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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COMPLEX ANALYSIS –SMT1602

UNIT – I – Analytic Function – SMT1602

I. Analytic Function

Introduction to Complex Numbers

A general form of a complex number is $z = x + iy$ when x and y are real and $i = \sqrt{-1}$. Here x is called the real part and y is the imaginary part of z.

A conjugate of a complex number z is $\bar{z} = x - iy$. Then

$$z + \bar{z} = 2x \Rightarrow x = \frac{1}{2}[z + \bar{z}]$$

$$z - \bar{z} = 2iy \Rightarrow y = \frac{1}{2i}[z - \bar{z}]$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

The complex number $z = x + iy$ can be represented by a point (x, y) in a complex plane. The modulus (absolute value) of z is given by

$$|z| = \sqrt{x^2 + y^2}$$

The distance between the points z_1 and z_2 is $|z_1 - z_2|$.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the distance

$$\begin{aligned} z_1 z_2 &= |z_1 - z_2| \\ &= |(x_1 + iy_1) - (x_2 + iy_2)\} \\ &= |(x_1 - x_2) + i(y_1 - y_2)| \end{aligned}$$

Polar form of a complex number : Let the polar coordinates of the point (x, y) be (r, θ), then

$$z = x + iy = r[\cos\theta + i\sin\theta] = r e^{i\theta}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

Squaring and adding, we get

$$x^2 + y^2 = r^2$$

$$\therefore r = \sqrt{x^2 + y^2}$$

Dividing the above results, we get

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The number r is called the modulus value of z and θ is called the amplitude (argument) of the complex number z.

Euler's Formula

We know $e^{in\theta} = \cos n\theta + i \sin n\theta$

Demoivre's theorem for positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

Note : $e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$

Functions of a Complex Variable

Let $z = x + iy$ and $\omega = u + iv$. If z and ω are two complex variables and if for each value of z in a complex plane there corresponds one or more values of ω, then ω is called to be a function of z.

We can write $\omega = f(z) = u + iv = u(x, y) + i\,v(x, y)$.

Here u and v are real functions of the real variables x and y.

For example
$$f(z) = z^2$$
$$= (x^2 - y^2) + i\,(2xy)$$

We can represent $z = x + iy$ and $\omega = u + iv$ on separate complex planes called z-plane and ω-plane respectively. The relation $\omega = f(z)$ gives the correspondence between the points (x, y) of the z-plane and the points (u, v) of the ω-plane.

Limits :

$$\text{Let } z = x + iy$$
$$\text{Let } z_0 = x_0 + i\,y_0$$
$$\underset{z \to z_0}{\text{Lt}}\ \omega = \underset{z \to z_0}{\text{Lt}}\ f(z) = \omega_0$$
$$\underset{z \to z_0}{\text{Lt}}\ f(z) = \underset{z \to z_0}{\text{Lt}}\ (u + iv) \qquad [\because f(z) = u + iv]$$
$$= \underset{\substack{x \to x_0 \\ y \to y_0}}{\text{Lt}}\ (u + iv) = u_0 + i\,v_0$$

In symbols, we write

$$\underset{z \to z_0}{\text{Lt}}\ f(z) = l$$

Continuity of f (z) :

A function $f(z)$ is said to be continuous at $z = z_0$ if

$$\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0).$$

If $f(z)$ is continuous in any region R of the z-plane, if it is continuous at every point of that region.

Derivatives of f (z)

Let $\omega = f(z)$ be a single-valued function of the variable z. The derivative of $f(z)$ is defined as

$$\frac{d\omega}{dz} = f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right] \text{ if limits exists.}$$

Partial derivative of u :

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right]$$

$$\frac{\partial u}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} \right]$$

Analytic Functions

A single valued function $f(z)$ which possesses a unique derivative with respect to z at all points of a region R is called an analytic function. It is also called a ***Regular function*** or ***Holomorphic*** function.

Singular Point : A point at which an analytic function $f(z)$ ceases to possess a derivative is called a singular point of the function or singularity of $f(z)$.

The necessary and sufficient conditions for the derivative of the function $f(z)$.

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (C-R equations).

Note :

(i) To check the given function is analytic or not, we can use the CR equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(ii) To find the derivative of $f(z)$, we can use

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(iii) To find $f(z)$ or $f'(z)$ in terms of z, we can substitute $x = z$ and $y = 0$ on both sides.

(iv) Recall the following formulae :

$$\begin{aligned}
\sin(ix) &= i\sinh x \\
\cos(ix) &= \cosh x \\
\sin(0) &= 0, \quad \cos(0) = 1 \\
\sinh(0) &= 0, \quad \cosh(0) = 1 \\
\frac{d}{dx}(\sin x) &= \cos x, \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\sinh x) &= +\cosh x \\
\frac{d}{dx}(\cosh x) &= +\sinh x \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y
\end{aligned}$$

Example 1 ***Prove that $f(z) = z^2$ is an analytic function.***

Solution : Given :

$$\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 + i^2y^2 + 2ixy \\
&= x^2 - y^2 + i2xy
\end{aligned}$$

$$\therefore \quad u = x^2 - y^2 \qquad\qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad\qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad\qquad \frac{\partial v}{\partial x} = 2x$$

$$\text{Here} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C.R. equations are satisfied.

$\therefore$ $f(z)$ is analytic function.

$\boxed{\textit{Example 2}}$ ***Test the analyticity of $f(z) = e^z$.***

Solution : Given : $e^z = e^{x+iy}$

$$= e^x e^{iy}$$

$$= e^x [\cos y + i \sin y]$$

$$= e^x \cos y + i e^x \sin y$$

Here $u = e^x \cos y \qquad\qquad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad\qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad\qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z) = e^z$ is analytic function.

Example 3 ***Test whether the function $f(z) = \cos z$ is analytic or not.***

Solution : Given :

$$\begin{aligned} f(z) &= \cos z \\ &= \cos(x+iy) \\ &= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\ &= \cos(x)\cosh y - \sin(x)\, i \sinh y \\ &= \cos x \cosh y + i(-\sin x \sinh y) \end{aligned}$$

Here

$u = \cos x \cosh y$	$v = -\sin x \sinh y$
$\dfrac{\partial u}{\partial x} = -\sin x \cosh y$	$\dfrac{\partial v}{\partial x} = -\cos x \sinh y$
$\dfrac{\partial u}{\partial y} = \cos x \sinh y$	$\dfrac{\partial v}{\partial y} = -\sin x \cosh y$

Here $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\therefore$ $f(z) = \cos z$ is analytic function.

Example 4 ***Discuss the analyticity of $f(z) = \log z$.***

Solution : We know $\log z = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\dfrac{y}{x}\right)$

[$\boldsymbol{\log z = \log(re^{i\theta})}$

$$\begin{aligned} &= \log r + \log(e^{i\theta}) \\ &= \log\sqrt{x^2+y^2} + i\theta \\ &= \log(x^2+y^2)^{1/2} + i\tan^{-1}\left(\frac{y}{x}\right) \\ &= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$]

$$u = \frac{1}{2}\log(x^2+y^2). \qquad v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \qquad = -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The partial derivatives are continuous except at $x = 0$, $y = 0$. CR equations are satisfied.

Its derivative is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{x-iy}{x^2+y^2} = \frac{(x-iy)}{(x-iy)(x+iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z}$$

Hence $f(z) = \log z$ is analytic everywhere except at $z = 0$, (at the origin).

Example 5 ***Prove that $f(z) = \sin z$ is analytic function and hence find the derivative.***

Solution : Given : $f(z) = \sin z = \sin(x+iy)$

$$= \sin(x+iy)$$

$$= \sin(x)\cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$ $\qquad\qquad$ $v = \cos x \sin\text{h}\, y$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \Bigg| \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \Bigg| \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$$

Here CR equations are satisfied.

Consider $\quad f'(z) = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}$

$$f'(z) = \cos x \cosh y + i(-\sin x \sinh y)$$

To find $f'(z)$ in terms of z, let us substitute $x = z$ and $y = 0$ on both sides,

$$f'(z) = \cos z \cdot 1 + i(-\sin z \cdot 0)$$

$$f'(z) = \cos z$$

Note : Here after we can use this method to find $f(z)$ or $f'(z)$ by substituting $x = z$ and $y = 0$.

$\boxed{\textit{Example 7}}$ ***Show that $f(z) = |z|^2$ is differentiable only at the origin.***

Solution : Given : $\quad f(z) = |z|^2$

$$= x^2 + y^2 \qquad [\because |z|^2 = z\bar{z} = x^2 + y^2]$$

$$\therefore u = x^2 + y^2, \qquad v = 0$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

Here CR equations are satisfied only when $x = 0$ and $y = 0$.

Note that CR equations are not satisfied for other values. Thus $f(z) = |z|^2$ is differentiable only at the origin.

Milne-Thomson Method to find $f(z)$

This method can be used to find an analytic function $f(z)$ when u or v is given.

Let us assume that the real part of $f(z)$ is given. Then we can find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

$$\begin{aligned} \text{Consider } f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right), \text{ using CR equations.} \\ &= \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} \end{aligned}$$

Put $x = z$ and $y = 0$ on both sides, we get

$$f'(z) = \frac{\partial}{\partial x}u(z,0) - i\,\frac{\partial u(z,0)}{\partial y} \qquad \ldots (1)$$

which is a function of z.

Integrating (1), we get $f(z)$ in terms of z.

Note : If the imaginary part of $f(z)$ is given, we can find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

For this consider

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x}, \text{ using CR equations.} \end{aligned}$$

Put $x = z$ and $y = 0$ on both sides, we get

$$f'(z) = \frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x}$$

Integrating (2), we get $f(z)$ in terms of z. This method is called Milne-Thomson method.

Method of find $f(z)$ when u is given

$\boxed{\textit{Example 1}}$ ***Find an analytic function $f(z)$ whose real part is given by $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.***

Solution : Given :

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial u}{\partial y} = 0 - 6xy + 0 - 6y + 0$$

$$= -6xy - 6y$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

Here u is given and using CR equations

$$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= [3x^2 - 3y^2 + 6x] + i[6xy + 6y]$$

Put $x = z$ and $y = 0$ on both sides

$$f'(z) = 3z^2 + 6z$$

Integrating, we get

$$f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + C$$

$$f(z) = z^3 + 3z^2 + C, \quad \text{C is a complex constant.}$$

Example 2 ***Find an analytic function $f(z)$ whose real part is given as $u = y + e^x \cos y$.***

Solution : Given :
$$u = y + e^x \cos y$$
$$\frac{\partial u}{\partial x} = e^x \cos y$$
$$\frac{\partial u}{\partial y} = 1 - e^x \sin y$$

Consider
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \left(\frac{\partial u}{\partial x}\right) + i\left(-\frac{\partial u}{\partial y}\right)$$
$$= e^x \cos y + i(-1 + e^x \sin y)$$

Put $x = z$ and $y = 0$ on both sides,

Put $x = z$ and $y = 0$ on both sides,

$$f'(z) = e^z - i$$

Integrating, we get
$$f(z) = e^z - iz + C$$

Example 3 ***Find an analytic function whose real part is given by $u = \frac{x}{x^2 + y^2}$.***

Solution : Given :
$$u = \frac{x}{x^2 + y^2}$$
$$u_x = \frac{(x^2 + y^2)\,1 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$u_y = \frac{0 - x\,(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

Let $f(z) = u + iv$

$$f'(z) = u_x + i\,v_x$$
$$= u_x - i\,u_y$$
$$= \frac{y^2 - x^2}{(x^2+y^2)^2} - i\,\frac{2\,xy}{(x^2+y^2)^2}$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\frac{z^2}{z^4} = -\frac{1}{z^2}$$

Integrating, we get $\quad f(z) = \frac{1}{z} + \mathrm{C}$

$\boxed{\textit{Example 4}}$ ***Find f(z) which is analytic, given***

$$\boldsymbol{u = \frac{1}{2}\,\log\,(x^2+y^2).}$$

Solution : Given : $\quad u = \frac{1}{2}\,\log\,(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{2\,x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{2\,y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= \frac{x}{x^2+y^2} + i\left(-\frac{y}{x^2+y^2}\right)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = \frac{z}{z^2} + i(0) = \frac{1}{z}$$

Integrating, we get $\quad f(z) = \log z + \mathrm{C}$

$\boxed{\textit{Example 5}}$ ***If*** $u = \dfrac{y}{x^2 + y^2}$ ***find an analytic function*** $f(z)$.

Solution : Given : $u = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial u}{\partial x} = \frac{0 - y\,(2\,x)}{(x^2 + y^2)^2} = \frac{-2\,xy}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2 + y^2)\,1 - y\,(2\,y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Consider
$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= \left[\frac{-2\,xy}{(x^2 + y^2)^2}\right] + i\left[\frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

Put $x = z,\ y = 0$, we get

$$f'(z) = i\left[\frac{-z^2}{z^4}\right] = i\left(-\frac{1}{z^2}\right)$$

Integrating, we get

$$f(z) = i \cdot \frac{1}{z} + C$$

$$= \frac{i}{z} + C \text{ where } C \text{ is complex constant}$$

Example 6 ***Find an analytic function $f(z) = u + iv$ if u is given by $u = \cos x \cosh y$.***

Solution : Given :

$$u = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y$$

Consider

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= -\sin x \cosh y + i\,(-\cos x \sinh y)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\sin z + 0$$

Integrating, we get

$$f(z) = \cos z + C$$

Example 7 ***Find an analytic function $f(z)$ whose real part is given by $u = e^{2x}\,[\,x \cos 2y - y \sin 2y\,]$.***

Solution : Given : $u = e^{2x}\,x \cos 2y - e^{2x}\,y \sin 2y$

$$\frac{\partial u}{\partial x} = [\,e^{2x} + 2\,x\,e^{2x}\,] \cos 2y - 2\,e^{2x}\,y \sin 2y$$

$$= e^{2x}\,[\,\cos 2y + 2\,x \cos 2y - 2\,y \sin 2y\,]$$

$$\frac{\partial u}{\partial y} = -2\,e^{2x}\,x \sin 2y - e^{2x}\,[\,\sin 2y + 2\,y \cos 2y\,]$$

$$= -e^{2x}\,[\,2\,x \sin 2y + \sin 2y + 2\,y \cos 2y\,]$$

$$\text{Consider } f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= e^{2x}\,[\,\cos 2y + 2x\cos 2y - 2y\sin 2y\,]$$

$$+\, i\,[\,e^{2x}\,(2x\sin 2y + \sin 2y + 2y\cos 2y)\,]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = e^{2z}\,[\,1 + 2z\,] + 0$$

Integrating, we get

$$f(z) = \int (2z+1)\,e^{2z}\,dz + \text{C}$$

For using Bernouli's formula

Put	$u = 2z+1$		$v = e^{2z}$
	$u' = 2$		$v_1 = \dfrac{e^{2z}}{2}$
	$u'' = 0$		$v_2 = \dfrac{e^{2z}}{4}$

$$\int uv\,dx = uv_1 - u'\,v_2 + u''\,v_3 - \ldots\ldots\ldots$$

$$f(z) = (2z+1)\,\frac{e^{2z}}{2} - 2\,\frac{e^{2z}}{4} + \text{C}$$

$$= z\,e^{2z} + \frac{1}{2}\,e^{2z} - \frac{1}{2}\,e^{2z} + \text{C}$$

$$f(z) = z\,e^{2z} + \text{C}$$

Example 8 ***Find the analytic function $f(z) = u + iv$ if***

$$u = e^{-x} [(x^2 - y^2) \cos y + 2xy \sin y].$$

Solution :

$$u_x = e^{-x} [2x \cos y + 2y \sin y] - e^{-x} [(x^2 - y^2) \cos y + 2xy \sin y].$$

$$u_y = e^{-x} [-2y \cos y - y^2 \sin y + 2x (y \cos y + \sin y)]$$

At $x = z,\ y = 0$,

$$u_x = e^{-z} [2z] - e^{-z} [(z^2)] = e^{-z} [2z - z^2]$$

$$u_y = e^{-z} [0]$$

$$\therefore F'(z) = u_x + i v_x$$

$$= u_x + i (-u_y)$$

$$F'(z) = e^{-z} [2z - z^2]$$

$$F(z) = \int (2z - z^2) e^{-z}\, dz + C$$

Using Bernouli's formula, we get

$u = 2z - z^2$	$v = e^{-z}$
$u' = 2 - 2z$	$v_1 = -e^{-z}$
$u'' = -2$	$v_2 = e^{-z}$
$u''' = 0$	$v_3 = -e^{-z}$

$$\therefore \int uv\, dx = u v_1 - u' v_2 + u'' v_3 - \ldots\ldots\ldots$$

$$\therefore F(z) = -(2z - z^2) e^{-z} - (2 - 2z) e^{-z} + 2 (e^{-z}) + C$$

$$= e^{-z}[-2z + z^2 - 2 + 2z + 2] + C$$

$$F(z) = z^2 e^{-z} + C$$

$\boxed{\textit{Example 9}}$ ***An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the complex potential function.***

Solution : Let $F(z) = \phi + i\psi$

$$\text{Given } \phi = 3x^2y - y^3$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy, \quad \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2$$

$$\text{Consider } F'(z) = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

$$= \frac{\partial\phi}{\partial x} + i\left(-\frac{\partial\phi}{\partial y}\right)$$

$$= 6xy - i(3x^2 - 3y^2)$$

Put $x = z,\ y = 0$, we get

$$F'(z) = -i\,3z^2$$

Integrating, we get $\quad F(z) = -iz^3 + C$

Note : If we take $F(z) = \phi + i\psi$ and it is analytic then the CR equations are

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$\boxed{\textit{Example 10}}$ ***Find an analytic function $f(z) = u + iv$, whose real part is given by $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.***

Solution : Let $f(z) = u + iv$, and $u_x = v_y,\ u_y = -v_x$

Given :
$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} \quad [\because \cos^2 2x + \sin^2 2x = 1]$$

$$u_y = \frac{0 - 2\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Consider
$$f'(z) = u_x + i\,v_x$$

$$= u_x - i\,u_y$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Put $x = z, y = 0$, we get

$$f'(z) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{(1 - \cos 2z)} = -\frac{1}{\sin^2 2z}$$

$$f'(z) = -\operatorname{cosec}^2 2z$$

Integrating, we get

$$f(z) = \cot z + C$$

Note : In the same way we can find $f(z)$, where

$$u = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x} \text{ is given.}$$

Method of Finding F(*z*) = *u* + *iv* when *v* is given

Example 1 ***Find an analytic function $f(z)$ where $v = 2xy$.***

Solution : Given : $v = 2xy$

$$\frac{\partial v}{\partial x} = 2y \text{ and } \frac{\partial v}{\partial y} = 2x$$

We know $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ [Here u is not given]

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = 2z$$

Integrating, we get

$$f(z) = z^2 + C$$

$\boxed{\textit{Example 2}}$ ***Find an analytic function f (z) whose imaginary part is given by $v = e^x \sin y$.***

Solution : Given :

$$v = e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \text{ and } \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \text{ Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i\, e^x \sin y$$

$$= e^x [\cos y + i \sin y]$$

Put $x = z$ and $y = 0$ on both sides,

$$f'(z) = e^z$$

Integrating, we get $\quad f(z) = e^z + C$

$\boxed{\textit{Example 3}}$ ***If $v = -\sin x \sinh y$, find a function foz which is regular.***

Solution : Given :

$$v = -\sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y, \; \frac{\partial v}{\partial y} = -\sin x \cosh y$$

$$\text{Consider } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= (-\sin x \cosh y) + i(-\cos x \sin \text{h}\, y)$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = -\sin z$$

Integrating, we get $\quad f(z) = \cos z + C$

$\boxed{\textit{Example 4}}$ ***Find an analytic function $f(z)$ whose imaginary part is $v = x^3 - 3xy^2 + 2x + 1$.***

Solution : Given :

$$v = x^3 - 3xy^2 + 2x + 1$$

$$v_x = 3x^2 - 3y^2 + 2$$

$$v_y = -6xy$$

$$v_x(z, 0) = 3z^2 + 2$$

$$v_y(z, 0) = 0$$

$$\text{Consider } F'(z) = u_x + i v_x$$

$$= v_y + i v_x$$

Putting $x = z, y = 0$, we get

$$F'(z) = v_y(z, 0) + i v_x(z, 0)$$

$$= 0 + i(3z^2 + 2)$$

Integrating, we get $\quad F(z) = i \int (3z^2 + 2)\, dz + C$

$$= i[z^3 + 2z] + C$$

$\boxed{\textit{Example 5}}$ ***If $u = \dfrac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$, then find the corresponding analytic function $f(z)$.***

[***Ans :*** $f(z) = \sec z + C$]

$\boxed{\textit{Example 6}}$ ***Find a regular f (z) whose imaginary part is given $v = e^{-x}[x \cos y + y \sin y]$.***

Solution : Given :

$$v = e^{-x}[x \cos y + y \sin y]$$

$$v_x = e^{-x}[\cos y] - e^{-x}[x \cos y + y \sin y]$$

$$= e^{-x}[\cos y - x \cos y - y \sin y]$$

$$v_y = e^{-x}[-x \sin y + y \cdot \cos y + \sin y]$$

Consider $F'(z) = u_x + i\, v_x$

$$= v_y + i\, v_x$$

At $x = z,\ y = 0$, we get

$$F'(z) = v_y(z, 0) + i\, v_x(z, 0)$$

$$= 0 + i\, e^{-z}[1 - z]$$

Integrating, we get

$$F(z) = i \int (1 - z)\, e^{-z}\, dz + C$$

$$= i[-(1 - z)\, e^{-z} - (-1)\, e^{-z}] + C$$

$$= i[-1\, e^{-z} + z\, e^{-z} + e^{-z}] + C$$

$$F(z) = i[z\, e^{-z}] + C$$

$\boxed{\textit{Example 7}}$ ***Find the regular function f(z) whose imaginary part is given by*** $v = e^{-x}[x \sin y - y \cos y]$.

Solution : Given :

$$v = e^{-x}[x \sin y - y \cos y]$$

$$\frac{\partial v}{\partial x} = e^{-x}[1 \cdot \sin y] - e^{-x}[x \sin y - y \cos y]$$

$$= e^{-x}[\sin y - x \sin y + y \cos y]$$

$$\frac{\partial v}{\partial y} = e^{-x}[x \cos y - \cos y + y \sin y]$$

Consider

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= e^{-x}[x \cos y - \cos y - y \sin y] + i\,e^{-x}[\sin y - x \sin y + y \cos y]$$

Put $x = z$ and $y = 0$, we get

$$f'(z) = e^{-z}[z-1] + i\,e^{-z}[0]$$

$$= (z-1)\,e^{-z}$$

Integrating, we get

$$f(z) = \int (z-1)\,e^{-z}\,dz$$

$$= -(z-1)\,e^{-z} - e^{-z} \cdot 1 + C$$

$$= -z\,e^{-z} + e^{-z} - e^{-z} + C$$

$$f(z) = -z\,e^{-z} + C$$

$$\boxed{\begin{array}{lll} u = z-1, & v = e^{-z} \\ u' = 1, & v_1 = -e^{-z} \\ u'' = 0, & v_2 = e^{-z} \end{array}}$$

Example 8 ***Find the analytic function whose imaginary part is*** $e^{x^2-y^2}$ ***sin (2 xy).***

Solution :

Given : $v = e^{x^2-y^2} \sin(2xy)$

$$\frac{\partial v}{\partial x} = e^{x^2-y^2}(2x)\sin(2xy) + e^{x^2-y^2}\cos 2xy\,(2y)$$

$$\frac{\partial v}{\partial y} = e^{x^2-y^2}(-2y)\sin 2xy + e^{x^2-y^2}\cos(2xy)(2x)$$

We know $f(z) = u + iv$

$$f'(z) = u_x + i\,v_x$$

$$= v_y + i\,v_x \qquad [\because u_x = v_y]$$

$$= 2\,e^{x^2-y^2}[-y\sin 2xy + x\cos 2xy]$$

$+ i\,2\,e^{x^2-y^2}[x\sin 2xy + y\cos 2xy]$

Put $x = z$ and $y = 0$,

$$f'(z) = 2\,e^{z^2}[0+z] + i\,2\,e^{z^2}[0]$$

$$f'(z) = 2z\,e^{z^2}$$

Integrating $f(z) = \int 2z\,e^{z^2}\,dz + C$

Put $z^2 = t$, $\therefore\ 2z\,dz = dt$

$$\therefore f(z) = \int e^t\,dt + C$$

$$\therefore f(z) = e^t + C$$

$$f(z) = e^{z^2} + C$$

Example 9 ***Construct the analytic function whose imaginary part is $e^{-x}[x\cos y + y\sin y]$ and which equals 1 at the origin.***

Solution : Given :.

$$v = e^{-x}[x\cos y + y\sin y]$$

$$v_x = e^{-x}[1\cdot\cos y + 0] - e^{-x}[x\cos y + y\sin y]$$

$$v_y = e^{-x}[-x\sin y + 1\cdot\sin y + y\cos y]$$

Consider

$$\begin{aligned} F'(z) &= u_x + i\,v_x \\ &= v_y + i\,v_x \\ &= e^{-x}[-x\sin y + \sin y + y\cos y] \\ &\qquad + i\,e^{-x}[\cos y - x\cos y - y\sin y] \end{aligned}$$

Put $x = z$ and $y = 0$, we get

$$F'(z) = e^{-z}[0] + i\,e^{-z}[1 - z]$$

Integrating, we get $\quad F(z) = i\displaystyle\int (1-z)\,e^{-z}\,dz + C$

Using integration by parts, we get

$$u = 1 - z, \qquad dv = e^{-z}\,dz$$

$$du = -\,dz, \qquad v = -e^{-z}$$

$$\begin{aligned} F(z) &= i\left[-(1-z)\,e^{-z} - \int -e^{-z}(-\,dz)\right] + C \\ &= i\,[-(1-z)\,e^{-z} + e^{-z}] + C \\ F(z) &= i\,z\,e^{-z} + C \end{aligned}$$

Given $F(0) = 1 \Rightarrow C = 1$

$$\therefore\ f(z) = i\,z\,e^{-z} + 1$$

Example 10 ***If $v = e^x\,[x \sin y + y \cos y]$ is an imaginary part of an analytic function $f(z)$, find $f(z)$ in terms of z.***

Solution : Given :

$$v = e^x (x \sin y + y \cos y)$$

$$v_x = e^x (x \sin y + y \cos y) + e^x (\sin y)$$

$$= e^x (x \sin y + y \cos y + \sin y)$$

$$v_y = e^x (x \cos y + \cos y - y \sin y)$$

Consider

$$f'(z) = u_x + i\, v_x$$

$$= v_y + i\, v_x$$

$$= e^x (x \cos y + \cos y - y \sin y) + i\, e^x (x \sin y + y \cos y + \sin y)$$

Put $x = z,\ y = 0$ on both sides,

$$f'(z) = e^z (z + 1)$$

Integrating, we get

$$f(z) = \int (z + 1)\, e^z\, dz$$

$$= (z + 1)\, e^z - e^z + C$$

$$f(z) = z\, e^z + C$$

Method of finding $f(z)$ when $u - v$ is given

Let $f(z) = u + iv$ and is an analytic function.

$$f(z) = u + iv \qquad \ldots (1)$$

$$i\, f(z) = iu - v \qquad \ldots (2)$$

Adding (i) and (ii), we get

$$(1 + i)\, f(z) = (u - v) + i\,(u + v) \qquad \ldots (3)$$

Let $U = u - v$, $V = u + v$ and $F(z) = (1 + i)\, f(z)$.

Then (iii) becomes,

$$F(z) = U + iV \qquad \dots (4)$$

If $u - v$ is given in the problem, then

(a) Substitute $u - v = U$. (Now U is known)

(b) Find $F(z)$ by usual method.

(c) Equate $F(z) = (1+i) f(z)$

$$\therefore f(z) = \frac{1}{1+i} F(z)$$

This is a procedure to find $f(z)$ if $u - v$ is given.

Note : If $u + v$ is given in the problem, we can use the similar ethod as above.

Let
$$f(z) = u + iv \qquad \dots (1)$$
$$i f(z) = iu - v \qquad \dots (2)$$

Adding (1) and (2),

$$(1+i) f(z) = (u - v) + i(u + v)$$
$$i.e., \quad F(z) = U + iV$$

Here $u + v$ is given. Then

(1) Substitute $u + v = V$ [V is known]

(2) Find $F(z)$ as usual method.

(3) Equate $F(z) = (1+i) f(z)$

$$\therefore f(z) = \frac{1}{1+i} F(z)$$

Note : If $F(z) = U + iV$ is analytic, then CR equations are

$$U_x = V_y$$
$$U_y = -V_x$$

Example 11 ***If $u - v = e^x [\cos y - \sin y]$, find the corresponding analytic function $f(z) = u + iv$.***

Solution : Consider
$$f(z) = u + iv \quad \ldots \text{(i)}$$
$$i f(z) = iu - v \quad \ldots \text{(ii)}$$

Adding (i) and (ii),
$$(1+i) f(z) = (u - v) + i(u + v)$$
$$\text{i.e., } F(z) = U + iV$$

Here
$$U = u - v = e^x [\cos y - \sin y] \text{ is given}$$
$$U_x = e^x [\cos y - \sin y]$$
$$U_y = e^x [-\sin y - \cos y]$$

Consider
$$F'(z) = U_x + iV_x$$
$$= U_x + i(-U_y)$$
$$= e^x [\cos y - \sin y] + i e^x [\sin y + \cos y]$$

Put $x = z$, $y = 0$, we get
$$F'(z) = e^z + i e^z$$
$$= (1+i) e^z$$

Integrating, we get
$$F(z) = (1+i) e^z + C$$
$$\text{i.e., } (1+i) f(z) = (1+i) e^z + C$$
$$f(z) = e^z + C_1$$

Example 12 ***Find an analytic function $f(z)$ if given $u + v = x^2 - y^2 + 2xy$.***

Solution : Consider

$$f(z) = u + iv \qquad \dots \text{(i)}$$

$$i\,f(z) = iu - v \qquad \dots \text{(ii)}$$

Adding

$$(1+i)\,f(z) = (u-v) + i\,(u+v) \qquad \dots \text{(iii)}$$

(iii) can be written as $\mathrm{F}(z) = \mathrm{U} + i\,\mathrm{V}$

where $u - v = \mathrm{U}$, $u + v = \mathrm{V}$, $(1+i)\,f(z) = \mathrm{F}(z)$.

Given

$$\mathrm{V} = u + v = x^2 - y^2 + 2\,xy$$

$$\mathrm{V}_x = 2\,x + 2\,y$$

$$\mathrm{V}_y = -2\,y + 2\,x$$

Consider

$$\begin{aligned} \mathrm{F}'(z) &= \mathrm{U}_x + i\,\mathrm{V}_x \\ &= \mathrm{V}_y + i\,\mathrm{V}_x \\ &= (-2\,y + 2\,x) + i\,(2\,x + 2\,y) \end{aligned}$$

Put $x = z$, $y = 0$ on both sides,

$$\begin{aligned} \mathrm{F}'(z) &= 2\,z + i\,2\,z \\ &= 2\,(1+i)\,z \end{aligned}$$

Integrating

$$\mathrm{F}(z) = (1+i)\,z^2 + c$$

i.e.,

$$(1+i)\,f(z) = (1+i)\,z^2 + c$$

$$\therefore f(z) = z^2 + \frac{c}{1+i}$$

$$f(z) = z^2 + c_1$$

Example 13 ***Find an analytic function***

$$f(z) = u + iv \text{ if } u - v = (x - y)\,(x^2 + 4\,xy + y^2).$$

Solution : Consider

$$\begin{aligned} f(z) &= u + iv \\ i\,f(z) &= iu - v \\ (1+i)\,f(z) &= (u - v) + i\,(u + v) \\ \mathrm{F}(z) &= \mathrm{U} + i\,\mathrm{V} \end{aligned}$$

Here let

$$\begin{aligned} \mathrm{U} &= (x-y)\,(x^2 + 4\,xy + y^2) \\ &= x^3 + 4\,x^2\,y + xy^2 - x^2y - 4\,xy^2 - y^3 \\ &= x^3 + 3\,x^2\,y - 3\,xy^2 - y^3 \\ \mathrm{U}_x &= 3\,x^2 + 6\,xy - 3\,y^2 \\ \mathrm{U}_y &= 3\,x^2 - 6\,xy - 3\,y^2 \end{aligned}$$

$$\begin{aligned} \mathrm{F}'(z) &= \mathrm{U}_x + i\,\mathrm{V}_x \\ &= \mathrm{U}_x - i\,\mathrm{U}_y \\ &= (3\,x^2 + 6\,xy - 3\,y^2) - i\,(3\,x^2 - 6\,xy - 3\,y^2) \end{aligned}$$

Put $x = z,\; y = 0$ on both sides,

$$\begin{aligned} \mathrm{F}'(z) &= 3\,z^2 - i\,3\,z^2 \\ &= 3\,(1-i)\,z^2 \end{aligned}$$

Integrating $\quad \mathrm{F}(z) = (1-i)\,z^3 + c$

i.e., $\quad (1+i)\,f(z) = (1-i)\,z^3 + c$

$$\therefore\; f(z) = \left(\frac{1-i}{1+i}\right)\cdot z^3 + \frac{c}{(1+i)}$$

Now $\quad \dfrac{1-i}{1+i} = \dfrac{(1-i)\,(1-i)}{(1+i)\,(1-i)} = \dfrac{1-i-i-1}{1+1}$

$$= \frac{-2i}{2} = -i$$

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = c_1$$

$$\therefore\ f(z) = -iz^3 + c_1$$

Harmonic Function

A function $f(x, y)$ is called Harmonic if it satisfies Laplace equation

$$f_{xx} + f_{yy} = 0$$

i.e., The solution of Laplace equation is called Harmonic function.

$\boxed{\textit{Example 1}}$ ***A function $f = x^2 - y^2$ is harmonic.***

Solution : Given :

$$f = x^2 - y^2 \qquad\qquad f = x^2 - y^2$$

$$f_x = 2x \qquad\qquad f_y = -2y$$

$$f_{xx} = 2 \qquad\qquad f_{yy} = -2$$

$$\therefore\ f_{xx} + f_{yy} = 2 + (-2) = 0$$

$\boxed{\textit{Example 2}}$ ***A function $f = \frac{1}{2}\log(x^2 + y^2)$ is harmonic.***

Solution : Given :

$$f = \frac{1}{2}\log(x^2 + y^2)$$

$$f_x = \frac{1}{2}\frac{1}{x^2 + y^2}(2x)$$

$$= \frac{x}{x^2 + y^2}$$

$$f_y = \frac{y}{x^2 + y^2}$$

$$f_{xx} = \frac{(x^2 + y^2)\cdot 1 - x\,(2x)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$f_{yy} = \frac{(x^2+y^2)\,1 - y\,(2\,y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f_{xx} + f_{yy} = 0 \Rightarrow f \text{ is harmonic function.}$$

$\boxed{\textit{Example 3}}$ ***Prove that $f = e^x \sin y$ satisfies Laplace equation.***

Solution : Given : $f = e^x \sin y$ $\qquad f_y = e^x \cos y$

$$f_{xx} = e^x \sin y \qquad f_{yy} = -e^x \sin y$$

$$f_{xx} + f_{yy} = e^x \sin y - e^x \sin y = 0$$

$\therefore$ f is harmonic function which satisfies Laplace equation.

$\boxed{\textit{Example 4}}$ ***Prove that the real part of an analytic function satisfies Laplace equation (Harmonic function).***

Solution : Proof : Given : $f(z) = u + iv$ is analytic.

$\therefore$ It satisfies CR equations.

$$U_x = V_y \qquad \dots \text{(i)}$$

$$U_y = -V_x \qquad \dots \text{(ii)}$$

Differentiating (i) partially with respect to x,

$$u_{xx} = v_{xy}$$

Differentiating (ii) partially with respect to y,

$$u_{yy} = -v_{yx}$$

Adding the above two equations, we get

$$u_{xx} + u_{yy} = 0$$

$\Rightarrow$ The real part u satisfies Laplace equation.

i.e., u is a harmonic function.

Note : If $f(z)$ is analytic function, then u is a harmonic function.

$\boxed{\textit{Example 5}}$ ***Prove that an imaginary part of an analytic function satisfies Laplace equation (harmonic function).***

Solution : Given : $f(z) = u + iv$ is an analytic function.

$$\therefore \; u_x = v_y \quad \ldots \text{(i)}$$

$$u_y = -v_x \quad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to y, we get

$$u_{yx} = v_{yy}$$

Differentiating (ii) partially with respect to x, we get

$$u_{xy} = -v_{xx}$$

$$-u_{xy} = v_{xx}$$

Adding the above two equation, we get

$$v_{xx} + v_{yy} = 0$$

v satisfies Laplace equation.

$\Rightarrow$ v is a harmonic function.

Note : If $f(z)$ is analytic then v is harmonic. The real and imaginary parts of an analytic functions are harmonic.

$\boxed{\textit{Example 9}}$ ***Prove that an analytic function with constant real part is constant.***

Solution : Given :

$$f(z) = u + iv \text{ is an analytic function.}$$

$$\text{Also given } u = \text{constant } (c_1)$$

$$u_x = 0$$

$$u_y = 0$$

Since $f(z)$ is analytic, then it satisfies

$$u_x = v_y \text{ and } u_y = -v_x$$

$$v_y = 0, \ v_x = 0 \qquad [\because u_x = u_y = 0]$$

$\Rightarrow$ V is constant (c_2).

$$\therefore f(z) = u + iv$$

$$= c_1 + i\,c_2$$

$$= \text{constant}$$

$\Rightarrow$ If u is constant then $f(z)$ is constant.

$\boxed{\textit{Example 11}}$ ***Prove that an analytic function with constant modulus is constant.***

Solution : Proof: Consider $f(z) = u + iv = u(x, y) + i\,v(x, y)$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\text{Given that } \sqrt{u^2 + v^2} = \text{constant } (c)$$

$$\text{Squaring } u^2 + v^2 = c^2 \qquad \ldots \text{(i)}$$

Differentiating (i) partially with respect to x,

$$2\,u\,u_x + 2\,v\,v_x = 0$$

$$u\,u_x + v\,v_x = 0 \qquad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to y,

$$
\begin{aligned}
2\,u\,u_y + 2\,v\,v_y &= 0 \\
u\,u_y + v\,v_y &= 0 \\
u\,(-v_x) + v\,u_x &= 0 && [\because \text{CR equation}] \\
v\,u_x + (-u)\,v_x &= 0 && \ldots \text{(iii)}
\end{aligned}
$$

For solving u_x and v_x from (ii) and (iii),

$$
\begin{aligned}
\begin{vmatrix} u & v \\ v & -u \end{vmatrix} &= -u^2 - v^2 = -(u^2+v^2) \\
&= -c^2, \text{ using (i)} \\
&\neq 0 \\
\therefore\ u_x &= 0 \text{ and } v_x = 0
\end{aligned}
$$

Since $f(z)$ is analytic, it satisfies

$$
\begin{aligned}
u_x &= v_y \text{ and } u_y = -v_x \\
\therefore\ v_y &= 0 \text{ and } u_y = 0 && [\because u_x = 0,\ v_x = 0]
\end{aligned}
$$

$\Rightarrow u_x = 0,\ u_y = 0,\ v_x = 0,\ v_y = 0.$

$\Rightarrow u = \text{constant } (c_1)$ and $v = \text{constant } (c_2)$

$$
\begin{aligned}
\therefore\ f(z) &= c_1 + i\,c_2 \\
&= \text{constant}
\end{aligned}
$$

$\boxed{\textit{Example 12}}$ ***Prove that*** $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$

if $f(z)$ ***is a regular function.***

Solution : Proof : We know that $f(z) = u + iv$

Then $|f(z)|^2 = u^2 + v^2$

$$\text{Also } f'(z) = u_x + i\,v_x$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

$$\text{Given } f(z) = u + iv \text{ is analytic, therefore}$$

$$u_x = v_y, \quad u_y = -v_x \text{ and}$$

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

Now consider $|f(z)|^2 = u^2 + v^2$ ... (1)

Differentiating (1) partially with respect to x,

$$\frac{\partial}{\partial x}|f(z)|^2 = 2\,u\,u_x + 2\,v\,v_x$$

$$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\,[\,u\,u_{xx} + u_x\,u_x + v\,v_{xx} + v_x\,v_x\,]$$

$$= 2\,[\,u\,u_{xx} + u_x^2 + v\,v_{xx} + v_x^2\,] \quad \ldots (2)$$

Similarly differentiating (1) partially with respect to y twice

$$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\,[\,u\,u_{yy} + u_y^2 + v\,v_{yy} + v_y^2\,] \quad \ldots (3)$$

Adding (2) and (3), using Laplace equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\,[\,u_x^2 + v_x^2 + u_y^2 + v_y^2\,]$$

$$\because\ u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$$

Using CR equations on RHS, we get

$$= 2\,[\,u_x^2 + v_x^2 + v_x^2 + u_x^2\,]$$

$$= 4\,[\,u_x^2 + v_x^2\,]$$

$$= 4\,|\,f(z)\,|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\,f(z)\,|^2 = 4\,|\,f'(z)\,|^2$$

HARMONIC CONJUGATES

We know that the real and imaginary parts of an analytic function $f(z) = u + iv$ are Harmonic Functions (satisfies Laplace equation). Here u and v are called Harmonic conjugates. *i.e.*, u is harmonic conjugate to v and v is harmonic conjugate to u.

Result (i) : If $f(z) = u + iv$ is analytic then u and v are harmonic functions.

For example, $f(z) = x^2 - y^2 + i\,2\,xy = z^2$ is analytic and $u = x^2 - y^2$, $v = 2\,xy$ are harmonic.

Result (ii) : If u and v are harmonic, then $f(z) = u + iv$ need not be harmonic. For example, $u = x^2 - y^2$, $v = e^x \sin y$ are harmonic but $u + iv = f(z)$ is not analytic.

Result (iii) : Since u is a function of x and y,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Similarly we can write $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Method of Finding Harmonic Conjugates

Given $f(z) = u + iv$ is analytic function, $u(x, y)$ is the real part of $f(z)$ and harmonic.

$$\therefore u_x = V_y, \quad u_y = -v_x, \quad u_{xx} + u_{yy} = 0.$$

Since v is a Harmonic conjugate and a function of x and y, we write,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using CR equations, we have

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Integrating, we get $\quad v = \int -\frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy + \text{constant}$

$$\text{Let } M = -\frac{\partial u}{\partial y},\ N = \frac{\partial u}{\partial x} \quad \text{... (i)}$$

$$V = \int M\,dx + \int N\,dy + C \quad \text{... (1)}$$

(i) Integrate M with respect to x by treating y as a constant.

(ii) Integrate N with respect to y by deleting the terms containing x.

In the same way we can find u if v is given.

$$u = \int M\,dx - \int N\,dy$$

(i) Integrate M with respect to x by treating y as a constant.

(ii) Integrate the second integral N with respect to y by deleting the terms which contains x.

This method is explained clearly by the following examples.

Example 1 ***If $u = x^2 - y^2$ is a real part of an analytic function $f(z)$, find its harmonic conjugate v.***

Solution : Given : $\quad u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y$$

Consider $\quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 2y\,dx + 2x\,dy$$

Example 2 ***Prove that $u = e^x \cos y$ is a harmonic function and find its harmonic conjugate.***

Solution : Given :

$$u = e^x \cos y$$

$$u_x = e^x \cos y, \qquad u_y = -e^x \sin y$$

$$u_{xx} = e^x \cos y, \qquad u_{yy} = -e^x \cos y$$

$$\therefore \; u_{xx} + u_{yy} = 0$$

$\Rightarrow$ u is a harmonic function.

To find its harmonic conjugate, consider

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= e^x \sin y \, dx + e^x \cos y \, dy$$

Integrating on both sides, we get

$$v = \sin y \int e^x \, dx + 0 \qquad \text{[by deleting the term containing } x\text{]}$$

$$v = e^x \sin y + c$$

Example 3 ***If $u = \frac{1}{2} \log (x^2 + y^2)$ is a real part of an analytic function $f(z)$, find v.***

Solution : Given : $u = \frac{1}{2} \log (x^2 + y^2)$

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}$$

Consider $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$= \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$= \frac{x\,dy - y\,dx}{x^2\left[1 + \left(\frac{y}{x}\right)^2\right]}$$

$$= \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} \qquad [\because \int \frac{1}{x^2 + a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)]$$

Integrating, we get $v = \tan^{-1}\left(\frac{y}{x}\right) + c$

Cauchy-Riemann Equations in Polar Form

Consider a function $f(z) = u + iv$ and $z = r\,e^{i\theta}$.

$$f(z) = f(r\,e^{i\theta}) = u(r, \theta) + i\,v(r, \theta) \quad \ldots (1)$$

Differentiating (1) partially, with respect to r, we get

$$f'(z)\,e^{i\theta} = \frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} \quad \ldots (2)$$

Differentiating (1) partially with respect to θ, we get

$$f'(z)\,r\,e^{i\theta}\,i = \frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}$$

$$f'(z)\,e^{i\theta} = \frac{1}{i\,r}\left[\frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{-i}{r}\left[\frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[-i\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\,\frac{\partial u}{\partial \theta}\right] \quad \ldots (3)$$

Equating (2) and (3) of RHS, we get

$$\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\,\frac{\partial u}{\partial \theta}\right]$$

Equating real and imaginary parts, we get

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}} \qquad \ldots (4)$$

The above equation given by (4) is called CR equations in polar form.

Note : Consider the equation (2),

$$f'(z)\ e^{i\theta} = \frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right] \qquad \ldots (5)$$

This equation can be used to find the derivative of $f(z)$.

This equation can be used to find the derivative of $f(z)$.

Example 11 ***Prove that the function $f(z) = z^n$ is analytic and hence find its derivative.***

Solution : Let $z = r\,e^{i\theta}$

$$z^n = (r\,e^{i\theta})^n = r^n\,e^{in\theta} = r^n\,[\cos n\theta + i\sin n\theta]$$

Here $u = r^n \cos n\theta$, $v = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = n\,r^{n-1}\cos n\theta \qquad \frac{\partial v}{\partial r} = n\,r^{n-1}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -n\,r^n \sin n\theta \qquad \frac{\partial v}{\partial \theta} = n\,r^n\cos n\theta$$

$$\therefore\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

CR equations in polar form satisfied.

$\therefore$ $f(z) = z^n$ is a regular function of z.

For derivative of $f(z)$, consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right]$$

$$= e^{-i\theta}\,[n\,r^{n-1}\cos n\theta + i\,n\,r^{n-1}\sin n\theta]$$

$$
\begin{aligned}
&= e^{-i\theta} \cdot n \; r^{n-1} \; [\cos n\theta + i \sin n\theta] \\
&= e^{-i\theta} \; n \; r^{n-1} \; e^{in\theta} \\
&= n \, (r \, e^{i\theta})^{n-1} \\
f'(z) &= n \; z^{n-1} \\
\therefore \frac{d}{dz} [z^n] &= n \, z^{n-1}
\end{aligned}
$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COMPLEX ANALYSIS –SMT1602

UNIT – II – Transformations – SMT1602

UNIT II

TRANSFORMATIONS

CONFORMAL MAPPING

Mapping (Transformation)

A curve C in the z-plane is mapped into the respective curve C_1 in the ω-plane by the given function $\omega = f(z)$ which defines a mapping (transformation) of the z-plane into the ω-plane.

Some standard transformations :

(i) Translation by $\omega = z + c$

(ii) Magnification and rotation by $\omega = cz$

(iii) Inversion and reflection by $\omega = \dfrac{1}{z}$

(iv) Bilinear transformation $\omega = \dfrac{az+b}{cz+d}$.

Here a, b, c, d are complex constants.

Conformal Mapping (Conformal Transformation)

Let two curves C_1 and C_2 in the z-plane intersect at the point P and the corresponding curves C_3 and C_4 in the ω-plane intersect at the point Q. If the angle of intersection of the curves at P and Q are the same in magnitude and sense, then the transformation is conformal or mapping is conformal.

Note : The transformation by the function (analytic) $\omega = f(z)$ is conformal if $f'(z) \neq 0$.

Critical point : A point at which the derivative of $f(z)$ equals to zero (the mapping is not conformal). *i.e.*, A point at which $f'(z) = 0$ is called a critical point of the transformation $\omega = f(z)$.

For example, consider $\omega = z^2$, then $\dfrac{d\omega}{dz} = 2z$.

$$\frac{d\omega}{dz} = 2z = 0$$

$$z = 0$$

$z = 0$ is a critical point of the transformation $\omega = z^2$.

Example : Consider
$$\omega = z + \frac{1}{z} = \frac{z^2+1}{z}$$
$$\frac{d\omega}{dz} = \frac{z(2z)-(z^2+1)}{z^2}$$
$$= \frac{z^2-1}{z^2}$$

The critical points are $\frac{d\omega}{dz} = 0$
$$z^2 - 1 = 0$$
$$z^2 = 1$$
$$z = \pm 1$$

Fixed Points (Invariant Points)

Fixed points of a mapping $\omega = f(z)$ are points that are mapped on to themselves (image is same as z).

Fixed points are obtained by $f(z) = z$.

Example 1 *Find the invariant points of* $\omega = \frac{1}{z-2i}$.

Solution :
$$\frac{1}{z-2i} = z$$
$$1 = z^2 - 2iz$$
$$z^2 = 2iz - 1 = 0$$
$$\therefore z = i, i$$

Example 2 *Find the points at which the transformation* $\omega = \sin z$ *is not conformal.*

Solution :
$$f'(z) = 0 \Rightarrow \cos z = 0$$
$$z = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots\ldots\ldots$$

$\boxed{Example\ 3}$ ***Find the invariant points of the transformation*** $\omega = \dfrac{1 + iz}{1 - iz}$.

Solution :

$$\frac{1 + iz}{1 - iz} = z$$

$$\therefore\ iz^2 + (i - 1)\, z + 1 = 0$$

$$\therefore\ z = \frac{1}{2}\,[\,1 + i \pm \sqrt{6\,i}\,]$$

$\boxed{Example\ 4}$ ***Consider*** $\omega = f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{1}{2}\,(z^2 + 1)$.

Solution : The invariants points are obtained from

$$f(z) = z$$

$$\frac{1}{2}\,(z^2 + 1) = z$$

$$z^2 + 1 = 2\,z^2$$

$$z^2 = 1$$

$$z = \pm\, 1$$

Isogonal Transformation (Isogonal Mapping)

If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves at Q of ω-plane only in magnitude then the transformation is called Isogonal.

$\boxed{Example\ 1}$ ***Discuss the transformation*** $\omega = f(z) = z^2$.

Solution : Given :

$$f(z) = z^2$$

$$u + iv = (x + iy)^2$$

$$= (x^2 - y^2) + i\,2\,xy$$

$$u = x^2 - y^2, \quad v = 2\,xy$$

Case (i) : Let $u =$ constant C_1

$\therefore\ x^2 - y^2 = C_1$ which is a rectangular hyperbola.

Similarly if $v = C_2$, then

$$2xy = C_2$$

$$xy = \frac{C_2}{2} \text{ which also represents rectangular hyperbola.}$$

$\therefore$ A pair of lines $u = C_1$, $v = C_2$ parallel to the axes in the ω-plane, mapping into the pair of orthogonal rectangular hyperbolas in the z-plane.

Case (ii) : Let $x = c$, a constant.

$$u = c^2 - y^2 \qquad v = 2cy$$

$$y^2 = c^2 - u \qquad y = \frac{v}{2c}$$

$$y^2 = \frac{v^2}{4c^2}$$

Eliminating y from the above equations,

$$c^2 - u = \frac{v^2}{4c^2}$$

$$v^2 = 4c^2(c^2 - u)$$

which represents a parabola.

Let $y = \text{constant } (k)$.

Then $\quad x^2 - k^2 = u, \qquad 2xk = v$

$$x^2 = u + k^2, \qquad x = \frac{v}{2k}$$

$$x^2 = \frac{v^2}{2k}$$

Eliminating x from the above equations, we get

$$u + k^2 = \frac{v^2}{2k^2}$$

$$v^2 = 2k^2 (u + k^2) \text{ which is also parabola.}$$

Here the pair of lines $x = c$ and $y = k$ parallel to the axes in the z-plane map into orthogonal parabolas in the ω-plane. The critical point of mapping $\omega = z^2$ is $z = 0$. (not conformal at $z = 0$).

$\boxed{\textit{Example 2}}$ ***Discuss the transformation*** $\omega = z + \dfrac{1}{z}$.

Solution : Let $z = r(\cos\theta + i\sin\theta)$ in polar form.

Given : $\omega = z + \dfrac{1}{z}$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r[\cos\theta + i \text{ isn } \theta]}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating $u = \left(r + \dfrac{1}{r}\right)\cos\theta$, $v = \left(r - \dfrac{1}{r}\right)\sin\theta$

$$\therefore \cos\theta = \frac{u}{\left(r + \frac{1}{r}\right)}, \quad \sin\theta = \frac{v}{\left(r - \frac{1}{r}\right)}$$

We know $\cos^2\theta + \sin^2\theta = 1$.

$$\therefore \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r + \frac{1}{r}\right)^2} = 1 \qquad \ldots (1)$$

For r = constant (c), the equation (1) represents an ellipse.

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Again consider $\omega = u + iv = r(\cos\theta + i\sin\theta)$.

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$r + \frac{1}{r} = \frac{u}{\cos\theta}, \qquad r - \frac{1}{r} = \frac{v}{\sin\theta}$$

$$\left(\frac{r^2+1}{r}\right) = \frac{u}{\cos\theta}, \qquad \left(\frac{r^2-1}{r}\right) = \frac{v}{\sin\theta}$$

$$\left(\frac{r^2+1}{r}\right)^2 = \frac{u^2}{\cos^2\theta}, \qquad \left(\frac{r^2-1}{r}\right)^2 = \frac{v^2}{\sin^2\theta}$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(\frac{r^4+1+2r^2}{r^2}\right) - \left(\frac{r^4+1-2r^2}{r^2}\right)^2$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(\frac{r^4+1+2r^2}{r^2}\right) - \left(\frac{r^4+1-2r^2}{r^2}\right)^2$$

$$= \frac{r^4+1+2r^2-r^4-1+2r^2}{r^2}$$

$$= 4$$

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad \ldots (2)$$

For θ = constant of the z-plane transforms into a family of hyperbolas.

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$$

$\boxed{\textit{Example 3}}$ ***Discuss the transformation*** $\omega = z + \frac{k^2}{z}$.

Solution : (Solve the problem as above.)

Example 4 ***Discuss the transformation $\omega = \cosh z$.***

Solution : Given : $\omega = f(z) = \cosh(z)$

$$u + iv = \cosh x \cos y + i \sinh x \sin y$$

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y \qquad \ldots (1)$$

$$\therefore \cosh x = \frac{u}{\cos y}, \qquad \sinh x = \frac{v}{\sin y}$$

We know that $\cosh^2 x - \sinh^2 x = 1$ (eliminating y).

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad \ldots (2)$$

i.e., The lines parallel to x-axis (y = constant) in the z-plane mapping into hyperbola.

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$$

We know that $\cos^2 y + \sin^2 y = 1$. For eliminating y from the given equation (1),

$$\cos y = \frac{u}{\cosh x}, \quad \sin y = \frac{v}{\sinh x}$$

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1 \qquad \ldots (3)$$

i.e., The lines parallel to Y-axis (x = constant) in the z-plane mapping into ellipse in the ω-plane.

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \qquad \ldots (4)$$

Example 5 ***Discuss the transformation*** $\omega = \frac{1}{z}$.

Solution : Given : $\omega = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2}, \qquad v = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{u}{v} = -\frac{x}{y} \Rightarrow y = -\frac{v}{u}x$$

Substituting the value of y in u,

$$u = \frac{x}{x^2 + \frac{v^2}{u^2}\cdot x^2} = \frac{u^2 x}{(u^2+v^2)\,x^2} = \frac{u^2}{(u^2+v^2)\,x}$$

$$\therefore x = \frac{u}{u^2+v^2}$$

$$\therefore y = -\frac{v}{u}\cdot x = \frac{-v}{u}\left(\frac{u}{u^2+v^2}\right) = -\left(\frac{v}{u^2+v^2}\right)$$

$$\therefore x = \frac{u}{u^2+v^2} \text{ and } y = -\frac{v}{u^2+v^2} \qquad \dots (1)$$

Now consider $\omega = \frac{1}{z}$. $\quad \therefore z = \frac{1}{\omega}$

$$x+iy = \frac{1}{(u+iv)}\,\frac{(u-iv)}{(u-iv)}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2} \text{ and } y = -\frac{v}{u^2+v^2} \qquad \dots (2)$$

Consider the equation,

$$a(x^2+y^2) + bx + cy + d = 0 \qquad \dots (3)$$

For $a = 0$, this represents a straight line and for $a \neq 0$, this represents a circle.

For the transformation $\omega = \frac{1}{z}$, we can substitute the value of x and y in (3).

$$a\left[\frac{1}{u^2+v^2}\right] + b\left[\frac{u}{u^2+v^2}\right] + c\left[\frac{-u}{u^2+v^2}\right] + d = 0$$

$$a + bu - cv + d(u^2+v^2) = 0$$

i.e.,

$$d(u^2+v^2) + bu - cv + a = 0 \qquad \dots (4)$$

If $d \neq 0$, this (4) represents a circle in the ω-plane.

If $d = 0$, it represents a straight line.

The transformation $\omega = \frac{1}{z}$ transforms circles into circles. It is called circular transformation.

$\boxed{\textit{Example 6}}$ ***Find the mapping of the circle $|z| = c$ by the transformation $\omega = 2z$.***

Solution : Given :

$$\omega = 2z = 2(x+iy) = 2x + i2y$$

$$u + iv = 2x + i2y$$

$$\therefore u = 2x, \quad v = 2y$$

Consider $|z| = c$.

$$\therefore \sqrt{x^2+y^2} = c$$

$$x^2+y^2 = c^2 \quad \text{(circle)}$$

$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = c^2$$

$$\frac{u^2}{4} + \frac{v^2}{4} = c^2$$

$$u^2 + v^2 = 4c^2$$
$$u^2 + v^2 = (2c)^2$$

This is an equation of the circle centre at the origin and radius $2c$.

$\boxed{\textit{Example 7}}$ ***Find the mapping of the circle $|z| = k$ by the transformation $f(z) = z + 2 + 3i$.***

Solution : Given :
$$\omega = z + 2 + 3i$$
$$u + iv = x + iy + 2 + 3i$$
$$u + iv = (x + 2) + i(y + 3)$$
$$u = x + 2, \qquad v = y + 3$$
$$\therefore x = u - 2, \qquad y = v - 3$$

Consider,
$$|z| = k \Rightarrow x^2 + y^2 = k^2$$
$$(u - 2)^2 + (v - 3)^2 = k^2$$

which is an equation of a circle with centre (2, 3) and radius k.

$\boxed{\textit{Example 8}}$ ***Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $\omega = \frac{1}{z}$.***

Solution :
$$\omega = \frac{1}{z}$$
$$u + iv = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$$
$$= \frac{x - iy}{x^2 + y^2}$$
$$u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}$$

The equation of the circle is $|z-1|=1$.

$$
\begin{aligned}
\text{i.e.,} \quad |x+iy-1| &= 1 \\
|(x-1)+iy| &= 1 \\
(x-1)^2+(y)^2 &= (1)^2 \\
x^2+1-2x+y^2 &= 1 \\
x^2+y^2 &= 2x \\
\frac{x}{x^2+y^2} &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{i.e.,} \quad u &= \frac{1}{2} \qquad \because \frac{x}{x^2+y^2} = u \\
2u &= 1 \\
2u-1 &= 0 \text{ which is a straight line.}
\end{aligned}
$$

$\boxed{\textit{Example 9}}$ ***Find the image of** $|z-2i|=2$ **under the mapping** $\omega = \frac{1}{z}$.*

Solution : Given : $\omega = \dfrac{1}{z}$

$$
u+iv = \frac{1}{x+iy} \qquad \therefore \; u = \frac{x}{x^2+y^2}
$$

$$
v = \frac{-y}{x^2+y^2}
$$

Also given

$$
\begin{aligned}
|z-2i| &= 2 \\
|x+iy-2i| &= 2 \\
|x+i(y-2)| &= 2 \\
x^2+(y-2)^2 &= 4
\end{aligned}
$$

$$x^2 + y^2 + 4 - 4y = 4$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 = 4y$$

$$4 = \frac{x^2 + y^2}{y}$$

$$\frac{1}{4} = \frac{y}{x^2 + y^2}$$

$$\frac{1}{4} = -v \qquad \left[\because v = \frac{-y}{x^2 + y^2}\right]$$

$4v + 1 = 0$ which is a straight line.

which is a straight line.

Example 10 ***Discuss the transformation $\omega = \sin z$.***

Solution : Given : $\omega = f(z) = \sin(z)$

$$u + iv = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\therefore u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

$$\sin x = \frac{u}{\cosh y}, \qquad \cos x = \frac{v}{\sinh y} \qquad \ldots (1)$$

We know $\sin^2 x + \cos^2 x = 1$.

$$\therefore \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

For $y = \text{constant}\ (c_1)$, say $\cosh^2(y) = a^2$, $\sinh^2(y) = b^2$,

then $$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad \text{(Ellipse)}$$

Similarly from (1),

$$\cosh y = \frac{u}{\sin x}, \quad \sinh y = \frac{v}{\cos x}$$

We know that $\cosh^2 y - \sinh^2 y = 1.$

$$\therefore \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

For $x = \text{constant } (c_2)$, say. $\sin^2 x = A^2$

$$\cos^2 x = B^2$$

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \text{ (Hyberbola)}$$

Example 11 ***Discuss the transformation*** $\omega = \cos z$.

Solution : Consider $\omega = \cos(z)$

$$u + iv = \cos(x + iy)$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$u = \cos x \cosh y, \qquad v = -\sin x \sinh y$$

$$\cos x = \frac{u}{\cosh y}, \qquad \sin x = -\frac{v}{\sinh y}$$

For eliminating x, consider $\cos^2 x + \sin^2 x = 1.$

$$\therefore \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

For $y = c$, $\cosh^2 y = a^2$ (say), $\sinh^2(y) = b^2$.

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad \text{(Ellipse)}$$

For eliminating y, consider $\cosh^2 y - \sinh^2 y = 1$.

$$\therefore \cosh y = \frac{u}{\cos x}, \quad \sinh y = \frac{-v}{\sin x}$$

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x}$$

$$\therefore \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

For x = constant, say $\cos^2 x = A^2$, $\sin^2 x = B^2$.

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \text{ (Hyperbola)}$$

Example 12 ***Discuss the transformation $\omega = \sinh z$.***

Solution : Given : $\omega = \sinh z = \sinh (x + iy)$

$$= \frac{1}{i} \sin (ix - y)$$

$$u + iv = \sinh x \cos y + i \cosh x \sin y$$

$$u = \sinh x \cos y \qquad v = \cosh x \sin y \dots \text{(i)}$$

$$\sinh x = \frac{u}{\cos y}, \qquad \cosh x = \frac{v}{\sin y}$$

We know $\cosh^2 x - \sinh^2 x = 1$ (for eliminating y)

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

For $y = c$,

$$\frac{v^2}{\sin^2 c} - \frac{u^2}{\cos^2 c} = 1$$

$$\frac{v^2}{a^2} - \frac{u^2}{b^2} = 1 \text{ for } a = \sin c\,;\ b = \cos c.$$

$$\boxed{\frac{v^2}{a^2} - \frac{u^2}{b^2} = 1}$$ which is a confocal hyperbola.

From (i) $\cos y = \dfrac{u}{\sinh x}, \quad \sin y = \dfrac{v}{\cosh x}$

We know $\cos^2 y + \sin^2 y = 1$

$$\frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1$$

For x = constant, say $\sinh x = A$, $\cosh x = B$.

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$ which is an ellipse.

Bilinear Transformation

The transformation of the form

$$\omega = \frac{az + b}{cz + d} \qquad \ldots (1)$$

where a, b, c, d are complex constants is known as Bilinear transformation if $ad - bc \neq 0$. It is also called Mobius transformation or Linear fractional transformation.

The condition $ad - bc \neq 0$ means that the transformation is conformal.

Note : $$\omega = \frac{az + b}{cz + d} \qquad \ldots (1)$$

$$\frac{d\omega}{dz} = \frac{(cz + d)\, a - (az + b)\, c}{(cz + d)^2}$$

$$= \frac{acz + ad - acz - bc}{(cz + d)^2}$$

$$= \frac{ad - bc}{(cz + d)^2}$$

The Bilinear transformation (1) is conformal if $\frac{d\omega}{dz} \neq 0$.

i.e., $ad - bc \neq 0$.

Note : If $ad - bc = 0$ then $\frac{d\omega}{dz} = 0$.

i.e., Every point of the z-plane is a critical point.

The inverse mapping of (1) is also bilinear transformation.

i.e.,
$$z = \frac{-d\omega + b}{c\omega - a}$$

The invariant points of a bilinear transformation,

$$z = \frac{az+b}{cz+d} \qquad [\because \omega = z \,;\; f(z) = z\,]$$

$$cz^2 + dz = az + b$$

$$cz^2 + (d-a)z - b = 0$$

The roots of this equation is invariant point or fixed point of the transformation.

Note :

(i) A bilinear transformation maps circles into circles.

(ii) A bilinear transformation preserves cross-ratio of four points.

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_1 - \omega_4)(\omega_3 - \omega_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

(OR)

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_4 - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

Example 13 ***Find the Mobius transformation that maps the points $z = 1, i, -1$ into the points $\omega = 2, i, -2$.***

Solution : Let $z_1 = 1, \quad z_2 = i, \quad z_3 = -1$

$$\omega_1 = 2, \quad \omega_2 = i, \quad \omega_3 = -2$$

We know $$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_4 - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)} \qquad \ldots (1)$$

Put $z_4 = z, \quad \omega_4 = \omega$, in (1),

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega)}{(\omega - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)}$$

$$\frac{(2-i)(-2-\omega)}{(\omega - 2)(i+2)} = \frac{(1-i)(-1-z)}{(z-1)(i+1)}$$

$$\frac{(\omega+2)}{(\omega-2)}\frac{(2-i)}{(2+i)} = \frac{(z+1)}{(z-1)}\frac{(1-i)}{(1+i)}$$

$$\frac{(\omega+2)}{(\omega-2)} = \frac{(z+1)}{(z-1)}\frac{(1-i)}{(1+i)}\frac{(2+i)}{(2-i)}$$

$$= \frac{(z+1)}{(z-1)}\frac{(2+i-2i+1)}{(2-i+2i+1)}$$

$$= \frac{(z+1)}{(z-1)}\frac{(3-i)}{(3+i)}$$

$$\frac{(\omega+2)}{(\omega-2)} = \frac{3z - iz + 3 - i}{3z + iz - 3 - i}$$

Using componendo and dividendo

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{a-c} = \frac{c+d}{c-d}\text{, we get}$$

$$\frac{(\omega+2)+(\omega-2)}{(\omega+2)-(\omega-2)} = \frac{(3z-iz+3-i)+(3z+iz-3-i)}{(3z-iz+3-i)-(3z+iz-3-i)}$$

$$\frac{2\omega}{4} = \frac{6z-2i}{-2iz+6}$$

$$\frac{\omega}{2} = \frac{2(3z-i)}{2(-iz+3)}$$

$$\omega = \frac{2[3z-i]}{[-iz+3]}$$

$$\text{i.e.,}\quad \omega = \frac{-6z+2i}{iz-3}$$

$\boxed{\textit{Example 14}}$ ***Find the invariant points of the transformation***

$$\omega = -\frac{2z+4i}{iz+1}.$$

Solution : $-\dfrac{2z+4i}{iz+1} = z$ $\qquad [\because \omega = z]$

$$2z+4i = -z(iz+1)$$

$$2z+4i = -iz^2-z$$

$$iz^2+3z+4i = 0$$

$$\therefore\ z = \frac{-3\pm\sqrt{9-4(i)(4i)}}{2i}$$

$$= \frac{-3\pm 5}{2i} = \frac{1}{i},\ \frac{-4}{i} = -i,\ 4i$$

Example 15 ***Find the bilinear transformation which maps the points $z = 1, i, -1$ into points $\omega = 0, 1, \infty$.***

Solution : We know that

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega)}{(\omega - \omega_1)(\omega_2 - \omega_3)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)} \qquad \ldots \text{(i)}$$

Here $\omega_3 = \infty$ is given. Equation (1) can be written as

$$\frac{(\omega_1 - \omega_2)\,\omega_3\left(1 - \dfrac{\omega}{\omega_3}\right)}{(\omega - \omega_1)\,\omega_3\left(\dfrac{\omega_2}{\omega_3} - 1\right)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)}$$

$$\frac{(\omega_1 - \omega_2)}{(\omega - \omega_1)(-1)} = \frac{(z_1 - z_2)(z_3 - z)}{(z - z_1)(z_2 - z_3)}$$

$$\frac{-(0 - 1)}{(\omega - 0)} = \frac{(1 - i)(-1 - z)}{(z - 1)(i + 1)}$$

$$+\frac{1}{\omega} = +\frac{(z + 1)}{(z - 1)}\,\frac{(1 - i)}{(1 + i)}$$

$$\frac{1}{\omega} = \frac{(z + 1)}{(z - 1)}\,\frac{(1 - i)}{(1 + i)}$$

$$\omega = \frac{(z - 1)}{(z + 1)}\,\frac{(1 + i)}{(1 - i)}$$

$$\omega = \frac{z + iz - 1 - i}{z - iz + 1 - i}$$

$$\omega = \frac{(1 + i)\,z - (1 + i)}{(1 - i)\,z + (1 - i)} \text{ which is of the form } \frac{az + b}{cz + d}.$$

Example 16 ***Find the linear fractional transformation which maps the points $z = -1, 0, 1$ into $\omega = 0, i, 3i$.***

Solution : We know that

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega-\omega_1)(\omega_2-\omega_3)} = \frac{(z_1-z_2)(z_3-z)}{(z-z_1)(z_2-z_3)}$$

$$\frac{(0-i)(3i-\omega)}{(\omega-0)(i-3i)} = \frac{(-1-0)(1-z)}{(z+1)(0-1)}$$

$$\frac{(-i)(3i-\omega)}{\omega(-2i)} = \frac{(-1)(1-z)}{(-1)(z+1)}$$

$$\frac{(3i-\omega)}{2\omega} = \frac{(1-z)}{(z+1)}$$

$$(z+1)(3i-\omega) = 2\omega(1-z)$$

$$3iz - z\omega + 3i - \omega = 2\omega - 2z\omega$$

$$3i(z+1) = 2\omega - 2z\omega + z\omega + \omega$$

$$= 3\omega - z\omega = \omega(3-z)$$

$$\therefore \omega = \frac{3i(z+1)}{(3-z)}$$

$$\omega = -3i\left(\frac{z+1}{z-3}\right)$$

Example 17 ***Find the Mobius transformation which maps from $(\infty, i, 0)$ into $(0, i, \infty)$.***

Solution : Substituting in the above formula,

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega)}{(\omega-\omega_1)(\omega_2-\omega_3)} = \frac{(z_1-z_2)(z_3-z)}{(z-z_1)(z_2-z_3)}$$

Taking z_1 and ω_3 outside and substitute, we get

$$\frac{(0-i)(1-0)}{(\omega-0)(-1)} = \frac{(1-0)(0-z)}{(0-1)(i-0)}$$

$$\frac{(-i)}{-\omega} = \frac{(1)(-z)}{(-i)}$$

$$\frac{i}{\omega} = \frac{z}{i}$$

$$\boxed{\omega = -\frac{1}{z}}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COMPLEX ANALYSIS –SMT1602

UNIT – III – Contour Integral – SMT1602

UNIT III

COMPLEX INTEGRATION

Introduction :

Consider a continuous function f (z) of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0),\ P_1(z_1),\ \ldots\ldots, P_i(z_i), \ldots\ldots, P_n(z_n) = B.$$

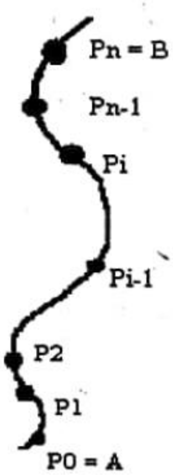
Let $\delta z_i = z_i - 1$ and ζ_i be any point on the arc $P_{i-1} P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the **line integral of f(z)** taken along the path C , i.e.

$$\int_C f(z)dz.$$

Writing $f(z) = u(x,y) + iv(x,y)$ and nothing that $dz = dx + i\,dy$,

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

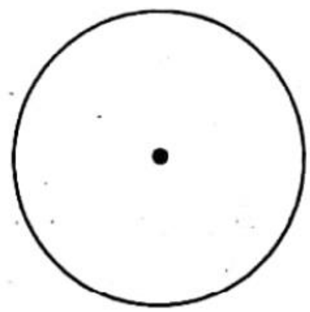


Note :

If $\omega = f(z) = u\,(x, y) + i\,v\,(x, y)$

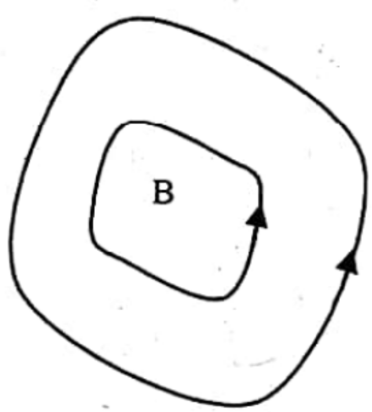
then
$$\int_C f(z)dz = \int_C (u + iv)\ d\,(x + iy)$$
$$= \int_C (u + iv)\ (dx + idy)$$
$$= \int_C (udx + vdy) + i\int_C (vdx + udy)$$

Simply connected Region: A simply connected region is one in which any closed curve lying entirely within it can be contracted to a point without passing out of the region.

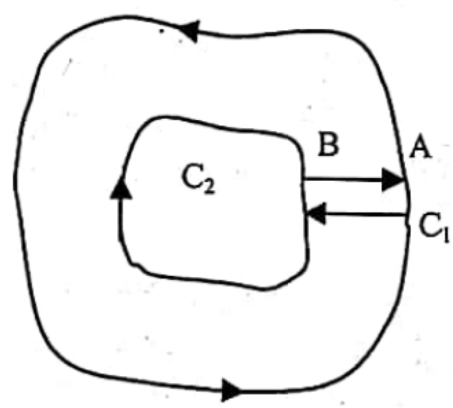


Simply Connected Region

Simply Connected Region



Multi-connected region

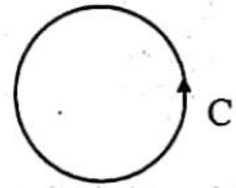


Simply connected region

CAUCHY'S THEOREM

Theorem :

If f(z) is an analytic function and f '(z) is continuous at each point within and an a closed curve C , then $\int_C f(z)\,dz = 0$.



Proof:

Consider f(z) = u(x,y) +iv(x,y) and z = x+iy , dz =dx + idy

$$\therefore \int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad \ldots\ldots(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. We know Green's theorem is

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

using this in (1)

$$\int_C f(z)\,dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\,dy + \iint \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \quad \ldots(2)$$

Now f(z) being analytic, u and v necessarily satisfy the Cauchy-Riemann equations

$$\text{i.e. } \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad \ldots\ldots\ldots(3)$$

Substituting (3) in (2) we have

$$\int_C f(z)\,dz=\iint_D\left(\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}\right)dx\,dy+i\iint_D\left(\frac{\partial v}{\partial y}-\frac{\partial v}{\partial y}\right)dx\,dy$$
$$=0$$

Hence $\int_C f(z)\,dz=0$

Extension of Cauchy's Theorem.

If f(z) is analytic in the region D between two simple closed curves C and C_1, then $\int_C f(z)\,dz=\int_{C_1} f(z)\,dz$.

To prove this, we need t o introduce the cross-cut AB. Then $\int f(z)dz=0$ where the path is as indicated by arrows in Fig.(1) .i.e. along AB-along C_1 in clockwise sense & along BA – along C in anti –clockwise sense

$$\text{i.e. } \int_{AB} f(z)\,dz+\int_{C_1} f(z)\,dz+\int_{BA} f(z)\,dz+\int_C f(z)\,dz=0.$$

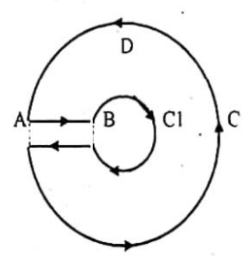


Fig.(1)

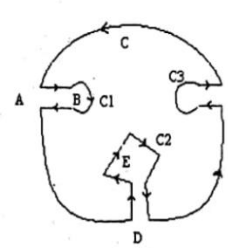


Fig.(2)

But, since the integrals along AB and along BA cancel , it follows that

$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz = 0.$$

Reversing the direction of the integral around C_1 and transposing , we get

$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$ each integration being taken in the anti-clockwise sense.

If $C_1, C_2, C_3, \ldots\ldots$ be any number of closed curves within C (Fig-2) then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \ldots..$$

CAUCHY'S INTEGRAL FORMULA

Theorem :

If f(z) is analytic within and on a closed curve and if a is any point within C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}.$

Proof :

Consider the function f(z) / (z-a) which is analytic at all points within C except at z = a. With the point a as center and radius r , draw a small circle C_1 lying entirely within C.

Now f(z) / (z-a) being analytic in the region enclosed by C and C1, we have by Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}.ire^{i\theta}\,d\theta = i\int_{C_1} f(a+re^{i\theta})\,d\theta. \qquad \ldots\ldots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a , i.e. as $r \to 0$, the integral (1) will approach to

$$i \int_{C_1} f(a)\, d\theta = i f(a) \int_{0}^{2\pi} d\theta = 2\pi i\, f(a). \text{thus} \int_{c} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

i.e. $$f(a) = \frac{1}{2\pi i} \int_{c} \frac{f(z)}{z-a} dz$$

which is the desired Cauchy's integral formula.

$$\Rightarrow \int_{c} \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

Cauchy's integral formula for derivative of an analytic function:-

We know Cauchy's integral formula is

$$F(a) = \frac{1}{2\pi i} \int_{c} \frac{f(z)}{z-a} dz.$$

Differentiating both sides of (2) w.r.t.a,

$$f'(a) = \frac{1}{2\pi i} \int_{c} \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right] dz = \frac{1}{2\pi i} \int_{c} \frac{f(z)}{(z-a)^2} dz \qquad \ldots\ldots(3)$$

similarly, $$f''(a) = \frac{2!}{2\pi i} \int_{c} \frac{f(z)}{(z-a)^3} dz \qquad \ldots\ldots(4)$$

and in general, $$f^{n}(a) = \frac{n!}{2\pi i} \int_{c} \frac{f(z)}{(z-a)^{n+1}} dz. \qquad \ldots\ldots(5)$$

thus it follows from the results (2) to (5) that if a function f(z) is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C. Incidently we have established a

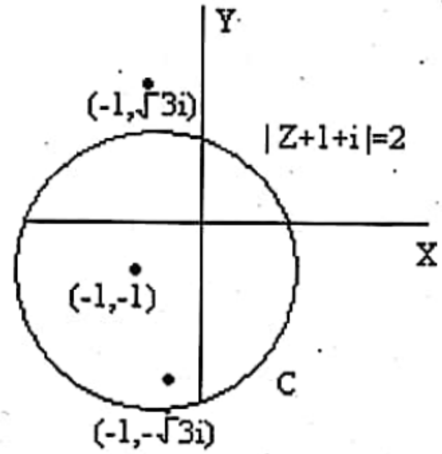
remarkable fact that **an analytic function possesses derivatives of all orders a[illegible] these are themselves all analytic.**

Example 1: Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dx$ where C is the circle

(i) $|z| = 1$, (ii) $|z| = \frac{1}{2}$.

(i) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since f(z) is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C.



$\therefore$ By Cauchy's Integral Formula $\frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} = f(a) = 1$ i.e. $\int_C \frac{z^2 - z + 1}{z - 1} dz = 2\pi i.$

(ii) In this case, $a = 1$ lies outside the circle $C : |z| = \frac{1}{2}$. So $\frac{(z^2 - z + 1)}{(z - 1)}$ analytic everywhere within C.

$\therefore$ By Cauchy's Theorem $\int_C \frac{z^2 - z + 1}{z - 1} dz = 0.$

Example 2:

Using Cauchy's integral formula, Evaluate $\int_C \frac{z + 1}{z^2 + 2z + 4} dz$ where c is the circle $|z + 1 + i| = 2$

Solution:

$|z + 1 + i| = |z - (-1 - i)|$ is the circle with centre at z = -1-I and radius 2 units

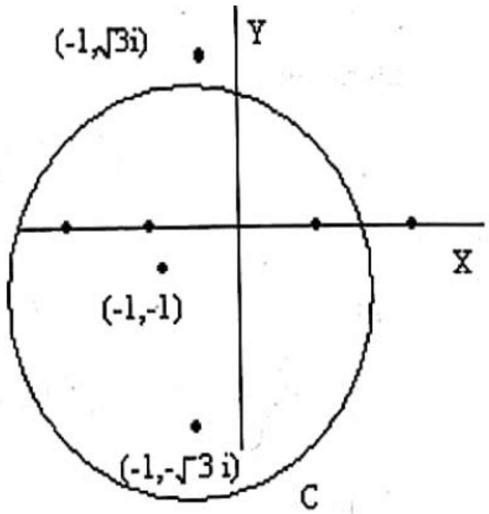
The function $\frac{z + 1}{z^2 + 2z + 4}$ will cease to be analytic where $z^2 + 2z + 4 = 0$

$$z = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm i2\sqrt{3}}{2}$$

$$z = -1 \pm i\sqrt{3}$$



$$z = -1 + i\sqrt{3}, -1 - i\sqrt{3}$$

$$\therefore \frac{(z+1)}{z^2+2z+4} = \frac{z+1}{(z+1-i\sqrt{3})(z+1+i\sqrt{3})}$$

The above function is analytic at all points except at the points $-1+i\sqrt{3}$ lies outside c and $-1-i\sqrt{3}$ lies inside c.

$\therefore$ we consider the function $f(z) = \dfrac{z+1}{z+1-i\sqrt{3}}$

by cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

here $a = -1 - i\sqrt{3}$, (lies inside c) $\therefore \displaystyle\int \frac{\frac{z+1}{z+1-i\sqrt{3}}}{z-(-1-i\sqrt{3})}\,dz = 2\pi i\, f(a)$

$$=2\pi i f(-1-i\sqrt{3})$$

$$= 2\pi i\left(\frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}}\right)$$

substitution in f(z)

$$= 2\pi i\left(\frac{-i\sqrt{3}}{-2i\sqrt{3}}\right) = \pi i$$

Example 3:

Using cauchy's integral formula evaluate $\int_c \frac{z+4}{z^2+2z+5}dz$ where c is circle $|z+1-i|=2$.

Solution :

$|z+1-i| = |z-(-1+i)|$ is the circle with center at (-1+i) and radius 2 units. The function $\frac{z+4}{z^2+2z+5}$ will cease to be regular where $z^2+2z+5=0$

$$\text{i.e., } z^2+2z+5=0$$

$$z = \frac{-2\pm\sqrt{4-20}}{2}$$

$$z = \frac{-2\pm\sqrt{-16}}{2} = -1\pm 2i$$

$$\therefore z = -1+2i, \quad -1-2i$$

$$\frac{z+4}{(z^2+2z+5)} = \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}$$

The above function is analytic at all points except at z=-1+2i which lies inside c and z = -1 --2i which lies outside c.

$\therefore$ We consider the function

$$f(z) = \frac{\frac{z+4}{[z-(-1-2i)]}}{[z-(-1+2i)]} = \frac{f(z)}{z-a}$$

$\therefore$ By cauchy integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Takng a = -1+2i (lies inside c)

$$\int_c \frac{\left(\frac{z+4}{z+1+2i}\right)}{[z-(-1+2i)]} dz = 2\pi i\, f(-1+2i)$$

$$= 2\pi i\left(\frac{-1+2i+4}{-1+2i+1+2i}\right)$$

$$= 2\pi i\left(\frac{2i+3}{4i}\right) = \frac{\pi}{2}(2i+3)$$

Example 4:

Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is $|z| = 3$ using cauchy integral formula.

Solution:

$|z| = 3$ is a circle with center at the origin and radius 3 units

consider

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

put $z = 1$ $A = -1$

put $z = 2$ $B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = -\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz + \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz$$

Since z=1, and z=2 lies inside c and $f(z) = \sin \pi z^2 + \cos \pi z^2$

By cauchy integral formula

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$
$$= -2\pi i\,(\sin \pi + \cos \pi) + 2\pi i\,(\sin 2\pi + \cos 2\pi)$$
$$= 2\pi i\,(1+1)$$
$$= 4\pi i$$

Example 5:

Using cauchy integral formula evaluate $\int_c \frac{dz}{(z^2+1)(z^2-4)}$

where c is $\int_c \frac{dz}{(z^2+1)(z^2-4)}$ where c is $|z| = \frac{3}{2}$

Solution :

$|z| = \frac{3}{2}$ is the circle with center at the origin and radius 3/2 units.

$$\frac{1}{(z^2+1)(z^2-4)}=\frac{1}{(z+i)(z-i)(z+2)(z-2)}$$

The above function is analytic at all points excepts at $z = i$, $-i$ which lies inside c and $z = \pm 2$ which lies outside c

$\therefore$ we consider the function

$$f(z)=\frac{\frac{1}{z^2-4}}{(z+i)(z-i)}$$

Now

$$\frac{1}{(z+1)(z-i)}=\frac{A}{(z+i)}+\frac{B}{(z-i)}$$

$$1 = A(z-i)+B(z+i)$$

$$\text{Put } z=i,\ B=\frac{1}{2i}=-\frac{i}{2}$$

$$\text{Put } z=-i,\ B=-\frac{1}{2i}=\frac{i}{2}$$

$$\therefore \frac{1}{(z+i)(z-i)}=\frac{\frac{i}{2}}{(z+i)}-\frac{\frac{i}{2}}{(z-i)}$$

$$\therefore \int_c \left[\frac{\frac{i}{2}}{(z+i)}-\frac{\frac{i}{2}}{(z-i)}\right]\frac{1}{z^2-4}\,dz=\frac{i}{2}\int_c \frac{\left(\frac{1}{z^2-4}\right)}{z+i}\,dz-\frac{i}{2}\int_c \frac{\left(\frac{1}{z^2-4}\right)}{z-i}\,dz$$

taking a = i , -i (which lie inside c)

By cauchy integral formula

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$= \left(\frac{i}{2}\right) 2\pi i\, f(-i) - \left(\frac{i}{2}\right) 2\pi i\, f(i)$$

$$= \left(\frac{i}{2}\right) 2\pi i \left[\frac{1}{-5} - \frac{1}{-5}\right]$$

$$= -\pi \left[-\frac{1}{5} + \frac{1}{5}\right]$$

$$= 0$$

Example 6: Evaluate $\int_c \frac{z^2\,dz}{(z-1)^2\,(z^2+1)}$ where c is $|z-2| = 2$. Using cauchy integral formula.

Solution :

$|z-2| = 2$ is a circle with center at 2 and radius 2 units consider.

$$\frac{z^2}{(z-1)^2(z^2-1)} = \frac{z^2}{(z-1)^3(z+1)}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-3)^3} + \frac{D}{(z+1)}$$

$$z^2 = A(z-1)^2(z+1) + B(z-1)(z+1) + C(z+1) + D(z-1)^3$$

put $z = 1$,

$$c = \frac{1}{2}$$

put $z = -1$

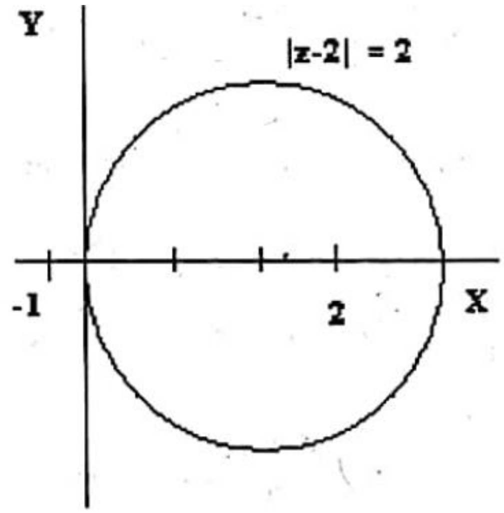
$$-8D = 1,$$

$$D = -\frac{1}{8}$$

Coefficient of z^3, $A + D = 0$

$$A = -D$$

$$A = \frac{1}{8}$$



equating constant coefficient

$$A - B + C - D = 0$$

$$B = \frac{1}{8} + \frac{1}{2} + \frac{1}{8}$$

$$= \frac{1+4+1}{8} = \frac{6}{8}$$

$$B = \frac{3}{4}$$

$$\int_c \frac{z^2}{(z-1)^2(z^2-1)}\,dz = \frac{1}{8}\int_c \frac{1}{(z-1)}\,dz + \frac{3}{4}\int_c \frac{1}{(z-1)^2} + \frac{1}{2}\int_c \frac{dz}{(z-1)^3} - \frac{1}{8}\int_c \frac{dz}{(z+1)}$$

Since the point z=1 lies inside c and z = -1 lies outside c. By cauchy integral formula & its derivatives we have

$$= \frac{1}{8} 2\pi i\, f(1) + \frac{3}{4}(2\pi i)\, f'(1) + \frac{1}{2}\frac{(2\pi i)\, f''(1)}{2!} + 0$$

$$= \frac{1}{8} 2\pi i + \frac{3}{4} 2\pi i + \frac{1}{2} \frac{(2\pi i)}{2!}$$

$$= \frac{\pi}{4} i + \frac{3}{2} \pi i + \frac{\pi}{2} i = \frac{\pi i + 6\pi i + 2\pi i}{4} = \frac{9\pi i}{4}$$

$$\left[\because f(z)=1 \quad f(1)=1 \quad f'(1)=1 \quad f''(1)=1\right]$$

Example 7: Evaluate using Cauchy's integral formula :

$\int_C \frac{e^{2z}}{(z-1)(z-2)}$, where C is the circle $|z| = 3$

Solution: $f(z) = e^{2z}$ is analytic within the circle $C : |z| = 3$ and the two singular points $a = 1$ and $a = 2$ lie inside C.

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz = \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz$$

$$= 2\pi i e^4 - 2\pi i e^2 = 2\pi I\,(e^4 - e^2)$$

[By Cauchy's integral formula]

Example 8:

Evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$.

Solution :

Here $|z| = 3$ is a circle with center at the origin and radius 3 units.

Also $f(z) = \cos \pi z^2$

and consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$1 = A(z-2) + B(z-1)$$

put $z = 1$, $A = -1$
put $z = 2$, $B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\int_C \frac{\cos \pi z^2}{(z-1)} dz + \int_C \frac{\cos \pi z^2}{(z-2)} dz$$

Since $z = 1$ and $z = 2$ lies inside c. By cauchy integral formula we have

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i\, [-\cos \pi + \cos 4\pi]$$

$$= 2\pi i\, [-(-1)+1] = 4\pi i$$

Example 8:

Evaluate $\int_C \frac{(z+1)\, dz}{(z^2+2z+4)^2}$ where c is $|z+1+i| = 2$ using cauchy integral formula.

Solution :

$|z + 1 + i| = 2$ is a circle with centre $(-1, -i)$ and radius 2 units.

$$\frac{z+1}{(z^2+2z+4)^2} = \frac{z+1}{\left[z-\left(-1-\sqrt{3}i\right)\right]^2\left[z-\left(-1+\sqrt{3}i\right)\right]^2}$$

The above function is analytic at all points except at $z = -1-\sqrt{3}$ I which lies inside c and $z = -1+\sqrt{3}$ I which lies outside c.

$\therefore$ Consider the function

$$f(z)=\frac{\dfrac{z+1}{\left[z-(-1+\sqrt{3}i\right]^2}}{\left[z-\left(-1-\sqrt{3}i\right)\right]^2}$$

$\therefore$ By cauchy integral formula for derivatives

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

taking $\quad a = -1-\sqrt{3}\,i$

$\therefore \qquad = 2\pi i\, f'(-1-\sqrt{3}i)$

But $\qquad f(z) = \dfrac{z+1}{\left[z-\left(-1+\sqrt{3}i\right)\right]^2}$

$$= \frac{z+1}{(z-\alpha)^2}\ \alpha = -1+\sqrt{3}i$$

$$f'(z) = \frac{(z-\alpha)^2 - 2(z+1)(z-\alpha)}{(z-\alpha)^4} = \frac{-(z+\alpha+2)}{(z-\alpha)^3}$$

$$f'(a) = f'\left(-1-\sqrt{3}i\right)$$

$$= \frac{-\left[-1-\sqrt{3}i-1+\sqrt{3}i+2\right]}{\left(-1-\sqrt{3}i+1-\sqrt{3}i\right)^3} = \frac{0}{-\left(2\sqrt{3}i\right)^3} = 0$$

$$\therefore \int_c \frac{(z+1)\,dz}{(z^2+2z+4)^2} = 2\pi i\ f'(-1-\sqrt{3}i)$$

$$= 0 \qquad\qquad \because f'(-1-\sqrt{3}i) = 0$$

Example 10 :

Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where c is $|z|=2$ using cauchy integral formula.

Solution :

$|z| = 2$ is a circle with centre at the origin and radius 2 units

Here $f(z) = e^{2z}$

Clearly $z = -1$ lies inside c

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = \int_c \frac{e^{2z}}{[z-(-1)]^4} dz$$

since $z = -1$ lies inside c

By cauchy integral formula for derivatives

$$f'''(a) = \frac{3!}{2\pi i} \int_c \frac{f(z)}{(z-a)^4} dz$$

$$\therefore \int_c \frac{e^{2z}}{[z-(-1)]^4} dz = \frac{2\pi i f'''(a)}{3!}$$

$$= 2\pi i f'''(-1) \qquad \text{...(1)}$$

since $\qquad f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$$f'''(z) = 8\,e^{2z}$$

$$f'''(-1) = 8e^{-2} \qquad \text{...(2)}$$

Therefore (2) in (1) we get

$$\int_c \frac{e^{2z}}{[z-(-1)]^4}\,dz = \frac{2\pi i \times 8e^{-2}}{6}$$

$$= \frac{8}{3}\pi i e^{-2}$$

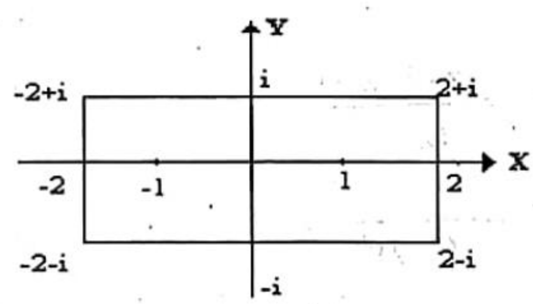
Example 11:

$$\int_C \frac{\cos \pi z}{z^2 - 1}\,dz \text{ around a rectangle with vertices } 2 \pm i, -2 \pm i.$$

Solution :

$f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points a=1 and a = -1 lie inside this rectangle.

$$\therefore \int_C \frac{\cos \pi z}{z^2 - 1}\,dz = \frac{1}{2}\int_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\, dz$$



$$= \frac{1}{2}\int_C \frac{\cos \pi z}{z-1}\,dz - \int_C \frac{\cos \pi z}{z+1}\,dz$$

$$= \frac{1}{2}\{2\pi i \cos \pi(1)\} - \frac{1}{2}\{2\pi i \cos \pi(-1)\} = 0.$$

[By Cauchy's integral formula]

Example 12:

Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)} dz$ where c is circle $|z-i| = 2$

Solution :

$|z-i| = 2$ is a circle with centre at i and radius 2 units.

Consider $\frac{z-1}{(z+1)^2(z-2)}$

The above function is analytic at all except at z= - 1 which lies inside 'c'.

$$\therefore \text{we consider } f(z) = \frac{z-1}{z-2}$$

$$\therefore \int_c \frac{\left(\frac{z-1}{z-2}\right)}{(z-(-1))^2} dz = 2\pi i f'(-1) \qquad \text{....(1)}$$

($\because$ using cauchy integral formula taking $a = -1$)

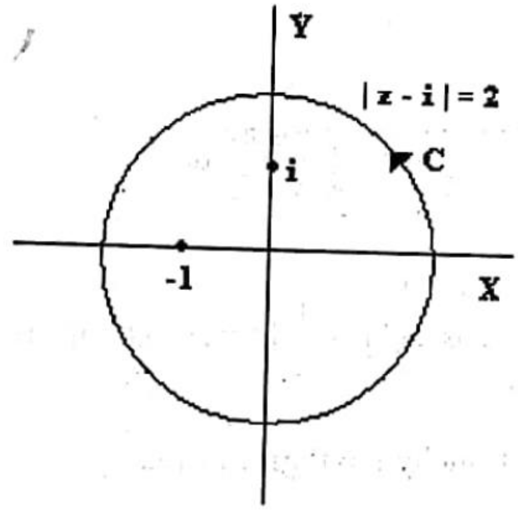
since $f(z) = \frac{z-1}{z-2}$

$$f'(z) = \frac{(z-2)-(z-1)}{(z-2)^2}$$

$$= \frac{z-2-z+1}{(z-2)^2}$$

$$= -\frac{1}{(z-2)^2}$$

$$\therefore f'(-1) = -\frac{1}{9} \qquad ......(2)$$



Substitute (2) in (1) we get

$$\int_C \frac{z-1}{(z+1)^2 (z-2)}\, dz = 2\pi i\, f'(-1)$$

$$= 2\pi i \left[-\frac{1}{9}\right]$$

$$= -\frac{2\pi i}{9}$$

Example 13:

Evaluate

(i) $\displaystyle\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\, dz$, where C is the circle $|z| = 1$.

(ii) $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, where C is the circle $|z| = 2$.

Solution:

(i) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (0.5 approx.) lies within C.

$\therefore$ By cauchy's integral formula $\displaystyle f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\, dz$,

We get $\displaystyle\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\, dz = \pi i \left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$

$$= \pi i\,(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i$$

(ii) $f(z) = e^{2z}$ is analytic within the circle C : | z | = 2. Also z = -1 lies inside C.

$\therefore$ By cauchy's integral formula : $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$

We get $\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{6}\left|\frac{d^3\left(e^{2z}\right)}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}\left[8e^{2z}\right]_{z=-1} = \frac{8\pi i}{3}e^{-2}$

Example 14:

Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, where c is | z | = 1

Solution :

Here $f(z) = \sin^6 z$ | z | = 1 is the circle with center at the origin and radius 1 units

clearly $z = \frac{\pi}{6}$ lies inside | z | =1

$\therefore$ By cauchy integral formula for derivatives

$$\int_C \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$

$$\therefore \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''(\pi/6) \quad \text{.....(1)}$$

But $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z \cos z$

$$f''(z) = 6\,[\,\sin^5 z\,(-\sin z) + \cos z\,(5\sin^4 z)]$$

$$= 6\,[-\sin^6 z + 5\cos z\,\sin^4 z]$$

$$\therefore f''\left(\frac{\pi}{6}\right) = 6\left[-\sin^6\left(\frac{\pi}{6}\right) + 5\cos\left(\frac{\pi}{6}\right) \text{x} \sin^4\left(\frac{\pi}{6}\right)\right]$$

$$= 6\left[-\frac{1}{64} + \frac{5}{16} \text{x} \frac{3}{4}\right] \qquad \text{.........}(2)$$

$$= \frac{21}{16}$$

Substitute (2) in (1) we have

$$\int_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3}\,dz = \frac{2\pi i}{2!}\left(\frac{21}{16}\right) = \frac{21\pi i}{16}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COMPLEX ANALYSIS –SMT1602

UNIT – IV – Taylors and Laurents Theorem – SMT1602

UNIT IV

TAYLORS AND LAURENTS THEOREM

Taylor's series:

If f(z) is analytic inside a circle C with centre at a , then for z inside C

$$, f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots\ldots\ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots\ldots\ldots\ldots(1)$$

Note: If a = 0 in Taylor's series we get Maclaurin's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{f^n(0)}{n!}$$

Note: Complex analytic functions can always be represented by power series of the form (1)

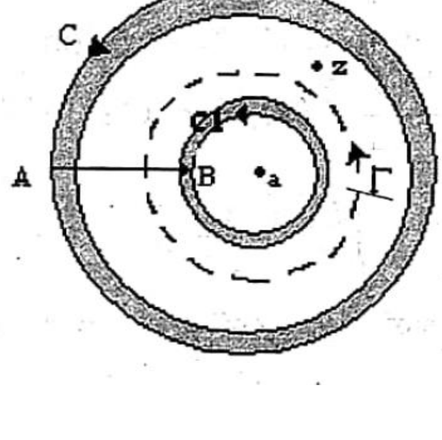
Complex analytic functions can always be represented by power series of the form (1)

Laurent's Series:

If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a , then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots\ldots + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \ldots\ldots$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$



Γ being any curve in R', encircling C_1

$$\text{Where } a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{-n+1}}\,dt$$

Note: $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called integral part and $\sum_{n=0}^{\infty} b_n (z-a)^{-n}$ is called principle part of the Laurents series.

Note:

(i) To obtain Taylor's series or Laurent's series simply expand f(z) by Binomial theorem.

(ii) Laurent's series of a given analytic function f(z) in its annulus of convergence is unique.

(iii) If $|z| < 1$, then (We Know)

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots\ldots\ldots\ldots$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots\ldots\ldots\ldots$$

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \ldots\ldots\ldots\ldots$$

$$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots\ldots\ldots\ldots$$

Example 1:

Find the Laurents series Expansion of $\frac{1}{z^2 - z - 2}$ in the region $1 < |z| < 2$

Solution: $f(z) = \frac{1}{z^2 - z - 2} = \frac{1}{(z+1)(z-2)}$

$$\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$$

$1 = A(z-2) + B(z+1)$

put $z = 2$ $\quad B = \frac{1}{3}$.

put z = -1 $\qquad A=-\frac{1}{3}$

$$f(z)=\frac{1}{z^2-z-2}=\frac{1}{(z+1)(z-2)}$$

$$\frac{1}{(z+1)(z-2)}=\frac{-1}{3(z+1)}+\frac{1}{3(z-2)}$$

$$=\frac{1}{3z(1+1/z)}-\frac{1}{6(1-z/2)}$$

$$=-\frac{1}{3z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$f(z)=-\frac{1}{3z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\dots\right)-\frac{1}{6}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)$$

In the first series the expansion in valid $\left|\frac{1}{z}\right|<1$, i.e. $1<|z|$

In the second series the expansion in valid $\left|\frac{z}{2}\right|<1$, $|z|<2$

$\therefore$ The series is valid when $1<|z|<2$.

Example 2: Obtain the expansion of the function $\frac{z-1}{z^2}$ in Taylors series of powers of (z - 1) and state the region of validity.

Solution: $f(z)=\frac{z-1}{z^2}$

$$=\frac{1}{z}-\frac{1}{z^2}$$

The Taylors series at z = 1 is

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^n(1) \qquad \ldots(1)$$

Now $f(z) = \frac{1}{z} - \frac{1}{z^2}$

$$f(1) = 0 \qquad \ldots(2)$$

$$f'(z) = -\frac{1}{z^2} + \frac{(-1)(-2)}{z^3}$$

$$f''(z) = \frac{(-1)(-2)}{z^3} + \frac{(-1)(-2)(-3)}{z^4}$$

....

....

$$f^n(z) = \frac{(-1)^n n!}{z^{n+1}} + \frac{(-1)^{n+1}(n+1)!}{z^{n+2}}$$

$$\therefore f^n(1) = (-1)^n n! + (-1)^{n+1}(n+1)!$$

$$=(-1)^n n![1-(n+1)]$$

$$=(-1)^n n!(-n)$$

$$f^n(1)=(-1)^{n+1} n.n! \qquad \ldots(3)$$

Substitute (2) & (3) in (1) we have

$$f(z) = \sum_{n=1}^{\infty} n(-1)^{n+1}(z-1)^n$$

f(z) is analytic at $z = 0$. Also $|z - 1| < 1$ is the region of converges.

Hence the region of validity $|z - 1| < 1$

Example 3: Obtain the Taylors series of expansion of $\log(1 + z)$ when $|z| < 1$.

Solution: Let $f(z) = \log(1 + z)$

$$f(0) = \log(1) = 0 \qquad \ldots(1)$$

$$f'(z) = \frac{1}{1+z}$$

$$f''(z) = -\frac{1}{(1+z)^2}$$

$$f'''(z) = \frac{(-1)(-2)}{(1+z)^3} = \frac{2!(-1)^2}{(1+z)^3}$$

....

....

$$f^n(z) = \frac{(-1)(-2)\ldots-(n-1)}{(1+z)^n} = \frac{(n-1)!(-1)^{n-1}}{(1+z)^n}$$

$$\therefore f^n(0) = (n-1)!\,(-1)^{n-1} \qquad \ldots(2)$$

The Taylors series at $z = 0$ is

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^n(0) \qquad \ldots(3)$$

substitute (1) & (2) we get

$$f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n!}(n-1)!(-1)^{n-1}$$

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n$$

Example 4: Expand cos z in a Taylors series about $z=\frac{\pi}{4}$

Solution:

$$f(z)=\cos z \quad f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'(z)=-\sin z \quad f'\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f''(z)=-\cos z \quad f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f'''(z)=\sin z \quad f'''\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

The Taylors series about $z=a$ is

$$f(z)=f(a)+\sum_{n=1}^{\infty}\frac{(z-a)^n}{n!}f^n(a)$$

$$=f(a)+\frac{\left(z-\frac{\pi}{4}\right)}{1!}\left(-\frac{1}{\sqrt{2}}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)+\ldots$$

$$\cos z=\frac{1}{\sqrt{2}}\left[1-\frac{\left(z-\frac{\pi}{4}\right)}{1!}-\frac{\left(z-\frac{\pi}{4}\right)}{2!}+\ldots\right]$$

Example 5: Find Taylors expansion of

(i) $f(z)=\dfrac{1}{(z+1)^2}$ about the point $z=-i$.

(ii) $f(z)=\dfrac{2z^3+1}{z^2+z}$ about the point $z=i$

(i) To expand f(z) about $z=-i$ i.e. in power of z+i, put z+i=t. Then

$$f(z)=\frac{1}{(t-i+1)^2}=(1-i)^{-2}[1+t/(1-i)]^{-2}$$

$$=\frac{i}{2}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}-\frac{4t^3}{(1-i)^3}+\ldots\right]$$

(Expanding by Binomial theorem)

$$=\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(ii) $f(z)=\dfrac{2z^3+1}{z(z+1)}=2z-2+\dfrac{2z+1}{z(z+1)}=(2i-2)+2(z-i)+\dfrac{1}{z}+\dfrac{1}{z+1}\ldots(1)$

(By partial fractions)

To expand 1/z and 1/(z+1) about z - i = t, so that

$$\frac{1}{z}=\frac{1}{(t+i)}=\frac{1}{i}\left(1+\frac{t}{i}\right)^{-1}$$ (Expanding by Binomial theorem)

$$=\frac{1}{i}\left[1-\frac{t}{i}+\frac{t^2}{i^2}-\frac{t^3}{i^3}+\frac{t^4}{i^4}-\ldots\infty\right]$$

$$=\frac{1}{i}+\frac{t}{1}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\frac{t^4}{i^5}-\ldots\infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \qquad \ldots(2)$$

and $\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1+\frac{t}{1+i}\right)^{-1}$ (Expanding by Binomial theorem)

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+i)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots\infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots\infty\right]$$

$$= \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}} \qquad \ldots(3)$$

Substituting from (2) and (3) in (1) we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

Example 6: Find the Laurents series expansion of $f(z) = \frac{e^{2z}}{(z-1)^3}$ above $z = 1$

Solution: $f(z) = \frac{e^{2z}}{(z-1)^3}$

Here we have to expand f(z) in Laurents series as powers of (z - 1)

Put $z - 1 = u$ i.e., $z = u + 1$

$$\therefore f(z)=\frac{e^{2u+2}}{u^3}=\frac{e^2}{u^3}\left[1+\frac{(2u)}{1!}+\frac{(2u)^2}{2!}+\ldots\right]$$

$$=e^2\left[\frac{1}{u^3}+\frac{2u}{u^3}+\frac{(2u)^2}{2u^3}+\frac{(2u)^3}{3!u^3}+\ldots.\right]$$

$$=e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)}+\frac{4}{3}+\frac{2}{3}(z-1)+\ldots\infty\right]$$

The series is valid when $|z-1|>0$

Example 7: Find the Laurents series of $f(z)=\dfrac{1}{(z-1)(z-2)}$ in $|z|>2$

Solution: $f(z)=\dfrac{1}{(z-1)(z-2)}$

$$f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)} \quad \text{(using partial fraction)}$$

In the region $|z|>2$ the Laurents series is

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z)=-\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\ldots\right)+\frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\ldots\right)$$

Example 8: Find the Laurents expansion of $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ in

(i) $|z|>3$ (ii) $2<|z|<3$

Solution:

$$f(z)=\frac{z^2}{(z+2)(z+3)}=A+\frac{B}{(z+2)}+\frac{C}{(z+3)}$$

$$z^2-1=A(z+2)(z+3)+B(z+3)+C(z+2)$$

put $z=-3$ $\qquad -C=8 \quad \therefore C=8$

put $z=-2$ $\qquad B=3$

Equating the coefficient of z^2, $\qquad A=1$

$$\therefore f(z)=1+\frac{3}{(z+2)}-\frac{8}{(z+3)}$$

(i) $|z|>3$

$$\therefore f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z)=1+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\ldots\right]-\frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\ldots\right]$$

In the above expansion the first series is valid when $\left|\frac{2}{z}\right| < 1$ i.e. $2 < |z|$

In the second series valid for $\left|\frac{3}{z}\right| < 1$ i.e. $3 < |z|$

$\therefore$ The whole expansion is valid when $|z| > 3$

(ii) $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right) - \ldots\right] - \frac{3}{8}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2 - \ldots\right]$$

In the above expansion the first series is valid when $\left|\frac{2}{z}\right| < 1$ i.e. $2 < |z|$

In the second expansion is valid when $\left|\frac{z}{3}\right| < 1$ i.e. $|z| < 3$

$\therefore$ The whole expansion is valid $2 < |z| < 3$

Example 9: Find the Laurents Expansion of the function $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$ in the annulus $1 < |z+1| < 3$

Solution: put $z + 1 = u$

$$z = u - 1$$

$$f(z) = \frac{7(u-1)-2}{(u-1)u(u-3)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

(using partial fraction), $1 < |u| < 3$

$$= -\frac{3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{2}{3\left(1-\frac{u}{3}\right)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u} + \left[1 + \frac{1}{u} + \frac{1}{u^2} + \ldots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \ldots\ldots\right]$$

$$= \left[\frac{-2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \ldots\ldots\ldots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \ldots\right]$$

$$=\left[\frac{-2}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots\ldots\right]-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\left(\frac{(z+1)}{3}\right)^2+\ldots\ldots\right]$$

clearly this series is valid in the region $1 < | z+1 | < 3$

(1) Zeroes of an analytic function

Def . A zero of an analytic function f(z) is that value of z for which f(z) = 0.

(2) Singularities of an analytic function

Def . A singular point of a function is the point at which the function ceases to be analytic.

(i) **Isolated Singularity** . It z =a is a singularity of f(z) such that f(z) is analytic at each point in its neighbourhood (i.e., there exists a circle with centre a which has no other singularity) , then z = a is called an **isolated singularity.**

In such a case, f(z) can be expanded in a Laurent's series around z = a, giving

$$f(z) = a_0 + a_1 (z - 1) + a_2 (z - a)^2 + \ldots\ldots + b_1 (z - 1)^{-1} + b_2 (z - a)^{-2} + \ldots\ldots \quad \ldots(1)$$

For example , f(z) = cot (π / z) is not analytic where tan (π / z)=0 i.e., at the points π / z = 4π or z = 1 / n (n = 1,2,3,....)

Thus z = 1 , ½ , 1/3 , ... are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large , z=0 is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus z = 0 is the *non – isolated singularity* of f(z).

(ii) **Removable Singularity .** If all the negative powers of (z – a) in (1) are zero , then

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. Here the singularity can be removed by defining f(z) at z = a in such a way that it becomes analytic at z = a. Such a singularity is called a *removable singularity*.

Thus if $\underset{x \to a}{\text{Lt}}\ f(z)$ exists finitely , then z = a is a removable singularity

(iii) **Poles .** If all the negative powers of (z – a) in (i) after the n^{th} are missing, then the singularity at z = a is called a **pole of order n**

A pole of first order is called a simple pole.

(iv) **Essential singularity .** If the number of negative powers of (z – a) in (1) is infinite , then z = a is called an essential singularity . In this case,

$\underset{x \to a}{\text{Lt}}\ f(z)$ does not exist.

Example 1 :

Find the nature of singularities of the function

(i) $\dfrac{z - \sin z}{z^2}$

Solution :

Here $z = 0$ is a singularity.

Also $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - .....$$

Since there are no negative powers of z in the expansion , z=0 is a *removable singularity*.

(ii) $(z+1)\sin\dfrac{1}{z-2}$

Solution :

$$(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t} \qquad \text{where } t = z - 2$$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} - ......\right\}$$

$$= \left(1 - \frac{1}{3!t^2} + \frac{1}{5!\,t^4} - .....\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!\,t^5} - ...\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - ......$$

$$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + ...$$

Since there are infinite number of terms in the negative powers of $(z-2)$ is an *essential singularity.*

(iii) $\dfrac{1}{\cos z - \sin z}$

Solution: Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero , i.e., by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi / 4$ is a simple pole of f(z).

Example :

What type of singularity have the following functions :

(i) $\dfrac{1}{1-e^z}$

Solution : Poles of $f(z) = \dfrac{1}{(1-e^z)}$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 =$ $e^{2n\pi I}$

$\therefore \quad z = 2n\pi i\,(n = 0, \pm 1, \pm 2, \ldots\ldots.)$

Clearly f(z) has a simple pole at $z = 2\pi i$.

(ii) $\dfrac{e^{2z}}{(z-1)^4}$

Solution :

$$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}.e^{2t} \qquad \text{where } t = z-1$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \ldots\ldots\right\}$$

$$= e^2 \left\{ \frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \ldots\ldots \right\}$$

$$= e^2 \left\{ \frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \ldots \right\}$$

since there are finite (4) number of terms containing negative powers of (z-1),

$\therefore$ z=1 is a pole of 4 th order.

(iii) ze^{1/z^2}

Solution : $f(z) = ze^{1/z^2} = z\left\{1 + \frac{1}{1!z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \ldots\right\}$

$$= z + z^{-1} + \frac{z^{-3}}{\ } + \frac{z^{-5}}{\ } + \ldots\infty$$

$$= z + z^{-1} + \frac{z^{-3}}{2} + \frac{z^{-5}}{6} + \ldots\infty$$

since there are infinite number of terms in the negative powers of z , therefore z = 0 is an essential singularity of f(z).

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

COMPLEX ANALYSIS –SMT1602

UNIT – V – Evaluation of Integral – SMT1602

UNIT V

EVALUATION OF INTEGRAL

RESIDUES

The co –efficient of $(z - a)^{-1}$ in the expansion of f(z) around an isolated singularity is called the **residue** of f(z) at that point. Thus in the Laurent's series expansion of f(z) around z = a i.e., $f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + a_{-1}(z - a)^{-2} + \ldots\ldots$, the residue of f(z) at z = a is a_{-1}.

$$\therefore \qquad \text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

i.e.,
$$\int_C f(z)\,dz = 2\pi i \text{ Res } f(a) \qquad \ldots\ldots\ldots\ldots(1)$$

CALCULATION OF RESIDUES

(1) If f(z) has a simple pole at z = a , then

$$\textbf{Res f(a)} = \underset{z \to a}{\text{Lt}}\left[(z-a)f(z)\right]$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z - a) + c_2(z - a)^2 \ldots\ldots\ldots + c_{-1}(z - a)^{-1}$$

Multiplying throughput by z – a , we have

$$(z - a)f(z) = c_0(z - a) + c_1(z - a)^2 + \ldots\ldots + c_{-1}$$

Taking limits as z → a , we get

$$\underset{z \to a}{\text{Lt}}\left[(z-a)f(z)\right] = c_{-1} = \text{Res } f(a)$$

(2) Another formula for Res f(a):

Let $f(z) = \phi(z) / \Psi(z)$, where $\Psi(z) = (z - a) F(z)$, $F(a) \neq 0$

Then $$\underset{z \to a}{\text{Lt}} \left[(z-a)\varphi(z)/\psi(z)\right] = \underset{z \to a}{\text{Lt}} \frac{(z-a)\left[\varphi(a) + (z-a)\varphi'(a) + \ldots\ldots\right]}{\psi(a) + (z-a)\psi'(a) + \ldots\ldots.}$$

$$= \underset{z \to a}{\text{Lt}} \frac{\varphi(a) + (z-a)\varphi'(a) + \ldots.}{\psi'(a) + (z-a)\psi''(a) + \ldots.}, \quad \text{since } \psi(a) = 0$$

Thus $$\textbf{Res f(a)} = \frac{\varphi(a)}{\psi'(a)}$$

(3) If f(z) has a pole of order n at $z = a$, then

$$\textbf{Res f(a)} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$$

Example :

Find the poles and residues of $f(z) = \dfrac{z}{z^2 - 3z + 2}$

Solution :

$$f(z) = \frac{z}{(z-2)(z-1)}$$

To find poles of f(z) put $Dr = 0$

(ie) $(z-2)(z-1) = 0$

$\therefore \quad z = 2, 1$ are two simple poles of f(z)

Residue of f(z) at z = 2

$$= \underset{z \to 2}{Lt}\left[(z-2)\frac{z}{(z-2)(z-1)}\right]$$

$$= \frac{2}{2-1} = 2$$

Residue of f(z) at z = 1

$$= \underset{z \to 1}{Lt}\left[(z-1)\frac{z}{(z-2)(z-1)}\right]$$

$$= \frac{1}{1-2} = -1$$

Example :

Find the poles and residues of f(z) = cot z.

Solution : f(z) = cot z

= cos z / sin z

This is of the form

$$f(z) = \phi(z) / \psi(z)$$

poles , sin z =0

$$z = n\pi \qquad z = 0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots$$

$$\therefore \varphi(a) \neq 0 \text{ and } \psi(a) = 0$$

$\therefore$ Residue at z = a is $\dfrac{\varphi(a)}{\Psi'(a)}$

$$z = a = 0, \pm\pi, \pm 2\pi, \ldots$$

$$\text{Residue of } f(z) = \frac{\cos z}{\frac{d}{dz}(\sin z)}$$

$$= \frac{\cos z}{\cos z}$$

$$=1$$

Example : Find the poles and residues of $f(z) = \dfrac{ze^z}{(z-a)^3}$

Solution: $f(z) = \dfrac{ze^z}{(z-a)^3}$

$\therefore z = a$ is a pole of order 3.

$$= \frac{1}{(m-1)} \underset{z\to a}{\mathrm{Lt}} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Here $m = 3$

$$= \frac{1}{2!} \underset{z\to a}{\mathrm{Lt}} \frac{d^2}{dz^2} (z-a)^3 \frac{ze^z}{(z-a)^3}$$

$$= \frac{1}{2!} \underset{z\to a}{\mathrm{Lt}} \frac{d^2}{dz^2} \left(ze^z\right)$$

$$= \frac{1}{2} \underset{z \to a}{Lt} \frac{d}{dz}\left(ze^z + e^z\right)$$

$$= \frac{1}{2} \underset{z \to a}{Lt} (e^z + ze^z + e^z)$$

$$= \frac{1}{2}(2e^a + ae^a)$$

$$= \frac{1}{2} e^a (2 + a)$$

Example: Evaluate the residue at the poles for the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2 + 4)}$$

Solution: $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2 + 4)}$ is pole of order 2

$z = 2i$ is a simple pole

Residue of f(z) at z = - 1 (pole of order 2)

$$\therefore \underset{z \to -1}{Lt} \frac{d}{dz}(z+1)^2 \frac{z^2 - 2z}{(z+1)^2 (z+4)}$$

$$= \underset{z\to -1}{\mathrm{Lt}} \frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$= \underset{z\to -1}{\mathrm{Lt}} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$$

$$= \frac{(5)(-4)-(3)(-2)}{25}$$

$$= \frac{14}{25}$$

Residue of f(z) at z = 2i (simple pole)

$$= \underset{z\to 2i}{\mathrm{Lt}} (z-2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \underset{z \to 2i}{Lt} \frac{z^2 - 2z}{(z+1)^2 (z-2i)(z+2i)}$$

$$= \frac{(2i)^2 - 2(2i)}{(2i+1)^2 (4i)}$$

$$= \frac{-4-4i}{(-4+1+4i)4i}$$

$$= \frac{-4(1+i)}{4i(-3+4i)}$$

$$= \frac{-(1+i)}{-3i-4} = \frac{(1+i)}{3i+4} \times \frac{(-3i+4)}{(-3i+4)}$$

$$= \frac{7+i}{25}$$

Residue of f(z) at z = -2i (simple pole)

$$= \underset{z \to -2i}{Lt} (z+2i) \frac{z^2 - 2z}{(z+1)^2 (z+2i)(z-2i)}$$

$$= \underset{z \to -2i}{Lt} \frac{z^2 - 2z}{(z+1)^2 (z-2i)}$$

$$= \frac{(-2i)^2 - 2(-2i)}{(-2i+1)^2 (-4i)}$$

$$= \frac{-4 - +4i}{(-4i)(-4-4i+1)}$$

$$= \frac{-4+4i}{(-4i)(-3-4i)}$$

$$= \frac{1-i}{(i)(3+4i)}$$

$$= \frac{(1-i)}{(3i-4)} x \frac{(-3i-4)}{(-3i-4)}$$

$$= \frac{(1-i)(3i+4)}{(3i-4)(3i+4)}$$

$$= \frac{(1-i)(3i+4)}{(3i-4)(3i+4)}$$

$$\frac{(1-i)(3i+4)}{(3i-4)(3i+4)}$$

Example: Find poles and residues of $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+1)}$

Solution: $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+1)}$

Poles of f(z) is

$z = -1$ is pole of order 2

$z = i$ is a simple pole

$z = -i$ is a simple pole

Residue at $z = -1$ (pole of order 2)

$$= \underset{z \to -1}{Lt} \frac{d}{dz}(z+1)^2 \frac{z^2 - 2z}{(z+1)^2 (z^2+1)}$$

$$= \underset{z \to -1}{Lt} \frac{d}{dz}\left(\frac{z^2 - 2z}{z^2+1}\right)$$

$$= \underset{z \to -1}{Lt} \frac{(z^2+1)(2z-2) - (z^2 2z)(2z)}{(z^2+1)^2}$$

$$= \frac{(2)(-4) - (3)(-2)}{4}$$

$$= \frac{-8+6}{4}$$

$$= -\frac{2}{4} = -\frac{1}{2}$$

Residue of f(z) at $z = i$ (simple pole)

$$= \underset{z \to -1}{Lt} (z - i) \frac{z^2 - 2z}{(z+1)^2 (z^2+1)(z-i)}$$

$$= \underset{z \to i}{\text{Lt}} \frac{z^2 - 2z}{(z+1)^2 (z+i)}$$

$$= \frac{(i)^2 - 2(i)}{(i+1)^2 (2i)}$$

$$= \frac{-(1+2i)}{-4} = \frac{1+2i}{4}$$

Residue of f(z) at z = I (simple pole)

$$= \underset{z \to -i}{\text{Lt}} (z+i) \frac{z^2 - 2z}{(z+1)^2 (z+i)(z-i)}$$

$$= \underset{z \to -i}{\text{Lt}} (z+i) \frac{z^2 - 2z}{(z+1)^2 (z+i)(z-i)}$$

$$= \frac{(-i)^2 - 2(-i)}{(-i+1)^2 (-2i)}$$

$$= \frac{-1+2i}{(-2i)(-2i)} = \frac{-1+2i}{-4} = \frac{1-2i}{4}$$

RESIDUE THEOREM

If f(z) is analytic in a closed curve C except at a finite number of singular points within C, then

$$\int_C f(z)dz = 2\pi i \text{ x (sum of the residues at the singular points within C)}$$

Let us surround each of the singular points $a_1, a_2, a_3, \ldots, a_n$ by a small circ such that it encloses no other singular point. Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which f(z) is analytic.

∴ Applying Cauchy's theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz \qquad \text{by (1)}$$

$$= 2\pi i[\operatorname{Res} f(a_1) + \operatorname{Res} f(a_2) + \ldots + \operatorname{Res} f(a_n)]$$

which is the desired result.

Example: Use residue theorem to evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz$

Where c is $|z-2| = 2$.

Solution: $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)}$

Poles are

$z = 1$ is a simple pole

$z = \pm 3i$ are two simple poles.

Here c is the circle $|z-2| = 2$.

∴ $z = 1$ is only pole lies inside c.

$\therefore$ By cauchy residue theorem

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residue of } f(z) \qquad \ldots(1)$$

at the poles which lies inside c)

$\therefore$ Residue of f(z) at z = 1 (simple pole)

$$= \underset{z\to 1}{\text{Lt}}(z-1)\frac{3z^2+2}{(z-1)(z^2+9)}$$

$$= \frac{5}{10} = \frac{1}{2}$$

$$\int_C f(z)dz = 2\pi(1/2)$$

$$= \pi i$$

Example: Determine poles and residues of $f(z) = \dfrac{z}{(1-z)^2(z+2)}$ and hence evaluate

$\int_C f(z)dz$ where c is the curve $|z| = 5/2$

Solution: $f(z) = \dfrac{z}{(1-z)^2(z+2)}$

$\therefore$ poles are z = 1 is poles of order 2 and z = -2 is a simple pole.

Here c is the circle $|z| = 5/2$.

$\therefore$ z = 1 and z = -2 are lying inside c.

$\therefore$ Residue of f(z) at z = 1 (pole of order 2)

$$= \underset{z \to 1}{Lt} \frac{d}{dz} (z-1)^2 \frac{z}{(1-z)^2 (z+2)}$$

$$= \underset{z \to 1}{Lt} \frac{d}{dz} \left(\frac{z}{z+2} \right)$$

$$= \underset{z \to 1}{Lt} \frac{(z+2)(1)-(z)(1)}{(z+2)^2}$$

$$\frac{3-1}{9} = \frac{2}{9}$$

Residue of f(z) at z = - 2 (simple pole)

$$= \underset{z \to -2}{Lt} (z+2) \frac{z}{(1-z)^2 (z+2)}$$

$$= \frac{(-2)}{(-2-1)^2} = -\frac{2}{9}$$

$\therefore$ By Cauchy residue theorem

$\int_C f(z)dz = 2\pi i$ (sum of the residues of f(z) at the poles which lies inside c)

$$= 2\pi i \left(\frac{2}{9} \right) - \left(\frac{2}{9} \right)$$

$$= 0$$

Example: Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where $|z| = 3$.

Solution: $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are

$z = 1$ simple pole

$z = 2$ simple pole

Here the circle is $|z| = 3$

$\therefore$ Both $z = 1$ & $z = 2$ lies inside c

$\therefore$ Residue of f(z) at $z = 1$

$$= \underset{z \to 1}{\mathrm{Lt}} (z-1) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin \pi + \cos \pi}{-1} = \frac{-1}{-1} = 1$$

Residue of f(z) at $z = 2$.

$$= \underset{z \to 2}{\mathrm{Lt}} (z-2) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1} = \frac{1}{1} = 1$$

$\therefore$ By residue theorem

$\int_C f(z) dz = 2\pi i$ (sum of the residues at the interior poles)

$= 2\pi i\,(1+i)$

$= 4\pi i$

Example :

Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where c is the circle $|z| = 3/2$

Solution :

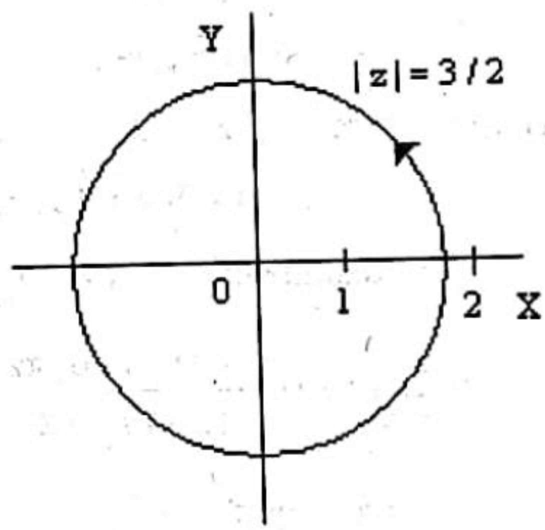
$$f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

$\therefore$ The poles are

$z = 0$ simple pole

$z = 1$ simple pole

$z = 2$ simple pole



Here the circle is $|z| = 3/2$

$\therefore$ $z = 0$ & $z = 1$ lie inside

c and $z = 2$ lies outside c.

$\therefore$ Residue of $f(z)$ at $z = 0$ simple pole

$$= \underset{z \to 0}{\text{Lt}}\ z \frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4}{2} = 2$$

Residue of f(z) at z =1 is

$$\underset{z \to 1}{Lt}\ (z-1)\frac{4-3z}{z(z-2)}$$

$$= \frac{1}{1(-1)} = -1$$

$\therefore$ By cauchy integral theorem

$$\int_C f(z)\,dz = 2\pi i\ (\text{sum of the residues at the interior poles})$$

$$= 2\pi i \quad (2-1)$$

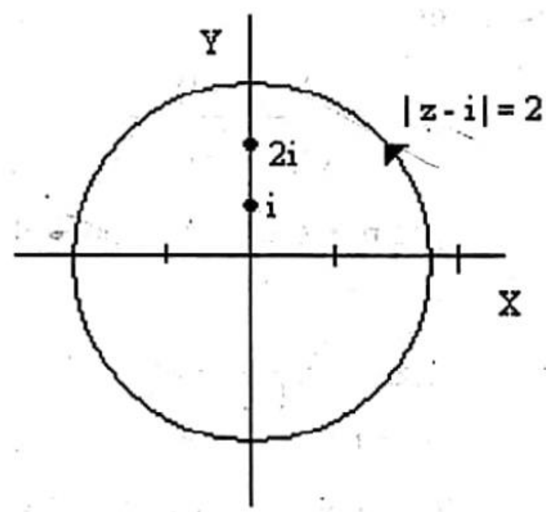
$$= 2\pi i$$

Example :

Evaluate $\displaystyle\int_C \frac{dz}{(x^2+4)^2}$ around the closed contour $|z-i| = 2$.

Solution :

$$f(z) = \frac{1}{(z^2+4)^2}$$



The poles are $z = \pm 2i$ are pole of order 2.

Here the circle is $|z-i| = 2$

$\therefore$ $z = 2i$ is the only pole lies inside c.

$\therefore$ Residue of f(z) at $z = 2i$ (pole of order 2)

$$= \underset{z \to 2i}{Lt} \frac{d}{dz}(z - $$

$$= \underset{z \to 2i}{Lt} \frac{(z+2i)^2(0) - 1(2)(z+2i)}{(z+2i)^4}$$

$$= \underset{z \to 2i}{Lt} \frac{-(2z+4i)}{(z+2i)^4} = \frac{-(4i+4i)}{(4i)^4} = -\frac{i}{32}$$

$\therefore$ By cauchy's integral theorem

$$\int_C f(z)\, dz = 2\pi i \quad (\text{sum of the residues at the interior poles})$$

$$= 2\pi i \left(\frac{-i}{32}\right) = \frac{\pi}{16}$$

Example : Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)}$ where c is the circle $|z-i| = 2$.

Solution :

$$f(z) = \frac{(z-1)}{(z+1)^2(z-2)}$$

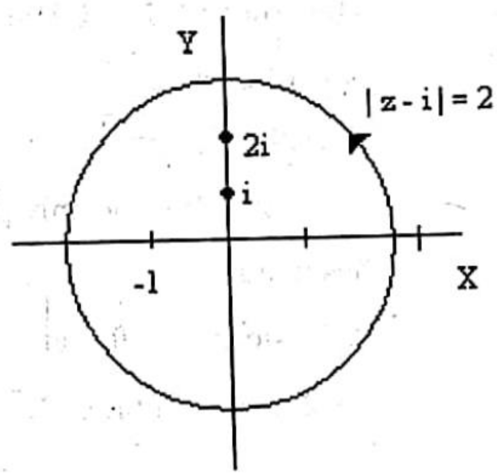
The poles are $z = -1$ pole of order 2

& $z = 2$ simple pole

Here the circle is $|z - i| = 2$

Therefore $z = -1$ is the only pole

Lies inside c.



Therefore Residue of f(z) at

$z = -1$ (pole of order 2)

$$= \underset{z \to -1}{Lt} \frac{d}{dz}(z+1)^2 \cdot \frac{(z-1)}{(z+1)^2(z-2)}$$

$$= \underset{z \to -1}{Lt} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \underset{z \to -1}{Lt} \frac{(z-2)(1)-(z-1)\,(1)}{(z-2)^2}$$

$$= \frac{(-3)-(-2)}{9} = -\frac{1}{9}$$

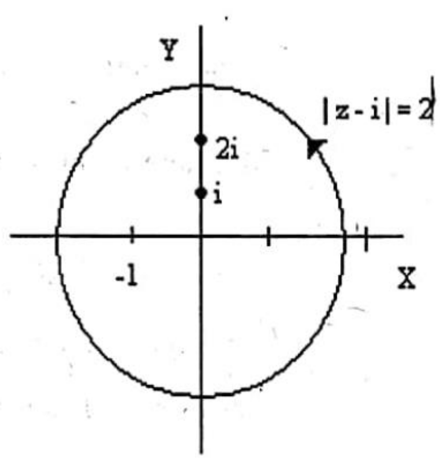
$\therefore$ By Residue theorem

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Example :

Find the poles and residues of

$$f(z) = \frac{z-3}{(z+1)^2(z-2)}$$



The poles are $z = -1$ pole of order 2

& $z = 2$ simple pole

Here the circle is $|z - i| = 2$

Here $z = -1$ is the only pole lies

Inside c.

Therefore Residue at $z = -1$

(pole of order 2)

Example 4 ***If $f(z) = \sin z$ is an analytic function, prove that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.***

Solution : Given :
$$f(z) = \sin z = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos(x)\sin(iy)$$
$$= \sin x \cosh y + i\cos x \sinh y$$

Consider $u(x,y) = c_1$
$$\sin x \cosh y = c_1 \qquad \ldots (1)$$

Differentiating (1) partially with respect to x, we get

$$\sin x \sinh y \frac{dy}{dx} + \cos x \cosh y = 0$$
$$\frac{dy}{dx} = -\frac{\cos x \cosh y}{\sin x \sinh y}$$
$$m_1 = -\cot x \coth y$$

Again consider $v(x,y) = c_2$
$$\cos x \sinh y = c_2. \qquad \ldots (2)$$

Differentiating partially with respect to x, we get

$$-\sin x \sinh y + \cos x \cosh y \frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{\sin x \sinh y}{\cos x \cosh y}$$
$$m_2 = \tan x \tanh y$$
$$\therefore m_1 m_2 = -1$$

$$u(x,y) = c_1 \text{ and } v(x,y) = c_2 \text{ are orthogonal.}$$

Note : For any analytic function $F(z) = u + iv$, the family of curves $u = c_1, v = c_2$ forms an orthogonal system.

Example :

Evaluate $\int_C \frac{z-3}{z^2+2z+5} dz$, where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

Solution:

The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by $z^2+2z+5=0$

i.e., by $$z = \frac{-2 \pm \sqrt{(4-20)}}{2} = -1 \pm 2i$$

(i) Both the poles $z = -1+2i$ and $z = -1-2i$ lie outside the circle $|z| = 1$.

Therefore , f(z) is analytic everywhere within C.

Hence by Cauchy's theorem, $\int_C \frac{z-3}{z^2+2z+5} dz = 0$

(ii) Here only one pole $z = -1+2i$ lies inside the circle C: $|z+1-i| = 2$.

Therefore , f(z) is analytic within C except at this pole.

$$\therefore \text{ Res } f(-1+2i) = \underset{z \to -1+2i}{Lt} [\{z-(-1+2i)\} f(z)] = \underset{z \to -1+2i}{Lt} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$$

$$= \lim_{z \to -1+2i} \frac{(z-3)}{z+1+2i} = \frac{-4+2i}{4i} = i+1/2$$

Hence by residue theorem $\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(-1+2i) = 2\pi i\,(i+1/2) = \pi\,(i-2)$

(iii) Here only one pole $z = -1 - 2i$ lies inside the circle C: $|z+1+i| = 2$.

Therefore, f(z) is analytic within C except at this pole.

$$\therefore \operatorname{Res} f(-1-2i) = \lim_{z \to -1-2i} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$$

$$= \lim_{z \to -1-2i} \frac{(z-3)}{z+1-2i} = \frac{-4-2i}{-4i} = 1/2 - i$$

Hence by residue theorem

$$\int_C f(z)\,dz$$

$$= 2\pi i \operatorname{Res} f(-1-2i)$$

$$= 2\pi i\,(1/2 - i)$$

$$= \pi\,(2+i)$$

CONTOUR INTEGRATION

VALUATION OF REAL DEFINITE INTEGRALS

any important definite integrals can be evaluated by applying the Residue theorem to operly chosen integrals.

a) Integration around the circle: An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form $\int_C f(z)dz$, where f(z) is a rational function of z and C is a unit circle $|z| = 1$.

Hence the integral is equal to $2\pi i$ times the sum of the residues at those poles of f(z) which are within C.

Procedure: Integrals of the form $\int_0^{2\pi} \varphi(\cos\theta, \sin\theta)d\theta$ where φ is a rational function of $\cos\theta$ and $\sin\theta$.

Working rule: put $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

since $z = e^{i\theta}$

$$dz = i e^{i\theta} d\theta$$

$$d\theta = \frac{dz}{ie^{i\theta}} \qquad = \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \varphi(\cos\theta, \sin\theta)d\theta = \int_C \varphi\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\frac{dz}{iz}\right]$$ where c is the unit circle

$|z| = 1$

$$= \int f(z)dz$$

$\therefore$ By cauchy residue theorem

$= 2\pi i$ (sum of the residues of f(z) at e poles which lies inside c)

Example: Using method of contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Solution: put $z = e^{i\theta}$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{dz/iz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$ where c is the unit circle $|z| = 1$

$$= \int_C \frac{dz/iz}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)}$$

$$= \int_C \frac{dz/iz}{\frac{4z+z^2+1}{2z}}$$

$$= \frac{2}{i} \int_C \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i} \int_C f(z)dz$$

By cauchy residue theorem

$$= \frac{2}{i} 2\pi i \text{ (sum of the residue of f(z) at the poles lies inside c)}$$

$$= 4\pi \text{ (sum of the residues of f(z) at the poles inside c)}$$

The poles of f(z) are given by the roots of $z^2 + 4z + 1 = 0$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

i.e., $z = -2 + \sqrt{3}$ & $z = 2 - \sqrt{3}$

i.e., $\alpha = -2 + \sqrt{3}$, $\beta = -2 - \sqrt{3}$

But $z = \alpha$ lies inside c

Residue of f(z) at $z = \alpha$.(simple pole).

Residue at the simple pole is given by $\underset{z \to \alpha}{\text{Lt}} (z-\alpha)f(z)$

Hence $\underset{z\to\alpha}{\mathrm{Lt}}\,(z-\alpha)\dfrac{1}{(z-\alpha)(z-\beta)}$

$$=\frac{1}{\alpha-\beta}$$

$$=\frac{1}{(-2+\sqrt{3}-(-2-\sqrt{3}))}$$

$$=\frac{1}{2\sqrt{3}}$$

$$\therefore \int_0^{2\pi}\frac{dz}{2+\cos\theta}=4\pi\left(\frac{1}{2\sqrt{3}}\right)$$

$$=\frac{2\pi}{\sqrt{3}}$$

Example: Evaluate $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

$$\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta=\int_0^{2\pi}\frac{\mathrm{R.P.}e^{i2\theta}}{5+4\cos\theta}\,d\theta$$

$$=\mathrm{R.P.}\int_0^{2\pi}\frac{(e^{i\theta})^2}{5+4\cos\theta}\,d\theta$$

put $z=e^{i\theta}$

$$d\theta=\frac{dz}{iz}$$

$$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$= R.P. \int_C \frac{z^2 dz / iz}{5 + 4\frac{1}{2}\left(z + \frac{1}{z}\right)}$$

$$= \frac{1}{i} R.P. \int_C \frac{z^2 dz}{5z + 2z^2 + 2}$$

$$= \frac{1}{i} R.P. \int_C f(z) dz$$

$$\text{where } f(z) = \frac{z^2}{2z^2 + 5z + 2}$$

$= \frac{1}{i}$ R.P. $2\pi i$ (sum of the residue of f(z) at its interior poles)

=R.P. 2π (sum of the residue of f(z) at its interior poles)

For poles of f(z) put Dr.=0

i.e., $2z^2 + 5z + 2 = 0$

$2z(z + 2) + 1(z + 2) = 0$

$(2z + 1)(z + 2) = 0$

$z = -2, -1/2$

But only $z = -1/2$ lies inside c,

Hence Residue of f(z) at $z = -1/2$ is

$$= \underset{z\to -\frac{1}{2}}{\mathrm{Lt}} \left(z+\frac{1}{2}\right) f(z)$$

$$= \underset{z\to -\frac{1}{2}}{\mathrm{Lt}} \left(z+\frac{1}{2}\right) \frac{z^2}{(z+2)(2z+1)}$$

$$= \underset{z\to -\frac{1}{2}}{\mathrm{Lt}} \left(z+\frac{1}{2}\right) \frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1/4}{(2)(3/2)} = \frac{1}{12}$$

$$\therefore \text{R.P.} \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta = 2\pi\left(\frac{1}{12}\right)$$

$$\text{R.P.} \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta} d\theta = \left(\frac{\pi}{6}\right)$$

$$\text{i.e., } \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \left(\frac{\pi}{6}\right)$$

Example: By integrating around a unit circle, evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$.

Putting $z = e^{i\theta}$, $d\theta = dz / iz$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

and $\cos 3\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right) = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

Hence the given integral $I = \int_C \frac{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)}{5 - 2\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$

$$= -\frac{1}{2i} \int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} dz = -\frac{1}{2i} \int_C \frac{(z^6 + 1)dz}{z^3(2z - 1)(z - 2)}$$

$= -\frac{2}{2i} \int f(z)dz$ where C is the unit circle $|z| = 1$.

Now f(z) has a pole of order 3 at $z = 0$ and simple poles at $z = ½$ and $z = 2$. Of these only $z = 0$ and $z = ½$ lie within the unit circle.

$$\therefore \operatorname{Res} f(1/2) = \operatorname*{Lt}_{z \to 1/2} \frac{(z - 1/2)(z^6 + 1)}{(2z - 1)(z - 2)} = \operatorname*{Lt}_{z \to 1/2} \left(\frac{z^6 + 1}{2z^3(z - 2)}\right) = -\frac{65}{24}$$

$$\operatorname{Res} f(0) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}[(z - 0)^n f(z)]\right)_{z=0}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\left(\frac{z^6 + 1}{2z^2 - 5z + 2}\right)\right]_{z=0} = \frac{d}{dz}\left[\frac{(2z^2 - 5z + 2)6z^5 - (z^6 + 1)(4z - 5)}{2(2z^2 - 5z + 2)^2}\right] \text{ at } z = 0$$

$$=\left[\frac{d}{dz}\left(\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^4}\right)\right]_{z=0}$$

$$=\left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2 5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$=\frac{4(-4)-5(-20)}{2x16}=\frac{84}{32}=\frac{21}{8}$$

Hence $I=-\frac{1}{2i}[2\pi i(\operatorname{Res} f(1/2)+\operatorname{Res} f(0))]=-\pi\left(-\frac{65}{24}+\frac{21}{8}\right)=-\pi\left(-\frac{1}{12}\right)=\frac{\pi}{12}.$