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SCHOOL OF SCIENCE AND HUMANITIES

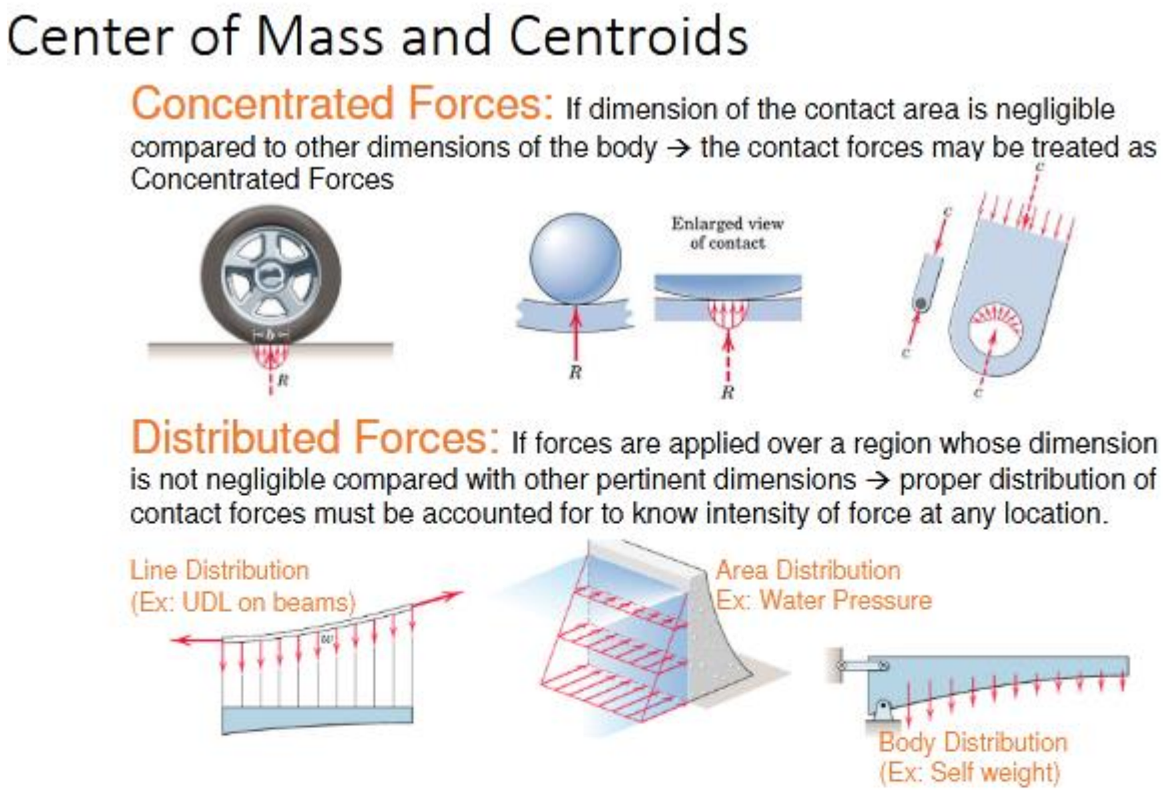
DEPARTMENT OF MATHEMATICS

UNIT – I –CENTRE OF GRAVITY- SMT1603

UNIT - I

Centre of Gravity

Centre of Gravity of areas –surfaces and volumes of solids of revolution –conditions of equilibrium.

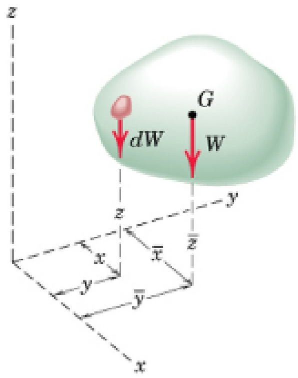


Centre of Gravity:

The centre of gravity (C.G.) of a body is the point about which the algebraic sum of moments of weights of all the particles constituting the body is zero. The entire weight of the body can be considered to act at this point howsoever the body is placed.

Determination of CG

- Apply Principle of Moments



Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.

Weight of the body $W = \int dW$

Moment of weight of an element (dW) @ x-axis = ydW

Sum of moments for all elements of body = $\int ydW$

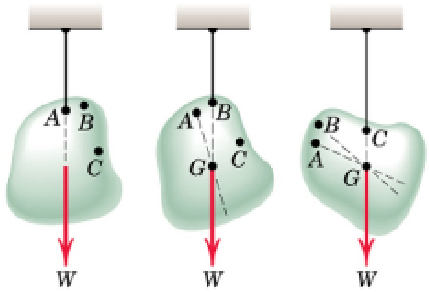
From Principle of Moments: $\int ydW = \bar{y}\, W$

$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$

→ Numerator of these expressions represents the sum of the moments;
Product of W and corresponding coordinate of G represents the moment of the sum → Moment Principle.

Center of Mass and Centroids

Center of Mass



A body of mass m in equilibrium under the action of tension in the cord, and resultant W of the gravitational forces acting on all particles of the body.

- The resultant is collinear with the cord

Suspend the body from different points on the body

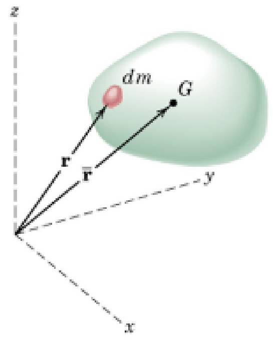
- Dotted lines show lines of action of the resultant force in each case.
- These lines of action will be concurrent at a single point G

  As long as dimensions of the body are smaller compared with those of the earth.
  - we assume uniform and parallel force field due to the gravitational attraction of the earth.

The unique **Point G** is called the Center of Gravity of the body (CG)

Center of Mass and Centroids

$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$



Determination of CG

Substituting $W = mg$ and $dW = gdm$

→ $\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$

In vector notations:

Position vector for elemental mass: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Position vector for mass center G: $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

→ $\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$ The above equations are the components of this single vector equation

Density ***ρ*** of a body = **mass per unit volume**

→ Mass of a differential element of volume dV → $dm = \rho dV$

→ ρ may not be constant throughout the body

$$\bar{x} = \frac{\int x\rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y\rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z\rho dV}{\int \rho dV}$$

Center of Mass and Centroids

Center of Mass: Following equations independent of *g*

$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m} \qquad \bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m} \qquad \bar{x} = \frac{\int x\rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y\rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z\rho dV}{\int \rho dV}$$

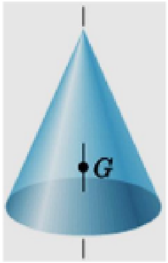
→ They define a unique point, which is a function of distribution of mass

→ This point is Center of Mass (CM)

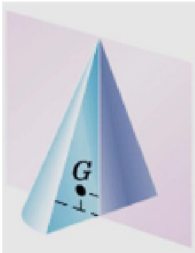
→ CM coincides with CG as long as gravity field is treated as uniform and parallel

→ CG or CM may lie outside the body

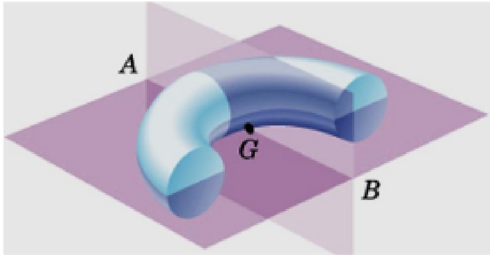
CM always lie on a line or a plane of symmetry in a homogeneous body



Right Circular Cone
CM on central axis



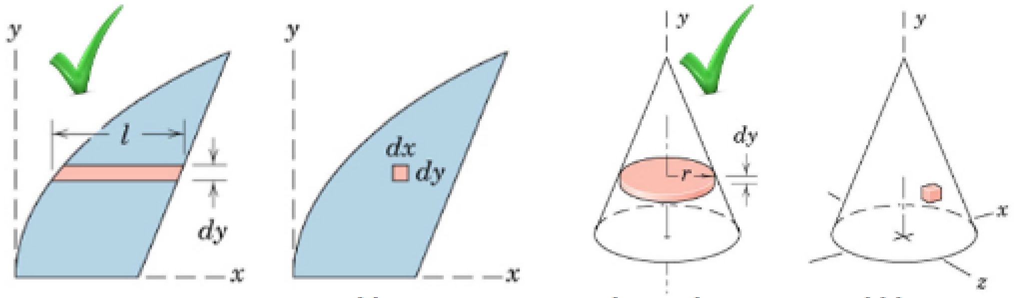
Half Right Circular Cone
CM on vertical plane of symmetry



Half Ring
CM on intersection of two planes of symmetry (line AB)

Centroids of Lines, Areas, and Volumes
Guidelines for Choice of Elements for Integration

- Order of Element Selected for Integration
  A first order differential element should be selected in preference to a higher order element → only one integration should cover the entire figure

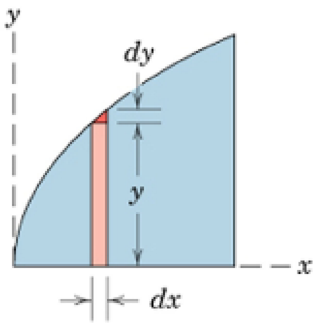


$A = \int dA = \int l\,dy$ $A = \int\int dx\,dy$ $V = \int dV = \int \pi r^2\,dy$ $V = \int\int\int dxdydz$

Centroids of Lines, Areas, and Volumes
Guidelines for Choice of Elements for Integration

- Discarding Higher Order Terms
  Higher order terms may always be dropped compared with lower order terms

Vertical strip of area under the curve is given by the first order term → $dA = ydx$
The second order triangular area $0.5dxdy$ may be discarded



Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- Choice of Coordinates
  Coordinate system should best match the boundaries of the figure
  → easiest coordinate system that satisfies boundary conditions should be chosen

Boundaries of this area (not circular) can be easily described in rectangular coordinates

Boundaries of this circular sector are best suited to polar coordinates

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

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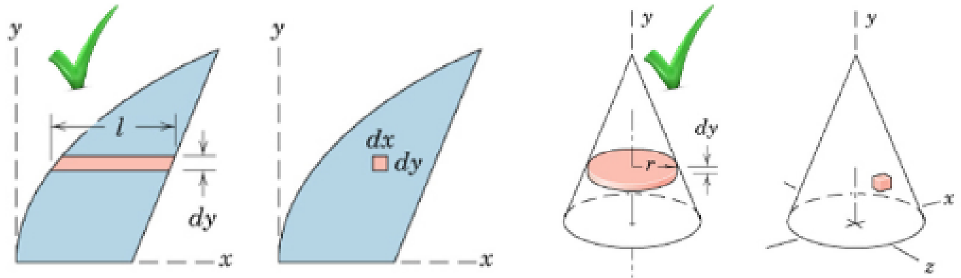


$A = \int dA = \int l dy$ $\quad A = \int\int dx\, dy$ $\quad V = \int dV = \int \pi r^2\, dy$ $\quad V = \int\int\int dxdydz$

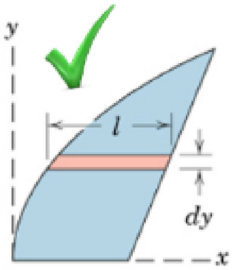
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

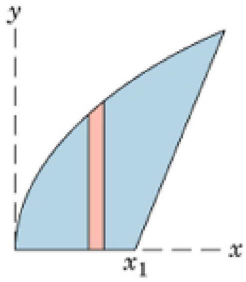
Guidelines for Choice of Elements for Integration

- Continuity

  Choose an element that can be integrated in one continuous operation to cover the entire figure → the function representing the body should be continuous → only one integral will cover the entire figure



Continuity in the expression for the width of the strip



Discontinuity in the expression for the height of the strip at $x = x_1$

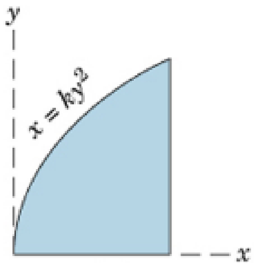
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

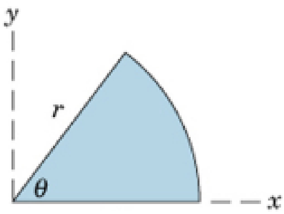
Guidelines for Choice of Elements for Integration

- Choice of Coordinates

  Coordinate system should best match the boundaries of the figure → easiest coordinate system that satisfies boundary conditions should be chosen



Boundaries of this area (not circular) can be easily described in rectangular coordinates



Boundaries of this circular sector are best suited to polar coordinates

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- Discarding Higher Order Terms
  Higher order terms may always be dropped compared with lower order terms

Vertical strip of area under the curve is given by the first order term → $dA = ydx$
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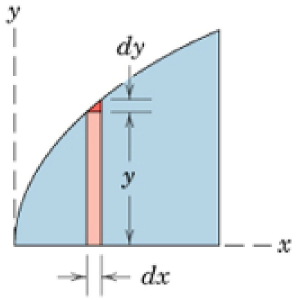
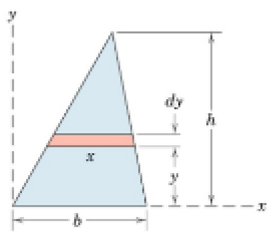


Center of Mass and Centroids

Examples: Centroids

Locate the centroid of the triangle along h from the base



Solution:

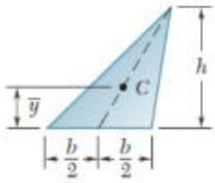
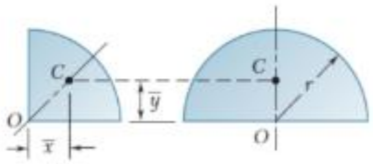
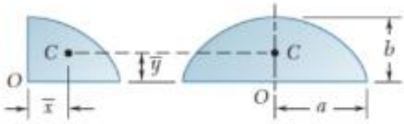
$dA = xdy$ $\qquad \dfrac{x}{(h-y)} = \dfrac{b}{h}$

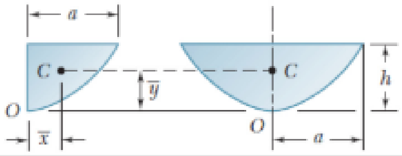
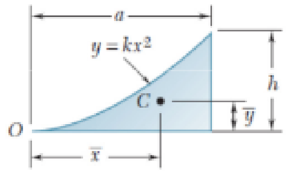
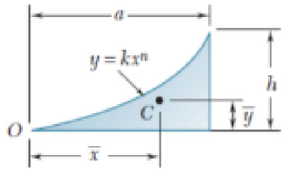
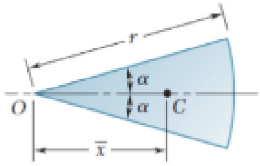
Total Area $A = \frac{1}{2}bh$ $\qquad y = y_c$

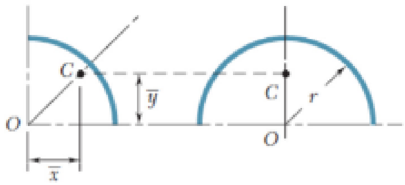
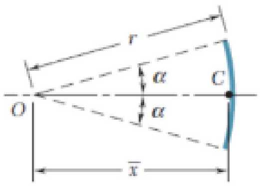
$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

$$A\bar{y} = \int y_c dA \quad \Rightarrow \frac{bh}{2}\bar{y} = \int_0^h y\frac{b(h-y)}{y}dy = \frac{bh^2}{6}$$

$$\boxed{\bar{y} = \frac{h}{3}}$$

Shape		$\overline{x}$	$\overline{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –II –Virtual Work– SMT1603

Unit –II Virtual Work

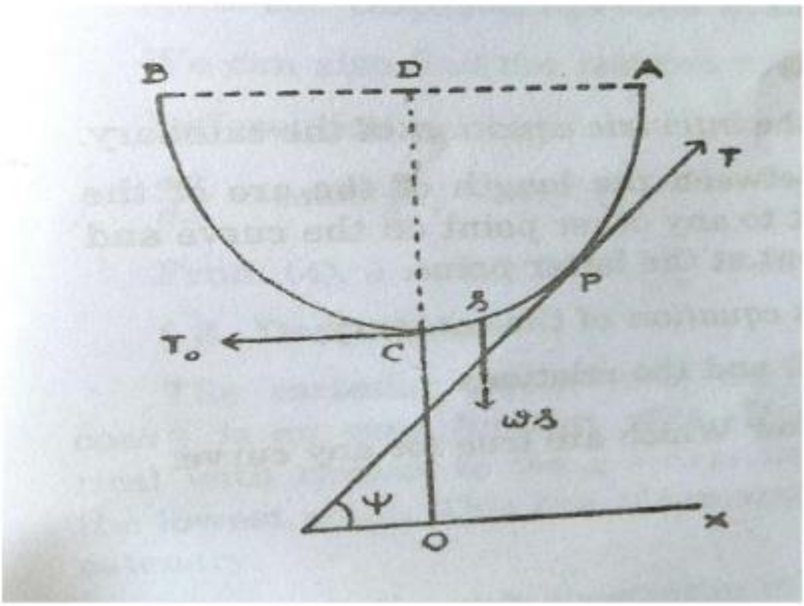
Virtual work – simple problems – equilibrium of strings and chains – common catenary – Suspension bridge

Equilibrium of Strings

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a *catenary*. If the weight per unit length of the chain or string is constant, the catenary is called the *uniform or common catenary*.

Equation of the common catenary:

A uniform heavy inextensible string hangs freely under the action of gravity; to find the equation of the curve which it forms.



Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest, of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the are CP = s

Let ω be the weight per unit length of the chain.

Consider the equilibrium of the portion CP of the chain.

The forces acting on it are:

(i) Tension T_0 acting along the tangent at C and which is therefore horizontal.

(ii) Tension T acting at P along the tangent at P making an angle Ψ with OX.

(iii) Its weight **ws** acting vertically downwards through the C.G. of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight **ws** must pass through the point of the intersection of T and T_o.

Resolving horizontally and vertically, we have

$$T\cos\Psi = T_o \ldots \ldots \quad (1)$$

$$\text{and } T\sin\Psi = \mathbf{ws} \ldots \ldots (2)$$

Dividing (2) by (1), $\tan\Psi = \frac{\mathbf{ws}}{T_0}$

Now it will be convenient to write the value of T_o the tension at the lowest point, as $T_o = wc \ldots \ldots (3)$ where c is a constant. This means that we assume T_o, to be equal to the weight of an unknown length c of the cable.

Then $\tan \Psi = \frac{ws}{wc} = \frac{s}{c}$

$\therefore S = c\tan\Psi \; \ldots \; \ldots \; \ldots \; (4)$

Equation (4) is called the *intrinsic* equation of the catenary.

It gives the relation between the length of the area of the curve from the lowest point to any other point on the curve and the inclination of the tangent at the latter point.

To obtain the *certesian equation* of the catenary,

We use the equation (4) and the relations

$\frac{dy}{ds} = \sin \Psi$ and $\frac{dy}{dx} = \tan \Psi$ which are true for any curve.

Now $\frac{dy}{d\Psi} = \frac{dy}{ds} \cdot \frac{ds}{d\Psi}$

$= \sin \Psi \frac{d}{d\Psi} c \tan \Psi$

$= \sin c\sec^2\Psi = c\sec \Psi \tan \Psi$

$\therefore y = \int c\sec \Psi \tan \Psi \, d\Psi + A$

$= c\sec \Psi + S$

If $y = c$ when $\Psi = 0$, then $c = c\sec 0 + A$

$\therefore A = 0$

Hence $y = c\sec \Psi \; \ldots \; \ldots \; \ldots \; (5)$

$\therefore y^2 = c^2 \sec \Psi = c^2 (1 + \tan^2 \Psi)$

$= c^2 + s^2 \; \ldots \; \ldots \; (6)$

$\frac{dy}{dx} = \tan \Psi = \frac{s}{c} = \frac{\sqrt{y^2 - c^2}}{c}$

$\therefore \frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}$

Integrating, $\cosh^{-1} \left(\frac{y}{c}\right) = \frac{x}{c} + B$

When $x = 0$, $y = c$

i.e. $\cosh^{-1} 1 = 0 + B$ or $B = 0$

$\therefore \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c}$

i.e. $y = c\cosh\left(\frac{x}{c}\right)$ … … (7)

(7) is the Cartesian equation to the catenary.

We can also find the relation connecting s and x.

Differentiating (7).

$$\frac{dy}{dx} = c\sinh\frac{x}{c}.\ \frac{1}{c} = \sinh\frac{x}{c}$$

From (4), $s = c\tan\Psi = c.\ \frac{dy}{dx} = c\sinh\frac{x}{c}$ … (8)

Definitions:

The Cartesian equation to the catenary is $y = c\cosh\frac{x}{c}$. $\cosh\frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the axis of the catenary.

Since c is the only constant, in the equation, it is called the *parameter* of the catenary and it determines the size of the curve.

The lowest point *C* is called the vertex of the catenary. The horizontal line at the depth *c* below the vertex (which is taken by us the x – axis) is called the directrix of the catenary.

If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the span and the distance CD (i.e. the depth of the lowest point C below AB) is called the sag.

Important Formulae:

The Cartesian coordinates of a point P on the catenary are (x, y) and its intrinsic coordinates are (s, Ψ). Hence there are four variable quantities we can have a relation connecting any two of them. There will be ${}_4C_2 = 6$ such relations, most of them having been already derived. We shall derive the remaining. It is worthwhile to collect these results for ready reference.

(i) The relation connecting x and y is

$y = c\cosh\frac{x}{c}$ … … (1)

and this is the Cartesian equation to the catenary.

(ii) The relation connecting s and Ψ is

$s = c\tan\Psi$ … … (2)

(iii) The relation connecting y and Ψ is

$y = c\sec\Psi$ … … … (3)

(iv) The relation connecting y and s is

$y^2 = c^2 + s^2$ …. … … (4)

(v) The relation connecting s and x is

$s = c\sinh\frac{x}{c}$

Integrating, x = $\int c\sec\Psi\, d\Psi + D$

$= c\log(\sec\Psi + \text{ran}\Psi) + D$

At the lowest point, $\Psi = 0$ and $x = 0$

$\therefore 0 = c\log(\sec 0 + \tan 0 + D$

i.e. $0 = D$

$\therefore x = c\log(\sec\Psi + \tan\Psi)$

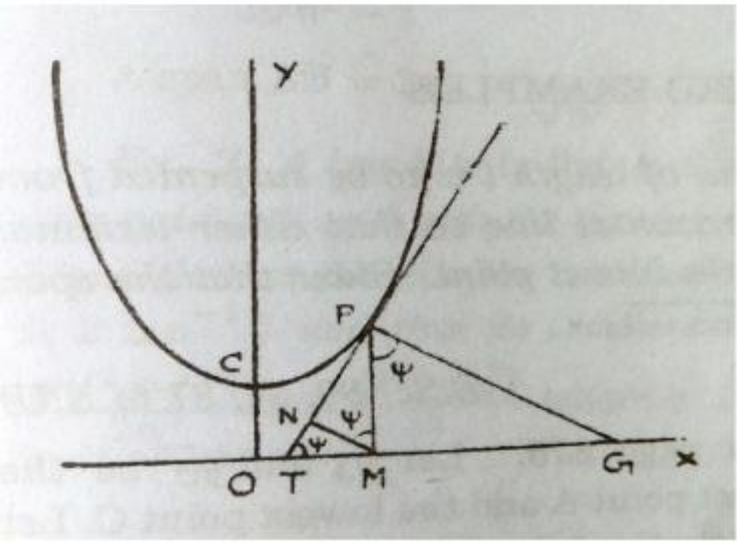
(vii) The tension at any point = wy … … (7), where y is the distance of the point from the directrix.

(viii) The tension at the lowest point = wc … … (8)

$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$

$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$

Geometrical Properties of the Common catenary:



Let P be any point on the catenary y = ccosh $\frac{x}{c}$.

PT is the tangent meeting the directrix (i.e. the x axis) at T.

angle PTX = Ψ

PM (=y) is the ordinate of P and PG is the normal at P.

Draw MN ⊥ to PT.

From ΔPMN. MN = PMcosΨ

$=y\cos\Psi$

$=c\sec\Psi\cos\Psi$

$=c$=constant

i.e. *The length of the perpendicular from the foot of the ordinate on the tangent at any point of the catenary is constant.*

Again $\tan\Psi = \frac{PN}{MN} = \frac{PN}{C}$

∴ PN = $c\tan\Psi = S$ arc CP

$PM^2 = NM^2 + PN^2$

∴ $y^2 = c^2+s^2$, a relation already obtained.

If is the radius of curvature of the catenary at P,

$P = \frac{ds}{d\Psi} = \frac{d}{d\Psi}(c\tan\Psi) = c\sec^2\Psi$

Let the normal at P cut the x axis at G.

If is the radius of curvature of the catenary at P,

$P = \frac{ds}{d\Psi} = \frac{d}{d\Psi}(c\tan\Psi) = c\sec^2\Psi$

Let the normal at P cut the x axis at G.

Then PG. $\cos\Psi$ = PM = y

∴ PG = $\frac{y}{\cos\Psi} = c\sec\Psi.\sec\Psi = c\sec^2\Psi$

∴ ρ = PG

Hence the radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1}$

Solution:

Tension at A = wy_A

And tension at C = $w.y_C$ since T = wy at any point

Now $w.y_A = n.w.y_C$

$\therefore y_A = ny_C = nc$

$$\text{or} \quad \frac{x_A}{c} = \cosh^{-1} n = \log(n+\sqrt{n^2-1})$$

$$\therefore x_A = c\log(n+\sqrt{n^2-1}) \ldots \ldots \ldots (1)$$

We have to find c.

$y^2_A = c^2+s^2_A$, s_A denoting the length of CA.

$$= c^2+\frac{l^2}{4} \text{ (as total length = l)}$$

$$\text{i.e. } n^2c^2 = c^2+\frac{l^2}{4}$$

$$\text{or } c^2 = \frac{l^2}{4(n^2-1)}$$

$$\therefore c = \frac{l^2}{2\sqrt{n^2-1}} \ldots \ldots (2)$$

Substituting (2) in (1),

$$x_A = \frac{l^2}{2\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$$

$$\therefore \text{span AB} = 2x_A = \frac{l}{\sqrt{n^2-1)}}\log(n+\sqrt{n^2-1})$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –III- Simple Harmonic Motion – SMT1603

Unit – 3

Simple Harmonic motion

Simple Harmonic motion and its application - Particle attached to an elastic string- composition of two simple harmonic motions - Simple pendulum - tangential and Normal velocity and acceleration of a particle along a curve.

Simple Harmonic Motion (S.H.M) is an interesting special type of motion in nature, having forward and backward oscillation (or) to and fro oscillation about a fixed point. The fixed point is known as the mean position or equilibrium position. When the oscillation is very small we prove the motion is simple harmonic. In this section we study about the resultant of two S.H.M'S of the same period in the same straight line and in two perpendicular lines. Also we find the periodic time of oscillation of a simple pendulum.

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Examples

Small oscillation of a cradle, simple pendulum, seconds pendulum, simple equivalent pendulum, transverse vibrations of a plucked violin string etc.

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.

Equation (1) is the fundamental differential equation representing a S.H.M.

If v is the velocity of the particle at time t (1) can be written as

$v\frac{dv}{dx} = -\mu x$ i.e. $\quad vdv = -\mu x\, dx \quad$ ....(2)

Integrating (2), we have $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c$ ....(3)

Initially let the particle starts from rest at the point A where OA = a

Hence when x=a, $v = 0 = \frac{dx}{dt}$

Putting these in (3), $0 = -\frac{\mu a^2}{2}$ +c or $c = \frac{\mu a^2}{2}$

Putting these in (3), $0 = -\frac{\mu a^2}{2}$ +c or $c = \frac{\mu a^2}{2}$

$\therefore v^2 = -\mu x^2 + \mu a^2 = \mu\left(a^2 - x^2\right)$

$\therefore v = \pm\sqrt{\mu\left(a^2 - x^2\right)}$ .... (4)

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence we take the negative sign in (4),

$\frac{dx}{dt}$ **= v =** $-\sqrt{\mu\left(a^2 - x^2\right)}$ **........(5)**

$$-\frac{dx}{\sqrt{(a^2-x^2)}} = \sqrt{\mu}\ dt$$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t + A$

Initially when $t = 0$, $x = a$, $\cos^{-1}1 = 0 + A \Rightarrow$ $\boxed{A=0}$

$\therefore \cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t$ or $\quad x = a\cos\sqrt{\mu}\ t \quad \ldots\ldots\ldots (6)$

To get the time from A to A^1, put $x = -a$ in (6)

We have $\cos\sqrt{\mu}\,t = -1 = \cos\pi$, $t = \frac{\pi}{\sqrt{\mu}}$

$\therefore$ The time from A to A' and back $= \frac{2\pi}{\sqrt{\mu}}$.

Equation (6) can be written as

$x = a\cos\sqrt{\mu}\ t = a\cos\ (\sqrt{\mu}\ t + 2\pi) = a\cos(\sqrt{\mu}\,t + 4\pi)$ etc

$$= a\cos\sqrt{\mu}\left(t+\frac{2\pi}{\sqrt{\mu}}\right) = a\cos\sqrt{\mu}\left(t+\frac{4\pi}{\sqrt{\mu}}\right) \text{ etc.}$$

Differentiating (6),

$$\frac{dx}{dt} = -a\sqrt{\mu}\ .\sin\sqrt{\mu}\ t$$

$$= -a\sqrt{\mu}\sin(\sqrt{\mu}\ t + 2\pi) = -a\sqrt{\mu}\ \sin(\sqrt{\mu}\ t + 4\pi) \text{ etc.}$$

$$= -a\sqrt{\mu}\ \sin\sqrt{\mu}\ (t + \frac{2\pi}{\sqrt{\mu}}) = -a\sqrt{\mu}\ \sin\sqrt{\mu}\,(t + \frac{4\pi}{\sqrt{\mu}}) \text{ etc.}$$

The values of $\frac{dx}{dt}$ are the same if t is increased by $\frac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\frac{2\pi}{\sqrt{\mu}}$. Hence after a time $\frac{2\pi}{\sqrt{\mu}}$ the particle is again at the same point moving with the same velocity in the same direction. Hence the particle has the period $\frac{2\pi}{\sqrt{\mu}}$.

T = $\frac{2\pi}{\sqrt{\mu}}$; frequency = $\frac{1}{T} = \frac{2\pi}{\sqrt{\mu}}$

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Amplitude = OA = OA' = a.

The periodic time = $\frac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Deductions : 1) Maximum acceleration $= \mu.a = \mu$. (amplitude)

2) Since v = $\sqrt{\mu(a^2 - x^2)}$, the greatest value of v is at x = 0 and its

Maximum velocity = a $\sqrt{\mu}$ = $\sqrt{\mu}$. (amplitude) at the centre

General solution of the S.H.M. equation

The S.H.M. equation is $\frac{d^2x}{d\,t^2} = -\mu\, x$

i.e. $\frac{d^2x}{d\,t^2} + \mu\, x = 0$ …….(1)

(1) is a differential equation of the second order with constant coefficients. Its general solution is of the form

x = A cos $\sqrt{\mu}$ t + B sin $\sqrt{\mu}$ t ……..(2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

x = C cos ($\sqrt{\mu}$ t + ε)…. (3) and x = D sin ($\sqrt{\mu}$ t + α) ………(4)

- If the solution of the S.H.M. equation is x = a cos ($\sqrt{\mu}$ t + ε), the quantity ε is called the **epoch**.

the epoch.

Definition

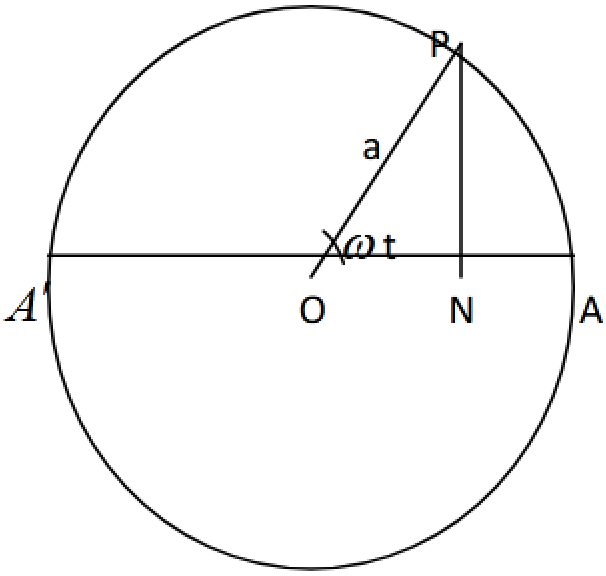
If two simple harmonic motions of the same period can be represented by

$$x_1 = a_1 \cos\left(\sqrt{\mu}\, t + \varepsilon_1\right) \text{and } x_2 = a_2 \cos\left(\sqrt{\mu}\, t + \varepsilon_2\right)$$

- The difference in phase = $\dfrac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$
- If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.
- If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phase.

4.2 Geometrical Representation of S.H.M

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H.M.



Let AA' be the diameter of the circle with centre O and P be the position of the particle at time t secs. Let N be the foot of the perpendicular drawn from P on the diameter AA'. P moves along the circumference of the circle with uniform speed and describes equal arcs in equal times. Let ω – be the angular velocity. $\therefore \angle AOP = \omega t$

If ON = x, Op = a, then, $x = a\cos(\omega t)$ ……………… (1)

$$\frac{dx}{dt} = -a\omega\sin(\omega t) \quad \text{………………..(2)}$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos(\omega t) = -\omega^2 x \quad \text{……………… (3)}$$

(3) shows that the motion of N is simple harmonic. When P moves along the circumference of the circle starting from A, N oscillates from A to A' and A' to A.

Problem 1

A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are x_1, x_2, x_3. Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$

Solution:

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that x = a cos $\sqrt{\mu}$ t ….. (1)

Let x_1, x_2, x_3 be the displacements at three consecutive seconds $t_1,\ t_1+1,\ t_1+2$.

Then $x_1 = a\cos\sqrt{\mu}\, t_1$ …… (2)

$x_2 = a\cos\sqrt{\mu}(t_1+1) = a\cos\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)$ …….(3)

$x_3 = a\cos\sqrt{\mu}(t_1+2) = a\cos\left(\sqrt{\mu}t_1 + 2\sqrt{\mu}\right)$ …….(4)

$$\therefore x_1 + x_3 = a\left[\cos\left(\sqrt{\mu}\, t_1 + 2\sqrt{\mu}\right) + \cos\left(\sqrt{\mu}\, t_1\right)\right]$$

$$= a.2\cos\frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2}\ .\cos\frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$$

$$= 2\,a\cos\left(\sqrt{\mu}\ t_1 + \sqrt{\mu}\right).\cos\sqrt{\mu} = 2x_2\ .\cos\sqrt{\mu}$$

$$\therefore \frac{x_1 + x_3}{2x_2} = \cos\sqrt{\mu}\ , \qquad \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

Problem 2

If the displacement of a moving point at any time be given by an equation of the form $x = a\cos\omega t + b\sin\omega t$, show that the motion is a simple harmonic motion.

If $a = 3$, $b=4$, $\omega = 2$ determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

Given x = a cos ωt + b sin ϖ t ……………… (1)

Differentiating (1) with respect to t,

$$\frac{dx}{dt} = -a\omega \sin \omega t + b\omega \cos \omega t \text{ ……………………(2)}$$

$$\frac{d^2x}{dt^2} = -\omega^2 \cos \omega t - b\,\omega^2 \sin \omega t$$

$$= -\omega^2(a \cos \omega t + b \sin \omega t) = -\omega^2 x \text{ ….(3)}$$

$\therefore$ The motion is simple harmonic.

The constant μ of the S,H.M. = ω^2.

$\therefore$ Period = $\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ secs.

Amplitude is the greatest value of x.

When x is maximum, $\frac{dx}{dt} = 0$.

$-a\omega \sin \omega t + b\omega \cos \omega t = 0$ i.e. a sin ωt = b cos ωt or tan ω t = $\frac{b}{a} = \frac{4}{3}$

When tan ωt = $\frac{4}{3}$, sin ωt = $\frac{4}{5}$ and cos ωt = $\frac{3}{5}$

Greatest value of x = a $\times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a+4b}{5} = \frac{3.3+4.4}{5} = 5$

Hence amplitude = 5.

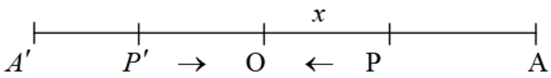
Max. acceleration = μ . Amplitude = 4 x 5 = 20

Max. velocity = $\sqrt{\mu}$. Amplitude = 2 x 5 =10

Problem 3

Show that the energy of a system executing S.H.M. is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration at a distance x from O = μx.

Force = mass $\times$acceleration = m μx

If the particle is given displacement dx from P,

work done against the force = m μ x. dx

Total work done in displacing the particle to a distance x

$$= \int_0^x m\ \mu x dx = m\ \mu \frac{x^2}{2} \qquad \text{.........(1)}$$

Work done = potential energy at P.

If v is the velocity at P. we know that $v^2 = \mu\left(a^2 - x^2\right)$,

$$\therefore \text{ Kinetic energy at P} = \frac{1}{2}\ mv^2 = \frac{1}{2} m\mu\left(a^2 - x^2\right) \qquad \text{.......... (2)}$$

The total energy at P = Potential energy + Kinetic energy

$$= \frac{m\mu x^2}{2} + \frac{m\mu}{2}\left(a^2 - x^2\right) = \frac{m\mu a^2}{2} \text{ .... (3)}$$

Total energy at P α a^2

If n is the frequency, we know that

$$n = \frac{1}{Period} = \frac{1}{\left(\dfrac{2\pi}{\sqrt{\mu}}\right)} = \frac{\sqrt{\mu}}{2\pi}$$

$$\therefore \sqrt{\mu} = 2\pi \text{ n} \quad \text{or} \quad \mu = 4\pi^2 n^2$$

$$\text{Total energy} = \frac{1}{2} m.\ 4\pi^2 n^2 a^2 = 2\pi^2 m a^2 n^2 \ \ \alpha\ n^2$$

Problem 4

A mass of 1 gm. Vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity,

Solution:

$$\text{Maximum velocity} \quad v = \sqrt{\mu}\,.\,a$$

$$\text{Given, frequency} = \frac{1}{T} \quad = 256 = \frac{\sqrt{\mu}}{2\pi}.$$

$$\therefore \sqrt{\mu} \quad = 2 \times 256 \times \pi.$$

$$\text{Given, amplitude} = a = 1 \text{ millimeter} = 1 \times 10^{-1} \text{ c.m.}$$

$$\therefore \text{Maximum velocity, } V = 2 \times 256 \times \pi \times \frac{1}{10} = \frac{256\ \pi}{5} \text{ cm/ sec}$$

Problem 5

In a S.H.M. if f be the acceleration and v the velocity at any time and T is the periodic time. Prove that $f^2T^2 + 4\pi^2 v^2$ is constant.

Solution:

$$\text{Velocity at any time, v} \quad = \sqrt{\mu\left(a^2 - x^2\right)}$$

Periodic time $\quad T = \frac{2\pi}{\sqrt{\mu}}, \ \frac{d^2x}{dt^2} = f.$

$$\text{For, S.H.M,} \ \frac{d^2x}{dt^2} = -\mu.x$$

$$\therefore f = -\mu.x$$

$$\therefore f^2T^2 + 4\pi^2v^2 = \mu^2x^2.\frac{4\pi^2}{\mu} + 4\pi \ \mu^2\left(a^2 - x^2\right)$$

$$= 4\pi^2\mu x^2 + 4\pi^2\mu a^2 - 4\pi^2\mu x^2$$

$$= 4\pi^2\mu a^2 \text{ (constant)}$$

Problem 6

A body moving with simple harmonic motion has an amplitude ‘a’ and period T. Show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$

Solution:

We know, $v^2 = \mu\left(a^2 - x^2\right)$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2}\left(a^2 - x^2\right)$$

$$\therefore v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV –Dynamics of particles– SMT1603

Unit – IV

Dynamics of a particle

Velocity and Acceleration of a particle in polar coordinates – circular, elliptic hyperbolic and parabolic orbits.

Circular Orbit $E = E_{\min}$

The lowest energy state, $E_{\min}$, corresponds to the minimum of the effective potential energy, $E_{\min} = (U_{\text{eff}})_{\min}$. We can minimize the effective potential energy

$$0 = \left. \frac{dU_{\text{eff}}}{dr} \right|_{r=r_0} = -\frac{L^2}{\mu r_0^{\ 3}} + \frac{G m_1 m_2}{r_0^{\ 2}} .$$

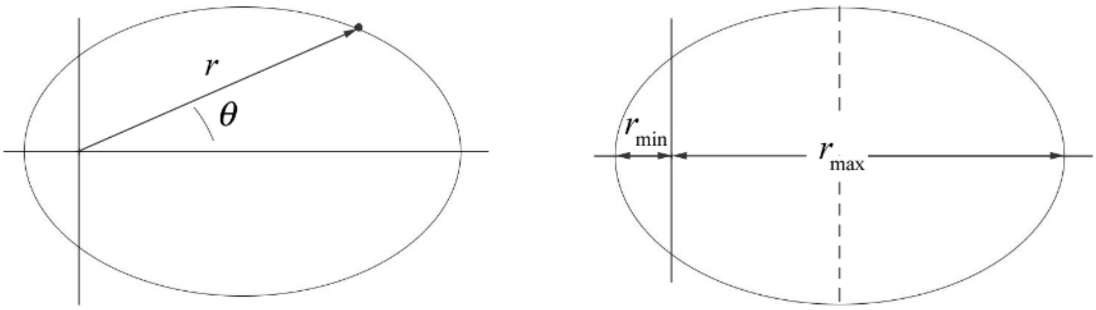
and solve Equation

$$r_0 = \frac{L^2}{G m_1 m_2} ,$$

For $E = E_{\min}$, $r = r_0$ which corresponds to a circular

Elliptic Orbit $E_{\min} < E < 0$

For $E_{\min} < E < 0$, there are two points $r_{\min}$ and $r_{\max}$ such that $E = U_{\mathrm{eff}}(r_{\min}) = U_{\mathrm{eff}}(r_{\max})$. At these points $K_{\mathrm{eff}} = 0$, therefore $dr / dt = 0$ which corresponds to a point of closest or furthest approach (Figure 25.6). This condition corresponds to the minimum and maximum values of r for an elliptic orbit.



(a) (b)

The energy condition at these two points

$$E = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}, \quad r = r_{\min} = r_{\max},$$

is a quadratic equation for the distance r,

$$r^2 + \frac{G m_1 m_2}{E} r - \frac{L^2}{2\mu E} = 0.$$

There are two roots

$$r = -\frac{G m_1 m_2}{2E} \pm \left(\left(\frac{G m_1 m_2}{2E} \right)^2 + \frac{L^2}{2\mu E} \right)^{1/2}.$$

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon) = \frac{r_0}{1 \mp \varepsilon} .$$

Substituting the last expression in (25.4.20) into Equation (25.4.19) gives for the points of closest and furthest approach,

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon) = \frac{r_0}{1 \mp \varepsilon} .$$

The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\text{min}} = \frac{r_0}{1+\varepsilon}$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\text{max}} = \frac{r_0}{1-\varepsilon} .$$

Parabolic Orbit $E = 0$

The effective potential energy, as given in Equation (25.4.1), approaches zero ($U_{\text{eff}} \to 0$) when the distance r approaches infinity ($r \to \infty$). When $E = 0$, as $r \to \infty$, the kinetic energy also approaches zero, $K_{\text{eff}} \to 0$. This corresponds to a parabolic orbit (see Equation (25.3.23)). Recall that in order for a body to escape from a planet, the body must have an energy $E = 0$ (we set $U_{\text{eff}} = 0$ at infinity). This *escape velocity* condition corresponds to a parabolic orbit. For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be found from the condition

$$E = U_{\text{eff}}(r_{\text{par}}) = \frac{L^2}{2\mu r_{\text{par}}{}^2} - \frac{G m_1 m_2}{r_{\text{par}}} = 0 .$$

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0 ;$$

/

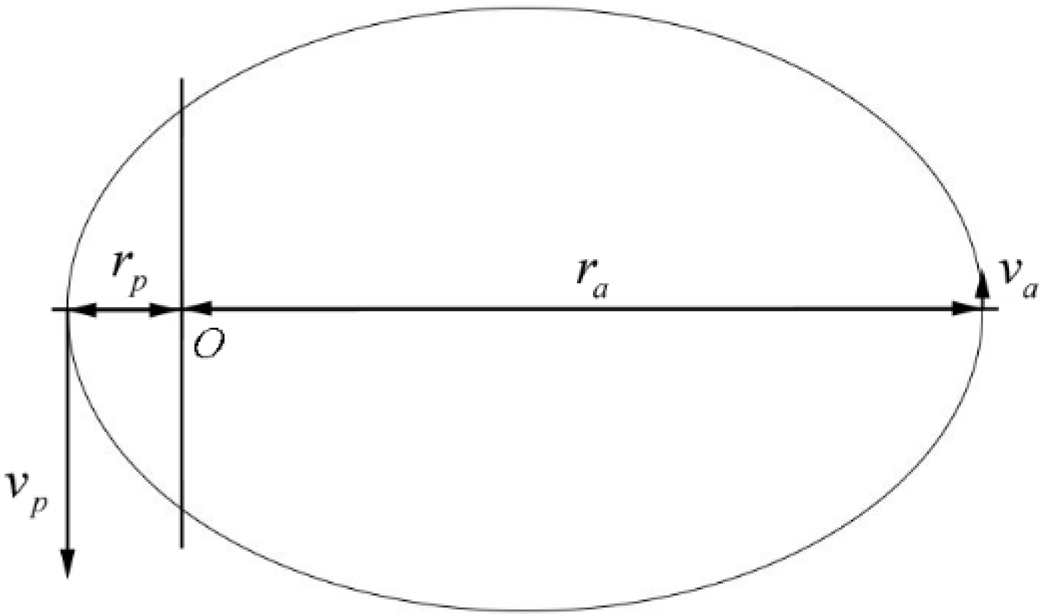
Hyperbolic Orbit $E > 0$

When $E > 0$, in the limit as $r \to \infty$ the kinetic energy is positive, $K_{\text{eff}} > 0$. This corresponds to a hyperbolic orbit (see Equation (25.3.24)). The condition for closest approach is similar to Equation (25.4.14) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (25.4.15), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1+\varepsilon} .$$

The constant r_0 is independent of the energy and from Equation (25.3.14) as the energy of the single body increases, the eccentricity increases, and hence from Equation (25.4.26), the distance of closest approach gets smaller (Figure 25.5).

A satellite of mass m_s is in an elliptical orbit around a planet of mass $m_p >> m_s$. The planet is located at one focus of the ellipse. The satellite is at the distance r_a when it is furthest from the planet. The distance of closest approach is r_p (Figure 25.11). What is (i) the speed v_p of the satellite when it is closest to the planet and (ii) the speed v_a of the satellite when it is furthest from the planet?



Solution: The angular momentum about the origin is constant and because $\vec{\mathbf{r}}_{O,a} \perp \vec{\mathbf{v}}_a$ and $\vec{\mathbf{r}}_{O,p} \perp \vec{\mathbf{v}}_p$, the magnitude of the angular momentums satisfies

$$L \equiv L_{O,p} = L_{O,a} \,. \qquad (25.6.1)$$

Because $m_s << m_p$, the reduced mass $\mu \cong m_s$ and so the angular momentum condition becomes

$$L = m_s r_p v_p = m_s r_a v_a \qquad (25.6.2)$$

We can solve for v_p in terms of the constants G, m_p, r_a and r_p as follows. Choose zero for the gravitational potential energy, $U(r=\infty)=0$. When the satellite is at the maximum distance from the planet, the mechanical energy is

$$E_a = K_a + U_a = \frac{1}{2} m_s v_a^{\ 2} - \frac{G m_s m_p}{r_a} \,. \qquad (25.6.3)$$

When the satellite is at closest approach the energy is

$$E_p = \frac{1}{2} m_s v_p{}^2 - \frac{G m_s m_p}{r_p}.$$

Mechanical energy is constant,

$$E \equiv E_a = E_p \ ,$$

therefore

$$E = \frac{1}{2} m_s v_p{}^2 - \frac{G m_s m_p}{r_p} = \frac{1}{2} m_s v_a{}^2 - \frac{G m_s m_p}{r_a}.$$

From Eq. (25.6.2) we know that

$$v_a = (r_p / r_a) v_p .$$

Substitute Eq. (25.6.7) into Eq. (25.6.6) and divide through by $m_s / 2$ yields

$$v_p{}^2 - \frac{2Gm_p}{r_p} = \frac{r_p{}^2}{r_a{}^2} v_p{}^2 - \frac{2Gm_p}{r_a}.$$

$$v_p{}^2\left(1-\frac{r_p{}^2}{r_a{}^2}\right)=2Gm_p\left(\frac{1}{r_p}-\frac{1}{r_a}\right)\Rightarrow$$

$$v_p{}^2\left(\frac{r_a{}^2-r_p{}^2}{r_a{}^2}\right)=2Gm_p\left(\frac{r_a-r_p}{r_pr_a}\right)\Rightarrow$$

$$v_p{}^2\left(\frac{(r_a-r_p)(r_a+r_p)}{r_a{}^2}\right)=2Gm_p\left(\frac{r_a-r_p}{r_pr_a}\right)\Rightarrow$$

$$v_p=\sqrt{\frac{2Gm_pr_a}{(r_a+r_p)r_p}}.$$

$$v_a=(r_p/r_a)v_p=\sqrt{\frac{2Gm_pr_p}{(r_a+r_p)r_a}}\ .$$

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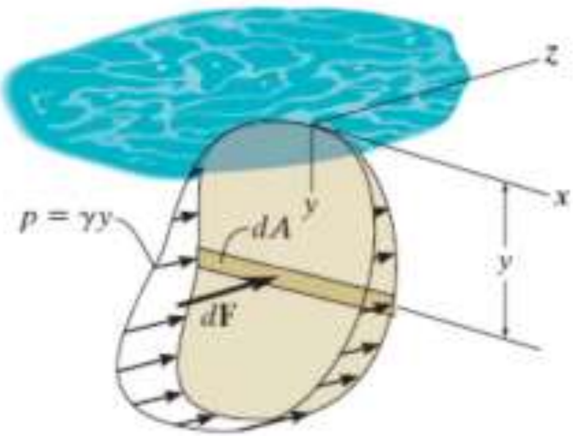
UNIT –V – Moment of Inertia – SMT1603

DEFINITION OF MOMENTS OF INERTIA FOR AREAS

- Centroid for an area is determined by the first moment of an area about an axis
- Second moment of an area is referred as the moment of inertia
- Moment of inertia of an area originates whenever one relates the normal stress σ or force per unit area

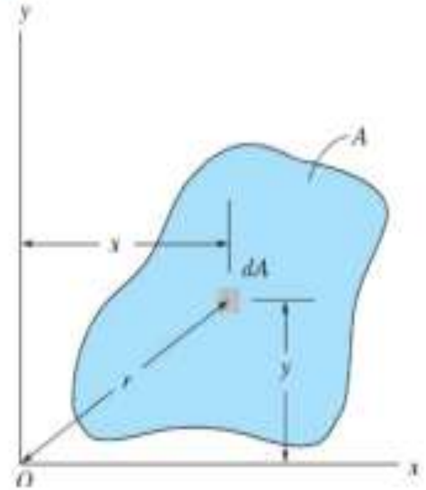


DEFINITION OF MOMENTS OF INERTIA FOR AREAS

Moment of Inertia

- Consider area A lying in the x-y plane
- Be definition, moments of inertia of the differential plane area dA about the x and y axes

$$dI_x = y^2 dA \quad dI_y = x^2 dA$$



- For entire area, moments of inertia are given by

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$

DEFINITION OF MOMENTS OF INERTIA FOR AREAS

Moment of Inertia

- Formulate the second moment of dA about the pole O or z axis
- This is known as the polar axis

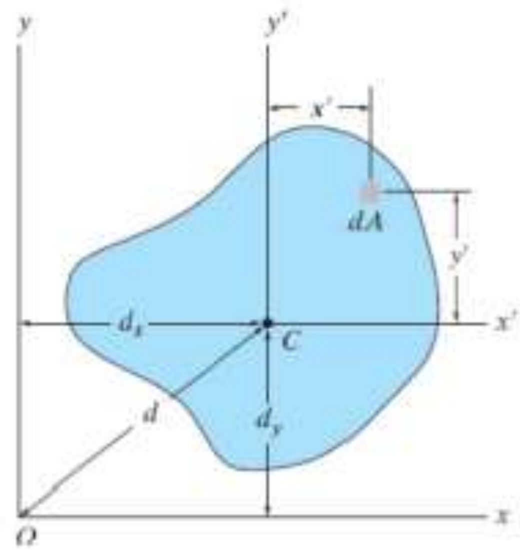
  $$dJ_O = r^2 dA$$

  where r is perpendicular from the pole (z axis) to the element dA
- Polar moment of inertia for entire area,

  $$J_O = \int_A r^2 dA = I_x + I_y$$

PARALLEL AXIS THEOREM FOR AN AREA

- For moment of inertia of an area known about an axis passing through its centroid, determine the moment of inertia of area about a corresponding parallel axis using the parallel axis theorem
- Consider moment of inertia of the shaded area
- A differential element dA is located at an arbitrary distance y′ from the centroidal x′ axis



PARALLEL AXIS THEOREM FOR AN AREA

- The fixed distance between the parallel x and x′ axes is defined as d_y
- For moment of inertia of dA about x axis

$$dI_x = \left(y' + d_y\right)^2 dA$$

- For entire area

$$I_x = \int_A \left(y' + d_y\right)^2 dA$$

$$= \int_A y'^2\, dA + 2d_y \int_A y'\, dA + d_y^2 \int_A dA$$

- First integral represent the moment of inertia of the area about the centroidal axis

PARALLEL AXIS THEOREM FOR AN AREA

- Second integral = 0 since x′ passes through the area′s centroid C $\int y' dA = \bar{y} \int dA = 0; \quad \bar{y} = 0$

- Third integral represents the total area A

$$I_x = \bar{I}_x + Ad_y^2$$

- Similarly

$$I_y = \bar{I}_y + Ad_x^2$$

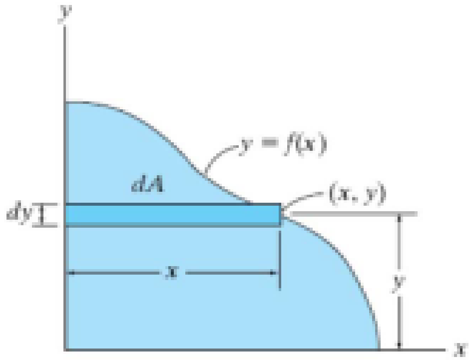
- For polar moment of inertia about an axis perpendicular to the x-y plane and passing through pole O (z axis)

$$J_O = \bar{J}_C + Ad^2$$

RADIUS OF GYRATION OF AN AREA

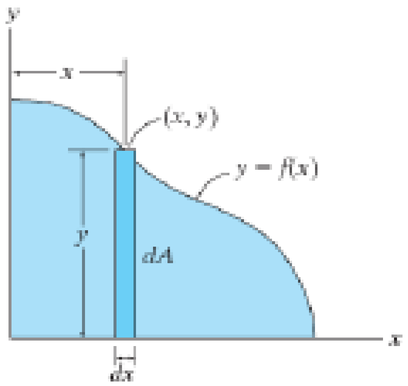
- Radius of gyration of a planar area has units of length and is a quantity used in the design of columns in structural mechanics
- For radii of gyration

$$k_x = \sqrt{\frac{I_x}{A}} \quad k_y = \sqrt{\frac{I_y}{A}} \quad k_z = \sqrt{\frac{J_O}{A}}$$



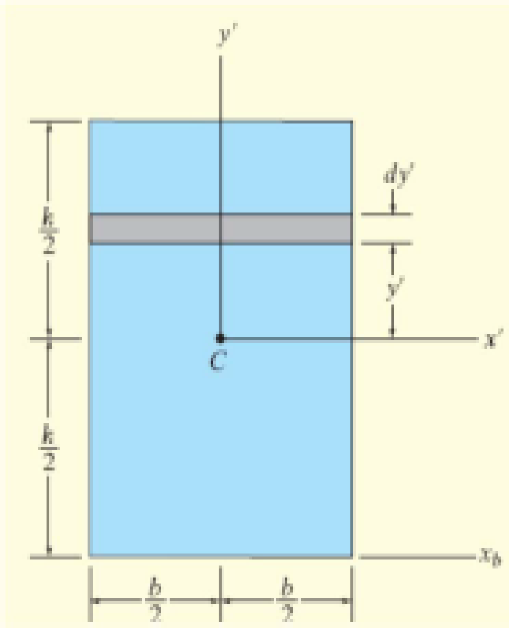
- Similar to finding moment of inertia of a differential area about an axis

$$I_x = k_x^2 A \quad dI_x = y^2 dA$$



EXAMPLE

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangular, and (c) the pole or z' axis perpendicular to the x'-y' plane and passing through the centroid C.



SOLUTION

Part (a)

Differential element chosen, distance y′ from x′ axis.

Since dA = b dy′,

$$\bar{I}_x = \int_A y'^2 \, dA = \int_{-h/2}^{h/2} y'^2 \,(b dy') = \int_{-h/2}^{h/2} y'^2 \, dy = \frac{1}{12} bh^3$$

Part (b)

By applying parallel axis theorem,

$$I_{x_b} = \bar{I}_x + Ad^2 = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3$$

SOLUTION

Part (c)

For polar moment of inertia about point C,

$$\bar{I}_{y'} = \frac{1}{12} hb^3$$

$$J_C = \bar{I}_x + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

MOMENTS OF INERTIA FOR COMPOSITE AREAS

- Composite area consist of a series of connected simpler parts or shapes
- Moment of inertia of the composite area = algebraic sum of the moments of inertia of all its parts

Procedure for Analysis

<u>Composite Parts</u>

- Divide area into its composite parts and indicate the centroid of each part to the reference axis

<u>Parallel Axis Theorem</u>

- Moment of inertia of each part is determined about its centroidal axis

MOMENTS OF INERTIA FOR COMPOSITE AREAS

Procedure for Analysis

Parallel Axis Theorem

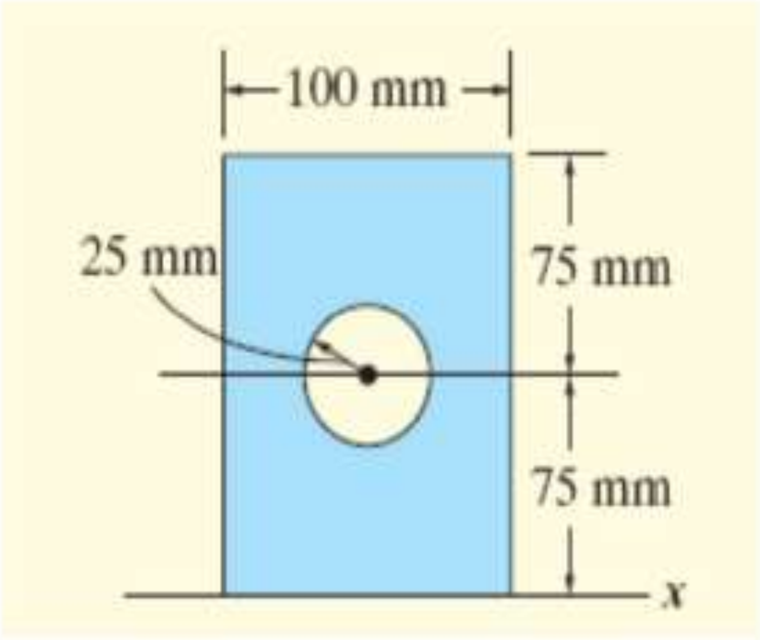
- When centroidal axis does not coincide with the reference axis, the parallel axis theorem is used

Summation

- Moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts

EXAMPLE

Compute the moment of inertia of the composite area about the x axis.

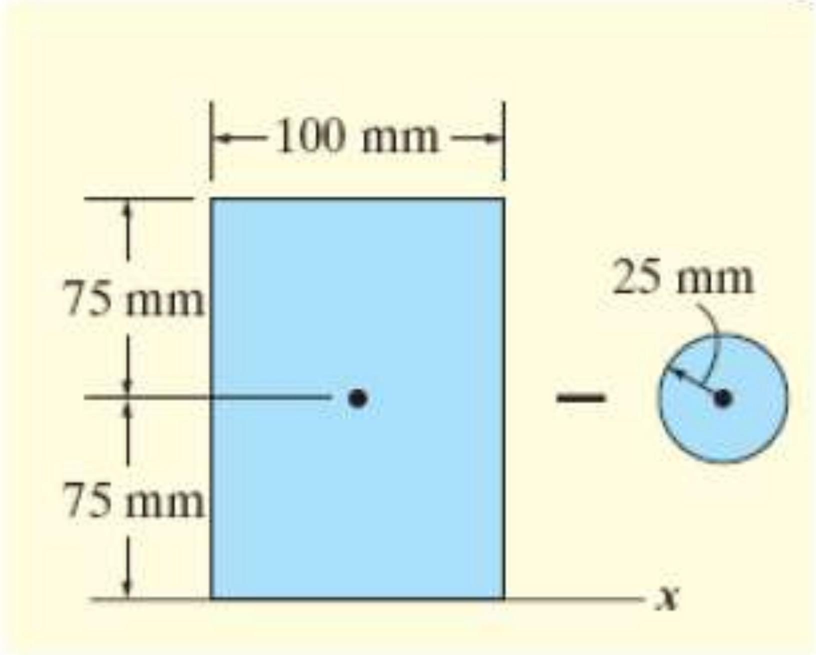


SOLUTION

<u>Composite Parts</u>

Composite area obtained by subtracting the circle form the rectangle.

Centroid of each area is located in the figure below.



SOLUTION

Parallel Axis Theorem

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4\left(10^6\right)mm^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5\left(10^6\right)mm^4$$

SOLUTION

Summation

For moment of inertia for the composite area,

$$I_x = -11.4\left(10^6\right) + 112.5\left(10^6\right)$$
$$= 101\left(10^6\right) mm^4$$

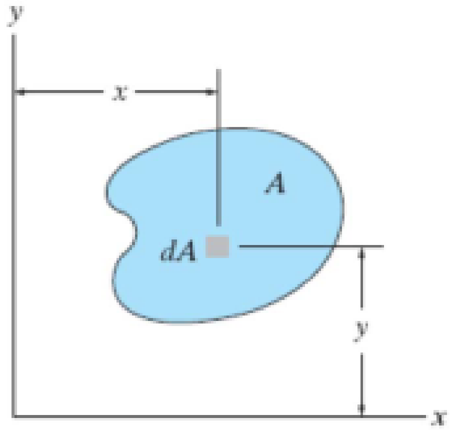
PRODUCT OF INERTIA FOR AN AREA

- Moment of inertia for an area is different for every axis about which it is computed
- First, compute the product of the inertia for the area as well as its moments of inertia for given x, y axes
- Product of inertia for an element of area dA located at a point (x, y) is defined as

$$dI_{xy} = xydA$$

- Thus for product of inertia,

$$I_{xy} = \int_A xydA$$

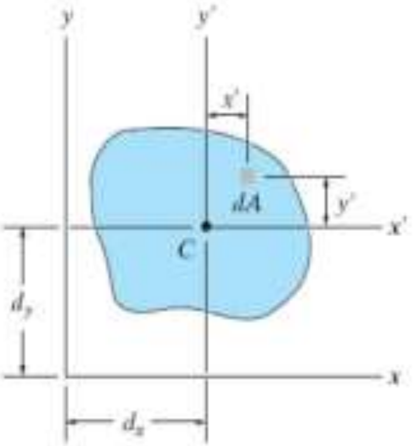


PRODUCT OF INERTIA FOR AN AREA

Parallel Axis Theorem

- For the product of inertia of dA with respect to the x and y axes

$$dI_{xy} = \int_A (x'+d_x)(y'+d_y)\,dA$$



- For the entire area,

$$dI_{xy} = \int_A (x'+d_x)(y'+d_y)\,dA$$

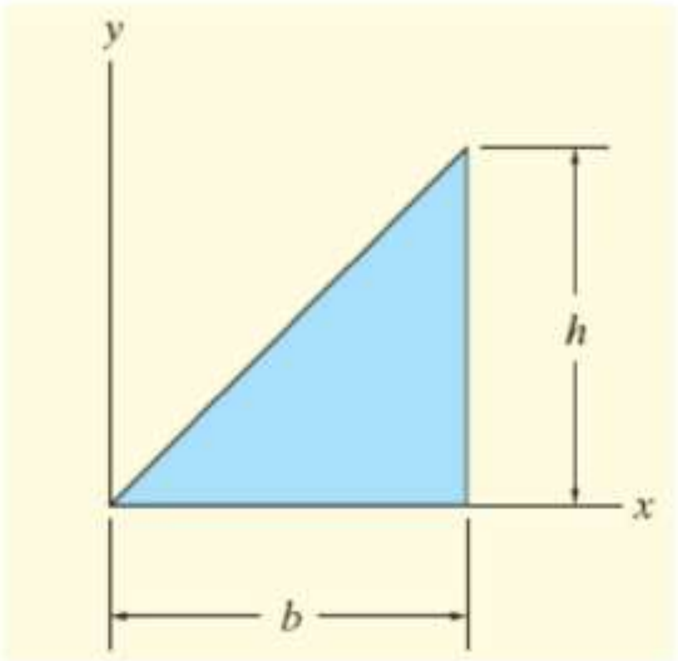
$$= \int_A x'y'\,dA + d_x\int_A y'\,dA + d_y\int_A x'\,dA + d_x d_y\int_A dA$$

- Forth integral represent the total area A,

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$

EXAMPLE

Determine the product I_{xy} of the triangle.



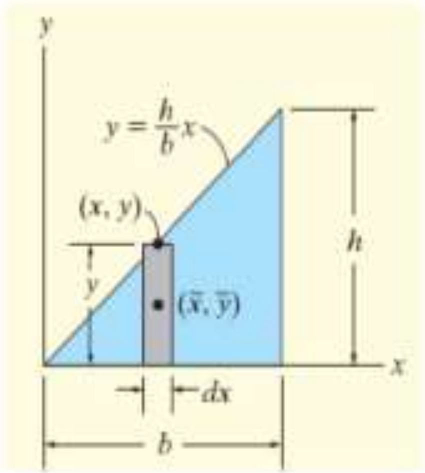
SOLUTION

Differential element has thickness dx and area dA = y dx

Using parallel axis theorem,

$$dI_{xy} = d\bar{I}_{xy} + dA\tilde{x}\tilde{y}$$

$(\tilde{x}, \tilde{y})$ locates centroid of the element or origin of x′, y′ axes



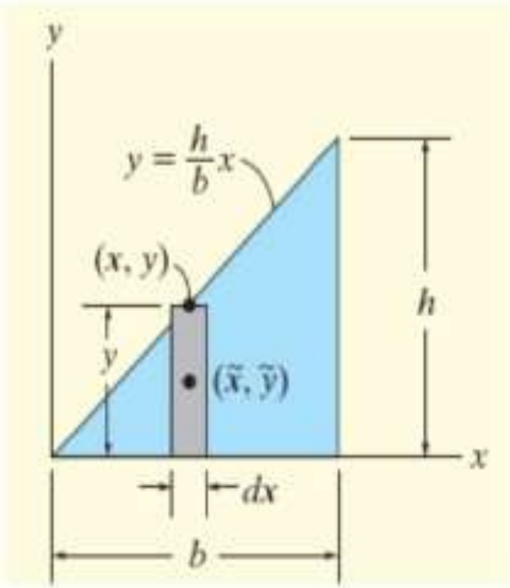
SOLUTION

Due to symmetry, $d\bar{I}_{xy} = 0 \quad \tilde{x} = x, \tilde{y} = y/2$

$$dI_{xy} = 0 + (ydx)x\left(\frac{y}{2}\right) = \left(\frac{h}{b}xdx\right)x\left(\frac{h}{2b}x\right) = \frac{h^2}{2b^2}x^3dx$$

Integrating we have

$$I_{xy} = \frac{h^2}{2b^2}\int_0^b x^3dx = \frac{b^2h^2}{8}$$



SOLUTION

Differential element has thickness dy and area dA = (b - x) dy.

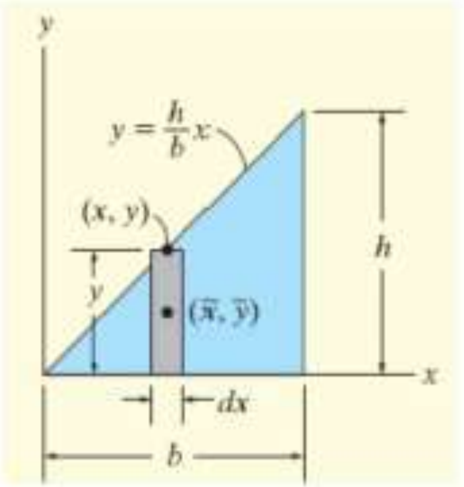
For centroid,

$$\tilde{x} = x + (b - x)/2 = (b + x)/2, \tilde{y} = y$$

For product of inertia of element

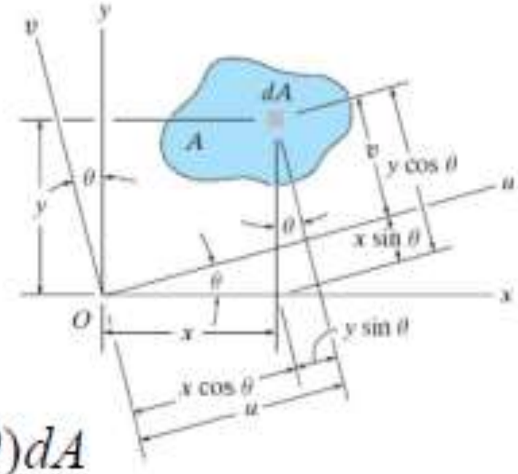
$$dI_{xy} = d\tilde{I}_{xy} + dA\tilde{x}\tilde{y} = 0 + (b - x)dy\left(\frac{b + x}{2}\right)y$$

$$= \left(b - \frac{b}{h}y\right)dy\left[\frac{b + (b/h)y}{2}\right]y = \frac{1}{2}y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy$$



MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

- In structural and mechanical design, necessary to calculate I_u, I_v and I_{uv} for an area with respect to a set of inclined u and v axes when the values of θ, I_x, I_y and I_{xy} are known
- Use transformation equations which relate the x, y and u, v coordinates



$$u = x\cos\theta + y\sin\theta$$

$$v = y\cos\theta - x\sin\theta$$

$$dI_u = v^2 dA = (y\cos\theta - x\sin\theta)^2 dA$$

$$dI_v = u^2 dA = (x\cos\theta + y\sin\theta)^2 dA$$

$$dI_{uv} = uv\,dA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA$$

MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

- Integrating,

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta - 2I_{xy} \sin\theta\cos\theta$$

$$I_v = I_x \sin^2\theta + I_y \cos^2\theta + 2I_{xy} \sin\theta\cos\theta$$

$$I_{uv} = I_x \sin\theta\cos\theta - I_y \sin\theta\cos\theta + 2I_{xy}(\cos^2\theta - \sin^2\theta)$$

- Simplifying using trigonometric identities,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

- We can simplify to

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + 2I_{xy}\cos 2\theta$$

- Polar moment of inertia about the z axis passing through point O is,

$$J_O = I_u + I_v = I_x + I_y$$

MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

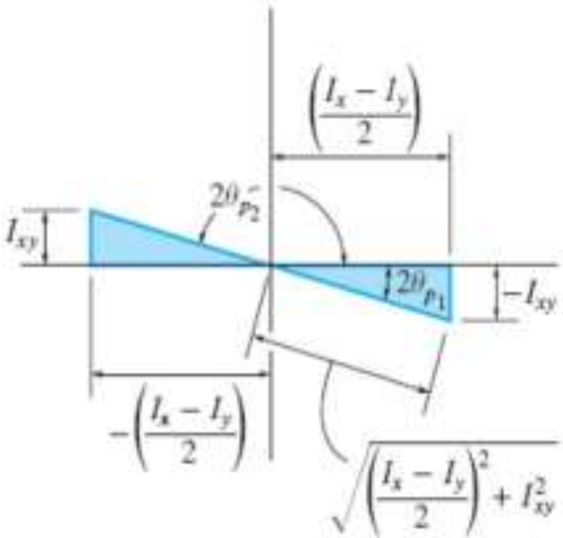
Principal Moments of Inertia

- I_u, I_v and I_{uv} depend on the angle of inclination θ of the u, v axes
- The angle $\theta = \theta_p$ defines the orientation of the principal axes for the area

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

$$\theta = \theta_p$$

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}$$



MOMENTS OF INERTIA FOR AN AREA ABOUT INCLINED AXES

Principal Moments of Inertia

- Substituting each of the sine and cosine ratios, we have

$$I_{\min}^{\max} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- Result can gives the max or min moment of inertia for the area
- It can be shown that $I_{uv} = 0$, that is, the product of inertia with respect to the principal axes is zero
- Any symmetric axis represent a principal axis of inertia for the area