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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS

UNIT – I – CURVES AND SURFACES – SMT5204

PARTIAL DIFFERENTIAL EQUATIONS:

- If a dependent variable is a function of two or more independent variables, then an equation involving partial differential coefficients is called a partial differential equation.
- PDEs are models of various physical and geometrical problems, arising when the unknown functions (the solutions) depend on two or more variables.
- The simplest physical systems can be modeled by ODEs, whereas most problems in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics require PDEs.
- The range of applications of PDEs is enormous, compared to that of ODEs.
- PDEs of applied mathematics, the wave equations governing the vibrating string and the vibrating membrane, the heat equation, and the Laplace equation.
- Solving initial and boundary value problems, that is methods of obtaining solutions satisfying conditions that are given by the physical situation.
- PDEs can also be solved by Fourier and Laplace transform methods.
- Boundary Conditions: The condition that the solution u assume given values on the boundary of the region R.
- Initial Conditions: When time t is one of the variables, u (or Ut = ∂u/ ∂t or both) may be prescribed at t = 0

A relation between the variables (including the dependent one) and the partial differential coefficients of the dependent variable with the two or more independent variables is called a partial differential equation (p.d.e.)

For example:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + xy \qquad \ldots(1)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = u \qquad \ldots(3)$$

with

$$\left.\begin{array}{llll} \dfrac{\partial u}{\partial x} = p, & \dfrac{\partial u}{\partial y} = q & & \\ \dfrac{\partial^2 u}{\partial x^2} = r, & \dfrac{\partial^2 u}{\partial x \partial y} = s, & \dfrac{\partial^2 u}{\partial y^2} = t & \\ \ldots & \ldots & \ldots & \text{etc} \end{array}\right\} \qquad \ldots(4)$$

The order of a partial differential equation is the order of the highest order differential coefficient occuring in the equation and the degree of the partial differential equation is the degree of the highest order differential coefficient occurring in the equation.

Equation (1) is of Ist order Ist degree, equation (2) is of 2nd order Ist degree whereas equation (3) is of 2nd order 3rd degree.

Types of Partial Differential Equations

1. **Linear equation:** A first order p.d.e. is said to be a linear equation if it is linear in p, q and z, i.e., if it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y).$$

Example: $yp - xq = xyz + x$.

2. **Semi-linear equation:** A first order p.d.e. is said to be a semi-linear equation if it is linear in p and q and the coefficients of p and q are functions of x and y only, i.e., if it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z).$$

Example: $e^x p - yxq = xz^2$.

3. **Quasi-linear equation:** A first order p.d.e. is said to be a quasi-linear equation if it is linear in p and q, i.e., if it is of the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z).$$

Example: $(x^2 + z^2)p - xyq = z^3 x + y^2$.

4. **Non-linear equation:** Partial differential equations of the form $f(x, y, z, p, q) = 0$ that do not come under the previous three types are said to be non-linear equations.
   Example: $pq = z$ does not belong to any of the first three types. So it is a non-linear first order p.d.e. Also refer to Example 1.2.6.

- Homogeneous and non-homogeneous

  A linear PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives. Otherwise we call the equation **non-homogeneous**.

  If each term of the equation contains either the dependent variable or one of its derivatives, it is said to be homogeneous, otherwise, non-homogeneous. For example, equation (2) is homogeneous, whereas equation (1) is non-homogeneous

- The partial differential equation is said to be linear if the differential co-efficients occurring in it are of the I^{st} order only or in other word if in each of the term, the differential coefficients are not in square or higher powers or their product, otherwise, non-linear. e.g. $x^2 p + y^2 q = z$ is a linear in z and of first order

- A PDE is **linear** if it is of the first degree in the unknown function u and its partial derivatives. Otherwise it is **nonlinear**.

(1)	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional wave equation*
(2)	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional heat equation*
(3)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	*Two-dimensional Laplace equation*
(4)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	*Two-dimensional Poisson equation*
(5)	$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	*Two-dimensional wave equation*
(6)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	*Three-dimensional Laplace equation*

FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

- These equations are formed either by the elimination of arbitrary constants or by the elimination of the arbitrary functions from a relation with one dependent variable and the rest two or more independent variables.
- Observations: When p.d.e. formed by elimination of arbitrary constants
- If the number of arbitrary constants are more than the number of independent variables in the given relations, the p.d.e. obtained by elimination will be of 2nd or higher order.
- If the number of arbitrary constants equals the number of independent variables in the given relation, the p.d.e. obtained by elimination will be of order one.
- Observations: When p.d.e. formed by elimination of arbitrary functions.

When n is the number of arbitrary functions, we may get several p.d.e., but out of which generally one with two least order is selected.

e.g. z = x f(y) + yg (x) involves two arbitrary functions, f and g.

xys = xp + yq – z (second order) …(ii) are the two p.d.e. are obtained by elimination of the arbitrary functions. However, 2nd equation being in lower in order to 1st is the desired p.d.e.

Example 1. Form a pde by eliminating the arbitrary constants a and b from
z = (x + a) (y + b)

Solution:

d(1) p.w. r to x,

∂z/∂x = 1(y+b) => p = y+b

d. p.w. r to y

∂z/∂y= (x+a) 1 => q = x+a

Sub (2) and (3) in (1), z = qp

Example 2. Form the pde by eliminating the constants a and b from
$\mathbf{z= (x^2+a^2)\ (y^2+b^2)}$

G.T. $z=(x^2+a^2)(y^2+b^2)$ ----------- (1)

d (1) p w.r to x , $\frac{\partial z}{\partial x} = p = 2x(y^2+b^2) \Rightarrow \frac{p}{2x} = y^2 + b^2$ ----------- (2)

d (1) p w.r to y, $\frac{\partial z}{\partial y} = q = 2y(x^2+a^2) \Rightarrow \frac{q}{2y} = x^2 + a^2$ ----------- (3)

Substitute

(2) & (3) in (1)

$z = \frac{q}{2y}\frac{p}{2x} \Rightarrow pq = 4xyz$

Example 3. Find the pde of all spheres having their centres on the z-axis.

Let the Centre of the sphere be (0, 0, c) point on the Z –axis and 'r' it's radius.

$\therefore (x-0)^2+(y-0)^2+(z-c)^2=r^2$ [Since centre lies on Z axis]

ie, $x^2+y^2+(z-c)^2 = r^2$ ------- 1

d (1) p.w.r. to x, [c&r arbitrary constants]

$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$

$\Rightarrow x + p(z - c) = 0$ ---------(2)

d (1) p.w.r. to y,

$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$

$\Rightarrow y + q(z - c) = 0$ ---------(3)

From(2) and (3)

(2) $\Rightarrow z - c = -x/p$

(3) $\Rightarrow z - c = -y/q$

$\therefore -x/p = -y/q$

$\Rightarrow qx = py$, which is the required p.d.e

Example 4. Form the pde from $(x - a)^2 + (y - b)^2 = r^2$

Given that $(x-a)^2+(y-b)^2+z^2 = r^2$ ---------- **(1)**

d.p.w.r to x,

$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$ [z is a fun of x and y]

$\Rightarrow (x-a) + zp = 0$ ---------- **(2)**

d.p.w.r. to y, $p=\frac{\partial z}{\partial x}$

$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$ $q=\frac{\partial z}{\partial y}$ $= (y-b) + zq = 0$

Eliminating a and b from 1, 2 and 3

$2 \Rightarrow x - a = -zp$

$3 \Rightarrow y - b = -zq$

$\therefore (1) \Rightarrow (-zp)^2 + (-zq)^2 + z^2 = r^2$

$\Rightarrow z^2p^2 + z^2q^2 + z^2 = r^2$

$\Rightarrow z^2(p^2+q^2+1) = r^2$

which is the required p.d.e

Example 5. Form a partial differential equation by eliminating a, b, c from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution: Clearly in the given equation a, b, c are three arbitrary constants and z is a dependent variable, depending on x and y. We can write the given relations as:

$$f(x, y, z) = \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) = 0 \qquad \ldots(1)$$

then differentiating (1) partially with respect to x and y respectively, we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0, \quad \left(\text{Keeping } \frac{\partial y}{\partial x}=0\right)$$

and
$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0, \quad \left(\text{Keeping } \frac{\partial x}{\partial y}=0\right)$$

or
$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \quad \Rightarrow \quad c^2x + a^2zp = 0 \qquad \ldots(2)$$

and
$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \quad \Rightarrow \quad c^2y + b^2zq = 0 \qquad \ldots(3)$$

Again differentiating (2) with respect to x, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0$$

On substituting $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ from (2) in above equation, we get

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

or
$$xz\cdot\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \qquad \ldots(4)$$

Similarly, differentiating (3) partially with respect to y and substituting the value of $\frac{c^2}{b^2}$ from (3) in the resultant equation, we have

$$yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \qquad \ldots(5)$$

Thus equations (4) and (5) are 'partial differential equations' of first degree and second order.

Example 6. Eliminate the arbitrary function 'f' from z = f (y/x) and form a pde.

Solution:

Given that $z = f(y/x)$ -----------(1)

d 1 p.w. r to x, $p = \frac{\partial z}{\partial x} = f^1(y/x)(-y/x^2)$ ---------- (2)

d 1 p.w.r to y, $q = \frac{\partial x}{\partial y} = f^1(y/x)(1/x)$ -----------(3)

Now,

$$\frac{\cancel{(2)}}{\cancel{(3)}} \Rightarrow \frac{p}{q} = \frac{f^1\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)}{f^1\left(\frac{y}{x}\right)(1/x)}$$

$$\Rightarrow \frac{p}{q} = -\frac{y}{x^2} X \frac{x}{1}$$

$$\Rightarrow \frac{p}{q} = -\frac{y}{x}$$

$\Rightarrow px = -qy$

is, $px + qy = 0$ is the required p.d.e.

Example 7. Form the pdeby eliminating the arbitrary function from $z^2 - xy = f(x/z)$

Solution:

G.T $z^2 - xy = f(x/z)$ -------- (1)

d (1) p.w.r. to x

$$2z\frac{\partial z}{\partial x} - y = f^1(x/z)\left|\frac{z(0) - x\frac{\partial z}{\partial x}}{Z^2}\right|$$

$$2zp - y = f^1(x/z)\left|\frac{z - xp}{Z^2}\right| \quad \text{-------------(2)}$$

d (1) p.w.r. to y

$$2z\frac{\partial z}{\partial y} - x = f^1(x/z)\left|\frac{z(01) - x\frac{\partial z}{\partial y}}{Z^2}\right|$$

$$2zq - x = f^1(x/z)\left|\frac{-xq}{Z^2}\right| \quad \text{-------------(3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2zp - y}{2zq - x} = \frac{z - xp}{-xq}$$

$\Rightarrow (-xq)(2zp-y) = (2zq-x)(z-xp)$

$\Rightarrow -2xzpq + xyq = 2\cancel{z}^2q - 2xzpq - xz + x^2p$

$\Rightarrow xyq = 2z^2q - xz + x^2p$

$\Rightarrow x^2p + 2z^2q - xyq = xz$

$\Rightarrow x^2p - (xy - 2z^2)q = xz$ is the required p.d.e

Example 8. Form the pde of all planes cutting equal intercepts from the x and y axes.

Solution:

The equation of such plane is

x/a + y/a + z/b = 1 --------- 1 (x and y have equal intercepts)

p.d.w.r. to x

$$\frac{1}{a} + \frac{1}{b}\frac{\partial z}{\partial x} = 0$$

$\frac{1}{a} + \frac{p}{b} = 0$, p= $-\frac{b}{a}$ -------------------- 2

p.d.w.r. to y

$$\frac{1}{a} + \frac{1}{b}\frac{\partial z}{\partial y} = 0$$

$\frac{1}{a} + \frac{q}{b} = 0$ q= $-\frac{b}{a}$ -------------------- (3)

From (2) and (3)

p = q

p –q = 0 is the required p.d.e.

Example 9. Form the pde by eliminating the function from z = f(x + t) + g (x – t)

Solution: z = f(x + t) + g (x – t)

d.p.w.r. to x, p $=\frac{\partial z}{\partial x}=f^1(x+t)+g^1(x-t)$ ------------- (1)

d.p.w.r to t, q$=\frac{\partial z}{\partial t}=f^1(x+t)-g^1(x-t)$ ------------- (2)

$\frac{\partial^2 z}{\partial x^2}=f''(x+t)+g''(x-t)$ --------------(3)

$\frac{\partial^2 z}{\partial t^2}=f''(x+t)+g''(x-t)$ -------------- (4)

From (3)*and* (4), $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$

Example 10. Form partial differential equation from $z = x f_1(x + t) + f_2(x + t)$.

Solution: Clearly z is a function of x and t

$$p = \frac{\partial z}{\partial x} = f_1(x+t) + x f_1'(x+t) + f_2'(x+t)$$

$$q = \frac{\partial z}{\partial t} = x f_1'(x+t) + f_2'(x+t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f_1'(x+t) + x f_1''(x+t) + f_1'(x+t) + f_2''(x+t)$$

$$= 2 f_1'(x+t) + x f_1''(x+t) + f_2''(x+t)$$

$$s = \frac{\partial}{\partial t}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x \partial t} = f_1'(x+t) + x f_1''(x+t) + f_2''(x+t)$$

$$t = \frac{\partial^2 z}{\partial t^2} = x f_1''(x+t) + f_2''(x+t)$$

Now $\quad (r + t) = 2 f_1'(x + t) + 2x f_1''(x + t) + 2 f_2''(x + t) = 2s$

or
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - 2\frac{\partial^2 z}{\partial x \partial t} = 0$$

COMPATIBLE SYSTEMS OF FIRST-ORDER PDEs

A system of two first-order PDEs f (x, y, u, p, q) = 0 (1) and g(x, y, u, p, q) = 0 (2) are said to be compatible if they have a common solution

Equations (1) and (2) are compatible on a domain D if

(i) J = ∂(f ,g)/ ∂(p,q) = 0 on D.

(ii) p and q can be explicitly solved from (1) and (2) as p = φ(x, y, u) and q = ψ(x, y, u). Further, the equation du = φ(x, y, u)dx + ψ(x, y, u)dy is integrable.

Theorem:

A necessary and sufficient condition for the integrability of the equation du = φ(x, y, u)dx + ψ(x, y, u)dy is

[f , g] ≡ ∂(f , g) /∂(x, p) + ∂(f , g) /∂(y, q) + p ∂(f , g) /∂(u, p) + q ∂(f , g) /∂(u, q) = 0. ----(3)

equations (1) and (2) are compatible iff (3) holds

Example: Show that the equations xp − yq = 0, xup + yuq = 2xy are compatible and solve them.

Solution. Take f ≡ xp − yq = 0, g ≡ u(xp + yq) − 2xy = 0. Then fx = p, fy = −q, fu = 0, fp = x, fq = −y, gx = up − 2y, gy = uq − 2x, gu = xp + yq, gp = ux, gq = uy.

Compute

$$J \equiv \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} = \begin{vmatrix} x & -y \\ ux & uy \end{vmatrix} = uxy + uxy = 2uxy \neq 0$$

for $x \neq 0$, $y \neq 0$, $u \neq 0$. Further,

$$\begin{aligned}
\frac{\partial(f,g)}{\partial(x,p)} &= \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} = \begin{vmatrix} p & x \\ up-2y & ux \end{vmatrix} = uxp - x(up-2y) = 2xy \\
\frac{\partial(f,g)}{\partial(u,p)} &= \begin{vmatrix} f_u & f_p \\ g_u & g_p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & ux \end{vmatrix} = 0 - x(xp+yq) = -x^2p - xyq \\
\frac{\partial(f,g)}{\partial(y,q)} &= \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} = \begin{vmatrix} -q & -y \\ uq-2x & uy \end{vmatrix} = -quy + y(uq-2x) = -2xy \\
\frac{\partial(f,g)}{\partial(u,q)} &= \begin{vmatrix} f_u & f_q \\ g_u & g_q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = y(xp+yq) = y^2q + xyp.
\end{aligned}$$

It is an easy exercise to verify that

$$\begin{aligned}
[f,g] &\equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(u,p)} + q\frac{\partial(f,g)}{\partial(u,q)} \\
&= 2xy - x^2p^2 - xypq - 2xy + y^2q^2 + xypq \\
&= y^2q^2 - x^2p^2 \\
&= 0.
\end{aligned}$$

So the equations are compatible.

Next step is to determine p and q from the two equations xp − yq = 0, u(xp + yq) = 2xy. Using these two equations, we have uxp + uyq − 2xy = 0 =⇒ xp + yq = 2xy u =⇒ 2xp = 2xy u =⇒ p = y u = φ(x, y, u). and xp − yq = 0 =⇒ q = xp y = xy yu =⇒ q = x u = ψ(x, y, u).
Substituting p and q in du = pdx + qdy, we get udu = ydx + xdy = d(xy), and hence integrating, we obtain u 2 = 2xy + k, where k is a constant.

CHARPIT'S METHOD

It is a general method for finding the general solution of a nonlinear PDE of first-order of the form f (x, y, u, p, q) = 0.

Basic Idea: To introduce another partial differential equation of the first order g(x, y, u, p, q, a) = 0 which contains an arbitrary constant a and is such that

(i) equations can be solved for p and q to obtain p = p(x, y, u, a), q = q(x, y, u, a).
(ii) The equation du = p(x, y, u, a)dx + q(x, y, u, a)dy is integrable.
(iii) The compatability of equations yields

$$[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(u,p)} + q\frac{\partial(f,g)}{\partial(u,q)} = 0.$$

auxiliary equations:

$$\boxed{\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{du}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_u)} = \frac{dq}{-(f_y + qf_u)}}$$

These equations are known as Charpit's equations. Once an integral g(x, y, u, p, q, a) of this kind has been found, the problem reduces to solving for p and q, and then integrating equation.

Example: Find a general solution of $p^2 x + q^2 y = u$.

Solution. To find a general solution, we proceed as follows: • Step 1: (Computing fx , fy , fu, fp, fq). Set f ≡ p 2 x + q 2 y − u = 0. Then fx = p 2 , fy = q 2 , fu = −1, fp = 2px, fq = 2qy, and hence, pfp + qfq = 2p 2 x + 2q 2 y, −(fx + pfu) = −p 2 + p, −(fy + qfu) = −q 2 + q

Step 2: (Writing Charpit's equations and finding a solution $g(x, y, u, p, q, a)$).
The Charpit's equations (or auxiliary) equations are:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{du}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_u)} = \frac{dq}{-(f_y + qf_u)}$$

$$\Longrightarrow \frac{dx}{2px} = \frac{dy}{2qy} = \frac{du}{2(p^2x + q^2y)} = \frac{dp}{-p^2 + p} = \frac{dq}{-q^2 + q}$$

From which it follows that

$$\frac{p^2dx + 2pxdp}{2p^3x + 2p^2x - 2p^3x} = \frac{q^2dy + 2qydq}{2q^3y + 2q^2y - 2q^3y}$$

$$\Longrightarrow \frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qydq}{q^2y}$$

On integrating, we obtain

$\log(p^2 x) = \log(q^2 y) + \log a$
$p^2 x = a q^2 y,$

$$
\begin{aligned}
& p^2 x + q^2 y = u, \quad p^2 x = a q^2 y \\
\Longrightarrow \quad & (a q^2 y) + q^2 y = u \Longrightarrow q^2 y(1+a) = u \\
\Longrightarrow \quad & q^2 = \frac{u}{(1+a)y} \Longrightarrow q = \left[\frac{u}{(1+a)y}\right]^{1/2}.
\end{aligned}
$$

$$
\begin{aligned}
& p^2 = a q^2 \frac{y}{x} = a \frac{u}{(1+a)y} \frac{y}{x} = \frac{au}{(1+a)x} \\
\Longrightarrow \quad & p = \left[\frac{au}{(1+a)x}\right]^{1/2}.
\end{aligned}
$$

$$
\begin{aligned}
& du = \left[\frac{au}{(1+a)x}\right]^{1/2} dx + \left[\frac{u}{(1+a)y}\right]^{1/2} dy \\
\Longrightarrow \quad & \left(\frac{1+a}{u}\right)^{1/2} du = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy.
\end{aligned}
$$

Integrate to have

$$[(1+a)u]^{1/2} = (ax)^{1/2} + (y)^{1/2} + b$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – CURVES AND SURFACES, FIRST ORDER PDE – SMT5204

JACOBI METHOD

The auxiliary equation for Jacobi's method is

$$\frac{dx}{-f_{u_x}} = \frac{dy}{-f_{u_y}} = \frac{dz}{-f_{u_z}} = \frac{d_{u_x}}{f_x} = \frac{d_{u_y}}{f_y} = \frac{d_{u_z}}{f_z}$$

and solve the equation

$$du = u_x dx + \ u_y dy + u_z dz$$

Ex.1) Solve or find complete integral or integral curves of

$$z^2 + zu_z - u_x^2 + u_y^2 = 0$$

by Jacobi's method.

Sol. Let

$$f\left(x, y, z, u_x, \ u_y, u_z\right) = z^2 + zu_z - u_x^2 + u_y^2 = 0 \qquad (1)$$

By Jacobi's method, the auxiliary equation is

$$\frac{dx}{-f_{u_x}} = \frac{dy}{-f_{u_y}} = \frac{dz}{-f_{u_z}} = \frac{d_{u_x}}{f_x} = \frac{d_{u_y}}{f_y} = \frac{d_{u_z}}{f_z}$$

$$\frac{dx}{2u_x} = \frac{dy}{-2u_y} = \frac{dz}{-z} = \frac{d_{u_x}}{0} = \frac{d_{u_y}}{0} = \frac{d_{u_z}}{2z + u_z}$$

Consider

$$\frac{d_{u_x}}{0} \Rightarrow d_{u_x} = 0$$

Integrating, we get

$$u_x = a$$

similarly consider

$$\frac{d_{u_y}}{0} \Rightarrow d_{u_y} = 0$$

Integrating, we get

$$u_y = b$$

Now from equation (1), we have

$$z^2 + zu_z - a^2 + b^2 = 0$$

$$zu_z = a^2 - b^2 - z^2$$

$$u_z = \frac{a^2 - b^2 - z^2}{z}$$

Integrating, we have

$$u = ax + by + (a^2 - b^2)logz - \frac{z^2}{2} + c$$

This is the general solution or complete integral or integral curves or primitives of given DE (1).

Ex.1) Solve or find complete integral or integral curves of

$$p^2x + q^2y = z$$

by Jacobi's method.

Sol. The given PDE is

$$p^2x + q^2y = z$$

where $p = -\frac{u_x}{u_z}$ and $q = -\frac{u_y}{u_z}$

$$\left(-\frac{u_x}{u_z}\right)^2 x + \left(-\frac{u_y}{u_z}\right)^2 y = z$$

$$xu_x^2 + yu_y^2 = zu_z^2$$

$$xu_x^2 + yu_y^2 - zu_z^2 = 0$$

Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = xu_x^2 + yu_y^2 - zu_z^2 = 0 \qquad (1)$$

By Jacobi's method, the auxiliary equation is

$$\frac{dx}{-f_{u_x}} = \frac{dy}{-f_{u_y}} = \frac{dz}{-f_{u_z}} = \frac{d_{u_x}}{f_x} = \frac{d_{u_y}}{f_y} = \frac{d_{u_z}}{f_z}$$

$$\frac{dx}{-2xu_x} = \frac{dy}{-2yu_y} = \frac{dz}{2zu_z} = \frac{d_{u_x}}{u_x^2} = \frac{d_{u_y}}{u_y^2} = \frac{d_{u_z}}{-u_z^2}$$

Consider

$$\frac{dx}{-2xu_x} = \frac{d_{u_x}}{u_x^2}$$

$$\frac{dx}{-2x} = \frac{d_{u_x}}{u_x}$$

$$2\frac{d_{u_x}}{u_x} + \frac{dx}{x} = 0$$

Integrating, we get

$$2logu_x + logx = loga$$

$$logu_x^2 + logx = loga$$

$$\log(xu_x^2) = loga$$

$$xu_x^2 = a$$

$$u_x^2 = \frac{a}{x}$$

$$u_x = \sqrt{\frac{a}{x}}$$

Similarly, consider

$$\frac{dy}{-2yu_y} = \frac{d_{u_y}}{u_y^2}$$

$$\frac{dy}{-2y} = \frac{d_{u_y}}{u_y}$$

$$2\frac{d_{u_y}}{u_y} + \frac{dy}{y} = 0$$

Integrating, we get

$$2logu_y + logy = logb$$

$$logu_y^2 + logy = logb$$

$$\log(yu_y^2) = logb$$

$$yu_y^2 = b$$

$$u_y^2 = \frac{b}{y}$$

$$u_y = \sqrt{\frac{b}{y}}$$

Now from equation (1), we have

$$x\frac{a}{x} + y\frac{b}{y} - zu_z^2 = 0$$

$$zu_z^2 = a + b$$

$$u_z^2 = \frac{a+b}{z}$$

$$u_z = \sqrt{\frac{a+b}{z}}$$

Consider equation

$$du = \sqrt{\frac{a}{x}}dx + \sqrt{\frac{b}{y}}dy + \sqrt{\frac{a+b}{z}}dz$$

Integrating, we have

$$u = 2\sqrt{ax} + 2\sqrt{by} + \sqrt{(a+b)z} + c$$

This is the general solution or complete integral or integral curves or primitives of given DE (1).

Integral surface passing through a given a curve

Consider the first order linear PDE $Pp + Qq = R$

We know that the auxiliary system associated with the given PDE is given by

$$\frac{dx}{Q} = \frac{dy}{P} = \frac{dz}{R}$$

Let $u(x, y, z) = c_1$ and $\mathrm{v}(x, y, z) = c_2$ represent the integral surface of the above system.

Suppose that $[x(t),\ y(t),\ z(t)]$ be the parametric form of the curve passing through the above integral surface

i.e., $u[x(t),\ y(t),\ z(t)] = 0$

and $\mathrm{v}[x(t),\ y(t),\ z(t)] = 0$

The general integral of the given PDE is $f(u, v) = 0$, subject to the condition that $f(c_1, c_2) = 0$

Exercise: Find the equation of the integral surface of the PDE

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

Sol: The auxiliary system is given by

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$$

Taking the 1st and 3rd fraction of (1.62a), we get

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\Rightarrow \qquad \frac{dx}{2(z-3)} = \frac{dz}{2x-3}$$

$$\Rightarrow \qquad (2x-3)dx = (2z-6)dz$$

Integrating we get

$$x^2 - 3x = z^2 - 6z + c_1$$

or

$$x^2 - 3x - z^2 + 6z = c_1$$

Using $(0,\ y, -1)$ as multipliers each fraction

$$\frac{ydy - dz}{2xy - yz - 2yx + 3y} = \frac{ydy - dz}{-yz + 3y} = \frac{ydy - dz}{y(3-z)}$$

Equating this expression with 1st fraction of

$$\frac{dx}{2y(z-3)} = \frac{ydy-dz}{y(3-z)}$$

$\Rightarrow$ $$\frac{dx}{2y(z-3)} = \frac{ydy-dz}{y(3-z)}$$

$\Rightarrow$ $$\frac{dx}{2} = \frac{ydy-dz}{-1}$$

$\Rightarrow$ $$dx + 2ydy - 2dz = 0$$

Integrating we get

$$x + y^2 - 2z = c_2$$

Now the given curve is $x^2 + y^2 = 2x, \quad z = 0,$

This equation can also be written as $(x-1)^2 + (y-0)^2 = 1$ which is circle with centre $(1,0)$ and radius 1. The corresponding parametric equation is

$$x - 1 = \cos\theta\,, \quad y - 0 = \sin\theta\,, \quad z = 0.$$

or $$x = 1 + \cos\theta\,, \quad y = \sin\theta\,, \quad z = 0.$$

By the given condition, the integral surface passes through the above circle. Therefore

$$1 + \cos\theta + \sin^2\theta - 2(0) = c_2$$

$\Rightarrow$ $$1 + \cos\theta + 1 - \cos^2\theta = c_2$$

$\Rightarrow$ $$2 + \cos\theta - \cos^2\theta = c_2$$

Also from (1.63), we get

$$(1 + \cos\theta)^2 - 3(1 + \cos\theta) - 0^2 + 6(0) = c_1$$

$\Rightarrow$ $$1 + 2\cos\theta + \cos^2\theta - 3 - 3\cos\theta = c_1$$

$\Rightarrow$ $$\cos^2\theta - 2 - \cos\theta = c_1$$

$\Rightarrow$ $$-(2 + \cos\theta - \cos^2\theta) = c_1$$

From (1.65) and (1.66), we get

$$c_1 = -c_2$$

or $$c_1 + c_2 = 0$$

$$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

$$x^2 + y^2 - z^2 - 2x + 4z = 0$$

<u>Integral surface orthogonal to given surface:</u>

Consider the linear partial differential equation

$$Pp + Qq = R$$

Let $$f(x, y, z) = c$$

be the integral surface of (1.67), also for any surface

$$z = g(x, y)$$

Exercises: Find the surface which is orthogonal to one parameter system $z = cxy(x^2+y^2)$ and passes through the hyperbola $x^2 - y^2 = a^2, \quad z = 0.$

Solution: The given one parameter system is $\frac{xy(x^2+y^2)}{z} = \frac{1}{c}$

Let $\qquad f(x,y,z) = \frac{xy(x^2+y^2)}{z}$

Now, $\qquad P = f_x = \frac{y(x^2+y^2)+2x^2y}{z}$

$$Q = f_y = \frac{x(x^2+y^2)+2xy^2}{z}$$

and $\qquad R = f_z = \frac{-xy(x^2+y^2)}{z^2}$

now auxiliary system of equations are

$$\frac{dx}{\frac{y(x^2+y^2)+2x^2y}{z}} = \frac{dy}{\frac{x(x^2+y^2)+2xy^2}{z}} = \frac{dz}{\frac{-xy(x^2+y^2)}{z^2}}$$

or
$$\frac{dx}{y(x^2+y^2)+2x^2y} = \frac{dy}{x(x^2+y^2)+2xy^2} = \frac{zdz}{-xy(x^2+y^2)}$$

Using multipliers $(x, y, 1)$, each ratio of (1.70) is equal to

$$\frac{xdx+ydy+zdz}{3x^3y+xy^3+x^3y+3xy^3-x^3y-xy^3} = \frac{xdx+ydy+zdz}{3xy(x^2+y^2)}$$

Equating this with 3rd term of (1.70), we get

$$\frac{xdx+ydy+zdz}{3xy(x^2+y^2)} = \frac{zdz}{-xy(x^2+y^2)}$$

$$\Rightarrow \qquad xdx + ydy + zdz = -3zdz$$

$$\Rightarrow \qquad xdx + ydy + 4zdz = 0$$

Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{4z^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow \qquad x^2 + y^2 + 4z^2 = c_1$$

Using multipliers $(x, y, 0)$ and $(x, -y, \ 0)$ and equating the two fractions we get

$$\frac{xdx+ydy}{3x^3y+xy^3+x^3y+3xy^3} = \frac{xdx-ydy}{3x^3y+xy^3-x^3y-3xy^3}$$

$$\Rightarrow \qquad \frac{xdx+ydy}{4x^3y+4xy^3} = \frac{xdx-ydy}{2x^3y-2xy^3}$$

$$\Rightarrow \qquad \frac{xdx+ydy}{x^2+y^2} = \frac{2(xdx-ydy)}{x^2-y^2}$$

Integrating, we get

$$\log(x^2+y^2) = 2\log(x^2-y^2) + \log c_2$$

$$\Rightarrow \qquad \frac{x^2+y^2}{(x^2-y^2)^2} = c_2$$

Now parametric systems of hyperbola is

$$x = a\sec\theta\,, \quad x = a\tan\,\theta\,, \quad z = 0$$

$\therefore$ from (1.72),

$$c_1 = a^2\ \sec^2\theta + a^2\tan^2\theta$$

$$\Rightarrow \qquad c_1 = a^2\,(\sec^2\theta + \tan^2\theta)$$

$$c_2 = \frac{a^2\ \sec^2\theta + a^2\tan^2\theta}{(a^2\ \sec^2\theta - a^2\tan^2\theta)^2}$$

$$\Rightarrow \qquad c_2 = \frac{\sec^2\theta + \tan^2\theta}{a^2(\sec^2\theta - \tan^2\theta)^2}$$

$$\Rightarrow \qquad c_2 = \frac{\sec^2\theta + \tan^2\theta}{a^2(1)^2}$$

$$\Rightarrow \qquad c_2 = \frac{\frac{c_1}{a^2}}{a^2(1)^2}$$

$$\Rightarrow \qquad c_2 = \frac{c_1}{a^4}$$

$$\Rightarrow \qquad a^4 c_2 = c_1$$

$$\Rightarrow \qquad c_1 = a^4 c_2$$

$\therefore$ The required surface orthogonal to the given system is

$$\frac{(x^2-y^2)^2(x^2+y^2+4z^2)}{x^2+y^2} = a^4$$

QUASILINEAR EQUATION

Quasilinear Equations: The Method of Characteristics

The quasilinear partial differential equation in two independent variables,

a(x, y, u)u_x + b(x, y, u)u_y − c(x, y, u) = 0.

dt = dx/ a = dy/ b = du / c .

Example: Find the general solution: $u_x + u_y - u = 0$.

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}.$$

/e can pair the differentials in three ways:

$$\frac{dy}{dx} = 1, \quad \frac{du}{dx} = u, \quad \frac{du}{dy} = u.$$

ı two of these relations are independent. We focus on the
he first equation gives the characteristic curves in the xy
sily solved to give

$$y = x + c_1.$$

second equation can be solved to give $u = c_2 e^x$.

general solution of the differential equation as

$$u(x, y) = G(y - x)e^x.$$

Find solutions of $u_x + u_y - u = 0$ subject to $u(x, 0) = 1$.

We found the general solution to the partial differential equation as $u(x, y) = G(y - x)e^x$. The side condition tells us that $u = 1$ along $y = 0$. This requires

$$1 = u(x, 0) = G(-x)e^x.$$

Thus, $G(-x) = e^{-x}$. Replacing x with $-z$, we find

$$G(z) = e^z.$$

Thus, the side condition has allowed for the determination of the arbitrary function $G(y - x)$. Inserting this function, we have

$$u(x, y) = G(y - x)e^x = e^{y-x}e^x = e^y.$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –III – SECOND ORDER PDE – SMT5204

Second order partial differential equations in two variables

The general second order partial differential equations in two variables is of the form

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}) = 0.$$

The equation is *quasi-linear* if it is linear in the highest order derivatives (second order), that is if it is of the form

$$a(x, y, u, u_x, u_y)u_{xx} + 2\, b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d(x, y, u, u_x, u_y)$$

We say that the equation is *semi-linear* if the coefficients a, b, c are independent of u. That is if it takes the form

$$a(x, y)\,)u_{xx} + 2b(x, y)\, u_{xy} + c(x, y)\, u_{yy} = d(x, y, u, u_x, u_y)$$

Finally, if the equation is semi-linear and d is a *linear* function of u, u_x and u_y, we say that the equation is *linear*. That is, when F is linear in u and all its derivatives.

Finally, if the equation is semi-linear and d is a *linear* function of u, u_x and u_y, we say that the equation is *linear*. That is, when F is linear in u and all its derivatives.

Let $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ be an invertible transformation of coordinates. That is,

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{vmatrix} \neq 0.$$

By the chain rule

$$u_x = u_\xi \phi_x + u_\eta \psi_x, \quad u_y = u_\xi \phi_y + u_\eta \psi_y$$

$$u_{xx} = u_\xi \phi_{xx} + \phi_x\left(u_{\xi\xi}\phi_x + u_{\xi\eta}\psi_x\right) + u_\eta \psi_{xx} + \psi_x\left(u_{\eta\xi}\phi_x + u_{\eta\eta}\psi_x\right)$$

$$= u_{\xi\xi}\phi_x^{\ 2} + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x^{\ 2} + \text{ first order derivatives of } u$$

Similarly,

$$u_{yy} = u_{\xi\xi}\phi_y^{\ 2} + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y^{\ 2} + \text{ first order derivatives of } u$$

$$u_{xy} = u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}\left(\phi_x\psi_y + \phi_y\psi_x\right) + u_{\eta\eta}\psi_x\psi_y + \text{ first order derivatives of } u$$

Substituting into the partial differential equation we obtain,

$$\boxed{A(\xi, \eta)u_{\xi\xi} + 2B(\xi, \eta)u_{\xi\eta} + C(\xi, \eta)u_{\eta\eta} = D(\xi, \eta, u, u_{\xi}, u_{\eta})}$$

where

$$A(\xi, \eta) = a\phi_x^{\,2} + 2b\phi_x\phi_y + c\phi_y^{\,2}$$

$$B(\xi, \eta) = a\phi_x\psi_x \; + b(\phi_x\psi_y + \psi_x\phi_y) + c\phi_y\psi_y$$

$$C(\xi, \eta) = a\psi_x^{\,2} + 2b\psi_x\psi_y + c\psi_y^{\,2}. \qquad B^2 - AC = (b^2 - ac)\left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right)^2.$$

Therefore $B^2 - AC$ has the same sign as $b^2 - ac$. We will now choose the new coordinates $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ to simplify the partial differential equation.
$\phi(x, y)$ = constant ,$\psi(x, y)$ = constant defines two families of curves in $\mathbf{R}^2$. On a member of the family $\phi(x, y)$ = constant, we have that

$$\frac{d\phi}{dx} = \phi_x + \phi_y\, y' = 0.$$

Therefore substituting in the expression for $A(\xi, \eta)$ we obtain

$$A(\xi, \eta) = a\,\phi_y^{\,2}\, y'^2 - 2b\,\phi_y^{\,2}\, y' + c\phi_y^{\,2}$$

$$= \phi_y^{\,2}[a\; y'^2 - 2b\,y' + c\,].$$

This nonlinear ordinary differential equation is called *the characteristic equation* of the partial differential equation and provided that $a \neq 0$, $b^2 - ac > 0$ it can be written as

$$\boxed{y' = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

For this choice of coordinates $A(\xi, \eta) = 0$ and similarly it can be shown that $C(\xi, \eta) = 0$ also. The partial differential equation becomes

$$2\,B(\xi, \eta)\,u_{\xi\eta} = D(\xi, \eta, u, u_{\xi}, u_{\eta})$$

where it is easy to show that $B(\xi, \eta) \neq 0$. Finally, we can write the partial differential equation in the *normal form*

$$\boxed{u_{\xi\eta} = D(\xi, \eta, u, u_\xi, u_\eta)}$$

The two families of curves $\phi(x, y)$ = constant, $\psi(x, y)$= constant obtained as solutions of the characteristic equation are called *characteristics* and the semi-linear partial differential equation is called *hyperbolic* if $b^2 - ac > 0$ whence it has two families of characteristics and a normal form as given above.

If $b^2 - ac < 0$, then the characteristic equation has *complex* solutions and there are no real characteristics. The functions $\phi(x, y)$, $\psi(x, y)$ are now complex conjugates. A change of variable to the *real* coordinates

$$\xi = \phi(x, y) + \psi(x, y), \quad \eta = -i(\phi(x, y) - \psi(x, y))$$

results in the partial differential equation where the *mixed* derivative term vanishes,

$$\boxed{u_{\xi\xi} + u_{\eta\eta} = D(\xi, \eta, u, u_\xi, u_\eta).}$$

In this case the semi-linear partial differential equation is called *elliptic* if $b^2 - ac < 0$. Notice that the left hand side of the normal form is the Laplacian. Thus Laplaces equation is a special case of an elliptic equation (with $D = 0$).

$$\boxed{u_{\xi\xi} = D(\xi, \eta, u, u_\xi, u_\eta)}$$

The partial differential equation is called *parabolic* in the case $b^2 - a = 0$. An example of a parabolic partial differential equation is the equation of *heat conduction*

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \text{ where } u = u(x, t).$$

Classify, reduce to normal form and obtain the general solution of the partial differential equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 4x^2$$

For this equation $b^2 - ac = (xy)^2 - x^2 y^2 = 0$ $\therefore$ the equation is parabolic everywhere in the plane (x, y). The characteristic equation is

$$y' = \frac{b}{a} = \frac{xy}{x^2} = \frac{y}{x}.$$

Therefore there is one family of characteristics $\frac{y}{x}$ = constant.

Let $\xi = x$ and $\eta = \frac{y}{x}$. Then using the chain rule,

$$u_x = u_\xi \ 1 + u_\eta \left(\frac{-y}{x^2} \right) = u_\xi - \frac{y}{x^2} u_\eta$$

$$u_y = u_\xi \ 0 + u_\eta \left(\frac{1}{x} \right) = \frac{1}{x} u_\eta$$

$$u_{xx} = u_{\xi\xi} \ 1 + u_{\xi\eta} \left(\frac{-y}{x^2} \right) + \frac{2y}{x^3} u_\eta - \frac{y}{x^2} \left(u_{\eta\xi} 1 + u_{\eta\eta} \left(\frac{-y}{x^2} \right) \right)$$

$$= u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta} + \frac{y^2}{x^4} u_{\eta\eta} + \frac{2y}{x^3} u_\eta$$

$$u_{yy} = \frac{1}{x} \left(u_{\eta\xi} 0 + u_{\eta\eta} \left(\frac{1}{x} \right) \right) = \frac{1}{x^2} u_{\eta\eta}$$

$$u_{yx} = -\frac{1}{x^2} u_\eta + \frac{1}{x} \left(u_{\eta\xi} 1 + u_{\eta\eta} \left(-\frac{y}{x^2} \right) \right)$$

$$= \frac{1}{x} u_{\xi\eta} - \frac{y}{x^3} u_{\eta\eta} - \frac{1}{x^2} u_\eta \ .$$

Substituting into the partial differential equation we obtain the normal form

$$u_{\xi\xi} = 4.$$

Integrating with respect to ξ

$$u_\xi = 4\xi + f(\eta)$$

where f is an arbitrary function of a real variable. Integrating again with respect to ξ

$$u(\xi, \eta) = 2\xi^2 + \xi f(\eta) + g(\eta),$$

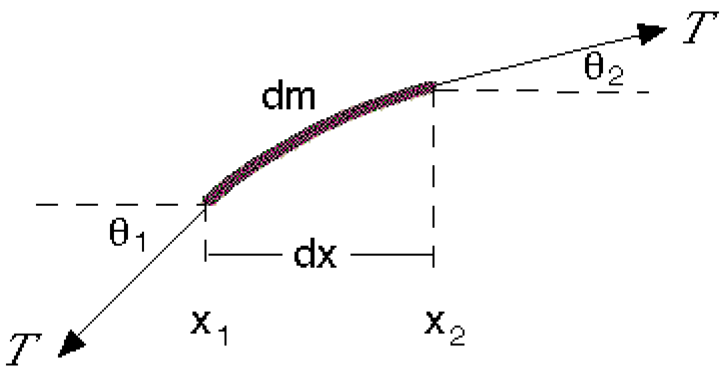
Therefore the general solution is given by

$$u(x, y) = 2x^2 + xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$

where f, g are arbitrary functions of a real variable.

Deriving the wave equation

Let's consider a string that has mass per unit length is μ. It is stretched by a tension T, which is much larger than the weight of the string and its equilibrium position is along the x axis. This diagram shows a short section of the string, stretched in the x direction, and the forces acting on it. Our analysis only applies for small deformations, for which the string is a linear medium, and we



neglect the gravitational force on the string (which in any case is constant).

One consequence of this restriction to small deformations is that the angle θ between the string and the x direction is much smaller than 1, so $\sin\theta \cong \theta$ and $\cos\theta \cong 1$. (On our diagram, however, the deformation has been exaggerated for clarity.) It also follows that the length of the segment shown is dx.

Let's apply Newton's second law in the vertical y direction:

$$F_y = ma_y.$$

The sum of forces in the y direction is

$$F_y = T\sin\theta_2 - T\sin\theta_1.$$

Using the small angle approximation, $\sin\theta \cong \tan\theta = \partial y/\partial x$. So we may write:

$$F_y = T\left(\frac{\partial y}{\partial x}\right)_2 - T\left(\frac{\partial y}{\partial x}\right)_1$$

So the total force depends on the difference in slope between the two ends: if the string were straight, no matter what its slope, the two forces would add up to zero. Now let's get quantitative. The mass per unit length is μ, so its mass dm = μdx. The acceleration in the y direction is the rate of change in the y velocity, so $a_y = \partial v_y/\partial t = \partial y^2/\partial t^2$. So we can write Newton's second law in the y direction as

$$F_y = T\left(\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right) = \mu\, dx\, \frac{\partial^2 y}{\partial t^2}$$

Rearranging this gives

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\,\frac{\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1}{dx}$$

Now we have been using the subscript 1 to identify the position x, and 2 to identify the position (x+dx). So the numerator in the last term on the right is difference between the (first) derivatives at these two points. When we divide it by dx, we get the rate of change of the first derivative with respect to x, which is, by definition, the second derivative, so we have derived the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\,\frac{\partial^2 y}{\partial x^2}$$

So the acceleration (on the left) is proportional to the tension *T* and inversely proportional to the mass per unit length μ. It is also proportional to $\partial y^2/\partial x^2$. So the a greater curvature in the string produces a greater acceleration and, as we have seen, a straight portion is not accelerated.

A solution to the wave equation

The wave equation is a partial differential equation. Sine waves can propagate in a one dimensional medium like a string. And we know that any function f(x − vt) is a wave travelling at speed v. In the first chapter on travelling waves, we saw that an elegant version of the general expression for a sine wave travelling in the positive x direction is y = A sin (kx − ωt + φ). A suitable choice of x or t axis allows us to set φ to zero, so let's look at the equation

$$y = A \sin (kx - \omega t)$$

to see whether and when this is a solution to the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}$$

SEMI-INFINITE STRING

Example *Consider the initial boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for} \quad x > 0,\, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for} \quad x > 0 \\ u(0,t) = 0 & \text{for } t \geq 0, \end{cases}$$

where $g(0) = 0 = h(0)$. If we extend g and h as **odd** *functions on $-\infty < x < \infty$, show that d'Alembert's formula gives the solution.*

Proof. Extend g and h as **odd** functions on $-\infty < x < \infty$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{for} \quad -\infty < x < \infty,\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x), \quad \tilde{u}_t(x,0) = \tilde{h}(x) & \text{for} \quad -\infty < x < \infty. \end{cases}$$

To show that d'Alembert's formula gives the solution to , we need to show that the solution given by d'Alembert's formula satisfies the boundary condition $\tilde{u}(0,t) = 0$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(\xi)\, d\xi, \\ \tilde{u}(0,t) &= \frac{1}{2}(\tilde{g}(ct) + \tilde{g}(-ct)) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{h}(\xi)\, d\xi \\ &= \frac{1}{2}(\tilde{g}(ct) - \tilde{g}(ct)) + \frac{1}{2c}(H(ct) - H(-ct)) \\ &= 0 + \frac{1}{2c}(H(ct) - H(ct)) = 0, \end{aligned}$$

where we used $H(x) = \int_0^x \tilde{h}(\xi)\, d\xi$; and since $\tilde{h}$ is **odd**, then H is **even**.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – BOUNDARY VALUE PROBLEMS – SMT5204

Initial Value Problem: An *initial value problem* is one in which the dependent variable and possibly its derivatives are specified initially (*i.e*. at time $t = 0$) or at the same value of independent variable in the equation. Initial value problems are generally time-dependent problems.

Pure Initial Value Problem (Cauchy Problem):

i) $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t \geq 0$$

along with the initial condition

$$u(x,0)=f(x)\ , \quad -\infty < x < \infty\ .$$

ii) $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t \geq 0$$

along with the initial conditions

$$u(x,0)=f(x)$$

$$u_t(x,0)=g(x)\ , \qquad -\infty < x < \infty$$

a) Initial Boundary Value Problem :

i) $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad x \geq 0\ , \ t \geq 0$$

along with the initial condition

$$u(x,0)=f(x) \qquad 0 \leq x < \infty$$

and the boundary condition

$$u(0,t)=\alpha_1(t)\ , \quad u_x(0,t)=\alpha_2(t)\ , \quad t \geq 0\ .$$

ii) $$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} \qquad 0 \leq x \leq 1\ , \ t \geq 0$$

along with the initial condition

$$u(x,0)=f(x) \qquad 0 \leq x \leq 1$$

and the boundary conditions

$$u(0,t)=\alpha_1(t)\ , \quad u(1,t)=\alpha_2(t), \quad t \geq 0\ .$$

Boundary Value Problems: A *boundary value problem* is one in which the dependent variable and possibly its derivatives are specified at the extreme of the independent variable. For steady state equilibrium problems, the auxiliary conditions consist of boundary conditions on the entire boundary of the closed solution domain. There are three types of boundary condition:

i. **Dirichlet boundary condition:** *Dirichlet boundary condition* is a type of boundary condition, named after Johann Peter Gustav Lejeune Dirichlet (1805–1859). When imposed on an ordinary or a partial differential equation, it specifies the values a solution needs to take on the boundary of the domain. For example if an iron rod had one end held at absolute zero then the value of the problem would be known at that point in space. The question of finding solutions to such equations is known as the *Dirichlet problem*. It is also called first type boundary condition.

ii. **Neumann boundary condition:** The *Neumann boundary condition* is a type of boundary condition, named after Carl Neumann. When imposed on an ordinary or a partial differential equation, it specifies the values that the derivative of a solution is to take on the boundary of the domain. For example if one iron rod had heater at one end then energy would be added at a constant rate but the actual temperature would not be known. It is also called second type boundary condition.

iii. **Mixed boundary condition:** The linear combination of Dirichlet and Neumann boundary conditions specified on the boundary is known as *mixed boundary condition*. Mixed boundary conditions are also known as Cauchy boundary conditions. A Cauchy boundary condition imposed on an ordinary or a partial differential equation specifies both the values a solution of a differential equation is to take on the boundary of the domain and the normal derivative at the boundary. It is also called Robin's boundary condition.

BOUNDARY VALUE PROBLEM

1. *Heat Equation:* $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$

In two dimensional $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$

2. *Wave Equation:* $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

3. *Laplace Equation:* $\nabla^2 u = 0$

Solution of one dimensional wave equation with initial value problem

Let problem be $u_{tt} - c^2 u_{xx} = 0,\ -\infty < x < \infty, t \geq 0$ ...(1)

Initial Conditions $u(x,0) = \eta(x), u_t(x,0) = v(x)$ ...(2)

Let characteristic lines be $\xi = x - ct, \eta = x + ct$

$\therefore$ *we have* $u_x = \mu_\xi \xi_x + u_\eta \eta_x = \mu_\xi + \mu_n$

$u_t = \mu_\xi \xi_t + u_\eta \eta_t = c(\mu_\eta - \mu_\xi)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(u_\xi + u_n) = \mu_{\xi\xi} + 2u_{\xi\eta} + \mu_{\eta\eta} \quad ...(3)$$

Similarly, $\frac{\partial^2 u}{\partial t^2} = c^2(\mu_{\xi\xi} - 2u_{\xi\eta} + \mu_{\eta\eta})$ ...(4)

Substituting (3) and (4) in equation (1), we get $4\mu_{\xi\eta} = 0$

Integrating, $\mu(\xi, \eta) = \phi(\xi) + \psi(\eta)$, *where* ϕ *and* ψ *are arbitrary functions*

$\therefore$ *the general solution is given by*

$\mu(x,t) = \phi(x - ct) + \psi(x + ct)$ ...(5)

Substituting initial conditions in (5), we - -

$\phi(x) + \psi(x) = \eta(x)$ ...(6)

$c[\phi'(x) - \psi'(x)] = v(x)$

Integrating, $\phi(x) - \psi(x) = \frac{1}{c}\int_0^x v(\xi)d\xi$ ...(7)

From equation (6) and (7), $\phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(\xi)d\xi$ *and* $\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(\xi)d\xi$

$\therefore$ *equation (5) gives* $u(x,t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi$,

which is known as D'Alembert's solution of one dimensional wave equation.

If v=0 then u(x,t)= $\frac{1}{2}[\eta(x+ct) - \eta(x-ct)]$

Solution of wave equation with initial and boundary conditions.

Consider one dimensional wave equation

$u_{tt} = c^2 u_{xx},\ 0 \leq x \leq L, t > 0$ ...(1)

Boundary Conditions: $u(0,t) = 0 = (L,t), t > 0$

Initial Conditions: $u(x,0) = f(x), u_t(x,0) = g(x)$

Let u(x,t) be solution of (1)

u (x, t) = X(x) T(t) and substituting into equation (1) we obtain

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

i.e., $\frac{d^2X/dx^2}{X} = \frac{d^2T/dt^2}{c^2T} = k$ ...(2)

***Case- I** When k>0, we have* $k=\lambda^2$. *Then* $\frac{d^2X}{dx^2}-\lambda^2X=0$ *and* $\frac{d^2T}{dt^2}-c^2\lambda^2T=0$

Their solution can be put in the form

$X=c_1e^{\lambda x}+c_2e^{-\lambda x}$

$T=c_3e^{c\lambda t}+c_4e^{-c\lambda t}$

Therefore, $u(x,t)=(c_1e^{\lambda x}+c_2e^{-\lambda x})(c_3e^{c\lambda t}+c_4e^{-c\lambda t})$ ...(3)

***Case II**: Let k=0. Then we have* $\frac{d^2X}{dx^2}=0, \frac{d^2T}{dt^2}=0$

Their solutions are found to be X=Ax + B, T= Ct + D

therefore, the required solution of the PDE is u(x, t) = (Ax + B)(Ct + D)

***Case III:** When k<0, say* $k=-\lambda^2$, *the differential equations are*

$$\frac{d^2X}{dx^2}+\lambda^2X=0, \frac{d^2T}{dt^2}+c^2\lambda^2T=0$$

Their general solutions give

$$u(x,t)=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos c\lambda t+c_4\sin c\lambda t)$$

***Example 1.** Let u(x,t) be solution of initial boundary value problem*

$$\frac{\partial^2u}{\partial t^2}=\frac{\partial^2u}{\partial x^2}, 0<x<\infty, t>0$$

$$u(x,0)=\cos\left(\frac{\pi x}{2}\right) \quad 0\le x<\infty$$

$$\frac{\partial u}{\partial t}(x,0)=0, 0\le x<\infty$$

$$\frac{\partial u}{\partial x}(0,t)=0, \quad t\ge 0$$

Solution: $u_{xx}=u_{tt}$,

$$u(x,0)=\eta(x)=\cos\frac{\pi x}{2}$$

$$u_t(x,0)=0=\nu(\xi)$$

$\therefore$ *By D' Alembert principle solution is given by*

$$u(x,t)=\frac{1}{2}[\eta(x+ct)-\eta(x-ct)]$$

$$u(x,t)=\frac{1}{2}\left[\cos\frac{\pi(x+ct)}{2}-\cos\frac{\pi(x-ct)}{2}\right]=-\sin\frac{\pi x}{2}\sin\frac{\pi ct}{2}$$

***Uniqueness Theorem.** The solution to the wave equation* $u_{tt}=c^2u_{xx}, 0<x<L, t>0$

satisfying the initial conditions

$u(x,0)=f(x), 0\le x\le L$

$u_t(x,0)=g(x), 0\le x\le L$

and boundary conditions u(0, t) = u(L, t) = 0 where u(x, t) is twice differentiable function w.r.t to x and t. is always unique

CHARACTERISTICS OF 'DOMAIN OF DEPENDENCE' AND THE 'RANGE OF INFLUENCE'

The solution $u(x, t)$ of the initial value problem depends on the values of α at the endpoints of the interval $[x - ct, x + ct]$ and on the values of β on this interval only, see Figure The interval $[x - ct, x + ct]$ is called *domain of dependence*.

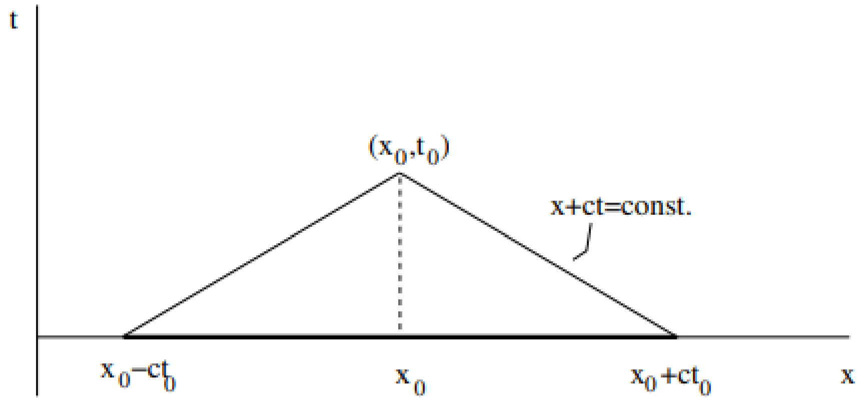


Figure : Interval of dependence

2. Let P be a point on the x-axis. Then we ask which points (x, t) need values of α or β at P in order to calculate $u(x, t)$? From the d'Alembert formula it follows that this domain is a cone, see Figure This set is called *domain of influence*.

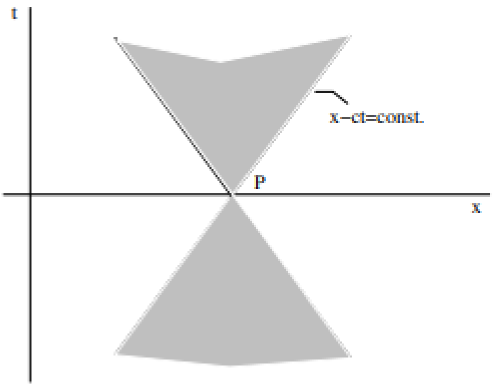


Figure : Domain of influence

Initial/Boundary Value Problem

Problem 1. *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u(0,t) = 0, \quad u(L,t) = 0 & t \geq 0. \end{cases}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L}.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$0 = u(0,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \quad \Rightarrow \quad a_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}.$$

- If we substitute into the equation $u_{tt} - c^2 u_{xx} = 0$, we get

$$\sum_{n=1}^{\infty} b_n''(t) \sin\frac{n\pi x}{L} + c^2 \sum_{n=1}^{\infty} \Big(\frac{n\pi}{L}\Big)^2 b_n(t) \sin\frac{n\pi x}{L} = 0, \quad \text{or}$$

$$b_n''(t) + \Big(\frac{n\pi c}{L}\Big)^2 b_n(t) = 0,$$

whose general solution is

$$b_n(t) = c_n \sin\frac{n\pi ct}{L} + d_n \cos\frac{n\pi ct}{L}.$$

Also, $b_n'(t) = c_n(\frac{n\pi c}{L}) \cos\frac{n\pi ct}{L} - d_n(\frac{n\pi c}{L}) \sin\frac{n\pi ct}{L}$.

- The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} g(x) = u(x,0) &= \sum_{n=1}^{\infty} b_n(0) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{L}, \\ h(x) = u_t(x,0) &= \sum_{n=1}^{\infty} b_n'(0) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}. \end{aligned}$$

By orthogonality, we may multiply by $\sin(m\pi x/L)$ and integrate:

$$\begin{aligned} \int_0^L g(x) \sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = d_m \frac{L}{2}, \\ \int_0^L h(x) \sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = c_m \frac{m\pi c}{L} \frac{L}{2}. \end{aligned}$$

Thus,

$$d_n = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi x}{L}\,dx, \qquad c_n = \frac{2}{n\pi c} \int_0^L h(x) \sin\frac{n\pi x}{L}\,dx.$$

Solution of Heat Equation

Let the problem be $C\left(\frac{\partial^2 u}{\partial x^2}\right)=\frac{\partial u}{\partial t}$ ...(1)

subject to boundary conditions $u(0,t)=0$ *and* $u(a,t)=0$ $\forall t$. *and* ...(2)

initial condition is $u(x,0)=f(x)$, $0<x<a$. ...(3)

Suppose that solution is of the form $u(x,t)=X(x)T(t)$ ...(4)

where X and T are respectively the functions of x and t alone. Using the values of (4) in (1), we get

$\frac{X''}{X}=\frac{T'}{CT}=\mu$ *(say)* ...(5)

where μ *is a separation constant.*

From equation (5), we can deduce that $X''-\mu X=0$ ...(6)

and $T'=\mu CT$ ...(7)

using (2) and (4), we get $X(0)=0$ *and* $X(a)=0$ ...(8)

Now, we want to solve (6) subject to the boundary condition. Hence, we have the following cases:

Case I: *Let* $\mu=0$. *Then solution of (6) is given by* $X(x)=Ax+B$

Using (8), we get $A=B=0 \Rightarrow X(x)=0 \Rightarrow u=0$, *which does not satisfy (3)*

Case II: *Let* $\mu=\lambda^2, \lambda\neq 0$. *In this case, solution of (6) is given by* $X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

Using (8), we get A+B=0 and $0=Ae^{a\lambda}+De^{-a\lambda} \Rightarrow A=B=0 \Rightarrow X=0 \Rightarrow u=0$

Thus, we reject this case also.

Case III: *Let* $\mu=-\lambda^2, \lambda\neq 0$. *In this case, solution of (6) is given by* $X(x)=A\cos\lambda x+B\sin\lambda x$

Using (8), we get

$A=0$ *and* $A\cos\lambda a+B\sin\lambda a=0$

Let $B\neq 0$, *then* $\sin\lambda a=0$

$\Rightarrow \lambda a=n\pi, n=1,2,...$

$\Rightarrow \lambda=\frac{n\pi}{a}, n=1,2,...$

Therefore, non zero solution of (6) is given

by $X_n(x)=B_n\sin\left(\frac{n\pi x}{a}\right)$

putting, $\lambda=\frac{n\pi}{a}$ *in (7), we get* $\frac{dT}{T}=-\frac{n^2\pi^2 C}{a^2}dt \Rightarrow \frac{dT}{T}=-C_n^2 dt$

whose solution is given by $T_n(t)=D_n e^{-C_n^2 t}$

Thus, we have $u_n(x,t)=X_n(x)T_n(t)=E_n\sin\left(\frac{n\pi x}{a}\right)e^{-C_n^2 t}$ ...(9)

$\therefore$ *The more general solution of (1) is given by*

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{a}\right)e^{-C_n^2 t} \quad ...(10)$$

Putting t = 0 in (10) and using (3) we get $f(x)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{a}\right)$

which is a fourier sine series, thus the constants E_n *are given by*

$$E_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx, n=1,2,3,...$$

Solve the equation in region $0 \le x \le \pi, t \ge 0$, *subject to conditions*

(1) *T remains finite as* $t \to \infty$

(2) *T =0 if x=0 and* π *for all t.*

(3) *At* $t = 0, T = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$

Solution: *The solution of equation is* $T(x,t) = \sum_{n=1}^{\infty} E_n \sin nx e^{-\alpha n^2 t}$

$$T(x,0) = \sum_{n=1}^{\infty} E_n \sin nx. \text{ where}$$

$$E_n = \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-x\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left((-(\pi - x))\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right] = \frac{4\sin(n\pi/2)}{n^2\pi}$$

$$\therefore T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha n^2 t}\sin(n\pi/2)}{n^2}\sin nx$$

MAXIMUM AND MINIMUM PRINCIPLES:

(Maximum principle). *Assume u is harmonic in a connected domain and achieves its supremum or infimum in Ω. Then $u \equiv$ const. in Ω.*

Proof. Consider the case of the supremum. Let $x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u(x) =: M.$$

Set $\Omega_1 := \{x \in \Omega : u(x) = M\}$ and $\Omega_2 := \{x \in \Omega : u(x) < M\}$. The set Ω_1 is not empty since $x_0 \in \Omega_1$. The set Ω_2 is open since $u \in C^2(\Omega)$. Consequently, Ω_2 is empty if we can show that Ω_1 is open. Let $\overline{x} \in \Omega_1$, then there is a $\rho_0 > 0$ such that $\overline{B_{\rho_0}(\overline{x})} \subset \Omega$ and $u(x) = M$ for all $x \in B_{\rho_0}(\overline{x})$.

If not, then there exists $\rho > 0$ and $\widehat{x}$ such that $|\widehat{x} - \overline{x}| = \rho$, $0 < \rho < \rho_0$ and $u(\widehat{x}) < M$. From the mean value formula, see Proposition 7.2, it follows

$$M = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} u(x)\ dS < \frac{M}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} dS = M,$$

which is a contradiction. Thus, the set Ω_2 is empty since Ω_1 is open.

Corollary. Assume Ω is connected and bounded, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then u achieves its minimum and its maximum on the boundary $\partial\Omega$.

Uniqueness

Sufficiently regular solutions of the initial-boundary value problem are uniquely determined since from

$$\begin{aligned} c_t &= D\triangle c \ \text{ in } \Omega \times (0,\infty) \\ c(x,0) &= 0 \\ \frac{\partial c}{\partial n} &= 0 \ \text{ on } \partial\Omega \times (0,\infty). \end{aligned}$$

it follows that for each $\tau > 0$

$$\begin{aligned} 0 &= \int_0^\tau \int_\Omega (c_t c - D(\triangle c)c)\ dxdt \\ &= \int_\Omega \int_0^\tau \frac{1}{2}\frac{\partial}{\partial t}(c^2)\ dtdx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt \\ &= \frac{1}{2}\int_\Omega c^2(x,\tau)\ dx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt. \end{aligned}$$

Problems: Heat Equation

Consider

$$\begin{cases} u_t = u_{xx} & \text{for } x > 0,\, t > 0 \\ u(x,0) = g(x) & \text{for } x > 0 \\ u(0,t) = 0 & \text{for } t > 0, \end{cases}$$

where g is continuous and bounded for $x \geq 0$ and $g(0) = 0$. Find a formula for the solution $u(x,t)$.

Proof. Extend g to be an **odd** function on all of $\mathbb{R}$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{for } x \in \mathbb{R},\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

The solution is given by: [60]

$$
\begin{aligned}
\tilde{u}(x,t) &= \int_{\mathbb{R}} K(x,y,t) g(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \\
&= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy \right] \\
&= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy - \int_0^{\infty} e^{-\frac{(x+y)^2}{4t}} g(y)\,dy \right] \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{\frac{-x^2+2xy-y^2}{4t}} - e^{\frac{-x^2-2xy-y^2}{4t}} \right) g(y)\,dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \left(e^{\frac{xy}{2t}} - e^{-\frac{xy}{2t}} \right) g(y)\,dy.
\end{aligned}
$$

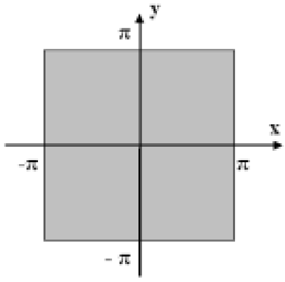
$$
\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \, 2\sinh\left(\frac{xy}{2t}\right) g(y)\,dy.}
$$

Since $\sinh(0) = 0$, we can verify that $u(0,t) = 0$.

Problem 1: The 2D LAPLACE Equation on a Square.
Let $\Omega = (0,\pi) \times (0,\pi)$, *and use separation of variables to solve the boundary value problem*

$$
\begin{cases}
u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\
u(0,y) = 0 = u(\pi, y) & 0 \le y \le \pi \\
u(x,0) = 0, \quad u(x,\pi) = g(x) & 0 \le x \le \pi,
\end{cases}
$$

where g *is a continuous function satisfying* $g(0) = 0 = g(\pi)$.



Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$
\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.
$$

- From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos nx + b_n \sin nx$. Boundary conditions give

$$
\begin{cases}
u(0,y) = X(0)Y(y) = 0 \\
u(\pi,y) = X(\pi)Y(y) = 0
\end{cases}
\Rightarrow \quad X(0) = 0 = X(\pi).
$$

Thus, $X_n(0) = a_n = 0$, and

$$
\begin{aligned}
&X_n(x) = b_n \sin nx, \quad n = 1, 2, \ldots . \quad \checkmark \\
&-n^2 b_n \sin nx + \lambda b_n \sin nx = 0, \\
&\lambda_n = n^2, \quad n = 1, 2, \ldots . \quad \checkmark
\end{aligned}
$$

- With these values of λ_n we solve $Y'' - n^2 Y = 0$ to find $Y_n(y) = c_n \cosh ny + d_n \sinh ny$.

Boundary conditions give

$$u(x,0) = X(x)Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0 = c_n.$$
$$Y_n(x) = d_n \sinh ny.$$

- By superposition, we write

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh ny,}$$

which satifies the equation and the three homogeneous boundary conditions. The boundary condition at $y = \pi$ gives

$$u(x,\pi) = g(x) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh n\pi,$$
$$\int_0^{\pi} g(x) \sin mx \, dx = \sum_{n=1}^{\infty} \tilde{a}_n \sinh n\pi \int_0^{\pi} \sin nx \sin mx \, dx = \frac{\pi}{2} \tilde{a}_m \sinh m\pi.$$

$$\boxed{\tilde{a}_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx \, dx.}$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – DIRICHLET AND HEAT PROBLEM – SMT5204

Inhomogeneous equation

Here we consider the initial value problem

$$\begin{aligned}\Box u &= w(x,t) \;\text{ on } x \in \mathbb{R}^n,\; t \in \mathbb{R}\\ u(x,0) &= f(x)\\ u_t(x,0) &= g(x),\end{aligned}$$

where $\Box u := u_{tt} - c^2 \triangle u$. We assume $f \in C^3$, $g \in C^2$ and $w \in C^1$, which are given.

Set $u = u_1 + u_2$, where u_1 is a solution of problem with $w := 0$ and u_2 is the solution where $f = 0$ and $g = 0$ in Since we have explicit solutions u_1 in the cases $n = 1$, $n = 2$ and $n = 3$, it remains to solve

$$\begin{aligned}\Box u &= w(x,t) \;\text{ on } x \in \mathbb{R}^n,\; t \in \mathbb{R}\\ u(x,0) &= 0\\ u_t(x,0) &= 0.\end{aligned}$$

The following method is called *Duhamel's principle* which can be considered as a generalization of the method of variations of constants in the theory of ordinary differential equations.

To solve this problem, we make the ansatz

$$u(x,t) = \int_0^t v(x,t,s)\; ds,$$

where v is a function satisfying

$$\Box v = 0 \;\text{ for all } s$$

and

$$v(x,s,s) = 0.$$

From ansatz and assumption we get

$$\begin{aligned}u_t &= v(x,t,t) + \int_0^t v_t(x,t,s)\; ds,\\ &= \int_0^t v_t(x,t,s).\end{aligned}$$

It follows $u_t(x,0) = 0$. The initial condition $u(x,t) = 0$ is satisfied because

$$\begin{aligned}u_{tt} &= v_t(x,t,t) + \int_0^t v_{tt}(x,t,s)\; ds,\\ \triangle_x u &= \int_0^t \triangle_x v(x,t,s)\; ds.\end{aligned}$$

$$\begin{aligned} u_{tt} - c^2 \triangle_x u &= v_t(x,t,t) + \int_0^t (\Box v)(x,t,s)\ ds \\ &= w(x,t). \end{aligned}$$

Thus necessarily $v_t(x,t,t) = w(x,t)$, We have seen that the ansatz provides a solution of if for all s

$$\Box v = 0, \quad v(x,s,s) = 0, \quad v_t(x,s,s) = w(x,s).$$

Let $v^*(x,t,s)$ be a solution of

$$\Box v = 0, \quad v(x,0,s) = 0, \quad v_t(x,0,s) = w(x,s),$$

then

$$v(x,t,s) := v^*(x,t-s,s)$$

is a solution of . In the case $n = 3$, where v^* is given by, see Theorem

$$v^*(x,t,s) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(x)} w(\xi,s)\ dS_\xi.$$

Then

$$\begin{aligned} v(x,t,s) &= v^*(x,t-s,s) \\ &= \frac{1}{4\pi c^2 (t-s)} \int_{\partial B_{c(t-s)}(x)} w(\xi,s)\ dS_\xi. \end{aligned}$$

from it follows

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t,s)\ ds \\ &= \frac{1}{4\pi c^2} \int_0^t \int_{\partial B_{c(t-s)}(x)} \frac{w(\xi,s)}{t-s}\ dS_\xi ds. \end{aligned}$$

Changing variables by $\tau = c(t-s)$ yields

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^{ct} \int_{\partial B_\tau(x)} \frac{w(\xi, t-\tau/c)}{\tau}\ dS_\xi d\tau \\ &= \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t-r/c)}{r}\ d\xi, \end{aligned}$$

where $r = |x - \xi|$.

Formulas for the cases $n = 1$ and $n = 2$ follow from formulas for the associated homogeneous equation with inhomogeneous initial values for these cases.

Theorem *The solution of*

$$\Box u = w(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

where $w \in C^1$, is given by:

Case $n = 3$:

$$u(x,t) = \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t - r/c)}{r}\, d\xi,$$

where $r = |x - \xi|$, $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$.

Case $n = 2$:

$$u(x,t) = \frac{1}{4\pi c} \int_0^t \left(\int_{B_{c(t-\tau)}(x)} \frac{w(\xi, \tau)}{\sqrt{c^2(t-\tau)^2 - r^2}}\, d\xi \right) d\tau,$$

$x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$.

Case $n = 1$:

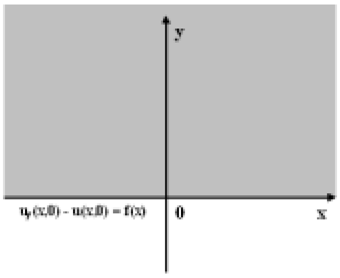
$$u(x,t) = \frac{1}{2c} \int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} w(\xi, \tau)\, d\xi \right) d\tau.$$

Problem **The 2D LAPLACE Equation in an Upper-Half Plane.**
Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad y > 0, \ -\infty < x < +\infty$$

$$\frac{\partial u(x,0)}{\partial y} - u(x,0) = f(x),$$

where $f(x) \in C_0^\infty(\mathbb{R}^1)$.



Find a bounded solution $u(x,y)$ and show that $u(x,y) \to 0$ when $|x| + y \to \infty$.

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

- Consider $X'' + \lambda X = 0$.

If $\lambda = 0$, $X_0(x) = a_0 x + b_0$.
If $\lambda > 0$, $X_n(x) = a_n \cos \sqrt{\lambda_n} x + b_n \sin \sqrt{\lambda_n} x$.
Since we look for bounded solutions as $|x| \to \infty$, we have $a_0 = 0$.

- Consider $Y'' - \lambda_n Y = 0$.

If $\lambda_n = 0$, $Y_0(y) = c_0 y + d_0$.
If $\lambda_n > 0$, $Y_n(y) = c_n e^{-\sqrt{\lambda_n} y} + d_n e^{\sqrt{\lambda_n} y}$.
Since we look for bounded solutions as $y \to \infty$, we have $c_0 = 0$, $d_n = 0$. Thus,

$$u(x,y) = \tilde{a}_0 + \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n} y}\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

Initial condition gives:

$$f(x) = u_y(x,0) - u(x,0) = -\tilde{a}_0 - \sum_{n=1}^{\infty}(\sqrt{\lambda_n}+1)\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

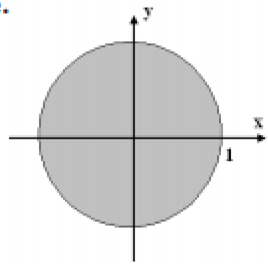
$f(x) \in C_0^{\infty}(\mathbb{R}^1)$, i.e. has compact support $[-L, L]$, for some $L > 0$. Thus the coefficients $\tilde{a}_n$, $\tilde{b}_n$ are given by

$$\begin{aligned}
\int_{-L}^{L} f(x)\cos\sqrt{\lambda_n}x\,dx &= -(\sqrt{\lambda_n}+1)\tilde{a}_n L.\\
\int_{-L}^{L} f(x)\sin\sqrt{\lambda_n}x\,dx &= -(\sqrt{\lambda_n}+1)\tilde{b}_n L.
\end{aligned}$$

Thus, $u(x,y) \to 0$ when $|x| + y \to \infty$.

The 2D LAPLACE Equation on a Circle.

Let Ω be the unit disk in $\mathbb{R}^2$ and consider the problem



$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = h & \text{on } \partial\Omega, \end{cases}$$

where h is a continuous function.

Proof. Use polar coordinates (r, θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) = h(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$.

$$\begin{aligned}
u_t &= u_r r_t = -e^{-t}u_r,\\
u_{tt} &= (-e^{-t}u_r)_t = e^{-t}u_r + e^{-2t}u_{rr} = ru_r + r^2 u_{rr}.
\end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
$\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have

$$\begin{aligned} u_0(r,\theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r,\theta) &= X_n(r)Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r,\theta) &= d_0 a_0, \\ u_n(r,\theta) &= d_n r^n (a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = h(\theta).$$

The coefficients a_n, b_n for $n \geq 1$ are determined from the Fourier series for $h(\theta)$.
a_0 is not determined by $h(\theta)$ and therefore may take an arbitrary value. Moreover, the constant term in the Fourier series for $h(\theta)$ must be zero [i.e., $\int_0^{2\pi} h(\theta)d\theta = 0$]. Therefore, the problem is **not** solvable for an arbitrary function $h(\theta)$, and when it is solvable, the solution is **not** unique.

Problem The 2D LAPLACE Equation on a Circle.
Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{(r,\theta) : 0 \leq r < 1,\ 0 \leq \theta < 2\pi\}$, *and use separation of variables* (r,θ) *to solve the Dirichlet problem*

$$\begin{cases} \triangle u = 0 & in\ \Omega \\ u(1,\theta) = g(\theta) & for\ 0 \leq \theta < 2\pi. \end{cases}$$

Proof. Use polar coordinates (r,θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } 0 \leq r < 1,\ 0 \leq \theta < 2\pi \\ u(1,\theta) = g(\theta) & \text{for } 0 \leq \theta < 2\pi. \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t),\theta)$.

$$\begin{aligned} u_t &= u_r r_t = -e^{-t} u_r, \\ u_{tt} &= (-e^{-t}u_r)_t = e^{-t}u_r + e^{-2t}u_{rr} = r u_r + r^2 u_{rr}. \end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

$\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.

- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.

If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.

- We have

$$\begin{aligned} u_0(r,\theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r,\theta) &= X_n(r)Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r,\theta) &= d_0 a_0, \\ u_n(r,\theta) &= d_n r^n (a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u(1,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = g(\theta).$$

$$\begin{aligned} \tilde{a}_0 &= \frac{1}{\pi} \int_0^{\pi} g(\theta)\, d\theta, \\ \tilde{a}_n &= \frac{2}{\pi} \int_0^{\pi} g(\theta) \cos n\theta\, d\theta, \\ \tilde{b}_n &= \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin n\theta\, d\theta. \end{aligned}$$