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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I - Probability – SMT5203

I Introduction

Probability theory has applications in many branches of Science and Engineering. Probability theory as a matter of fact, is study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a **Deterministic experiment**.

Although all possible outcomes of an experiment may be known in advance the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a **Random experiment**.

(e.g.) Whenever a fair dice is thrown, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be.

Sample Space

The set of all possible outcomes which are assumed equally likely.

Event

A sub-set of S consisting of possible outcomes.

Mathematical definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A. then the probability of event A occurring is denoted as P(A), is denoted by

$$P(A) = \frac{n(A)}{n(S)}$$

Note: 1. It is obvious that $0 \leq P(A) \leq 1$.

2. If A is an impossible event, P(A) = 0.

3. If A is a certain event , P(A) = 1.

A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$ A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$$

Axiomatic definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, P(A) is defined as a real number satisfying the following axioms.

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and
4. If $A_1, A_2\ A_3, \ldots, A_n, \ldots$ are mutually exclusive events,
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n, \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) \ldots$$

Important Theorems

Theorem 1: Probability of impossible event is zero.

Proof: Let S be sample space (certain events) and ϕ be the impossible event.
Certain events and impossible events are mutually exclusive.

$P(S \cup \phi) = P(S) + P(\phi)$ (Axiom 3)

$S \cup \phi = S$

$P(S) = P(S) + P(\phi)$

$P(\phi) = 0$, hence the result.

Theorem 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A}) = 1 - P(A) \leq 1$.

Proof: Let A be the occurrence of the event
$\overline{A}$ be the non-occurrence of the event .
Occurrence and non-occurrence of the event are mutually exclusive.

$$P(A \cup \overline{A}) = P(A) + P(\overline{A})$$
$$A \cup \overline{A} = S \quad \Rightarrow \quad P(A \cup \overline{A}) = P(S) = 1$$
$$\therefore \quad 1 = P(A) + P(\overline{A})$$
$$P(\overline{A}) = 1 - P(A) \leq 1.$$

Theorem 3: (Addition theorem)

If A and B are any 2 events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Proof: We know, $A = A\overline{B} \cup AB$ and $B = \overline{A}B \cup AB$

$$\therefore \quad P(A) = P(A\overline{B}) + P(AB) \text{ and } P(B) = P(\overline{A}B) + P(AB) \qquad \text{(Axiom 3)}$$

$$P(A) + P(B) = P(A\overline{B}) + P(AB) + P(\overline{A}B) + P(AB)$$

$$= P(A \cup B) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B).$$

Note: The theorem can be extended to any 3 events, A,B and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Theorem 4: If $B \subset A$, $P(B) \le P(A)$.

Proof: A and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$

$$\therefore \quad P(B \cup A\overline{B}) = P(A)$$

$$P(B) + P(A\overline{B}) = P(A) \qquad \text{(Axiom 3)}$$

$$P(B) \le P(A)$$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \ne 0$$

Product theorem of probability

Rewriting the definition of conditional probability, We get

$$P(A \cap B) = P(A)P(A/B)$$

The product theorem can be extended to 3 events, A, B and C as follows:

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

Note: 1. If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$.

2. If $B \subset A$, $P(B/A) \ge P(B)$, since $A \cap B = B$, and $\dfrac{P(B)}{P(A)} \ge P(B)$,

As $P(A) \leq P(S) = 1$.

3. If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$.
4. If $P(A) > P(B)$, $P(A/B) > P(B/A)$.
5. If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$.

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

If the two events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$, Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events are said to be independent (pair wise independent).

The product theorem can be extended to any number of independent events, If A_1 A_2 A_3 ….. A_n are n independent events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n) = P(A_1) \times P(A_2) \times P(A_3) \times \ldots\ldots \times P(A_n)$$

Theorem 4:

If the events A and B are independent, the events $\overline{A}$ and B are also independent.

Proof:

The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\overline{A} \cap B) = B$

$$\therefore \quad P(A \cap B) + P(\overline{A} \cap B) = P(B)$$

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)\,P(B) \qquad (\because \text{A and B are independent})$$

$$= P(B)\,[1 - P(A)]$$

$$= P(\overline{A})\,P(B).$$

Theorem 5:

If the events A and B are independent, the events $\overline{A}$ and $\overline{B}$ are also independent.

Proof:

$$P(\overline{A} \cap \overline{B}) = P\left(\overline{A \cup B}\right) = 1 - P(A \cup B)$$

$$= 1 - [\,P(A) + P(B) - P(A \cap B)] \qquad \text{(Addition theorem)}$$

$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$

$$= P(\overline{A})P(\overline{B}).$$

Problem 1:
From a bag containing 3 red and 2 black balls, 2 ball are drawn at random. Find the probability that they are of the same colour.

Solution :
Let A be the event of drawing 2 red balls

B be the event of drawing 2 black balls.

$\therefore$ $P(A \cup B) = P(A) + P(B)$

$$= \frac{3C_2}{5C_2} + \frac{2C_2}{5C_2} = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$$

Problem 2:
When 2 cards are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit?

Solution :
Let A be the event of drawing 2 spade cards

B be the event of drawing 2 claver cards

C be the event of drawing 2 hearts cards

D be the event of drawing 2 diamond cards.

$$\therefore P(A \cup B \cup C \cup D) = 4\frac{13C_2}{52C_2} = \frac{4}{17}.$$

Problem 3:
When A and B are mutually exclusive events such that P(A) = 1/2 and P(B) = 1/3, find $P(A \cup B)$ and $P(A \cap B)$.

Solution :
$P(A \cup B) = P(A) + P(B) = 5/6$; $P(A \cap B) = 0$.

Problem 4:
If P(A) = 0.29, P(B) = 0.43, find $P(A \cap \overline{B})$, if A and B are mutually exclusive.

Solution :

We know A $\cap$ $\overline{B}$ = A

$$P(A \cap \overline{B}) = P(A) = 0.29$$

Problem 5:

A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution :

Let A be the event of drawing a spade

B be the event of drawing a ace

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}.$$

Problem 6:

If P(A) = 0.4, P(B) = 0.7 and P(A $\cap$ B) = 0.3, find $P(\overline{A} \cap \overline{B})$.

Solution :

$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 0.2$

Problem 7:

If P(A) = 0.35, P(B) = 0.75 and P(A $\cup$ B) = 0.95, find $P(\overline{A} \cup \overline{B})$.

Solution :

$P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B) = 1 - [P(A) + P(B) - P(A \cup B)] = 0.85$

Problem 8:

A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random(with out replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.

Solution :

(i) P(both are good) = $\frac{10C_2}{16C_2} = \frac{3}{8}$

(ii) P(both have major defects) = $\frac{2C_2}{16C_2} = \frac{1}{120}$

(iii) P(at least 1 is good) = $\frac{10C_1 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$

(iv) P(at most 1 is good) = $\frac{10C_0 6C_2 + 10C_1 6C_1}{16C_2} = \frac{5}{8}$

(v) P(exactly 1 is good) = $\frac{10C_1 6C_1}{16C_2} = \frac{1}{2}$

(vi) P(neither has major defects) = $\frac{14C_2}{16C_2} = \frac{91}{120}$

(vii) P(neither is good) = $\frac{6C_2}{16C_2} = \frac{1}{8}$.

Problem 9:
If A, B and C are any 3 events such that P(A) = P(B) = P(C) = 1/4, P(A ∩ B) = P(B ∩ C) = 0; P(C ∩ A) = 1/8. Find the probability that at least 1 of the events A, B and C occurs.

Solution :

Since P(A ∩ B) = P(B ∩ C) = 0; P(A ∩ B ∩ C) = 0

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}.$$

Problem 10:
A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution :

Let A be a good tube drawn and B be an other good tube drawn.

P(both tubes drawn are good) = P(A ∩ B) = $\frac{6C_2}{10C_2} = \frac{1}{3}$

P(B/A) = $\frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$ (By conditional probability)

Problem 11:
In shooting test, the probability of hitting the target is 1/2, for a, 2/3 for B and ¾ for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.

Solution :

Let A, B and C be the event of hitting the target .
P(A) = 1/2, P(B) = 2/3, P(C) = 3/4
P($\overline{A}$) = 1/2, P($\overline{B}$) = 1/3, P($\overline{C}$) = 1/4

P(none of them hits) = P($\overline{A} \cap \overline{B} \cap \overline{C}$) = P($\overline{A}$) × P($\overline{B}$) × P($\overline{C}$) = 1/24

P(at least one hits) = 1 – P(none of them hits)
= 1 – (1/24) = 23/24.

Problem 12:
A and B alternatively throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.

Solution :

Let A be the event of throwing 6

B be the event of throwing 7.

P(throwing 6 with 2 dice) = 5/36 P(throwing 7 with 2 dice) = 1/6
P(not throwing 6) = 31/36 P(not throwing 7) = 5/6

A plays in I, III, V,……trials.
A wins if he throws 6 before Be throws 7.

$$P(\text{A wins}) = P(A \cup \bar{A}\,\bar{B}\,A \cup \bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A \cup \ldots\ldots)$$
$$= P(A) + P(\bar{A}\,\bar{B}\,A) + P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A) + \ldots\ldots$$
$$= \frac{5}{36} + \left(\frac{31}{36}\times\frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36}\times\frac{5}{6}\right)^2 \frac{5}{36} + \ldots$$
$$= \frac{30}{61}$$

Problem 13:
A and B toss a fair coin alternatively with the understanding that the first who obtain the head wins. If A starts, what is his chance of winning?

Solution :
P(getting head) = 1/2 , P(not getting head) = 1/2

A plays in I, III, V,……trials.
A wins if he gets head before B.

$$P(\text{A wins}) = P(A \cup \bar{A}\,\bar{B}\,A \cup \bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A \cup \ldots\ldots)$$
$$= P(A) + P(\bar{A}\,\bar{B}\,A) + P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A) + \ldots\ldots$$
$$= \frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)^2 \frac{1}{2} + \cdots$$
$$= \frac{2}{3}$$

Problem 14:
A problem is given to 3 students whose chances of solving it are 1/2 , 1/3 and 1/4 . What is the probability that (i) only one of them solves the problem and (ii) the problem is solved.

Solution :
P(A solves) = 1/2 P(B) = 1/3 P(C) = 1/4
$P(\bar{A})$ = 1/2, $P(\bar{B})$ = 2/3, $P(\bar{C})$ = 3/4

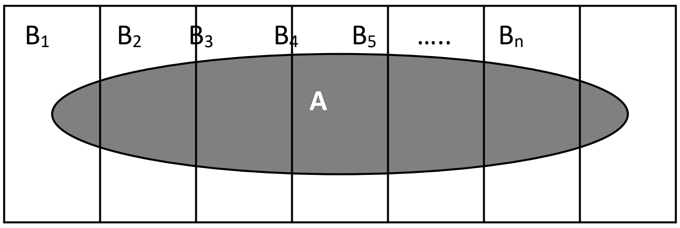
P(none of them solves) = $P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$ = 1/4
P(at least one solves) = 1 – P(none of them solves)
= 1 – (1/4) = 3/4 .

Baye's Theorem

Statement: If B_1, B_2, B_3, ….B_n be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with B_i, then

$$P(B_i / A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}$$

Proof :



The shaded region represents the event A, A can occur along with B_1, B_2, B_3, ….B_n that are mutually exclusive.

$\therefore$ AB_1, AB_2, AB_3, …, AB_n are also mutually exclusive.

Also $A = AB_1 \cup AB_2 \cup AB_3 \cup \ldots \cup AB_n$

$P(A) = P(AB_1) + P(AB_2) + P(AB_3) + \ldots + P(AB_n)$

$$= \sum_{i=1}^{n} P(AB_i)$$

$$= \sum_{i=1}^{n} P(B_i) \times P(A/B_i) \quad \text{(By conditional probability)}$$

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}.$$

Problem 15:

Ina bolt factory machines A, B, C manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the produce and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C.

Solution :

Let B_1 be bolt produced by machine A

B_2 be bolt produced by machine B

B_3 be bolt produced by machine C

Let A/B_1 be the defective bolts drawn from machine A

A/B_2 be the defective bolts drawn from machine B

A/B_3 be the defective bolts drawn from machine C.

$P(B_1) = 0.25$	$P(A/B_1) = 0.05$
$P(B_2) = 0.35$	$P(A/B_2) = 0.04$
$P(B_3) = 0.40$	$P(A/B_3) = 0.02$

Let B_1/A be defective bolts manufactured by machine A

B_2/A be defective bolts manufactured by machine B

B_3/A be defective bolts manufactured by machine C

$$P(A)=\sum_{i=1}^{3}P(B_i)\times P(A/B_i) = (0.25)\times(0.05)+(0.35)\times(0.04)+(0.4)\times(0.02)$$

$$= 0.0345$$

$$P(B_1/A) = \frac{P(B_1)\times P(A/B_1)}{P(A)} = 0.3623$$

$$P(B_2/A) = \frac{P(B_2)\times P(A/B_2)}{P(A)} = 0.405$$

$$P(B_3/A) = \frac{P(B_3)\times P(A/B_3)}{P(A)} = 0.231\ .$$

Problem 16 :

The first bag contains 3 white balls, 2 red balls and 4 black balls. Second bag contains 2 white, 3 red and 5 black balls and third bag contains 3 white, 4 red and 2 black balls. One bag is chosen at random and from it 3 balls are drawn. Out of three balls two balls are white and one is red. What are the probabilities that they were taken from first bag, second bag and third bag.

Solution :

Let P(selecting the bag) = P(A_i) = 1/3 , i = 1, 2, 3.

$$P(A/B_1) = \frac{3C_2\, 2C_1}{9C_3} = \frac{6}{84} \qquad P(A)=\sum_{i=1}^{3}P(B_i)\times P(A/B_i) = 0.0746$$

$$P(A/B_2) = \frac{2C_2\, 3C_1}{10C_3} = \frac{3}{120}$$

$$P(A/B_3) = \frac{3C_2\, 4C_1}{9C_3} = \frac{12}{84}$$

$$P(B_1/A) = \frac{P(B_1)\times P(A/B_1)}{P(A)} = 0.319$$

$$P(B_2/A) = \frac{P(B_2)\times P(A/B_2)}{P(A)} = 0.4285$$

$$P(B_3/A) = \frac{P(B_3)\times P(A/B_3)}{P(A)} = 0.638$$

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UNIT – II – Random Variables – SMT5203

I Introduction

A real variable X whose value is determined by the outcome of a random experiment is called a **Random variable.**

e.g., consider random experiment of throwing a die. Then X the number of points on the die is a random variable, since X takes the values 1, 2, 3, 4, 5 and 6 each with the probability 1/6.

Discrete Random Variable

If the random variable taken the values only on the set {0, 1, 2, 3, ….. n} is called a Discrete random variable.

e.g., The number of printing mistakes in each page of a book, the number of telephone calls received by the telephone operator.

Continuous Random Variable

If a random variable takes on all values within a certain interval, then the random variable is called Continuous random variable.

e.g., The height, age and weight of individuals, the amount of rainfall on a rainy day.

Distribution Function of the Random Variable X

The distribution function of a random variable X defined in $(-\infty,\infty)$ is given by

$$F(x) = P(X \leq x)$$

Properties of the Distribution function

1. $P(a < X \leq b) = F(b) - F(a)$
2. $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$
3. $P(a < X < b) = F(b) - F(a) - P(X = b)$
4. $P(a \leq X < b) = F(b) - F(a) - P(X = b) + P(X = a)$

Probability Mass Function

Let X be a one dimensional discrete random variable which takes the values x_1 ,x_2 ,x_3,…. Then $P(X = x_i)$ satisfies the following conditions

1. $P(x_i) \geq 0$
2. $\sum_{i=1}^{\infty} P(x_i) = 1$

Probability Density Function

If X is a continuous r.v. then $f(x)$ sis called the probability density function of X provided $f(x)$ satisfies the following conditions;

(i) $f(x) \geq 0, \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Note :

(a) 1. $P(a \le X \le b) = \int_a^b f(x)dx$

2. When X is continuous r.v. $P(X = a) = 0$

$\therefore P(a < X \le b) = P(a \le X \le b) = P(a < X < b) = P(a \le X < b)$.

(b) If $F(x)$ is the distribution function of one dimensional random variables, then

1. $0 \le F(x) \le 1$
2. If $x < y$, thenc $F(x) \le F(y)$
3. $F(-\infty) = 0$, $F(\infty) = 1$.
4. If X is discrete r.v. taking values $x_1, x_2, x_3, \ldots$ where $x_1 < x_2 < x_3 < \ldots$ then

   $P(X = x_i) = F(x_i) - F(x_{i-1})$.
5. If X is continuous r.v., then $\frac{dF(x)}{dx} = f(x)$.

Problem 1:

A random variable X has the following probability function

Value of X, x_i	0	1	2	3	4	5	6	7	8
Probability P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(ii) Determine the value of 'a'.

(iii) Find $P(X < 3)$, $P(X \ge 3)$. $P(0 < X < 5)$.

(iv) Find the distribution function of X.

Solution :

(i) Since $\sum_{i=1}^{\infty} P(x_i) = 1$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$a = \frac{1}{81}$$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 1/9$

$P(X \ge 3) = 1 - P(X < 3) = 8/9$

$P(0 < X < 5) = 24/81$

(iii)

x_i	$P(x_i)$	$F(x)$
0	1/81	1/81
1	3/81	4/81
2	5/81	9/81

3	7/81	16/81
4	9/81	25/81
5	11/81	36/81
6	13/81	49/81
7	15/81	64/81
8	17/81	1

Problem 2:A random variable X has the following probability distribution.

X	–2	–1	0	1	2	3
P(x)	0.1	K	0.2	2K	0.3	3K

(a) Find K, (b) Evaluate P(X < 2) and P(–2 < X <2), (c) find the c.d.f. of X and (d) evaluate the mean of X.

Solution:

(a) Since $\sum_{i=1}^{\infty}$ P(x_i) = 1

0.1 + K + 0.2 + 2K + 0.3 + 3K = 1

K = 1/15

(b) P(X < 2) = 0.1 + 1/15 + 0.2 + 2/15 + = 1/2

P(–2 < X < 2) = 1/15 + 0.2 + 2/15 = 2/5

(c)

x_i	P(x_i)	F(x)
–2	1/10	1/10
–1	1/15	1/6
0	2/10	11/30
1	2/15	1/2
2	3/10	4/5
3	3/15	1

(d) The mean of X is defined as E(X) = $\sum xP(x)$

Mean of X = –2 × (1/10) + (–1) × (1/15) + 0 × (1/5) + 1 × (2/15)
+ 2 × (3/10) + 3 × (1/5) = 16/15

Problem 3:

If the random variable X takes the values 1, 2, 3 and 4 such that 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)., find the probability
distribution and cumulative distribution function of X.

Solution :

Since $\sum_{i=1}^{\infty} P(x_i) = 1$

2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = K

$\frac{K}{2}+\frac{K}{3}+\frac{K}{1}+\frac{K}{5}=1, \quad \therefore K = 30/61$

x_i	$P(x_i)$	$F(x)$
1	15/61	15/61
2	10/61	25/61
3	30/61	55/61
4	6/61	1

Problem 4:

A random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

Find (i) the value of K, (ii) P(1.5 < X 4.5 / X > 2) and (ii) the smallest value of λ for which P(X ≤ λ) > 1/2.

Solution :

Since $\sum_{i=1}^{\infty} P(x_i) = 1$

$10K^2 + 9K = 1$

K = 1/10 or −1. As K= −1 is meaningless, K = 1/10

$$P(1.5 < X\, 4.5 / X > 2) = \frac{P[(1.5 < X\, 4.5) \cap (X > 2)]}{P(X > 2)}$$

$$= \frac{P(X=3)+P(X=4)}{1-[P(X=0)+P(X=1)+P(X=2)} = \frac{5}{7}$$

P(X ≤ 0) = 0; P(X ≤ 1) = 0.1; P(X ≤ 2) = 0.3; P(X ≤ 3) = 0.5 and P(X ≤ 4) = 0.8

λ = 1/2 for which P(X ≤ λ) > 1/2 .

Problem 5: If the density function of a continuous r.v. X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

(i) find the value of a

(ii) find the c.d.f. of X

(iii) P(X > 1.5)

Solution :

(i) Since $f(x)$ is a p.d.f. $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^3 f(x)dx = \int_0^1 axdx + \int_1^2 adx + \int_2^3 (3a - ax)dx = 1$$

$$a = 1/2$$

(ii) $$F(x) = P(X \le x) = \begin{cases} 0, & x > 0 \\ \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{4} + \frac{x-1}{2}, & 1 \le x \le 2 \\ \frac{-5}{4} + \frac{6x - x^2}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

(iv) $\quad$ $P(X > 1.5) = \int_{1.5}^{3} f(x)dx = \int_{1.5}^{2} \frac{1}{2}dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx = \frac{1}{2}$

Problem 6: A continuous r.v. has a p.d.f. $f(x) = kx^2 e^{-x}, x \ge 0$. Find k, mean and variance.

Solution :

$\int_0^{\infty} f(x)dx = 1$. $\therefore$ k = ½

Mean = $\int_0^{\infty} xf(x)dx = \int_0^{\infty} x^3 e^{-x} dx = \frac{1}{2}[-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}] = 3$

Var(X) = $E(X^2) - [E(X)]^2$

$E[X^2] = \int_0^{\infty} x^4 e^{-x} dx = 12$

$\therefore$ Var(X) = 3

Problem 7: A continuous r.v. has a p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that

(i) $\quad$ $P(X \le a) = P(X > a)$ and (ii) $P(X > b) = 0.05$.

Solution :

(i) $\quad$ $P(X \le a) = P(X > a)$

$$\int_0^a 3x^2 dx = \int_a^1 3x^2 dx$$

$$a^3 = \frac{1}{2}; \qquad \therefore\ a = 0.7937$$

(ii) P(X > b) = 0.05

$$\int_b^1 3x^2 dx = 0.05 \qquad b^3 = 95; \qquad b = 0.9830.$$

Problem 8: If the probability distribution of X is given as:

x	:	1	2	3	4
P(x)	:	0.4	0.3	0.2	0.1

Find P(1/2 < X < 7/2 / X >1)

Problem 9: If the c.d.f. of a r.v. is given by F(x) = 0, for x < 0; F(x) = x^2/16 for 0 ≤ x ≤ 4 and F(x) = 1, for 4 ≥ x, find P(X > 1 / X < 3).

Problem 10: For the following density function $f(x) = ae^{-|x|}, -\infty < x < \infty$, find (i) the value of a, (ii) mean and variance of X. [1/2, 0, 2]

Problem 11: A r.v. X has the p.d.f. $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$ find (i) $p\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$ and (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$. [1/4, 3/16, 7/12]

Chebyshev Inequality

If X is a random variable with mean μ and variance σ^2, than for any positive number K, we have

$$P\{|X - \mu| \geq K\sigma\} \leq \frac{1}{K^2}$$

(or)

$$P\{|X - \mu| < K\sigma\} \geq 1 - \frac{1}{K^2}$$

Solution: We know that

$$\sigma^2 = E[X - E(X)]^2 = E[X - \mu]^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\mu - K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu - K\sigma}^{\mu + K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu + K\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

Form first integral

$$x \leq \mu - K\sigma$$

$$-(x-\mu) \geq K\sigma$$

$$(x-\mu)^2 \geq K^2\sigma^2$$

Form second integral

$$x \geq \mu + K\sigma$$

$$(x-\mu) \geq K\sigma$$

$$(x-\mu)^2 \geq K^2\sigma^2$$

$$\sigma^2 \geq \int_{-\infty}^{\mu-K\sigma} K^2\sigma^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} K^2\sigma^2 f(x)dx$$

$$1 \geq \int_{-\infty}^{\mu-K\sigma} K^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} K^2 f(x)dx$$

$$1 \geq K^2\{P[X \leq \mu - K\sigma] + P[X \geq \mu + K\sigma]$$

$$1 \geq K^2\{P[X - \mu \leq -K\sigma] + P[X - \mu \geq K\sigma]$$

$$1 \geq K^2\{P[|X - \mu| \geq K\sigma]$$

$$\therefore \quad P\{|X-\mu| \geq K\sigma\} \leq \frac{1}{K^2}$$

Since the probability is 1, we have

$$P[|X-\mu| \geq K\sigma] + P[|X-\mu| < K\sigma] = 1$$

$$P[|X-\mu| < K\sigma] = 1 - P[|X-\mu| \geq K\sigma] \geq 1 - \frac{1}{K^2}$$

Problem 1: Let X be a continuous RV whose probability density function given by $f(x) = e^{-x}$ $0 \leq x \leq \infty$. Using Chebyshev inequality verify $P[|X-\mu| > 2] \leq \frac{1}{4}$ and show that actual probability is e^{-3}.

Solution: E(X) = $\int_0^\infty xe^{-x}dx = 1$; E(X^2) = $\int_0^\infty x^2e^{-x}dx = 2$; Var(X) = 1.

We know, $P\{|X-\mu| \geq K\sigma\} \leq \frac{1}{K^2}$

$$K\sigma = 2; \; K = 2, \quad \because \sigma = 1$$

$$\therefore \quad P[|X-\mu| > 2] \leq \frac{1}{4}$$

$$P[|X-1| > 2] = P(-\infty \leq X - 1 \leq -2) + P(2 \leq X - 1 \leq \infty)$$

$$= 0 + \int_{3}^{\infty} f(x)dx$$

$$= e^{-3}$$

Problem 2: If X is the number on a die when it is thrown, prove that
P[|X - μ| > 2.5] < 0.47 where μ is the mean.

Solution:

X	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \Sigma\, x\, p(x) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

$$E(X^2) = \Sigma\, x^2\, p(x) = \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) = \frac{91}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = 2.9167$$

$\sigma = 1.707$

By Chebyshev's inequality, $P[|X - \mu| \geq K\sigma] \leq 1/K^2$

Given $P[|X - \mu| > 2.5] < 0.47$

Comparing. $K\sigma = 2.5$ $K = 1.46$ $\therefore \sigma = 1.707$

$\therefore P[|X - \mu| > 2.5] = (1/(1.46))^2 < 0.47$

Problem 3: A discrete RV X takes the values −1,0,1 with probabilities 1/8, 3/4, 1/8 respectively. Evaluate P[|X − μ| ≥ 2σ] and compare it with the upper bound given by chebyshev's inequality.

Solution:

$$E(X) = -1\times\frac{1}{8} + 0\times\frac{3}{4} + 1\times\frac{1}{8} = 0$$

$$E(X^2) = 1\times\frac{1}{8} + 0\times\frac{3}{4} + 1\times\frac{1}{8} = \frac{1}{4}$$

$$\therefore\ Var(X) = \frac{1}{4}$$

$P[|X - \mu| \geq 2\sigma] = P[|X| \geq 1 = P(X = -1) + P(X = 1) = 1/8 + 1/8 = 1/4$

By Chebychev's inequality, $P[|X - \mu| \geq 2\sigma] \leq \frac{1}{2^2} = \frac{1}{4}$.

Problem 4: A RV X takes the values {−1,1,3,5} with associated probability {1/6, 1/6, 1/6, 1/2} .Find an upper bound to the probability P[|X − 3| ≥ 1] by applying Chebychev's inequality. [E(X) = 3,Var(X) = 16/3; Prob. 16/3]

Problem 5: If x has a distribution with p.d.f of $f(x) = e^{-x}$,$0 \le x \le \infty$. Using Chebychev's inequality to obtain a lower bound to the probability $P(-1 \le x \le 3)$ and compare it with actual values. [E(X) = 1, Var(X) = 1, $P(-1 \le x \le 3) \ge 3/4$, Actual prob. 0.95]

Problem 6: If S denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for $p[|X - 7| \ge 4]$. Compare with the exact probability.

Solution: Let X_1 X_2 denote the outcomes of the first and second dice respectively.

$E(X_1) = E(X_2) = 7/2$; $E(X_1^2) = E(X_2^2) = 91/6$

$Var(X_1) = Var(X_2) = 35/12$

$E(X) = E(X_1 + X_2) = 7$

$Var(X) = Var(X_1 + X_2) = (35/12) + (35/12) = 35/6$

By Chebychev's inequality, $P\{|X-\mu| \ge K\sigma\} \le \dfrac{1}{K^2}$

$$P\{|X-7| \ge 4\} \le \frac{35}{96}$$

$$P[|X - 7| \ge 4] = P\{X = 2,3,11,12\} = \frac{1}{36} + \frac{2}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}.$$

Problem 7: A RV has mean $\mu = 12, \sigma^2 = 9$ and an unknown probability distribution. Find the probability of $P(6 < X < 18)$. [$P(6 < X < 18) \ge 3/4$]

Problem 8: A RV X is exponentially distributed with parameter 1. Use Chebychev's inequality to show that $P[-1 \le X \le 3] \ge 3/4$. Find the actual probability also.

Solution: X is Exponentially distributed with parameter $\lambda = 1$. $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$

$\therefore \mu = 1, \sigma = 1$

By Chebychev's inequality, $P[|X - \mu| \le K\sigma] \ge 1 - (1/K^2)$

$P[-1 \le X \le 3] = P[-2 \le X - 1 \le 2] = P[\,|X - 1| \le 2]$

Comparing, $K\sigma = 2$, as $\sigma = 1$, $K = 2$

$P[-1 \le X \le 3] = P[\,|X - 1| \le 2] \ge 1 - (1/4)$

$\therefore P[-1 \le X \le 3] \ge 3/4$.

Problem 9: Use Chebychev's inequality to find how many times a fair coin must be tossed in order that the probability that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55 will be at least 0.95.

Solution: Let X be the number of heads obtained when a fair coin is tossed *n* times.

Then, X follows binomial distribution $B(np, \sqrt{npq})$ where $p = q = 1/2$

i.e., $X \sim B(np, \sqrt{npq})$

$\therefore X/n \sim B(p, \sqrt{(pq/n)})$

i.e, $X/n \sim B(1/2, 1/2\sqrt{n})$

By Chebychev's inequality, $P\left\{\left|\frac{X}{n} - \frac{1}{2}\right| \le K\sigma\right\} \ge 1 - \frac{1}{K^2}$

i.e, $P[0.5 - K\sigma \le X/n \le 0.5 + K\sigma] \ge 1 - 1/K^2$

Given, $P[\,0.45 \le X/n \le 0.55] \ge 0.95$

Comparing, $1 - \frac{1}{K^2} = 0.95$

$$\frac{1}{K^2} = 0.05$$

$$K^2 = \frac{1}{0.05} = 20$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III - Probability Distribution– SMT5203

I Introduction

Moments

If X is a discrete or continuous r.v., $E[X^n]$ is called nth order raw moment of X about the orgin and denoted by μ'_n .

$E[(X - \mu_x)^n]$ is called the nth order central moment of X and denoted by μ_n.

Note : 1. First order moment about about origin is Mean
2. Second order moment about mean is Variance.

Moment Generating Function

Moment generating function (MGF) of a r.v. X (discrete or continuous) is defined as $E[e^{tX}]$, where t is a real variable and denoted as M(t).

If X is discrete, then $M(t) = \sum_r e^{tx_r} p_r$

If X is continuous r.v. with density function $f(x)$, then $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$.

Properties of Moment Generating Function

1. $M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$.
2. $\mu'_n = E(X^n) = \left[\frac{d^n}{dt^n} M(t)\right]_{t=0}$.
1. If the MGF of X is $M_x(t)$ and if Y = aX + b, then $M_y(t) = e^{bt} M_x(at)$.
2. If X and Y are independent random variables and Z = X + Y, then $M_z(t) = M_x(t) M_y(t)$.

Characteristic function

Characteristic function of a r.v. X (discrete or continuous) is defined as $E(e^{i\omega X})$ and denoted by $\phi(\omega)$.

If X is discrete, then $\phi(\omega) = \sum_r e^{i\omega x_r} p_r$

If X is continuous r.v. with density function $f(x)$, then $\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx$.

Properties of Characteristic Function

1. $\phi(\omega) = \sum_{n=0}^{\infty} \frac{i^n \omega^n}{n!} \mu'_n$.

2. $\mu_n' = \left[\dfrac{d^n}{d\omega^n}\phi(\omega)\right]_{\omega=0}$.

3. If the characteristic function of a r.v. X is $f_x(\omega)$ and if Y = *a*X + *b* , then

$$\phi_y(\omega) = e^{ib\omega}\phi_x(a\omega) .$$

4. If X and Y are independent random variables and Z = X + Y, then

$$\phi_z(\omega) = \phi_x(\omega)\phi_y(\omega) .$$

Problem 1: If X represents the outcome, when a fair die is tossed, find the MGF of X and hence find E(X) and Var(X).

Solution :

The probability distribution of X is given by

P(X = *i*) = 1/6, *i* = 1,2,3,4,5,6

$$M(t) = \sum_r e^{tx_r} p_r = \sum_{i=1}^{6} e^{it} p_i = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$E(X) = [M'(t)]_{t=0}$ (by property 2 of MGF)

$$= \frac{1}{6}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}) = \frac{7}{2}$$

$$E(X^2) = [M''(t)]_{t=0} = \frac{1}{6}(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t})_{t=0} = \frac{91}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} .$$

Problem 2: Find the MGF of a r.v. X whose probability function is $P(x) = \frac{1}{2^x}, x = 1,2,3,....$

Hence find its mean.

Solution :

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \frac{(e^t)^2}{2^2} + \frac{(e^t)^3}{2^3} + ...$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= e^t(2 - e^t)^{-1}$$

$$M'_X(t) = -e^t(2-e^t)^{-2} + (2-e^t)^{-1}e^t$$
$$Mean = M'_X(0) = 0.$$

Problem 3: Find the MGF of the r.v. X having p.d.f. $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$

$$\left[\frac{(e^t-1)^2}{t^2}\right]$$

Some Special Probability Distributions

Discrete distributions

Some of the discrete distributions are

- Binomial distribution
- Poisson distribution
- Geometric distribution
- Negative binomial distribution

Binomial Distribution

If X is discrete r.v. which can take values 0,1,2,3,…,n such that

$$P(X=r) = nC_r\, p^r q^{n-r},\ r = 0,1,2,\ldots,n \quad \text{where } p+q=1$$

then X is said to follow a Binomial distribution with parameters n and p.

Mean and Variance of the Binomial Distribution

$$\begin{aligned}
E(X) = \sum_r x_r p_r &= \sum_{r=0}^{n} r\, nC_r\, p^r q^{n-r} \\
&= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r q^{n-r} \\
&= np\sum_{r=1}^{n} r \frac{(n-1)!}{(r-1)!\{(n-1)-(r-1)\}!} p^{r-1} q^{(n-1)-(r-1)} \\
&= np\sum_{r=1}^{n} (n-1)C_{r-1}\, p^{r-1} q^{(n-1)-(r-1)} \\
&= np(q+p)^{n-1} = np
\end{aligned}$$

$$\begin{aligned}
E(X^2) = \sum_r x_r^2 p_r &= \sum_{r=0}^{n} r^2 p_r \\
&= \sum \{r(r-1)+r\} \frac{n!}{r!(n-r)!} p^r q^{n-r}
\end{aligned}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} (n-2)C_{r-2}\, p^{r-2} + np$$
$$= n(n-1)p^2 (q+p)^{n-2} + np$$
$$Var(X) = E(X^2) - [E(X)]^2 = n(n-1)p^2 + np - n^2 p^2 = npq$$

Moment Generating Function of Binomial Distribution

The binomial distribution is given by

$P(X = r) = nC_r\, p^r q^{n-r} \quad , r = 0,1,2,\ldots,n$ where $p + q = 1$

$$M(t) = \sum_{r=0}^{n} e^{tr} p_r = \sum_{r=0}^{n} e^{tr}\, nC_r\, p^r q^{n-r}$$
$$= \sum_{r=0}^{n} nC_r (pe^t)^r q^{n-r} = (pe^t + q)^n$$
$$M'(t) = n(pe^t + q)^{n-1} \times pe^t$$
$$M''(t) = np[(pe^t + q)^{n-1} \times e^t + (n-1)(pe^t + q)^{n-2} pe^{2t}]$$

$$E(X) = M'(0) = np$$
$$E(X^2) = M''(0) = np[1 + (n-1)p]$$
$$Var(X) = E(X^2) - [E(X)]^2 = npq$$

Poisson Distribution

If X is a discrete r.v. that can assume the values 0,1,2,… such that its probability mass function is given by

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0,1,2,\ldots;\ \lambda > 0.$$

Then X is said to follow a Poisson distribution with parameter λ.

Note: Poisson distribution is a limiting case of binomial distribution under the following assumptions.

(i) The number of trials 'n' should be indefinitely large. i.e., $n \to \infty$.
(ii) The probability of successes 'p' for each trial is indefinitely small.
(iii) $np = \lambda$, should be finite where λ is a constant.

Mean and Variance of Poisson Distribution

$$E(X) = \sum_r x_r p_r = \sum_{r=0}^{\infty} r \frac{e^{-\lambda}\lambda^r}{r!}$$
$$= \lambda e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$
$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(X^2) = \sum_r x_r^2 p_r = \sum_{r=0}^{\infty}[r(r-1)+r]\frac{e^{-\lambda}\lambda^r}{r!}$$
$$= \lambda^2 e^{-\lambda}\sum_{r=2}^{\infty}\frac{\lambda^{r-2}}{(r-2)!} + \lambda$$
$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$
$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Moment Generating Function of Poisson Distribution

The Poisson distribution is given by

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad , \; r = 0,1,2,\ldots.; \; \lambda > 0$$
$$M(t) = \sum_{r=0}^{\infty} e^{tr} p_r = \sum_{r=0}^{\infty} e^{tr} \frac{e^{-\lambda}\lambda^r}{r!}$$
$$= \sum_{r=0}^{\infty} e^{-\lambda} \frac{(e^t \lambda)^r}{r!}$$
$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{(e^t \lambda)^r}{r!}$$
$$= e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)}$$

Additive property of independent Poisson Variates

Let X and X be independent r.v.'s that follow Poisson distributions with parameters λ and λ respectively. Let X = X + X ,

$$P(X = n) = P[X_1 + X_2 = n]$$
$$= \sum_{r=0}^{n} P(X_1 = r)\cdot P(X_2 = n-r)$$
$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1}\lambda_1^r}{r!}\cdot\frac{e^{-\lambda_2}\lambda_2^r}{(n-r)!}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\lambda_1^r \cdot \lambda_2^{n-r}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n} nC_r \lambda_1^r \cdot \lambda_2^{n-r}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$
$$= \frac{e^{-\lambda}}{n!}\lambda^n .$$

Thus the sum of 2 independent random variables with parameters λ_1 and λ_2 Is also a Poisson variable with parameter ($\lambda_1 + \lambda_2$).

This property can be extended to any finite number of independent Poisson variables is known as the Reproductive property of Poisson r.v.'s.

Theorem 1:
If X and Y are independent r.v.'s, show that the conditional distribution of X, given the value of X + Y, is a binomial distribution.

Proof :

Let X and Y follow Poisson distribution with parameters λ_1 and λ_2 respectively.

$$P[X = r/(X+Y) = n] = \frac{P[x = r\, and\, (X+Y) = n]}{P[(X+Y) = n]}$$

$$= \frac{P[X = r\, and\, Y = n-r]}{P[(X+Y) = n]}$$

$$= \frac{\{e^{-\lambda_1}\lambda_1^r / r!\}\cdot\{e^{-\lambda_2}\lambda_2^{n-r} / (n-r)!\}}{e^{-(\lambda_1+\lambda_2)}\cdot(\lambda_1+\lambda_2)^n / n!}$$

$$= \frac{n!}{r!(n-r)!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$$= nC_r\, p^r q^{n-r}$$

Hence the result.

Geometric Distribution

Let the r.v. X denote the number of trials of a random experiment required to obtain the first success. Obviously X can take the values 1,2,3,….

The first (r – 1) trials result in failure and the rth trial results in success. Hence

$$P(X = r) = q^{r-1}p, \quad r = 1,2,3,\ldots\infty \quad \text{where } p + q = 1.$$

Then X is said to follow a geometric distribution.

Mean and Variance of Geometric Distribution

$$E(X) = \sum_r r p_r = \sum_{r=1}^{\infty} r q^{r-1} p$$

$$= p[1 + 2q + 3q^2 + \cdots + \infty]$$

$$= p(1-q)^{-2}$$

$$= \frac{1}{q}$$

$$E(X^2) = \sum_r r^2 p_r = \sum_{r=1}^{\infty} r^2 q^{r-1} p$$

$$= p\sum_{r=1}^{\infty}[r(r+1)-r]q^{r-1}$$

$$= p[\{1\times 2+2\times 3q+3\times 4q+\cdots+\infty\}-\{1+2q+3q^2+\cdots+\infty\}]$$

$$= p[2(1-q)^{-3}-(1-q)^{-2}]$$

$$= p\left(\frac{2}{p^3}-\frac{1}{p^2}\right)=\frac{1}{p^2}(1+q)$$

$$Var(X)=E(X^2)-[E(X)]^2=\frac{1}{p^2}(1+q)-\frac{1}{p^2}=\frac{q}{p^2}$$

Note: Some times the probability mass function of a geometric r.v. X is taken as $P(X=r)=q^r p;\ r=0,1,2,3....$ where $p+q=1$.

Moment Generating Function of Geometric Distribution

$$M_X(t)=E(e^{tx})=\sum_{x=1}^{\infty}e^{tx}q^{x-1}p$$

$$=\frac{p}{q}\sum_{x=1}^{\infty}(e^t q)^x$$

$$=\frac{p}{q}[e^t q+(e^t q)^2+(e^t q)^3+....]$$

$$= pe^t[1+e^t q+(e^t q)^2+...]$$

$$= pe^t[1-e^t q]^{-1}$$

$$=\frac{pe^t}{1-e^t q}$$

Negative Binomial Distribution

Let $p(x)$ be the probability that exactly x + r trails will be required to produce r success. Clearly the last trial must be a success and the probability is p. In the remaining x + r – 1 trials, there must be r – 1 successes and the probability of this is given by

$$p(X=x)=\begin{cases}(x+r-1)C_{r-1}p^r q^x, & x=0,1,2,3,...\\ 0, & otherwise\end{cases}$$

Also, We know $nC_r=nC_{n-r}$

$\therefore \quad (x+r-1)C_{r-1}=(x+r-1)C_x$

$$= \frac{(x+r-1)(x+r-2)\cdots(r+1)r}{x!}$$

$$= \frac{(-1)^x(-r)(-r-1)\ldots(-r-x+2)(-r-x+1)}{x!}$$

$$= (-1)^x(-r)C_x$$

$$\therefore\ p(x) = \begin{cases} -rC_x Q^{-r}\left(\dfrac{-P}{Q}\right)^x, & x = 0,1,2,.... \\ 0, & otherwise \end{cases} \qquad \text{where } p = \frac{1}{Q},\ q = \frac{P}{Q}$$

Moment Generating Function of Negative Binomial Distribution

The M.G.F. of a negative Binomial distribution is

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} (-r)Cx\ Q^{-r}\left(\frac{-P}{Q}\right)^x$$

$$= \sum_{x=0}^{\infty} (-r)Cx\ Q^{-r}\left(\frac{-Pe^t}{Q}\right)^x$$

$$= \sum_{x=0}^{\infty} (-r)Cx Q^{-r-x}\left(-Pe^t\right)^x$$

$$= \left(Q - Pe^t\right)^{-r}$$

$$Mean = \mu_1^{/} = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}(Q - Pe^t)^{-r}\right]_{t=0}$$

$$= rP = \frac{rq}{p}$$

$$\mu_2^{/} = \left[\frac{d^2 M_X(t)}{dt^2}\right]_{t=0}$$

$$= rP + r(r+1)P^2$$

Variance(X) = $rPQ = \frac{rq}{p^2}$

Problem 1: The mean and variance of a Binomial distribution are 4 and 4/3.
Find $P(X \geq 1)$.

Solution :

Mean is np and variance is npq.

$np = 4$ and $npq = 4/3$ $\therefore q = 1/3$

Also $p + q = 1$ $\therefore p = 1 - q = 2/3$ and $n = 6$.

$\therefore P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - 6C_0\,(2/3)^0\,(1/3)^{6-0} = 0.998$.

Problem 2: Find the Binomial distribution for which the mean is 4 and variance 3.

Solution :

$p = 1/4$ $q = 3/4$ and $n = 16$ $p(x) = 16C_x(1/4)^x(3/4)^{n-x}$.

Problem 3: Ten coins are thrown simultaneously. Find the probability of getting atleast 7 heads.

Solution :

$p = 1/2$ $q = 1/2$ $n = 10$ and X be event of getting heads

Then $P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$

$= 10C_7(1/2)^7(1/2)^3 + 10C_8(1/2)^8(1/2)^2 + 10C_9(1/2)^9(1/2)^1 + 10C_{10}(1/2)^{10}(1/2)^0$

$= 176/2^{10} = 0.171875$

Problem 4: In a large consignment of electric bulbs 10 % are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good bulbs (ii) at most 3 are defective bulbs (iii) Exactly there are three defective bulbs.

Solution : Let X be the event of defective bulbs,

$p = 0.1$ $q = 0.9$ $n = 20$

(i) $P(X = 0) = 20C_0\,(0.1)^0\,(0.9)^{20} = 0.1216$

(ii) $P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$

$= 20C_0\,(0.1)^0\,(0.9)^{20} + 20C_1\,(0.1)^1\,(0.9)^{19} + 20C_2\,(0.1)^3\,(0.9)^{18} + 20C_3\,(0.1)^3\,(0.9)^{17}$

$= 0.8666$

(iii) $P(X = 3) = 20C_3\,(0.1)^3\,(0.9)^{17} = 0.19$.

Problem 5: Assuming that half the population are consumers of rice, so that the chance of an individual being a consumer is 1/2 and assuming that 100 investigator each take 10 individuals to see whether these are consumers, how many investigators would you expect report that 3 people or less were consumers.

Solution : Let X be the event of rice consumers

$N = 100$, $n = 10$, $p = 1/2$, $q = 1/2$

$P(X \leq 3) = p(0) + p(1) + p(2) + p(3) = 0.1718$

So for 100 investigators, $N\,p(x) = 100 \times 0.1718 = 17$ approx.

Problem 6: In a 256 sets of twelve tosses of a coin, in how many cases may one expect eight heads and four tails?

Solution : Let X be Event of 8 heads and 4 tails,

N = 256, n = 12, p = 1/2 , q = 1/2

P(8 heads and 4 tails) = $12C_8(1/2)^8(1/2)^4$

In 256 sets N $p(x)$ = 256 × $12C_8(1/2)^8(1/2)$ = 31 times.

Problem 7: With usual notation find 'p' for a binomial random variable 'X' if n = 6 and if P(X = 4) = P(X = 2). [p = 0.25]

Problem 8: If X and Y are independent Poisson variate such that P(X = 1) = P(X = 2) and P(Y = 2) = P(Y = 3), find the variance of X – 2Y. [$\lambda_1 = 2, \lambda_2 = 3; Var. = 14$]

Problem 9: A manufacturer of cotterpins knows that 5% of his product is defective. If he sells cotterpins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is the approximate probability that a box will fail to meet the guaranteed quality?

[n =100, p = 0.05, $\lambda = np$ =5, P(X > 10) = 1 – P(X ≤ 10) = 0.014]

Problem 10: If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test (a) on the fourth trial and (b) in fewer than 4 trials?

Solution : Let X denote the number of trials required to achieve the first success. Then X follows a geometric distribution given by P(X = r) = $q^{r-1}p$; r = 1,2,3,….

Here p =0.8 and q = 0.2

(a) P(X = 4) = 0.8 × $(0.2)^{4-1}$ = 0.0064

(b) P(X < 4) = $\sum_{r=1}^{3}(0.8)\times(0.2)^{r-1}$ = 0.9984 .

Problem 11: A die is thrown until 6 appear. What is the probability that it must be thrown more than 5 times. [p = 1/6 q = 5/6;P(X > 5) = 1 – P(X ≤ 5) = 0.401]

Problem 12: A fair die is thrown several times. Find the probability that 3 appear before 4.

Solution : Die will be thrown until 3 or 4 occurs.

Failure is getting 1,2,5,6 = 4/6; success is getting 3 =1/6

P(X = r) = $\sum_{r=1}^{\infty} pq^{r-1} = \frac{1}{6}\left(1-\frac{4}{6}\right)^{-1} = \frac{1}{2}$

Problem 13: The probability of a student passing a subject is 0.8. What is the probability that he will pass the subject (a) On his third attempt (b) before the third attempt.
[p = 0.8 q = 0.2;P(X = 3) = 0.032; P(X < 3) = 0.96]

Problem 14: If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6^{th} attempt? [P(X = 6) = $q^{6-1}p = (0.5)^6$]

Problem 15: An item is produced in large numbers. The machine is known to produce 5% defective. A quality control inspector is examining the items by taking them at random. What is the probability that atleast 4 items are to be examined to get 2 defectives?
[$p = 0.05$ $q = 0.95$, $x + r \geq 4$ $P(X \geq 2) = 1 - P(X < 2) = 0.99275$]

Problem 16: Find the probability that in tossing 4 coins one will get either all heads or all tails for the third time on the seventh toss.

Solution : P(H H H H) = 1/16; P(T T T T) = 1/16

P(all head $\cup$ all tail) = 1/16 + 1/16 = 1/8

$\therefore$ $p = 1/8$ $q = 7/8$; $x + r = 7$ $r = 3$

$$P(X = x) = (x + r - 1)C_{r-1} p^r q^x$$

$$P(X = 4) = 7 - 1C_{3-1}\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^4$$

$$= 6C_2\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^4$$

$$= 0.0169$$

Problem 17: In a company 5% defective components are produced. What is the probability that atleast 5 components are to be examined in order to get three defectives?

Solution : $p = 0.05$ $q = 0.95$ $x + r \geq 5$ $r = 3$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1)$$
$$= 1 - 2C_2\,(0.05)^3\,(0.95)^0 - 3C_2\,(0.05)^3\,(0.95)^1$$
$$= 0.9995.$$

Problem 18: In a colony the probability that a person will own a car is 0.4. Find the probability that the 8th person randomly checked be the 4th one to own a car.
[$p = 0.4$ $q = 0.6$; $x + r = 8$ $r = 4$ $P(X = 4) = 7C_3\,(0.4)^4\,(0.6)^4 = 0.116$]

Problem 19: In a box of different coloured balls, Balls are taken one at a time until a red coloured ball is taken. If the probability of picking up a red ball is 0.5, what is the probability that a first red ball is picked up in 5th trial? [$p = 0.5$ $q = 0.5$; $x + r = 5$ $r = 1$, $P(X = 4) = 4C_0\,(0.5)^1\,(0.5)^4 = 0.03125$]

Special Continuous Distributions

Some of the continuous distributions are

- Uniform distribution
- Exponential distribution
- Gamma distribution
- Weibull distribution

Uniform distribution

A continuous random variable X is said to follow a uniform distribution ina any finite interval if its probability density function is constant in that interval.

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$$

Mean and Variance of Uniform distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{a}^{b} \frac{x dx}{b-a}$$

$$= \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{a}^{b} \frac{1}{b-a} x^2 dx$$

$$= \frac{1}{b-a} \frac{b^3 - a^3}{3}$$

$$= \frac{1}{b-a} \frac{(b-a)(a^2 + ab + b^2)}{3}$$

$$= \frac{(a^2 + ab + b^2)}{3}$$

$$Var(X) = \frac{(a^2 + ab + b^2)}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(a-b)^2}{12}$$

Moment Generating Function of Uniform distribution

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{e^{tb} - e^{ta}}{t} \right]$$

$$M_X(t) = \frac{1}{b-a} \left[\frac{e^{tb} - e^{ta}}{t} \right]$$

Exponential Distribution

A continuous r.v. X is said to follow an exponential distribution or negative exponential distribution with parameter λ > 0. Its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

Moment Generating Function of Exponential distribution

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{\lambda - t}$$

$$M_X(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

We can write MGF

$$M_X(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{1}{\lambda}t + \frac{1}{\lambda^2}t^2 + \ldots\ldots$$

$$= 1 + \frac{1}{\lambda}\frac{t}{1!} + \frac{2}{\lambda^2}\frac{t^2}{2!} + \frac{3}{\lambda^3}\frac{t^3}{3!} + \ldots.$$

Mean = E(X) = Co-efficient of $\frac{t}{1!} = \frac{1}{\lambda}$

E(X^2) = Co-efficient of $\frac{t^2}{2!} = \frac{2}{\lambda^2}$

Var(X) = $\frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Gamma Distribution

A continuous r.v. X is said to follow gamma distribution with parameter $k > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \frac{x^{k-1}e^{-x}}{\Gamma k}, x \geq 0,\ k > 0 \\ 0, \qquad otherwise \end{cases}$$

Mean and Variance of Gamma Distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \frac{x^{k-1}e^{-x}}{\Gamma k} dx$$

$$= \frac{1}{\Gamma k}\int_0^{\infty} x^k e^{-x} dx$$

$$= \frac{1}{\Gamma k}\Gamma(k+1) \qquad\qquad \int_0^{\infty} x^{n-1}e^{-ax}dx = \frac{\Gamma n}{a^n}$$

$$= \frac{k!}{(k-1)!} = k$$

$$E(X) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \frac{x^{k-1}e^{-x}}{\Gamma k} dx$$

$$= \frac{1}{\Gamma k}\int_0^{\infty} x^{k+1} e^{-x} dx$$

$$= \frac{\Gamma(k+2)}{\Gamma k}$$

$$= (k+1)k$$

Var(X) = E(X^2) – [E(X)]^2 = $(k + 1)k - k^2 = k$

Moment Generating Function of Gamma Distribution

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{tx} \frac{x^{k-1}e^{-x}}{\Gamma k} dx$$

$$= \frac{1}{\Gamma k}\int_0^{\infty} x^{k-1}e^{-(1-t)x} dx$$

$$= \frac{1}{\Gamma k}\frac{\Gamma k}{(1-t)^k}$$

$$= (1-t)^{-k}$$

Weibull Distribution

A continuous r.v. X has a Weibull distribution if its density function is given by

$$f(x) = \begin{cases} \alpha\beta\ x^{\beta-1}e^{-\alpha x^{\beta}} & , x > 0, \alpha, \beta > 0 \\ 0, & otherwise \end{cases}$$

if β = 1 it becomes Exponential distribution with parameter α.

Mean and Variance of Weibull distribution

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\alpha\beta\ x^{\beta-1}e^{-\alpha x^{\beta}} dx$$

$$= \alpha\beta\int_0^{\infty} x^{\beta}e^{-\alpha x^{\beta}} dx \qquad y = \alpha x^{\beta}; x = \left(\frac{y}{\alpha}\right)^{1/\beta}$$

$$= \alpha\beta\int_0^{\infty} \frac{y}{\alpha}e^{-y}\frac{dy}{\alpha\beta x^{\beta-1}}$$

$$= \int_0^{\infty}\left(\frac{y}{\alpha}\right)^{1/\beta} e^{-y}dy$$

$$= \frac{1}{\alpha^{1/\beta}}\int_0^{\infty} y^{1/\beta}e^{-y}dy$$

$$= \frac{1}{\alpha^{1/\beta}}\Gamma\left(1/\beta + 1\right)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \alpha\beta\, x^{\beta-1} e^{-\alpha x^{\beta}}\, dx$$

$$= \alpha\beta \int_0^{\infty} x^{\beta+1} e^{-\alpha x^{\beta}}\, dx$$

$$= \int_0^{\infty} \left(\frac{y}{\alpha}\right)^{2/\beta} e^{-y} dy$$

$$= \frac{1}{\alpha^{2/\beta}} \Gamma\left(\frac{2}{\beta}+1\right)$$

Var (X) = $\frac{1}{\alpha^{2/\beta}} \Gamma\left(\frac{2}{\beta}+1\right) - \frac{1}{\alpha^{2/\beta}} \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2$.

Moment Generating Function of Weibull distribution

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{tx} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx \qquad e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$$

$$= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \right) \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx$$

$$= \int_0^{\infty} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx + \frac{t}{1!} \int_0^{\infty} x \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx$$

$$+ \frac{t^2}{2!} \int_0^{\infty} x^2 \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx + \ldots\ldots + \frac{t^n}{n!} \int_0^{\infty} x^n \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx + \ldots.$$

$$\because E[X^n] = \int_X x^n f(x)dx = \alpha^{-n/\beta} \Gamma\left(\frac{n}{\beta}+1\right)$$

$$M(t) = 1 + \frac{t}{1!} E(X) + \frac{t^2}{2!} E(X^2) + \ldots + \frac{t^n}{n!} E(X^n) + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$$

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \alpha^{-n/\beta} \Gamma\left(\frac{n}{\beta}+1\right)$$

Problem 1: Buses arrive at a specified stop at 15 minutes intervals starting at 7:00 a.m. (i.e.) they arrive at 7:00, 7:15, 7:30, … If the passenger arrives at the stop at a random time (i.e.) Uniformly distributed between 7:00 and 7:30 a.m. Find the probability that he waits (i) less than 5 minutes for a bus (ii) atleast 12 minutes for a bus.

Solution: a = 7:00 and b = 7.30

$$f(x) = \frac{1}{b-a} = \frac{1}{30} \text{ minutes} \quad 0 < x < 30$$

P(X < 5) = P(10 < X < 15) + P(25 < X < 30)

$$= \int_{10}^{15} f(x)dx + \int_{25}^{30} f(x)dx$$

$$= \frac{1}{30}[15-10] + \frac{1}{30}[30-25] = \frac{1}{3}$$

P(waits atleast 12 minutes) = P(0 < X < 3) + P(15 < X < 18)

$$= \int_{0}^{3} \frac{1}{30} dx + \int_{15}^{18} \frac{1}{30} dx = \frac{1}{5}$$

Problem 2: The mileage which car owners get with a certain kind of radial tire is a r.v. having an exponential distribution with mean 40,000 km. Find the probabilities that one of these tires will last (i) at least 20,000 km and (ii) at most 30,000 km.

Solution: $$f(x) = \frac{1}{40{,}000} e^{-x/40{,}000}, \quad x > 0$$

$$P(X \geq 20{,}000) = \int_{20{,}000}^{\infty} \frac{1}{40{,}000} e^{-x/40{,}000} dx = e^{-0.5} = 0.6065$$

$$P(X \leq 30{,}000) = \int_{0}^{30{,}000} \frac{1}{40{,}000} e^{-x/40{,}000} dx = 0.527$$

Problem 3: Suppose that the life of a certain kind of an emergency lamp back up battery in hours is a r.v. X having the Weibull distribution α = 0.1 , β = 0.5 . Find (i) mean life time of these batteries (ii) the probability that such a battery will last more than 300 hours.

Solution: Here X is life time of batteries.

$$E(X) = \alpha^{-1/\beta}\, \Gamma\left(1/\beta + 1\right) = 200$$

∴ life time of the batteries is 200 hours.

$$P(X > 300) = \int_{300}^{\infty} f(x)dx$$

$$= \int_{300}^{\infty} \alpha\beta\, x^{\beta-1} e^{-\alpha x^{\beta}}\, dx \qquad\qquad y = \alpha x^{\beta};\ x = \left(\frac{y}{\alpha}\right)^{1/\beta}$$

$$= \int_{\sqrt{3}}^{\infty} e^{-y} dy = 0.176$$

Problem 4: Each of the 6 tubes of a radio set has a life length (in years) which may be considered as a r.v. that follows a Weibull distribution with parameters $\alpha = 25$, $\beta = 2$. If these tubes function independently of one another, What is the probability that no tube will have to be replaced during the first 2 months of service?

Solution: X is the length of life of each tube.

$$f(x) = 50xe^{-25x^2},\ \ x > 0$$

P(a tube is not to be replaced during the first two month) is

$$P(X > 1/6) = \int_{1/6}^{\infty} 50xe^{-25x^2} dx = e^{-25/36}$$

P(all the 6 tubes are not to be replaced during the first 2 months)

$$= \left(e^{-25/36}\right)^6 = e^{-25/6} = 0.0155$$

Problem 5: If the life X (in years) of a certain type of a car has a Weibull distribution with the parameter $\beta = 2$, fins the value of the parameter α, given that probability that the life of the car exceeds 5 years is $e^{-0.25}$. For these values of α and β, find the mean and variance of X.

Solution: $f(x) = 2\alpha\, x e^{-\alpha x^2}, x > 0$

$$P(X > 5) = \int_{5}^{\infty} 2\alpha\, x e^{-\alpha x^2} dx = \left[-e^{-\alpha x^2}\right]_5^{\infty} = e^{-25\alpha}$$

Given $P(X > 5) = e^{-0.25}$ $\qquad\therefore\ e^{-25\alpha} = e^{-0.25}$

$\therefore\ \alpha = 1/100.$

$E(X) = 5\sqrt{\pi}$ $\qquad Var(X) = 100\left(1 - \frac{\pi}{4}\right).$

Chebyshev Inequality

If X is a random variable with mean μ and variance σ^2, than for any positive number K, we have

$$P\{|X-\mu|\geq K\sigma\}\leq\frac{1}{K^2}$$

(or)

$$P\{|X-\mu|<K\sigma\}\geq 1-\frac{1}{K^2}$$

Solution: We know that

$$\sigma^2=E[X-E(X)]^2=E[X-\mu]^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

$$=\int_{-\infty}^{\mu-K\sigma}(x-\mu)^2 f(x)dx+\int_{\mu-K\sigma}^{\mu+K\sigma}(x-\mu)^2 f(x)dx+\int_{\mu+K\sigma}^{\infty}(x-\mu)^2 f(x)dx$$

$$\geq\int_{-\infty}^{\mu-K\sigma}(x-\mu)^2 f(x)dx+\int_{\mu+K\sigma}^{\infty}(x-\mu)^2 f(x)dx$$

Form first integral	Form second integral
$x\leq\mu-K\sigma$	$x\geq\mu+K\sigma$
$-(x-\mu)\geq K\sigma$	$(x-\mu)\geq K\sigma$
$(x-\mu)^2\geq K^2\sigma^2$	$(x-\mu)^2\geq K^2\sigma^2$

$$\sigma^2\geq\int_{-\infty}^{\mu-K\sigma}K^2\sigma^2 f(x)dx+\int_{\mu+K\sigma}^{\infty}K^2\sigma^2 f(x)dx$$

$$1\geq\int_{-\infty}^{\mu-K\sigma}K^2 f(x)dx+\int_{\mu+K\sigma}^{\infty}K^2 f(x)dx$$

$$1\geq K^2\{P[X\leq\mu-K\sigma]+P[X\geq\mu+K\sigma]$$

$$1\geq K^2\{P[X-\mu\leq -K\sigma]+P[X-\mu\geq K\sigma]$$

$$1\geq K^2\{P[|X-\mu|\geq K\sigma]$$

$$\therefore\quad P\{|X-\mu|\geq K\sigma\}\leq\frac{1}{K^2}$$

Since the probability is 1, we have

$$P[|X-\mu|\geq K\sigma]+P[|X-\mu|<K\sigma]=1$$

$$P[|X-\mu|<K\sigma]=1-P[|X-\mu|\geq K\sigma]\geq 1-\frac{1}{K^2}$$

Problem 1: Let X be a continuous RV whose probability density function given by $f(x)=e^{-x}$ $0 \le x \le \infty$. Using Chebyshev inequality verify $P[|X-\mu|>2] \le \frac{1}{4}$ and show that actual probability is e^{-3}.

Solution: $E(X) = \int_0^\infty x e^{-x} dx = 1$; $E(X^2) = \int_0^\infty x^2 e^{-x} dx = 2$; Var(X) = 1.

We know, $P\{|X-\mu| \ge K\sigma\} \le \frac{1}{K^2}$

$K\sigma = 2;\ K = 2, \quad \because \sigma = 1$

$\therefore \quad P[|X-\mu|>2] \le \frac{1}{4}$

$P[|X-1|>2] = P(-\infty \le X-1 \le -2) + P(2 \le X-1 \le \infty)$

$= 0 + \int_3^\infty f(x)dx$

$= e^{-3}$

Problem 2: If X is the number on a die when it is thrown, prove that
P[|X - μ| > 2.5] < 0.47 where μ is the mean.

Solution:

X	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

$E(X) = \Sigma\, x\, p(x) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$

$E(X^2) = \Sigma\, x^2\, p(x) = \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) = \frac{91}{6}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = 2.9167$

$\sigma = 1.707$

By chebyshev's inequality, $P[|X-\mu| \ge K\sigma] \le 1/K^2$

Given $P[|X-\mu| > 2.5] < 0.47$

Comparing. $K\sigma = 2.5$ $K = 1.46$ $\therefore \sigma = 1.707$

$\therefore P[|X-\mu| > 2.5] = (1/(1.46)^2 < 0.47$

Problem 3: A discrete RV X takes the values –1,0,1 with probabilities 1/8, 3/4, 1/8 respectively. Evaluate $P[|X-\mu| \ge 2\sigma]$ and compare it with the upper bound given by chebyshev's inequality.

Solution:

$E(X) = -1\times\frac{1}{8}+0\times\frac{3}{4}+1\times\frac{1}{8}=0$

$E(X^2) = 1\times\frac{1}{8}+0\times\frac{3}{4}+1\times\frac{1}{8}=\frac{1}{4}$

$\therefore$ Var(X) = $\frac{1}{4}$

P[|X − μ| ≥ 2σ] = P[|X| ≥ 1 = P(X = −1) +P(X = 1) = 1/8 + 1/8 =1/4

By Chebychev's inequality, P[|X − μ| ≥ 2σ] ≤ $\frac{1}{2^2}=\frac{1}{4}$.

Problem 4: A RV X takes the values {−1,1,3,5} with associated probability {1/6, 1/6, 1/6, 1/2} .Find an upper bound to the probability P[|X − 3| ≥ 1] by applying Chebychev's inequality. [E(X) = 3,Var(X) = 16/3; Prob. 16/3]

Problem 5: If x has a distribution with p.d.f of $f(x) = e^{-x}$,$0 \le x \le \infty$. Using Chebychev's inequality to obtain a lower bound to the probability $P(-1 \le x \le 3)$ and compare it with actual values. [E(X) = 1, Var(X) = 1, $P(-1 \le x \le 3) \ge 3/4$, Actual prob. 0.95]

Problem 6: If S denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for p[|X − 7| ≥ 4]. Compare with the exact probability.

Solution: Let X_1 X_2 denote the outcomes of the first and second dice respectively.

$E(X_1) = E(X_2) = 7/2$; $E(X_1^2) = E(X_2^2) = 91/6$

$Var(X_1) = Var(X_2) = 35/12$

$E(X) = E(X_1+ X_2) = 7$

$Var(X) = Var(X_1+ X_2) = (35/12)+ (35/12) = 35/6$

By Chebychev's inequality, $P\{|X-\mu|\ge K\sigma\}\le\frac{1}{K^2}$

$$P\{|X-7|\ge 4\}\le\frac{35}{96}$$

P[|X − 7| ≥ 4] = P{X = 2,3,11,12} = $\frac{1}{36}+\frac{2}{36}+\frac{2}{36}+\frac{1}{36}=\frac{1}{6}$.

Problem 7: A RV has mean $\mu=12, \sigma^2=9$ and an unknown probability distribution. Find the probability of P(6 < X < 18). [P(6 < X < 18) ≥ 3/4]

Problem 8: A RV X is exponentially distributed with parameter 1. Use Chebychev's inequality to show that P[−1 ≤ X ≤ 3] ≥ 3/4. Find the actual probability also.

Solution: X is Exponentially distributed with parameter λ = 1. E(X) = 1/λ and

$Var(X) = 1/\lambda^2$

$\therefore \mu = 1, \sigma = 1$

By Chebychev's inequality, $P[|X - \mu| \leq K\sigma] \geq 1 - (1/K^2)$

$P[-1 \leq X \leq 3] = P[-2 \leq X - 1 \leq 2] = P[\,|X - 1| \leq 2]$

Comparing, $K\sigma = 2$, as $\sigma = 1$, $K = 2$

$P[-1 \leq X \leq 3] = P[\,|X - 1| \leq 2] \geq 1 - (1/4)$

$\therefore\ P[-1 \leq X \leq 3] \geq 3/4$.

Problem 9: Use Chebychev's inequality to find how many times a fair coin must be tossed in order that the probability that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55 will be at least 0.95.

Solution: Let X be the number of heads obtained when a fair coin is tossed *n* times.

Then, X follows binomial distribution $B(np, \sqrt{npq})$ where $p = q = 1/2$

i.e., $X \sim B(np, \sqrt{npq})$

$\therefore X/n \sim B(p, \sqrt{(pq/n)})$

i.e, $X/n \sim B(1/2, 1/2\sqrt{n})$

By Chebychev's inequality, $$P\left\{\left|\frac{X}{n} - \frac{1}{2}\right| \leq K\sigma\right\} \geq 1 - \frac{1}{K^2}$$

i.e, $P[0.5 - K\sigma \leq X/n \leq 0.5 + K\sigma] \geq 1 - 1/K^2$

Given, $P[\,0.45 \leq X/n \leq 0.55] \geq 0.95$

Comparing, $$1 - \frac{1}{K^2} = 0.95$$

$$\frac{1}{K^2} = 0.05$$

$$K^2 = \frac{1}{0.05} = 20$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Stochastic Convergence – SMT5203

I. Introduction

Law of large numbers : In statistics, the theorem that, as the number of identically distributed, randomly generated variables increases, their sample mean (average) approaches their theoretical mean.The law of large numbers was first proved by the Swiss mathematician Jakob Bernoulli in 1713. He and his contemporaries were developing a formal probability theory with a view toward analyzing games of chance. Bernoulli envisaged an endless sequence of repetitions of a game of pure chance with only two outcomes, a win or a loss. Labeling the probability of a win *p*, Bernoulli considered the fraction of times that such a game would be won in a large number of repetitions. It was commonly believed that this fraction should eventually be close to *p*. This is what Bernoulli proved in a precise manner by showing that, as the number of repetitions increases indefinitely, the probability of this fraction being within any prespecified distance from *p* approaches 1.

There is also a more general version of the law of large numbers for averages, proved more than a century later by the Russian mathematician Pafnuty Chebyshev.The law of large numbers is closely related to what is commonly called the law of averages. In coin tossing, the law of large numbers stipulates that the fraction of heads will eventually be close to $^1/2$. Hence, if the first 10 tosses produce only 3 heads, it seems that some mystical force must somehow increase the probability of a head, producing a return of the fraction of heads to its ultimate limit of $^1/2$. Yet the law of large numbers requires no such mystical force. Indeed, the fraction of heads can take a very long time to approach $^1/2$. For example, to obtain a 95 percent probability that the fraction of heads falls between 0.47 and 0.53, the number of tosses must exceed 1,000. In other words, after 1,000 tosses, an initial shortfall of only 3 heads out of 10 tosses is swamped by results of the remaining 990 tosses.

The laws of large numbers are a collection of theorems that establish the convergence, in some of the ways already studied. These theorems are classified as weak or strong laws, depending on whether the convergence is in probability or almost surely.

Weak laws of large numbers: (Chebychev's Theorem) Let {Xn}n∈IN be a sequence of independent r.v.s (not necessarily identically distributed) such that V (Xn) ≤ M < ∞, ∀n ∈ IN. Then, 1 n X n i=1 Xi − 1 n X n i=1 E(Xi) P → 0.

Chebychev's Theorem for R.V s with equal mean) :In the conditions of Theorem 3.1, if $E(X_n) = \mu$, $\forall n \in N$, then X n ,i=1 Xi P $\rightarrow \mu$.

Strong Law Of Large Numbers :Thus, the sample mean converges weakly to the population mean. Historically, the next corollary was the first law of large numbers.

Bernouilli's Theorem: Let $\{X_n\}n \in IN$ be a sequence of i.i.d. r.v.s distributed as Bern(p). Then, 1 n X n i=1 Xi P $\rightarrow$ p. The next theorem does not require the existence of the variances, but in turn requires the r.v.**s to be identically distributed.**

Khintchine's weak law of large numbers : Let $\{X_n\},n \in N$ be a sequence of i.i.d. r.v.s with mean $E(X_n) = \mu \in (-\infty,\infty)$. Then, 1 n X n i=1 Xi P $\rightarrow \mu$.

Strong law of large numbers - Kolmogorov's Inequality: Let $\{X_n\},n \in IN$ be a sequence of independent r.v.s with mean $E(X_n) = \mu n$ and $V (X_n) = \sigma$ 2 n , both finite. Let Sn = Pn i=1 Xi and c 2 P n = n i=1 σ 2 i . Then, it holds that for all H > 0, P [n k=1 $\{\omega \in \Omega : |S_k(\omega) - E(S_k)|$ $\geq Hc_n\}$! $\leq$ 1 H2 .

Kolmogorov's strong law of large numbers : Let $\{X_n\}n \in IN$ be a sequence of independent r.v.s with mean $E(X_n)$,by strong law of large numbers μn and $V (X_n) = \sigma^{2 n}$, both finite. If X tends to ∞, n= $\sigma^{2 n} < \infty$, then X n, i=1 Xi − 1 n X n i=1 μi a.s. $\rightarrow 0$.

Borel-Cantelli's Theorem : Let $\{X_n\}n \in IN$ be a sequence of i.i.d. r.v.s distributed as Bern(p). Then, 1 n X n, i=1 Xi a.s. $\rightarrow$ p. This theorem says that the relative frequency of a dichotomic event goes almost surely to the probability of the event. Finally, the next strong law does not require anything to the variances but it assumes that the r.v.s are i.i.d.

Khintchine's strong law of large numbers: Let $\{X_n\}n \in IN$ be a sequence of i.i.d. r.v.s with $E(X_n) = \mu < \infty$. Then 1 n X n i=1 Xi a.s. $\rightarrow \mu$

Law of Large Numbers for Discrete Random Variables We are now in a position to prove our first fundamental theorem of probability. We have seen that an intuitive way to view the probability of a certain outcome is as the frequency with which that outcome occurs in the long run, when the experiment is repeated a large number of times. We have also defined probability mathematically as a value of a distribution function for the random variable representing the experiment. The Law of Large Numbers, which is a theorem proved about the mathematical model of probability, shows that this model is consistent with the frequency interpretation of probability. This theorem is sometimes called the law of averages. To find out what would happen if this law were not true, see the article by Robert M. Coates.1 Chebyshev Inequality To discuss the Law of Large Numbers, we first need an important inequality called the Chebyshev Inequality.

Chebyshev Inequality: Let X be a discrete random variable with expected value μ = E(X), and let ² > 0 be any positive real number. Then P(|X − μ| ≥ ²) ≤ V (X) ² .

Proof. Let m(x) denote the distribution function of X. Then the probability that X differs from μ by at least ² is given by P(|X − μ| ≥ ²) = X |x−μ|≥² m(x) . We know that V (X) = X x (x − μ) 2m(x) , and this is clearly at least as large as X |x−μ|≥² (x − μ) 2m(x) , since all the summands are positive and we have restricted the range of summation in the second sum. But this last sum is at least X |x−μ|≥² ² 2m(x) = ² 2 X |x−μ|≥² m(x) = ² 2P(|X − μ| ≥ ²) . So, P(|X − μ| ≥ ²) ≤ V (X) X in the above theorem can be any discrete random variable.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Central Limit Theorems – SMT5203

I. Introduction

The Central Limit Theorem (for the mean)

If random variable X is defined as the average of n If random variable X is defined as the average of n independent and identically distributed random independent and identically distributed random variables, X variables, X1, X2, …, Xn; with mean, ; with mean, μ, and Sd, σ. Then, for large enough n (typically n≥30), Xn is approximately Normally distributed with parameters: $\mu_x = \mu$ and $\sigma_x = \sigma / \sqrt{n}$.

Again, this result holds regardless of the shape of the X distribution (i.e. the Xs don't have to be The central limit theorem in statistics states that, given a sufficiently large sample size, the sampling distribution of the mean for a variable will approximate a normal distribution regardless of that variable's distribution in the population.

Distribution of the Variable in the Population

Part of the definition for the central limit theorem states, "regardless of the variable's distribution in the population." In a population, the values of a variable can follow different probability distributions. These distributions can range from normal, left-skewed, right-skewed, and uniform among others.This part of the definition refers to the distribution of the variable's values in the population from which you draw a random sample.

The central limit theorem applies to almost all types of probability distributions, but there are exceptions. For example, the population must have a finite variance. That restriction rules out the Cauchy distribution because it has infinite variance.

How the Central Limit Theorem is used in practice

In practice, the CLT is used as follows:

1. we observe a sample consisting of observations , , , ;
2. if is large enough, then a standard normal distribution is a good approximation of the distribution of the standardized sample mean;
3. therefore, we pretend that

$$\frac{\bar{X}_n - \mathrm{E}[\bar{X}_n]}{\sqrt{\mathrm{Var}[\bar{X}_n]}} \sim N(0,1)$$

4. as a consequence, the distribution of the sample mean is

$$\bar{X}_n \sim N(\mathrm{E}[\bar{X}_n], \mathrm{Var}[\bar{X}_n])$$

1. Lindeberg-Lévy Central Limit Theorem

The best known Central Limit Theorem is probably Lindeberg-Lévy CLT:

Proposition (Lindeberg-Lévy CLT) Let be an IID sequence of random variables such that:

where . Then, a Central Limit Theorem applies to the sample mean :

$$\mathrm{E}[X_n] = \mu < \infty, \forall n \in \mathbb{N}$$

$$\mathrm{Var}[X_n] = \sigma^2 < \infty, \forall n \in \mathbb{N}$$

where $\sigma^2 > 0$. Then, a Central Limit Theorem applies to the sample mean $\overline{X}_n$:

$$\sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right) \xrightarrow{d} Z$$

where Z is a standard normal random variable and $\xrightarrow{d}$ denotes convergence in distribution.

Proof:

We will just sketch a proof. For a detailed and rigorous proof see, for example: Resnick (1999) and Williams (1991). First of all, denote by the sequence whose generic term is

$$Z_n = \sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right)$$

$$\begin{aligned}
\varphi_{Z_n}(t) &= \mathrm{E}[\exp(itZ_n)] \\
&= \mathrm{E}\left[\exp\left(it\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma}\right)\right] \\
&= \mathrm{E}\left[\exp\left(it\sqrt{n}\frac{1}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\frac{t}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i - \mu}{\sigma}\right)\right] \\
&= \mathrm{E}\left[\prod_{i=1}^{n}\exp\left(i\frac{t}{\sqrt{n}}\frac{X_i - \mu}{\sigma}\right)\right] \\
&= \prod_{i=1}^{n}\mathrm{E}\left[\exp\left(i\frac{t}{\sqrt{n}}\frac{X_i - \mu}{\sigma}\right)\right] && \text{(by independence)} \\
&= \prod_{i=1}^{n}\mathrm{E}\left[\exp\left(i\frac{t}{\sqrt{n}}Y_i\right)\right] && \text{(defining: } Y_i = \frac{X_i - \mu}{\sigma}\text{)} \\
&= \prod_{i=1}^{n}\varphi_{Y_i}\left(\frac{t}{\sqrt{n}}\right) && (\varphi_{Y_i} \text{ is the characteristic function of } Y_i) \\
&= \left[\varphi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n && \text{(all the } Y_i \text{ have the same distribution)}
\end{aligned}$$

$$
\begin{aligned}
\varphi_{Y_1}(s) &= \mathrm{E}[\exp(isY_1)] \\
&= \mathrm{E}[\exp(isY_1)]|_{s=0} + \frac{d}{ds}(\mathrm{E}[\exp(isY_1)])\Big|_{s=0} s + \frac{1}{2}\frac{d^2}{ds^2}(\mathrm{E}[\exp(isY_1)])\Big|_{s=0} s^2 + o(s^2) \\
&= \mathrm{E}[\exp(isY_1)]|_{s=0} + \left(\mathrm{E}\left[\frac{d}{ds}\exp(isY_1)\right]\right)\Big|_{s=0} s + \frac{1}{2}\left(\mathrm{E}\left[\frac{d^2}{ds^2}\exp(isY_1)\right]\right)\Big|_{s=0} s^2 + o(s^2) \\
&= \mathrm{E}[\exp(isY_1)]|_{s=0} + (\mathrm{E}[iY_1\exp(isY_1)])|_{s=0} s + \frac{1}{2}\left(\mathrm{E}\left[-Y_1^2\exp(isY_1)\right]\right)\Big|_{s=0} s^2 + o(s^2) \\
&= 1 + i\mathrm{E}[Y_1]s - \frac{1}{2}\mathrm{E}\left[Y_1^2\right]s^2 + o(s^2) \\
&= 1 - \frac{1}{2}\mathrm{Var}[Y_1]s^2 + o(s^2) \qquad \text{(because } \mathrm{E}[Y_1] = 0\text{)} \\
&= 1 - \frac{1}{2}s^2 + o(s^2) \qquad \text{(because } \mathrm{Var}[Y_1] = 1\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \varphi_{Z_n}(t) &= \lim_{n\to\infty}\left[\varphi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n \\
&= \lim_{n\to\infty}\left[1 - \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + o\left(\left(\frac{t}{\sqrt{n}}\right)^2\right)\right]^n \\
&= \lim_{n\to\infty}\left[1 - \frac{1}{2}\frac{t^2}{n} + o\left(\frac{t^2}{n}\right)\right]^n \\
&= \exp\left(-\frac{1}{2}t^2\right) = \varphi_Z(t)
\end{aligned}
$$

$$\lim_{n\to\infty} \varphi_{Z_n}(t) = \varphi_Z(t)$$

$$\varphi_Z(t) = \exp\left(-\frac{1}{2}t^2\right)$$

is the characteristic function of a standard normal random variable (see the lecture entitled Normal distribution). A theorem, called Lévy continuity theorem, which we do not cover in these lectures, states that if a sequence of random variables is such that their characteristic functions converge to the characteristic function of a random variable , then the sequence converges in distribution to . Therefore, in our case the sequence converges in distribution to a standard normal distribution.

2. De Moivre-Laplace Central Limit Theorem

We are interested in the natural random variation of Sn around its mean. From the Weak Law of Large Numbers, we know that ℙn |Sn n − p| > ϵ ⟶ 0. From the Large Deviations result we also know that ℙn |Sn n − p| > ϵ ≤e−nh+(ϵ) + e−nh−(ϵ). Equivalently, we can say that Sn willfall outside the range np(1 ± ϵ) with probability near 0. Finally, note that E (Sn − np)2 = np(1 − p). We ask, "How large a fluctuation or deviation of Sn from np should be surprising?". We want a function ψ(n) with

$\lim n\to\infty\mathbb{P}n\ Sn - np > \psi(n) = \alpha$, for $0 < \alpha < 1$.

To measure the surprise of a fluctuation, we specify α, then ask what is the order of $\psi(n)$ as a function of n. Small but fixed values of α would indicate large surprise, i.e. unlikely deviations, and so we expect $\psi(n)$ to grow but more slowly than ϵn.

Take $p = 1⁄2$ to simplify the calculations for the discovery oriented proof in this subsection. We can make some useful guesses about $\psi(n)$. Interpret the probability on the left in as the area in the histogram for the binomial distribution of Sn. From the expression of Wallis' Formula for the central term in the binomial distribution, the maximum height of the histogram bars is of the order $1\ n\pi$, see Wallis' Formula.. That means that to get a fixed area α around that central term requires an interval of width at least a multiple of n. If we take $\psi(n) = xnn⁄2$ (with the factor $1⁄2$ put in to make variances cancel nicely), then we are looking for a sequence xn which will make

$$\lim n\to\infty\mathbb{P}n\ Sn - n⁄2 > xnn⁄2 = \alpha \text{ as } n \to\infty$$

true for $0 < \alpha < 1$. By Chebyshev's Inequality, we can estimate this probability as

$$\mathbb{P}n\ Sn⁄n - 1⁄2 > xn⁄(2n) \leq 1⁄xn2.$$

If $\limsup n\to\infty xn = \infty$, we could only obtain $\alpha = 0$, so xn is bounded above.
If $\liminf n\to\infty xn = 0$ then for a fixed $\epsilon > 0$ and some subsequence nm such that for sufficiently large m

$$\mathbb{P}nm\ Snm⁄nm - 1⁄2 > \epsilon > xnm⁄(2nm) \to 0.$$

which is also contradiction to the assumption $\alpha > 0$. Hence xn is bounded below by a positive value. We guess that $xn = x$ so $\psi(n) = xn⁄2$ for all values of n.

Proof :
To simplify the calculations, take the number of trials to be even and $p = 1⁄2$. Then the expression we want to evaluate and estimate is

$\mathbb{P}_{2n=}\ S_{2n} - n < x2n⁄2$.

This is evaluated as $\sum |k-n|<xn⁄22-2n2n\ k = \sum |j|<xn⁄22-2n\ 2n\ n + j$.

Let $Pn = 2-2n2n\ n$ be the central binomial term and then write each binomial probability in terms of this central probability Pn, specifically

$2-2n\ 2n\ n + j = Pn \cdot n(n - 1)\cdots(n - j + 1)\ (n + j)\cdots(n + 1)$.

Name the fractional factor above as Dj,n and rewrite it as
$Dj,n = 1\ (1 + j⁄n)(1 + j⁄(n - 1))\cdots(1 + j⁄(n - j + 1))$ and then

$\log(Dj,n) = -\sum k=0j-1 \log(1 + j⁄(n - k))$.Now use the common two-term asymptotic expansion for the logarithm function $\log(1 + x) = x(1 + \epsilon1(x))$. Note that

$\epsilon1(x) = \log(1 + x)\ x - 1 = \sum k=2n(-1)k+1xk\ k$

so $-x/2 < \epsilon 1(x) < 0$ and $\lim x \to 0 \epsilon 1(x) = 0$.$\log(Dj,n) = -\sum k=0j-1\ j\ n-k\ 1+\epsilon 1\ j\ n-k$ Let

$\epsilon 1,j,n \sum k=0j-1\ j\ n-k = \sum k=0j-1\ j\ n-k\epsilon 1\ j\ n-k$.

Then we can write $\log(Dj,n) = -(1+\epsilon 1,j,n) \sum k=0j-1\ j\ n-k$. and

then $\epsilon 1,j,n = \max |j|<xn/2\epsilon 1\ j\ n-k \to 0$ as $n \to \infty$.

Write $j\ n-k = j\ n \cdot 1\ 1-k/n$ and then

expand $1\ 1-k/n = 1+\epsilon 2(k/n)$ where $\epsilon 2(x) = 1/(1-x)-1 = \sum k=1\infty xk$ so $\epsilon 2(x) \to 0$ as $x \to 0$.

Once again k is restricted to the range $|k|\leq|j| < xn/2$ so $k\ n < xn/2\ n = x\ 2n$

$\epsilon 2,j,n = \max |k|<xn/2\epsilon 2\ k\ n \to 0$ as $n \to \infty$.

Then we can write

$\log(Dj,n) = -(1+\epsilon 1,j,n)(1+\epsilon 2,j,n) \sum k=0j-1\ j\ n$.

$\log(Dj,n) = -(1+\epsilon 3,j,n) \sum k=0j-1\ j\ n = -(1+\epsilon 3,j,n)j2\ n$

where $\epsilon 3,j,n=\epsilon 1,j,n+\epsilon 2,j,n+\epsilon 1,j,n\cdot\epsilon 2,j,n$. Therefore $\epsilon 3,j,n \to 0$ as $n \to \infty$ uniformly in j

Exponentiating ,$Dj,n = e-j2/n(1+\Delta\ j,n)$

where $\Delta j,n \to 0$ as $n \to 0$ uniformly in j.

Using Stirling's Formula, $Pn = 2-2n\ (2n)!\ n!\ ,n! = 1\ n\pi(1+\delta n)$.

Summarizing,$\mathbb{P}2n\ S2n-n < x2n/2 = \sum |j|<xn/22-2n\ 2n\ n+j = \sum |j|<xn/2Pn \cdot Dj,n = \sum |j|<xn/2$

$Pn \cdot e-j2/n(1+\Delta\ j,n) = (1+\delta n) \sum |j|<xn/2\ 1\ 2\pi \cdot e-j2/n\ 2\ n$

3.Levy-Cramer theorem

If the sum of two independent non-constant random variables is normally distributed, then each of the summands is normally distributed. This result was stated by P. Lévy [1] and proved by H. Cramer [2]. Equivalent formulations are:

1) if the convolution of two proper distributions is a normal distribution, then each of them is a normal distribution; and

2) if $\phi 1(t)\phi 1(t)$ and $\phi 2(t)\phi 2(t)$ are characteristic functions and

if $\phi 1(t)\phi 2(t)=\exp(-\gamma t2+i\beta t$

then $\phi j(t)=\exp(-\gamma jt2+i\beta t),\gamma j\geq 0,-\infty<\beta<\infty.\phi j(t)=\exp(-\gamma jt2+i\beta t),\gamma j\geq 0,-\infty<\beta<\infty$.

In formulation 1), the Levy–Cramer theorem admits a generalization to the convolution of two signed measures with restrictions on their negative variation; in formulation 2) it admits a generalization to the case when instead of condition ,one considers the condition

$$\prod_{j=1}^{m}\{\phi_j(t)\}^{\alpha_j}=\exp(-\gamma t^2+i\beta t),\prod_{j=1}^{m}\{\phi_j(t)\}^{\alpha_j}=\exp(-\gamma t^2+i\beta t),$$

$$\gamma\geq 0,-\infty<\beta<\infty,t\in E,\gamma\geq 0,-\infty<\beta<\infty,t\in E,$$

where $\phi_1(t),\dots,\phi_m(t)\phi_1(t),\dots,\phi_m(t)$ are characteristic functions, $\alpha_1,\dots,\alpha_m\alpha_1,\dots,\alpha_m$ are positive numbers and EE is a set of real numbers with a limit point at the origin. There are generalizations of the Lévy–Cramér theorem to random variables in Euclidean spaces and in locally compact Abelian groups.The Levy–Cramer theorem has the following stability property. Closeness of the distribution of a sum of independent random variables to the normal distribution implies closeness of the distribution of each of the summands to the normal distribution; qualitative estimates of the stability are known.Theorems analogous to the Lévy–Cramér theorem have been obtained for the Poisson distribution (Raikov's theorem), for the convolution of a Poisson and a normal distribution, and for other classes of infinitely-divisible distributions.