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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Introduction to Operations Research – SMT1604

GAME THEORY

Over view of Operations Research - games and strategies - two person zero sum games - maximin minimax principle - games without saddle points - mixed strategies - graphical solution - dominance property – solution using L.P.P.

A competitive situation in business can be treated similar to a game. There are two or more players and each player uses a strategy to out play the opponent. A strategy is an action plan adopted by a player in-order to counter the other player.In our of game theory we have two players namely Player A and Player B. The basic objective would be that

Player A – plays to Maximize profit (offensive) - Maxi (min) criteria Player B – plays to Minimize losses (defensive) - Mini (max) criteria The Maxi (Min) criteria is that – Maximum profit out of minimum possibilities The Mini (max) criteria is that – Minimze losses out of maximum possibilities. Game theory helps in finding out the best course of action for a firm in view of the anticipated counter-moves from the competing organizations.

Characteristics of a game

A competitive situation is a competitive game if the following properties hold good

1. The number of competitors is finite, say N.

2. A finite set of possible courses of action is available to each of the N competitors.

3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result no competitor will be in a position to know the choices of his competitors.

4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

TERMINOLOGIES

Zero Sum game

Because the Gain of A – Loss of B = 0. In other words, the gain of Player A is the Loss of Player B.

Pure strategy

If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to minimize the gain. Therefore the pure strategy is a decision rule always to select a particular course of action.

Mixed strategy

If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Optimal strategy

The strategy that puts the player in the most preferred position irrespective of the strategy of his opponents is called an optimal strategy Any deviation from this strategy would reduce his payoff.

Saddle Point

If the Maxi (min) of A = Mini (max) of B then it is known as the Saddle Point Saddle point is the number, which is lowest in its row and highest in its column. When minimax value is equal to maximin value , the game is said to have saddle point. It is the cell in the payoff matrix which satisfies minimax to maximin value.

Value of the Game

It is the average wining per play over a long no. of plays. It is the expected pay off when all the players adopt their optimum strategies .If the value of game is zero it is said to be a fair game , If the value of game is not zero it is said to be a unfair game . In all problems relating to game theory, first look for saddle point, then check out for rule of dominance and see if you can reduce the matrix.

Rule of Dominance

The dominance and modified dominance principles and their applications for reducing the size of a game with or without a saddle point. If every value of one strategy of A is lesser than that of the other strategy of A, Then A will play the strategy with greater values and remove the strategy with the lesser payoff values. If every value of one strategy of B is greater than that of other strategy of B, B will play the lesser value strategy and remove the strategy with higher payoff values.

Dominance rule for the row

If all the elements in a particular row is lower than or equal to all the elements in another row, then the row with the lower items are said to be dominated by row with higher ones, Then the row with lower elements will be eliminated.

Dominance rule for the column

If all the elements in a particular column is higher than or equal to all the elements in another column, then the column with the higher items are said to be dominated by column with lower ones, Then the column with higher elements will be eliminated.

Modified Dominance Rule

In few cases, if the given strategy is inferior to the average of two or more pure strategies, then the inferior strategy is deleted from the pay-off matrix and the size of the matrix is reduced considerably. In other words, if a given row has lower elements than the elements of average of two rows then particular row can be eliminated. Similarly if a given column has higher elements than the elements of average of two columns then particular column can be eliminated. Average row/column cannot be eliminated under any circumstances.. This type of dominance property is known as the modified dominance property

Graphical Method

If one of the players, play only two strategies or if the game can be reduced such that one of the players play only two strategies. Then the game can be solved by the graphical method. In case the pay-off matrix is of higher order (say m x n), then we try to reduce as much as possible using dominance and modified dominance ,f we get a pay-off matrix of order 2 x n or n x 2 we try to reduce the size of the pay-off matrix to that of order 2 x 2 with the graphical method so that the value of game could be obtained.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1. Analysis of the market strategies of a business organization in the long run.

2. Evaluation of the responses of the consumers to a new product.

3. Resolving the conflict between two groups in a business organization.

4. Decision making on the techniques to increase market share.

5. Material procurement process.

6. Decision making for transportation problem.

7. Evaluation of the distribution system.

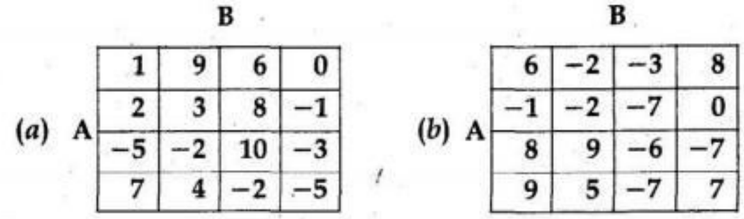
8. Evaluation of the location of the facilities.

9. Examination of new business ventures and

10. Competitive economic environment

Problems

1. Find the value of games shown below also indicator whether they are fair or strictly determinable.



(a) A, B

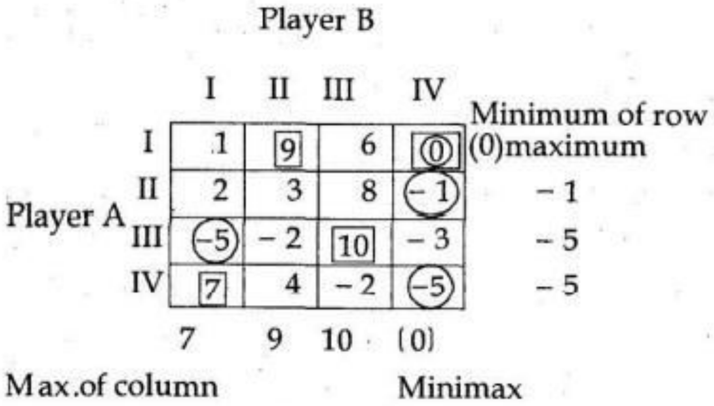
1	9	6	0
2	3	8	−1
−5	−2	10	−3
7	4	−2	−5

(b) A, B

6	−2	−3	8
−1	−2	−7	0
8	9	−6	−7
9	5	−7	7

Solution.

(a)



Player B

		I	II	III	IV	Minimum of row
Player A	I	1	$\boxed{9}$	6	$\boxed{0}$	(0)maximum
	II	2	3	8	−1	−1
	III	−5	−2	$\boxed{10}$	−3	−5
	IV	$\boxed{7}$	4	−2	−5	−5
Max.of column		7	9	10	(0) Minimax	

Saddle point = (I, IV)
Game value 0 Strategy of A = AI
Strategy of B = B IV
Since maximum = Minimax = 0
So game = Fair.

(*b*)

$$\begin{array}{c|cccc|l}
 & & & B & & \text{Minimum of row} \\
\hline
 & 6 & -2 & \boxed{\enclose{circle}{-3}} & \boxed{8} & (-3)\ \text{maximin} \leftarrow \\
 & -1 & -2 & \enclose{circle}{-7} & 0 & -7 \\
A & 8 & \boxed{9} & -6 & \enclose{circle}{-7} & -7 \\
 & \boxed{9} & 5 & \enclose{circle}{-7} & 7 & -7 \\
\hline
\text{Max. of column} & 9 & 9 & (-3) & 8 & \\
 & & & \text{Minimax} & & \\
 & & & \uparrow & &
\end{array}$$

Saddle point = A1, B3

Game value (V) = -3

Strategy of A = A1

Strategy of B = B 3

Maximum = minimax = -3 ≠ 0

Hence Game = Strictly determinable but not fair.

2. In a game of matching coins, player A wins Rs. 2. If there are two heads, win nothing if there are two tails and loses Rs. 1. When there are one head and one tail. Determine the pay off matrix, best strategies for each player and the value of game to A.

Solution. The payoff matrix for A will be

$$\begin{array}{cc|c|c|}
 & & H & T \\
\cline{3-4}
\text{Player A} & H & 2 & -1 \\
\cline{3-4}
 & T & -1 & 0 \\
\cline{3-4}
\end{array}$$

There is no saddle point
By Arithmetic method

$$\begin{array}{cc|c|c|l}
 & & H & T & \\
\cline{3-4}
 & H & 2 & -1 & 1(0.25) \\
\cline{3-4}
A & T & -1 & 0 & 3(0.75) \\
\cline{3-4}
 & & 1 & 3 & \\
 & & (0.25) & (0.75) &
\end{array}$$

Player A best strategy (0.25, 0.75)
Player B best strategy (0.25, 0.75)
Game value
Let B plays H; Value of the game

$$(V) = \text{Rs.}\left(\frac{1\times2-3\times1}{3+1}\right) = \text{Rs.}\left(-\frac{1}{4}\right).$$

3. Two players P and Q play a game. Each of them has to choose one of three colours, white (W) Black (B) and Red (R) independently of the other. There after the colours are compared. If both P and Q have choosen white (W,W) neither win anything. If player P selects white and player Q black (W, B), player P loses Rs. 2 or player Q wins the same amount and so on. The complete payoff table is shown. Find the optimum strategies for P and Q and the value of the game.

		W	**B**	**R**
	W	0	−2	7
(P)	**B**	2	5	6
	R	3	−3	8

Solution.

Colour choosen by Q

		W	B	R
	W	0	−2	7
Colour choosen by P	B	2	5	6
	R	3	−3	8

There is no saddle point
By dominance rule for column, 3' column may be removed. The resulting matrix is

(Q)

		W	B
	W	0	−2
(P)	B	2	5
	R	3	−3

By dominance rule for row, row may be removed The resulting matrix (2 x 2) is

	W	B
B	2	5
R	3	−3

Applying Arithmatic method

$$\begin{array}{cc|cc|cc} & & \multicolumn{2}{c}{Q} & & \\ & & W & B & & \\ \hline P & B & 2 & 5 & 6 & \frac{6}{9} \\ & R & 3 & -3 & 3 & \frac{3}{9} \\ \hline & & 8 & 1 & & \\ & & \frac{8}{9} & \frac{1}{9} & & \end{array}$$

Optimum strategies for P

$$= \left(0, \frac{6}{9}, \frac{3}{9}\right)$$

Optimum strategies for Q $= \left(\frac{8}{9}, \frac{1}{9}, 0\right)$

$$\text{Game value} = \frac{2\times6+3\times3}{9} = \frac{12+9}{9} = \frac{21^{7}}{9_{3}}$$

(Let Q plays W)

$$\text{Game value (V)} = \frac{7}{3}.$$

4. Solve the following games by reducing them to 2 x 2 games by graphical method.

$$(a)\quad \begin{array}{c|ccccc} & & & B & & \\ \hline A & 3 & 0 & 6 & -1 & 7 \\ & -1 & 5 & -2 & 2 & 1 \end{array} \qquad (b)\quad \begin{array}{c|cc} & \multicolumn{2}{c}{B} \\ \hline & -4 & 3 \\ & -7 & 1 \\ A & -2 & -4 \\ & -5 & -2 \\ & -1 & -6 \end{array}$$

Solution.

$$(a)\quad \begin{array}{c|ccccc} & & & B & & \\ \hline A & 3 & 0 & 6 & -1 & 7 \\ & -1 & 5 & -2 & 2 & 1 \end{array}$$

— reduce by dominance rule for column 2 and 5th column may be deleted as dominated by 4th and 3' column.

The resulting matrix is

1. No saddle point

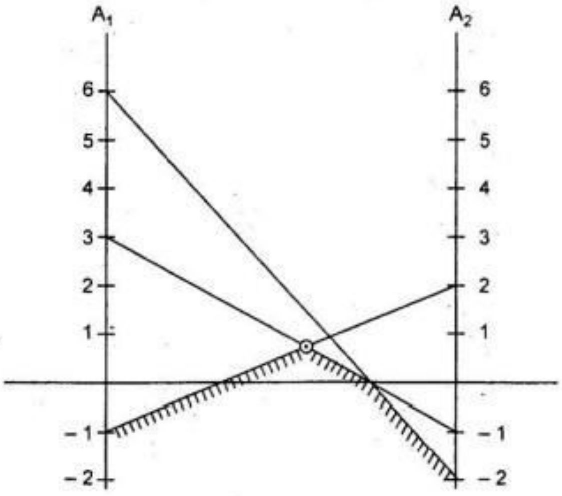
— reduce by dominance rule for column 2 and 5th column may be deleted as dominated by 4th and 3' column.

The resulting matrix is

1. No saddle point

$$\begin{array}{c|c|c|c|} & B_1 & B_3 & B_4 \\ \cline{2-4} A & 3 & 6 & -1 \\ \cline{2-4} & -1 & -2 & 2 \\ \cline{2-4} \end{array}$$

Since player A wishes to minimize his minimum expected payoff, the two lines which intersect at highest point of lower bound show the two cause of action B should choose in his best strategy.



Resulting matrix

$$\begin{array}{cc|c|c|cc} & & B_1 & B_4 & & \\ \cline{3-4} A & A_1 & 3 & -1 & 3 & \frac{3}{7} \\ \cline{3-4} & A_2 & -1 & 2 & 4 & \frac{4}{7} \\ \cline{3-4} & & 3 & 4 & & \\ & & \frac{3}{7} & \frac{4}{7} & & \end{array}$$

Ans. Optimum st. for $A = \left(\frac{3}{7}, \frac{4}{7}\right)$

Optimum st. for $B = \left(\frac{3}{7}, 0, 0, \frac{4}{7}\right)$

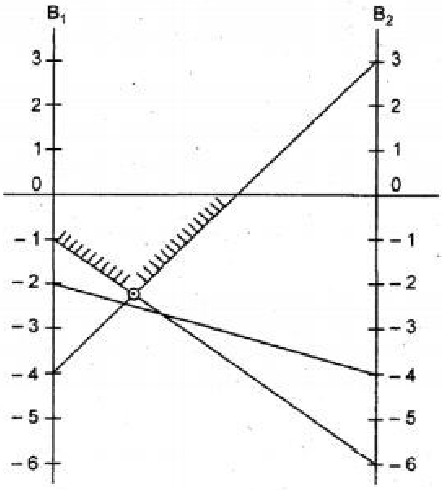
$$\text{Game value} = \frac{3 \times 3 + (-1) \times 4}{7} = \frac{5}{7}$$

Let B plays B_1.

$$(b)\quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{B} \\ \hline & -4 & 3 \\ \hline & -7 & 1 \\ \hline A & -2 & -4 \\ \hline & -5 & -2 \\ \hline & -1 & -6 \\ \hline \end{array}$$

– Reduced by dominance rule for row. 2^{nd} and 4^{th} row may be removed as it is dominated by 1^{st} row. The resulting matrix is

$$\begin{array}{cc|c|c|} & & B_1 & B_2 \\ \hline & A_1 & -4 & 3 \\ \hline A & A_3 & -2 & -4 \\ \hline & A_5 & -1 & -6 \\ \hline \end{array}$$



Since B wishes to minimize his minimum expected payoff the two lines which intersect at lowest pomt of upper bound show the two course of action A should choose in his best strategy.

The resulting matrix is

$$\begin{array}{cc|c|c|cc} & & B_1 & B_2 & & \\ \hline & A_1 & -4 & 3 & 5 & \frac{5}{12} \\ \hline A & A_5 & -1 & -6 & 7 & \frac{7}{12} \\ \hline & & 9 & 3 & & \\ & & \frac{9}{12} & \frac{3}{12} & & \end{array}$$

$$\text{Optimum strategy for A} = \left(\frac{5}{12}, 0, 0, 0, \frac{7}{12}\right)$$

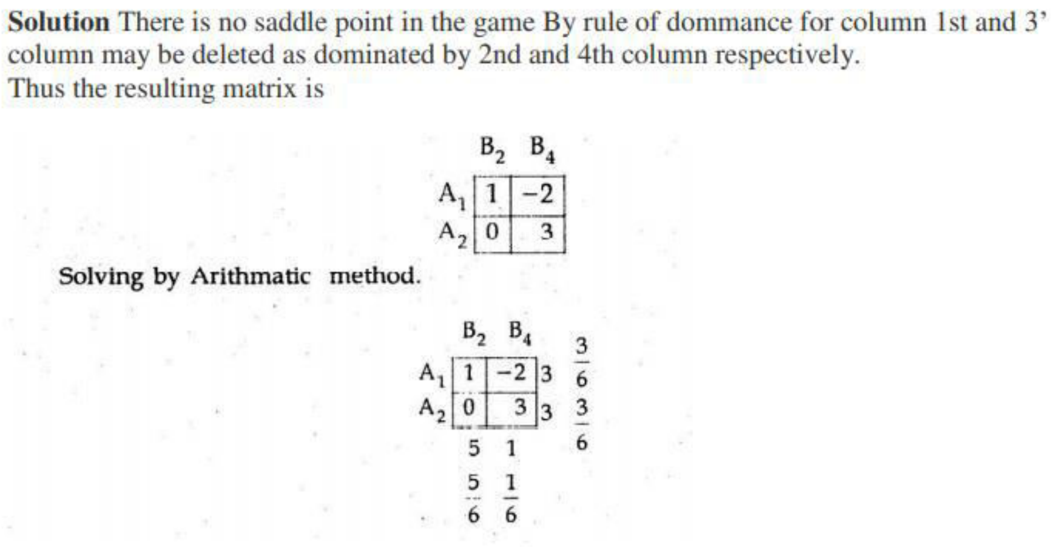
$$\text{Optimum strategy for B} = \left(\frac{9}{12}, \frac{3}{12}\right)$$

$$\text{Game value (V)} = \frac{-4 \times 5 + (-1) \times 7}{12} = \frac{-20-7}{12} = \frac{-27}{12} = \frac{-9}{4}$$

$$V = -\frac{9}{4}.$$

5. Solve the following game.

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{\textbf{(B)}} \\ & & B_1 & B_2 & B_3 & B_4 \\ \hline \textbf{(A)} & A_1 & 2 & 1 & 0 & -2 \\ \hline & A_2 & 1 & 0 & 3 & 3 \\ \hline \end{array}$$

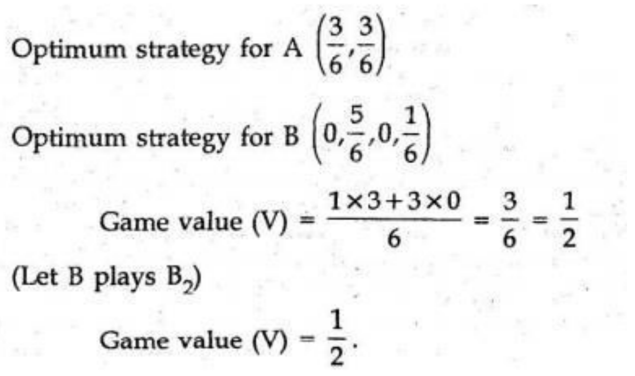


Solution There is no saddle point in the game By rule of dommance for column 1st and 3' column may be deleted as dominated by 2nd and 4th column respectively.
Thus the resulting matrix is

$$\begin{array}{c|cc|} & B_2 & B_4 \\ \hline A_1 & 1 & -2 \\ A_2 & 0 & 3 \\ \hline \end{array}$$

Solving by Arithmatic method.

$$\begin{array}{c|cc|cc} & B_2 & B_4 & & \\ \hline A_1 & 1 & -2 & 3 & \frac{3}{6} \\ A_2 & 0 & 3 & 3 & \frac{3}{6} \\ \hline & 5 & 1 & & \\ & \frac{5}{6} & \frac{1}{6} & & \end{array}$$



Optimum strategy for A $\left(\frac{3}{6}, \frac{3}{6}\right)$

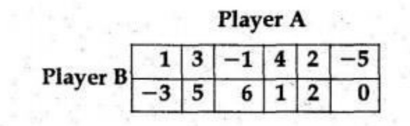
Optimum strategy for B $\left(0, \frac{5}{6}, 0, \frac{1}{6}\right)$

$$\text{Game value (V)} = \frac{1\times3+3\times0}{6} = \frac{3}{6} = \frac{1}{2}$$

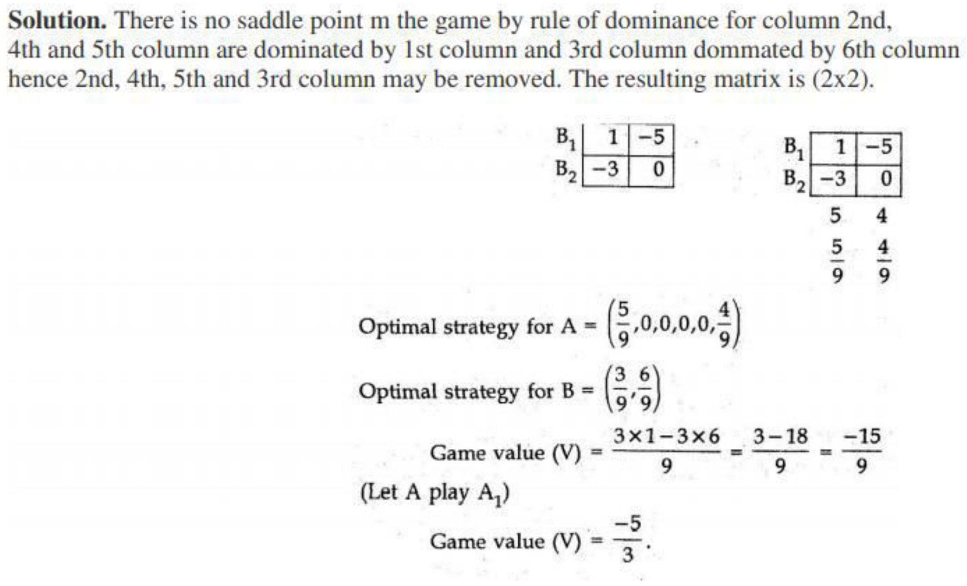
(Let B plays B_2)

$$\text{Game value (V)} = \frac{1}{2}.$$

6. Obtain the optimal strategies for both persons and the value of the game for zero sum two person game whose payoff matrix is given as follows:



$$\begin{array}{c|cccccc|} & \multicolumn{6}{c}{\textbf{Player A}} \\ \hline \textbf{Player B} & 1 & 3 & -1 & 4 & 2 & -5 \\ & -3 & 5 & 6 & 1 & 2 & 0 \\ \hline \end{array}$$



Solution. There is no saddle point m the game by rule of dominance for column 2nd, 4th and 5th column are dominated by 1st column and 3rd column dommated by 6th column hence 2nd, 4th, 5th and 3rd column may be removed. The resulting matrix is (2x2).

$$\begin{array}{c|cc|} B_1 & 1 & -5 \\ B_2 & -3 & 0 \\ \hline \end{array} \qquad \begin{array}{c|cc|} B_1 & 1 & -5 \\ B_2 & -3 & 0 \\ \hline & 5 & 4 \\ & \frac{5}{9} & \frac{4}{9} \end{array}$$

Optimal strategy for A = $\left(\frac{5}{9}, 0, 0, 0, 0, \frac{4}{9}\right)$

Optimal strategy for B = $\left(\frac{3}{9}, \frac{6}{9}\right)$

$$\text{Game value (V)} = \frac{3\times1-3\times6}{9} = \frac{3-18}{9} = \frac{-15}{9}$$

(Let A play A_1)

$$\text{Game value (V)} = \frac{-5}{3}.$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – Replacement Model – SMT1604

REPLACEMENT MODEL

Replacement policy for items whose maintenance cost increases with time- Consideration of time value of money - Replacement policy- Individual, Group replacement of items that fail completely and suddenly.

REPLACEMENT MODEL

If any equipment or machine is used for a long period of time, due to wear and tear, the item tends to worsen. A remedial action to bring the item or equipment to the original level is desired. Then the need for replacement becomes necessary. This may be due physical impairment, due to normal wear and tear, obsolescence etc. The resale value of the item goes on diminishing with the passage of time. The depreciation of the original equipment is a factor, which is responsible not to favor replacement because the capital is being spread over a long time leading to a lower average cost. Thus there exists an economic trade-off between increasing and decreasing cost functions. We strike a balance between the two opposing costs with the aim of obtaining a minimum cost. Replacement model aims at identifying the time at which the assets must be replaced in order to minimize the cost.

REASONS FOR REPLACEMENT OF EQUIPMENT:

1. Physical impairment or malfunctioning of various parts refers to the physical condition of the equipment itself. Leads to a decline in the value of service rendered by the equipment. Increasing operating cost of the equipment. Increased maintenance cost of the equipment. Or a combination of the above.

2. Obsolescence of the equipment, caused due to improvement in the existing tools and machinery mainly when the technology becomes advanced.

3. When there is sudden failure or breakdown.

REPLACEMENT MODELS

Assets that fails Gradually: Certain assets wear and tear as they are used. The efficiency of the assets decline with time. The maintenance cost keeps increasing as the years pass by eg. Machinery, automobiles, etc.

1. Gradual failure without taking time value of money into consideration

2. Gradual failure taking time value of money into consideration. Assets which fail suddenly Certain assets fail suddenly and have to be replaced from time to time eg. bulbs. 1. Individual Replacement policy (IRP) 2. Group Replacement policy (GRP)

Assests that fails Gradually

Gradual failure without taking time value of money into consideration As mentioned earlier the equipments, machineries and vehicles undergo wear and tear with the passage of time. The cost of operation and the maintenance are bound to increase year by year. A stage may be reached that the maintenance cost amounts prohibitively large that it is better and economical to replace the equipment with a new one. We also take into account the salvage value of the items in assessing the appropriate or opportune time to replace the item. We assume that the details regarding the costs of operation, maintenance and the salvage value of the item are already known.

Procedure for replacement of an asset that fails gradually (without considering Time value of money)

a) Note down the years b) Note down the running cost ÄRÅ (Running cost or operating cost or Maintenance cost or other expenses) c) Calculate Cumulative the running cost Ä∑RÅ d) Note down the capital cost ÄCÅ e) Note down the scrap or resale value ÄSÅ f) Calculate Depreciation = Capital Cost – Resale value g) Find the Total Cost Total Cost = Cumulative Running cost + Depreciation h) Find the average cost Average cost = Total cost/No. of corresponding year i) Replacement decision: Average cost is minimum (Average cost will decrease and reach minimum, later it will increase).

Gradual failure taking time value of money into consideration

In the previous section we did not take the interest for the money invested, the running costs and resale value. If the effect of time value of money is to be taken into account, the analysis must be based on an equivalent cost. This is done with the present value or present worth analysis. For example, suppose the interest rate is given as 10% and Rs. 100 today would amount to Rs. 110 after a year's time. In other words the expenditure of Rs. 110 in year's time is equivalent to Rs. 100 today. Likewise one rupee a year from now is equivalent to (1.1)-1 rupees today and one-rupee in 'n' years from now is equivalent to (1.1)-n rupees today. This quantity (1.1)-n is called the present value or present worth of one rupee spent 'n' years from now.

Procedure for replacement of an asset that fails gradually (with considering Time value of money)

Assumption: i. Maintenance cost will be calculated at the beginning of the year ii. Resale value at the end of the year Procedure: a) Note down the years b) Note down the running cost ÄRÅ (Running cost or operating cost or Maintenance cost or other expenses) c) Write the present value factor at the beginning for running cost d) Calculate present value for Running cost e) Calculate Cumulative the running cost Ä∑ RÅ f) Note down the capital cost ÄCÅ g) Note down the scrap or resale value ÄSÅ h) Write the present value factor at the end of the year and also calculate present value for salvage or scrap or resale value. i) Calculate Depreciation = Capital Cost – Resale value j) Find the Total Cost = Cumulative Running cost + Depreciation k) Calculate annuity factor (Cumulative present value factor at the beginning). l) Find the Average cost = Total cost / Annuity m) Replacement decision: Average cost is minimum (Average cost will decrease and reach minimum, later it will increase).

ITEMS THAT FAIL COMPLETELY AND SUDDENLY

There is another type of problem where we consider the items that fail completely. The item fails such that the loss is sudden and complete. Common examples are the electric bulbs, transistors and replacement of items, which follow sudden failure mechanism.

INDIVIDUAL REPLACEMENT POLICY (IRP):

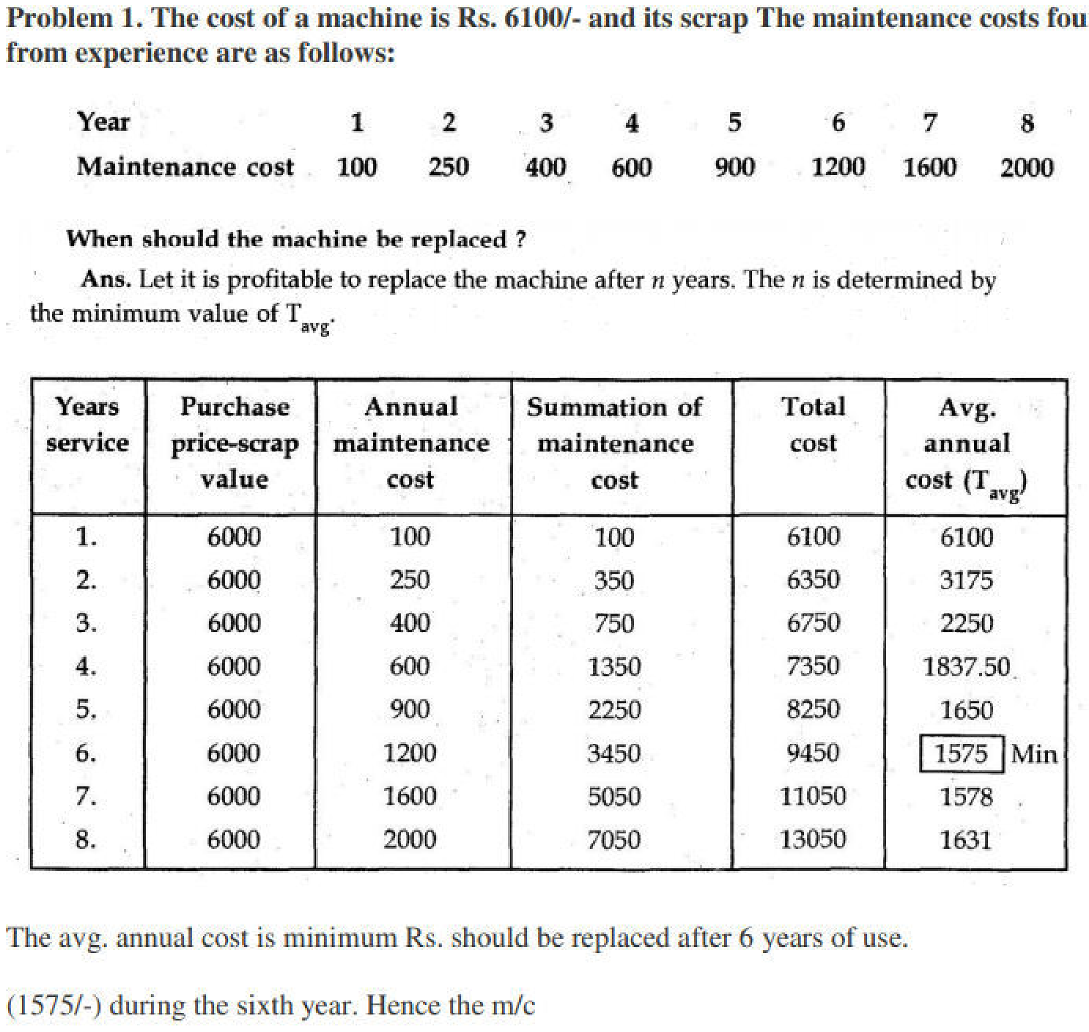
Under this strategy equipments or facilities break down at various times. Each breakdown can be remedied as it occurs by replacement or repair of the faulty unit. Examples: Vacuum tubes, transistors Calculation of Individual Replacement Policy (IRP): n Average life of an item = ∑i * Pi i-1 Pi denotes Probability of failure during that week i denotes no. of weeks No. of failures = Total no. of items Average life of an item Total IRP Cost = No. of failures * IRP cost

GROUP REPLACEMENT

As per this strategy, an optimal group replacement period 'P' is determined and common preventive replacement is carried out as follows. (a) Replacement an item if it fails before the optimum period 'P'.6 (b) Replace all the items every optimum period of 'P' irrespective of the life of individual item. Examples: Bulbs, Tubes, and Switches. Among the three strategies that may be adopted, the third one namely the group replacement policy turns out to be economical if items are supplied cheap when purchased in bulk quantities. With this policy, all items are replaced at certain fixed intervals.

Procedure for Group Replacement Policy (GRP)

1. Write down the weeks 2. Write down the individual probability of failure during that week 3. Calculate No. of failures: N0 - No. of items at the beginning N1 - No. of failure during 1st week (N0P1) N2 - No. of failure during 2nd week (N0P2 + N1P1) N3 - No. of failure during 3rd week (N0P3 + N1P2 + N2P1) 4. Calculate cumulative failures 5. Calculate IRP Cost = Cumulative no. of failures * IRP cost 6. Calculate and write down GRP Cost = Total items * GRP Cost 7. Calculate Total Cost = IRP Cost + GRP Cost 8. Calculate Average cost = Total cost / no. of corresponding year.



Problem 1. The cost of a machine is Rs. 6100/- and its scrap The maintenance costs found from experience are as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost	100	250	400	600	900	1200	1600	2000

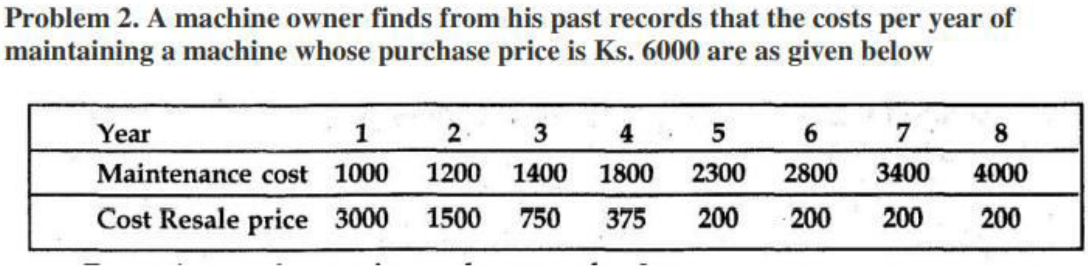
When should the machine be replaced ?

Ans. Let it is profitable to replace the machine after *n* years. The *n* is determined by the minimum value of T_{avg}.

Years service	Purchase price-scrap value	Annual maintenance cost	Summation of maintenance cost	Total cost	Avg. annual cost (T_{avg})
1.	6000	100	100	6100	6100
2.	6000	250	350	6350	3175
3.	6000	400	750	6750	2250
4.	6000	600	1350	7350	1837.50
5.	6000	900	2250	8250	1650
6.	6000	1200	3450	9450	$\boxed{1575}$ Min
7.	6000	1600	5050	11050	1578
8.	6000	2000	7050	13050	1631

The avg. annual cost is minimum Rs. should be replaced after 6 years of use.

(1575/-) during the sixth year. Hence the m/c



Problem 2. A machine owner finds from his past records that the costs per year of maintaining a machine whose purchase price is Ks. 6000 are as given below

Year	1	2	3	4	5	6	7	8
Maintenance cost	1000	1200	1400	1800	2300	2800	3400	4000
Cost Resale price	3000	1500	750	375	200	200	200	200

Determine at what age is a replacement due?

Ans. Capital cost C = 6000/-. Let it be profitable to replace the. machine after n years. Then n should be determined by the minimum value of Tav•

Year of service	Resale value	Purchase Price Resale value	Annual Maintenance cost	Summation of maintenance cost	Total Cost	Average annual cost
1.	3000	3000	1000	1000	4000	4000
2.	1500	4500	1200	2200	6700	3350
3.	750	5250	1400	3600	8850	2950
4.	375	5625	1800	5400	11025	2756.25
5.	200	5800	2300	7700	13500	$\boxed{2700}$
6.	200	5800	2800	10500	16300	2716.66
7.	200	5800	3400	13900	19700	2814.28
8.	200	5800	3400	17300	23100	2887.5

We observe from the table that avg. annual cost is minimum (Rs. 2700/-). Hence the m/c should replace at the end of 5th year.

Replacement of items whose maintenance costs increase with time and value of money also changes with time.

The machine should be replaced if the next period's cost is greater than weighted average of previous cost. Discount rate [Present worth factor (PWF)]

$$V = \frac{1}{1+i}$$

$$V_n = (V)^{n-1}$$

n - no. of year

i - annual interest rate

V_n - PWF of n^{th} year.

Problem 3. A machine costs Rs. 500/— Operation and Maintenance cost are zero for the first year and increase by Rs. 100/— every year. If money. is worth 5% every year, determine the best age at which the machine should be replaced. The resale value of the machine is negligible small. What is the weighted average cost of owning and operating the machine?

Ans. Discount rate $V = \frac{1}{1+i} = \frac{1}{1+0.05} = 0.9524$

Discount rate for Ist year $V_n = \left(\frac{1}{1+i}\right)^{n-1}$

$$V_1 = (0.9524)^0 = 1$$
$$2^{nd} \text{ year } V_2 = (0.9524)^1 = 0.9524$$
$$3^{rd} \text{ year } V_3 = (0.9524)^2 = 0.9070$$
$$4^{th} \text{ year } V_4 = (0.9524)^3 = 0.8638$$
$$5^{th} \text{ year } V_5 = (0.9524)^4 = 0.8227$$

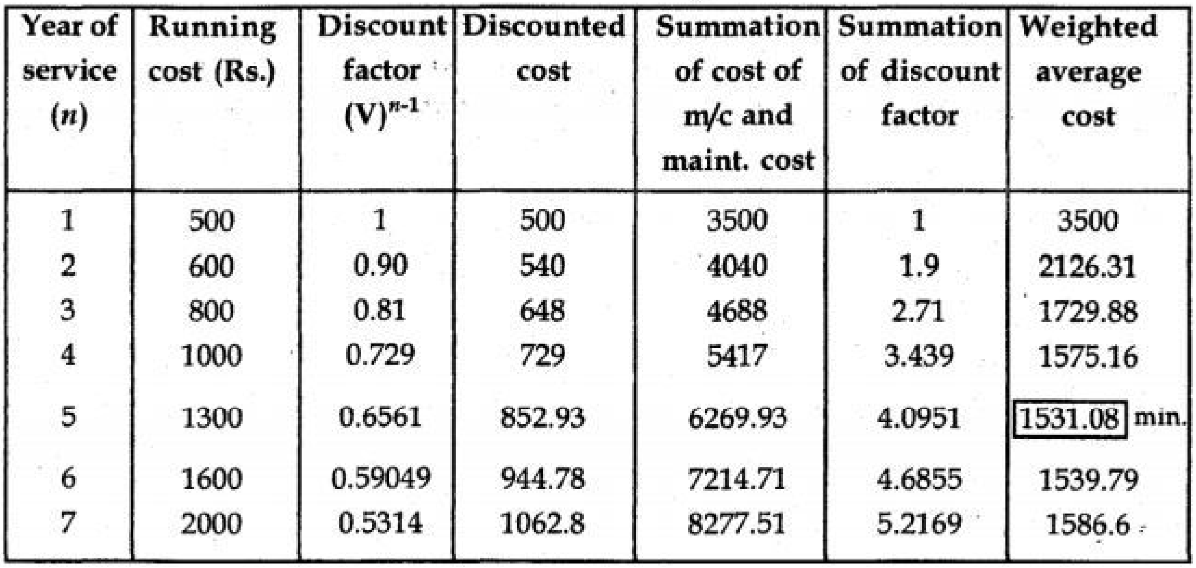
Years of service (n)	Maintenance cost (Rs)	Discount factor $(V)^{n-1}$	Discounted cost	Summation of cost of m/c and maint. Cost	Summation of discount factor	Weighted average cost
1	0	1.0000	0.00	500.00	1.0000	500
2	100	0.9524	95.24	595.24	1.9524	304.88
3	200	0.9070	181.40	776.64	2.8594	$\boxed{217.61}$ min
4	300	0.8638	259.14	1035.78	3.7232	278.20
5	400	0.8227	329.08	1364.86	4.5459	300.25

M/c su1d be replaced at the end of 3C1 year.

Problem 4: Purchase price of a machine is Rs. 3000/— and its running cost is given in the table below. If should be replaced. the discount rate is 0.90. Find at what age the machine

Year	1	2	3	4	5	6	7
Running cost (Rs.)	500	600	800	1000	1300	1600	2000

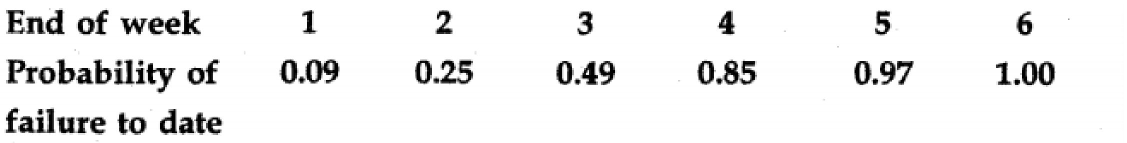
Ans. V (Discount rate) = 0.90



Year of service (n)	Running cost (Rs.)	Discount factor $(V)^{n-1}$	Discounted cost	Summation of cost of m/c and maint. cost	Summation of discount factor	Weighted average cost
1	500	1	500	3500	1	3500
2	600	0.90	540	4040	1.9	2126.31
3	800	0.81	648	4688	2.71	1729.88
4	1000	0.729	729	5417	3.439	1575.16
5	1300	0.6561	852.93	6269.93	4.0951	$\boxed{1531.08}$ min.
6	1600	0.59049	944.78	7214.71	4.6855	1539.79
7	2000	0.5314	1062.8	8277.51	5.2169	1586.6

M/c should be replaced at the end of 5th year.

Problem 5: The following mortality ratio have been observed for a certain type of light bulbs in an installation with 1000 bulb



End of week	1	2	3	4	5	6
Probability of failure to date	0.09	0.25	0.49	0.85	0.97	1.00

There are a large no. of such bulbs which are to be kept in working order. If a bulb fails in service, it cost Rs. 3 to replace but if all the bulbs all replaced in the same operation it can be done for only Rs. 0.70/— a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and continue replacing burnt out bulb as they fail.

(a) What is the best interval between group replacement?

(b) Also establish if the policy, as determined by you is superior to the policy of replacing bulbs as and when they, fail, there being nothing like group replacement.

(c) At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to the adopted policy?

Solution : Let p. be the probability that a new light bulbs fail during the 1th wek of the life.

$P_1 = 0.09$

$P_2 = 0.25 - 0.09 = 0.16$

$P_3 = 0.49 - 0.25 = 0.24$

$P_4 = 0.85 - 0.49 = 0.36$

$P_5 = 0.97 - 0.85 = 0.12$

$P_6 = 1.00 - 0.97 = 0.03$

Week	**Expected no. of failure (N)**	
0	$N_0 = N_0$	
1	$N_1 = 1000 \times 0.09$	$= 90$
2	$N_2 = 1000 \times 0.16 + 90 \times 0.09$	$= 168$
3	$N_3 = 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09$	$= 269$
4	$N_4 = 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269 \times 0.09$	$= 432$
5	$N_5 = 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09$	$= 274$
6.	$N_6 = 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24 + 432 \times 0.16 + 274 \times 0.09$	$= 260$
and so on		

(*a*) Determination of optimum group replacement interval

Week	Total cost of group replacement	Avg cost/week
1.	$1000 \times 0.70 + 90 \times 3 = 970$	970.00
2.	$1000 \times 0.70 + 3\,(90+168) = 1474$	$\boxed{737.00}$
3.	$1000 \times 0.70 + 3\,(90 + 168 + 269) = 2281$	760.33

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – Inventory Control – SMT1604

INVENTORY CONTROL

Inventory Control - Various Types of inventory models - deterministic inventory models - Production model, Purchase model -with and without shortage - Economic Order Quantity (EOQ) - Buffer stock -Shortage quantity - Probabilistic inventory models – Quantity Discount and Price Breaks.

INVENTORY

Inventory may be defined a stock of goods, commodities or other economic resources that are stored or reserved for smooth and efficient running of business.

The inventory may be kept in any one of the following forms:

1. Raw material

2. Work-in progress

3. Finished goods If an order for a product is receive, we should have sufficient stock of materials required for manufacturing the item in order to avoid delay in production and supply. Also there should not be over stock of materials and goods as it involves storage cost and wastage in storing. Therefore inventory control is essential to promote business.

Maintaining inventory helps to run the business smoothly and efficiently and also to provide adequate service to the customer. Inventory control is very useful to reduce the cost of transportation and storage. A good inventory system, one has to address the following questions quantitatively and qualitatively. • What to order? • When to order? • How much to order? • How much to carry in an inventory?

Objectives of inventory management/Significance of inventory management

To maintain continuity in production. To provide satisfactory service to customers. To bring administrative simplicity. To reduce risk. To eliminate wastage. To act as a cushion against high rate of usage. To avoid accumulation of inventory. To continue production even if there is a break down in few machinery. To ensure proper execution of policies. To take advantages of price fluctuations and buy economically.

Costs involved in inventory

1. Holding Cost (Carrying or Storage Cost) It is the cost associated with the carrying or holding the goods in stock. It includes storage cost, depreciation cost, rent for godown, interest on investment

locked up, record keeping and administrative cost, taxes and insurance cost, deterioration cost, etc. It is denoted by ÅCÇ.

2. Setup Cost/ Ordering Cost Ordering cost is associated with cost of placing orders for procurement of material or finished goods from suppliers. It includes, cost of stationery, postage, telephones, travelling expenses, handling of materials, etc. (Purchase Model)Setup cost is associated with production. It includes, cost involved in setting up machines for production run. (Production Model). Both are denoted by ÅSÇ.

3. Purchase Cost/Production Cost When the organization purchases materials from other suppliers, the actual price paid for the material will be called the purchase cost. When the organization produces material in the factory, the cost incurred for production of material is called as production cost. Both are denoted by ÅPÇ.

4. Shortage Cost If the inventory on hand is not sufficient to meet the demand of materials or finished goods, then it results in shortage of supply. The cost may include loss of reputation, loss of customer, etc. Total incremental cost = Holding Cost + Setup Cost/ Ordering Cost Total Cost = Purchase Cost/ Production Cost + Shortage Cost + Total Incremental cost.

Demand is one of the most important aspects of an inventory system. Demand can be classified broadly into two categories: Deterministic i.e., a situation when the demand is known with certainty. And, deterministic demand can either be static (where demand remains constant over time) or it could be dynamic (where the demand, though known with certainty, may change with time).

Probabilistic (Stochastic) refers to situations when the demand is random and is governed by a probability density function or probability mass function. Probabilistic demand can also be of two types - stationary(in which the demand probability density function remains unchanged over time),and non-stationary, where the probability densities vary over time. Deterministic Inventory Models i. Model I: Purchasing model without shortages ii. Model II: Production model without shortages iii. Model III: Purchasing model with shortages4 iv. Model IV: Production model with shortages .

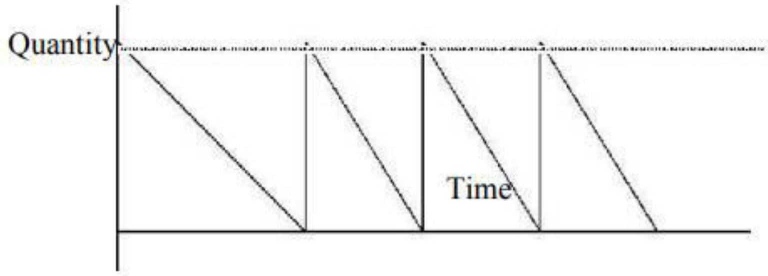
Model I: Purchasing model without shortages

Assumptions

- Demand(D) per year is known and is uniform

- Ordering cost(S) per order remains constant

• Carrying cost(C) per unit remains constant • Purchase price(P) per unit remains constant • No Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back. Lead time is Zero.

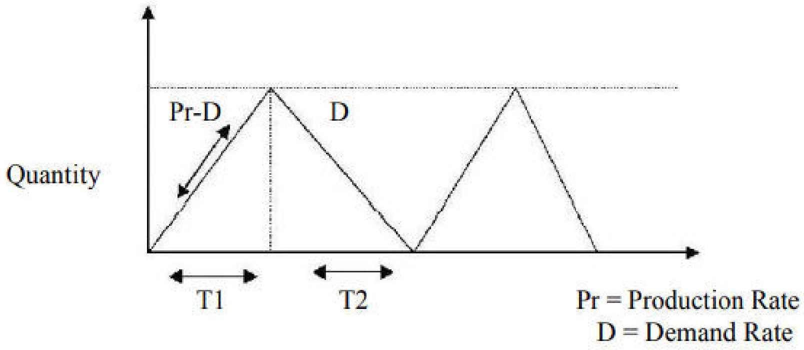


Inventory decreases at the rate of ÅDÇ As soon as the level of inventory reaches zero, the inventory is replenished back.

Model II: Production model without shortages

Assumptions

- Demand(D) per year is known and is uniform
- Setup cost (S) per production run remains constant
- Carrying cost(C) per unit remains constant
- Production cost per unit(P) per unit remains constant
- No Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back.



T1 is the time taken when manufacturing takes place at the rate of Pr and demand at the rate of

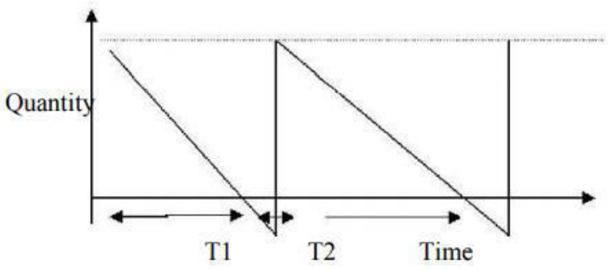
D. So the stock is built up at the rate of (Pr – D). During t2 there is no production only usage of stock. Hence, stock is decreased at the rate of ÅDÇ. At the end of t2, stock will be nil.

Model III: Purchasing model with shortages

Assumptions

- Demand(D) per year is known and is uniform
- Ordering cost(S) per order remains constant
- Carrying cost(C) per unit remains constant
- Purchase price(P) per unit remains constant
- Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back with lead time

. • Shortage cost (sh) per unit remains constant



T1 is the time during which stock is nil.During T2 shortage occur and at the end of T2 stock is replenished back.

Model IV: Production model with shortages

Assumptions

Demand(D) per year is known and is uniform.

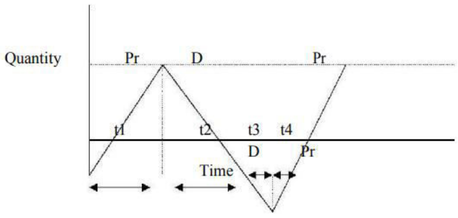
Setup cost (S) per production run remains constant.

Carrying cost(C) per unit remains constant.

Production cost per unit(P) per unit remains constant.

Shortages are allowed. As soon as the level of inventory reaches zero, the inventory is replenished back with lead time.

Shortage cost (Sh) per unit remains constant.



T1 is the time taken when manufacturing takes place at the rate of Pr and demand at the rate of D. So the stock is built-up at the rate of (Pr – D). During t2 there is no production only usage of stock. Hence, stock is decreased at the rate of ÅDÇ. At the end of t2, stock will be nil. During T3 shortage exists at the rate of ÅDÇ. During T4 production begins stock builds and shortage decreases at the rate of ÅPr-DÇ.

Inventory basic terminologies

- EOQ- Economic order quantity – The optimum order per order quantity for which total inventory cost is minimum.

- EBQ- Economic batch quantity – The optimum manufacturing quantity in one batch for which total inventory cost is minimum.

- Demand Rate – rate at which items are consumed

- Production rate- rate at which items are produced

- Stock replenishment rate o Finite rate – the inventory builds up slowly /step by step(production model) o Instantaneous rate – rate at which inventory builds up from minimum to maximum instantaneously (purchasing model)

- Lead time- Time taken by supplier to supply goods

- Lead time demand it is the demand for goods in the organization during lead time

. • Reorder level- the level between maximum and minimum inventory at which purchasing or manufacturing activities must start from replenishment. Reorder level = Buffer stock+ Lead time demand

- Buffer stock- to face the uncertainties in consumption rate and lead time , an extra stock is maintained. This is termed as buffer stock: Buffer stock = (Maximum Lead time – Average Lead time) x Demand per month

- Maximum Inventory Level: Maximum quantity that can be allowed in the stock: Maximum Inventory = EOQ + Buffer stock

- Minimum Inventory Level is the level that is expected to be available when thee supply is due: Minimum Inventory level = Buffer stock

- Average Inventory = (Minimum Inventory + Maximum Inventory)/2

- Order cycle is the period of time between two consecutive placements of orders

Problem 1. A particular item has a demand of 9000 units/year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortance are allowed.

Determine

1. Economic lot size.

2. The no. of order per year.

3. The time between orders.

4. Total cost per year if the cost of one unit is Rs. 1.

Solution.

$$R = 9000 \text{ unit/year}$$

$$C_3 = \text{Rs. } 100/\text{procurement}$$

$$C_1 = \text{Rs. } 2.40/\text{unit/year}$$

1.

$$q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times100\times9000}{2.40}} = 866 \text{ units/Procurement.}$$

2.

$$n_0 = \frac{1}{t_0} = \sqrt{\frac{C_1R}{2C_3}} = \sqrt{\frac{2.40\times9000}{2\times100}} = \sqrt{108} = 10.4 \text{ order/year.}$$

3.

$$t_0 = \frac{1}{n_0} = \frac{1}{10.4} = 0.0962 \text{ years} = 1.15 \text{ months between procurement.}$$

4.

$$\begin{aligned} C_0 &= 9000\times1+\sqrt{2C_1C_3R} \\ &= 9000+\sqrt{2\times2.40\times100\times9000} \\ &= \text{Rs. } 11080 \text{ per year.} \end{aligned}$$

Problem 2. A manufacturing company purchases 9000 parts of a machine for its annual requirements, ordering one month usuage at a time. Each part cost Rs. 20. The order cost per ordering is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it same the company per year.

Solution.

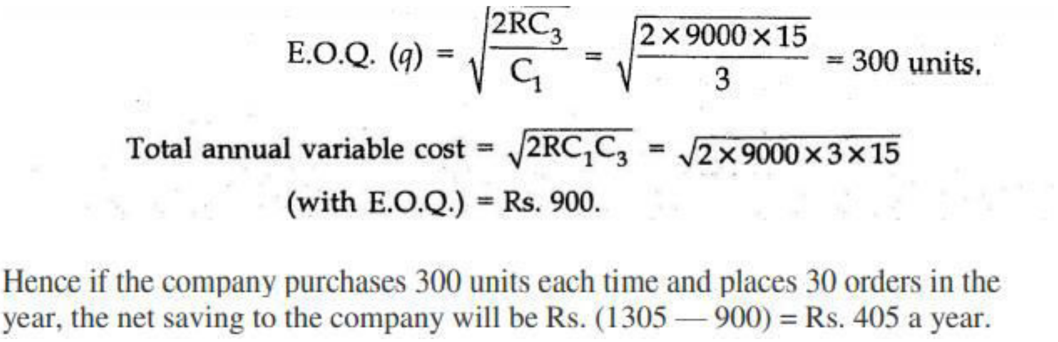
$$R = 9000 \text{ parts/year.}$$

$$q = \frac{9000}{12} = 750 \text{ parts}$$

$$C = \text{Rs. } 20/ \text{ parts, } C_3 = \text{Rs. } 15/\text{order.}$$

$$C_1 = \text{Rs.}20\times\frac{15}{100} = \text{Rs. } 3/\text{part/year}$$

$$\begin{aligned} \text{Total annual variable cost} &= \frac{q}{2}.C_1+\frac{R}{q}.C_3 \\ &= \text{Rs.}\left[\frac{750}{2}\times3+\frac{9000}{750}\times15\right] \\ &= \text{Rs.}1305. \end{aligned}$$



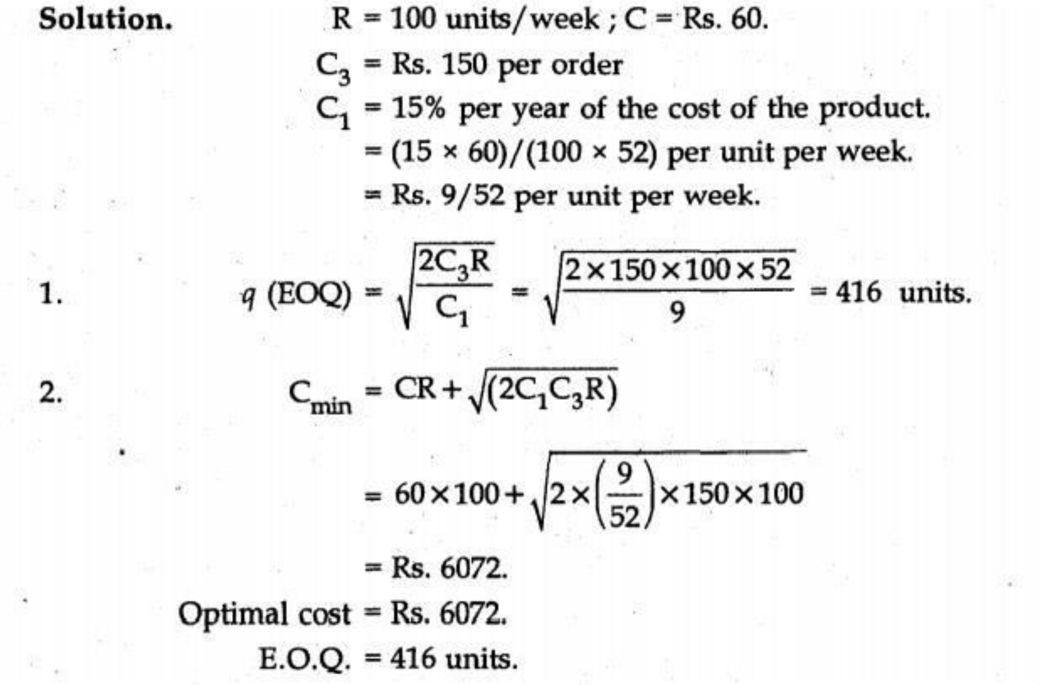
$$\text{E.O.Q. } (q) = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 9000 \times 15}{3}} = 300 \text{ units.}$$

$$\text{Total annual variable cost} = \sqrt{2RC_1C_3} = \sqrt{2 \times 9000 \times 3 \times 15}$$

$$\text{(with E.O.Q.)} = \text{Rs. } 900.$$

Hence if the company purchases 300 units each time and places 30 orders in the year, the net saving to the company will be Rs. (1305 — 900) = Rs. 405 a year.

Problem 3. You have to supply your customers 100 units of a certain product every monday- You obtain the product from a local supplier at Rs. 60 per unit. The cost of ordering and transportation from the supplier are Rs. 150 per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried. 1. Find the lot size which will minimize the cost of the system. 2. Determine the optimal cost.



Solution. $R = 100$ units/week ; $C =$ Rs. 60.

$C_3 =$ Rs. 150 per order

$C_1 =$ 15% per year of the cost of the product.

$= (15 \times 60)/(100 \times 52)$ per unit per week.

$=$ Rs. 9/52 per unit per week.

1. $$q \text{ (EOQ)} = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 150 \times 100 \times 52}{9}} = 416 \text{ units.}$$

2. $$C_{\min} = CR + \sqrt{(2C_1C_3R)}$$

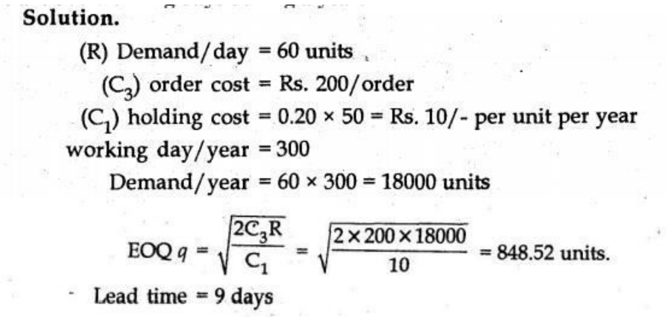
$$= 60 \times 100 + \sqrt{2 \times \left(\frac{9}{52}\right) \times 150 \times 100}$$

$$= \text{Rs. } 6072.$$

Optimal cost = Rs. 6072.

E.O.Q. = 416 units.

Problem 4. Daily demand for a product is normally distributed with mean, 60 units and a standard deviation of 6 units. The lead time is constant at 9 days. The cost of placing an order is Rs. 200, and the annual holding costs are 20% of the unit price of Rs. 50. A 95% service level is desired for the customers, who place orders during the reorder period. Determine the order quantity and the reorder level for the item in question, assuming that there are 300 working days during a year.



Solution.

(R) Demand/day = 60 units

(C_3) order cost = Rs. 200/order

(C_1) holding cost = 0.20 × 50 = Rs. 10/- per unit per year

working day/year = 300

Demand/year = 60 × 300 = 18000 units

$$\text{EOQ } q = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 200 \times 18000}{10}} = 848.52 \text{ units.}$$

Lead time = 9 days

Standard deviation of daily demand = 6 units

Now variance of demand during the lead time is equal to the sum of variance of daily demand during the lead time period

$$\text{variance} = 6^2 + 6^2 + 6^2 + \ldots.. + 6^2 \text{ (9 times)}$$

$$= 9 \times 6^2 = 324$$

Standard deviation of demand during the lead time

$$\text{Period} = \sqrt{324} = 18.$$

With E.O.Q. of 848.52 the no. of order during the year

$$(n_0) = \frac{18000}{848.52} = 21.21$$

Service level = 0.95

the normal deviate Z as found from probability table is 1.65.

Safety stock = Z × standard deviation

= 1.65 × 18 = 29.7 units.

Reorder level = Expected demand during lead time + safety stock

= 60 × 9 + 29.7 = 569.7 units.

Ans. Economic order quantity EOQ = 848.52 units.

Reorder level = 569.7 units.

Problem 5. The demand per month for a product is distributed normally with a mean of 100 and standard deviation 25. The lead time distribution is given below. What service level will be offorded by a reorder level of 500 units?

Lead time (months) :	1	2	3	4	5
Probability :	0.10	0.20	0.40	0.20	0.10

Solution. It is given that the demand is distributed normally with

Mean $(\overline{D})$ = 100 units SD (σd) = 25 units

lead time (L) = 1, 2, 3, 4 and 5

Reorder level (M) = 500 units

We shall use iterative method of computing service level for the reorder level policy when the demand per unit time is distributed normally and distribution of lead time is known

$$Z = \frac{M - L\overline{D}}{\sigma d\sqrt{L}}$$

By iterative method

Lead time	Value of Z when M = 500	Probability of not running out of stock corresponding to the value of Z (from table)	Probability of this particular lead time occuring	Conditional Probability of not running out of stock
1.	$\frac{500-100\times1}{25\sqrt{1}} = 16.0$	100	0.10	10
2.	$\frac{500-100\times2}{25\sqrt{2}} = 8.49$	100	0.20	20
3.	$\frac{500-100\times3}{25\sqrt{3}} = 4.49$	100	0.40	40
4.	$\frac{500-100\times4}{25\sqrt{4}} = 2.00$	97.7	0.20	19.5
5.	$\frac{500-100\times5}{25\sqrt{5}} = 0.00$	50.0	0.10	5.0

Total conditional probability of not running out of stock =10 + 20+ 40+ 19.5 + 5= 94.5. Hence a reorder level of 500 units will give 94.5% service level.

Problem 6. The annual demand for a product is 500 units. The cost of storage per unit per year is 10% of the unit cost, The ordering cost is Rs. 180 for each order. The unit cost depends upon the amount ordered. The range of amount ordered and the unit cost price are as follows

Range of amount ordered	$0 \le Q_1 \le 500$	$0 \le Q_2 \le 1500$	$1500 \le Q_3 \le 3000$	$3000 < Q_4$
Unit cost (Rs.)	25.00	24.80	24.60	24.4

Solution. Here $R = 500$ units

$I = 0.10$

$C_3 = \text{Rs. } 180$

$$\text{EOQ for unit price of Rs. 24.40} = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times180\times500}{24.40\times0.10}}$$

$$= 271.6 \text{ units.}$$

But this is not feasible because the unit price of Rs. 24.40 is not available for an order size of 271.6 units.

$$\text{EOQ for unit price of Rs. 24.60} = \sqrt{\frac{2\times180\times500}{24.60\times0.10}} = 270.5 \text{ units (infeasible)}$$

$$\text{EOQ for unit price of Rs. 24.80} = \sqrt{\frac{2\times180\times500}{24.80\times0.10}} = 269.4 \text{ units (infeasible)}$$

$$\text{EOQ for unit price of Rs. 25.00} = \sqrt{\frac{2\times180\times500}{25\times0.10}} = 268.3. \text{ units (feasible)}$$

Total annual cost for order quantity of 268.3 units (optimal size)

$$= \sqrt{2C_1C_3R} + CR$$

$$= \sqrt{2\times25\times0.10\times180\times500} + 25\times500$$

$$= \text{Rs. } 13170.82.$$

Total annual cost for order quantity corresponding to cut off point of 500 units.

$$= \frac{q}{2}C_1 + C_3\frac{R}{q} + CR$$

$$= \frac{500}{2}\times24.80\times0.10 + 180\times\frac{500}{500} + 24.80\times500$$

$$= \text{Rs. } 13200.$$

Total annual cost for order quantity corresponding to cut of point of 3000 units.

$$= \text{Rs.}\left(\frac{1500}{2}\times24.60\times0.10 + 180\times\frac{500}{1500} + 24.60\times500\right)$$

$$= \text{Rs. } 14745.$$

Total annual cost for order quantity corresponding to cut off point of 3000 units.

$$= \text{Rs.}\left(\frac{3000}{2}\times24.40\times0.10 + 180\times\frac{500}{3000} + 24.40\times500\right)$$

$$= \text{Rs. } (3660 + 30 + 12200) = \text{Rs. } 15890.$$

Since the total cost is minimum at $q_0 = 271.6$ units. It represents the optimal order quantity.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Decision Theory – SMT1604

DECISION THEORY

Introduction to Decision Making process - Decision making under uncertainty - Maximin and Maximax criteria - Hurwicz criterion – Laplace criterion – Minimax Regret criterion – Decision tree analysis - Simulation - Nature and need for simulation - Monte Carlo method.

Decision theory deals with methods for determining the optimal course of action when a number of alternatives are available and their consequences cannot be forecast with certainty. It is difficult to imagine a situation which does not involve such decision problems, but we shall restrict ourselves primarily to problems occurring in business, with consequences that can be described in dollars of profit or revenue, cost or loss.

For these problems, it may be reasonable to consider as the best alternative that which results in the highest profit or revenue, or lowest cost or loss, on the average, in the long run. This criterion of optimality is not without shortcomings, but it should serve as a useful guide to action in repetitive situations where the consequences are not critical. (Another criterion of optimality, the maximization of expected ìutility,î provides a more personal and subjective guide to action for a consistent decision-maker.) The simplest decision problems can be resolved by listing the possible monetary consequences and the associated probabilities for each alternative, calculating the expected monetary values of all alternatives, and selecting the alternative with the highest expected monetary value.

The determination of the optimal alternative becomes a little more complicated when the alternatives involve sequences of decisions. In another class of problems, it is possible to acquireñoften at a certain costñadditional information about an uncertain variable. This additional information is rarely entirely accurate. Its valueñhence, also the maximum amount one would be willing to pay to acquire itñshould depend on the difference between the best one expects to do with the help of this information and the best one expects to do without it. These are, then, the types of problems which we shall now begin to examine in more detail.

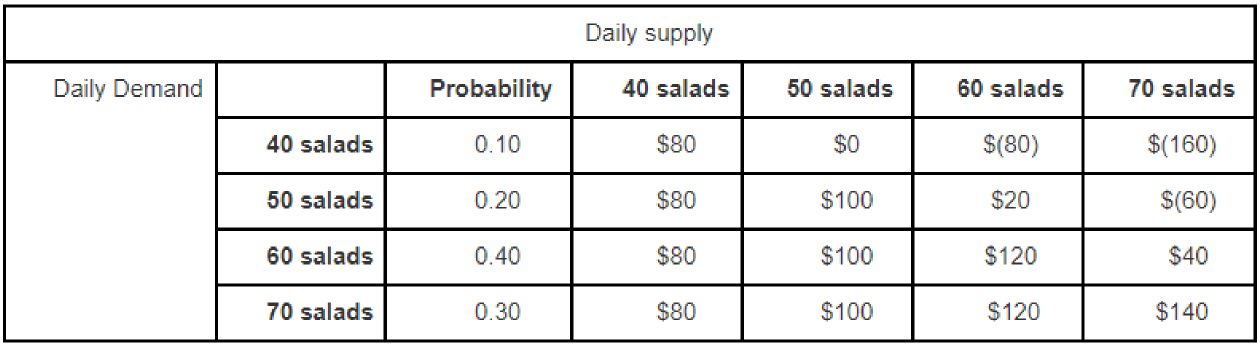
Very simply, the decision problem is how to select the best of the available alternatives. The elements of the problem are the possible alternatives (actions, acts), the possible events (states, outcomes of a random process), the probabilities of these events, the consequences associated with each possible alternative-event combination, and the criterion (decision rule) according to which the best alternative is selected.

Thepayoff (profit or loss) for the range of possible outcomes based on two factors:

1. Different decision choices
2. Different possible real world scenarios

For example, suppose Geoffrey Ramsbottom is faced with the following pay-off table. He has to choose how many salads to make in advance each day before he knows the actual demand.

- His choice is between 40, 50, 60 and 70 salads.
- The actual demand can also vary between 40, 50, 60 and 70 with the probabilities as shown in the table - e.g. P(demand = 40) is 0.1.
- The table then shows the profit or loss - for example, if he chooses to make 70 but demand is only 50, then he will make a loss of $60.



Daily supply						
Daily Demand		**Probability**	**40 salads**	**50 salads**	**60 salads**	**70 salads**
	40 salads	0.10	$80	$0	$(80)	$(160)
	50 salads	0.20	$80	$100	$20	$(60)
	60 salads	0.40	$80	$100	$120	$40
	70 salads	0.30	$80	$100	$120	$140

Maximax

The maximax rule involves selecting the alternative that maximises the maximum payoff available.

This approach would be suitable for an optimist, or 'risk-seeking' investor, who seeks to achieve the best results if the best happens. The manager who employs the maximax criterion is assuming that whatever action is taken, the best will happen; he/she is a risk-taker. So, how many salads will Geoffrey decide to supply?

Looking at the payoff table, the highest maximum possible pay-off is $140. This happens if we make 70 salads and demand is also 70. Geoffrey should therefore decide to supply 70 salads **every** day.

Minimax

The maximin rule involves selecting the alternative that maximises the minimum pay-off achievable. The investor would look at the worst possible outcome at each supply level, then selects the highest one of these. The decision maker therefore chooses the outcome which is guaranteed to minimise his losses. In the process, he loses out on the opportunity of making big profits.

This approach would be appropriate for a pessimist who seeks to achieve the best results if the worst happens.

So, how many salads will Geoffrey decide to supply? Looking at the payoff table,

- If we decide to supply 40 salads, the minimum pay-off is $80.
- If we decide to supply 50 salads, the minimum pay-off is $0.
- If we decide to supply 60 salads, the minimum pay-off is ($80).
- If we decide to supply 70 salads, the minimum pay-off is ($160).

The highest minimum payoff arises from supplying 40 salads. This ensures that the worst possible scenario still results in a gain of at least $80.

The miimax regret

The minimax regret strategy is the one that minimises the maximum regret. It is useful for a risk-neutral decision maker. Essentially, this is the technique for a 'sore loser' who does not wish to make the wrong decision.

'Regret' in this context is defined as the opportunity loss through having made the wrong decision.

To solve this a table showing the size of the regret needs to be constructed. This means we need to find the biggest pay-off for each demand row, then subtract all other numbers in this row from the largest number.

For example, if the demand is 40 salads, we will make a maximum profit of $80 if they all sell. If we had decided to supply 50 salads, we would achieve a nil profit. The difference or 'regret' between that nil profit and the maximum of $80 achievable for that row is $80.

Regrets can be tabulated as follows :

		Daily supply			
Daily Demand		**40 salads**	**50 salads**	**60 salads**	**70 salads**
	40 salads	$0	$80	$160	$240
	50 salads	$20	$0	$80	$160
	60 salads	$40	$20	$0	$80
	70 salads	$60	$40	$20	$0

The maximum regrets for each choice are thus as follows (reading down the columns):

- If we decide to supply 40 salads, the maximum regret is $60.
- If we decide to supply 50 salads, the maximum regret is $80.
- If we decide to supply 60 salads, the maximum regret is $160.
- If we decide to supply 70 salads, the maximum regret is $240.

A manager employing the minimax regret criterion would want to minimise that maximum regret, and therefore supply 40 salads only.

Note that the above techniques can be used even if we do not have probabilities. To calculate excepted values, for example, we will need probabilities.

DECISION-TREE ANALYSIS

A *decision-tree* is a graphic display of various decision alternatives and the sequence of events as if they were branches of a tree. In constructing a tree diagram, it is a convention to use the symbol '□' to indicate the *decision point* and 'O' to denote the situation of uncertainty or '*event*'. Branches coming out of a decision point are nothing but representation of immediate mutually exclusive alternative *acts* (alternative options) open to the decision-maker. Branches emanating from the 'event' point 'O' represent all possible situations (events). These events are not fully under the control of the decision-maker and may represent consumer demand, etc. The basic advantage of a tree diagram is that another act (called *second* act) subsequent to the happening of each event may also be represented. The resulting outcome (payoff) for each act-event combination may be indicated in the tree diagram at the outer end of each branch. A decision-tree diagram is illustrated below:

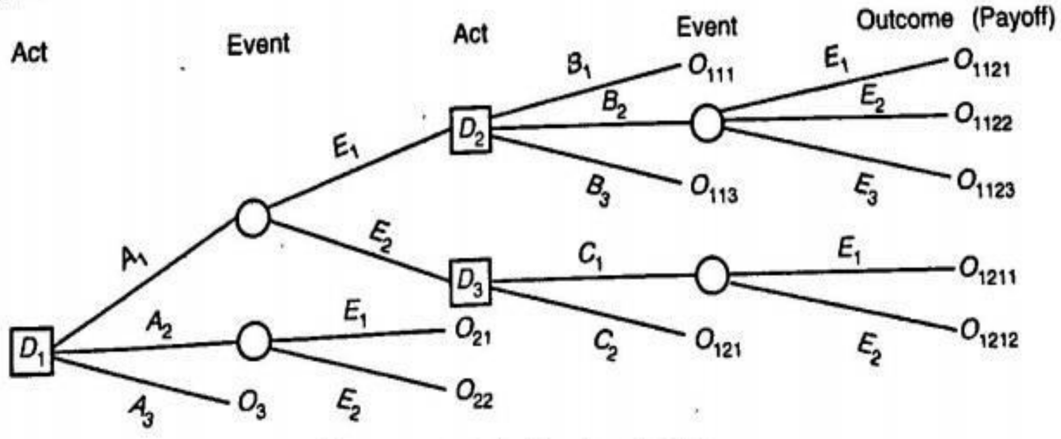


Fig. 19.1. A decision-tree diagram

For example, O_{1211} represents the Payoff of the act-event combination $A_1 - E_2 - C_1 - E_1$.

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DEPARTMENT OF MATHEMATICS

UNIT – V – Queuing Theory – SMT1604

QUEUING THEORY

Queuing theory – characteristics – Poisson and Exponential distributions – transient and steady state – Poisson process – finite and infinite queues – M/M/I and M/M/C models.

Queuing theory concerns the mathematical study of queues or waiting lines (seen in banks, post offices, hospitals, airports etc.). The formation of waiting lines usually occurs whenever the current demand for a service exceeds the current capacity to provide that service. The objective of the waiting line model is to minimize the cost of idle time & the cost of waiting time.

IDLE TIME COST

If an organization operates with many facilities and the demand from customers is very low, then the facilities are idle and the cost involved due to the idleness of the facilities is the idle time cost. The cost of idle service facilities is the payment to be made to the services for the period for which they remain idle.

WAITING TIME COST

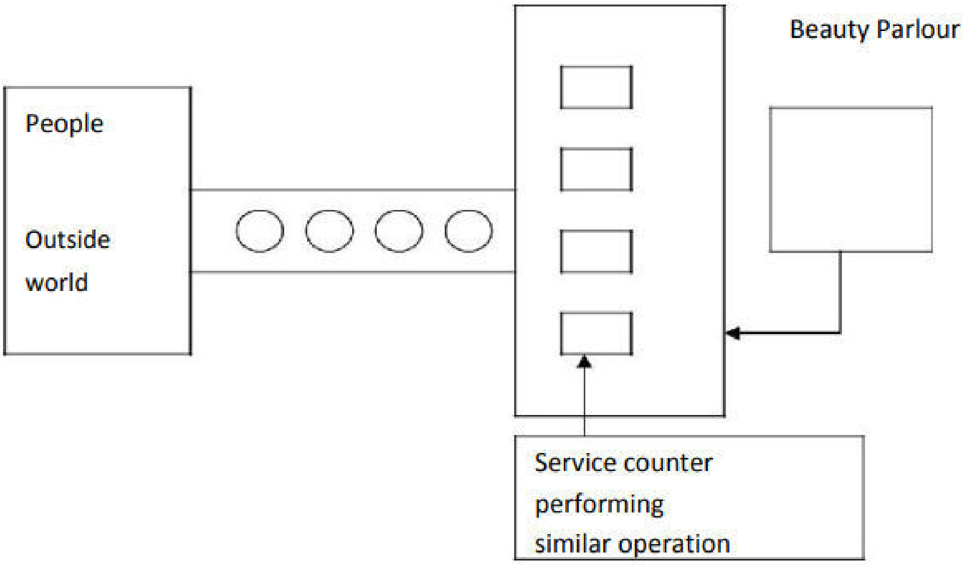
If an organization operates with few facilities and the demand from customer is high and hence the customer will wait in queue. This may lead to dissatisfaction of customers, which leads to waiting time cost. The cost of waiting generally includes the indirect cost of lost business.

TYPE OF QUEUE

a) Parallel queues. b) Sequential queues.

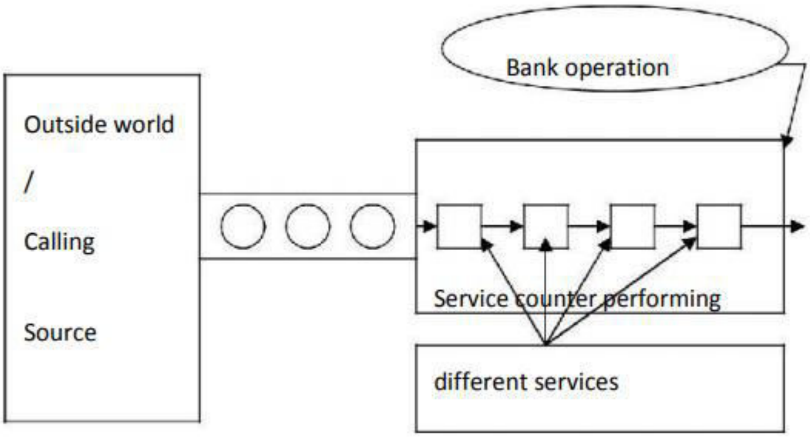
PARALLEL QUEUES

If there is more than one server performing the same function, then queues are parallel.



SEQUENTIAL QUEUES

If there is one server performing one particular function or many servers performing sequential operations then the queue will be sequential.



a. **Limited Queue**
In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number below less the limiting value. (Number of appointments in hospitals)

b. **Unlimited Queue**
In some facilities, there is no limit to the number of customer allowed in the system. (Entertainment centers).

a. Infinite queue: If the customer who arrives and forms the queue from a very large population the queue is referred to as infinite queue.

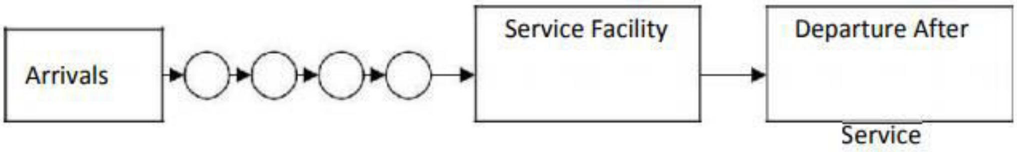
b. Finite Queue: if the customer who arrives and forms the queue from a small population then the queue is referred to as finite queue.

DEFINITIONS

1. The customer: The arriving unit that requires some service to be provided.
2. Server: A server is one who provides the necessary service to the arrived customer.
3. Queue (Waiting line): The number of customers, waiting to be serviced. The queue does not include the customer being serviced.
4. Service channel: The process or system, which performs the service to the customer. Based on the number of servers available.

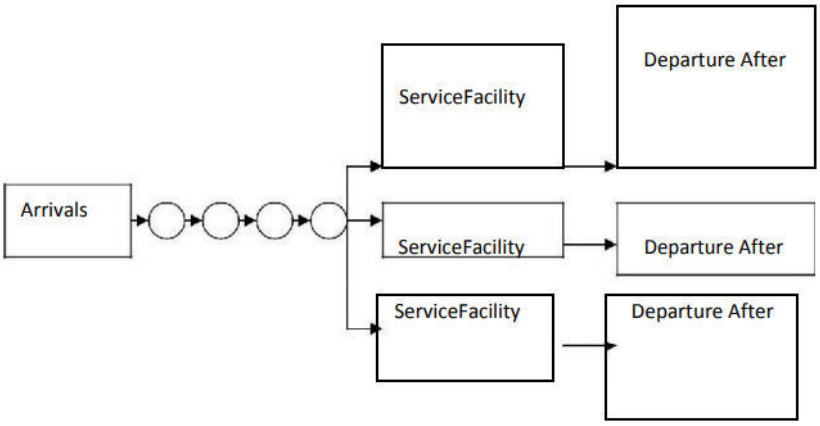
4A. Single Channel:
If there is a single service station, customer arrivals from a single line to be serviced then the channel is said to Single Channel Model or Single Server Model.

Eg. Doctor's clinic



B. Multiple Channel Waiting Line Model: If there are more than one service station to handle customer who arrive then it is called Multiple Channel Model. Symbol “c” is used

E.g., Barber shop



5. Arrival rate: The rate at which the customers arrive to be serviced. It is denoted by λ. λ indicates take average number of customer arrivals per time period.

6. Service rate: The rate at which the customers are actually serviced. It in indicated by µ . µ indicates the mean value of customer serviced per time period.

7. Infinite queue: If the customers who arrive and form the queue from a very large population the queue is referred to as infinite queue.

8. Priority: This refers to method of deciding as to which customer will be serviced. Priority is said to occur when an arriving customer is chosen for service ahead of some other customer already in the queue.

9. Expected number in the queue“Lq”: This is average or mean number of customer waiting to be serviced. This is indicated by “Lq”. ServiceFacility Departure.

10. Expected number in system Ls.: This is average or mean number of customer either waiting to be serviced or being serviced. This is denoted by Ls.

11. Expected time in queue Wq”.: This is the expected or mean time a customer spends waiting in the queue. This is denoted by “Wq”.

12. The Expected time in the system “WsÇ: This is the expected time or mean time customers spends for waiting in the queue and for being serviced. This is denoted by “WsÇ.

13. Expected number in a non-empty queue: Expected number of customer waiting in the line excluding those times when the line is empty.

14. System utilization or traffic intensity: This is ratio between arrival and service rate.

15. Customer Behaviour:

The customer generally behaves in 4 ways:

a) Balking: A customer may leave the queue, if there is no waiting space or he has no time to wait.

b) Reneging: A customer may leave the queue due to impatience

c) Priorities: Customers are served before others regardless of their arrival

d) Jockeying: Customers may jump from one waiting line to another.

Transient and Steady State

A system is said to be in Transient state when its operating characteristics are dependent on time.A system is said to be in Steady state when its operating characteristics are not dependent on time.

CHARACTERISTICS OF QUEUING MODELS

a) Input or arrival (inter –arrival) distribution. b) Output or Departure (Service) distribution. c) Service channel d) Service discipline. e) Maximum number of customers allowed in the system. f) Calling source or Population.

ARRIVAL DISTRIBUTION

It represents the rate in which the customer arrives at the system. Arrival rate/interval rate: ►Arrival rate is the rate at which the customers arrive to be serviced per unit of time. ►Inter-arrival time is the time gap between two arrivals. ►Arrival may be separated 1) By equal interval of time 2) By unequal interval of time which is definitely known. 3) Arrival may be unequal interval of time whose probability is known.

Arrival rate may be

1. Deterministic (D)
2. Probabilistic a. Normal (N) b. Binomial (B)
c. Poisson (M/N) d. Beta (β) e. Gama (g) f. Erlongian

The typical assumption is that arrival rate is randomly distributed according to Poisson distribution it is denoted by λ. λ indicates average number of customer arrival per time period. b) SERVICE OR DEPARTURE DISTRIBUTON: It represents the pattern in which the customer leaves the system. Service rate at which the customer are actually serviced. It indicated by μ. μ indicates the mean value of service per time period. Interdeparture is the rate time between two departures.23 Service time may be ► Constant. ► Variable with definitely known probability. ► Variable with known probability. Service Rate Or

Departure Rate may be: 1. Deterministic 2. Probabilistic. a. Normal (N) b. Binomial (B) c. Poisson (M/N) d. Beat (β) e. Gama (g) f. Erlongian (Ek) g. Exponential (M/N)

The typical assumption used is that service rate is randomly distributed according to exponential distribution. Service rate at which the customer are actually serviced. It indicated by μ . μ indicates the mean value of serv ice per time period.

SERVICE CHANNELS

The process or system, which is performing the service to the customer. Based on the number of channels: Single channel If there is a single service station and customer arrive and from a single line to be serviced, the channel is said to single channel.

Single Channel

1. Multiple channel If there is more than one service station to handle customer who arrive, then it is called multiple channel model. Multiple Channel - C. d) SERVICE DISCIPLINE: Service discipline or order of service is the rule by which customer are selected from the queue for service.
2. FIFO: First In First Out – Customer are served in the order of their arrival. Eg. Ticket counter, railway station, banks.
3. LIFO: Last In First Out – Items arriving last come out first. Priority: is said to occur when a arriving customer is chosen ahead of some other customer for service in the queue.
4. SIRO: Service in random order Here the common service discipline “First Come, First Served”.

MAXIMUM NUMBER OF CUSTOMER ALLOWED IN THE SYSTEM

Maximum number of customer in the system can be either finite or finite. a. Limited Queue: In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number below less the limiting value. (Number of appointments in hospitals) b. Unlimited Queue: In some facilities, there is no limit to the number of customer allowed in the system. (Entertainment centers).

POPULATION

The arrival pattern of the customer depends upon the source, which generates them.

a. Finite population

If there are a few numbers of potential customers the calling source is finite.

b. Infinite calling source or population:

If there are large numbers of potential customer, it is usually said to be infinite

Kendel’s Notation: a/b/c:

d/e/f. Where, a – Arrival rate.

b – Service rate.

c – Number of service s 1 or c.

d – Service discipline (FIFO)

e - Number of persons allowed in the queue (N or∞)

f - Number of people in the calling source (∞ or N)

1. **M/M/1, FIFO/∞/∞:**

Means Poisson arrival rate, Exponential service rate/one server /FIFO service discipline/Unlimited queues & Unlimited queue in the calling source.

2. **M/M/C, FIFO/∞/∞:**

Poisson arrival rate, Exponential service rate, more than one server, FIFO service discipline Unlimited queues and unlimited persons in the calling source.

3. **M/M/I, FIFO/N/∞:**

Means Poisson arrival rate, Exponential service rate, One server, FIFO, Limited queue & Unlimited population.

SINGLE CHANNEL /MULTIPLE CHANNELPOPULATION MODEL

1. Find an expression for probability of n customer in the system at time (Pn) in terms of λ and μ
2. Find an expression for probability of zero customers in the system at time t.(Po)
3. Having known Pn, find out the expected number of units in the Queue (Lq)
4. Find out the expected number of units in the system (Ls)
5. Expected waiting time in system (Ws)

Formulas

1. Expected number of units m the system (waiting + being served) (or) Length of the system

$$\boxed{L_s = \frac{\lambda}{\mu - \lambda}}$$

λ - mean arrival rate (no. of arrivals per unit of time)

μ - mean service rate per server (no. of customer served per unit time).

2. Expected number of units in the queue

Length of the queue $\boxed{L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu-\lambda}\right).}$

3. Expected time per unit in the system
(Expected time a unit spends in the system)

$$\boxed{W_s = \frac{1}{\mu-\lambda}.}$$

4. Expected time per unit in the queue (Expected time a unit spends in the queue)

$$\boxed{W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right).}$$

5. Avg. length of non empty queue (length of the queue that is formed from time to time)

$$\boxed{L_n = \frac{\mu}{\mu-\lambda}}$$

6. Avg. waiting time is non empty queue Avg. waitmg time of an arrival who waits

$$\boxed{W_n = \frac{1}{\mu-\lambda}}$$

7. Traffic Intensity/Equipment utilization $= \frac{\lambda}{\mu}$.

8. Probability that a person will have to wait $= \frac{\lambda}{\mu}$.

9. Probability that equipment or service (person) remain idle $= 1 - \frac{\lambda}{\mu}$.

10. Probability that n customers (units) in the system

$$\boxed{p(n) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n}$$

n = no. of customers in the system.

11. Probability that there are more than (n) customer in the system = 1-(P_0 + P_1 + ...+P_n).

12. Probability of n customers arriving in time $t = \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}$.

13. Proabability [Waiting time $\geq t$].

$$= \int_t^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu-\lambda)t}dt$$

14. Probability [time in system $\geq t$]

$$= \int_t^{\infty} (\mu - \lambda)e^{-(\mu-\lambda)t}\,dt.$$

Problem 1

Auto vehicles arrive at a petrol pump, having one petrol unit, in poission fashion an avg. of 10 units per hour. The service is distributed exponentially with a mean of 3 minutes. Find the following: (a) Avg. number of units in the system. (b) Avg. waiting time for customer. (c) Avg. length of queue (d) Probability that a customer arriving at the pump will have to wait. (e) The utilisation factor for the pump unit. (t) Probability that the number of customers in the system is 2.

Ans.

Arrival rate (λ) = 10 units per hour

Service time = 3 minutes

$$\text{Service rate } (\mu) = \frac{1}{3} \times 60 = 20 \text{ units per hour.}$$

(*a*) Avg. number of units in the system

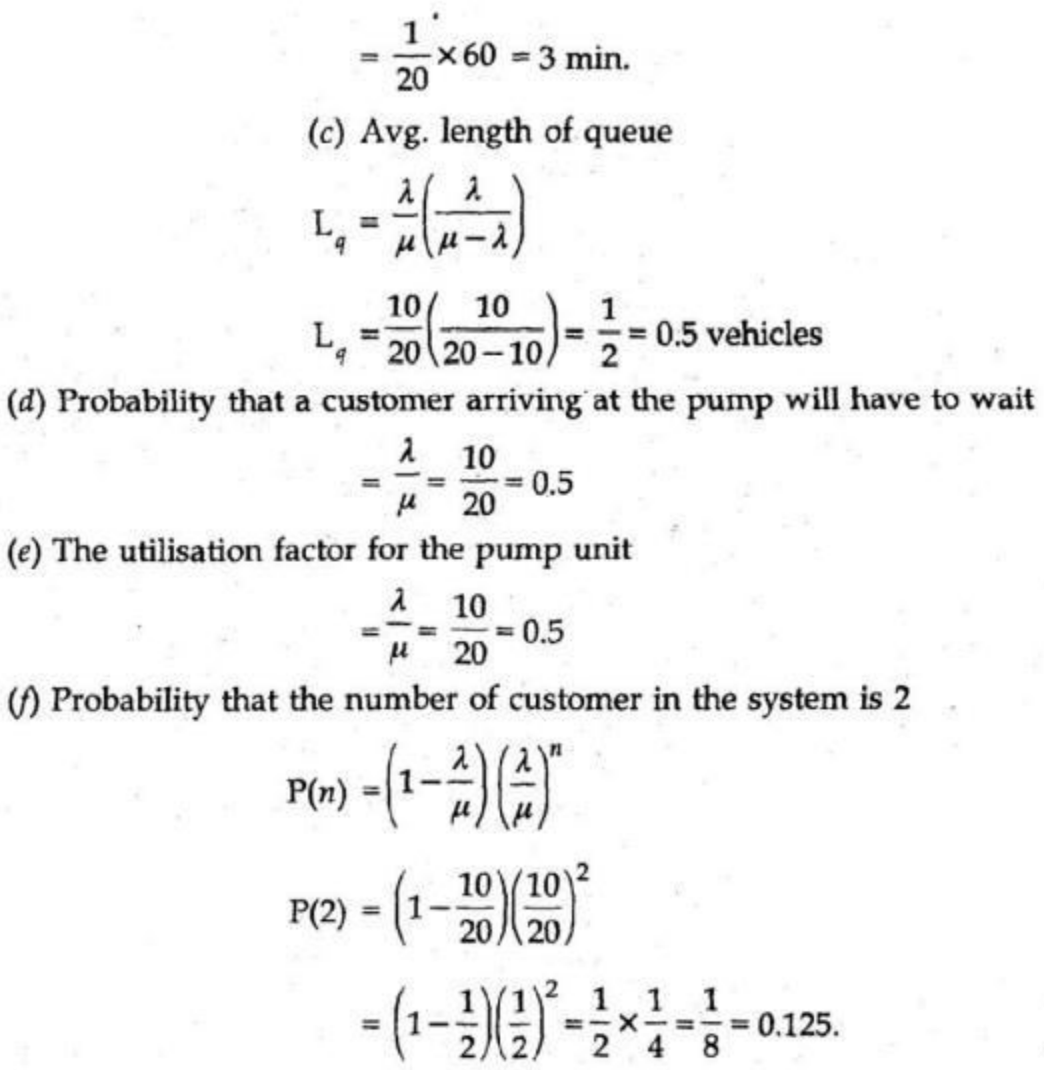
$$L_s = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{10}{20 - 10} = 1 \text{ auto vehicle.}$$

(*b*) Avg. waiting time for customer

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu - \lambda}\right)$$

$$= \frac{10}{20}\left(\frac{1}{20 - 10}\right) = \frac{1}{2} \times \frac{1}{10} = \frac{1}{20} \text{ hours}$$



$$= \frac{1}{20} \times 60 = 3 \text{ min.}$$

(*c*) Avg. length of queue

$$L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu - \lambda}\right)$$

$$L_q = \frac{10}{20}\left(\frac{10}{20-10}\right) = \frac{1}{2} = 0.5 \text{ vehicles}$$

(*d*) Probability that a customer arriving at the pump will have to wait

$$= \frac{\lambda}{\mu} = \frac{10}{20} = 0.5$$

(*e*) The utilisation factor for the pump unit

$$= \frac{\lambda}{\mu} = \frac{10}{20} = 0.5$$

(*f*) Probability that the number of customer in the system is 2

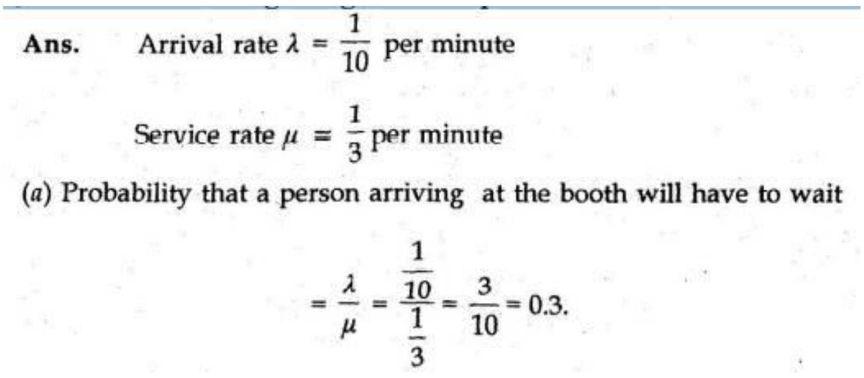
$$P(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$P(2) = \left(1 - \frac{10}{20}\right)\left(\frac{10}{20}\right)^2$$

$$= \left(1 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} = 0.125.$$

Problem 2

Arrival at a telephone booth are considered are to be poission, with an average time of 10 minutes between one arrival and next. The length of phone call assumed to be distributed exponentially with mean 3 minutes then (a) What is. the probability that a person arriving at the booth will have to wait? (b) What is the average length of the queues that form from time to time.



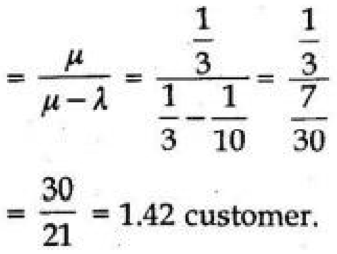
Ans. Arrival rate $\lambda = \frac{1}{10}$ per minute

Service rate $\mu = \frac{1}{3}$ per minute

(*a*) Probability that a person arriving at the booth will have to wait

$$= \frac{\lambda}{\mu} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10} = 0.3.$$

(b) Average queue length that is formed from time to time

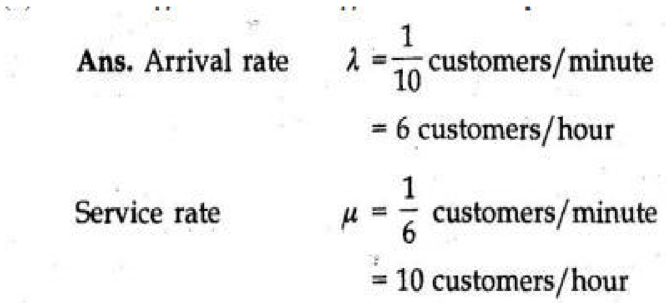


$$= \frac{\mu}{\mu-\lambda} = \frac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{10}} = \frac{\frac{1}{3}}{\frac{7}{30}}$$

$$= \frac{30}{21} = 1.42 \text{ customer.}$$

Problem 3

Customers arrive at one-window drive according to a poission distribution with mean of 10 mm and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have wait outside the space. Calculate. (a) Probability that an arriving customer can drive directly to the space in front of the window. (b) Probability that an arriving customer will have to wait outside the directed space. - (c) How long is an arriving customer expected to wait before starting service?

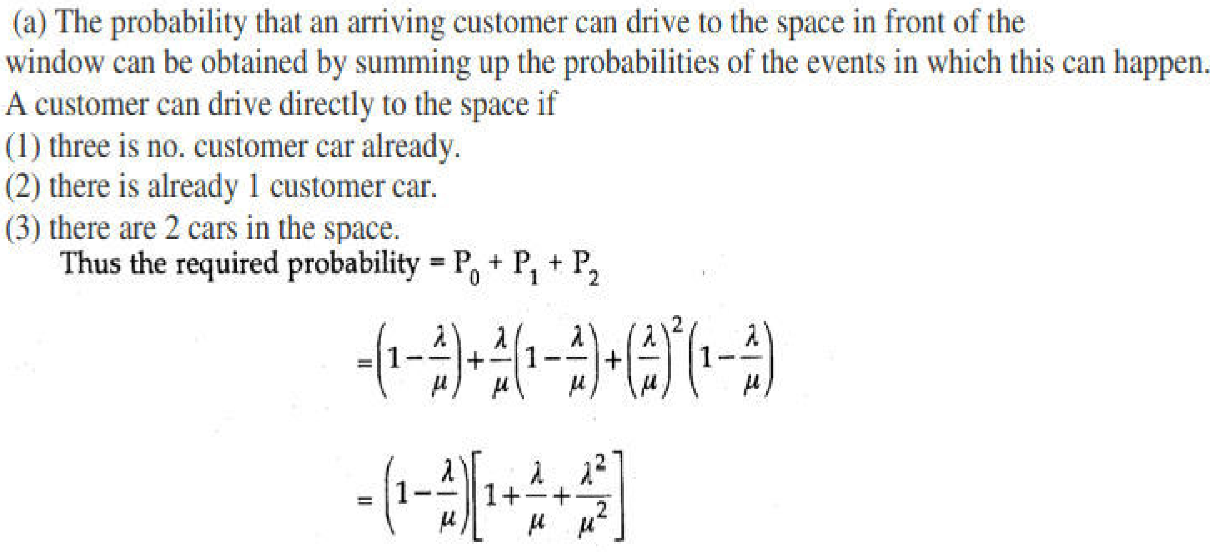


Ans. Arrival rate $\lambda = \frac{1}{10}$ customers/minute

$= 6$ customers/hour

Service rate $\mu = \frac{1}{6}$ customers/minute

$= 10$ customers/hour



(a) The probability that an arriving customer can drive to the space in front of the window can be obtained by summing up the probabilities of the events in which this can happen. A customer can drive directly to the space if

(1) three is no. customer car already.

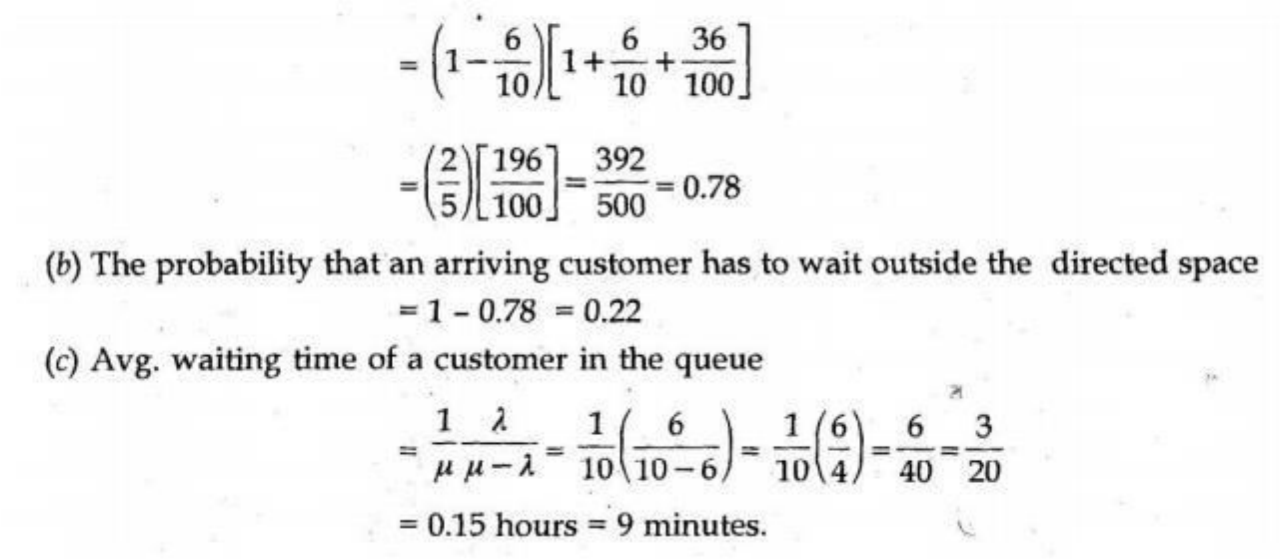
(2) there is already 1 customer car.

(3) there are 2 cars in the space.

Thus the required probability = $P_0 + P_1 + P_2$

$$= \left(1-\frac{\lambda}{\mu}\right)+\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right]$$



$$= \left(1-\frac{6}{10}\right)\left[1+\frac{6}{10}+\frac{36}{100}\right]$$

$$= \left(\frac{2}{5}\right)\left[\frac{196}{100}\right] = \frac{392}{500} = 0.78$$

(*b*) The probability that an arriving customer has to wait outside the directed space

$$= 1 - 0.78 = 0.22$$

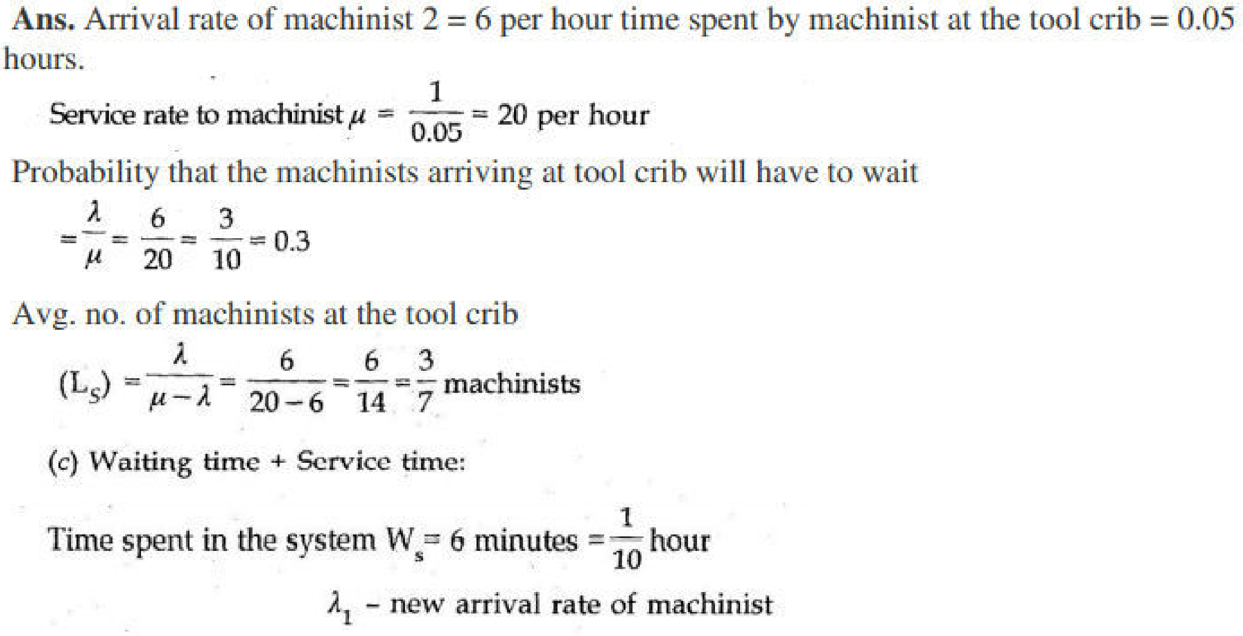
(*c*) Avg. waiting time of a customer in the queue

$$= \frac{1}{\mu}\frac{\lambda}{\mu-\lambda} = \frac{1}{10}\left(\frac{6}{10-6}\right) = \frac{1}{10}\left(\frac{6}{4}\right) = \frac{6}{40} = \frac{3}{20}$$

$$= 0.15 \text{ hours} = 9 \text{ minutes.}$$

Problem 4

. Arrival of machinists at a tool crib are considered to be poission distribution at an avg. rate of 6 per hour. The length of time the machinists must remain at the tool crib is exponentially distributed with an average time being 0.05 hours. (a) What is the probability that the machinists arriving at tool crib will have to wait. (b) What is the average number of machinists at the tool crib. (c) The company will install a second tool crib when convinced that a machinist would expect to have spent at least 6 mins waiting and being serviced at the tool crib. By how much must the flow of machinists to toolcrib increase to justify the addition of second tool crib?



Ans. Arrival rate of machinist 2 = 6 per hour time spent by machinist at the tool crib = 0.05 hours.

Service rate to machinist $\mu = \frac{1}{0.05} = 20$ per hour

Probability that the machinists arriving at tool crib will have to wait

$$= \frac{\lambda}{\mu} = \frac{6}{20} = \frac{3}{10} = 0.3$$

Avg. no. of machinists at the tool crib

$$(L_S) = \frac{\lambda}{\mu-\lambda} = \frac{6}{20-6} = \frac{6}{14} = \frac{3}{7} \text{ machinists}$$

(*c*) Waiting time + Service time:

Time spent in the system W_s = 6 minutes $= \frac{1}{10}$ hour

λ_1 – new arrival rate of machinist

$$W_s = \frac{1}{\mu - \lambda_1} = \frac{1}{20 - \lambda_1}$$

$$\frac{1}{10} = \frac{1}{20 - \lambda_1} \Rightarrow 20 - \lambda_1 = 10 \qquad \Rightarrow \lambda_1 = 10 \text{ machinist/hour}$$

Increase in the flow of machinists to toolcrib increase to justify the addition of a second tool crib =10 — 6 = 4/hour.

Problem 5

On an average 96 patients per 24 hours day require the service of an emergency clinic. Also an average a patient requires 10 miii. of active attention. Assume that the facility can handle one emergency at a time. Suppose that it cost the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in his average time would cost Rs. 10/-per patient treated. How much would have to be budgeted by the clinic to decrease the average size of

the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient.

Ans. $$\lambda = \frac{96}{24} = 4 \text{ patients/hour}$$

$$\mu = \frac{1}{10} \times 60 = 6 \text{ patients/hour}$$

Avg. no. of patients in the queue.

$$L_q = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu - \lambda}\right) = \frac{4}{6}\left(\frac{4}{6-4}\right)$$

$$L_q = \frac{4}{2}(2) = \frac{8}{6} = 1\frac{1}{3}$$

This number is to be reduced from $1\frac{1}{3}$ to $\frac{1}{2}$.This can be achieved by increasing the service rate to say μ'

$$L_q' = \frac{\lambda}{\mu'}\left(\frac{\lambda}{\mu' - \lambda}\right)$$

$$\frac{1}{2} = \frac{4}{\mu'}\left(\frac{4}{\mu' - 4}\right)$$

$$\mu'^2 - 4\mu' - 32 = 0 \text{ or } (\mu' - 8)(\mu' + 4) = 0$$

$\mu' = 8$ patients/hour ($\mu' = -4$ is illogical and hence neglected)

Avg. time required by each patient $= \frac{1}{8} hr$

$$= \frac{15}{2} \text{ minutes}$$

Therefore the budget required for each patient

$$= \text{Rs. } (100 + \frac{5}{2} \times 10) = \text{Rs. } 125/-$$

Thus to decrease the size of the queue, the budget per patient should be increased from Rs. 100 to Rs. 125/—