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INSTITUTE OF SCIENCE AND TECHNOLOGY

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMT5201 – Foundation of Mathematics

UNIT – I – Set Theory – SMT5201

Course Material

SMT5201 - Foundation of mathematics

Unit - I

Set Theory

Introduction to Set theory, Laws of set theory, Venn diagram, Partition of Sets, Cartesian of Sets, basic theorems in set.

The concept of a set is used in various disciplines and particularly in computers.

Basic Definition:

1. "A collection of well defined objects is called a set".

The capitals letters are used to denote sets and small letters are used for denote objects of the set. Any object in the set is called element or member of the set. If x is an element of the set X, then we write $x \in X$, to be read as 'x *belongs to X'* , and if x is not an element of X, the we write $x \notin X$ to be read as *' x does not belongs to X'*.

2. The number of elements in the set A is called *cardinality* of the set A, denoted by $|A|$ or $n(A)$. We note that in any set the elements are distinct. The collection of sets is also a set.

$S = \{P_1, \{P_2, P_3\}, P_4, P_5\}$

Here $\{P_2, P_3\}$ itself one set and it is one element of S and $|S|$=4.

3. Let A and B be any two sets. If every element of A is an element of B, then A is called a *subset* of B is denote by $'A \subseteq B'$.

We can say that A *contained (included)* in B, (or) B *contains (includes)* A.

Symbolically, $A \subseteq B$ (or) $B \supseteq A$

Logically, $A \subseteq B = (x\forall)\{x \in A \rightarrow x \in B\}$

Let $A = \{1,2,3,4,5\},\ B = \{1,2,4\},\ C = \{1,5\},\ D = \{2\},\ E = \{1,4,2\}$

Then $B \subseteq A,\ C \subseteq A,\ D \subseteq A, D \subseteq B$

$C \not\subseteq B$, since $5 \in C \Rightarrow 5 \notin B,\ E \subseteq B$ and $B \subseteq E$.

Some of the important properties of set inclusion.

For any sets A, B and C

$A \subseteq A$ (Reflexive)

$(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$ (Transitive)

Note that $A \subseteq B$ does not imply $B \subseteq A$ except for the following case.

4. Two sets A and B are said to be *equal* if and only if $A \subseteq B$ and $B \subseteq A$,

$i.e., \quad A = B \Leftrightarrow (A \subseteq B\ and\ B \subseteq C)$

Example $\{1,2,4\} = \{4,1,2\}$ and $P = \{\{1,2\},4\},\ Q = \{1,2,4\}$ then $P \neq Q$

Since $\{1,2\} \in P$ and $\{1,2\} \notin Q$ eventhough $1,2 \in Q$.

The equality of sets is reflexive, symmetric, and transitive.

5. A set A is said to be a *proper subset* of a set B if $A \subseteq B$ and $A \neq B$. Symbolically it is written as $A \subset B. i.e.,\ A \subset B \Leftrightarrow (A \subseteq B\ \wedge\ A \neq B)$

$\subset$ is also called a *proper inclusion.*

6. A set is said to be *universal set* if it includes every set under our discussion. A universal set is denoted by ∪ or E.

In other words, if *p(x)* is a predicate.$E = \{x | p(x) \vee \urcorner p(x)\}$

One can observe that universal set contains all the sets.

7. A set is said to be *empty set* or *null set* if it does not contain any element, which id denoted by $\emptyset$.

In other words, if *p(x)* is a predicate.$\emptyset = \{x | p(x) \vee \neg p(x)\}$

One can observe that null set is a subset for all sets.

8. For a set A, the set of all subsets of A is called the *power set* of A. The power set of A is denoted by $\rho(A)$ or $2^{\wedge}$ $i.e.,\ \rho(A) = \{S \mid S \subseteq A\}$

Example, Let $A = \{a, b, c\}$

Then $\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\},\ A\}$

Then set $\emptyset$ and A are called *improper subsets* of A and the remaining sets are called *proper subsets* of A.

One can easily note that the number of elements of $\rho(A)$ is $2^{|A|}.i.e.,\ |\rho(A)| = 2^{|A|}$

SOME OPERATIONS ON SETS

1. Intersection of sets

Definition:

Let A and B be any two sets, the *intersection* of A and B is written as $A \cap B$ is the set of all elements which belong to both A and B.

Symbolically

$A \cap B = \{\, x \mid x \in A \text{ and } x \in B\}$

Example$A = \{1,2,3,4,5,6\},\ B = \{2,4,6,8\}$ then $A \cap B = \{2,4,6\}$. From the definition of intersection it follows that for any sets A,B,C and universal set E.

$A \cap A = A$ $\quad$ $A \cap B = B \cap A$ $\quad$ $A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap E = A$ $\quad$ $A \cap \emptyset = \emptyset$

2. Disjoint sets

Definition:

Two set A and B are called *disjoint* if and only if $A \cap B = \emptyset$, that is, A and B have no element in common.

Example $A = \{1,2,3\}$ $B = \{5,7,9\}$ $C = \{3,4\}$

$A \cap B = \emptyset,\ A \cap C = \{3\},\ B \cap C = \emptyset$

A and B are disjoint and B and C also, but A and C are not disjoint.

3. Mutually disjoint sets

Definition:

A collection of sets is called a *disjoint collection,* if for every pair of sets in the collection, are disjoint. The elements of a disjoint collection are said to be *mutually disjoint.*

Let $A = \{A_i\}_{i \in I}$ be an indexed set, A is mutually disjoint if and only if

$A_i \cap A_j = \emptyset$ for all $i, j \in I, i \neq j$.

Example

$A_1 = \{\{1,2\},\{3\}\}$, $A_2 = \{\{1\},\{2,3\}\}$, $A_3 = \{\{1,2,3\}\}$

Then $A = \{A_1, A_2, A_3\}$ is a disjoint collection of sets.

$A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$

4. Unions of sets

Definition:

The *union* of two sets A and B, written as $A \cup B$, is the set of all elements which are elements of A or the elements of B or both.

Symbolically $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Example Let $A = \{1,2,3,4,5,6\}$ $B = \{2,4,6,8\}$ then $A \cup B = \{1,2,3,4,5,,6,8\}$

From the union, it is clear that, for any sets A, B,C, and universal set E.

$A \cup A = A \qquad A \cup B = B \cup A \qquad A \cup (B \cup C) = (A \cup B) \cup C$

$A \cup E = E \qquad A \cup \emptyset = A$

5. Relative complement of a set

Definition:

Let A and B are any two sets. The *relative complement* of B in A, written $A - B,$ is the set of elements of A which are not elements of B.

Symbolically $A - B = \{ x \mid x \in A \; or \; x \notin B\}$

Note that $A - B = A \cap \bar{B}$.

Example Let $A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A - B = \{1,3,5\}$

$B - A = \{8\}$

It is clear from the definition that, for any set A and B.

$A - B = \emptyset$

$A - B \neq B - A$

$A - \emptyset = A$

6. Complement of a set

Definition:

Let A be any set, and E be universal. The relative complement of A in E is called *absolute complement or complement* of A. The complement of A is denoted by $\bar{A}$ (or $A^C \; or \; \backsim A$)

Symbolically

$E - A = \bar{A} = \{\, x \mid x \in E \text{ and } x \notin A\}$

Example Let $E = \{1,2,3,4, \ldots\}$ be universal set and

$A = \{2,4,6,8, \ldots\}$ be any set in E.

Then

$\bar{A} = \{1,3,5,7, \ldots\}$

From the definition, for any sets A$\bar{\bar{A}} = A \quad \bar{\emptyset} = E$

$\bar{E} = \emptyset \quad A \cup \bar{A} = EA \cap \bar{A} = \emptyset$

7. Boolean sum of sets

Definition:

Let A and B are any two sets. The *symmetric difference or Boolean sum* of A and B is the set A+B defined by

$A + B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$

(or) $A + B = \{\, x \mid x \in A \text{ and } x \notin B\} \cup \{\, x \mid x \in B \text{ and } x \notin A\}$

Example Let

$A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A + B = \{1,3,5,8\}$ From the definition, for any sets A and B.

$A + A = \emptyset, \;\; A + \emptyset = A$

$A + E = \bar{A}, \;\; A + B = B + A$ and

$A + (B + C) = (A + B) + C$

8. The principle of duality

If we interchange the symbols $\cap$, $\cup$, E and $\emptyset$, $\subseteq$ and $\supseteq$, $\subset$ and $\supset$, in a set equation or expression. We obtain a new equation or expression is said to be *dual* of the original on *(primal)*.

“ If T is any theorem expressed in terms of $\cap$,$\cup$ and $-$ deducible from the given basic laws, then the dual of T is also a theorem”.

Note that, the theorem T is proved in *m* steps, then dual of T also proved in *m* step.

Example The dual of $A \cap \overline{A} = \emptyset$ is given by $A \cup \overline{A} = E$.

Remark: Dual (Dual T) =T.

Identities on sets

$A \cup A = A$	Idempotent laws
$A \cap A = A$	
$A \cup B = B \cup A$	Commutative laws
$A \cap B = B \cap A$	
$(A \cup B) \cup C = A \cup (B \cup C)$	Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$A \cup (A \cap B) = A$	Absorption laws
$A \cap (A \cup B) = A$	
$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	De Morgan’s laws

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

$A \cup \emptyset = A$ $\quad$ $A \cap \emptyset = \emptyset$

$A \cup E = E$ $\quad$ $A \cap E = A$

$A \cup \bar{A} = E$ $\quad$ $A \cap \bar{A} = \emptyset$

$\bar{\emptyset} = E$ $\quad$ $\bar{E} = \emptyset$ $\quad$ $\bar{\bar{A}} = A$

PROBLEMS

1.$S = \{a, b, p, q\}$, $Q = \{a, p, t\}$. Find $S \cup Q$ and $S \cap Q$?

Solution:

$S \cup Q = \{a, b, p, q, t\}$

$S \cap Q = \{a, p\}$

2. If $A = \{a, b, c\}$. Find $\rho(A)$?

Solution:

$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$ and

$|A| = 3$

$|\rho(A)| = 2^3 = 8$

3. Write all proper subsets of $A = \{a, b, c\}$.

Solution:

The proper subsets are

$\rho(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

4. Show that $A \subseteq B \Leftrightarrow A \cap B = A$.

Solution:

If $A \subseteq B$, then $\forall\, x \in A \Longrightarrow x \in B$

Now , let

$x \in A \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow\ x \in A \cap B$

$A = A \cap B$

If $A \cap B = A$, then

Let $x \in A,\quad x \in A \cap B\ \Longrightarrow x \in B$

Therefore $A \subseteq B$.

5. If $A = \{2,5,6,7\}, B = \{1,2,3,4\},\ C = \{1,3,5,7\}$. Find $A - B, A - C, C - B$ and $B - C$.

Solution:

$A - B = \{5,6,7\}$

$A - C = \{2,6\}$

$C - B = \{5,7\}$

$B - C = \{2,4\}$

6. If $A = \{2,3,4\}, B = \{1,2\},\ C = \{4,5,6\}$. Find $A + B, B + C, A + C, A + B + C$ and $(A + B) + (B + C)$.

Solution:

$A + B = \{1,3,4\}$

$B + C = \{1,2,4,5,6\}$

$A + C = \{2,3,5,6\}$

$A + B + C = \{1,3,5,6\}$

$(A + B) + (B + C) = \{2,3,5,6\}$

Note that

$A + (B + B) + C = A + (\emptyset) + C = A + C = \{2,3,5,6\}$

7. Show that $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Solution:

Let

$x \in A \implies x \in A\ (or)\ x \in B$

$\implies x \in A \cup B$

$\implies A \subseteq A \cup B$

Now let $x \in A \cap B \implies x \in A\ and\ x \in B$

$\implies x \in A$

$A \cap B \subseteq A$

Hence $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Remark: $B \subseteq A \cup B, A \cap B \subseteq B$ and $A \cap B \subseteq A \cup B$.

8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.

Solution:

$x \in A - (A \cap B) \Leftrightarrow x \in A\ and\ x \notin (A \cap B)$

$\Leftrightarrow x \in A\ and\ \{x \notin A\ or\ x \notin B\}$

$\Leftrightarrow \{x \in A \; and \; x \notin A\}(or) \; \{x \in A \; and \; x \notin B\}$

$\Leftrightarrow \; \emptyset \; (or)\{x \in A \; and \; x \notin B\}$

$\Leftrightarrow x \in A \; and \; x \notin B$

$A - (A \cap B) \subseteq A - B$ and $A - B \subseteq A - (A \cap B)$

Therefore $A - (A \cap B) = A - B$.

9. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution:

$x \in A \cup (B \cap C) \Leftrightarrow x \in A \; or \; x \in B \cap C$

$\Leftrightarrow x \in A \; or \; \{x \in B \; and \; x \in C\}$

$\Leftrightarrow \{x \in A \; or \; x \in B\} \; and \; \{x \in A \; or \; x \in C\}$

$\Leftrightarrow \{x \in A \cup B\} \; and \; \{x \in A \cup C\}$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

10. Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

Solution:

Let $x \in \; \overline{(A \cup B)} \; \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \; and \; x \notin B$

$\Leftrightarrow x \in \bar{A} \; and \; x \in \bar{B}$

$\Leftrightarrow x \in \; \bar{A} \cap \bar{B}$

Therefore $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

11. Show that $(A - B) - C = A - (B \cup C)$.

Solution:

$(A - B) - C = (A - B) \cap \bar{C}$ $\qquad$ $(P - Q = P \cap \bar{Q})$

$= (A \cap \bar{B}) \cap \bar{C}$

$= A \cap (\bar{B} \cap \bar{C})$ $\qquad$ (Associative)

$= A \cap (\overline{B \cup C})$ $\qquad$ (De Morgan's law)

12. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

Solution:

Let $(A \cap B) - (A \cap C)$

$= (A \cap B) \cap (\overline{A \cap C})$

$= (A \cap B) \cap (\bar{A} \cup \bar{C})$

$= (A \cap B \cap \bar{A}) \cup (A \cap B \cap \bar{C})$

$= ((A \cap \bar{A}) \cap B) \cup (A \cap B \cap \bar{C})$

$= (\emptyset \cap B) \cup (A \cap B \cap \bar{C})$

$= \emptyset \cup (A \cap B \cap \bar{C})$

$= A \cap (B \cap \bar{C})$

$= A \cap (B - C)$

ASSIGNMENT PROBLEMS

Part –A

1. Define a set
2. Define subset of a set. What is mean by proper subset?

(i) Find all subset of $A = \{1,2,3\}$

(ii)Find all proper subsets of A.

3. Define power set.
4. Define disjoint sets with example?
5. If $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8,10\}$. Find $A \cup B, A \cap B, a - B, B - A,$

   $A + B,$ and $B + A$?

6. Which of the following sets are empty?
7. $\{x \mid x \in R, x + 6 = 6\}$
8. $\{x \mid x\ is\ a\ real\ integer\ such\ that\ x^2 + 1 = 0\}$
9. $\{x \mid x\ is\ a\ real\ integer\ and\ x^2 - 4 = 0\}$
10. State duality principle in set theory.
11. Define cardinality of a set.
12. If a set A has *n* elements, then the number of elements of power set of A is……..
13. Find the intersection of the following sets

    (i) $\{x \mid x^2 - 1 = 0\}, \{x \mid x^2 + 2x + 1 = 0\}$
14. Write the dual of $A \cap \bar{A} = \emptyset$.
15. Let A, B and C sets, such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, can we conclude that B=C.
16. State De Morgan's Laws.
17. Whether the union of sets is commutative or not?

PART –B

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. Verify the De Morgan's laws

   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
3. Show that the intersection of sets is associative.
4. Show that $A - (B - C) = (A - B) \cup (A \cap C)$.
5. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. Let $A_i = \{1,2,3, \dots\}$ for $i = 1,2,3, \dots$ find (a) $\cup_{i=1}^{n} A_i$ (b) $\cap_{i=1}^{n} A_i$
7. Prove that $A - (A - B) \subset B$.
8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
9. Prove that $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$.
10. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that B=C.(cancelation law)
11. Show that $A - (B \cup C) = (A - B) \cap (A - C)$.
12. Show that $A + A = \emptyset$, where + is the symmetric difference of sets.
13. Show that $(R \subseteq S)$ and $(S \subset Q)$ imply $R \subset Q$.
14. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$.

CARTESIAN PRODUCT OF SETS

The *Cartesian product* of the sets A and B, is written an $A \times B$, is the set of all ordered pairs in which the first elements are in A and the second elements are in B.

$i.e.\ A \times B = \{\langle x, y \rangle | x \in A \ and \ x \in B\}$

For example

Let $A = \{1,2\}, B = \{a, b, c\},\ c = \{\alpha, \beta\}$

Now

$A \times B = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 1,c\rangle\langle 2,a\rangle, \langle 2,b\rangle, \langle 3,c\rangle\}$

$A \times C = \{\langle 1,\alpha\rangle, \langle 1,\beta\rangle, \langle 2,\alpha\rangle, \langle 2,\beta\rangle\}$

$A \times B = \{\langle \alpha,a\rangle, \langle \alpha,b\rangle, \langle \alpha,c\rangle\langle \beta,a\rangle, \langle \beta,b\rangle, \langle \beta,c\rangle\}$

It is clear from the definition

$A \times B \neq B \times A$ and $\langle\langle a,b\rangle, c\rangle \in (A \times B) \times C$, is an ordered triple then $\langle a,b\rangle \in A \times B$ and $c \in C$.

Now , $A \times (B \times C) = \{\langle a, \langle b,c\rangle\rangle \,|\, a \in A \; and \; \langle b,c\rangle \in \langle B,C\rangle\}$

Note that $\langle a, \langle b,c\rangle\rangle$ is not an ordered triple.

This fact show that $(A \times B) \times C \neq A \times (B \times C)$

i.e. Cartesian product is not associative.

Now

$A \times A = A^2 = \{\langle x,y\rangle, \forall x,y \in A\}$ and $A^n = A^{n-1} \times A$.

Note that if A has *n* elements and B has *m* elements $A \times B$ has *nm* elements.

PROBLEMS

1.If $A = \{1,2,3\}$, $B = \{a,b\}$. Find $A \times B, B \times A$ and $A \times A$ and $A^2 \times B$

Solution :

$A \times B = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 2,b\rangle, \langle 3,a\rangle, \langle 3,b\rangle\}$

$B \times A = \{\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle b,3\rangle\}$

$A^2 = A \times A = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

$A^2 \times B = \{\langle 1,1,a\rangle, \langle 1,1,b\rangle, \langle 1,2,a\rangle, \langle 1,2,b\rangle, \langle 1,3,a\rangle, \langle 1,3,b\rangle, \langle 2,1,a\rangle, \langle 2,1,b\rangle,$

$\langle 2,2,a\rangle, \langle 2,2,b\rangle, \langle 2,3,a\rangle, \langle 2,3,b\rangle, \langle 3,1,a\rangle, \langle 3,1,b\rangle, \langle 3,2,a\rangle, \langle 3,2,b\rangle, \langle 3,3,a\rangle, \langle 3,3,b\rangle\}$

2.Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (B \cap C) \Leftrightarrow x \in A \; and \; y \in B \cap C$

$\Leftrightarrow x \in A \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{x \in A \; and \; y \in B\} \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times B\} \; and \; \{\langle x, y\rangle \in A \times C\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3.Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (A \cap B) \times (C \cap D) \Leftrightarrow x \in (A \cap B) \; and \; y \in (C \cap D)$

$\Leftrightarrow \{x \in A \; and \; x \in B\} \; and \; \{y \in C \; and \; y \in D\}$

$\Leftrightarrow \{x \in A \; and \; y \in C\} \; and \; \{x \in B \; and \; y \in D\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times C\} \; and \; \{\langle x, y\rangle \in B \times D\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times C) \cap (B \times D)\}$.

ASSIGNMENT PROBLEMS

Part A

1. Define Cartesian product of sets? Given an example?
2. If $A = \{0,1\}$, find A^2.
3. If $A = \{1,2,3\}$ and $B = \{a, b\}$, find $A \times B, B \times A, A^2$.
4. True or False
   I. If $A = \{1,3,5,7,9\}$, the $\{\forall x \in A, x + 2\ is\ a\ prime\ number\}$
   II. If $A = \{1,2,3,4,5\}$, the $\{\ni x \in A, x + 3 = 10\}$
5. If $A \times B = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$

Part B

6. If A,B and C are sets, prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. Prove that $(A \times C) - (B \times C) = (A - B) \times C$.
8. If $A = \{a, b\}$ and $B = \{1,2\}$,and $C = \{2,3\}$, find
   I. $A \times (B \cup C)$
   II. $(A \times B) \cup (A \times C)$
   III. $A \times (B \cap C)$
   IV. $(A \times B) \cap (A \times C)$
9. Show that the Cartesian product is not commutative? It is commutative only for equality of sets?

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMT5201 – Foundation of Mathematics

UNIT – II – Relations – SMT5201

Course Material

SMT5201-Foundation of Mathematics

Unit II

Relations

Relations: Product set, Relations (Directed graph of relations on set is omitted). Composition of relations, Types of relations, Partitions, Equivalence relations with example of congruence modulo relation, Partial ordering relations, n-ary relations.

If a set A is given explicitly, it is immaterial in which order the elements of A are listed, e.g. the set {x,y} is the same as the set {y,x}. In many instances, however, one would like, and, indeed, needs, to have some order in the appearance of the elements. As an example, consider a coordinate plane with an x-axis and a y-axis; then we can identify any point in the plane by its coordinates <x, y>. If you wanted to find the point, <a,b>, you would move on the x-axis a units to the right or to the left from the origin (depending on the sign of a), and then you would move b units up or down. If a and b are different, then <a,b> and <b,a> denote different points. So, in this example the order in which the elements appear is relevant.

The decisive property of ordered pairs is that two ordered pairs are equal if the respective components are the same.

Introduction to Relations

Sometimes it is necessary not to look at the full Cartesian product of two sets A and B, but rather at a subset of the Cartesian product. This leads to the following Definition. Any subset of $A \times B$ is called a relation between A and B. Any subset of $A \times A$ is called a relation on A.

In other words, if A is a set, any set of ordered pairs with components in A is a relation on A. Since a relation R on A is a subset of $A \times A$, it is an element of the powerset of $A \times A$, i.e. $R \subseteq P(A \times A)$. If R is a relation on A and <x,y> ∈ R, then we also write xRy, read as "x is in R-relation to y", or simply, x is in relation to y, if R is understood.

If x and y are binary related, under the relation R, the we write $\langle x, y\rangle \in R$ or xRy. If not the case we write $\langle x, y\rangle \notin R$.

1. Example $F = \{\langle x, y\rangle \mid x\ is\ the\ father\ of\ y\}$

$L = \{\langle x, y\rangle \mid x\ and\ y\ are\ real\ number\ and\ x < y\}$

Then F, L are binary relations.

2.Example Let A and B be any two sets, then any non empty subset R of $A \times B$ is called a *binary relation.*

Now

$A = \{1,2,3\}$

$B = \{a, b\}$ then

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

Let

$R_1 = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$R_2 = \{\langle 1, b\rangle, \langle 3, a\rangle\}$

$R_3 = \{\langle 2, a\rangle\}$

Then R_1,R_2 and R_3 are binary relations A to B.

Let S be any binary relation. The *domain* of S is the set of all elements x such that for some y, $\langle x, y\rangle \in S$.

$D(S) = \{x \mid \langle x, y\rangle \in S, for\ some\ y\}$

Similarly, the *range* of S is the set of all elements y such that, for some x, $\langle x, y\rangle \in S$

$i.e.$ $\quad R(S) = \{y \mid \langle x, y\rangle \in S, for\ some\ x\}$

Let

$S = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, b\rangle, \langle 3, a\rangle\}$

$D(S) = \{1,2,3\}$

$R(S) = \{a, b\}$

If $S \subseteq X \times Y$, then clearly $D(S) \subseteq X$ and $R(S) \subseteq Y$.

In case of $X = Y$, then the relation defined on $X \times X$ is called *an universal relation* in X.

If $X = \emptyset$, then a relation on $X \times X$ is called *void relation* in X.

Since relations are sets, then we can have their union and intersection and so on.

$R \cup S = \{\langle x, y\rangle \mid xRy \text{ or } xSy\}$

$R \cap S = \{\langle x, y\rangle \mid xRy \text{ and } xSy\}$

$R - S = \{\langle x, y\rangle \mid xRy \text{ and } \langle x, y\rangle \notin S\}$

$R + S = \{\langle x, y\rangle \mid \langle x, y\rangle \text{ is either in } R \text{ or in } S \text{ but not in both}\}$

Properties of Binary relations

1. Reflexive

Let R be a binary relation defined on X.

Then R is *reflexive* if, for every $x \in X, \langle x, y\rangle \in R$.

Example:

Let

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 2,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ are defined on X.

Then R is reflexive, but S is not reflexive. Since $\langle 2,2\rangle \notin S$ and $2 \in X$.

2. Symmetric

A relation R from X to Y is *symmetric* if every $x \in X$ and $y \in Y$, whenever $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

That is, if $xRy \Rightarrow yRx$, then R is symmetric

Example:

Let

$X = \{1,2\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$ are defined on X.

Then R is symmetric, but S is not symmetric. Since $\langle 1,2\rangle \in S$ but $\langle 1,2\rangle \notin S$.

3. Transitive

A relation R is *transitive* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.

That is, if $xRy \wedge yRz$, then R is transitive.

Example:

Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,3\rangle, \langle 2,1\rangle\}$

Then R is transitive, but S is not transitive. Since $\langle 2,1\rangle \in S$ and $\langle 1,2\rangle \in S$ but $\langle 2,2\rangle \notin S$.

4.Irreflexive

A relation R in a set X is *irreflexive* if, for every $x \in X, \langle x, x\rangle \notin R$.

Example:

Let

$A = \{1,2,3\}$

$R = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 2,2\rangle, \langle 1,3\rangle\}$

Then R is irreflexive, but S is not reflexive. Since $\langle 3,3\rangle \notin S$ and $\langle 1,1\rangle \in S$.

5. Antisymmetric

A relation R in a set X is *antisymmetric* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then$x = y$.

That is, if $xRy \wedge yRx \Rightarrow x = y$, then R is antisymmetric.

Example:

Let

X be the set of all subsets of E.

R be the inclusion relation ($\subseteq$) defined on X.

$A \subseteq B \wedge B \subseteq A \Rightarrow A = B$

Therefore R is antisymmetric in X.

6. Relation matrix

Let $X = \{x_1, x_2, \dots x_m\}$, $Y = \{y_1, y_2, \dots y_m\}$ are ordered sets, R be a relation defined from X to Y, then the *relation matrix* of R, is defined as

$M_R = (r_{ij})\ i : 1 \rightarrow m, j : 1 \rightarrow n$

Example 1:

Let $X = \{1,2,3\}$ $Y = \{a, b\}$

$R = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 3,b\rangle\}$ be a relation from X to Y. Then $M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$ be a relation on $X = \{1,2,3\}$. Then $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

7. Composition of Binary Relations

The concept of composition of relation is different from union and intersection of two relations.

Definition:

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite $R \circ S$ is a relation from X to Z defined by

The operation $\circ$ in $R \circ S$ is called *" composition of relations".*

Example.

Let

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$

$S = \{\langle 2,3\rangle, \langle 4,1\rangle, \langle 4,3\rangle, \langle 2,1\rangle\}$. Then

$R \circ S = \{\langle 1,3\rangle, \langle 1,1\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$

$S \circ R = \{\langle 2,4\rangle, \langle 4,2\rangle, \langle 4,4\rangle, \langle 2,2\rangle\}$

Note that

$R \circ R = R^2$

$R \circ R \circ R = R^2 \circ R = R^3$

$R^{n-1} \circ R = R^n$ etc.,

Definition:

The relation matrix for $R \circ S$ is given by $M_{R \circ S} = M_R \odot M_S$ where $\odot$ is defined as follows.

$M_R \odot M_S = \langle m_{ij} \rangle$ where $m_{ij}(\langle i,j \rangle th\ \ element)$ is 1 if and only if row i of M_R and column *j* of M_S have a 1 in the same relative position *k,* for some *k.*

Example:

Let

$R = \{\langle 1,2 \rangle, \langle 1,5 \rangle, \langle 2,2 \rangle, \langle 3,4 \rangle, \langle 5,1 \rangle, \langle 5,5 \rangle\}$

$S = \{\langle 1,3 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 4,2 \rangle, \langle 4,4 \rangle, \langle 5,2 \rangle, \langle 5,3 \rangle\}$. Then

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
M_{R \circ S} &= M_R \odot M_S \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
\text{and} & \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$R^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \langle 5,5\rangle\}$

Definition

Let R be a relation from X to Y. The *converse* of R, is written as $\tilde{R}$, is a relation from Y to X such that$xRy \Leftrightarrow x\tilde{R}y$.

Example:

If $R = \{\langle 1,a\rangle, \langle 2,b\rangle, \langle 2,a\rangle, \langle b,3\rangle$

$\tilde{R} = \{\langle a,1\rangle, \langle b,2\rangle, \langle a,2\rangle, \langle b,3\rangle$

Also it is clear that

1. $R = S \Leftrightarrow \tilde{R} = \tilde{S}$
2. $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$
3. $\widetilde{R \cup S} = \tilde{R} \cup \tilde{S}$

Result: The relation matrix $M_{\tilde{R}}$ is the transpose of the relation M_R.

$i.e. M_{\tilde{R}} = transpose\ of\ M_R$

Example:

Let

$R = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle$

$\tilde{R} = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,3\rangle$

We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[M_R]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{\tilde{R}}$$

EQUIVALENCE RELATION

Definition:

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 1:

Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$ is an equivalence relation on X.

Example 2:

Equality of subsets on a universal set is an equivance relation.

Example 3:

Let

$X = \{1,2,3, \ldots 7\}$

$R = \{\langle x, y\rangle \mid x - y \text{ is divisible by } 3\}$

Now, $\forall x \in X, x - x = 0$ is divisible by 3.

Therefore $\forall x \in X, \langle x, x\rangle \in R$ (reflexive)

For any $x, y \in X$

Let $\langle x, x\rangle \in R \Rightarrow x - y$ is divisible by 3 we have $-(x - y) = y - x$ is also divisible by 3.

$\langle y, x\rangle \in R$ (symmetric)

Let $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R$

$\Rightarrow x - y$ is divisible by 3 and $y - z$ is divisible by 3.

$\Rightarrow (x - y) + (y - z)$ is divisible by 3.

$\Rightarrow x - z$ is divisible by 3.

Therefore $\langle x, y \rangle \in R$ (Transitive)

Therefore R is an equivalence relation on X.

EQUIVALENCE CLASSES

Definition:

Let R be an equivalence relation on a set X. For any $x \in X$, the set $[x]_R \subseteq X$ given by

$[x]_R = \{y \mid xRy \; for \; y \in X\}$

is called an R-*equivalence class* generated by $x \in X$.

Therefore, an equivalence class $[x]_R$ of $x \in X$ is the set of all elements which are related to *x* by an equivalence relation R on X.

Example:

Let Z be the set of all integers and R be the relation called *"congruence modulo 4"* defined by

$R = \{\langle x, y \rangle \mid (x - y) \; is \; divisible \; by \; 4, for \; x \; and \; y \in Z\}$ (or $x \equiv y (mod \; 4)$)
Now, we determine the equivalence classes generated by R.

$[0]_R = \{\ldots - 8, -4, 0, 4, 8 \ldots\}$

$[1]_R = \{\ldots - 7, -3, 1, 5, 9 \ldots\}$

$[2]_R = \{\ldots - 6, -2, 2, 6, 10 \ldots\}$

$[3]_R = \{\ldots - 5, -1, 3, 7, 11 \ldots\}$

Note that

$[0]_R = [4]_R, [1]_R = [5]_R, \dots etc.$

Therefore $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R, [3]_R\}$

In a similar manner, we get the equivalence classed generated by the relation *"congruence modulo m"* for any integer *m.*

Therefore, an equivalence relation R on X, will divide the set X into an *equivalence classes,* and they are called *portion* of X.

PARTIAL ORDERED RELATION

A relation R on a set X is said to be a partial ordered relation, if R satisfies reflexive, antisymmetric, and transitive.

Example:

Let $\rho(A)$ be the power set of a set A.

Define a subset relation $(\subseteq)$ on $\rho\ (A)$, then $\subseteq$ is a partial ordered relation.

Usually we denote the partial ordered relations as $' \leq '$ is said to be *partially ordered set* (or) *poset,* which is denoted by $\langle X, \leq \rangle$. We will study more about posets in the subsequent sections.

1. Closures of a relation

Let R be a relation on the set X.

2. Reflexive closure

We have the relation R is reflexive if and only if the relation.

$R = \{\langle x, y \rangle \mid \forall\, x \in X\}$ is contained in R.

i.e. R is reflexive $\Leftrightarrow I \subset R$.

Definition:

Let R be a relation on X, then the smallest reflexive relation on X, containing R, is called *reflexive closure* of R.

Therefore $R_1 = R \cup I$ is the reflexive closure of R.

3. Symmetric closure

We have, the relation R is symmetric if $\langle x, y \rangle \in R \Leftrightarrow \langle y, x \rangle \in \tilde{R}$

$i.e. \tilde{R} = \{\langle y, x \rangle \mid \langle x, y \rangle \in R\}$

Definition:

Let R be a relation X, then smallest symmetric relation on X, containing R, is called the *symmetric closure* of R.

Therefore $R \cup \tilde{R}$ is the symmetric of R.

4. Transitive closure

We have, the relation R is transitive, if $\langle x, y \rangle \in R\ and\ \langle y, z \rangle \in R$ then $\langle x, z \rangle \in R$.

Definition:

A relation R^+ is said to be the *transitive closure* of the relation R on X if R^+ is the smallest transitive relation on X, containing R,

i.e R^+ is the transitive closure of R, if

I. $R \subseteq R^+$
II. R^+ is transitive on X
III. There is no transitive relation R_1 on X, such that $R \subset R_1 \subset R^+$

Remarks:

1. The transitive closure of R can be obtained by

$$R^+ = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{i=1}^{\propto} R^i$$

2. We know that $\langle x,z\rangle \in R^2$ if and only if there is an element y such that $\langle x,y\rangle \in R$ and $\langle y,z\rangle \in R$.

   Therefore $\langle a,b\rangle \in R^n$ if and only if we can find a sequence $x_1, x_2, \ldots . x_{n-1}$ in X such that $\langle a,x_1\rangle, \langle x_1,x_2\rangle, \ldots \langle x_{n-1},b\rangle$ are all in R.

   The sequence $a, x_1, x_2, \ldots . x_{n-1}, b$ is said to be a *chain* of length *n* from a to b in R. Here $x_1, x_2, \ldots . x_{n-1}$ are called interval vertices of the chain in R. Note that the interval vertices need not be distinct.

PROBLEMS

1. If $P = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$, $Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

Find (i) $P \cup Q, P \cap Q, \tilde{P}, \tilde{P} \cup Q$ (ii) domains of $P, P \cup Q, P \cap Q$ and (iii) ranges of $Q, P \cup Q, P \cap Q$.

Solution:

$P \cup Q = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$

$P \cap Q = \{\langle 2,4\rangle\}$

$\tilde{P} = \{\langle 2,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

$\tilde{P} \cup Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 2,1\rangle, \langle 4,3\rangle\}$

Domain of $P = \{1,2,3\}$

Domain of $(P \cup Q) = D(P \cup Q) = \{1,2,3,4\}$

Domain of $(P \cap Q) = D(P \cap Q) = \{2\}$

Range of $Q = R(Q) = \{2,3,4\}$

Range of $(P \cup Q) = R(P \cup Q) = \{2,3,4\}$

Range of $(P \cap Q) = R(P \cap Q) = \{4\}$

It is clear that

$D(P \cup Q) = D(P) \cup D(Q)$ and

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

In general $D(P) = R(\tilde{P})$ and $R(P) = D(\tilde{P})$.

2.Let $X = \{1,2,3,4\}$ and $R = \{\langle x, y\rangle \mid x, y \in X \; and \; (x - y) \; is \; an \; integeral$

$non \; zeromultiple \; of \; 2\}$ $S = \{\langle x, y\rangle \mid x, y \in X \; and \; (x - y) \; is \; an \; integeral$

$non \; zeromultiple \; of \; 3\}$. Find $R \cup S$ and $R \cap S$?

Solution:

Given that $R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$ and

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$ $R \cup S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,1\rangle, \langle 4,2\rangle\}$

$R \cap S = \emptyset$

Remarks:

$D(R) = \{1,2,3,4\}$

$R(R) = \{1,2,3,4\}$

$D(S) = \{1,4\}$

$R(S) = \{1,4\}$

3.Let $S = \{\langle x, x^2\rangle \mid x \in N\}$ and $T = \{\langle x, 2x\rangle \mid x \in N\}$, where $= \{0,1,2, \ldots .\}$. Find the range of S and T, find $S \cup T$ and $S \cap T$?

Solution:

$S = \{\langle x, x^2 \rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \langle 3,9\rangle, \langle 4,16\rangle, \dots\dots\}$ and

$T = \{\langle x, 2x\rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 4,8\rangle, \dots\dots\}$

$R(S) = \{x^2 \mid x \in N\}$

$= \{0,1,4,9,16,25 \dots\dots\}$

$R(T) = \{2x \mid x \in N\}$

$= \{0,2,4,6,8,10, \dots\dots\}$

$S \cup T = \{\langle x, x^2\rangle \mid x \in N\} \cup \{\langle x, 2x\rangle \mid x \in N\}$

$= \{\langle x, y\rangle \mid x, y \in N, such\ that\ y = x^2\ (or) 2x\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 3,9\rangle, \dots\dots\}$

$S \cap T = \{\langle x, y\rangle \mid x, y \in N, such\ that\ y = 2x\ and\ y = x^2\}$

(Now $y = 2x$ and $y = x^2 \Rightarrow 2x = x^2\ i.e.\ x = 0\ or\ x = 2$

$x = 0\ y = 0\ and\ x = 2 \Rightarrow y = 4$)

$S \cap T = \{\langle 0,0\rangle, \langle 2,4\rangle\}$

4. Given an example which is neither reflexive nor irreflexive?

Solution:

Let $X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}$

Then R is not reflexive, since $\langle 2,2\rangle \notin R$, for $2 \in X$ and R is not irreflexive, since $1 \in X$, and $\langle 1,1\rangle \in R$.

5. Test whether the following relations are transitive or not on

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle\}$

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$

$T = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$.

Solution: The relation R and T are transitive.

Since, in R, we have $\langle 1,1\rangle \in R$, then check any other pair starting with $\langle 1,z\rangle \in R$, then we must have $1R1 \wedge 1Rz \Rightarrow 1Rz$ i.e., $\langle 1,z\rangle \in R$, but there is no pair staring with 1. So, pass on to next pair $\langle 2,2\rangle$ then we check any other pair starting with 2, and so on.

In T, we have $\langle 1,1\rangle \in T$, then there are two pairs $\langle 1,2\rangle$ and $\langle 1,3\rangle$ must be the transitive of $\langle 1,1\rangle \in T$, then we must have $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in T. Then pass to $\langle 1,2\rangle \in T$ the transitive pairs are $\langle 2,1\rangle, \langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must have the pairs $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle$ in T.

Then pass to $\langle 1,3\rangle \in T$, find the transitive pairs of $\langle 1,3\rangle$ and so on, for all pairs in T. Hence T is a transitive relation.

The relation S is not transitive, since for $\langle 1,2\rangle \in S$, the transitive pairs are $\langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in S but $\langle 1,3\rangle \notin S$.

6. Let R denotes a relation on the set of pairs of positive $N \times N$ integers such that $\langle x,y\rangle R \langle u,v\rangle$ if and only if $xv = yu$. Show that R is an equivalence relations.

Solution:

Let

$P = \{\langle x,y\rangle \mid x \text{ and } y \text{ are positive integer}\}$

Now R is a relation defined on P as

$\langle x, y\rangle R \langle u, v\rangle \Leftrightarrow xv = yu$ for $\langle x, y\rangle, \langle u, v\rangle \in P$.

Let $\langle x, y\rangle, \langle u, v\rangle$ and $\langle m, n\rangle \in P$.

I. R is reflexive:

We have

$\langle x, y\rangle R \langle x, y\rangle \Leftrightarrow xy = yx$ (RHS) is true.

II. R is symmetric:

Let $\langle x, y\rangle R \langle u, v\rangle \Leftrightarrow xv = yu$

$\Leftrightarrow yu = xv$

$\Leftrightarrow uy = vx$

$\Leftrightarrow \langle u, v\rangle R \langle x, y\rangle$

III. R is transitive:

Let $\langle x, y\rangle R \langle u, v\rangle$ and $\langle u, v\rangle R \langle m, n\rangle$

$\Leftrightarrow (xv = yu)$ and $(un = vm)$

$\Leftrightarrow (xv = yu)$ and $(u = \frac{vm}{n})$

$\Leftrightarrow xv = y(\frac{vm}{n})$

$\Leftrightarrow xn = ym$

$\Leftrightarrow \langle u, v\rangle R \langle m, n\rangle$

Therefore R is reflexive, symmetric , and transitive.

Hence R is an equivalence relation.

7. Let R and S are equivalence relations on X, show that $R \cap S$ also equivalent? Whether $R \cup S$ is also an equivalent relation. If not given an example.

Solution:

Given let R and S are equivalence relations on X.

Let x, y and $z \in X$.

(i) We have $\langle x, x\rangle \in R$ and $\langle x, x\rangle \in S \Rightarrow \langle x, x\rangle \in R \cap S, \ \forall x \in X$.

Therefore $R \cap S$ is reflexive.

(ii)Let $\langle x, y\rangle \in R \cap S \Rightarrow \langle x, y\rangle \in R$ and $\langle x, y\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R$ and $\langle y, x\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R \cap S$

Therefore $R \cap S$ is symmetric.

(iii) Let $\langle x, y\rangle \in R \cap S$ and $\langle y, z\rangle \in R \cap S$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle x, y\rangle \in S)$ and $(\langle y, z\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$ and $(\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow \langle x, y\rangle \in R$ and $\langle x, z\rangle \in S$

$\Rightarrow \langle x, z\rangle \in R \cap S$

Therefore $R \cap S$ is transitive.

Hence $R \cap S$ is equivalence.

8. Prove that the relation *"congruence modulo m"* over the set of positive integers is an equivalence relation?

Show also that if $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) = (y_1 + y_2)$.

Solution:

Let N be the set of all positive integers we have *"congruence modulo m"* relation on N as $x \equiv y (mod\, m) \Leftrightarrow m |\, x - y$, for $x, y \in N$.

Let $x, y, z \in N$

(i) We have

$x - x = 0 = 0m$

Therefore $x \equiv x \ (mod\ m)$ for $x \in N$.

"Congruence modulo m" is reflexive.

(ii) Let

$x \equiv y (mod\ m)$

$\Rightarrow\ m |\ x - y$

$\Rightarrow x - y = km$, for some integer $k \in Z$

$\Rightarrow y - x = (-k)m$, for some integer $-k \in Z$

$\Rightarrow y \equiv x\ (mod\ m)$

"congruence modulo m" is symmetric on N.

(iii) Let

$x \equiv y (mod\ m)$ and $y \equiv z\ (mod\ m)$

$\Rightarrow x - y = k_1 m$, and $y - x = k_2 m$ for some integer $k_1, k_2 \in Z$

$\Rightarrow (x - y) + (y - z) = (k_1 + k_2)m$

$\Rightarrow x - z = (k_1 + k_2)m$ for some integer $k_1 + k_2$

$\Rightarrow x \equiv z\ (mod\ m)$

"Congruence modulo m" is transitive on N.

Hence *"congruence modulo m"* is an equivalence relation.

Let $x_1 \equiv y_1 (mod\ m)$ and $x_2 \equiv y_2 (mod\ m)$.

Then $m |\ x_1 - y_1$ and $m |\ x_2 - y_2$

i.e., $x_1 - y_1 = k_1 m$ and $x_2 - y_2 = k_2 m$

Now

$(x_1 - y_1) + (x_2 - y_2) = k_1 m + k_2 m$

$(x_1 + x_2) - (y_1 + y_2) = (k_1 + k_2)m$

$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$

$(x_1 + x_2) \equiv (y_1 + y_2)(mod\, m)$

9. Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$ be a relation defined on A. Find the transitive closure of R?

Solution:

The matrix of the relation R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \odot M_R$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R^3} = M_{R^2} \odot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R^4} = M_{R^3} \odot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

As $|A| = 4$, we get

$$M_{R^+} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Hence

$R^+ = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$

ASSIGNMENT PROBLEMS

Part -A

1. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$ and $S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 1,3\rangle, \langle 3,3\rangle\}$ be any relations on $X = \{1,2,3\}$. Find $R \cup S, R \cap S, \tilde{R}, R(R), R(\tilde{S}), D(R \cup S), R(R \cap S)$.
2. Give an example for reflexive, symmetric, transitive and irreflexive relations.
3. Give an example of a relation which is neither reflexive nor irreflexive.
4. Give an example of a relation which is neither symmetric not antisymmetric?
5. Find the graph of the relation $R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

6. Find the relation matrix of

   $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$

7. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$ and $= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$. Find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $R \circ R \circ S$ and $S \circ S \circ S$?

8. Define equivalence relation and equivalence classes?

9. Define Poset?

10. Define reflexive closure?

11. Define transitive closure of the relation R?

12. Let $R = \{\langle 1,2\rangle, \langle 3,5\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$ be a relation $A = \{1,2,3,4,5,6\}$. Identify the root of the tree of R.

13. Determine whether the relation R is a partial ordered on the set Z, where Z is set of positive integer, and aRb if and only if a=2b.

14. The following relations are on $\{1,3,5\}$. Let R be a relation, xRy if and only if $y = x + 2$, and let S be a relation, xSy if and only if $x \leq y$. Find $R \circ S$ and $S \circ R$?

15. True or False: The relation $<$ on Z^+ is not a partial order since it is not reflexive.

Part B

1. Show that the intersection of equivalence relations is an equivalence relation.

2. Determine whether the relations represented by the following zero-one matrices are equivalence relations.

$$a)\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix} \qquad b)\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix}$$

3. If R and S are symmetric, show that $R \cup S$ and $R \cup S$ are symmetric.
4. Let L be set of all straight lines in the Euclidean plane and R be the relation in L defined by $xRy \Leftrightarrow x$ is perpendicular to y. Is R is Reflexive? Symmetric? Antisymmetric? Transive?
5. Consider the subsets $A = \{1,7,8\}$, $B = \{1,6,9,10\}$ and $C = \{1,9,10\}$ where $E = \{1,2,3 \dots \dots 10\}$ is an universal set. List the non empty minsets generated by A,B and C . Do they form a partition on E?
6. Let $X = \{1,2,3, \dots \dots 20\}$ and $R = \{\langle x, y\rangle \,|x - y \text{ is divisible by } 5\}$ be a relation on X. Show that R is an equivalent relation and find the partition of X induced by R.
7. If R is an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes constitute a partition on A.
8. Given the relation matrix M_R and M_S. Explain how to find $M_{R \circ S}$, $M_{S \circ R}$and M_{R^2}?
9. Let A be s set of books. Let R be a relation on A such that $\langle a, b\rangle \in R$ if ' book a' with cost more and contains fever pages then ' book b'. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?
10. Let R be a binary relation on the set of all positive integers such that $R = \{\langle a, b\rangle \,|a = b^2\}$. Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation?

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMT5201 – Foundation of Mathematics

UNIT – III – Functions – SMT5201

Course Material

SMT5201 – Foundation of Mathematics

Unit III

Functions

A relation is a function if for every x in the domain there is exactly one y in the codomain. A vertical line through any element of the domain should intersect the graph of the function exactly once. (one to one or many to one but not all the Bs have to be busy.

A function is injective if for every y in the codomain B there is at most one x in the domain. A horizontal line should intersect the graph of the function at most once (i.e.not at all or once). (one to one only but not all the Bs have to be busy.

A function is surjective if for every y in the codomain B there is at least one x in the domain. A horizontal line intersects the graph of the function at least once (i.e.once or more). The range and the codomain are identical. (one to one or many to one and all the Bs must be busy).

A function is bijective if for every y in the codomain there is exactly one x in the domain. A horizontal line through any element of the range should intersect the graph of the function exactly once. (one to one only and all the Bs must be busy).

Let $f:A\to B$ be a one-to-one correspondence (bijection). Then the inverse function of f, $f^{-1}:B\to A$, is defined by: $f^{-1}(b)$ = that unique element $a\in A$ such that $f(a)=b$. We say that f is invertible.

Example

If $f(x) = 4x - 1$ and $g(x) = x^2$. Then g(x), for $g : R \to R$ is not a bijection, so it cannot have an inverse. Now f(x) is a bijection, so we can compute its inverse. Suppose that $y = f(x)$, then $y = 4x -$

$1 \iff y + 1 = 4x \iff x = y + 1\ 4$, and $f^{-1}(y) = y+1\ 4$. We saw that for the notion of inverse f^{-1} to be defined, we need f to be a bijection. The next result shows that f^{-1} is a bijection as well.

Proposition 1.

If $f : X \to Y$ is a one-to-one correspondence, then $f^{-1} : Y \to X$ is a one-to-one correspondence.

Proof.

To prove this, we just apply the definition of bijection, namely, we need to show that f^{-1} is an injection, and a surjection. Let us start with injection. f^{-1} is an injection: we have to prove that if $f^{-1}(y1) = f^{-1}(y2)$, then $y1 = y2$. All right, then $f^{-1}(y1) = f^{-1}(y2) = x$ for some x in X. But $f^{-1}(y1) = x$ means that $y1 = f(x)$, and $f^{-1}(y2) = x$ means that $y2 = f(x)$, by definition of the inverse of function. But this shows that $y1 = y2$, as needed. f^{-1} is an surjection: by definition, we need to prove that any $x \in X$ has a preimage, that is, there exists y such that $f^{-1}(y) = x$. Because f is a bijection, there is some y such that $y = f(x)$, therefore $x = f^{-1}(y)$

Example

Consider $f : Z \to Z$ and $g : Z \to Z$ defined by $f(n) = 2n+3$, $g(n) = 3n + 2$. We have $(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n + 2) + 3 = 6n + 7$, while $(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n + 3) + 2 = 6n + 11$.

Suppose now that you compose two functions f, g, and both of them turn out to be injective. The next result tells us that the combination will be as well.

Proposition 2.

Let $f : X \to Y$ and $g : Y \to Z$ be two injective functions. Then $g \circ f$ is also injective.

Proof.

What we need to do is check the injectivity of a function, so we do this as usual: we check that $g \circ f(x1) = g \circ f(x2)$ implies $x1 = x2$. Typically, to be able to prove this, you will have to keep in mind assumptions, namely that both f and g are injective. So let us start. We have $g \circ f(x1) = g \circ f(x2)$ or equivalently $g(f(x1)) = g(f(x2))$. But we know that g is injective, so this implies $f(x1) = f(x2)$. Next we use that f is injective, thus $x1 = x2$, as needed! Let us ask the same question with surjectivity, namely whether the composition of two surjective functions gives a function which is surjective.

Proposition 3.

Let $f : X \to Y$ and $g : Y \to Z$ be two surjective functions. Then $g \circ f$ is also surjective.

Proof.

The codomain of $g \circ f$ is Z, therefore we need to show that every $z \in Z$ has a preimage x, namely that there always exists an x such that $g \circ f(x) = z$. Again, we keep in mind that f and g are both surjective. Since g is surjective, we know there exists $y \in Y$ such that $g(y) = z$. Now again, since f is surjective, we know there exists $x \in X$ such that $f(x) = y$. Therefore there exist x, y such that $z = g(y) = g(f(x))$ as needed.

Complexity of Algorithm

Complexity of an algorithm is a measure of the amount of time and/or space required by an algorithm for an input of a given size (n).

What effects run time of an algorithm?

(a) computer used, the harware platform
(b) representation of abstract data types (ADT's)
(c) efficiency of compiler
(d) competence of implementer (programming skills)
(e) complexity of underlying algorithm
(f) size of the input
We will show that of those above (e) and (f) are generally the most significant

Time for an algorithm to run t(n)

A function of input. However, we will attempt to characterise this by the size of the input. We will try and estimate the WORST CASE, and sometimes the BEST CASE, and very rarely the AVERAGE CASE.

What do we measure?

In analysing an algorithm, rather than a piece of code, we will try and predict the number of times "the principle activity" of that algorithm is performed. For example, if we are analysing a sorting algorithm we might count the number of comparisons performed, and if it is an algorithm to find some optimal solution, the number of times it evaluates a solution. If it is a graph colouring algorithm we might count the number of times we check that a coloured node is compatible with its neighbours.

Worst Case

Worse case is the maximum run time, over all inputs of size n, ignoring effects (a) through (d) above. That is, we only consider the "number of times the principle activity of that algorithm is performed".

Best Case

In this case we look at specific instances of input of size n. For example, we might get best behaviour from a sorting algorithm if the input to it is already sorted.

Average Case

Arguably, average case is the most useful measure. It might be the case that worst case behaviour is pathological and extremely rare, and that we are more concerned about how the algorithm runs in the general case. Unfortunately this is typically a very difficult thing to measure. Firstly, we must in some way be able to define by what we mean as the "average input of size n". We would need to know a great deal about the distribution of cases throughout all data sets of size n. Alternatively we might make a possibly dangerous assumption that all data sets of size n are equally likely. Generally, in order to get a feel for the average case we must resort to an empirical study of the algorithm, and in some way classify the input (and it is only recently with the advent of high performance, low cost computation, that we can seriously consider this option).

The Growth rate of t(n)

Suppose the worst case time for algorithm A is

$$t(n) = 60*n*n + 5*n + 1$$

for input of size n.

Assume we have differing machine and compiler combinations, then it is safe to say that

$$t(n) = n*n + 5*n/60 + 1/60$$

That is, we ignore the coefficient that is applied to the most significant (dominating) term in t(n). Consequently this only affects the "units" in which we measure. It does not affect how the worst case time grows with n (input size) but only the units in which we measure worst case time Under these assumptions we can say ...

"t(n) grows like n*n as n increases"

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UNIT – IV – Basic Logic I – SMT5201

Course Material

SMT5201-Foundation of Mathematics

UNIT IV: BASIC LOGIC I

Basic Logic-1 Introduction, propositions, truth table, negation, conjunction and disjunction. Implications, biconditional propositions, converse, contra positive and inverse propositions and precedence of logical operators. Propositional equivalence: Logical equivalences. Predicates and quantifiers: Introduction, Quantifiers, Binding variables and Negations

Propositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- "Man is Mortal", it returns truth value"TRUE"
- "12 + 9 = 3 − 2", it returns truth value "FALSE"

The following is not a Proposition−

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true orfalse.

Connectives

In propositional logic generally we use five connectives which are − OR (∨), AND (∧), Negation/ NOT (¬), Implication / if-then (→), If and only if (↔).

OR (∨) : The OR operation of two propositions A and B (written as A ∨B) is true if at least any of the propositional variable A or B istrue.
The truth table is as follows −

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

AND (∧): The AND operation of two propositions A and B (written as A ∧ B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A ∧ B
True	True	False
True	False	False
False	True	False
False	False	True

Negation (¬) : The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows –

A	¬A
True	False
False	True

Implication / if-then (→): An implication A→B is False if A is true and B is false. The rest of the cases are true.

The truth table is as follows −

A	B	A→ B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (↔) :A↔B is bi-conditional logical connective which is true when p and q are both false or both are true.

The truth table is as follows −

A	B	A↔B
True	True	True
True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.**Example**− Prove [(A → B) ∧ A] → B is a tautology

The truth table is as follows −

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A → B) ∧ A] → B is “True”, it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove (A ∨ B) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows −

A	B	A ∨ B	¬A	¬B	(¬A) ∧ (¬B)	(A ∨ B) ∧ [(¬A) ∧ (¬B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A ∨ B) ∧[(¬A) ∧ (¬B)] is "False", it is a contradiction

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove (A ∨ B∨) ∧ (¬A) a contingency

The truth table is as follows −

A	B	A ∨ B	¬A	(A ∨ B) ∧ (¬A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A ∨ B) ∧ (¬A) has both "True" and "False",it is a contingency.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions −

- The truth tables of each statement have the same truthvalues.
- The bi-conditional statement X ↔Y is atautology.

Example − Prove ¬(A ∨ B) and [(¬A) ∧ (¬B)] are equivalent

Testing by 1st method (Matching truth table)

A	B	A ∨ B	¬ (A ∨ B)	¬A	¬B	[(¬A) ∧ (¬B)]
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of ¬ (A ∨ B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality)

A	B	¬ (A ∨ B)	[(¬A) ∧ (¬B)]	[¬ (A ∨B)] ⇔ [(¬A) ∧ (¬B)]
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬ (A ∨ B)] ⇔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

EQUIVALENT LAWS

Equivalence	Name of Identity
p$\wedge T \equiv p$ p$\vee F \equiv p$	Identity Laws
p$\wedge F \equiv F$ p$\vee T \equiv T$	Domination Laws
p$\wedge p \equiv p$ p$\vee p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
p$\wedge q \equiv q \wedge p$ p$\vee q \equiv q \vee p$	Commutative Laws
(p$\wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (p$\vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws
p$\wedge(q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ p$\vee(q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Ditributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
p$\wedge(p \vee q) \equiv p$ p$\vee(p \wedge q) \equiv p$	Absorption Laws
p$\wedge\neg p \equiv F$ p$\vee\neg p \equiv T$	Negation Laws

Logical Equivalences involving Conditional Statements

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Logical Equivalences involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

A conditional statement has two parts − **Hypothesis** and **Conclusion**.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

Inverse, Converse, andContra-positive

Inverse –An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q". The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

Converse−The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the inverse willbe "If *q*, then*p*". The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do not do your homework".

Contra-positive –The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the inverse will be "If not *q*, then not *p*". The Contra-positive of "If you do your homework, you will not be punished" is "If you will be punished, you do yourhomework".

Example:

Give the converse and the Contra positve of the implication " If it is raining then I get wet".

Solution :

P : It is raining Q : I get wet

Converse : $Q \rightarrow P$: If I get wet, then it is raining.

Contrapositive : $\neg Q \rightarrow \neg P$: If I do not get wet, then it is not raining

DUALITY PRINCIPLE

Duality principle set states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said **self-dual**statement.

Examples :i) The dual of (A ∩ B) ∪ C is (A ∪ B) ∩ C

ii) The dual of P∧Q∧F is P∨Q∨ T

Example : 1

Construct a truth table for $(p \rightarrow q) \rightarrow (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \rightarrow (q \rightarrow q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Example 2: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

.

Solution : The truth tables for these compound proposition is as follows.

1	2	3	4	5	6	7	8
P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$6 \leftrightarrow 7$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

We can observe that the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q.

Example 3: Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution : The truth tables for these compound proposition as follows.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As the truth values of $p \to q$ and $\neg p \vee q$ are logically equivalent.

Example 4 : Determine whether each of the following form is a tautology or a contradiction or neither :

i) $(P \wedge Q) \to (P \vee Q)$

ii) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

iii) $(\neg P \wedge \neg Q) \to (P \to Q)$

iv) $(P \to Q) \wedge (P \wedge \neg Q)$

v) $[P \wedge (P \to \neg Q) \to Q]$

Solution:

i) The truth table for $(p \wedge q) \to (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Here all the entries in the last column are 'T'.

$\therefore (p \wedge q) \to (p \vee q)$ is a tautology.

ii) The truth table for $(p \vee q) \wedge (\neg p \wedge \neg q)$ is

1	2	3	4	5	6	
p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg P \wedge \neg q$	$3 \wedge 6$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

The entries in the last column are 'F'. Hence $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

iii) The truth table is as follows.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \rightarrow q$	$(\neg p \wedge \neg q) \rightarrow (p \rightarrow q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Here all entries in last column are 'T'.

$\therefore$ $(\neg p \wedge \neg q) \rightarrow (p \rightarrow q)$ is a tautology.

iv) The truth table is as follows.

p	q	$\neg q$	$p \wedge \neg q$	$p \rightarrow q$	$(p \rightarrow q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

All the entries in the last column are 'F'. Hence it is contradiction.

v) The truth table for $[p \wedge (p \rightarrow \neg q) \rightarrow q]$

p	q	$\neg q$	$p \rightarrow \neg q$	$p \wedge (p \rightarrow \neg q)$	$[p \wedge (p \rightarrow \neg q) \rightarrow q]$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

The last entries are neither all 'T' nor all 'F'.

$\therefore$ $[p \wedge (p \rightarrow \neg q) \rightarrow q]$ is a neither tautology nor contradiction. It is a Contingency.

Example 5: Symbolize the following statement

Let p, q, r be the following statements:
p: I will study discrete mathematics
q: I will watch T.V.
r: I am in a good mood.
Write the following statements in terms of p, q, r and logical connectives.
(1) If I do not study and I watch T.V., then I am in good mood.
(2) If I am in good mood, then I will study or I will watch T.V.
(3) If I am not in good mood, then I will not watch T.V. or I will study.
(4) I will watch T.V. and I will not study if and only if I am in good mood.
Solution:

(1) $(\neg p \wedge q) \rightarrow r$

(2) $r \rightarrow (p \vee q)$

(3) $\neg r \rightarrow (\neg q \vee p)$

(4) $(q \wedge \neg p) \leftrightarrow r$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

SMT5201 – Foundation of Mathematics

UNIT – V – Basic Logic II – SMT5201

Course Material

SMT5201-Foundation of Mathematics

UNIT V: BASIC LOGIC II

Basic Logic-2 Methods of proof: Rules of inference, valid arguments, methods of proving theorems; direct proof, proof by contradiction, proof by cases, proofs by equivalence, existence proofs, uniqueness proofs and counter examples.

Inference Theory

The theory associated with checking the logical validity of the conclusion of the given set of premises by using Equivalence and implication

Inference Theory

The theory associated with checking the logical validity of the conclusion of the given set of premises by using Equivalence and Implication rule is called **Inference theory**

Direct Method

When a conclusion is derived from a set of premises by using the accepted rules of reasoning is called **direct method**.

Indirect method

While proving some results regarding logical conclusions from the set of premises, we use negation of the conclusion as an additional premise and try to arrive at a contradiction is called **Indirect method**

Consistency and Inconsistency of Premises

A set of formular $H_1,H_2,\ldots,H_m$ is said to be **inconsistent** if their conjunction implies Contradiction.
A set of formular $H_1,H_2,\ldots,H_m$ is said to be **consistent** if their conjunction implies Tautology.

Rules of Inference

Rule P: A premise may be introduced at anypoint in the derivation
Rule T: A formula S may be introduced at any point in a derivation if S is tautologically implied by any one or more of the preceedingformula.
Rule CP: If S can be derived from R and set of premises , then R S can be derived from the set ofpremises alone.

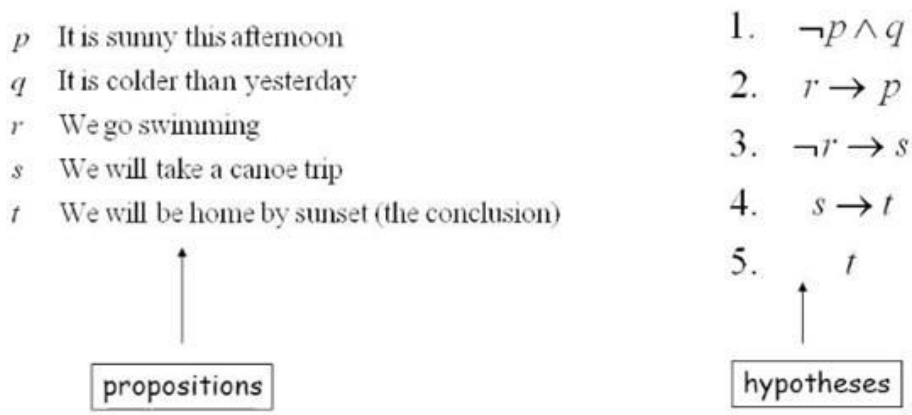
Rules of Inference

TABLE 1 Rules of Inference.		
Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Rule of inference to build arguments

Example:

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny.
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. We will be home by sunset



Example 1.Show that R is logically derived from $P \rightarrow Q$, $Q \rightarrow R$, and P

Solution.	{1}	(1)	$P \rightarrow Q$	Rule P
	{2}	(2)	P	Rule P
	{1, 2}	(3)	Q	Rule (1), (2) and I11
	{4}	(4)	$Q \rightarrow R$	Rule P
	{1, 2, 4}	(5)	R	Rule (3), (4) and I11.

Example 2.Show that S V R tautologically implied by (P V Q) ^ ($P \rightarrow R$) ^ ($Q \rightarrow S$).

Solution .	{1}	(1)	P V Q	Rule P
	{1}	(2)	$7P \rightarrow Q$	T, (1), E1 and E16
	{3}	(3)	$Q \rightarrow S$	P
	{1, 3}	(4)	$7P \rightarrow S$	T, (2), (3), and I13
	{1, 3}	(5)	$7S \rightarrow P$	T, (4), E13 and E1
	{6}	(6)	$P \rightarrow R$	P
	{1, 3, 6}	(7)	$7S \rightarrow R$	T, (5), (6), and I13
	{1, 3, 6)	(8)	S V R	T, (7), E16 and E1

Example 3. Show that 7Q, P→ Q => 7P

Solution .

{1}	(1) P → Q	Rule P
{1}	(2) 7P → 7Q	T, and E 18
{3}	(3) 7Q	P
{1, 3}	(4) 7P	T, (2), (3), and I11 .

Example 4 .Prove that R ^ (P V Q) is a valid conclusion from the premises PVQ , Q → R, P → M and 7M.

Solution .

{1}	(1) P → M	P
{2}	(2) 7M	P
{1, 2}	(3) 7P	T, (1), (2), and I12
{4}	(4) P V Q	P
{1, 2 , 4}	(5) Q	T, (3), (4), and I10.
{6}	(6) Q → R	P
{1, 2, 4, 6}	(7) R	T, (5), (6) and I11
{1, 2, 4, 6}	(8) R ^ (PVQ)	T, (4), (7), and I9.

Example 5 .Show that R → S can be derived from the premises P → (Q → S), 7R V P , and Q.

Solution.

{1}	(1) 7R V P	P
{2}	(2) R	P, assumed premise
{1, 2}	(3) P	T, (1), (2), and I10
{4}	(4) P → (Q → S)	P
{1, 2, 4}	(5) Q → S	T, (3), (4), and I11
{6}	(6) Q	P
{1, 2, 4, 6}	(7) S	T, (5), (6), and I11
{1, 4, 6}	(8) R → S	CP.

Example 6.Show that P → S can be derived from the premises, 7P V Q, 7Q V R, and R → S .

Solution.

{1}	(1)	7P V Q	P
{2}	(2)	P	P, assumed premise
{1, 2}	(3)	Q	T, (1), (2) and I11
{4}	(4)	7Q V R	P
{1, 2, 4}	(5)	R	T, (3), (4) and I11
{6}	(6)	R → S	P
{1, 2, 4, 6}	(7)	S	T, (5), (6) and I11
{2, 7}	(8)	P → S	CP

Predicate Logic

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

Eg.
" x is a Man"
Here **Predicate** is " is a Man" and it is denoted by M and **subject** "x" is denoted by x.
Symbolic form is M(x).

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol ∀.

∀x P(x) is read as for every value of x, P(x) is true.

Example − "Man is mortal" can be transformed into the propositional form ∀x P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.∃x P(x) is read as for some values of x, P(x) is true.

Example − "Some people are dishonest" can be transformed into the propositional form ∃x P(x) where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Nested Quantifiers

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Eg.2.

"Every apple is red".

The above statement can be restated as follows

For all x, if x is an apple then x is red

Now, we will translate it into symbolic form using universal quantifier.

Define A (x) : x is an apple.

R (x) : x is red.

∴ We write (*) into symbolic form as

$$(\forall x)\ (A(x) \rightarrow R(x))$$

Eg.3. ***"Some men are clever".***

The above statement can be restated as

"there is an x such that x is a man and x is clever".

We will translate it into symbolic form using Existential quantifier.

Let M (x) : x is a man

and C (x) : x is clever

∴ We write (B) into symbolic form as

$$(\exists x)\ (M(x) \wedge C(x))$$

Inference theory for Predicate calculus

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(y)}$	Rule US: Universal Specification
$\dfrac{P(c) \text{ for any c}}{\therefore \forall x P(x)}$	Rule UG: Universal Generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for any c}}$	Rule ES: Existential Specification
$P(c)$ for any c $\therefore \exists x P(x)$	Rule EG: Existential Generalization

Problem : Show that $(\exists x)\ M(x)$ follows logically from the premises $(x)\ (H(x) \rightarrow M(x))$ and $(\exists x)\ H(x)$

Solution : 1)	$(\exists x)\ H(x)$	rule P
2)	$H(y)$	ES
3)	$(x)\ (H(x) \rightarrow M(x))$	P
4)	$H(y) \rightarrow M(y)$	US
5)	$M(y)$	T, (2)
6)	$(\exists x)\ M(x)$	EG

Symbolize the following statements:

(a) All men are mortal

(b) All the world loves alover

(c) X is the father of mother of Y

(d)No cats has atail

(e) Some people who trust others are rewarded

Solution:

(a) Let M(x): x is a man H(x): x is Mortal

$(\forall x)\ (M(x) \rightarrow H(x))$

(b) Let P(x): x is a person L(x): x is a lover R(x,y): x loves y

$(x)\ (P(x) \rightarrow (y)\ (P(y) \wedge L(y) \rightarrow R(x,y)))$

(c) Let P(x): x is a person F(x,y): x is the father of y

M(x,y): x is the mother of y $(\exists z)\ (P(z) \wedge F(x,z) \wedge M(z,y))$

(d) Let C(x): x is a cat T(x): x has a tail

$(\forall x)\ (C(x) \rightarrow \neg T(x))$

(e) Let P(x): x is a person T(x): x trust others R(x): x is rewarded

$(\exists x)\ (P(x) \wedge T(x) \wedge R(x))$

Use the indirect method to prove that the conclusion $\exists z Q(z)$ follows from the premises $\forall x(P(x) \to Q(x))$ and $\exists y P(y)$

Solution:

1	$\neg \exists z Q(z)$	P(assumed)
2	$\forall z \neg Q(z)$	T, (1)
3	$\exists y P(y)$	P
4	$P(a)$	ES, (3)
5	$\neg Q(a)$	US, (2)
6	$P(a) \wedge \neg Q(a)$	T, (4),(5)
7	$\neg(P(a) \to Q(a))$	T, (6)
8	$\forall x(P(x) \to Q(x))$	P
9	$P(a) \to Q(a)$	US, (8)
10	$P(a) \to Q(a) \wedge \neg(P(a) \to Q(a))$	T,(7),(9) contradiction

Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$

Solution:

1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
2) $P(a) \wedge Q(a)$	ES, 1
3) $P(a)$	Rule T, 2
4) $Q(a)$	Rule T, 2
5) $(\exists x) P(x)$	EG, 3
6) $(\exists x) Q(x)$	EG, 4
7) $(\exists x) P(x) \wedge (\exists x) Q(x)$	Rule T, 5, 6