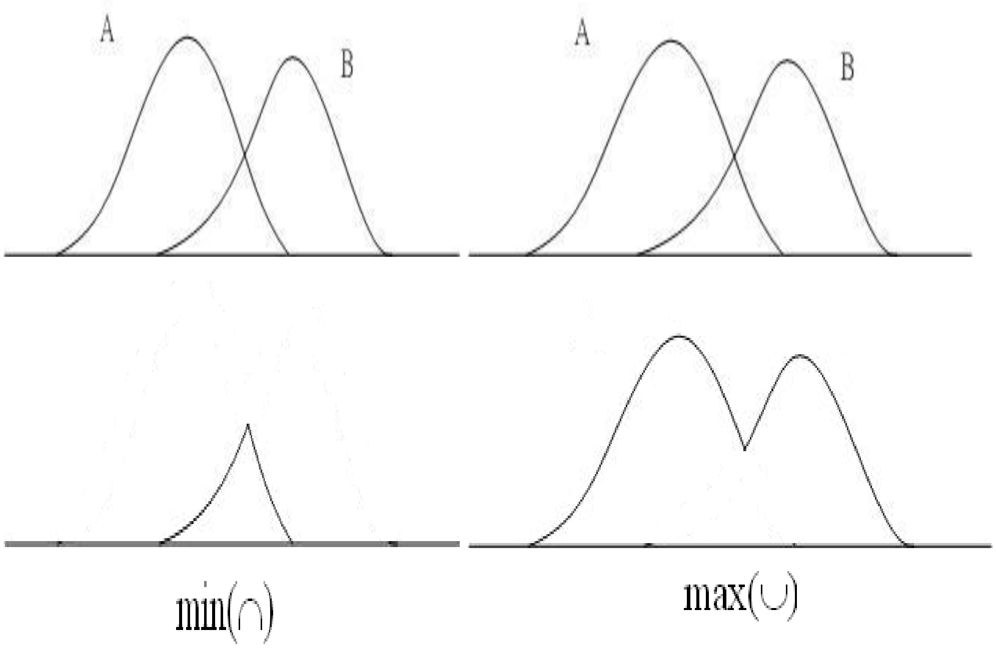
.Standard complement： $\overline{A}(x) = 1 - A(x)$

.Equilibrium points： $A(x) = \overline{A}(x)$

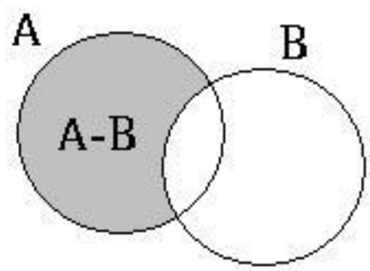
$$A(x) = \overline{A}(x) = 1 - A(x) \Rightarrow 2A(x) = 1 \quad \therefore A(x) = \overline{A}(x) = 0.5$$

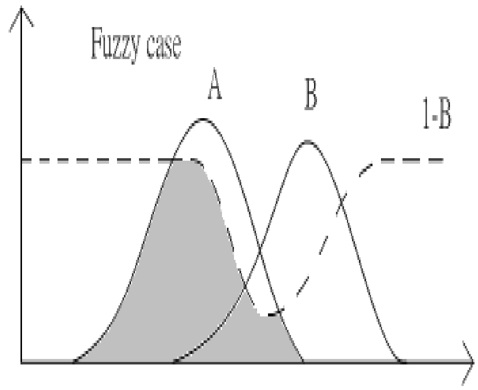
.Standard intersection： $(A \cup B)(x) = \min[A(x), B(x)]$

.Standard union: $(A \cup B)(x) = \max[A(x), B(x)]$



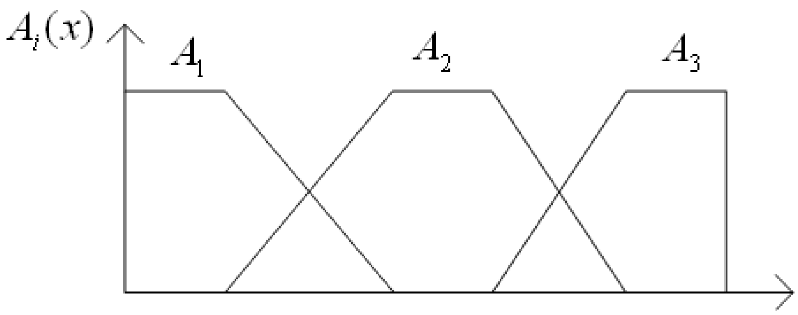
.Difference $A - B = A \cap \overline{B} = \min(A(x), 1 - B(x))$

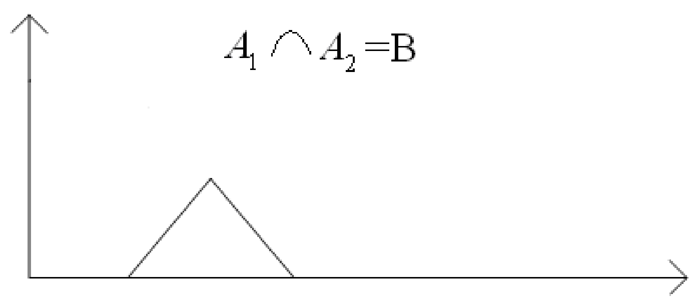


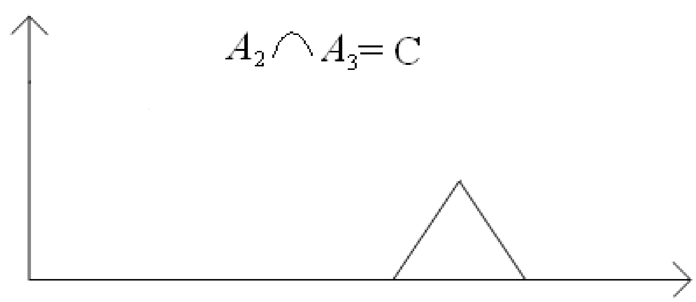


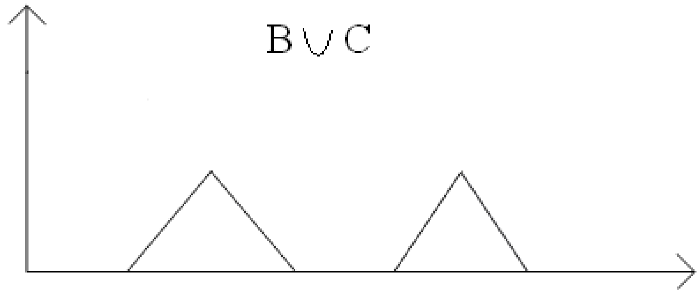
Symmetric difference $A \triangle B = (A-B) \cup (B-A)$

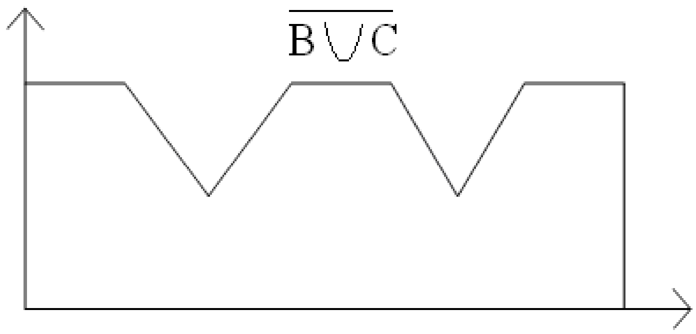
○ Example:



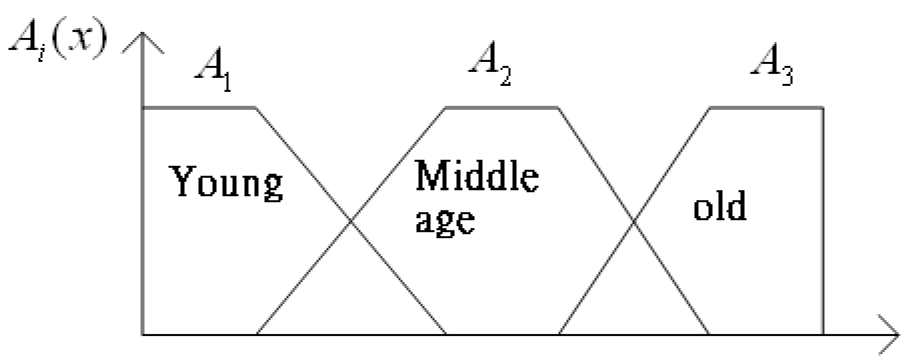


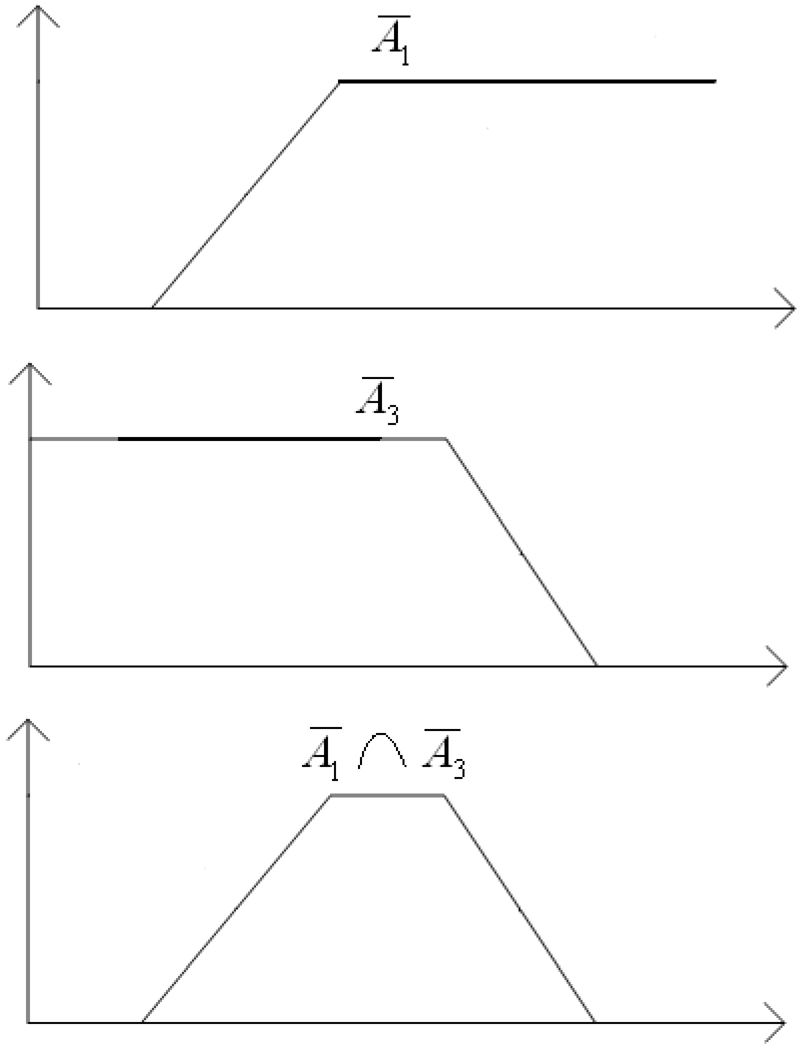






○ Example: $\overline{A_1} \cap \overline{A_3}$?





$\overline{A_1} \cap \overline{A_3}$: not young and not old

A_2 : middle age

○ Any fuzzy power set $P(X)$ with $\subseteq$ form a lattice, referred to as a De Morgan lattice

(De Morgan algebra)

In such a lattice ,

$\forall A, B \in P(X)$, $\exists$

join : $A \cup B$ (LUB, supremum)

meet : $A \cup B$ (GLB, infimum)

This lattice possesses all the properties (Table 1.1) of the Boolean lattice (or Boolean algebra) except the laws of contradiction ($A \cap \overline{A} = \Phi$) and exclusive middle ($A \cup \overline{A} = X$)

.Verify $A \cap \overline{A} = \Phi$ (law of contradiction) is violated for fuzzy sets,

i.e., Show $\exists x \ \min\{A(x), 1 - A(x)\} \neq 0$

e.g.,

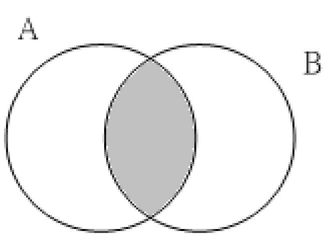
$$A(x) = 0.3 \quad \Rightarrow \quad 1 - A(x) = 0.7$$

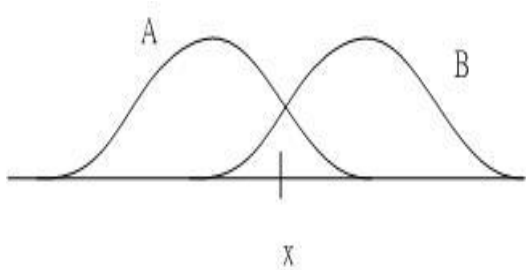
$$\min\{0.3,\ 0.7\} = 0.3 \neq 0$$

.Verify $A \cup (A \cap B) = A$ (law of absorption)

i.e., Show

$$\forall x \ \max\{A(x), \min\{A(x), B(x)\}\} = A(x)$$





$\forall x$

i, if $A(x) \le B(x)$,

$\Rightarrow \min[A(x)\ , B(x)] = A(x)$ and

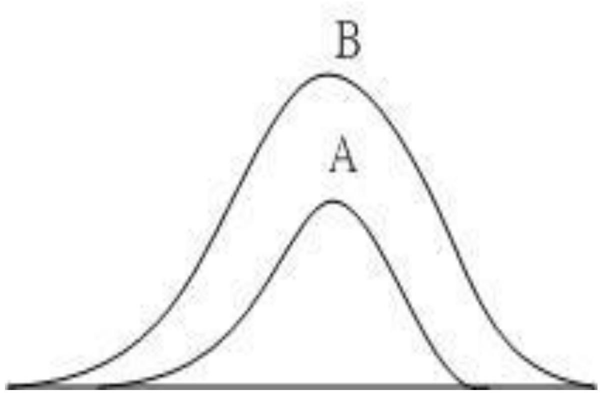
$\max[A(x), B(x)] = A(x)$

ii, if $A(x) > B(x)$,

$\Rightarrow \min[A(x)\ , B(x)] = B(x)$ and

$\max[A(x)\ , B(x)] = A(x)$

◎ Fuzzy set inclusion (subset) $\subseteq$



$A \subseteq B$ iff $\forall x$, $A(x) \le B(x)$

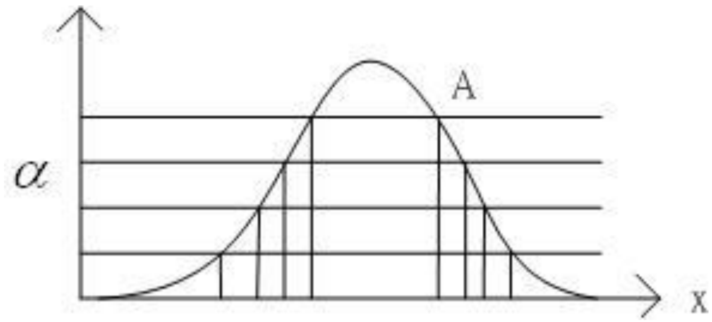
$$\Leftrightarrow \quad A \cap B = A,\ A \cup B = B$$

○ Description of fuzzy sets with finite supports

i, Finite universersal set X (discrete case)

$$A = \frac{a_1}{x_1} + \frac{a_2}{x_2} + ... + \frac{a_n}{x_n}$$

$$\text{or} \quad A = \sum_{x_i \in Supp(X)} \frac{a_i}{x_i}\ , \ a_i = A(x_i)$$

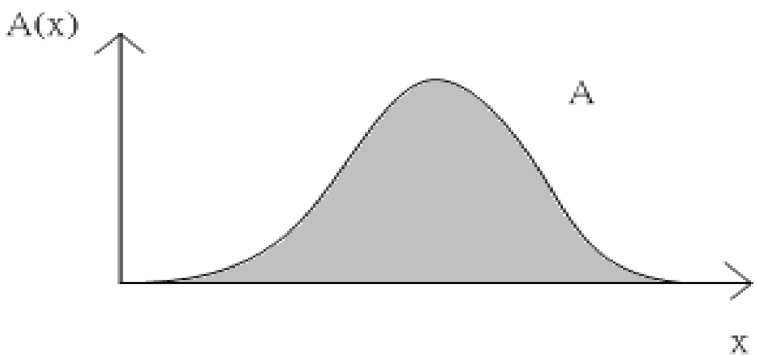


ii, X is an interval of real numbers (continuous case)

$$A = \int_X \frac{A(x)}{x}$$

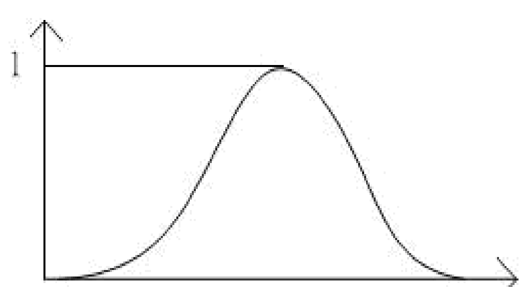
◎ Scalar cardinality (or sigma count) $|A|$

$$|A| = \sum_{x \in X} A(x)$$



◎ Fuzzy cardinality

.Fuzzy number: convex, normalized fuzzy set



.Fuzzy cardinality $|A|$

--- a fuzzy number define on N whose

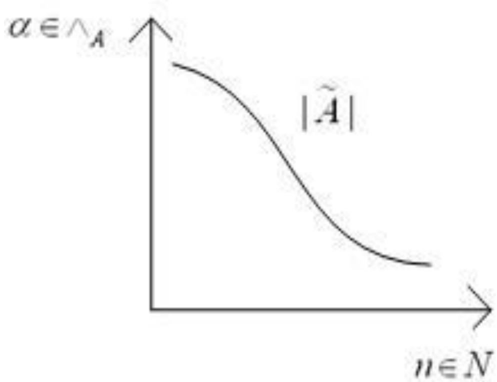
membership function is

$$\forall \alpha \in \wedge_A \quad |A|(|{}^{\alpha}A|) = \alpha$$

or $|A| = \sum_{\alpha \in \wedge_A} \frac{\alpha}{|{}^{\alpha}A|}$

$\frac{\alpha}{|{}^{\alpha}A|}$: the degree to which fuzzy set A contain the number of members ,

$|{}^{\alpha}A|$, is α



○ Example

X : crisp universal set

X = {5 , 10 , 20 , 30 , 40 , 50 , 60 , 70 , 80}

Fuzzy sets labeled as

"infant" , "adult" , "young" , "old"

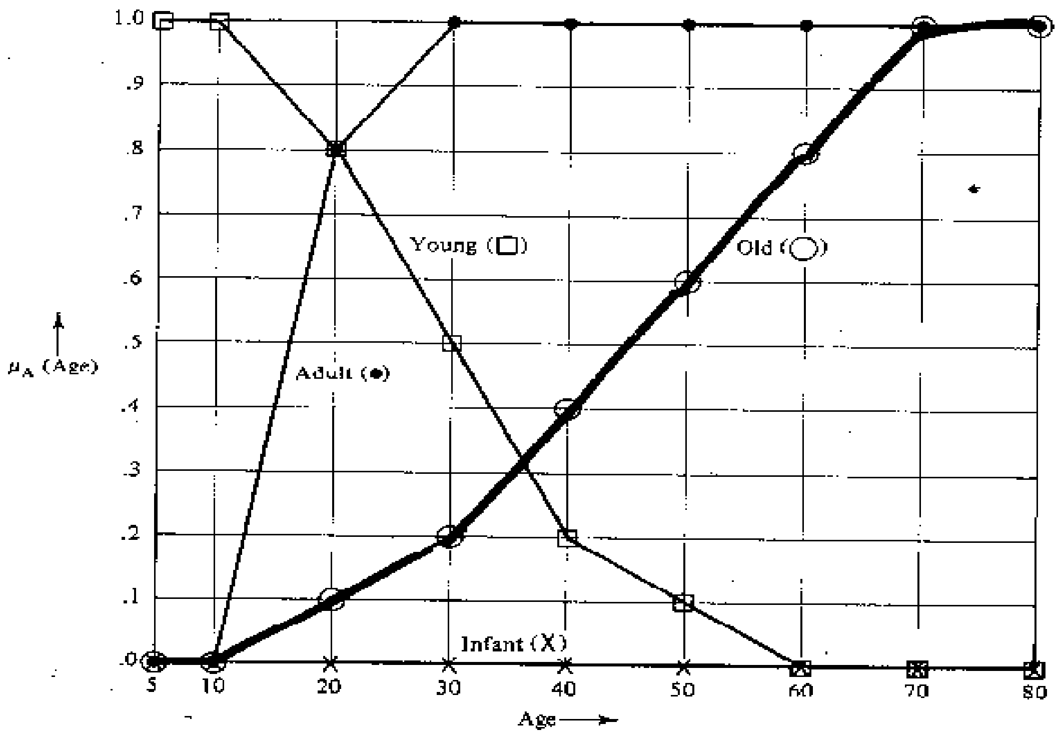


Figure 1.4. Examples of fuzzy sets defined in Table 1.2 ($A \in$ {*infant, young, adult, old*}).

Elements (ages)	Infant	Adult	Young	Old
5	0	0	1	0
10	0	0	1	0
20	0	.8	.8	.1
30	0	1	5	.2
40	0	1	.2	.4
50	0	1	.1	.6
60	0	1	0	.8
70	0	1	0	1
80	0	1	0	1

Consider Fuzzy set labeled "old"

$\Rightarrow$ Scalar cardinaliity:

$$|\text{old}| = 0 + 0 + 0.1 + 0.2 + 0.4$$
$$+ 0.6 + 0.8 + 1 + 1 = 4.1$$

$\Rightarrow$ Fuzzy cardinality:

$$\because \wedge_{\text{old}} = \{0, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$$

when

$$\alpha = 0.1, \quad {}^{0.1}old = \{20, 30, 40, 50, 60, 70, 80\}$$
$$\therefore |{}^{0.1}\text{old}| = 7$$

$$\alpha = 0.2, \quad {}^{0.2}old = \{30, 40, 50, 60, 70, 80\}$$
$$\therefore |{}^{0.2}\text{old}| = 6$$

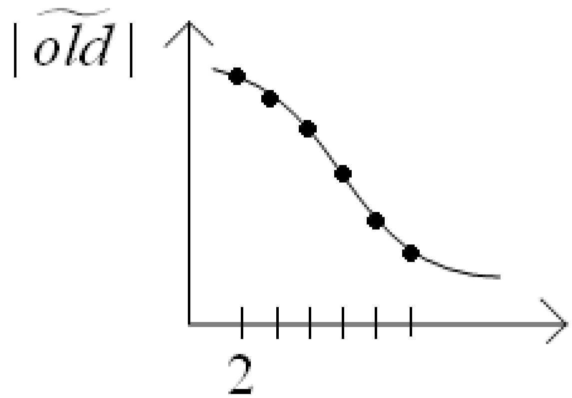
$$\alpha = 0.4, \quad {}^{0.4}old = \{40, 50, 60, 70, 80\}$$
$$\therefore |{}^{0.4}\text{old}| = 5$$

$$\alpha = 0.6, \quad {}^{0.6}old = \{, 50, 60, 70, 80\}$$
$$\therefore |{}^{0.6}\text{old}| = 4$$

$$\alpha = 0.8, \quad {}^{0.8}old = \{60, 70, 80\}$$
$$\therefore |{}^{0.8}\text{old}| = 3$$

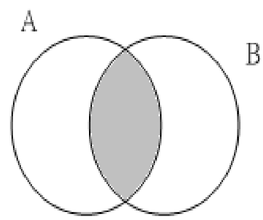
$$\alpha = 1, \quad {}^{1}old = \{70, 80\}$$
$$\therefore |{}^{1}\text{old}| = 2$$

$$|\,old\,| = \frac{0.1}{7} + \frac{0.2}{6} + \frac{0.4}{5} + \frac{0.6}{4} + \frac{0.8}{3} + \frac{1}{2}$$



◎ Degree of subsethood , *S*(*A*,*B*) , of *A* in *B*

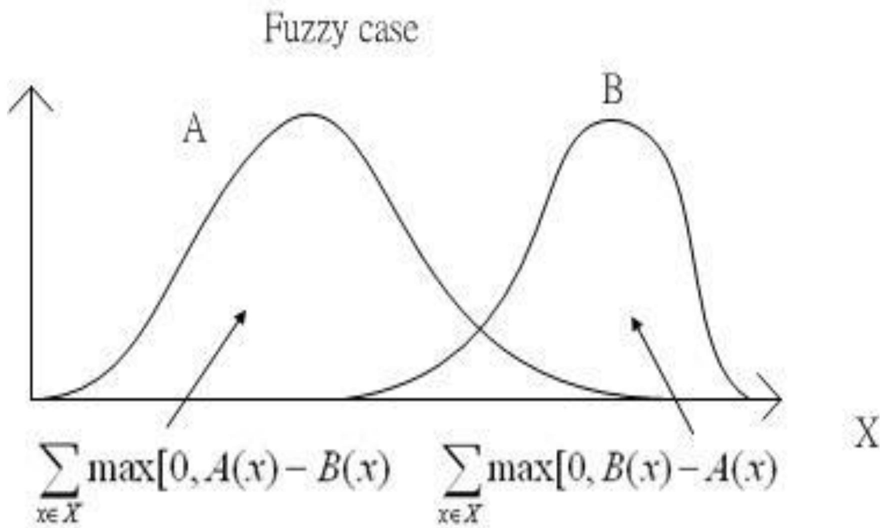
$$S(A,B)=\frac{|A\cap B|}{|A|}$$



$$S(A,B)=\frac{1}{|A|}(|A|-\sum_{x\in X}\max\{0,A(x)-B(x)\})$$

$$=\frac{1}{|A|}(|B|-\sum_{x\in X}\max\{0,B(x)-A(x)\})$$

$$=\frac{1}{|A|}(\sum_{x\in X}\min\{A(x),B(x)\})$$



◎ Distances between fuzzy sets

X ∶ universal set containing n elements

A, *B* ∶ fuzzy sets defined on *X*

$$A = \frac{a_1}{x_1} + \frac{a_1}{x_2} + \cdots\cdots + \frac{a_n}{x_n}$$

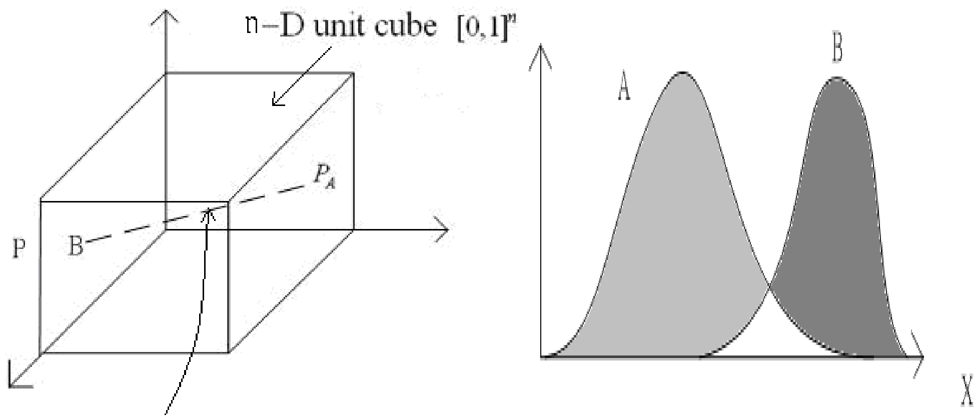
$$B = \frac{b_1}{x_1} + \frac{b_2}{x_2} + \cdots\cdots + \frac{b_n}{x_n}$$

$$0 \le a_i , b_i \le 1$$

From $A \Rightarrow P_A = (a_1, a_2, \cdots\cdots, a_n)$

From $B \Rightarrow P_B = (b_1, b_2, \cdots\cdots, b_n)$

In an n-D space ,



$$d(A,B)=\sum_{x\in X}|A(x)-B(x)|$$

$$\therefore d(A,B)=d(B,A)$$

The *n*-cube represents the fuzzy power set $\Im(X)$

The vertices represents the crisp power set *P*(*X*)

※ Scalar cardinality |*A*| = *d*(*A*,*Φ*):

Probability distributions are represented by sets whose cardinality is 1 $(\because \sum P_i = 1)$ the set of all probability distributions is represented by a (*n*-1)-*D* simplex of the *n*-cube $(\because \sum P_i = 1)$

Representations of fuzzy sets

◎ Representations of fuzzy sets by crisp sets (decomposition)

e.g. $A=\frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1.0}{x_5}$

This can be represented by itsα-cut

α-cuts

$$^{0.2}A = \{x_1, x_2, x_3, x_4, x_5\}$$
$$^{0.4}A = \{x_2, x_3, x_4, x_5\}$$
$$^{0.6}A = \{x_3, x_4, x_5\}$$
$$^{0.8}A = \{x_4, x_5\}$$
$$^{1.0}A = \{x_5\}$$

Define a fuzzy set ${}_{\alpha}A$ for eachα-cut as

$${}_{\alpha}A = \sum_{x \in {}^{\alpha}A} \frac{\alpha}{x} \quad \text{fuzzy}\alpha\text{-cut}$$

$${}_{0.2}A = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5}$$
$${}_{0.4}A = \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5}$$
$${}_{0.6}A = \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$$
$${}_{0.8}A = \frac{0.8}{x_4} + \frac{0.8}{x_5}$$
$${}_{1.0}A = \frac{1.0}{x_5}$$
$$\because\ A = \bigcup_{\alpha \subset \Lambda} {}_{\alpha}A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1.0}{x_5}$$

◎ Decomposition theorems of fuzzy sets

● **Theorem 2.5** (First decomposition Theorem)

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A, \ where \ {}_{\alpha}A = \sum_{x \in {}^{\alpha}A} \frac{\alpha}{x}$$

$$\text{proof: } \forall x \in X,\ \text{Let } A(x) = a$$

$$\Rightarrow (\bigcup_{\alpha \in [0,1]} {}_{\alpha}A)(x) = \sup_{\alpha \in [0,1]} {}_{\alpha}A(x)$$

$$\Rightarrow \max[\sup_{\alpha \in [0,a]} {}_{\alpha}A(x), \sup_{\alpha \in [a,1]} {}_{\alpha}A(x)]$$

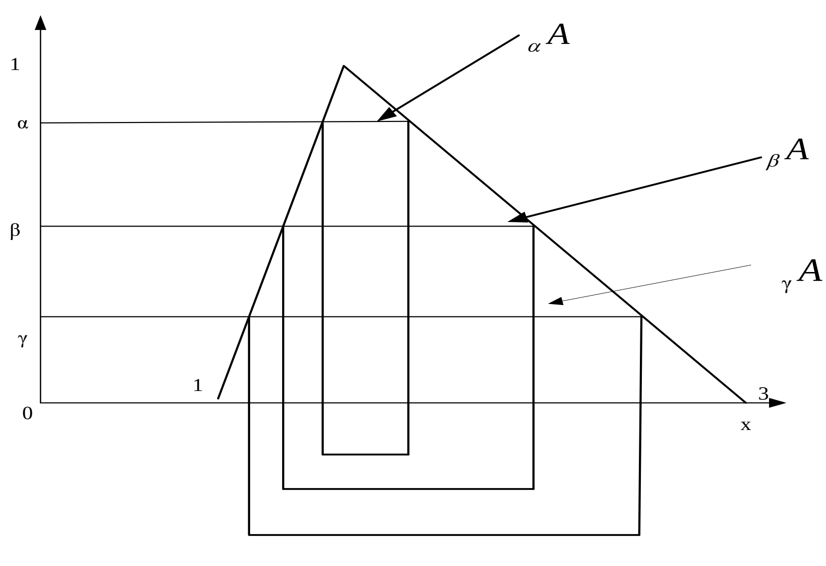
$$\begin{pmatrix} \forall \alpha \in (a,1], A(x) = a < 2, \therefore x \notin {}^{\alpha}A \Rightarrow {}_{\alpha}A(x) = 0 \\ \forall \alpha \in (0,a], A(x) = a \geq 2, \therefore x \in {}^{\alpha}A \Rightarrow {}_{\alpha}A(x) = \alpha \end{pmatrix}$$

$$\Rightarrow \max[\sup_{\alpha \in [0,a]} \alpha, 0] = \max[a, 0] = a$$

$$\therefore \bigcup_{\alpha \in [0,1]} {}_{\alpha}A = A$$

Example :

A: a fuzzy set with membership function



$$A(x)=\begin{cases} x-1 & x\in[1,2] \\ 3-x & x\in[2,3] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \forall\alpha\in(0,1],$$

$$\alpha-cut\ \ {}_{\alpha}A=\begin{cases} 2 & x\in[\alpha+1,3-\alpha] \\ 0 & \text{otherwise} \end{cases}$$

according to theorem 2.5

$$A=\bigcup_{\alpha\in[0,1]} {}_{\alpha}A$$

。**Theorem 2.6** (Second decomposition Theorem)

$$A=\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A,\quad {}_{\alpha+}A=\sum_{x\in{}^{\alpha+}A}\frac{\alpha}{x}$$

proof: $\forall x\in X$, Let $A(x)=a$

$$\Rightarrow (\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A)(x)=\sup_{\alpha\in[0,1]} {}_{\alpha+}A(x)$$

$$\Rightarrow \max[\sup_{\alpha\in[0,a]} {}_{\alpha+}A(x),\ \sup_{\alpha\in[a,1]} {}_{\alpha+}A(x)]$$

$$\sup_{\alpha\in[0,a]}\alpha=a=A(x)$$

∘ **Theorem 2.7** (Third decomposition Theorem)

$$A = \bigcup_{\alpha \subset \Lambda} {}_{\alpha}A, \quad \Lambda(A): \text{level set}$$

Extension Principle for Fuzzy Sets

--- a principle for fuzzifying crisp functions

concerning sets to power sets

⊙ Crisp case:

a crisp function-

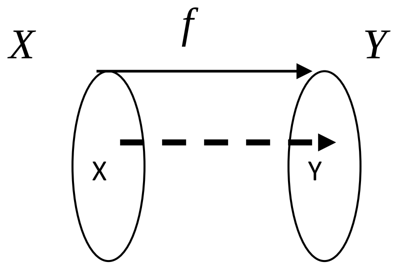
f: X --> Y , X,Y : crisp sets defined on univesal sets U,V

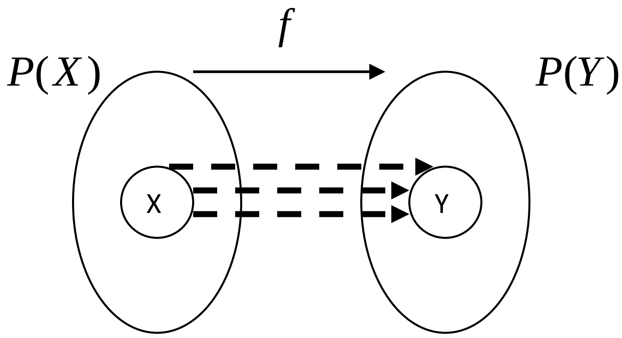
an extension

f : P(X) → P(Y)

P(X),P(Y): Crisp power set of X,Y

f^{-1} : P(Y) → P(X)



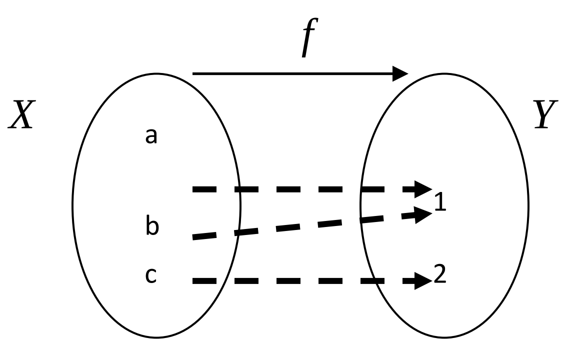


$$Let\ A \in P(X) \Rightarrow B = f(A) = \{y \mid y = f(x), x \in A\}$$

$$Let\ \mathrm{B} \in P(Y) \Rightarrow A = f^{-1}(B) = \{x \mid f(x) \in B\}$$

Example:

X={a,b,c} , Y={1,2}

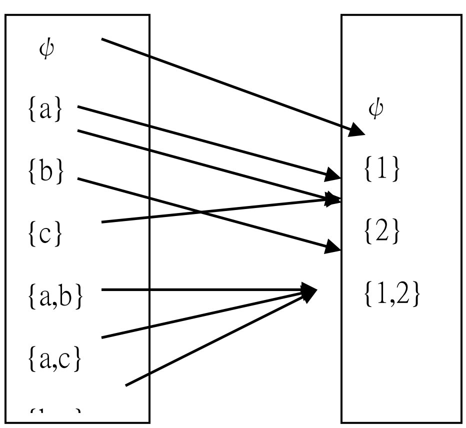


Extension f: p(X) ——→ p(Y)

Where

$$p(X) = \{\Phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

$$p(y) = \{\Phi, \{1\}, \{2\}, \{1,2\}\}$$



F(A)={y | y=f(x), $x \in A$}

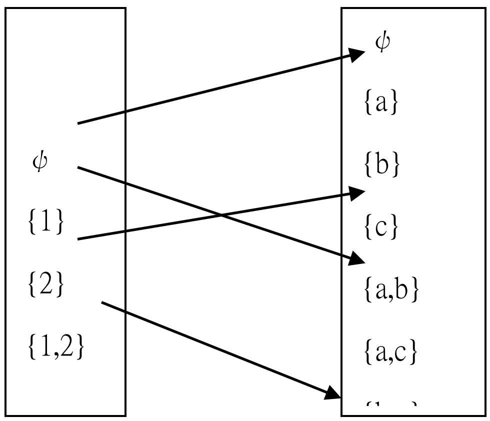
e.g

A={a,c}

⇨ $\mathrm{f(A)} = \mathrm{f}(\{\mathrm{a,c}\}) = \{1,2\}$

A={a,b}

=> $\mathrm{F(A)} = \mathrm{f}(\{\mathrm{a,b}\}) = \{1\}$



$$F^{-1}(B) = \{x \mid f(x) \in B\}$$

e.g

B={1}

$\Rightarrow$ $f^{-1}(A) = f^{-1}(\{1\}) = \{a,b\}$

B={1,2}

=> $f^{-1}(B) = f^{-1}(\{1,2\}) = \{a,b,c\}$

Fuzzy case:

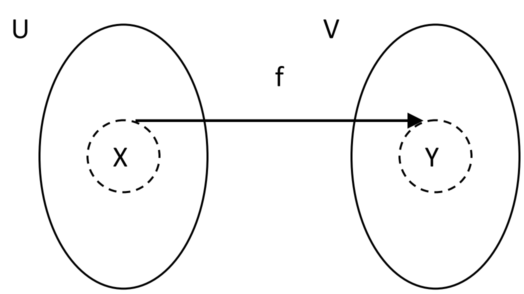
Given a fuzzy function f: X => Y

X,Y: fuzzy sets defined on crisp universal sets U,V

An extension

$$f : f(X) -> F(Y)$$

$$f^{-1} : F(Y) -> F(X)$$



F(X),F(Y): Fuzzy power sets of X,Y

$$\forall a \in F(X),\ Let\ B = f(A) \in f(Y)$$

The membership function of fuzzy set B

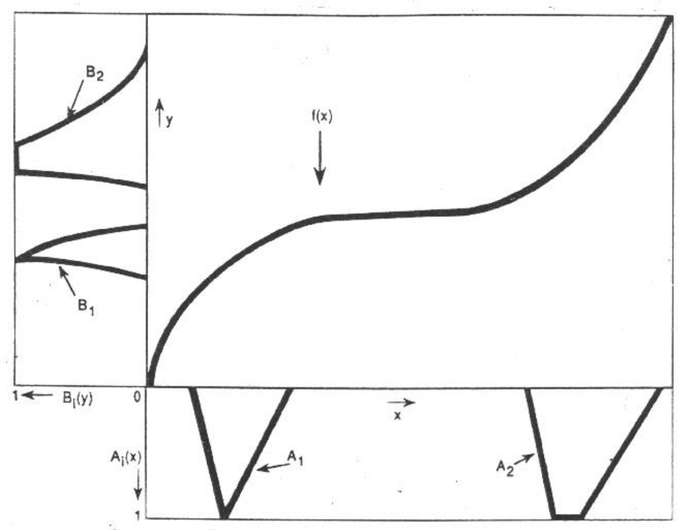
$$B(Y) = [f(A)](y) = \sup_{x \mid f(x)=y} A(X)$$

The membership function of fuzzy set A

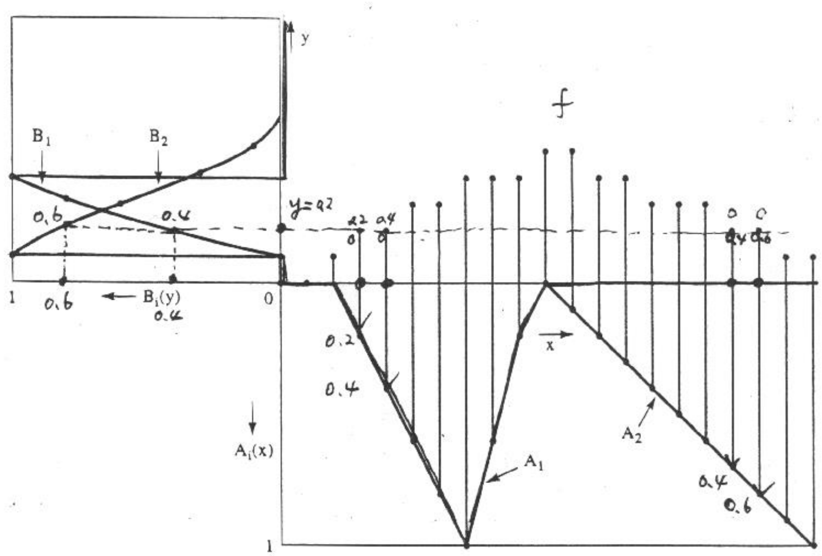
$$A(x) = [f^{-1}(B)](x) = B(\mathrm{f(x)})$$

Example : Function Extension

(a) Continuous case



(b) Discrete case



$$B_1(y) = [f(A_1)](y) = \sup_{x|y=f(x)} A_1(x) = \max[0.2, 0.4, 0, 0] = 0.4$$

$$B_2(y) = [f(A_2)](y) = \sup_{x|y=f(x)} A_2(x) = \max[0, 0, 0.4, 0.6] = 0.6$$

$f : X_1 \times X_2 \times \ldots \times X_n \quad -> Y$ where

X_1, X_2, X_n : crisp set

Let fuzzy set $A_1, A_2 \ldots\ldots\ldots A_n$ defined on

$X_1, X_2, \ldots X_n$ respectively

if $f\,(x_1^k, x_2^k, \ldots) = y \quad k = 1\ldots m$

$$\frac{\mu}{y} = \frac{\sup_k \min\{A_1\,(x_1^k), A2(x_2^{k)}), \ldots, A_n(x_n^k)\}}{y}$$

- Example: Fuzzy Mapping (Multivariants)

$X_1 = \{a, b, c\}$, $X_2 = \{x, y\}$, $Y = \{p, q, r\}$,

$f : X_1 \times X_2 \rightarrow Y$

Where

$$f : \begin{array}{c|cc} & x & y \\ \hline a & (p) & (p) \\ b & q & r \\ c & r & (p) \end{array}$$

Let A_1, A_2 ; Fuzzy sets defined on X_1, X_2

$$A_1 = \frac{0.3}{a} + \frac{0.9}{b} + \frac{0.5}{c} \quad A_2 = \frac{0.5}{x} + \frac{1.0}{y} \quad F(Y)$$

Let $B = f(A_1, A_2) \in F(Y)$

(*a*,*x*) (*a*,*y*) (*c*,*y*)

B(*p*) = *max*{*min*{0.3,0.5},*min*{0.3,0.5},*min*{0.3,0.5}}

= *max*{0.3,0.3,0.5} = 0.5

B(*q*) = *max*{*min*{0.9,0.5} = 0.5

B(*r*) = *max*{*min*{0.9,1}, *min*{0.5,0.5}}

= *max*{0.9,0.5} = 0.9

$$B = f(A_1, A_2) = \frac{0.5}{p} + \frac{0.5}{q} + \frac{0.9}{r}$$

FUZZY COMPLEMENTS

Let A be a fuzzy set on X. Then, by definition, $A(x)$ is interpreted as *the degree to which x belongs to A*. Let cA denote a fuzzy *complement* of A of type c. Then, $cA(x)$ may be interpreted not only as the degree to which x belongs to cA, but also as *the degree to which x does not belong to A*. Similarly, $A(x)$ may also be interpreted as the degree to which x does not belong to cA.

As a notational convention, let a complement cA be defined by a function

$$c : [0, 1] \rightarrow [0, 1],$$

which assigns a value $c(A(x))$ to each membership grade $A(x)$ of any given fuzzy set A. The value $c(A(x))$ is interpreted as the value of $cA(x)$. That is,

$$c(A(x)) = cA(x)$$

for all $x \in X$ by definition. Given a fuzzy set A, we obtain cA by applying function c to values $A(x)$ for all $x \in X$.

To produce meaningful fuzzy complements, function c must satisfy at least the following two axiomatic requirements:

Axiom c1. $c(0) = 1$ and $c(1) = 0$ (*boundary conditions*).

Axiom c2. For all $a, b \in [0, 1]$, if $a \leq b$, then $c(a) \geq c(b)$ (*monotonicity*).

Axiom c3. c is a continuous function.

Axiom c4. c is *involutive*, which means that $c(c(a)) = a$ for each $a \in [0, 1]$.

Theorem 3.1. Let a function $c : [0, 1] \rightarrow [0, 1]$ satisfy Axioms c2 and c4. Then, c also satisfies Axioms c1 and c3. Moreover, c must be a bijective function.

Proof:

(i) Since the range of c is $[0, 1]$, $c(0) \leq 1$ and $c(1) \geq 0$. By Axiom c2, $c(c(0)) \geq c(1)$; and, by Axiom c4, $0 = c(c(0)) \geq c(1)$. Hence, $c(1) = 0$. Now, again by Axiom c4, we have $c(0) = c(c(1)) = 1$. That is, function c satisfies Axiom c1.

(ii) To prove that c is a bijective function, we observe that for all $a \in [0, 1]$ there exists $b = c(a) \in [0, 1]$ such that $c(b) = c(c(a)) = a$. Hence, c is an onto function. Assume now that $c(a_1) = c(a_2)$; then, by Axiom c4,

$$a_1 = c(c(a_1)) = c(c(a_2)) = a_2.$$

That is, c is also a one-to-one function; consequently, it is a bijective function.

(iii) Since c is bijective and satisfies Axiom c2, it cannot have any discontinuous points. To show this, assume that c has a discontinuity at a_0, as illustrated in Fig. 3.1. Then, we have

$$b_0 = \lim_{a \rightarrow a_0-} c(a) > c(a_0)$$

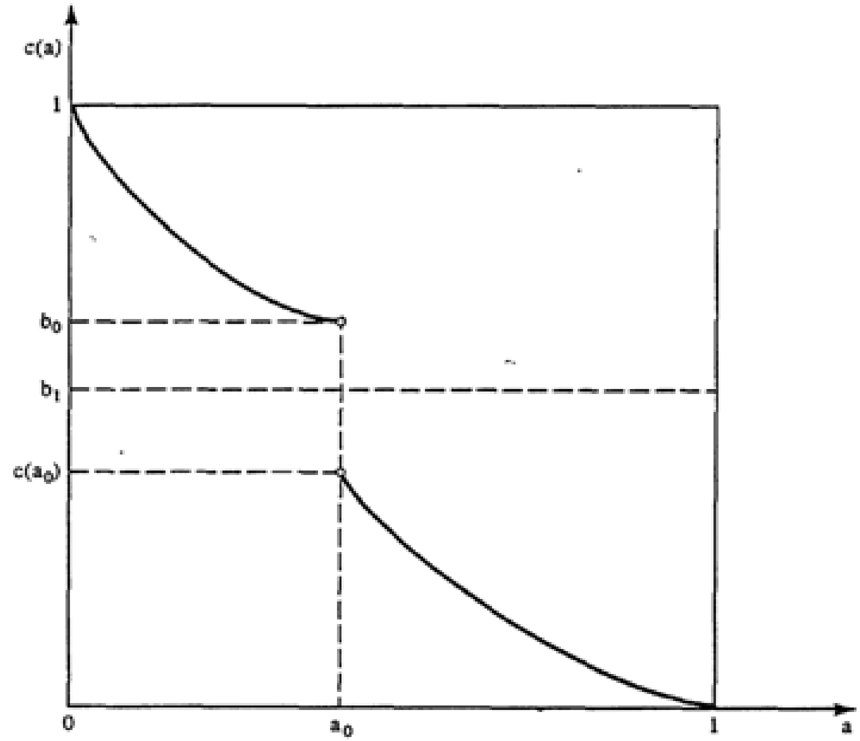


Figure 3.1 Illustration to Theorem 3.1.

and, clearly, there must exist $b_1 \in [0, 1]$ such that $b_0 > b_1 > c(a_0)$ for which no $a_1 \in [0, 1]$ exists such that $c(a_1) = b_1$. This contradicts the fact that c is a bijective function. ■

Theorem 3.2. Every fuzzy complement has at most one equilibrium.

Proof: Let c be an arbitrary fuzzy complement. An equilibrium of c is a solution of the equation

$$c(a) - a = 0,$$

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where b is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that a_1 and a_2 are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2.$$

However, because c is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts $c(a_1) - a_1 = c(a_2) - a_2$.

thus demonstrating that the equation must have at most one

Theorem 3.3. Assume that a given fuzzy complement c has an equilibrium e_c, which by Theorem 3.2 is unique. Then

$$a \leq c(a) \text{ iff } a \leq e_c$$

and

$$a \geq c(a) \text{ iff } a \geq e_c.$$

Proof: Let us assume that $a < e_c, a = e_c$, and $a > e_c$, in turn. Then, since c is monotonic nonincreasing by Axiom c2, $c(a) \geq c(e_c)$ for $a < e_c, c(a) = c(e_c)$ for $a = e_c$, and $c(a) \leq c(e_c)$ for $a > e_c$. Because $c(e_c) = e_c$, we can rewrite these expressions as $c(a) \geq e_c, c(a) = e_c$, and $c(a) \leq e_c$, respectively. In fact, due to our initial assumption we

can further rewrite these as $c(a) > a, c(a) = a$, and $c(a) < a$, respectively. Thus, $a \leq e_c$ implies $c(a) \geq a$ and $a \geq e_c$ implies $c(a) \leq a$. The inverse implications can be shown in a similar manner. ■

Theorem 3.4. If c is a continuous fuzzy complement, then c has a unique equilibrium.

Proof: The equilibrium e_c of a fuzzy complement c is the solution of the equation $c(a) - a = 0$. This is a special case of the more general equation $c(a) - a = b$, where $b \in [-1, 1]$ is a constant. By Axiom c1, $c(0) - 0 = 1$ and $c(1) - 1 = -1$. Since c is a continuous complement, it follows from the intermediate value theorem for continuous functions that for each $b \in [-1, 1]$, there exists at least one a such that $c(a) - a = b$. This demonstrates the necessary existence of an equilibrium value for a continuous function, and Theorem 3.2 guarantees its uniqueness. ■

If we are given a fuzzy complement c and a membership grade whose value is represented by a real number $a \in [0, 1]$, then any membership grade represented by the real number ${}^d a \in [0, 1]$ such that

$$c({}^d a) - {}^d a = a - c(a) \tag{3.8}$$

is called a *dual point* of a with respect to c.

It follows directly from the proof of Theorem 3.2 that (3.8) has at most one solution for ${}^d a$ given c and a. There is, therefore, at most one dual point for each particular fuzzy complement c and membership grade of value a. Moreover, it follows from the proof of Theorem 3.4 that a dual point exists for each $a \in [0, 1]$ when c is a continuous complement.

Theorem 3.5. If a complement c has an equilibrium e_c, then

$${}^d e_c = e_c.$$

Proof: If $a = e_c$, then by our definition of equilibrium, $c(a) = a$ and thus $a - c(a) = 0$. Additionally, if ${}^d a = e_c$, then $c({}^d a) = {}^d a$ and $c({}^d a) - {}^d a = 0$. Therefore,

$$c({}^d a) - {}^d a = a - c(a).$$

This satisfies (3.8) when $a = {}^d a = e_c$. Hence, the equilibrium of any complement is its own dual point. ∎

Theorem 3.6. For each $a \in [0, 1]$, ${}^d a = c(a)$ iff $c(c(a)) = a$, that is, when the complement is involutive.

Proof: Let ${}^d a = c(a)$. Then, substitution of $c(a)$ for ${}^d a$ in (3.8) produces

$$c(c(a)) - c(a) = a - c(a).$$

Therefore, $c(c(a)) = a$. For the reverse implication, let $c(c(a)) = a$. Then substitution of $c(c(a))$ for a in (3.8) yields the functional equation

$$c({}^d a) - {}^d a = c(c(a)) - c(a).$$

for ${}^d a$ whose solution is ${}^d a = c(a)$. ∎

Theorem 3.7 (First Characterization Theorem of Fuzzy Complements). Let c be a function from $[0, 1]$ to $[0, 1]$. Then, c is a fuzzy complement (involutive) iff there exists a

continuous function g from $[0, 1]$ to $\mathbb{R}$ such that $g(0) = 0$, g is strictly increasing, and

$$c(a) = g^{-1}(g(1) - g(a)) \tag{3.9}$$

for all $a \in [0, 1]$.

(i) First, we prove the inverse implication $\Leftarrow$. Let g be a continuous function from $[0, 1]$ to $\mathbb{R}$ such that $g(0) = 0$ and g is strictly increasing. Then the pseudoinverse of g, denoted by $g^{(-1)}$, is a function from $\mathbb{R}$ to $[0,1]$ defined by

$$g^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ g^{-1}(a) & \text{for } a \in [0, g(1)] \\ 1 & \text{for } a \in (g(1), \infty), \end{cases}$$

where g^{-1} is the ordinary inverse of g.

Let c be a function on $[0, 1]$ defined by (3.9). We now prove that c is a fuzzy complement. First, we show that c satisfies Axiom c2. For any $a, b \in [0, 1]$, if $a < b$, then $g(a) < g(b)$, since g is strictly increasing. Hence, $g(1) - g(a) > g(1) - g(b)$ and, consequently, $c(a) = g^{-1}[g(1) - g(a)] > g^{-1}[g(1) - g(b)] > c(b)$. Therefore, c satisfies Axiom c2. Second, we show that c is involutive. For any $a \in [0, 1]$, $c(c(a)) = g^{-1}[g(1) - g(c(a))] = g^{-1}[g(1) - g(g^{-1}(g(1) - g(a)))] = g^{-1}[g(1) - g(1) + g(a)] = g^{-1}(g(a)) = a$. Thus, c is involutive (i.e., c satisfies Axiom c4).

It follows from Theorem 3.1 that c also satisfies Axiom c2 and c3. Therefore, c is a fuzzy complement.

(ii) Now, we prove the direct implication $\Rightarrow$. Let c be a fuzzy complement satisfying Axioms c1–c4. We need to find a continuous, strictly increasing function g that satisfies (3.9) and $g(0) = 0$.

It follows from Theorem 3.4 that c must have a unique equilibrium, let us say e_c; that is, $c(e_c) = e_c$, where $e_c \in (0, 1)$. Let $h : [0, e_c] \to [0, b]$ be any continuous, strictly increasing bijection such that $h(0) = 0$ and $h(e_c) = b$, where b is any fixed positive real number. For example, function $h(a) = ba/e_c$ is one instance of this kind of function. Now we define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(a) = \begin{cases} h(a) & a \in [0, e_c] \\ 2b - h(c(a)) & a \in (e_c, 1]. \end{cases}$$

Obviously, $g(0) = h(0) = 0$ and g is continuous as well as strictly increasing since h is continuous and strictly increasing. It is easy to show that the pseudoinverse of g is given by

$$g^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ h^{-1}(a) & \text{for } a \in [0, b] \\ c(h^{-1}(2b - a)) & \text{for } a \in [b, 2b] \\ 1 & \text{for } a \in (2b, \infty]. \end{cases}$$

Now, we show that g satisfies (3.9). For any $a \in [0, 1]$, if $a \in [0, e_c]$, then $g^{-1}[g(1) - g(a)] = g^{-1}[g(1) - h(a)] = g^{-1}[2b - h(a)] = c(h^{-1}[2b - (2b - h(a))]) = c(a)$; if $a \in (e_c, 1]$, then $g^{-1}[g(1) - g(a)] = g^{-1}[2b - (2b - h(c(a)))] = g^{-1}[h(c(a))] = h^{-1}[h(c(a))] = c(a)$. Therefore, for any $a \in [0, 1]$, $c(a) = g^{-1}[g(1) - g(a)]$ (i.e., (3.9) holds). ■

Theorem 3.8 (Second Characterization Theorem of Fuzzy Complements). Let c be a function from $[0, 1]$ to $[0, 1]$. Then c is a fuzzy complement iff there exists a continuous function f from $[0, 1]$ to $\mathbb{R}$ such that $f(1) = 0$, f is strictly decreasing, and

$$c(a) = f^{-1}(f(0) - f(a)) \tag{3.15}$$

for all $a \in [0, 1]$.

Proof: According to Theorem 3.7, function c is a fuzzy complement iff there exists an increasing generator g such that $c(a) = g^{-1}(g(1) - g(a))$. Now, let $f(a) = g(1) - g(a)$. Then, $f(1) = 0$ and, since g is strictly increasing, f is strictly decreasing. Moreover,

$$\begin{aligned} f^{-1}(a) &= g^{-1}(g(1) - a) \\ &= g^{-1}(f(0) - a) \end{aligned}$$

since $f(0) = g(1) - g(0) = g(1)$, $f(f^{-1}(a)) = g(1) - g(f^{-1}(a)) = g(1) - g(g^{-1}(g(1) - a)) = a$, and $f^{-1}(f(a)) = g^{-1}(g(1) - f(a)) = g^{-1}(g(1) - (g(1) - g(a))) = g^{-1}(g(a)) = a$. Now,

$$\begin{aligned} c(a) &= g^{-1}(g(1) - g(a)) \\ &= f^{-1}(g(a)) \\ &= f^{-1}(g(1) - (g(1) - g(a))) \\ &= f^{-1}(f(0) - f(a)). \end{aligned}$$

If a decreasing generator f is given, we can define an increasing generator g as

$$g(a) = f(0) - f(a).$$

Then, (3.15) can be rewritten as

$$\begin{aligned} c(a) &= f^{-1}(f(0) - f(a)) \\ &= g^{-1}(g(1) - g(a)). \end{aligned}$$

Hence, c defined by (3.15) is a fuzzy complement. ■

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

FUZZY ANALYSIS

UNIT – II – FUZZY ARITHMETIC – SMT5205

FUZZY NUMBERS

To qualify as a *fuzzy number*, a fuzzy set A on $\mathbb{R}$ must possess at least the following three properties:

(i) A must be a normal fuzzy set;
(ii) ${}^{\alpha}A$ must be a closed interval for every $\alpha \in (0, 1]$;
(iii) the support of A, ${}^{0+}A$, must be bounded.

Special cases of fuzzy numbers include ordinary real numbers and intervals of real numbers, as illustrated in Fig. 4.1: (a) an ordinary real number 1.3; (b) an ordinary (crisp) closed interval [1.25, 1.35]; (c) a fuzzy number expressing the proposition "close to 1.3;" and (d) a fuzzy number with a flat region (a fuzzy interval).

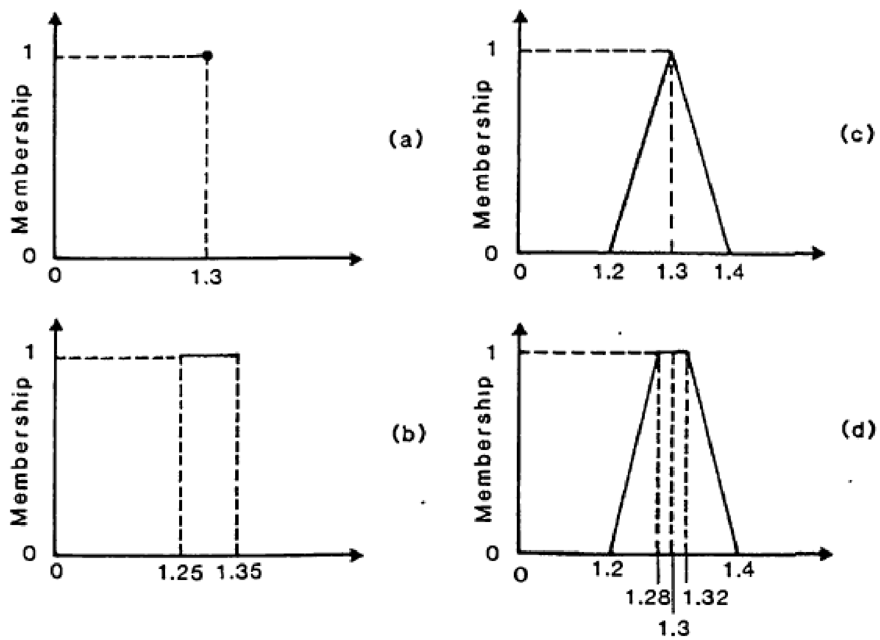


Figure 4.1 A comparison of a real number and a crisp interval with a fuzzy number and a fuzzy interval, respectively.

Although the triangular and trapezoidal shapes of membership functions shown in Fig. 4.1 are used most often for representing fuzzy numbers, other shapes may be preferable in some applications. Furthermore, membership functions of fuzzy numbers need not be symmetric as are those in Fig. 4.1. Fairly typical are so-called "bell-shaped" membership functions, as exemplified by the functions in Fig. 4.2a (symmetric) and 4.2b (asymmetric). Observe that membership functions which only increase (Fig. 4.2c) or only decrease (Fig. 4.2d) also qualify as fuzzy numbers. They capture our conception of a *large number* or a *small number* in the context of each particular application.

Theorem 4.1. Let $A \in \mathcal{F}(\mathbb{R})$. Then, A is a fuzzy number if and only if there exists a closed interval $[a, b] \neq \varnothing$ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ l(x) & \text{for } x \in (-\infty, a) \\ r(x) & \text{for } x \in (b, \infty), \end{cases} \tag{4.1}$$

where l is a function from $(-\infty, a)$ to $[0, 1]$ that is monotonic increasing, continuous from the right, and such that $l(x) = 0$ for $x \in (-\infty, \omega_1)$; r is a function from (b, ∞) to

$[0, 1]$ that is monotonic decreasing, continuous from the left, and such that $r(x) = 0$ for $x \in (\omega_2, \infty)$.

Proof: Necessity. Since A is a fuzzy number, ${}^{\alpha}A$ is a closed interval for every $\alpha \in (0, 1]$. For $\alpha = 1$, ${}^{1}A$ is a nonempty closed interval because A is normal. Hence, there exists a pair $a, b \in \mathbb{R}$ such that ${}^{1}A = [a, b]$, where $a \leq b$. That is, $A(x) = 1$ for $x \in [a, b]$ and $A(x) < 1$ for $x \notin [a, b]$. Now, let $l(x) = A(x)$ for any $x \in (-\infty, a)$. Then, $0 \leq l(x) < 1$ since $0 \leq A(x) < 1$ for every $x \in (-\infty, a)$. Let $x \leq y < a$; then

$$A(y) \geq \min[A(x), A(a)] = A(x)$$

by Theorem 1.1 since A is convex and $A(a) = 1$. Hence, $l(y) \geq l(x)$; that is, l is monotonic increasing.

Assume now that $l(x)$ is not continuous from the right. This means that for some $x_0 \in (-\infty, a)$ there exists a sequence of numbers $\{x_n\}$ such that $x_n \geq x_0$ for any n and

$$\lim_{n \to \infty} x_n = x_0,$$

but

$$\lim_{n \to \infty} l(x_n) = \lim_{n \to \infty} A(x_n) = \alpha > l(x_0) = A(x_0).$$

Now, $x_n \in {}^{\alpha}A$ for any n since ${}^{\alpha}A$ is a closed interval and hence, also $x_0 \in {}^{\alpha}A$. Therefore, $l(x_0) = A(x_0) \geq \alpha$, which is a contradiction. That is, $l(x)$ is continuous from the right.

The proof that function r in (4.1) is monotonic decreasing and continuous from the left is similar.

Since A is a fuzzy number, ${}^{0+}A$ is bounded. Hence, there exists a pair $\omega_1, \omega_2 \in \mathbb{R}$ of finite numbers such that $A(x) = 0$ for $x \in (-\infty, \omega_1) \cup (\omega_2, \infty)$.

Sufficiency. Every fuzzy set A defined by (4.1) is clearly normal, and its support, ${}^{0+}A$, is bounded, since ${}^{0+}A \subseteq [\omega_1, \omega_2]$. It remains to prove that ${}^{\alpha}A$ is a closed interval for any $\alpha \in (0, 1]$. Let

$$x_\alpha = \inf\{x | l(x) \geq \alpha, x < a\},$$

$$y_\alpha = \sup\{x | r(x) \geq \alpha, x > b\}$$

for each $\alpha \in (0, 1]$. We need to prove that ${}^{\alpha}A = [x_\alpha, y_\alpha]$ for all $\alpha \in (0, 1]$.

For any $x_0 \in {}^{\alpha}A$, if $x_0 < a$, then $l(x_0) = A(x_0) \geq \alpha$. That is, $x_0 \in \{x | l(x) \geq \alpha, x < a\}$ and, consequently, $x_0 \geq \inf\{x | l(x) \geq \alpha, x < a\} = x_\alpha$. If $x_0 > b$, then $r(x_0) = A(x_0) \geq \alpha$; that is, $x_0 \in \{x | r(x) \geq \alpha, x > b\}$ and, consequently, $x_0 \leq \sup\{x | r(x) \geq \alpha, x > b\} = y_\alpha$. Obviously, $x_\alpha \leq a$ and $y_\alpha \geq b$; that is, $[a, b] \subseteq [x_\alpha, y_\alpha]$. Therefore, $x_0 \in [x_\alpha, y_\alpha]$ and hence, ${}^{\alpha}A \subseteq [x_\alpha, y_\alpha]$. It remains to prove that $x_\alpha, y_\alpha \in {}^{\alpha}A$.

By the definition of x_α, there must exist a sequence $\{x_n\}$ in $\{x | l(x) \geq \alpha, x < a\}$ such that $\lim_{n\to\infty} x_n = x_\alpha$, where $x_n \geq x_\alpha$ for any n. Since l is continuous from the right, we have

$$l(x_\alpha) = l(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} l(x_n) \geq \alpha.$$

Hence, $x_\alpha \in {}^{\alpha}A$. We can prove that $y_\alpha \in {}^{\alpha}A$ in a similar way. ■

LINGUISTIC VARIABLES

The concept of a fuzzy number plays a fundamental role in formulating *quantitative fuzzy variables*. These are variables whose states are fuzzy numbers. When, in addition, the fuzzy numbers represent linguistic concepts, such as *very small, small, medium*, and so on, as interpreted in a particular context, the resulting constructs are usually called *linguistic variables*.

Each linguistic variable the states of which are expressed by linguistic terms interpreted as specific fuzzy numbers is defined in terms of a *base variable*, the values of which are real numbers within a specific range. A base variable is a variable in the classical sense, exemplified by any physical variable (e.g., temperature, pressure, speed, voltage, humidity, etc.) as well as any other numerical variable, (e.g., age, interest rate, performance, salary, blood count, probability, reliability, etc.). In a linguistic variable, linguistic terms representing approximate values of a base variable, germane to a particular application, are captured by appropriate fuzzy numbers.

Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the *name* of the variable, T is the set of *linguistic terms* of v that refer to a base variable whose values range over a universal set X, g is a *syntactic rule* (a grammar) for generating linguistic terms, and m is a *semantic rule* that assigns to each linguistic term $t \in T$ its *meaning*, $m(t)$, which is a fuzzy set on X (i.e., $m : T \to \mathcal{F}(X)$).

An example of a linguistic variable is shown in Fig. 4.4. Its name is performance. This variable expresses the performance (which is the base variable in this example) of a goal-oriented entity (a person, machine, organization, method, etc.) in a given context by five basic linguistic terms—*very small, small, medium, large, very large*—as well as other linguistic terms generated by a syntactic rule (not explicitly shown in Fig. 4.4), such as *not very small, large or very large, very very small*, and so forth. Each of the basic linguistic terms is assigned one of five fuzzy numbers by a semantic rule, as shown in the figure. The fuzzy numbers, whose membership functions have the usual trapezoidal shapes, are defined on the interval $[0, 100]$, the range of the base variable. Each of them expresses a fuzzy restriction on this range.

ARITHMETIC OPERATIONS ON INTERVALS

Fuzzy arithmetic is based on two properties of fuzzy numbers: (1) each fuzzy set, and thus also each fuzzy number, can fully and uniquely be represented by its α-cuts (2) α-cuts of each fuzzy number are closed intervals of real numbers for all $\alpha \in (0, 1]$. These properties enable us to define arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α-cuts (i.e., arithmetic operations on closed intervals).

Let $*$ denote any of the four arithmetic operations on closed intervals: *addition* $+$, *subtraction* $-$, *multiplication* $\cdot$, and *division* $/$. Then,

$$[a, b] * [d, e] = \{f * g | a \leq f \leq b, d \leq g \leq e\} \tag{4.2}$$

is a general property of all arithmetic operations on closed intervals, except that $[a, b]/[d, e]$ is not defined when $0 \in [d, e]$. That is, the result of an arithmetic operation on closed intervals is again a closed interval.

The four arithmetic operations on closed intervals are defined as follows:

$$[a, b] + [d, e] = [a + d, b + e], \tag{4.3}$$

$$[a, b] - [d, e] = [a - e, b - d], \tag{4.4}$$

$$[a, b] \cdot [d, e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)], \tag{4.5}$$

and, provided that $0 \notin [d, e]$,

$$\begin{aligned} [a, b]/[d, e] &= [a, b] \cdot [1/e, 1/d] \\ &= [\min(a/d, a/e, b/d, b/e), \max(a/d, a/e, b/d, b/e)]. \end{aligned} \tag{4.6}$$

The following are a few examples illustrating the interval-valued arithmetic operations defined by (4.3)–(4.6):

$$[2,5]+[1,3]=[3,8] \qquad [0,1]+[-6,5]=[-6,6],$$
$$[2,5]-[1,3]=[-1,4] \qquad [0,1]-[-6,5]=[-5,7],$$
$$[-1,1]\cdot[-2,-0.5]=[-2,2] \qquad [3,4]\cdot[2,2]=[6,8],$$
$$[-1,1]/[-2,-0.5]=[-2,2] \qquad [4,10]/[1,2]=[2,10].$$

Arithmetic operations on closed intervals satisfy some useful properties. To overview them, let $A=[a_1,a_2]$, $B=[b_1,b_2]$, $C=[c_1,c_2]$, $\mathbf{0}=[0,0]$, $\mathbf{1}=[1,1]$. Using these symbols, the properties are formulated as follows:

1. $A+B=B+A$,
   $A\cdot B=B\cdot A$ (*commutativity*).
2. $(A+B)+C=A+(B+C)$
   $(A\cdot B)\cdot C=A\cdot(B\cdot C)$ (*associativity*).
3. $A=\mathbf{0}+A=A+\mathbf{0}$
   $A=\mathbf{1}\cdot A=A\cdot\mathbf{1}$ (*identity*).
4. $A\cdot(B+C)\subseteq A\cdot B+A\cdot C$ (*subdistributivity*).
5. If $b\cdot c\geq 0$ for every $b\in B$ and $c\in C$, then $A\cdot(B+C)=A\cdot B+A\cdot C$ (*distributivity*). Furthermore, if $A=[a,a]$, then $a\cdot(B+C)=a\cdot B+a\cdot C$.
6. $0\in A-A$ and $1\in A/A$.
7. If $A\subseteq E$ and $B\subseteq F$, then:

$$A+B\subseteq E+F,$$
$$A-B\subseteq E-F,$$
$$A\cdot B\subseteq E\cdot F,$$
$$A/B\subseteq E/F \text{ (\textit{inclusion monotonicity})}.$$

Most of these properties follow directly from (4.3)–(4.6). As an example, we prove only the less obvious properties of subdistributivity and distributivity. First, we have

$$\begin{aligned} A\cdot(B+C) &= \{a\cdot(b+c)|a\in A, b\in B, c\in C\}\\ &= \{a\cdot b+a\cdot c|a\in A, b\in B, c\in C\}\\ &\subseteq \{a\cdot b+a'\cdot c|a,a'\in A, b\in B, c\in C\}\\ &= A\cdot B+A\cdot C. \end{aligned}$$

Hence, $A\cdot(B+C)\subseteq A\cdot B+A\cdot C$.

Assume now, without any loss of generality, that $b_1 \geq 0$ and $c_1 \geq 0$. Then, we have to consider the following three cases:

1. If $a_1 \geq 0$, then

$$\begin{aligned} A \cdot (B + C) &= [a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2)] \\ &= [a_1 \cdot b_1, a_2 \cdot b_2] + [a_1 \cdot c_1 + a_2 \cdot c_2] \\ &= A \cdot B + A \cdot C. \end{aligned}$$

2. If $a_1 < 0$ and $a_2 \leq 0$, then $-a_2 \geq 0$, $(-A) = [-a_2, -a_1]$, and

$$(-A) \cdot (B + C) = (-A) \cdot B + (-A) \cdot C.$$

   Hence, $A \cdot (B + C) = A \cdot B + A \cdot C$.

3. If $a_1 < 0$ and $a_2 > 0$, then

$$\begin{aligned} A \cdot (B + C) &= [a_1 \cdot (b_2 + c_2), a_2 \cdot (b_2 + c_2)] \\ &= [a_1 \cdot b_2, a_2 \cdot b_2] + [a_1 \cdot c_2, a_2 \cdot c_2] \\ &= A \cdot B + A \cdot C. \end{aligned}$$

To show that distributivity does not hold in general, let $A = [0, 1]$, $B = [1, 2]$, $C = [-2, -1]$. Then, $A \cdot B = [0, 2]$, $A \cdot C = [-2, 0]$, $B + C = [-1, 1]$, and

$$A \cdot (B + C) = [-1, 1] \subset [-2, 2] = A \cdot B + A \cdot C.$$

ARITHMETIC OPERATIONS ON FUZZY NUMBERS

Let A and B denote fuzzy numbers and let $*$ denote any of the four basic arithmetic operations. Then, we define a fuzzy set on $\mathbb{R}$, $A * B$, by defining its α-cut, ${}^\alpha(A * B)$, as

$$^\alpha(A * B) = {}^\alpha A * {}^\alpha B \tag{4.7}$$

for any $\alpha \in (0, 1]$. (When $* = /$, clearly, we have to require that $0 \notin {}^\alpha B$ for all $\alpha \in (0, 1]$.) Due to Theorem 2.5, $A * B$ can be expressed as

$$A * B = \bigcup_{\alpha \in [0,1]} {}_\alpha(A * B). \tag{4.8}$$

Since ${}^\alpha(A * B)$ is a closed interval for each $\alpha \in (0, 1]$ and A, B are fuzzy numbers, $A * B$ is also a fuzzy number.

As an example of employing (4.7) and (4.8), consider two triangular-shape fuzzy numbers A and B defined as follows:

$$A(x) = \begin{cases} 0 & \text{for } x \le -1 \text{ and } x > 3 \\ (x+1)/2 & \text{for } -1 < x \le 1 \\ (3-x)/2 & \text{for } 1 < x \le 3, \end{cases}$$

$$B(x) = \begin{cases} 0 & \text{for } x \le 1 \text{ and } x > 5 \\ (x-1)/2 & \text{for } 1 < x \le 3 \\ (5-x)/2 & \text{for } 3 < x \le 5. \end{cases}$$

Their α-cuts are:

$$\begin{aligned} {}^\alpha A &= [2\alpha - 1, 3 - 2\alpha], \\ {}^\alpha B &= [2\alpha + 1, 5 - 2\alpha]. \end{aligned}$$

Using (4.3)–(4.7), we obtain

$$\begin{aligned} {}^\alpha(A + B) &= [4\alpha, 8 - 4\alpha] \quad \text{for } \alpha \in (0, 1], \\ {}^\alpha(A - B) &= [4\alpha - 6, 2 - 4\alpha] \quad \text{for } \alpha \in (0, 1], \\ {}^\alpha(A \cdot B) &= \begin{cases} [-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15] & \text{for } \alpha \in (0, .5] \\ [4\alpha^2 - 1, 4\alpha^2 - 16\alpha + 15] & \text{for } \alpha \in (.5, 1], \end{cases} \\ {}^\alpha(A/B) &= \begin{cases} [(2\alpha - 1)/(2\alpha + 1), (3 - 2\alpha)/(2\alpha + 1)] & \text{for } \alpha \in (0, .5] \\ [(2\alpha - 1)/(5 - 2\alpha), (3 - 2\alpha)/(2\alpha + 1)] & \text{for } \alpha \in (.5, 1]. \end{cases} \end{aligned}$$

The resulting fuzzy numbers are then:

$$(A + B)(x) = \begin{cases} 0 & \text{for } x \le 0 \quad \text{and } x > 8 \\ x/4 & \text{for } 0 < x \le 4 \\ (8 - x)/4 & \text{for } 4 < x \le 8, \end{cases}$$

$$(A-B)(x) = \begin{cases} 0 & \text{for } x \le -6 \quad \text{and } x > 2 \\ (x+6)/4 & \text{for } -6 < x \le -2 \\ (2-x)/4 & \text{for } -2 < x \le 2, \end{cases}$$

$$(A \cdot B)(x) = \begin{cases} 0 & \text{for } x < -5 \quad \text{and } x \ge 15 \\ \left[3-(4-x)^{1/2}\right]/2 & \text{for } -5 \le x < 0 \\ (1+x)^{1/2}/2 & \text{for } 0 \le x < 3 \\ \left[4-(1+x)^{1/2}\right]/2 & \text{for } 3 \le x < 15, \end{cases}$$

$$(A/B)(x) = \begin{cases} 0 & \text{for } x < -1 \quad \text{and } x \ge 3 \\ (x+1)/(2-2x) & \text{for } -1 \le x < 0 \\ (5x+1)/(2x+2) & \text{for } 0 \le x < 1/3 \\ (3-x)/(2x+2) & \text{for } 1/3 \le x < 3. \end{cases}$$

Let $*$ denote any of the four basic arithmetic operations and let A, B denote fuzzy numbers. Then, we define a fuzzy set on $\mathbb{R}$, $A * B$, by the equation

$$(A * B)(z) = \sup_{z=x*y} \min[A(x), B(y)] \tag{4.9}$$

for all $z \in \mathbb{R}$. More specifically, we define for all $z \in \mathbb{R}$:

$$(A + B)(z) = \sup_{z=x+y} \min[A(x), B(y)], \tag{4.10}$$

$$(A - B)(z) = \sup_{z=x-y} \min[A(x), B(y)], \tag{4.11}$$

$$(A \cdot B)(z) = \sup_{z=x \cdot y} \min[A(x), B(y)], \tag{4.12}$$

$$(A / B)(z) = \sup_{z=x/y} \min[A(x), B(y)]. \tag{4.13}$$

Although $A * B$ defined by (4.9) is a fuzzy set on $\mathbb{R}$, we have to show that it is a fuzzy number for each $* \in \{+, -, \cdot, /\}$. This is a subject of the following theorem.

Theorem 4.2. Let $* \in \{+, -, \cdot, /\}$, and let A, B denote continuous fuzzy numbers. Then, the fuzzy set $A * B$ defined by (4.9) is a continuous fuzzy number.

Proof: First, we prove (4.7) by showing that ${}^\alpha(A * B)$ is a closed interval for every $\alpha \in (0, 1]$. For any $z \in {}^\alpha A * {}^\alpha B$, there exist some $x_0 \in {}^\alpha A$ and $y_0 \in {}^\alpha B$ such that $z = x_0 * y_0$. Thus,

$$\begin{aligned}(A * B)(z) &= \sup_{z=x*y} \min[A(x), B(y)] \\ &\ge \min[A(x_0), B(y_0)] \\ &\ge \alpha.\end{aligned}$$

Hence, $z \in {}^{\alpha}(A * B)$ and, consequently,

$$^{\alpha}A * {}^{\alpha}B \subseteq {}^{\alpha}(A * B).$$

For any $z \in {}^{\alpha}(A * B)$, we have

$$(A * B)(z) = \sup_{z=x*y} \min[A(x), B(y)] \geq \alpha.$$

Moreover, for any $n > [1/\alpha] + 1$, where $[1/\alpha]$ denotes the largest integer that is less than or equal to $1/\alpha$, there exist x_n and y_n such that $z = x_n * y_n$ and

$$\min[A(x_n), B(y_n)] > \alpha - \frac{1}{n}.$$

That is, $x_n \in {}^{\alpha-1/n}A$, $y_n \in {}^{\alpha-1/n}B$ and we may consider two sequences, $\{x_n\}$ and $\{y_n\}$. Since

$$\alpha - \frac{1}{n} \leq \alpha - \frac{1}{n+1},$$

we have

$$^{\alpha-1/(n+1)}A \subseteq {}^{\alpha-1/n}A, \; {}^{\alpha-1/(n+1)}B \subseteq {}^{\alpha-1/n}B.$$

Hence, $\{x_n\}$ and $\{y_n\}$ fall into some ${}^{\alpha-1/n}A$ and ${}^{\alpha-1/n}B$, respectively. Since the latter are closed intervals, $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Thus, there exists a convergent subsequence $\{x_{n,i}\}$ such that $x_{n,i} \to x_0$. To the corresponding subsequence $\{y_{n,i}\}$, there also exists a convergent subsequence $\{y_{n,i,j}\}$ such that $y_{n,i,j} \to y_0$. If we take the corresponding subsequence, $\{x_{n,i,j}\}$, from $\{x_{n,i}\}$, then $x_{n,i,j} \to x_0$. Thus, we have two sequences, $\{x_{n,i,j}\}$ and $\{y_{n,i,j}\}$, such that $x_{n,i,j} \to x_0$, $y_{n,i,j} \to y_0$, and $x_{n,i,j} * y_{n,i,j} = z$.

Now, since $*$ is continuous,

$$z = \lim_{j\to\infty} x_{n,i,j} * y_{n,i,j} = (\lim_{j\to\infty} x_{n,i,j}) * (\lim_{j\to\infty} y_{n,i,j}) = x_0 * y_0.$$

Also, since $A(x_{n,i,j}) > \alpha - \dfrac{1}{n_{i,j}}$ and $B(y_{n,i,j}) > \alpha - \dfrac{1}{n_{i,j}}$,

$$A(x_0) = A(\lim_{j\to\infty} x_{n,i,j}) = \lim_{j\to\infty} A(x_{n,i,j}) \geq \lim_{j\to\infty} (\alpha - \frac{1}{n_{i,j}}) = \alpha$$

and

$$B(y_0) = B(\lim_{j\to\infty} y_{n,i,j}) = \lim_{j\to\infty} B(y_{n,i,j}) \geq \lim_{j\to\infty} (\alpha - \frac{1}{n_{i,j}}) = \alpha.$$

Therefore, there exist $x_0 \in {}^{\alpha}A$, $y_0 \in {}^{\alpha}B$ such that $z = x_0 * y_0$. That is, $z \in {}^{\alpha}A * {}^{\alpha}B$. Thus,

$$^{\alpha}(A * B) \subseteq {}^{\alpha}A * {}^{\alpha}B,$$

and, consequently,

$$^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B.$$

Now we prove that $A * B$ must be continuous. By Theorem 4.1, the membership function of $A * B$ must be of the general form depicted in Fig. 4.3. Assume $A * B$ is not continous at z_0; that is,

$$\lim_{z \to z_{0^-}} (A * B)(z) < (A * B)(z_0) = \sup_{z_0 = x * y} \min[A(x), B(y)].$$

Then, there must exist x_0 and y_0 such that $z_0 = x_0 * y_0$ and

$$\lim_{z \to z_{0^-}} (A * B)(z) < \min[A(x_0), B(y_0)]. \tag{4.14}$$

Since the operation $* \in \{+, -, \cdot, /\}$ is monotonic with respect to the first and the second arguments, respectively, we can always find two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$, and $x_n * y_n < z_0$ for any n. Let $z_n = x_n * y_n$; then $z_n \to z_0$ as $n \to \infty$. Thus,

$$\lim_{z \to z_{0^-}} (A * B)(z) = \lim_{n \to \infty} (A * B)(z_n) = \lim_{n \to \infty} \sup_{z_n = x * y} \min[A(x), B(y)]$$

$$\geq \lim_{n \to \infty} \min[A(x_n), B(y_n)] = \min[A(\lim_{n \to \infty} x_n), B(\lim_{n \to \infty} y_n)] = \min[A(x_0), B(y_0)].$$

This contradicts (4.14) and, therefore, $A * B$ must be a continuous fuzzy number. This completes the proof. ■

LATTICE OF FUZZY NUMBERS

As is well known, the set $\mathbb{R}$ of real numbers is linearly ordered. For every pair of real numbers, x and y, either $x \leq y$ or $y \leq x$. The pair $(\mathbb{R}, \leq)$ is a lattice, which can also be expressed in terms of two lattice operations,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y \leq x, \end{cases} \tag{4.15}$$

$$\max(x, y) = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y \leq x \end{cases} \tag{4.16}$$

for every pair $x, y \in \mathbb{R}$. The linear ordering of real numbers does not extend to fuzzy numbers, but we show in this section that fuzzy numbers can be ordered partially in a natural way and that this partial ordering forms a distributive lattice.

To introduce a meaningful ordering of fuzzy numbers, we first extend the lattice operations min and max on real numbers, as defined by (4.15) and (4.16), to corresponding operations on fuzzy numbers, MIN and MAX. For any two fuzzy numbers A and B, we define

$$\text{MIN}(A, B)(z) = \sup_{z=\min(x,y)} \min[A(x), B(y)], \tag{4.17}$$

$$\text{MAX}(A, B)(z) = \sup_{z=\max(x,y)} \min[A(x), B(y)] \tag{4.18}$$

for all $z \in \mathbb{R}$.

Observe that the symbols MIN and MAX, which denote the introduced operations on fuzzy numbers, must be distinguished from the symbols min and max, which denote the usual operations of minimum and maximum on real numbers, respectively. Since min and max are continuous operations, it follows from (4.17), (4.18), and the proof of Theorem 4.2 that MIN (A, B) and MAX (A, B) are fuzzy numbers.

It is important to realize that the operations MIN and MAX are totally different from the standard fuzzy intersection and union, min and max. This difference is illustrated in Fig. 4.6, where

$$A(x) = \begin{cases} 0 & \text{for } x < -2 \text{ and } x > 4 \\ (x+2)/3 & \text{for } -2 \le x \le 1 \\ (4-x)/3 & \text{for } 1 \le x \le 4, \end{cases}$$

$$B(x) = \begin{cases} 0 & \text{for } x < 1 \text{ and } x > 3 \\ x-1 & \text{for } 1 \le x \le 2 \\ 3-x & \text{for } 2 \le x \le 3, \end{cases}$$

$$\text{MIN}(A, B)(x) = \begin{cases} 0 & \text{for } x < -2 \text{ and } x > 3 \\ (x+2)/3 & \text{for } -2 \le x \le 1 \\ (4-x)/3 & \text{for } 1 < x \le 2.5 \\ 3-x & \text{for } 2.5 < x \le 3, \end{cases}$$

$$\text{MAX}(A, B)(x) = \begin{cases} 0 & \text{for } x < 1 \text{ and } x > 4 \\ x-1 & \text{for } 1 \le x \le 2 \\ 3-x & \text{for } 2 < x \le 2.5 \\ (4-x)/3 & \text{for } 2.5 < x \le 4. \end{cases}$$

Let $\mathcal{R}$ denote the set of all fuzzy numbers. Then, operations MIN and MAX are clearly functions of the form $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$. The following theorem, which establishes basic properties of these operations, ensures that the triple $\langle \mathcal{R}, \text{MIN}, \text{MAX} \rangle$ is a distributive lattice, in which MIN and MAX represent the meet and join, respectively.

Theorem 4.3. Let MIN and MAX be binary operations on $\mathcal{R}$ defined by (4.17) and (4.18), respectively. Then, for any $A, B, C \in \mathcal{R}$, the following properties hold:

(a) $\text{MIN}(A, B) = \text{MIN}(B, A)$,
$\text{MAX}(A, B) = \text{MAX}(B, A)$ (*commutativity*).

(b) $\text{MIN}[\text{MIN}(A, B), C] = \text{MIN}[A, \text{MIN}(B, C)]$,
$\text{MAX}[\text{MAX}(A, B), C] = \text{MAX}[A, \text{MAX}(B, C)]$ (*associativity*).

(c) $\text{MIN}(A, A) = A$,
$\text{MAX}(A, A) = A$ (*idempotence*).

(d) $\text{MIN}[A, \text{MAX}(A, B)] = A$,
$\text{MAX}[A, \text{MIN}(A, B)] = A$ (*absorption*).

(e) $\text{MIN}[A, \text{MAX}(B, C)] = \text{MAX}[\text{MIN}(A, B), \text{MIN}(A, C)]$,
$\text{MAX}[A, \text{MIN}(B, C)] = \text{MIN}[\text{MAX}(A, B), \text{MAX}(A, C)]$ (*distributivity*).

Proof: We focus only on proving properties (b), (d), and (e); proving properties (a) and (c) is rather trivial.

(b) For all $z \in \mathbb{R}$,

$$
\begin{aligned}
\mathrm{MIN}\,[A, \mathrm{MIN}\,(B,C)](z) &= \sup_{z=\min(x,y)} \min[A(x), \mathrm{MIN}\,(B,C)(y)] \\
&= \sup_{z=\min(x,y)} \min[A(x), \sup_{y=\min(u,v)} \min[B(u), C(v)]] \\
&= \sup_{z=\min(x,y)} \sup_{y=\min(u,v)} \min[A(x), B(u), C(v)] \\
&= \sup_{z=\min(x,u,v)} \min[A(x), B(u), C(v)] \\
&= \sup_{z=\min(s,v)} \sup_{s=\min(x,u)} \min[A(x), B(u), C(v)] \\
&= \sup_{z=\min(s,v)} \min[\sup_{s=\min(x,u)} \min[A(x), B(u)], C(v)] \\
&= \sup_{z=\min(s,v)} \min[\mathrm{MIN}\,(A,B)(s), C(v)] \\
&= \mathrm{MIN}\,[\mathrm{MIN}\,(A,B), C](z).
\end{aligned}
$$

The proof of the associativity of MAX is analogous.

(d) For all $z \in \mathbb{R}$,

$$
\begin{aligned}
\mathrm{MIN}\,[A, \mathrm{MAX}\,(A,B)](z) &= \sup_{z=\min(x,y)} \min[A(x), \mathrm{MAX}\,(A,B)(y)] \\
&= \sup_{z=\min(x,y)} \min[A(x), \sup_{y=\max(u,v)} \min[A(u), B(v)]] \\
&= \sup_{z=\min(x,\max(u,v))} \min[A(x), A(u), B(v)].
\end{aligned}
$$

Let M denote the right-hand side of the last equation. Since B is a fuzzy number, there exists $v_0 \in \mathbb{R}$ such that $B(v_0) = 1$. By $z = \min[z, \max(z, v_0)]$, we have

$$M \geq \min[A(z), A(z), B(v_0)] = A(z).$$

On the other hand, since $z = \min[x, \max(u, v)]$, we have

$$\min(x, u) \leq z \leq x \leq \max(x, u).$$

By the convexity of fuzzy numbers,

$$
\begin{aligned}
A(z) &\geq \min[A[\min(x,u)], A[\max(x,u)]] \\
&= \min[A(x), A(u)] \\
&\geq \min[A(x), A(u), B(v)].
\end{aligned}
$$

Thus, $M = A(z)$ and, consequently, $\mathrm{MIN}[A, \mathrm{MAX}(B, C)] = A$. The proof of the other absorption property is similar.

(e) For any $z \in \mathbb{R}$, it is easy to see that

$$\mathrm{MIN}\,[A, \mathrm{MAX}\,(B, C)](z) = \sup_{z=\min[x, \max(u,v)]} \min[A(x), B(u), C(v)], \tag{4.19}$$

$$\mathrm{MAX}\,[\mathrm{MIN}\,(A, B), \mathrm{MIN}\,(A, C)](z) = \sup_{z=\max[\min(m,n), \min(s,t)]} \min[A(m), B(n), A(s), C(t)]. \tag{4.20}$$

To prove that (4.19) and (4.20) are equal, we first show that $E \subseteq F$, where

$$E = \{\min[A(x), B(u), C(v)] \mid \min[x, \max(u, v)] = z\},$$

$$F = \{\min[A(m), B(n), A(s), C(t)] \mid \max[\min(m, n), \min(s, t)] = z\}.$$

For every $a = \min[A(x), B(u), C(v)]$ such that $\min[x, \max(u, v)] = z$ (i.e., $a \in E$), there exists $m = s = x$, $n = u$, and $t = v$ such that

$$\begin{aligned} \max[\min(m, n), \min(s, t)] &= \max[\min(x, u), \min(x, v)] \\ &= \min[x, \max(u, v)] = z; \end{aligned}$$

hence, $a = \min[A(x), B(u), A(x), C(v)] = \min[A(m), B(n), A(s), C(t)]$. That is, $a \in F$ and, consequently, $E \subseteq F$. This means that (4.20) is greater than or equal to (4.19). Next, we show that these two functions are equal by showing that for any number b in F, there exists a number a in E such that $b \leq a$.

For any $b \in F$, there exist m, n, s, and t such that

$$\max[\min(m,n), \min(s,t)] = z,$$
$$b = \min[A(m), B(n), A(s), C(t)].$$

Hence, we have

$$z = \min[\max(s,m), \max(s,n), \max(t,m), \max(t,n)].$$

Let $x = \min[\max(s,m), \max(s,n), \max(t,m)]$, $u = n$, and $v = t$. Then, we have $z = \min[x, \max(u,v)]$. On the other hand, it is easy to see that

$$\min(s,m) \leq x \leq \max(s,m).$$

By convexity of A,

$$A(x) \geq \min[A(\min(s,m)), A(\max(s,m))]$$
$$= \min[A(s), A(m)].$$

Hence, there exists $a = \min[A(x), B(u), C(v)]$ with $\min[x, \max(u,v)] = z$ (i.e., $a \in F$), and

$$a = \min[A(x), B(u), C(v)] \geq \min[A(s), A(m), B(n), C(t)] = b.$$

That is, for any $b \in F$, there exists $a \in F$ such that $b \leq a$. This implies that

$$\sup F \leq \sup E.$$

This inequality, together with the previous result, ensure that (4.19) and (4.20) are equal. This concludes the proof of the first distributive law. The proof of the second distributive law is analogous. ■

FUZZY EQUATIONS

One area of fuzzy set theory in which fuzzy numbers and arithmetic operations on fuzzy numbers play a fundamental role are *fuzzy equations*.
$A + X = B$ and $A \cdot X = B$, where A and B are fuzzy numbers, and X is an unknown fuzzy number for which either of the equations is to be satisfied.

Equation $A + X = B$

The difficulty of solving this fuzzy equation is caused by the fact that $X = B - A$ is not the solution. To see this, let us consider two closed intervals, $A = [a_1, a_2]$ and $B = [b_1, b_2]$, which may be viewed as special fuzzy numbers. Then, $B - A = [b_1 - a_2, b_2 - a_1]$ and

$$\begin{aligned} A + (B - A) &= [a_1, a_2] + [b_1 - a_2, b_2 - a_1] \\ &= [a_1 + b_1 - a_2, a_2 + b_2 - a_1] \\ &\neq [b_1, b_2] = B, \end{aligned}$$

whenever $a_1 \neq a_2$. Therefore, $X = B - A$ is not a solution of the equation.

Let $X = [x_1, x_2]$. Then, $[a_1 + x_1, a_2 + x_2] = [b_1, b_2]$ follows immediately from the equation. This results in two ordinary equations of real numbers,

$$\begin{aligned} a_1 + x_1 &= b_1, \\ a_2 + x_2 &= b_2, \end{aligned}$$

whose solution is $x_1 = b_1 - a_1$ and $x_2 = b_2 - a_2$. Since X must be an interval, it is required that $x_1 \leq x_2$. That is, the equation has a solution iff $b_1 - a_1 \leq b_2 - a_2$. If this inequality is satisfied, the solution is $X = [b_1 - a_1, b_2 - a_2]$.

This example illustrates how to solve the equation when the given fuzzy numbers A and B are closed intervals. Since any fuzzy number is uniquely represented by its α-cuts (Theorem 2.5), which are closed intervals, the described procedure can be applied to α-cuts of arbitrary fuzzy numbers. The solution of our fuzzy equation can thus be obtained by solving a set of associated interval equations, one for each nonzero α in the level set $\Lambda_A \cup \Lambda_B$.

For any $\alpha \in (0, 1]$, let ${}^\alpha A = [{}^\alpha a_1, {}^\alpha a_2]$, ${}^\alpha B = [{}^\alpha b_1, {}^\alpha b_2]$, and ${}^\alpha X = [{}^\alpha x_1, {}^\alpha x_2]$ denote, respectively, the α-cuts of A, B, and X in our equation. Then, the equation has a solution iff:

(i) ${}^\alpha b_1 - {}^\alpha a_1 \leq {}^\alpha b_2 - {}^\alpha a_2$ for every $\alpha \in (0, 1]$, and
(ii) $\alpha \leq \beta$ implies ${}^\alpha b_1 - {}^\alpha a_1 \leq {}^\beta b_1 - {}^\beta a_1 \leq {}^\beta b_2 - {}^\beta a_2 \leq {}^\alpha b_2 - {}^\alpha a_2$.

Property (i) ensures that the interval equation

$${}^\alpha A + {}^\alpha X = {}^\alpha B$$

has a solution, which is ${}^\alpha X = [{}^\alpha b_1 - {}^\alpha a_1, {}^\alpha b_2 - {}^\alpha a_2]$. Property (ii) ensures that the solutions of the interval equations for α and β are nested; that is, if $\alpha \leq \beta$, then ${}^\beta X \subseteq {}^\alpha X$. If a solution ${}^\alpha X$ exists for every $\alpha \in (0, 1]$ and property (ii) is satisfied, then by Theorem 2.5, the solution X of the fuzzy equation is given by

$$X = \bigcup_{\alpha \in (0,1]} {}_\alpha X.$$

To illustrate the solution procedure, let A and B in our equation be the following fuzzy numbers:

$$\begin{aligned} A &= .2/[0, 1) + .6/[1, 2) + .8/[2, 3) + .9/[3, 4) + 1/4 + .5/(4, 5] + .1/(5, 6], \\ B &= .1/[0, 1) + .2/[1, 2) + .6/[2, 3) + .7/[3, 4) + .8/[4, 5) + .9/[5, 6) \\ &\quad + 1/6 + .5/(6, 7] + .4/(7, 8] + .2/(8, 9] + .1/(9, 10]. \end{aligned}$$

TABLE 4.1 α-CUTS ASSOCIATED WITH THE DISCUSSED FUZZY EQUATION OF TYPE $A + X = B$

α	${}^{\alpha}A$	${}^{\alpha}B$	${}^{\alpha}X$
1.0	[4,4]	[6,6]	[2,2]
0.9	[3,4]	[5,6]	[2,2]
0.8	[2,4]	[4,6]	[2,2]
0.7	[2,4]	[3,6]	[1,2]
0.6	[1,4]	[2,6]	[1,2]
0.5	[1,5]	[2,7]	[1,2]
0.4	[1,5]	[2,8]	[1,3]
0.3	[1,5]	[2,8]	[1,3]
0.2	[0,5]	[1,9]	[1,4]
0.1	[0,6]	[0,10]	[0,4]

All relevant α-cuts of A, B, and X are given in Table 4.1. The solution of the equation is the fuzzy number

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X = .1/[0,1) + .7/[1,2) + 1/2 + .4/(2,3] + .2/(3,4].$$

Equation $A \cdot X = B$

Let us assume, for the sake of simplicity, that A, B are fuzzy numbers on $\mathbb{R}^+$. It is easy to show that $X = B/A$ is not a solution of the equation. For each $\alpha \in (0,1]$, we obtain the interval equation

$${}^{\alpha}A \cdot {}^{\alpha}X = {}^{\alpha}B.$$

Our fuzzy equation can be solved by solving these interval equations for all $\alpha \in (0,1]$. Let ${}^{\alpha}A = [{}^{\alpha}a_1, {}^{\alpha}a_2]$, ${}^{\alpha}B = [{}^{\alpha}b_1, {}^{\alpha}b_2]$, and ${}^{\alpha}X = [{}^{\alpha}x_1, {}^{\alpha}x_2]$. Then, the solution of the fuzzy equation exists iff:

(i) ${}^{\alpha}b_1/{}^{\alpha}a_1 \leq {}^{\alpha}b_2/{}^{\alpha}a_2$ for each $\alpha \in (0,1]$, and

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}b_1/{}^{\alpha}a_1 \leq {}^{\beta}b_1/{}^{\beta}a_1 \leq {}^{\beta}b_2/{}^{\beta}a_2 \leq {}^{\alpha}b_2/{}^{\alpha}a_2$.

If the solution exists, it has the form

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X.$$

As an example, let A and B in our equation be the following triangular-shape fuzzy numbers:

$$A(x) = \begin{cases} 0 & \text{for } x \le 3 \text{ and } x > 5 \\ x - 3 & \text{for } 3 < x \le 4 \\ 5 - x & \text{for } 4 < x \le 5 \end{cases}$$

$$B(x) = \begin{cases} 0 & \text{for } x \le 12 \text{ and } x > 32 \\ (x - 12)/8 & \text{for } 12 < x \le 20 \\ (32 - x)/12 & \text{for } 20 < x \le 32. \end{cases}$$

Then, ${}^{\alpha}A = [\alpha + 3, 5 - \alpha]$ and ${}^{\alpha}B = [8\alpha + 12, 32 - 12\alpha]$. It is easy to verify that

$$\frac{8\alpha + 12}{\alpha + 3} \le \frac{32 - 12\alpha}{5 - \alpha};$$

consequently,

$${}^{\alpha}X = \left[\frac{8\alpha + 12}{\alpha + 3}, \frac{32 - 12\alpha}{5 - \alpha}\right]$$

for each $\alpha \in (0, 1]$. It is also easy to check that $\alpha \le \beta$ implies ${}^{\beta}X \subseteq {}^{\alpha}X$ for each pair α, $\beta \in (0, 1]$. Therefore, the solution of our fuzzy equation is

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X = \begin{cases} 0 & \text{for } x \le 4 \text{ and } x \ge 32/5 \\ \dfrac{12 - 3x}{x - 8} & \text{for } 4 < x \le 5 \\ \dfrac{32 - 5x}{12 - x} & \text{for } 5 \le x \le 32/5. \end{cases}$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

FUZZY ANALYSIS

UNIT –III - Fuzzy LOGIC – SMT5205

FUZZY PROPOSITIONS

The fundamental difference between classical propositions and fuzzy propositions is in the range of their truth values. While each classical proposition is required to be either true or false, the truth or falsity of fuzzy propositions is a matter of degree. Assuming that truth and falsity are expressed by values 1 and 0, respectively, the degree of truth of each fuzzy proposition is expressed by a number in the unit interval [0, 1].

we classify into the following four types:

1. unconditional and unqualified propositions;
2. unconditional and qualified propositions;
3. conditional and unqualified propositions;
4. conditional and qualified propositions.

Unconditional and Unqualified Fuzzy Propositions

The canonical form of fuzzy propositions of this type, p, is expressed by the sentence

$$p : \mathcal{V} \text{ is } F, \tag{8.4}$$

where $\mathcal{V}$ is a variable that takes values v from some universal set V, and F is a fuzzy set on V that represents a fuzzy predicate, such as tall, expensive, low, normal, and so on. Given a particular value of $\mathcal{V}$ (say, v), this value belongs to F with membership grade $F(v)$. This membership grade is then interpreted as the degree of truth, $T(p)$, of proposition p. That is,

$$T(p) = F(v) \tag{8.5}$$

for each given particular value v of variable $\mathcal{V}$ in proposition p. This means that T is in effect a fuzzy set on [0, 1], which assigns the membership grade $F(v)$ to each value v of variable $\mathcal{V}$.

To illustrate the introduced concepts, let variable $\mathcal{V}$ be the air temperature at some particular place on the Earth (measured in °F) and let the membership function shown in Fig. 8.1a represent, in a given context, the predicate *high*. Then, assuming that all relevant measurement specifications regarding the temperature are given, the corresponding fuzzy proposition, p, is expressed by the sentence

$$p : \textit{temperature } (\mathcal{V}) \text{ is } \textit{high } (F).$$

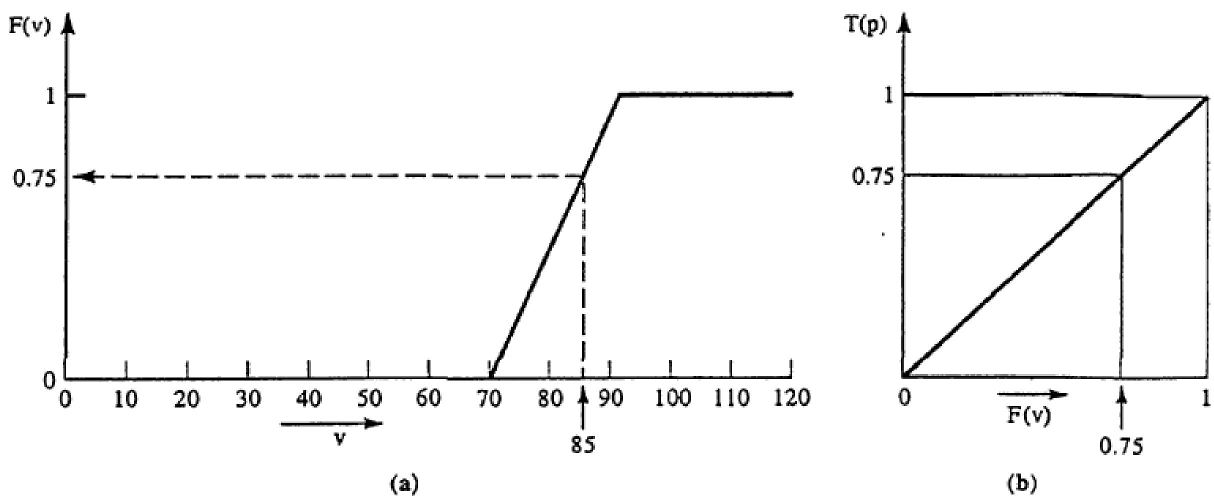


Figure 8.1 Components of the fuzzy proposition p: Temperature ($\mathcal{V}$) is high (F).

The degree of truth, $T(p)$, depends on the actual value of the temperature and on the given definition (meaning) of the predicate *high*; it is defined by the membership function T in Fig. 8.1b, which represents (8.5). For example, if $v = 85$, then $F(85) = 0.75$ and $T(p) = 0.75$.

We can see that the role of function T is to provide us with a bridge between fuzzy sets and fuzzy propositions. Although the connection between grades of membership in F and degrees of truth of the associated fuzzy proposition p, as expressed by (8.5), is numerically trivial for unqualified propositions, it has a conceptual significance.

In some fuzzy propositions, values of variable $\mathcal{V}$ in (8.4) are assigned to individuals in a given set I. That is, variable $\mathcal{V}$ becomes a function $\mathcal{V} : I \rightarrow V$, where $\mathcal{V}(i)$ is the value of $\mathcal{V}$ for individual i in V. The canonical form (8.4) must then be modified to the form

$$p : \mathcal{V}(i) \text{ is } F, \tag{8.6}$$

where $i \in I$.

Consider, for example, that I is a set of persons, each person is characterized by his or her *Age*, and a fuzzy set expressing the predicate *Young* is given. Denoting our variable by *Age* and our fuzzy set by *Young*, we can exemplify the general form (8.6) by the specific fuzzy proposition

$$p : Age(i) \text{ is } Young.$$

The degree of truth of this proposition, $T(p)$, is then determined for each person i in I via the equation

$$T(p) = Young\,(Age(i)).$$

As explained in Sec. 7.4, any proposition of the form (8.4) can be interpreted as a possibility distribution function r_F on V that is defined by the equation

$$r_F(v) = F(v)$$

for each value $v \in V$. Clearly, this interpretation applies to propositions of the modified form (8.6) as well.

Unconditional and Qualified Propositions

Propositions p of this type are characterized by either the canonical form

$$p : \mathcal{V} \text{ is } F \text{ is } S, \tag{8.7}$$

or the canonical form

$$p : \text{Pro}\{\mathcal{V} \text{ is } F\} \text{ is } P, \tag{8.8}$$

where $\mathcal{V}$ and F have the same meaning as in (8.4), Pro $\{\mathcal{V}$ is $F\}$ is the probability of fuzzy event "$\mathcal{V}$ is F," S is a fuzzy truth qualifier, and P is a fuzzy probability qualifier. If desired, $\mathcal{V}$ may be replaced with $\mathcal{V}(i)$, which has the same meaning as in (8.6). We say that the proposition (8.7) is *truth-qualified*, while the proposition (8.8) is *probability-qualified*. Both S and P are represented by fuzzy sets on [0,1].

An example of a truth-qualified proposition is the proposition "Tina is young is very true," where the predicate *young* and the truth qualifier *very true* are represented by the respective fuzzy sets shown in Fig. 8.2. Assuming that the age of Tina is 26, she belongs to the set representing the predicate *young* with the membership grade 0.87. Hence, our proposition belongs to the set of propositions that are very true with membership grade 0.76, as illustrated in Fig. 8.2b. This means, in turn, that the degree of truth of our truth-qualified proposition is also 0.76. If the proposition were modified by changing the predicate (e.g., to *very young*) or the truth qualifier (e.g., to *fairly true*, *very false*, etc.), we would obtain the respective degrees of truth of these propositions by the same method.

In general, the degree of truth, $T(p)$, of any truth-qualified proposition p is given for each $v \in V$ by the equation

$$T(p) = S(F(v)). \tag{8.9}$$

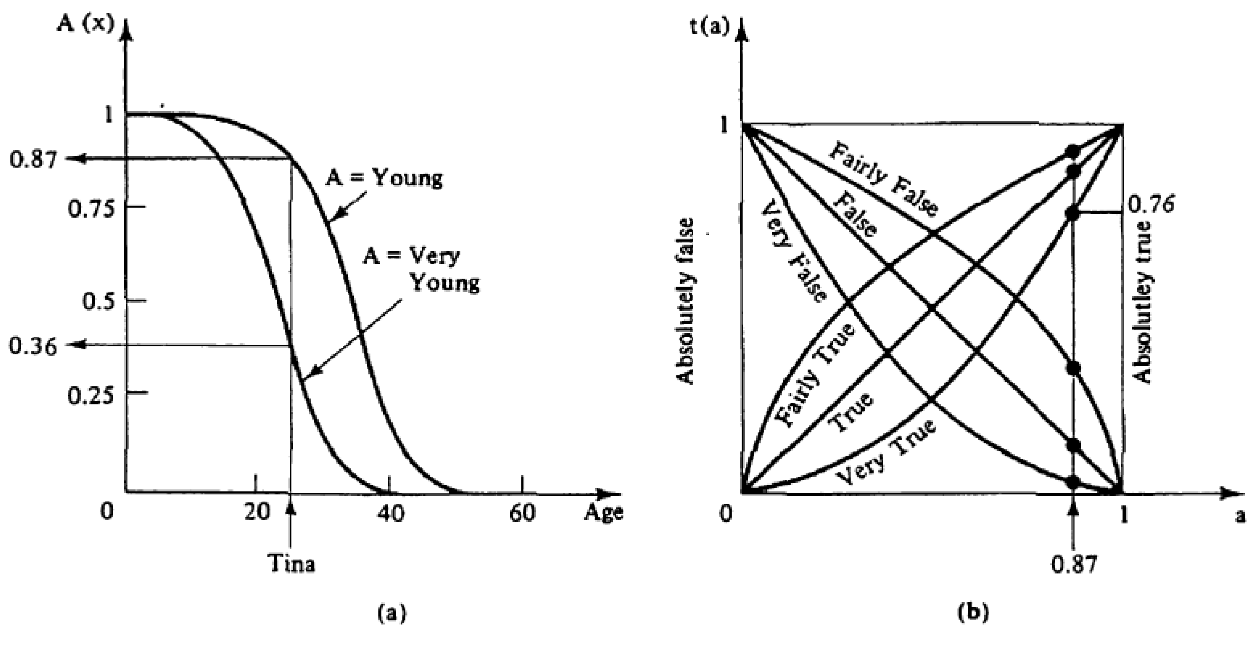


Figure 8.2 Truth values of a fuzzy proposition.

Viewing the membership function $G(v) = S(F(v))$, where $v \in V$, as a simple predicate, we can interpret any truth-qualified proposition of the form (8.7) as the unqualified proposition "$\mathcal{V}$ is G."

Observe that unqualified propositions are, in fact, special truth-qualified propositions, in which the truth qualifier S is assumed to be *true*. As shown in Figs. 8.1b and 8.2b, the membership function representing this qualifier is the identity function. That is, $S(F(v)) = F(v)$ for unqualified propositions; hence, S may be ignored for the sake of simplicity.

Let us discuss now probability-qualified propositions of the form (8.8). Each proposition of this type describes an elastic restriction on possible probability distributions on V. For any given probability distribution f on V, we have

$$\text{Pro}\{\mathcal{V} \text{ is } F\} = \sum_{v \in V} f(v) \cdot F(v); \tag{8.10}$$

and, then, the degree $T(p)$ to which proposition p of the form (8.8) is true is given by the formula

$$T(p) = P(\sum_{v \in V} f(v) \cdot F(v)). \tag{8.11}$$

As an example, let variable $\mathcal{V}$ be the average daily temperature t in °F at some place on the Earth during a certain month. Then, the probability-qualified proposition

p : Pro {temperature t (at given place and time) is around 75°F} is likely

may provide us with a meaningful characterization of one aspect of climate at the given place and time and may be combined with similar propositions regarding other aspects, such as humidity, rainfall, wind speed, and so on. Let in our example the predicate "around 75°F" be represented by the fuzzy set $\tilde{A}$ on $\mathbb{R}$ specified in Fig. 8.3a and the qualifier "likely" be expressed by the fuzzy set on [0, 1] defined in Fig. 8.3b.

Assume now that the following probability distribution (obtained, e.g., from relevant statistical data over many years) is given:

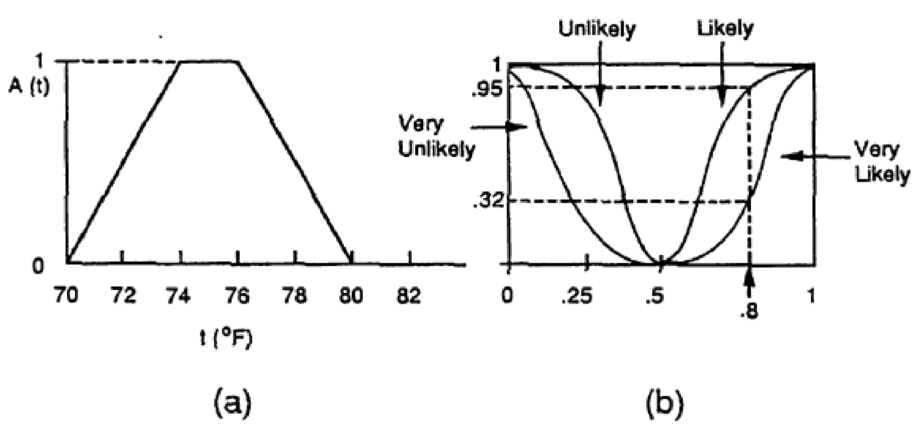


Figure 8.3 Example of a probability-qualified proposition.

t	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83
$f(t)$	.002	.005	.005	.01	.04	.11	.15	.21	.16	.14	.11	.04	.01	.005	.002	.001

Then, using (8.10), we obtain

$$\text{Pro}\,(t \text{ is close to } 75°\text{F}) = .01 \times .25 + .04 \times .5 + .11 \times .75 + .15 \times 1 + .21 \times 1$$
$$+ .16 \times 1 + .14 \times .75 + .11 \times .5 + .04 \times .25 = .8,$$

and, applying this result to the fuzzy probability *likely* in Fig. 8.3b (according to (8.11)), we find that $T(p) = .95$ for our proposition. That is, given the definitions of *around 75* and *likely* in Fig. 8.3, it is true with the degree of .95 that it is *likely* that the temperature (at a given place, time, etc.) is *around* 75°F. Due to this high degree of truth, we may conclude that our proposition is a good characterization of the actual situation. However, if we replaced the qualification *likely* in our proposition with *very likely* (as also defined in Fig. 8.3b), the degree of truth of the new proposition would be only .32. This low degree of truth would not make the new proposition a good description of the actual situation.

Observe that the degree of truth depends on the predicate F, the qualifier P, and the given probability distribution. Replacing, for example, our fuzzy predicate *around 75* with a crisp predicate *in the 70s*, we obtain

$$\text{Pro}\,\{t \text{ is in the 70s}\} = \sum_{t=70}^{79} f(t) = .98,$$

and $T(p)$ becomes practically equal to 1 even if we apply the stronger qualifier *very likely*.

Conditional and Unqualified Propositions

Propositions p of this type are expressed by the canonical form

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B, \tag{8.12}$$

where $\mathcal{X}$, $\mathcal{Y}$ are variables whose values are in sets X, Y, respectively, and A, B are fuzzy sets on X, Y, respectively. These propositions may also be viewed as propositions of the form

$$\langle \mathcal{X}, \mathcal{Y} \rangle \text{ is } R, \tag{8.13}$$

where R is a fuzzy set on $X \times Y$ that is determined for each $x \in X$ and each $y \in Y$ by the formula

$$R(x, y) = \mathcal{J}[A(x), B(y)],$$

where $\mathcal{J}$ denotes a binary operation on [0, 1] representing a suitable *fuzzy implication.*

Fuzzy implications are discussed in detail in the context of approximate reasoning in Secs. 11.2 and 11.3. Here, let us only illustrate the connection between (8.13) and (8.12) for one particular fuzzy implication, the Lukasiewicz implication

$$\mathcal{J}(a, b) = \min(1, 1 - a + b). \tag{8.14}$$

Let $A = .1/x_1 + .8/x_2 + 1/x_3$ and $B = .5/y_1 + 1/y_2$. Then

$$R = 1/x_1, y_1 + 1/x_1, y_2 + .7/x_2, y_1 + 1/x_2, y_2 + .5/x_3, y_1 + 1/x_3, y_2.$$

This means, for example, that $T(p) = 1$ when $\mathcal{X} = x_1$ and $\mathcal{Y} = y_1$; $T(p) = .7$ when $\mathcal{X} = x_2$ and $\mathcal{Y} = y_1$; and so on.

Conditional and Qualified Propositions

Propositions of this type can be characterized by either the canonical form

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \text{ is } S \tag{8.15}$$

or the canonical form

$$p: \text{Pro}\,\{\mathcal{X} \text{ is } A \,|\, \mathcal{Y} \text{ is } B\} \text{ is } P, \tag{8.16}$$

where Pro $\{\mathcal{X}$ is $A | \mathcal{Y}$ is $B\}$ is a conditional probability.

LINGUISTIC HEDGES

Linguistic hedges (or simply hedges) are special linguistic terms by which other linguistic terms are modified. Linguistic terms such as *very, more or less*, *fairly*, or *extremely* are examples of hedges. They can be used for modifying fuzzy predicates, fuzzy truth values, and fuzzy probabilities. For example, the proposition "x is young," which is assumed to mean "x is young is true," may be modified by the hedge *very* in any of the following three ways:

"x is very young is true,"
"x is young is very true,"
"x is very young is very true."

Similarly, the proposition "x is young is likely" may be modified to "x is young is very likely," and so forth.

In general, given a fuzzy proposition

$$p : x \text{ is } F$$

and a linguistic hedge, H, we can construct a modified proposition,

$$Hp : x \text{ is } HF,$$

where HF denotes the fuzzy predicate obtained by applying the hedge H to the given predicate F. Additional modifications can be obtained by applying the hedge to the fuzzy truth value or fuzzy probability employed in the given proposition.

It is important to realize that linguistic hedges are not applicable to crisp predicates, truth values, or probabilities. For example, the linguistic terms *very horizontal*, *very pregnant*, *very teenage*, or *very rectangular* are not meaningful. Hence, hedges do not exist in classical logic.

Any linguistic hedge, H, may be interpreted as a unary operation, h, on the unit interval [0, 1]. For example, the hedge *very* is often interpreted as the unary operation $h(a) = a^2$, while the hedge *fairly* is interpreted as $h(a) = \sqrt{a}$ ($a \in [0, 1]$). Let unary operations that represent linguistic hedges be called *modifiers*.

Given a fuzzy predicate F on X and a modifier h that represents a linguistic hedge H, the modified fuzzy predicate HF is determined for each $x \in X$ by the equation

$$HF(x) = h(F(x)).$$

This means that properties of linguistic hedges can be studied by studying properties of the associated modifiers.

Any modifier h is an increasing bijection. If $h(a) < a$ for all $a \in [0, 1]$, the modifier is called *strong*; if $h(a) > a$ for all $a \in [0, 1]$, the modifier is called *weak*. The special (vacuous) modifier for which $h(a) = a$ is called an *identity modifier*.

A strong modifier strengthens a fuzzy predicate to which it is applied and, consequently, it reduces the truth value of the associated proposition. A weak modifier, on the contrary, weakens the predicate and, hence, the truth value of the proposition increases. For example, consider three fuzzy propositions:

p_1 : John is young,
p_2 : John is very young,
p_3 : John is fairly young,

and let the linguistic hedges *very* and *fairly* be represented by the strong modifier a^2 and the weak modifier $\sqrt{a}$. Assume now that John is 26 and, according to the fuzzy set YOUNG representing the fuzzy predicate *young*, YOUNG (26) = 0.8. Then, VERY YOUNG (26) = 0.8^2 = 0.64 and FAIRLY YOUNG (26) = $\sqrt{0.8}$ = 0.89. Hence, $T(p_1) = 0.8$, $T(p_2) = 0.64$, and $T(p_3) = 0.89$. These values agree with our intuition: the stronger assertion is less true and vice versa.

8.6 INFERENCE FROM CONDITIONAL FUZZY PROPOSITIONS

As explained in Sec. 8.1, inference rules in classical logic are based on the various tautologies. These inference rules can be generalized within the framework of fuzzy logic to facilitate approximate reasoning. In this section, we describe generalizations of three classical inference rules, *modus ponens*, *modus tollens*, and *hypothetical syllogism*. These generalizations are based on the so-called compositional rule of inference.

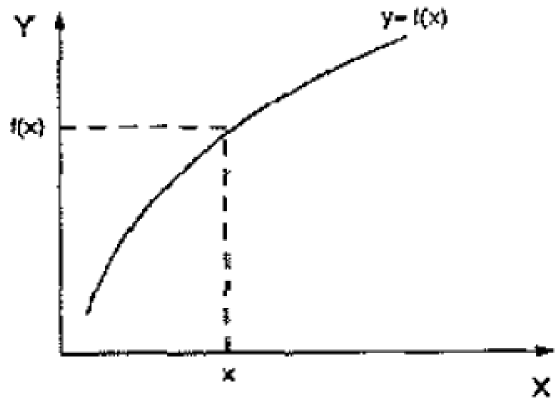
Consider variables $\mathcal{X}$ and $\mathcal{Y}$ that take values from sets X and Y, respectively, and assume that for all $x \in X$ and all $y \in Y$ the variables are related by a function $y = f(x)$. Then, given $\mathcal{X} = x$, we can infer that $\mathcal{Y} = f(x)$, as shown in Fig. 8.6a. Similarly, knowing that the value of $\mathcal{X}$ is in a given set A, we can infer that the value of $\mathcal{Y}$ is in the set $B = \{y \in Y | y = f(x), x \in A\}$, as shown in Fig. 8.6b.

Assume now that the variables are related by an arbitrary relation on $X \times Y$, not necessarily a function. Then, given $\mathcal{X} = u$ and a relation R, we can infer that $\mathcal{Y} \in B$, where $B = \{y \in Y | \langle x, y \rangle \in R\}$, as illustrated in Fig. 8.7a. Similarly, knowing that $\mathcal{X} \in A$, we can infer that $\mathcal{Y} \in B$, where $B = \{y \in Y | \langle x, y \rangle \in R, x \in A\}$, as illustrated in Fig. 8.7b. Observe that this inference may be expressed equally well in terms of characteristic functions χ_A, χ_B, χ_R of sets A, B, R respectively, by the equation

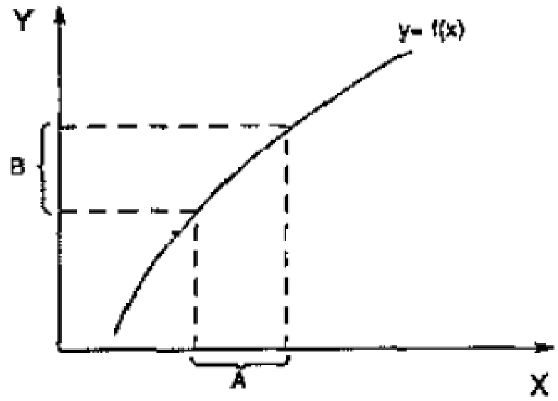
$$\chi_B(y) = \sup_{x \in X} \min[\chi_A(x), \chi_R(x, y)] \tag{8.38}$$

for all $y \in Y$.

Let us proceed now one step further and assume that R is a fuzzy relation on $X \times Y$, and A', B' are fuzzy sets on X and Y, respectively. Then, if R and A' are given, we can obtain B' by the equation



(a)



(b)

Figure 8.6 Functional relation between two variables: (a) $x \to y$, where $y = f(x)$; (b) $A \to B$, where $B = \{y \in Y | y = f(x), x \in A\}$.

$$B'(y) = \sup_{x \in X} \min[A'(x), R(x, y)] \tag{8.39}$$

for all $y \in Y$, which is a generalization of (8.38) obtained by replacing the characteristic functions in (8.38) with the corresponding membership functions. This equation, which can also be written in the matrix form as

$$\mathbf{B}' = \mathbf{A}' \circ \mathbf{R},$$

is called the *compositional rule of inference*. This rule is illustrated in Fig. 8.8.

The fuzzy relation employed in (8.39) is usually not given directly, but in some other form. In this section, we consider the case in which the relation is embedded in a single conditional fuzzy proposition. A more general case, in which the relation emerges from several conditional fuzzy propositions, is discussed in Chapter 11.

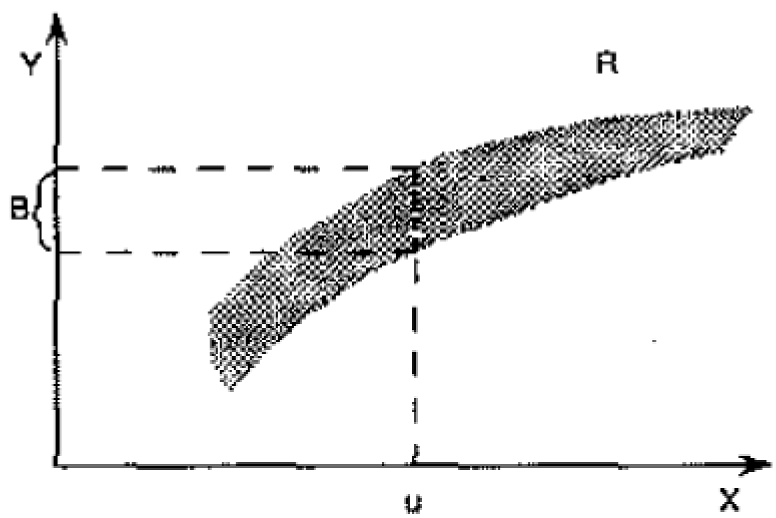
As explained in Sec. 8.3, relation R that is embedded in a conditional fuzzy proposition p of the form

$$p: \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B$$

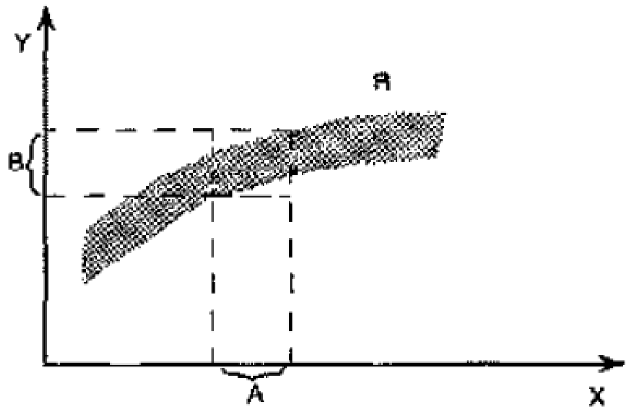
is determined for all $x \in X$ and all $y \in Y$ by the formula

$$R(x, y) = \mathcal{J}[A(x), B(y)], \tag{8.40}$$

where $\mathcal{J}$ denotes a fuzzy implication



(a)



(b)

Figure 8.7 Inference expressed by (8.38).

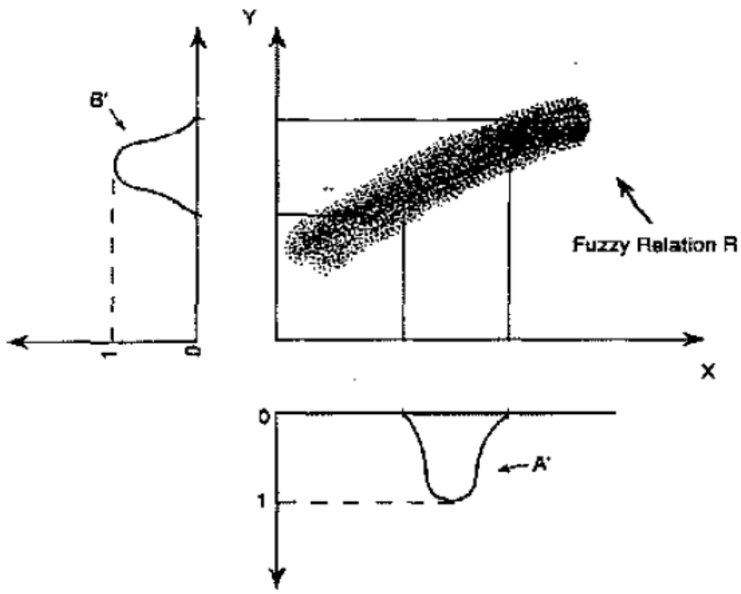


Figure 8.8 Compositional rule of inference expressed by (8.39).

Using relation R obtained from given proposition p by (8.40), and given another proposition q of the form

$$q : \mathcal{X} \text{ is } A',$$

we may conclude that $\mathcal{Y}$ is B' by the compositional rule of inference (8.39). This procedure is called a *generalized modus ponens*.

Viewing proposition p as a rule and proposition q as a fact, the generalized modus ponens is expressed by the following schema:

$$\begin{array}{ll} \text{Rule :} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \\ \text{Fact :} & \mathcal{X} \text{ is } A' \\ \hline \text{Conclusion :} & \mathcal{Y} \text{ is } B' \end{array} \tag{8.41}$$

In this schema, B' is calculated by (8.39), and R in this equation is determined by (8.40). Observe that (8.41) becomes the classical modus ponens when the sets are crisp and $A' = A, B' = B$.

Example 8.1

Let sets of values of variables $\mathcal{X}$ and $\mathcal{Y}$ be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively. Assume that a proposition "if $\mathcal{X}$ is A, then $\mathcal{Y}$ is B" is given, where $A = .5/x_1 + 1/x_2 + .6/x_3$ and $B = 1/y_1 + .4/y_2$. Then, given a fact expressed by the proposition "x is A'," where $A' = .6/x_1 + .9/x_2 + .7/x_3$, we want to use the generalized modus ponens (8.41) to derive a conclusion in the form "$\mathcal{Y}$ is B'."

Using, for example, the Lukasiewicz implication (8.14), we obtain

$$R = 1/x_1, y_1 + .9/x_1, y_2 + 1/x_2, y_1 + .4/x_2, y_2 + 1/x_3, y_1, +.8/x_3, y_2$$

by (8.40). Then, by the compositional rule of inference (8.39), we obtain

$$\begin{aligned} B'(y_1) &= \sup_{x \in X} \min[A'(x), R(x, y_1)] \\ &= \max[\min(.6, 1), \min(.9, 1), \min(.7, 1)] \\ &= .9 \\ B'(y_2) &= \sup_{x \in X} \min[A'(x), R(x, y_2)] \\ &= \max[\min(.6, .9), \min(.9, .4), \min(.7, .8)] \\ &= .7 \end{aligned}$$

Thus, we may conclude that $\mathcal{Y}$ is B', where $B' = .9/y_1 + .7/y_2$.

Another inference rule in fuzzy logic, which is a *generalized modus tollens*, is expressed by the following schema:

$$\begin{array}{ll} \text{Rule :} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \\ \text{Fact :} & \mathcal{Y} \text{ is } B' \\ \hline \text{Conclusion :} & \mathcal{X} \text{ is } A' \end{array}$$

In this case, the compositional rule of inference has the form

$$A'(x) = \sup_{y \in Y} \min[B'(y), R(x, y)], \tag{8.42}$$

and R in this equation is again determined by (8.40). When the sets are crisp and $A' = \overline{A}, B' = \overline{B}$, we obtain the classical *modus tollens*.

Example 8.2

Let $X, Y, \mathcal{J}, A$, and B are the same as in Example 8.1. Then, R is also the same as in Example 8.1. Assume now that a fact expressed by the proposition "$\mathcal{Y}$ is B'" is given, where $B' = .9/y_1 + .7/y_2$. Then, by (8.42),

$$\begin{aligned} A'(x_1) &= \sup_{y \in Y} \min[B'(y), R(x_1, y)] \\ &= \max[\min(.9, 1), \min(.7, .9)] = .9, \\ A'(x_2) &= \sup_{y \in Y} \min[B'(y), R(x_2, y)] \\ &= \max[\min(.9, 1), \min(.7, .4)] = .9, \\ A'(x_3) &= \sup_{y \in Y} \min[B'(y), R(x_3, y)] \\ &= \max[\min(.9, 1), \min(.7, .8)] = .9. \end{aligned}$$

Hence, we conclude that $\mathcal{X}$ is A' where $A' = .9/x_1 + .9/x_2 + .9/x_3$.

Finally, let us discuss a generalization of hypothetical syllogism, which is based on two conditional fuzzy propositions. The *generalized hypothetical syllogism* is expressed by the following schema:

$$\begin{array}{ll} \text{Rule 1:} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \\ \text{Rule 2:} & \text{If } \mathcal{Y} \text{ is } B, \text{ then } \mathcal{Z} \text{ is } C \\ \hline \text{Conclusion:} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Z} \text{ is } C \end{array} \tag{8.43}$$

In this case, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are variables taking values in sets X, Y, Z, respectively, and A, B, C are fuzzy sets on sets X, Y, Z, respectively.

For each conditional fuzzy proposition in (8.43), there is a fuzzy relation determined by (8.40). These relations are determined for each $x \in X, y \in Y$, and $z \in Z$ by the equations

$$\begin{aligned} R_1(x, y) &= \mathcal{J}[A(x), B(y)], \\ R_2(y, z) &= \mathcal{J}[B(y), C(z)], \\ R_3(x, z) &= \mathcal{J}[A(x), C(z)]. \end{aligned}$$

Given R_1, R_2, R_3, obtained by these equations, we say that the generalized hypothetical syllogism holds if

$$R_3(x, z) = \sup_{y \in Y} \min[R_1(x, y), R_2(y, z)], \tag{8.44}$$

which again expresses the compositional rule of inference. This equation may also be written in the matrix form

$$\mathbf{R}_3 = \mathbf{R}_1 \circ \mathbf{R}_2. \tag{8.45}$$

Example 8.3

Let X, Y be the same as in Example 8.1, and let $Z = \{z_1, z_2\}$. Moreover, let $A = .5/x_1 + 1/x_2 + .6/x_3$, $B = 1/y_1 + .4/y_2$, $C = .2/z_1 + 1/z_2$, and

$$\mathcal{J}(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b. \end{cases}$$

Then, clearly,

$$\mathbf{R}_1 = \begin{bmatrix} 1 & .4 \\ 1 & .4 \\ 1 & .4 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} .2 & 1 \\ .2 & 1 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} .2 & 1 \\ .2 & 1 \\ .2 & 1 \end{bmatrix}$$

The generalized hypothetical syllogism holds in this case since $\mathbf{R}_1 \circ \mathbf{R}_2 = \mathbf{R}_3$.

8.7 INFERENCE FROM CONDITIONAL AND QUALIFIED PROPOSITIONS

The inference rule of our concern in this section involves conditional fuzzy propositions with fuzzy truth qualifiers. Given a conditional and qualified fuzzy proposition p of the form

$$p : \text{ If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \text{ is } S, \tag{8.46}$$

where S is a fuzzy truth qualifier, and a fact is in the form "$\mathcal{X}$ is A'," we want to make an inference in the form "$\mathcal{Y}$ is B'."

One method developed for this purpose, called a *method of truth-value restrictions*, is based on a manipulation of linguistic truth values. The method involves the following four steps.

Step 1. Calculate the relative fuzzy truth value of A' with respect to A, denoted by $RT(A'/A)$, which is a fuzzy set on the unit interval defined by

$$RT(A'/A)(a) = \sup_{x: A(x)=a} A'(x), \tag{8.47}$$

for all $a \in [0, 1]$. The relative fuzzy truth value $RT(A'/A)$ expresses the degree to which the fuzzy proposition (8.46) is true given the available fact "$\mathcal{X}$ is A'."

Step 2. Select a suitable fuzzy implication $\mathcal{J}$ by which the fuzzy proposition (8.46) is interpreted. This is similar to the selection of fuzzy implication in Sec. 8.6, whose purpose is to express a conditional but unqualified fuzzy proposition as a fuzzy relation.

Step 3. Calculate the relative fuzzy truth value $RT(B'/B)$ by the formula

$$RT(B'/B)(b) = \sup_{a \in [0,1]} \min[RT(A'/A)(a), S(\mathcal{J}(a, b))] \tag{8.48}$$

for all $b \in [0, 1]$, where S is the fuzzy qualifier in (8.46). Clearly, the role of the qualifier S is to modify the truth value of $\mathcal{J}(a, b)$. Note that when S stands for *true* (i.e., $S(a) = a$)

for all $a \in [0, 1]$, then $S(\mathcal{J}(a, b)) = \mathcal{J}(a, b)$, The relative fuzzy truth value $RT(B'/B)$ expresses the degree to which the conclusion of the fuzzy proposition (8.46) is true.

Step 4. Calculate the set B' involved in the inference "$\mathcal{Y}$ is B'" by the equation

$$B'(y) = RT(B'/B)(B(y)), \tag{8.49}$$

for all $y \in Y$.

Example 8.4

Suppose we have a fuzzy conditional and qualified proposition,

$$p: \text{ If } \mathcal{X} \text{ is } A \text{ then } \mathcal{Y} \text{ is } B \text{ is very true,}$$

where $A = 1/x_1 + .5/x_2 + .7/x_3$, $B = .6/y_1 + 1/y_2$, and S stands for *very true*; let $S(a) = a^2$ for all $a \in [0, 1]$. Given a fact "$\mathcal{X}$ is A'," where $A' = .9/x_1 + .6/x_2 + .7/x_3$, we conclude that "$\mathcal{Y}$ is B'," where B' is calculated by the following steps.

Step 1. We calculate $RT(A'/A)$ by (8.47):

$$\begin{aligned} RT(A'/A)(1) &= A'(x_1) = .9, \\ RT(A'/A)(.5) &= A'(x_2) = .6, \\ RT(A'/A)(.7) &= A'(x_3) = .7, \\ RT(A'/A)(a) &= 0 \text{ for all } a \in [0, 1] - \{.5, .7, 1\}. \end{aligned}$$

Step 2. We select the Lukasiewicz fuzzy implication $\mathcal{J}$ defined by (8.14).

Step 3. We calculate $RT(B'/B)$ by (8.48):

$$\begin{aligned} RT(B'/B)(b) &= \max\{\min[.9, S(\mathcal{J}(.9, b))], \min[.6, S(\mathcal{J}(.6, b))], \\ &\qquad \min[.7, S(\mathcal{J}(.7, b))]\} \\ &= \begin{cases} (.4 + b)^2 & \text{for } b \in [0, .375) \\ .6 & \text{for } b \in [.375, .475) \\ (.3 + b)^2 & \text{for } b \in [.475, .537) \\ .7 & \text{for } b \in [.537, .737) \\ (.1 + b)^2 & \text{for } b \in [.737, .849) \\ .9 & \text{for } b \in [.849, 1] \end{cases} \end{aligned}$$

A graph of this function $RT(B'/B)$ is shown in Fig. 8.9.

Step 4. We calculate B' by (8.49):

$$\begin{aligned} B'(y_1) &= RT(B'/B)(B(y_1)) = RT(B'/B)(.6) = .7, \\ B'(y_2) &= RT(B'/B)(B(y_2)) = RT(B'/B)(1) = .9. \end{aligned}$$

Hence, we make the inference "$\mathcal{Y}$ is B'," where $B' = .7/y_1 + .9/y_2$.

When S in (8.46) stands for *true* (i.e., S is the identity function), the method of truth-value restrictions is equivalent to the generalized modus ponens under a particular condition, as stated in the following theorem.

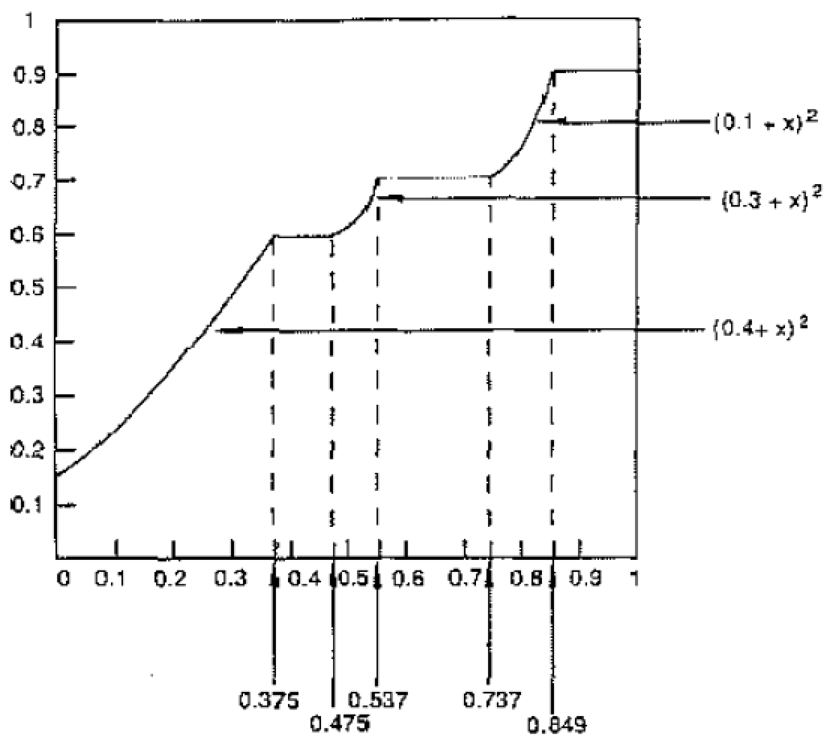


Figure 8.9 Function $RT(B/B')$ in Example 8.3.

Theorem 8.1. Let a fuzzy proposition of the form (8.46) be given, where S is the identity function (i.e., S stands for *true*), and let a fact be given in the form "$\mathcal{X}$ is A'," where

$$\sup_{x:A(x)=a} A'(x) = A'(x_0) \tag{8.50}$$

for all $a \in [0, 1]$ and some x_0 such that $A(x_0) = a$. Then, the inference "$\mathcal{Y}$ is B'" obtained by the method of truth-value restrictions is equal to the one obtained by the generalized modus ponens (i.e., (8.41) and (8.49) define the same membership function B'), provided that we use the same fuzzy implication in both inference methods.

Proof: When $S(a) = a$ for all $a \in [0, 1]$, B', defined by (8.49), becomes

$$B'(y) = \sup_{a\in[0,1]} \min[RT(A'/A)(a), \mathcal{J}(a, B(y))] \tag{8.51}$$

for all $y \in Y$. Using the same fuzzy implication $\mathcal{J}$, B', defined by (8.41), becomes

$$B'(y) = \sup_{x\in X} \min[A'(x), \mathcal{J}(A(x), B(y))] \tag{8.52}$$

for all $y \in Y$. To prove the theorem, we have to show that (8.51) and (8.52) define the same membership function B'. To facilitate the proof, let B_1', B_2' denote the functions defined by (8.51) and (8.52), respectively. Since

$$A'(x) \le \sup_{x':A(x')=A(x)} A'(x') = RT(A'/A)(A(x))$$

for all $x \in X$, we have

$$\min[A'(x), \mathcal{J}(A(x), B(y))] \le \min[RT(A'/A)(A(x)), \mathcal{J}(A(x), B(y))]$$

for all $y \in Y$. Hence,

$$\begin{aligned} B_2'(y) &= \sup_{x \in X} \min[A'(x), \mathcal{J}(A(x), B(y))] \\ &\leq \sup_{x \in X} \min[RT(A'/A)(A(x)), \mathcal{J}(A(x), B(y))] \\ &\leq \sup_{a \in [0,1]} \min[RT(A'/A)(a), \mathcal{J}(a, B(y))] \\ &= B_1'(y) \end{aligned}$$

for all $y \in Y$. On the other hand, by condition (8.50), we have

$$\begin{aligned} \min[RT(A'/A)(a), \mathcal{J}(a, B(y))] &= \min[\sup_{x:A(x)=a} A'(x), \mathcal{J}(a, B(y))] \\ &= \min[A'(x_0), \mathcal{J}(A(x_0), B(y))] \\ &\leq \sup_{x \in X} \min[A'(x), \mathcal{J}(A(x), B(y))] \\ &= B_2'(y) \end{aligned}$$

for all $y \in Y$. Thus,

$$B_1'(y) = \sup_{a \in [0,1]} \min[RT(A'/A)(a), \mathcal{J}(a, B(y))] \leq B_2'(y)$$

for all $y \in Y$ and, consequently, $B_1' = B_2'$. ■

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

FUZZY ANALYSIS

UNIT – IV – FUZZY DECISION MAKING – SMT5205

INDIVIDUAL DECISION MAKING

In the first paper on fuzzy decision making, Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set A of *possible actions*;
- a set of *goals* $G_i (i \in \mathbb{N}_n)$, each of which is expressed in terms of a fuzzy set defined on A;
- a set of *constraints* $C_j (j \in \mathbb{N}_m)$, each of which is also expressed by a fuzzy set defined on A.

It is common that the fuzzy sets expressing goals and constraints in this formulation are not defined directly on the set of actions, but indirectly, through other sets that characterize relevant states of nature. Let G'_i and C'_j be fuzzy sets defined on sets X_i and Y_j, respectively, where $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$. Assume that these fuzzy sets represent goals and constraints expressed by the decision maker. Then, for each $i \in \mathbb{N}_n$ and each $j \in \mathbb{N}_m$, we describe the meanings of actions in set A in terms of sets X_i and Y_j by functions

$$g_i : A \to X_i,$$
$$c_j : A \to Y_j,$$

and express goals G_i and constraints C_j by the compositions of g_i with G'_i and the compositions of c_j and C'_j; that is,

$$G_i(a) = G'_i(g_i(a)), \tag{15.1}$$
$$C_j(a) = C'_j(c_j(a)) \tag{15.2}$$

for each $a \in A$.

Given a decision situation characterized by fuzzy sets A, $G_i (i \in \mathbb{N}_n)$, and $C_j (j \in \mathbb{N}_m)$, a *fuzzy decision*, D, is conceived as a fuzzy set on A that simultaneously satisfies the given goals G_i and constraints C_j. That is,

$$D(a) = \min[\inf_{i \in \mathbb{N}_n} G_i(a), \inf_{j \in \mathbb{N}_m} C_j(a)] \tag{15.3}$$

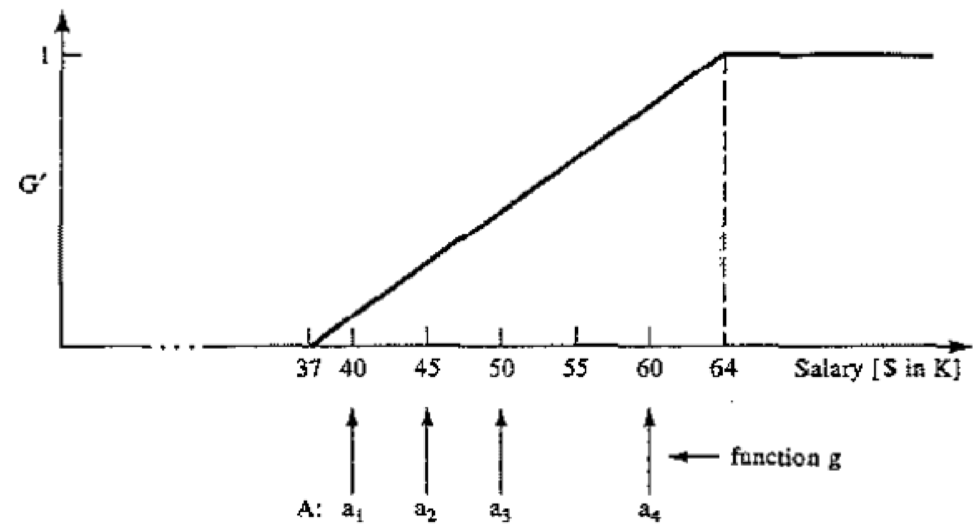
for all $a \in A$, provided that the standard operator of fuzzy intersection is employed.

Once a fuzzy decision has been arrived at, it may be necessary to choose the "best" single crisp alternative from this fuzzy set. This may be accomplished in a straightforward manner by choosing an alternative $\hat{a} \in A$ that attains the maximum membership grade in D. Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations.

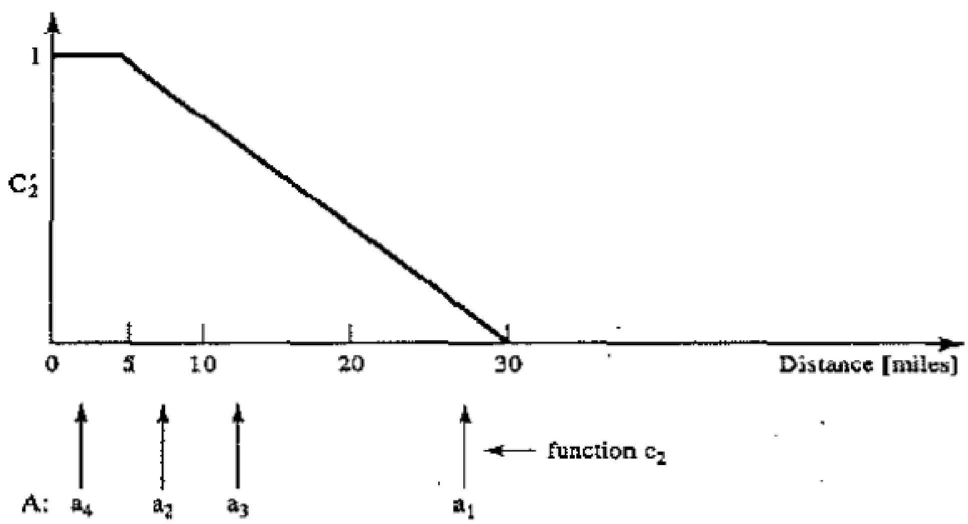
Example 15.1

Suppose that an individual needs to decide which of four possible jobs, a_1, a_2, a_3, a_4, to choose. His or her goal is to choose a job that offers a high salary under the constraints that the job is interesting and within close driving distance. In this case, $A = \{a_1, a_2, a_3, a_4\}$, and the fuzzy sets involved represent the concepts of *high salary, interesting job,* and *close driving distance*. These concepts are highly subjective and context-dependent, and must be defined by the individual in a given context. The goal is expressed in monetary terms, independent of the jobs available. Hence, according to our notation, we denote the fuzzy set expressing the goal by G'. A possible definition of G' is given in Fig. 15.1a, where we assume, for convenience, that the underlying universal set is $\mathbb{R}^+$. To express the goal in terms of set A, we need a function g: $A \to \mathbb{R}^+$, which assigns to each job the respective salary. Assume the following assignments:

$$g(a_1) = \$40{,}000,$$
$$g(a_2) = \$45{,}000,$$
$$g(a_3) = \$50{,}000,$$
$$g(a_4) = \$60{,}000.$$



(a) Goal G′: High salary.



(b) Constraint C_2': Close driving distance.

Figure 15.1 Fuzzy goal and constraint (Example 15.1): (a) goal G': high salary; (b) constraint C_2': close driving distance.

Composing now functions g and G', according to

(15.1), we obtain the fuzzy set

$$G = .11/a_1 + .3/a_2 + .48/a_3 + .8/a_4,$$

which expresses the goal in terms of the available jobs in set A.

The first constraint, requiring that the job be interesting, is expressed directly in terms of set A (i.e., c_1, in (15.2) is the identity function and $C_1 = C_1'$). Assume that the individual assigns to the four jobs in A the following membership grades in the fuzzy set of interesting jobs:

$$C_1 = .4/a_1 + .6/a_2 + .2/a_3 + .2/a_4.$$

The second constraint, requiring that the driving distance be close, is expressed in terms of the driving distance from home to work. Following our notation, we denote the fuzzy set expressing this constraint by C_2'. A possible definition of C_2' is given in Fig. 15.1b, where distances of the four jobs are also shown. Specifically,

$$c_2(a_1) = 27 \text{ miles},$$
$$c_2(a_2) = 7.5 \text{ miles},$$
$$c_2(a_3) = 12 \text{ miles},$$
$$c_2(a_4) = 2.5 \text{ miles}.$$

By composing functions c_2 and C_2', according to (15.2), we obtain the fuzzy set

$$C_2 = .1/a_1 + .9/a_2 + .7/a_3 + 1/a_4,$$

which expresses the constraint in terms of the set A.

Applying now formula (15.3), we obtain the fuzzy set

$$D = .1/a_1 + .3/a_2 + .2/a_3 + .2/a_4,$$

which represents a fuzzy characterization of the concept of *desirable job*. The job to be chosen is $\hat{a} = a_2$; this is the most desirable job among the four available jobs under the given goal G and constraints C_1, C_2, provided that we aggregate the goal and constraints as expressed by (15.3).

FUZZY RANKING METHODS

The first method is based upon defining the *Hamming distance* on the set $\mathcal{R}$ of all fuzzy numbers. For any given fuzzy numbers A and B, the Hamming distance, $d(A, B)$, is defined by the formula

$$d(A, B) = \int_{\mathbb{R}} |A(x) - B(x)| dx. \tag{15.16}$$

For any given fuzzy numbers A and B, which we want to compare, we first determine their least upper bound, $\mathrm{MAX}(A, B)$, in the lattice. Then, we calculate the Hamming distances $d(\mathrm{MAX}(A, B), A)$ and $d(\mathrm{MAX}(A, B), B)$, and define

$$A \leq B \text{ if } d(\mathrm{MAX}(A, B), A) \geq d(\mathrm{MAX}(A, B), B).$$

If $A \preceq B$ (i.e., fuzzy numbers are directly comparable), then $\mathrm{MAX}(A, B) = B$ and, hence, $A \leq B$. That is, the ordering defined by the Hamming distance is compatible with the ordering of comparable fuzzy numbers in $\mathcal{R}$. Observe that we can also define a similar ordering of fuzzy numbers A and B via the greatest lower bound $\mathrm{MIN}(A, B)$.

The second method is based on α-cuts.

Given fuzzy numbers A and B to be compared, we select a particular value of $\alpha \in [0, 1]$ and determine the α-cuts ${}^{\alpha}A = [a_1, a_2]$ and ${}^{\alpha}B = [b_1, b_2]$. Then, we define

$$A \leq B \text{ if } a_2 \leq b_2.$$

This definition is, of course, dependent on the chosen value of α. It is usually required **that $\alpha > 0.5$.**

The third method is based on the extension principle. This method can be employed for ordering several fuzzy numbers, say $A_1, A_2, \ldots, A_n$. The basic idea is to construct a fuzzy set P on $\{A_1, A_2, \ldots, A_n\}$, called a *priority set*, such as $P(A_i)$ is the degree to which A_i is ranked as the greatest fuzzy number. Using the extension principle, P is defined for each $i \in \mathbb{N}_n$ by the formula

$$P(A_i) = \sup \min_{k \in \mathbb{N}_n} A_k(r_k), \tag{15.17}$$

where the supremum is taken over all vectors $\langle r_1, r_2, \ldots, r_n \rangle \in \mathbb{R}^n$ such that $r_i \geq r_j$ for all $j \in \mathbb{N}_n$.

Example 15.6

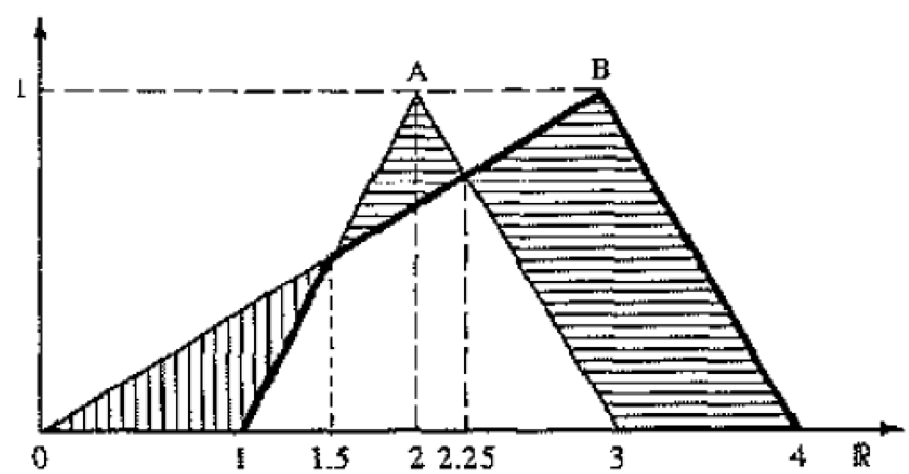
In this example, we illustrate and compare the three fuzzy ranking methods. Let A and B be fuzzy numbers whose triangular-type membership functions are given in Fig. 15.5a. Then, $\text{MAX}(A, B)$ is the fuzzy number whose membership function is indicated in the figure in bold. We can see that the Hamming distances $d(\text{MAX}(A, B), A)$ and $d(\text{MAX}(A, B), B)$ are expressed by the areas in the figure that are hatched horizontally and vertically, respectively. Using (15.16), we obtain

$$\begin{aligned} d(\text{MAX}(A,B),A) &= \int_{1.5}^{2}[x-1-\frac{x}{3}]dx + \int_{2}^{2.25}[-x+3-\frac{x}{3}]dx \\ &\quad + \int_{2.25}^{3}[\frac{x}{3}+x-3]dx + \int_{3}^{4}[4-x]dx \\ &= \frac{1}{12}+\frac{1}{24}+\frac{3}{8}+\frac{1}{2} = 1 \\ d(\text{MAX}(A,B),B) &= \int_{0}^{1.5}\frac{x}{3}dx - \int_{1}^{1.5}[x-1]dx \\ &= \frac{3}{8}-\frac{1}{8} = 0.25. \end{aligned}$$

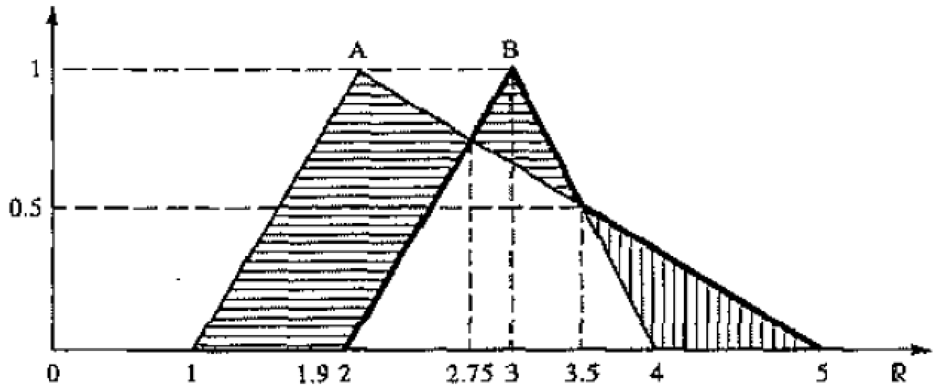
Since $d(\text{MAX}(A, B), A) > d(\text{MAX}(A, B), B)$, we may conclude that, according to the first ranking method, $A \leq B$. When applying the second method to the same example, we can easily find, from Fig. 15.5a, that $A \leq B$ for any $\alpha \in [0, 1]$. According to the third method, we construct the priority fuzzy set P on $\{A, B\}$ as follows:

$$P(A) = \sup_{r_1 \geq r_2} \min[A(r_1), B(r_2)] = 0.75,$$

$$P(B) = \sup_{r_2 \geq r_1} \min[A(r_1), B(r_2)] = 1.$$



(a)



(b)

Figure 15.5 Ranking of fuzzy members (Example 15.6).

Hence, again, we conclude that $A \leq B$.

Consider now the fuzzy numbers A and B whose membership functions are given in Fig. 15.5b. The horizontally and vertically hatched areas have the same meaning as before. We can easily find that

$$d(\text{MAX}(A, B), A) = 1, d(\text{MAX}(A, B), B) = 0.25.$$

Hence, $A \leq B$ according to the first method. The second method gives the same result only for $\alpha > 0.5$. This shows that the method is inconsistent. According to the third method, we again obtain $P(A) = 0.75$ and $P(B) = 1$; hence, $A \leq B$.

FUZZY LINEAR PROGRAMMING

The most general type of fuzzy linear programming is formulated as follows:

$$\begin{aligned} &\max \sum_{j=1}^{n} C_j X_j \\ &\text{s.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \quad (i \in \mathbb{N}_m) \\ &X_j \geq 0 \quad (j \in \mathbb{N}_n), \end{aligned} \tag{15.19}$$

where A_{ij}, B_i, C_j are fuzzy numbers, and X_j are variables whose states are fuzzy numbers ($i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$); the operations of addition and multiplication are operations of fuzzy arithmetic, and $\leq$ denotes the ordering of fuzzy numbers.

Case 1. Fuzzy linear programming problems in which only the right-hand-side numbers B_i are fuzzy numbers:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq B_i \quad (i \in \mathbb{N}_m) \\
& x_j \geq 0 \quad (j \in \mathbb{N}_n).
\end{aligned}
$$

In general, fuzzy linear programming problems are first converted into equivalent crisp linear or nonlinear problems, which are then solved by standard methods. The final results of a fuzzy linear programming problem are thus real numbers, which represent a compromise in terms of the fuzzy numbers involved.

fuzzy numbers $B_i (i \in \mathbb{N}_m)$ typically have the form

$$
B_i(x) = \begin{cases} 1 & \text{when } x \leq b_i \\ \dfrac{b_i + p_i - x}{p_i} & \text{when } b_i < x < b_i + p_i \\ 0 & \text{when } b_i + p_i \leq x, \end{cases}
$$

where $x \in \mathbb{R}$ (Fig. 15.7a). For each vector $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we first calculate the degree, $D_i(\mathbf{x})$, to which $\mathbf{x}$ satisfies the ith constraint ($i \in \mathbb{N}_m$) by the formula

$$
D_i(\mathbf{x}) = B_i\left(\sum_{j=1}^{n} a_{ij} x_j\right).
$$

These degrees are fuzzy sets on $\mathbb{R}^n$, and their intersection, $\bigcap_{i=1}^{m} D_i$, is a *fuzzy feasible set.*

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values, z_l, is obtained by solving the standard linear programming problem:

$$
\begin{aligned}
\max \quad & z = \mathbf{cx} \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i \in \mathbb{N}_m) \\
& x_j \geq 0 \quad (j \in \mathbb{N}_n);
\end{aligned}
$$

the upper bound of the optimal values, z_u, is obtained by a similar linear programming problem in which each b_i is replaced with $b_i + p_i$:

$$\begin{aligned} \max \quad & z = \mathbf{cx} \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i \quad (i \in \mathbb{N}_m) \\ & x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

Then, the fuzzy set of optimal values, G, which is a fuzzy subset of $\mathbb{R}^n$, is defined by

$$G(\mathbf{x}) = \begin{cases} 1 & \text{when } z_u \leq \mathbf{cx} \\ \dfrac{\mathbf{cx} - z_l}{z_u - z_l} & \text{when } z_l \leq \mathbf{cx} \leq z_u \\ 0 & \text{when } \mathbf{cx} \leq z_l. \end{cases}$$

Now, the problem (15.20) becomes the following classical optimization problem:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \lambda(z_u - z_l) - \mathbf{cx} \leq -z_l \\ & \lambda p_i + \sum_{i=1}^{n} a_{ij}x_j \leq b_i + p_i \quad (i \in \mathbb{N}_m) \\ & \lambda, x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

The above problem is actually a problem of finding $\mathbf{x} \in \mathbb{R}^n$ such that

$$[(\bigcap_{i=1}^{m} D_i) \cap G](\mathbf{x})$$

reaches the maximum value; that is, a problem of finding a point which satisfies the constraints and goal with the maximum degree.

Example 15.8

Assume that a company makes two products. Product P_1 has a \$0.40 per unit profit and product P_2 has a \$0.30 per unit profit. Each unit of product P_1 requires twice as many labor hours as each product P_2. The total available labor hours are at least 500 hours per day, and may possibly be extended to 600 hours per day, due to special arrangements for overtime work. The supply of material is at least sufficient for 400 units of both products, P_1 and P_2, per day, but may possibly be extended to 500 units per day according to previous experience. The problem is, how many units of products P_1 and P_2 should be made per day to maximize the total profit?

Let x_1, x_2 denote the number of units of products P_1, P_2 made in one day, respectively. Then the problem can be formulated as the following fuzzy linear programming problem:

$$\begin{aligned} \max \quad & z = .4x_1 + .3x_2 \text{ (profit)} \\ \text{s.t.} \quad & x_1 + x_2 \leq B_1 \quad \text{(material)} \\ & 2x_1 + x_2 \leq B_2 \quad \text{(labor hours)} \\ & x_1, x_2 \geq 0, \end{aligned}$$

where B_1 is defined by

$$B_1(x) = \begin{cases} 1 & \text{when } x \leq 400 \\ \dfrac{500 - x}{100} & \text{when } 400 < x \leq 500 \\ 0 & \text{when } 500 < x, \end{cases}$$

and B_2 is defined by

$$B_2(x) = \begin{cases} 1 & \text{when } x \leq 500 \\ \dfrac{600 - x}{100} & \text{when } 500 < x \leq 600 \\ 0 & \text{when } 600 < x. \end{cases}$$

First we need to calculate the lower and upper bounds of the objective function. By solving the following two classical linear programming problems, we obtain $z_l = 130$ and $z_u = 160$.

$$\begin{aligned} (P_1)\ \max\quad & z = .4x_1 + .3x_2 \\ \text{s.t.}\quad & x_1 + x_2 \leq 400 \\ & 2x_1 + x_2 \leq 500 \\ & x_1, x_2 \geq 0. \\ (P_2)\ \max\quad & z = .4x_1 + .3x_2 \\ \text{s.t.}\quad & x_1 + x_2 \leq 500 \\ & 2x_1 + x_2 \leq 600 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Then, the fuzzy linear programming problem becomes:

$$\begin{aligned} \max\quad & \lambda \\ \text{s.t.}\quad & 30\lambda - (.4x_1 + .3x_2) \leq -130 \\ & 100\lambda + x_1 + x_2 \leq 500 \\ & 100\lambda + 2x_1 + x_2 \leq 600 \\ & x_1, x_2, \lambda \geq 0. \end{aligned}$$

Solving this classical optimization problem, we find that the maximum, $\lambda = 0.5$, is obtained for $\hat{x}_1 = 100, \hat{x}_2 = 350$. The maximum profit, $\hat{z}$, is then calculated by

$$\hat{z} = .4\hat{x}_1 + .3\hat{x}_2 = 145.$$

Case 2. Fuzzy linear programming problems in which the right-hand-side numbers B_i and the coefficients A_{ij} of the constraint matrix are fuzzy numbers:

$$\begin{aligned} \max\quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\quad & \sum_{j=1}^{n} A_{ij} x_j \leq B_i \quad (i \in \mathbb{N}_m) \qquad (15.21) \\ & x_j \geq 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

In this case, we assume that all fuzzy numbers are triangular. Any triangular fuzzy number A can be represented by three real numbers, s, l, r,

Using this representation, we write $A = \langle s, l, r\rangle$. Problem (15.21) can then be rewritten as

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \langle s_{ij}, l_{ij}, r_j\rangle x_{ij} \le \langle t_i, u_i, v_i\rangle \quad (i \in \mathbb{N}_m) \\ & x_j \ge 0 \quad (j \in \mathbb{N}_n), \end{aligned}$$

where $A_{ij} = \langle s_{ij}, l_{ij}, r_{ij}\rangle$ and $B_i = \langle t_i, u_i, v_i\rangle$ are fuzzy numbers. Summation and multiplication are operations on fuzzy numbers, and the partial order $\le$ is defined by $A \le B$ iff $\text{MAX}(A, B) = B$. It is easy to prove that for any two triangular fuzzy numbers $A = \langle s_1, l_1, r_1\rangle$ and $B = \langle s_2, l_2, r_2\rangle$, $A \le B$ iff $s_1 \le s_2, s_1 - l_1 \le s_2 - l_2$ and $s_1 + r_1 \le s_2 + r_2$. Moreover, $\langle s_1, l_1, r_1\rangle + \langle s_2, l_2, r_2\rangle = \langle s_1 + s_2, l_1 + l_2, r_1 + r_2\rangle$ and $\langle s_1, l_1, r_1\rangle x = \langle s_1 x, l_1 x, r_1 x\rangle$ for any non-negative real number x. Then, the problem can be rewritten as

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} s_{ij} x_j \le t_i \\ & \sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \le t_i - u_i \\ & \sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \le t_i + v_i \quad (i \in \mathbb{N}_m) \\ & x_j \ge 0 \quad (j \in \mathbb{N}_n). \end{aligned}$$

However, since all numbers involved are real numbers, this is a classical linear programming problem.

Example 15.9

Consider the following fuzzy linear programming problem:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & \langle 4, 2, 1\rangle x_1 + \langle 5, 3, 1\rangle x_2 \leq \langle 24, 5, 8\rangle \\ & \langle 4, 1, 2\rangle x_1 + \langle 1, .5, 1\rangle x_2 \leq \langle 12, 6, 3\rangle \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can rewrite it as

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 4x_1 + 5x_2 \leq 24 \\ & 4x_1 + x_2 \leq 12 \\ & 2x_1 + 2x_2 \leq 19 \\ & 3x_1 + 0.5x_2 \leq 6 \\ & 5x_1 + 6x_2 \leq 32 \\ & 6x_1 + 2x_2 \leq 15 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solving this problem, we obtain $\hat{x}_1 = 1.5$, $\hat{x}_2 = 3$, $\hat{z} = 19.5$.

Notice that if we defuzzified the fuzzy numbers in the constraints of the original problem by the maximum method, we would obtain another classical linear programming problem:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 4x_1 + 5x_2 \leq 24 \\ & 4x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

We can see that this is a classical linear programming problem with a smaller number of constraints than the one converted from a fuzzy linear programming problem.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

FUZZY ANALYSIS

UNIT – V –FUZZY RELATIONS – SMT5205

Relations

A classical relation can be considered as a set of tuples, where a tuple is an ordered pair. A binary tuple is denoted by (x,y), an example of a ternary tuple is (x,y,z) and an example of *n*-ary tuple is $(x_1,...,x_n)$.

Example: Let U be the domain of man {John, Charles, James} and V the domain of women {Diana, Rita, Eva}, then the relation "married to" on U ×V is, for example

{(Charles, Diana), (John, Eva), (James, Rita) }

Definition: (classical n-ary relation) Let $X_1,...,X_n$ be classical(crisp) sets. The subsets of the Cartesian product $X_1 \times\cdots\times X_n$ are called n-ary relations. If $X_1 = \cdots = X_n$ and $R \subseteq U^n$ then R is called an n-ary relation (operation) in U.

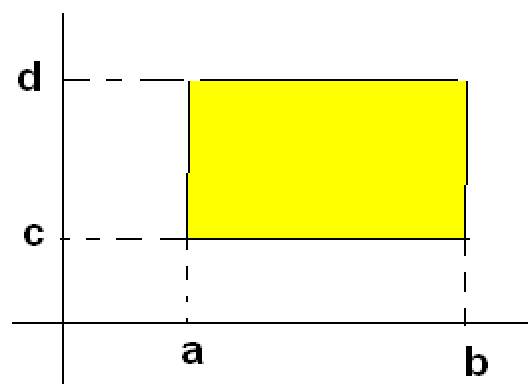
Let R be a binary relation in R. Then the characteristic function of R is defined as

$$\chi_R(x,y) = \begin{cases} 1, (x,y) \in R \\ 0, (x,y) \notin R \end{cases}$$

Example: Consider the following relation

$$(x,y) \in R \Leftrightarrow x \in \langle a,b \rangle \wedge y \in \langle c,d \rangle$$

$$\chi_R(x,y) = \begin{cases} 1, (x,y) \in \langle a,b \rangle \times \langle c,d \rangle \\ 0, (x,y) \notin \langle a,b \rangle \times \langle c,d \rangle \end{cases}$$



Let *R* be a binary relation in a classical set *X*. Then

Fig.12: Graph relation R

Definition. *(***reflexivity**) R is reflexive if $(x,x) \in R$ for all $x \in U$.

Definition. ***(anti-reflexivity)*** *R is anti-reflexive if* f $(x,x) \notin R$ for all $x \in U$.

Definition. ***(symmetricity)*** R is symmetric if from (x,y) $\in$ R $\Rightarrow$ (y,x)$\in$ R for all x,y$\in$U.

Definition. ***(anti-symmetricity)*** R is anti-symmetric if (x,y) $\in$ R and (y,x) $\in$ R then x=y for all x,y$\in$U.

Definition. ***(transitivity)*** *R is transitive if* (x*, y*)$\in$ *R and* (y*,z*)R *R then* (x*, z*) $\in$ *R,* for all x,y,z$\in$U.

Example. Consider the classical inequality relations on the real line R. It is clear that ≤ is reflexive, anti-symmetric and transitive, < is anti-reflexive, antisymmetric and transitive.

Other binary relations are

Definition. (equivalence) R is an equivalence relation if R is reflexive, symmetric and transitive

Example.

The relation = on natural numbers is equivalence relation.

Definition. (partial order) R is a partial order relation if it is reflexive, antsymmetric and transitive.

Definition. (total order) R is a total order relation if it is partial order and for all x,y$\in$U (x,y)$\in$R or (y,x)$\in$R.

Example. Let us consider the binary relation ”subset of”. It is clear that we have a partial order relation.

The relation ≤ on natural numbers is a total order relation.

Fuzzy relation

Definition of fuzzy relation. Let U and V be nonempty sets. A fuzzy relation R is a fuzzy subset of U × V .

In other words, $R \in \mathcal{F}(U \times V)$, $\mu_R : U \times V \to \langle 0,1 \rangle$

It is often used equivalence notation $\mu_R(x,y) = R(x,y)$.

If U =V then we say that R is a binary fuzzy relation in U.

Let *R* be a binary fuzzy relation on R. Then *R*(x,y) is interpreted as the degree of membership of the ordered pair (x,y) in *R*.

Example. A simple example of a binary fuzzy relation on

U = {1, 2, 3},

called ”approximately equal” can be defined as

R(1, 1) = R(2, 2) = R(3, 3)=1,R(1, 2) = R(2, 1) = R(2, 3) = R(3, 2)=0.8 ,

R(1, 3) = R(3, 1)=0.3

In matrix notation it can be represented as $\begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{pmatrix}$

Operations on fuzzy relations

The intersection

Fuzzy relations are very important because they can describe nteractions between variables. Let *R* and *S* be two binary fuzzy relations on *X* × *Y* .

Definition: The **intersection** of R and S is defined by

$$(R \wedge S)(x,y) = \min\{R(x,y),S(x,y)\}.$$

Note that $R: U \times V \rightarrow$ <0, 1>, i.e. *R* the domain of *R* is the whole Cartesian product $U \times V$.

Definition: *The union of R and S is defined by*

$$(R \vee S)(x,v) = \max\{R(x, z), S(x, z)\}$$

Example: Let us define two binary relations

R = "x is considerable larger than y"= $\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$

S = "x is very close to y"= $\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4, & 0 & 0.9 & 0.6 \\ x_2 & 0.9 & 0.4 & 0.5 & 0.7 \\ x_3 & 0.3 & 0 & 0.8 & 0.5 \end{pmatrix}$

The intersection of R and S means that "x is considerable larger than y" **and** *„is very close to y".*

$$(R \wedge S)(x,y) = \min\{R(x,y),S(x,y)\} = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0 & 0.1 & 0.6 \\ x_2 & 0 & 0.4 & 0 & 0 \\ x_3 & 0.3 & 0 & 0.7 & 0.5 \end{pmatrix}$$

The union of R and S means that "x is considerable larger than y" **or** *"x is very close to y".*

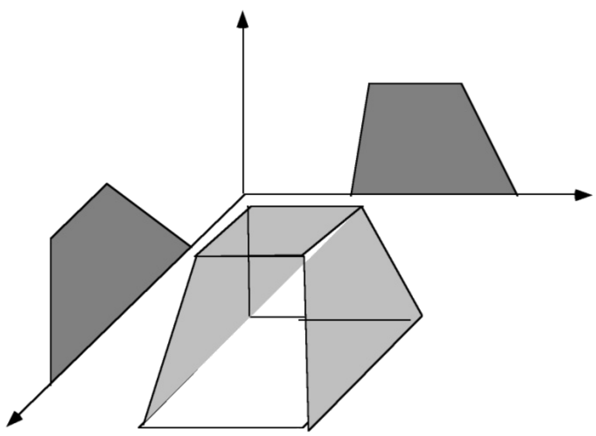
$$\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.8 & 0.5 & 0.7 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{pmatrix}$$

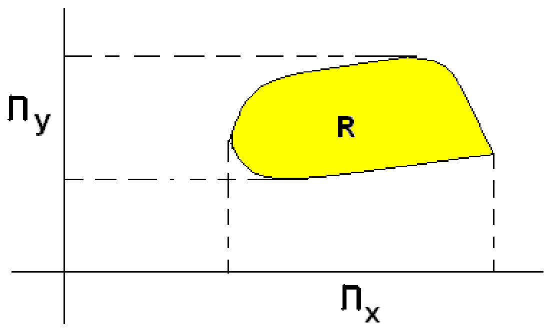
$(R \vee S)(x, y)=$

Projections of fuzzy relation

Consider a classical relation *R* on R.

Fig. 13. Projections on axis.





$$R(x, y) = \begin{cases} 1, (x, y) \\ 0, (x, y) \end{cases}$$

I

t is clear that the *projection* (or **shadow**) of *R* on the *X*-axis is the closed interval <*a, b*> and its projection on the *Y* -axis is <c,d>.

Definition: If *R* is a classical relation in $U \times V$ then

$$\Pi_X = \{x \in U | \exists y \in V : (x, y) \in R\}$$

$$\Pi_Y = \{y \in V | \exists x \in U : (x, y) \in R\}$$

where Π_X denotes **projection on U** and $\mathbf{\Pi_Y}$ **denotes projection on V** .

Definition: *Let R be a fuzzy binary fuzzy relation on U × V . The projection of R on U is defined as*

$\Pi_X(x) = \sup\{R(x, y) \mid y \in V\}$

and the projection of R on Y is defined as

$\Pi_Y(y) = \sup\{R(x, y) \mid x \in U\}$

Example: *Consider the relation*

$$R = \text{"x is considerable larger than y"} = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

then the projection on X means that

- x_1 *is assigned the highest membership degree from the tuples* (x_1,y_1), (x_1,y_2), (x_1,y_3), (x_1,y_4), *i.e.* $\Pi_X(x_1)=1$, *which is the maximum of the first row.*
- x_2 *is assigned the highest membership degree from the tuples* (x_2,y_1), (x_2,y_2), (x_2,y_3), (x_2,y_4), *i.e.* $\Pi_X(x_2)=0.8$, *which is the maximum of the second row.*
- x_3 *is assigned the highest membership degree from the tuples* (x_3,y_1), (x_3,y_2), (x_3,y_3), (x_3,y_4), *i.e.* $\Pi_X(x_3)=1$, *which is the maximum of the third row.*

Cartesian product of fuzzy sets

It is clear that Cartesian product of two fuzzy sets is a fuzzy relation.

If A and B are normal then $\Pi_Y(A \times B) = B$ and $\Pi_X(A \times B) = A$.

Really,

$\Pi_X(x) = \sup\{(A \times B)(x, y) \mid y\}$

$= \sup\{A(x) \wedge B(y) \mid y\} = \min\{A(x), \sup\{B(y)\} \mid y\}$

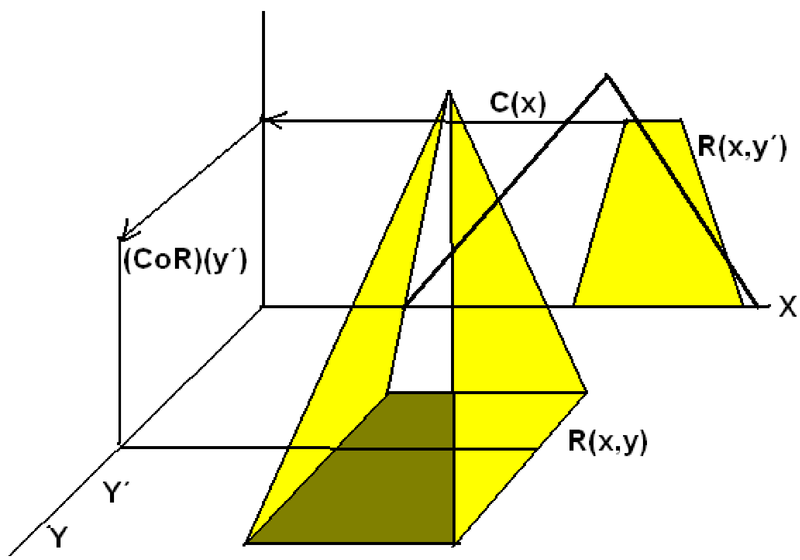
$= \min\{A(x), 1\} = A(x).$

Definition: The sup-min composition of a fuzzy set $\tilde{C} \in \mathscr{F}(U)$ and a fuzzy relation R $\in \mathscr{F}(U \times V)$ is defined as

$$(\tilde{C} \circ R)(y) = \sup_{x \in U} \{\min\{C(x), R(x, y)\}\}$$

for all y∈V .

The composition of a fuzzy set $\tilde{C}$ and a fuzzy relation *R* can be considered as the shadow of the relation *R* on the fuzzy set $\tilde{C}$.



Example: *Let* $\tilde{A}$ *and* $\tilde{B}$ *fuzzy sets and let*

$$\mu_A(x)=\begin{cases}\dfrac{x-a}{b-a}, x\in\langle a,b\rangle\\ \dfrac{c-x}{c-b}, x\in\langle b,c\rangle\\ 0, \quad x\notin\langle a,c\rangle\end{cases}\qquad \mu_B(x)=\begin{cases}\dfrac{x-e}{f-e}, x\in\langle e,f\rangle\\ \dfrac{g-x}{g-f}, x\in\langle f,g\rangle\\ 0, \quad x\notin\langle e,g\rangle\end{cases}$$

Let R = $\tilde{A}\times\tilde{B}$ *Is fuzzy relation.*

Observe the following property of composition $\tilde{A}\circ R=\tilde{A}\circ(\tilde{A}\times\tilde{B})=\tilde{A}$,

$$\tilde{B}\circ R=\tilde{B}\circ(\tilde{A}\times\tilde{B})=\tilde{B}.$$

Example: *Let* $\tilde{C}$ *be a fuzzy set in the universe of discourse {1, 2, 3} and let R be a binary fuzzy relation in {1, 2, 3}. Assume that*

$$\tilde{C}=\{(1,0.2),(2,1)(3,0.3)\}\ \text{ and }\ R=\begin{pmatrix}1 & 0.8 & 0.3\\ 0.8 & 1 & 0.8\\ 0{,}3 & 0{,}8 & 1\end{pmatrix}$$

Using the definition of sup-min composition we get

$$\tilde{C}\circ R=(0.2,1,0.3)\circ\begin{pmatrix}1 & 0.8 & 0.3\\ 0.8 & 1 & 0.8\\ 0{,}3 & 0{,}8 & 1\end{pmatrix}=(\max\{\min\{0.2,1\},\min\{1,0.8\},\min\{0.3,0.3\}\},$$

$$\max\{\min\{0.2,0.8\},\min\{1,1\},\min\{0.3,0.8\}\},\max\{\min\{0.2,0.3\},\min\{1,0.8\},\min\{0.3,1\}\}=$$

=(0.8,1,0.8).

Example: Let $\tilde{C}$ be a fuzzy set in the universe of discourse <0, 1> and let R be a binary fuzzy relation in <0, 1>. Assume that C(x)= x and R(x, y)=1-|x-y|.

Using the definition of sup-min composition we get

. $(\tilde{C} \circ R)(y) = \sup_{x\in\langle 0,1\rangle} \min\{x, 1-|x-y|\} = \frac{1+y}{2}$

for all y∈<0,1>

Sup-min composition of fuzzy relations

Definition: (**sup-min composition of fuzzy relations**) Let R $\in\mathcal{F}(U \times V)$ and S $\in\mathcal{F}(V \times T)$. The sup-min composition of R and S, denoted by R°S is defined as

$$(R \circ S)(x,z) = \sup_{y\in V} \min\{R(x,y), S(y,z)\}$$

It is clear that $R \circ S$ is a binary fuzzy relation in U×*T*.

Example: Consider two fuzzy relations

R = ”x is considerable larger than y”=

$$\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

S = ”y is very close to z” =

$$\begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix}$$

Then their composition is

$$R\circ S = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix} \circ \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{0.4,0,0.1,0.6\} & \max\{0.8,0.1,0.1,0.7\} & \max\{0.3,0,0.1,0.5\} \\ \max\{0,0,0,0\} & \max\{0,0.4,0,0\} & \max\{0,0,0,0\} \\ \max\{0.4,0,0.7,0.6\} & \max\{0.9,0.4,0.5,0.7\} & \max\{0.3,0,0.7,0.5\} \end{pmatrix} =$$

$$= \begin{pmatrix} 0.6 & 0.8 & 0.5 \\ 0 & 0.4 & 0 \\ 0.7 & 0.9 & 0.7 \end{pmatrix}$$

i.e., the composition of R and S is nothing else, but the classical product of the matrices R and S with the difference that instead of addition we use maximum and instead of multiplication we use minimum operator.

Sup-product composition of fuzzy relations

Definition: (**sup-product composition of fuzzy relations**) Let R $\in \mathcal{F}$(U × V) and S $\in \mathcal{F}$(V × T). The sup-product composition of R and S, denoted by R°S is defined as

$$(R\circ S)(x,z) = \sup_{y\in V}\{R(x,y).S(y,z)\}$$

It is clear that $R\circ S$ is a binary fuzzy relation in U×T.

Example: Consider two fuzzy relations

R = ”x is considerable larger than y”=

S = ”y is very close to z” =

$$\begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix}$$

Then their sup-product composition is

$$R \circ S = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix} \circ \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{0.32,0,0.09,0.42\} & \max\{0.72,0.04,0.5,0.49\} & \max\{0.24,0,0.08,0.35\} \\ \max\{0,0,0,0\} & \max\{0,0.72,0,0\} & \max\{0,0,0,0\} \\ \max\{0.36,0,0.63,0.48\} & \max\{0.81,0.4,0.35,0.56\} & \max\{0.27,0,0.56,0.4\} \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{0.32,0,0.09,0.42\} & \max\{0.72,0.04,0.5,0.49\} & \max\{0.24,0,0.08,0.35\} \\ \max\{0,0,0,0\} & \max\{0,0.72,0,0\} & \max\{0,0,0,0\} \\ \max\{0.36,0,0.63,0.48\} & \max\{0.81,0.4,0.35,0.56\} & \max\{0.27,0,0.56,0.4\} \end{pmatrix} =$$

$$\begin{pmatrix} 0.42 & 0.72 & 0.35 \\ 0 & 0.72 & 0 \\ 0.63 & 0.81 & 0.56 \end{pmatrix}$$

If possible to define composition fuzzy of relations in another manner. For instance, operator max we can replace any t-conorm and min any t-norm.

Fuzzy relation is

Reflexive if R(x,x)=1 for all x$\in$U.

Symmetric if R(x,y)=R(y,x) for all (x,y)$\in$R

Transitive if $$\mathrm{R(x,\ y)} \geq \sup_{z \in U} \{R(x,z).R(z,y)\}$$

Total if for all x$\in$U R(x,y) >0 or R(y,x)>0.

Anti symmetric if R(x,y) >0 and R(y,x)>0 implies x=z.

Strongly fuzzy transitive if $$\mu(x,y) \geq \bigvee_{z \in U} \mu(x,z) \wedge \mu(z,y)$$

for all (x,y)$\in$R

It is clear there exist a fuzzy transitive relations R^* that R^* is strongly transitive and $R^*(x,y) \geq R(x,y)$(for example $R^*(x,y)=1$).

The fuzzy transitive closer of R

Let R^* is strongly transitive relations and $R^*(x,y) \geq R(x,y)$ and for any strongly transitive transitive relation S,$S(x,y) \geq R(x,y)$ $S(x,y) \geq R^*(x,y)$, then R^* is**.**

If U is reflexive, transitive and has n elements, then $\mathbf{R}^{\mathbf{n}-1} = \underbrace{\mathbf{R} \circ \mathbf{R} \circ \ldots \circ \mathbf{R}}_{(n-1)\times}$ is **fuzzy transitive closer of R** transitive closer of R.

Proof: Is evident**.** We leave it to reader.

Example: Let

$$R = \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix}$$

$$R^2 = \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} =$$

$$=\begin{pmatrix} \max\{1,.2,.2,.6\} & \max\{.2,.2,.5,.2\} & \max\{.5,.2,.5,.4\} & \max\{.7,.2,.5,.7\} \\ \max\{.3,.3,.2,.6\} & \max\{.2,1,.5,.2\} & \max\{.3,.5,.5,.4\} & \max\{.3,.7,.5,.7\} \\ \max\{.2,.3,.2,.6\} & \max\{.2,.5,.5,.4\} & \max\{.2,.5,1.4\} & \max\{.2,.5,.7,.7\} \\ \max\{.6,.2,.2,.6\} & \max\{.2.2,.4,.2\} & \max\{.5,.2,.4,.4\} & \max\{.6,.2,.4,1\} \end{pmatrix} =$$

$$=\begin{pmatrix} 1 & 0.5 & 0.5 & 0.7 \\ 0.6 & 1 & 0.5 & 0.7 \\ 0.6 & 0.5 & 1 & 0.7 \\ 0.6 & 0.4 & 0.5 & 1 \end{pmatrix}$$

$$R^3 = R^2 \circ R = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.7 \\ 0.6 & 1 & 0.5 & 0.7 \\ 0.6 & 0.5 & 1 & 0.7 \\ 0.6 & 0.4 & 0.5 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} =$$

$$=\begin{pmatrix} \max\{1,.3,.2,.6\} & \max\{.2,.5,.5,.2\} & \max\{.5,.5,.5,.4\} & \max\{.7,.5,.5,.7\} \\ \max\{.6,.3,.2,.6\} & \max\{.2,1,.5,.2\} & \max\{.5,.5,.5,.4\} & \max\{.6,.7,.5,.7\} \\ \max\{.6,.3,.2,.6\} & \max\{.2,.5,.5,.2\} & \max\{.5,.5,1.4\} & \max\{.6,.5,.7,.7\} \\ \max\{.6,.3,.2,.6\} & \max\{.2.4,.5,.2\} & \max\{.5,.4,.5,.4\} & \max\{.6,.4,.5,1\} \end{pmatrix} =$$

Let R^* is $= \begin{pmatrix} 1 & 0.5 & 0.5 & 0.7 \\ 0.6 & 1 & 0.5 & 0.7 \\ 0.6 & 0.5 & 1 & 0.7 \\ 0.6 & 0.5 & 0.5 & 1 \end{pmatrix}$ reflexive, symmetric relation then R^* is **fuzzy similarity relation.**

Example: The relation $\mathbf{R} = \begin{pmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 1 & 0 \\ 0.7 & 0 & 1 \end{pmatrix}$ is reflexive(R(x,x)=1 for all x) and symmetric(R(1,2)=R(2,1)=0.5, R(1,3)=R(3,1)=0.7, R(2,3)=R(3,2)=0) and so is is fuzzy similarity reletion.

The **converse fuzzy relation** is usually denoted as R^c is defined as

$$R^c(x,y)=R(y,x)$$

For all $x,y \in U$

Identity relation

$$I(x,x)=1 \text{ for all } x \in U$$

$$I(x,y)=0 \text{ for all } x \neq y \in U$$

Zero relation

$$o(x,y)=0 \text{ for all } x,y\in U$$

Universe relation

$$u(x,y)=1 \text{ for all } x,y\in U$$

Example: The following are examples of these relations

$$\mathbf{R}=\begin{pmatrix} 1 & 0.5 & 0.7 \\ 0.2 & 1 & 0 \\ 0.1 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{R}^{c}=\begin{pmatrix} 1 & 0.2 & 0.1 \\ 0.5 & 1 & 0 \\ 0.7 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}=\begin{pmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 1 & 0 \\ 0.7 & 0 & 1 \end{pmatrix} \quad O=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad U=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The Fuzzy equivalence relation.

Let R^* is reflexive, symmetric and is strongly fuzzy transitive relation then R^* is fuzzy similarity relation often called **fuzzy equivalence relation.**

Theorem: R is fuzzy equivalence relation if and only if its α-cut R_α is relation equivalence for all $\alpha\in\langle 0,1\rangle$.

Proof: Let R is fuzzy relation equivalence. Then R is fuzzy reflexive (R(x,y)=1) and so R_α(x,y)=1 and R_α is reflexive. R is symmetric(R(x,y)=R(y,x)). It implies $R_\alpha(x,y)=R_\alpha(y,x)$ and R_α is symmetric. R is transitive and so R_α is transitive too and $R\alpha$ is relation equivalence.

Let R_α is relation equivalence for all $\alpha\in\langle 0,1\rangle$. Then R is fuzzy reflexive, symmetric and transitive. It implies R is fuzzy relation equivalence.

Example: Let fuzzy relation is defined by its α*-cuts

$$R_{0.4}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\quad R_{0.5}=\begin{pmatrix}1&1&0&1\\1&1&0&1\\0&0&1&0\\1&1&0&1\end{pmatrix}\quad R_{0.8}=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$R_{0.9}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

All α-cuts are relations equivalence and so R is fuzzy relation equivalence.

The basic properties of fuzzy relations

We wil now try to give some basic properties of compositions of fuzzy relations which plays a major role in areas such as fuzzy control, fuzzy diagnosis and fuzzy expert systems.

1. $R\circ I=I\circ R=R$
2. $R\circ O=O\circ R=O$

3. In general $R \circ S \neq S \circ R$

4. $R^{m+1} = R^{m} \circ R = R$

5. $R^{m} \circ R^{n} = R^{n+m}$

6. $\left(R^{m}\right)^{n} = R^{mn}$

7. $(R \circ S) \circ T = R \circ (S \circ T)$

8. $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$

9. $R \circ (S \cap T) = (R \circ S) \cap (R \circ T)$

10. $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$

Fort inverse relarions

11. $(R \cup S)^{c} = R^{c} \cup S^{c}$

$(R \cap S)^{c} = R^{c} \cap S^{c}$

$(R \circ S)^{c} = R^{c} \circ S^{c}$

12. $\left(R^{c}\right)^{c} = R$

13. $R \subseteq S \Rightarrow R^{c} \subseteq S^{c}$

Minimum fuzzy equivalence closer of R.

Let R^{*} I fuzzy equivalence relation and $R^{*}(x,y) \geq R(x,y)$ and for any fuzzy equivalence relation S, S(x,y)≥R^*(x,y), then R^* is **minimum fuzzy equivalence closer of R.**

Example: Let

$$R = \begin{pmatrix} 0.9 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 0.4 & 0.7 \\ 0.6 & 0.2 & 0.4 & 0.8 \end{pmatrix}$$

What is minimum fuzzy equivalence closer of R?

The minimum fuzzy equivalence closer of R is fuzzy reflexive relation. The fuzzy relation is reflexive if for all $x \in U$ R(x,x)=1. The minimum reflexive relation $R^* \supseteq R$ is relation $R^*(x,x)=1$ and $R^*(x,y)=R(x,y)$ for all $x \neq y$. Hence

$$R^* = \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix}$$

The fuzzy relation is symmetric if for all $x,y \in U$ R(x,y)=R(y,x). The minimum symmetric relation $R^* \supseteq R$ is relation $R^*(x,y)=\max\{R(x,y),R(z,x)\}$ for all $x \neq y$. Hence

$$R^* = \begin{pmatrix} 1 & \max\{0.2,0.3\} & \max\{0.2,0.5\} & \max\{0.6,0.7\} \\ \max\{0.2,0.3\} & 1 & \max\{0.2,0.5\} & \max\{0.2,0.7\} \\ \max\{0.2,0.5\} & \max\{0.5,0.5\} & 1 & \max\{0.4,0.7\} \\ \max\{0.6,0.7\} & \max\{0.2,0.7\} & \max\{0.4,0.7\} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.5 & 0.5 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & \end{pmatrix}$$

The minimum fuzzy transitive relation fuzzy closer of R and if U is finite then $R^{*}=R^{n-1}$.

Hence

$$R^2 = \begin{pmatrix} 1 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.5 & 0.5 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.5 & 0.5 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{1,.3,.5,.7\} & \max\{.3,.3,.5,.7\} & \max\{.5,.3,.5,.7\} & \max\{.7,.3,.5,.7\} \\ \max\{.3,.3,.5,.7\} & \max\{.3,1,.5,.7\} & \max\{.3,.5,.5,.7\} & \max\{.3,.7,.5,.7\} \\ \max\{.5,.3,.5,.7\} & \max\{.3,.5,.5,.7\} & \max\{.5,.5,1.7\} & \max\{.5,.5,.7,.7\} \\ \max\{.7,.3,.5,.7\} & \max\{.3,.7,.5,.7\} & \max\{.5,.5,.7,.7\} & \max\{.7,.7,.7,1\} \end{pmatrix} =$$

$$=\begin{pmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix}$$

$$R^3 = R^2 \circ R = \begin{pmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.5 & 0.5 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix} \equiv$$

$$= \begin{pmatrix} \max\{1,.3,.5,.7\} & \max\{.3,.3,.5,.7\} & \max\{.5,.3,.5,.7\} & \max\{.7,.7,.7,.7\} \\ \max\{.3,.3,.5,.7\} & \max\{.3,1,.5,.7\} & \max\{.5,.5,.7,.7\} & \max\{.7,.7,.7,.7\} \\ \max\{.5,.3,.5,.7\} & \max\{.5,.5,.7,.7\} & \max\{.5,.5,1.7\} & \max\{.7,.7,.7,.7\} \\ \max\{.7,.7,.7,.7\} & \max\{.7,.7,.7,.7\} & \max\{.7,.7,.7,.7\} & \max\{.7,.7,.7,1\} \end{pmatrix} \equiv$$

$$=\begin{pmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{pmatrix}$$

If fuzzy relations is not symmetric then for symmetric closer of R pay

$R^*(x,y) \geq R(x,y)$ and $R^*(x,y) \geq R(y,x)$. At first we take $R^*(x,y)=\max\{ R(y,x), R(x,y) \}$. It can be interesting to take $R^*(x,y)=\min\{ R(y,x), R(x,y) \}$.

Example: Let

$$R = \begin{pmatrix} 1 & 0.2 & 0.5 & .7 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.7 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix}$$

Then the first estimation of R^* is

$$R' = \begin{pmatrix} 1 & 0.2 & 0.2 & 0.6 \\ 0.2 & 1 & 0.5 & 0.2 \\ 0.2 & 0.5 & 1 & 0.4 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix}$$

The minimum fuzzy transitive relation fuzzy closer of R´, f U is finite, is $R^*=R^{n-1}$. Hence

$$R^2 = \begin{pmatrix} 1 & 0.2 & 0.2 & 0.6 \\ 0.2 & 1 & 0.5 & 0.2 \\ 0.2 & 0.5 & 1 & 0.4 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.2 & 0.2 & 0.6 \\ 0.2 & 1 & 0.5 & 0.2 \\ 0.2 & 0.5 & 1 & 0.4 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \max\{1,.2,.2,.6\} & \max\{.2,.2,.2,.2\} & \max\{.2,.2,.2,.4\} & \max\{.6,.2,.2,.6\} \\ \max\{.2,.2,.2,.2\} & \max\{.2,1,.5,.2\} & \max\{.2,.5,.5,.2\} & \max\{.2,.2,.4,.4\} \\ \max\{.2,.2,.2,.4\} & \max\{.2,.5,.5,.2\} & \max\{.2,.5,1.4\} & \max\{.2,.2,.4,.4\} \\ \max\{.6,.2,.2,.6\} & \max\{.2,.2,.4,.4\} & \max\{.2,.2,.4,.4\} & \max\{.6,.2,.4,1\} \end{pmatrix} =$$

$$
= \begin{pmatrix} 1 & 0.2 & 0.4 & 0.6 \\ 0.2 & 1 & 0.5 & 0.4 \\ 0.4 & 0.5 & 1 & 0.4 \\ 0.6 & 0.4 & 0.4 & 1 \end{pmatrix}
$$

$$
R^3 = \begin{pmatrix} 1 & 0.2 & 0.4 & 0.6 \\ 0.2 & 1 & 0.5 & 0.4 \\ 0.4 & 0.5 & 1 & 0.4 \\ 0.6 & 0.4 & 0.4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0.2 & 0.2 & 0.6 \\ 0.2 & 1 & 0.5 & 0.2 \\ 0.2 & 0.5 & 1 & 0.4 \\ 0.6 & 0.2 & 0.4 & 1 \end{pmatrix} =
$$

$$
= \begin{pmatrix} 1 & 0.2 & 0.4 & 0.6 \\ 0.2 & 1 & 0.5 & 0.4 \\ 0.4 & 0.5 & 1 & 0.4 \\ 0.6 & 0.4 & 0.4 & 1 \end{pmatrix}
$$

As it is well known, within a classical context, an equivalence relation in a set defines a partition or a classification in it, and viceversa.

Fuzzy partial ordered relations

The fuzzy relation is **fuzzy partial ordered relation** if it satisfy following conditions

a) is reflexive(R(x,x)=1 for all $x\in U$)
b) is symmetric(If $R(x,y)\rangle 0 \Rightarrow R(y,x)=0$ for all $x\neq y$)
c) is transitive($R(x,z)\geq \sup\{\min\{R(x,y),R(y,z)\}$ for all $x,z\in U\}$

Example: Fuzzy relation $R=\begin{pmatrix}1 & 0,5 & 0.6 & 0.8\\ 0 & 1 & 0.7 & 0.9\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$ is fuzzy partial ordered relation

Note: Fuzzy relation R is fuzzy partial ordered relation if ad only if its α-cut is patial ordered relation for all $\alpha\in\langle 0,1\rangle$.