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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

III YEAR B.SC MATHEMATICS

SMT5103 MECHANICS 1

Unit 1- Statics – SMT1503

Introduction

"***Mathematics is the Queen of the Sciences and Number Theory is the Queen of Mathematics***" - Gauss.

Mechanics is a branch of Science which deals with the action of forces on bodies. Mechanics has two branches called Statics and Dynamics.

Statics is the branch of Mechanics which deals with bodies remain at rest under the influence of forces.

Dynamics is the branch of Mechanics which deals with bodies in motion under the action of forces.

Definitions:

Space: The region where various events take place is called a space.

Body: A portion of a matter is called a body.
Rigid body: A body consists of innumerable particles in which the distance between any two particles remains the same in all positions of the body is called a rigid body.

Particle: A particle is a body which is very small whose position at any time coincides with a point.

Motion: If a body changes its position under the action of forces, then it is said to be in motion.

Path of a particle: It is the curve joining the different positions of the particle in space while in motion.

Speed: The rate at which the body describes its path. It is a scalar quantity.

Displacement (vector quantity): It is the change in the positions of a particle in a certain interval.
Velocity (vector quantity): It is the rate of change of displacement.

Acceleration (vector quantity): It is the rate of change of velocity.
Equilibrium: A body at rest under the action of any number of forces on it is said to be in equilibrium.
Equilibrium of two forces

$Q \longleftarrow\!\!\!\longrightarrow P$

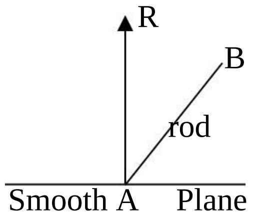
If two forces P, Q act on a body such that they have equal magnitude, opposite directions, same line of action then they are in equilibrium.

Force (vector): Force is any cause which produces or tends to produce a change in the existing state of rest of a body or of its uniform motion in a straight line. Force is represented by a straight line (through the point of application) which has both magnitude and direction.

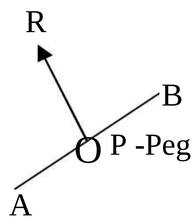
Types of forces: Weight, attraction, repulsion, tension, thrust, friction etc.
By Newton's third law, action and reaction are always equal and opposite.

Directions of Normal Reaction 'R' at the point of contact.

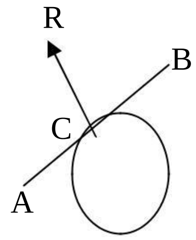
1. When a rod AB is in contact with a smooth plane, R is perpendicular to the plane at the point of contact A.



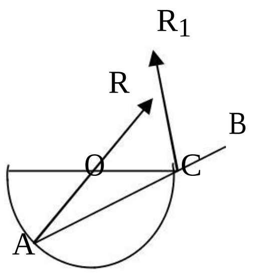
2. When a rod AB is resting on a smooth peg P, R is perpendicular to the rod at the point of contact P.



3. When a rod AB is resting on a smooth sphere, R is normal to the sphere at the point of contact C.



4. When a rod AB is resting on the rim of a hemisphere, with one end A in contact with the inner surface and C in contact with the rim. Then the normal reactions R at A is normal to the spherical surface and passes through the centre O, R_1 at C is perpendicular to the rod.



Regular polygon is a polygon with equal sides. Its vertices lie on a circle.

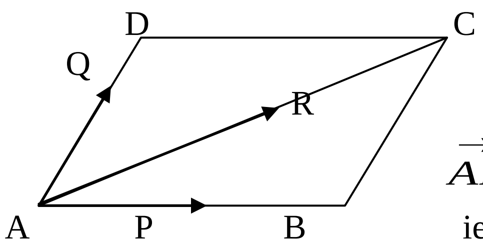
UNIT I
Forces Acting at a Point

Introduction

Forces are represented by straight lines with magnitude and direction. Forces acting on a rigid body may be represented by straight lines with magnitude and direction passing through the same point and we say the forces are acting at a point. If P_1, P_2, P_3 …….. are the forces acting on a rigid body it is easy to find a single force whose effect is same as the combined effect of P_1, P_2, P_3 …….. Then the single force is called the resultant. P_1, P_2, P_3 ….. are called the components of the resultant. In this section we study some theorems and methods to find the resultant of two or more forces acting at a point.

1.1 Parallelogram law of forces (Fundamental theorem in statics)

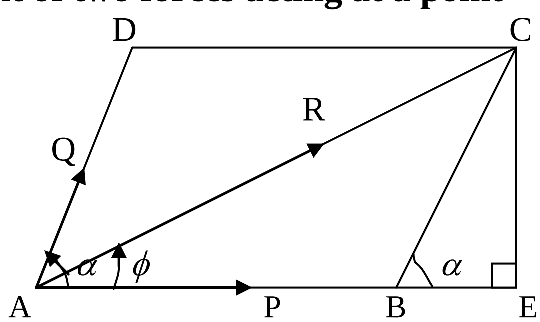
If two forces acting at a point be represented in magnitude and direction by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.



$$\vec{AB} + \vec{AD} = \vec{AC}$$

ie) P + Q = R

The resultant of two forces acting at a point



Let the two forces P and Q acting at A be represented by AB and AD. Let α be the angle between them.

i.e. $\angle BAD = \alpha$

Complete the parallelogram ABCD.

Then the diagonal AC will represent the resultant.

Let $\angle$CAB $= \varphi$

Draw CE $\perp r$ to AB. Now BC = AD = Q.

From the right angled ΔCBE,

$$\sin C\hat{B}E = \frac{CE}{BC} \quad \text{i.e. } \sin\alpha = \frac{CE}{Q}$$

$\therefore$ CE = Q $\sin\alpha$ .... (i)

$$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$\therefore$ BE = Q $\cos\alpha$ .... (ii)

$R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$

$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$

$= P^2 + 2PQ\cos\alpha + Q^2$

$$\boxed{\therefore R = \sqrt[2]{\left(P^2 + 2PQ\cos\alpha + Q^2\right)}}$$

$$\boxed{\tan\varphi = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha}}$$

Result 1 If the forces P and Q are at right angles to each other, then $\alpha = 90^{o}$;

$$R = \sqrt{P^2 + Q^2} \qquad \tan\varphi = \frac{Q}{P}$$

Result 2 If the forces are equal (i.e.) Q = P, then

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1 + \cos\alpha)}$$

$$= \sqrt{2P^2.2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

$$\tan\varphi = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2}$$

ie) $\boxed{\varphi = \frac{\alpha}{2}}$

Thus the resultant of two equal forces P, P at an angle α is 2 P cos $\frac{\alpha}{2}$ in a direction bisecting the angle between them.

Result 3 Resultant R is greatest when cos α is greatest.

i.e. when cos α = 1 or $\alpha = 0^{o}$.

ie) Greatest value of R is R = P +Q.

R is least when cos α is least.

i.e. when cos $\alpha = -1$ or $\alpha = 180^{o}$. Least value of R is P~Q.

Problem 1

The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y, Prove that.

$$P = (X^2 + QR)^{1/2} = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

Prove also that, if P + Q + R = 0, Y = X.

Solution:

Let α be the angle between P and Q

Given

$X^2 = P^2 + Q^2 + 2PQ\cos\alpha$ ……..... (1)

$X^2 = P^2 + R^2 + 2PR\cos\alpha$ .... (2)

$Y^2 = Q^2 + R^2 + 2QR\cos\alpha$ .... (3)

(1) – (2) gives $0 = Q^2 - R^2 + 2P\cos\alpha\,(Q-R)$

i.e. $0 = (Q-R)(Q+R+2P\cos\alpha)$

But $Q \neq R$ and so $Q - R \neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

$$\cos\alpha = -\frac{Q+R}{2P} \qquad \text{........ (4)}$$

Substitute (4) in (1),

$$X^2 = P^2 + Q^2 + 2PQ\left[-\left(\frac{Q+R}{2P}\right)\right] = P^2 + Q^2 - Q^2 - QR$$

$P^2 = X^2 + QR$. i.e. $P = (X^2 + QR)^{1/2}$

Substitute (4) in (3),

$$Y^2 = Q^2 + R^2 + 2QR\left[-\left(\frac{Q+R}{2P}\right)\right]$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

$$P = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

If P + Q + R = 0, then Q + R = $-P$,

$$\therefore \text{From (4), } \cos\alpha = -\frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$$

$$\cos\alpha = \frac{1}{2} \Rightarrow$$

$$X^2 = P^2 + R^2 + PR \ldots \quad \ldots \quad \ldots (5)$$

$$Y^2 = Q^2 + R^2 + QR \ldots \quad \ldots \quad \ldots (6)$$

(5) – (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$

$$= (P - Q)(P + Q + R)$$

$$= (P - Q).0 = 0$$

$$\therefore X = Y$$

Problem 2

Two forces of given magnitude P and Q act at a point at an angle α. What will be the maximum and minimum value of the resultant?

Solution:

i. Maximum value of the resultant = P + Q

ii. Minimum value of the resultant = P~ Q.

Problem 3

The greatest and least magnitudes of the resultant of two forces of constant magnitudes are R and S respectively. Prove that, when the forces act at an angle 2φ, the resultant is of magnitude $\sqrt{R^2\cos^2\varphi + S^2\sin^2\varphi}$

Solution:

Given, R = P + Q, S = P-Q, where P and Q are two forces.

When P and Q are acting at an angle 2φ

$$\text{Resultant} = \sqrt{P^2+Q^2+2PQ.\cos 2\varphi}$$

$$= \sqrt{\left(P^2+Q^2\right)+2PQ\left(\cos^2\varphi-\sin^2\varphi\right)}$$

$$= \sqrt{\left(P^2+Q^2\right)\left(\sin^2\varphi+\cos^2\varphi\right)+2PQ\left(\cos^2\varphi-\sin^2\varphi\right)}$$

$$= \sqrt{\left(P^2+Q^2+2PQ\right)\cos^2\varphi+\left(P^2+Q^2-2PQ\right)\sin^2\varphi}$$

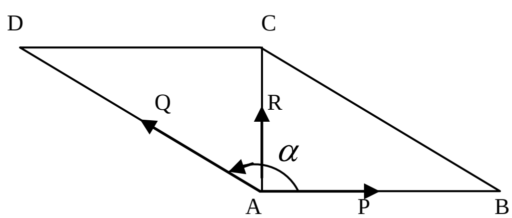
$$= \sqrt{R^2\cos^2\varphi+S^2\sin^2\varphi}.$$

Problem 4

The resultant of two forces P and Q is at right angles to P. Show that the angle between the forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$

Solution:

Let α be the angle between the two forces P and Q. Given φ = 90°.



We know, $\tan\varphi = \dfrac{Q\sin\alpha}{P+Q\cos\alpha}$

i.e. $\tan 90^{o} = \dfrac{Q\sin\alpha}{P+Q\cos\alpha}$

$$\frac{1}{0} = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$\therefore P+Q\cos\alpha = 0$$

$$\therefore \cos\alpha = -\frac{P}{Q}$$

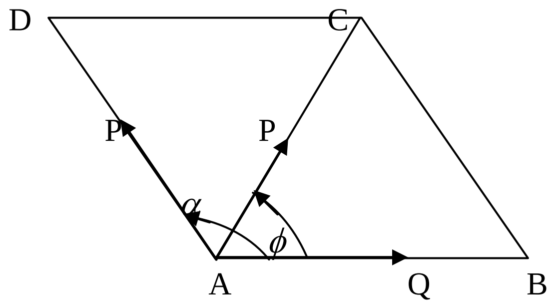
$$\boxed{\therefore \alpha = \cos^{-1}\left(-\frac{P}{Q}\right)}$$

Problem 5

The resultant of two forces P and Q is of magnitude P. Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2-Q^2}$.

Solution:

Let α be the angle between P and Q



Given, $P^2 = P^2+Q^2+2PQ\cos\alpha$.

$\therefore$ Q (Q+2Pcosα) = 0

$$\boxed{\therefore \cos\alpha = -\frac{Q}{2P}}$$

If **P is doubled,** let R be the new resultant, and φ be the angle between Q and R.

$$\therefore R^2 = (2P)^2+Q^2+2(2P)Q.\cos\alpha$$

$$= 4P^2+Q^2+4PQ\left(-\frac{Q}{2P}\right)$$

$$= 4P^2+Q^2-2Q^2 = 4P^2-Q^2$$

$$\boxed{\therefore R = \sqrt{4P^2-Q^2}}$$

$$\tan\varphi = \frac{(2P)\sin\alpha}{Q+(2P)\cos\alpha} = \frac{2P\sin\alpha}{Q+2P\left(-\frac{Q}{2P}\right)}$$

$$\text{i.e. } \tan\varphi = \frac{2P\sin\alpha}{0}$$

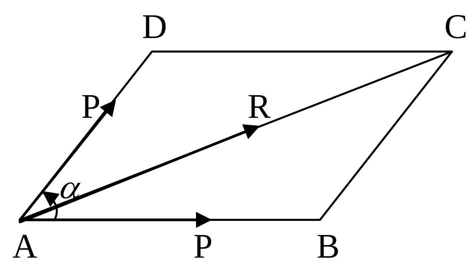
$$\therefore \cos\varphi = 0 \quad \Rightarrow \quad \varphi = 90^0$$

$\therefore$ Q is at right angles to R.

Problem 6

Two equal forces act on a particle, find the angle between them when the square of their resultant is equal to three times their product.

Solution:



Let α be the angle between the two equal forces P, P, and let R be their resultant.

$$\therefore R^2 = P^2 + P^2 + 2P.P.\cos\alpha$$

$$= 2P^2(1+\cos\alpha) = 2P^2 \times 2\cos^2\frac{\alpha}{2}$$

$$\text{i.e. } R^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \quad \boxed{R = 2P\cos\frac{\alpha}{2}}$$

$$\text{Given, } R^2 = 3\times P\times P = 3P^2$$

$$\therefore 3P^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \cos^2\frac{\alpha}{2} = \frac{3}{4} \quad \Rightarrow \quad \cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{\alpha}{2} = 30^{o}$$

$$\Rightarrow \alpha = 60^{0}$$

Problem 7

If the resultant of forces 3P, 5P is equal to 7P find

i. the angle between the forces

ii. the angle which the resultant makes with the first force.

Solution:

Let α be the angle between 3P, 5P

i. Given $(7P)^2 = (3P)^2 + (5P)^2 + 2\,(3P)\,(5P)\,.\cos\alpha$

$$49P^2 = 9P^2 + 25P^2 + 30P^2\cos\alpha$$

$$\therefore 15P^2 = 30P^2\cos\alpha$$

$$\therefore \cos\alpha = \frac{1}{2} \quad \Rightarrow \alpha = 60^{0}$$

ii. Let φ be the angle between the resultant and 3P.

$$\therefore \tan\varphi = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$= \frac{5P.\sin\alpha}{3P + 5P.\cos\alpha}$$

$$= \frac{5P.\sin 60^\circ}{3P + 5P.\cos 60^\circ}$$

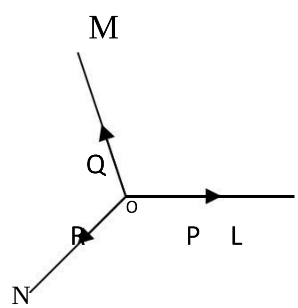
$$= \frac{5 \times \frac{\sqrt{3}}{2}}{3 + \left(5 \times \frac{1}{2}\right)}$$

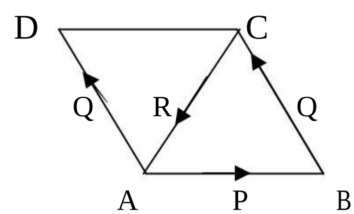
$$\tan\varphi = \frac{5\sqrt{3}}{11}$$

$$\therefore \varphi = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

1.2 Triangle of forces

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.





Let the forces, P,Q,R act at a point O and be represented in magnitude and direction by the sides AB,BC,CA of the triangle ABC.

To prove : They will be in equilibrium.

Complete the parallelogram BADC.

P+Q $= \overline{AB} + \overline{AD} = \overline{AB} + \overline{BC}$

$= \overline{AC}$

ie) The resultant of the forces P, Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

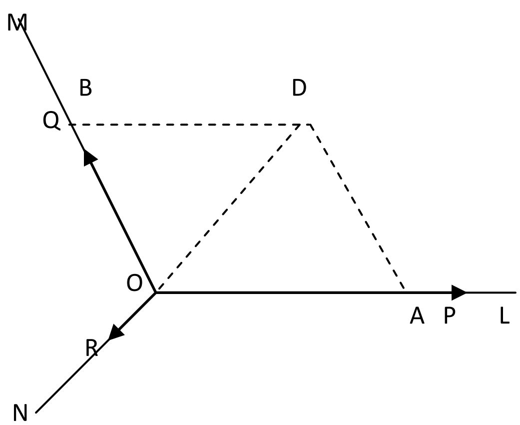
Hence P+Q+R $= \overline{AC} + \overline{CA} = \overline{0}$

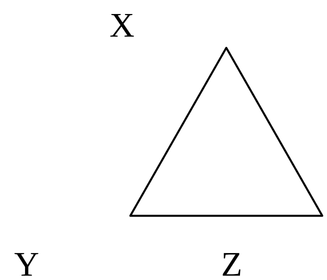
Principle

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

1.3 Lami's Theorem

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.





Proof:

By converse of the triangle of forces, the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.

By sine rule in ΔOAD , we have

$$\frac{OA}{\sin\angle ODA}=\frac{AD}{\sin\angle DOA}=\frac{DO}{\sin\angle OAD} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

But $\angle OAD = alt.\angle BOD = 180^0 - \angle MON$

$$\therefore \sin\angle ODA = \sin\left(180^0 - \angle MON\right) = \sin\angle MON \quad \ldots\ldots\ (2)$$

Also $\angle DOA = 180^0 - \angle NOL$

$$\therefore \sin\angle DOA = \sin\left(180^0 - \angle NOL\right) = \sin\angle NOL \quad \ldots\ldots\ (3)$$

And $\angle OAD = 180^0 - \angle BOA = 180^0 - \angle LOM$

$\therefore \sin\angle OAD = \sin\left(180^0 - \angle LOM\right) = \sin\angle LOM$ ……. (4)

Substitute (2), (3), (4) in (1),

$$\frac{OA}{\sin\angle MON} = \frac{AD}{\sin\angle NOL} = \frac{DO}{\sin\angle LOM}$$

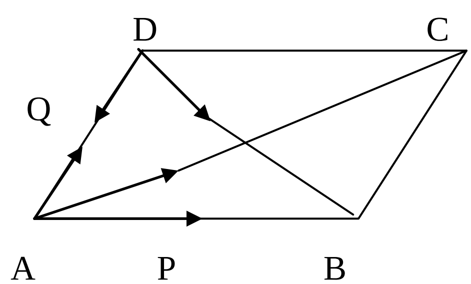
i.e. $\dfrac{P}{\sin\angle MON} = \dfrac{Q}{\sin\angle NOL} = \dfrac{R}{\sin\angle LOM}$

$$\frac{P}{\sin(Q.R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$$

Problem 8

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.

Solution:



Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD. Complete the parallelogram BAD.

Then P+Q= $\overline{AB} + \overline{AD} = \overline{AC}$

P-Q $= \overline{AB} - \overline{AD}$

$= \overline{AB} + \overline{DA}$

$= \overline{DA} + \overline{AB}$

$= \overline{DB}$

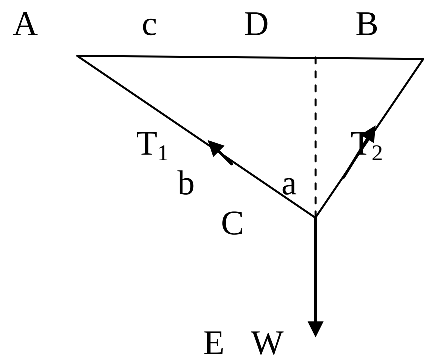
Given $\overline{AC}$ and $\overline{DB}$ are at right angles.
The diagonals AC and BD cut at right angles.

$\therefore$ ABCD must be a rhombus.
$\therefore$ AB = AD.
P = Q.

Problem 9

Let A and B two fixed points on a horizontal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b(a^2+c^2-b^2):a(b^2+c^2-a^2)$

Solution



Forces T_1, T_2, W are acting at C.

By Lami's theorem,

$$\frac{T_1}{\sin\angle ECB}=\frac{T_2}{\sin\angle ECA}\text{........(1)}$$

Now $\sin\ \angle ECB=\sin\left(180^0-\angle DCB\right)$

$= \sin\ \angle DCB$

$= \sin\ \left(90^0-\angle ABC\right)=\cos\angle ABC$

$\sin\ \angle ECA=\sin\left(180^0-\angle ACD\right)$

$= \sin\ \angle ACD$

$= \sin\ \left(90^0-\angle BAC\right)=\cos\angle BAC$

$$\frac{T_1}{\cos \angle ABC} = \frac{T_2}{\cos \angle BAC} \quad \therefore \quad \frac{T_1}{T_2} = \frac{\cos B}{\cos A} = \frac{\left(\dfrac{c^2 + a^2 - b^2}{2ca}\right)}{\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)}$$

$$\therefore \frac{T_1}{T_2} = \left(\frac{c^2 + a^2 - b^2}{2ca}\right) \times \left(\frac{2bc}{b^2 + c^2 - a^2}\right) = \frac{b\left(c^2 + a^2 - b^2\right)}{a\left(b^2 + c^2 - a^2\right)}$$

Problem 10

ABC is a given triangle. Forces P,Q,R acting along the lines OA,OB,OC are in equilibrium. Prove that

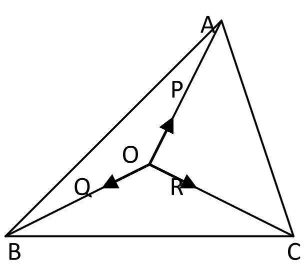
(i)P : Q : R= $a^2\left(b^2 + c^2 - a^2\right) : b^2\left(c^2 + a^2 - b^2\right) : c^2\left(a^2 + b^2 - c^2\right)$ if O is the cicumcentre of the triangle.

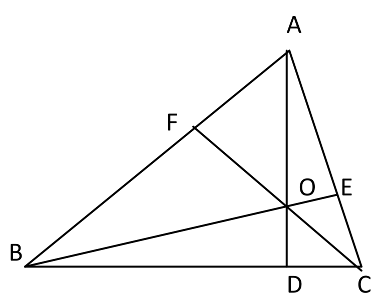
(ii) P : Q : R= $= \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$ if O is the incentre of the triangle.

(iii) P : Q : R= a:b:c if O is the ortho centre of the triangle.

(iv) P : Q : R=OA : OB : OC if O is the centroid of the triangle,

Solution:





By Lami's theorem,

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

(i) O is the circumcentre of the Δ ABC

$$\angle BOC = 2\angle BAC = 2A; \angle COA = 2B \ \ and \ \ \angle AOB = 2C$$

$$\therefore (1) \quad \Rightarrow \quad \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

i.e. $\dfrac{P}{2\sin A\cos A} = \dfrac{Q}{2\sin B\cos B} = \dfrac{R}{2\sin C\cos C}$ ……. (2)

But $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and $\sin A = \dfrac{2\Delta}{bc}$

where Δ is the area of the triangle ABC

$$\therefore 2\sin A\cos A = 2\,\frac{2\Delta\left(b^2 + c^2 - a^2\right)}{bc \quad 2bc}$$

$$= \frac{2\Delta\left(b^2 + c^2 - a^2\right)}{b^2c^2}$$

Similarly $2\sin B\cos B = \dfrac{2\Delta\left(c^2 + a^2 - b^2\right)}{c^2a^2}$

$$2\sin C\cos C = \frac{2\Delta\left(a^2 + b^2 - c^2\right)}{a^2b^2}$$

Substitute in (2)

$$\frac{P.b^2c^2}{2\Delta\left(b^2 + c^2 - a^2\right)} = \frac{Q.c^2a^2}{2\Delta\left(c^2 + a^2 - b^2\right)} = \frac{Ra^2b^2}{2\Delta\left(a^2 + b^2 - c^2\right)}$$

Divide by $\dfrac{a^2b^2c^2}{2\Delta}$

$$\frac{P}{a^2\left(b^2 + c^2 - a^2\right)} = \frac{Q}{b^2\left(c^2 + a^2 - b^2\right)} = \frac{R}{c^2\left(a^2 + b^2 - c^2\right)}$$

(ii) O is the in-centre of the triangle,

OB and OC are the bisectors of $\angle$B and $\angle$C

$$\therefore \angle BOC = 180^0 - \frac{B}{2} - \frac{C}{2} = 180^0 - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180^0 - \left(90^0 - \frac{A}{2}\right) = 90^0 + \frac{A}{2}$$

Similarly $\angle$COA = $90^0 + \frac{B}{2}$, $\angle AOB = 90^0 + \frac{C}{2}$

$$(1) \Rightarrow \frac{P}{\sin\left(90^0 + \frac{A}{2}\right)} = \frac{Q}{\sin\left(90^0 + \frac{B}{2}\right)} = \frac{R}{\sin\left(90^0 + \frac{C}{2}\right)}$$

i.e. $\dfrac{P}{\cos\frac{A}{2}} = \dfrac{Q}{\cos\frac{B}{2}} = \dfrac{R}{\cos\frac{C}{2}}$

(iii) O is the ortho-centre of the triangle

AD, BE, CF are the altitudes of the triangle

AFOE is a cyclic quadrilateral.

$\therefore \angle FOE + A = 180^0$, $\therefore \angle FOE = 180^0 - A$

$\therefore \angle BOC = 180^0 - A$

Similarly, $\angle COA = 180^0 - B$, $\angle AOB = 180^0 - C$

Hence (1) becomes

$$\frac{P}{\sin(180^0 - A)} = \frac{Q}{\sin(180^0 - B)} = \frac{R}{\sin(180^0 - C)}$$

i.e. $\dfrac{P}{\sin A} = \dfrac{Q}{\sin B} = \dfrac{R}{\sin C}$

i.e. $\dfrac{P}{a} = \dfrac{Q}{b} = \dfrac{R}{c}$ $\left(\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\right)$

(iv) O is the centroid of the triangle

Δ BOC = Δ COA = Δ AOB = $\frac{1}{3}\Delta ABC$

Δ BOC = $\frac{1}{2}OB.OC\sin\angle BOC = \frac{1}{3}\Delta ABC$

$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3OB.OC}$

Similarly, $\sin\angle COA = \frac{2\Delta ABC}{3OC.OA}$, $\sin\angle AOB = \frac{2\Delta ABC}{3OA.OB}$

Hence (1) becomes $\frac{P.3OB.OC}{2\Delta ABC} = \frac{Q.3OC.OA}{2\Delta ABC} = \frac{R.3OA.OB}{2\Delta ABC}$

i.e. P.OB.OC = Q.OC.OA = R.OA.OB

Dividing by OA.OB.OC, we get $\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$.

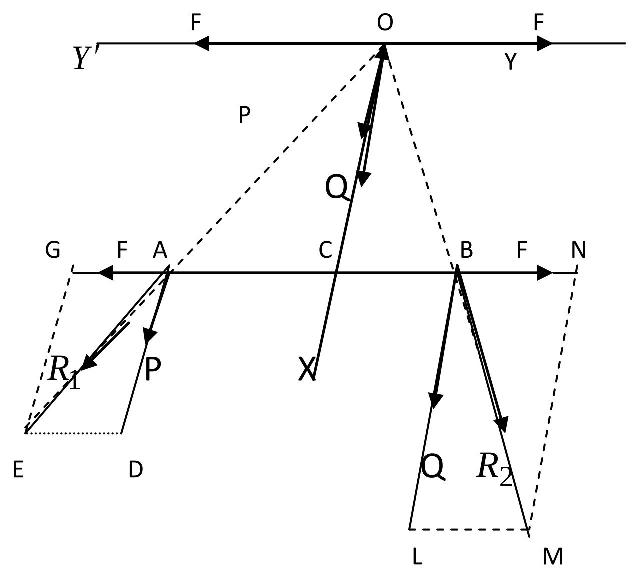
1.4 Parallel forces:

Forces acting along parallel lines are called parallel forces. There are two types of parallel forces known as like and unlike parallel forces. Since the parallel forces do not meet at a point, in this chapter we study methods to find the resultant of two like parallel and unlike parallel forces. Parallel forces acting on a rigid body have a tendency to rotate it about a fixed point. Such tendency is known as moment of the parallel forces. Here we study the theorem on moments of forces about a point.

Definition:

Two parallel forces are said to be **like** if they act in the same direction, they are said to be **unlike** if they act in opposite parallel directions.

The resultant of two like parallel forces acting on a rigid body



Proof:

Let P and Q be two like parallel forces acting at A and B along the lines AD and BL.At A and B, introduce two equal and opposite forces F along AG and BN. These two forces F balance each other and will not affect the system.

Now, R_1 is the resultant of P and F at A and R_2 is the resultant of Q and F at B as in the diagram.

Produce EA and MB to meet at O. At O, draw YOY^1 parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their original components. R_1 at O is equal to F along OY^1 and P along OX. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. The remaining two forces P and Q acting along OX have the resultant **P+Q (sum)** along OX.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles, OAC and AED are similar.

$$\therefore \ \frac{OC}{AD} = \frac{AC}{ED} \text{ ie) } \frac{OC}{P} = \frac{AC}{F}$$

$$\therefore \ F.OC \ = P.AC \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Triangles OCB and BLM are similar.

$$\therefore \ \frac{OC}{BL} = \frac{CB}{LM} \text{ ie) } \frac{OC}{Q} = \frac{CB}{F}$$

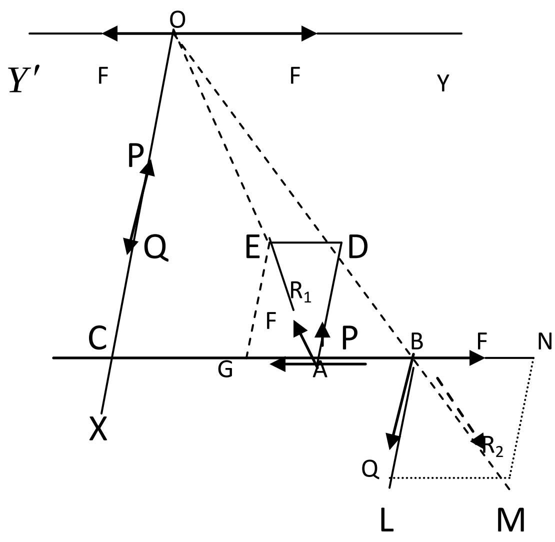
$$\therefore \ F.OC = Q.CB \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

(1) & (2) $\Rightarrow$ $\boxed{P.AC = Q.CB}$

ie) $\frac{AC}{CB} = \frac{Q}{P}$

ie) 'C' divides AB internally in the inverse ratio of the forces.

The resultant of two unlike and unequal parallel forces acting on a rigid body:



Proof:

Let P and Q at A and B be two unequal unlike parallel forces acting along AD and BL.

Let P > Q.

At A and B introduce two equal and opposite forces F along AG and BN. These two balances each other and will not affect the system.

Let R_1 be the resultant of F and P at A and R_2 be the resultant of F and Q at B. as in the diagram.

Produce EA and MB to meet at O. At O, draw Y'OY parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their components. R_1 at O is equal to F along OY' and P along XO. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. Now, the remaining forces are P and Q along the same line but opposite directions.

Hence the resultant is **P ~ Q (difference)** along XO.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles OCA and EGA are similar.

$$\therefore \frac{OC}{EG} = \frac{CA}{GA}, \text{ ie) } \frac{OC}{P} = \frac{CA}{F}$$

$$F.OC = P.AC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM}, \text{ ie) } \frac{OC}{Q} = \frac{CB}{F}$$

$$\therefore F.OC = Q.CB \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

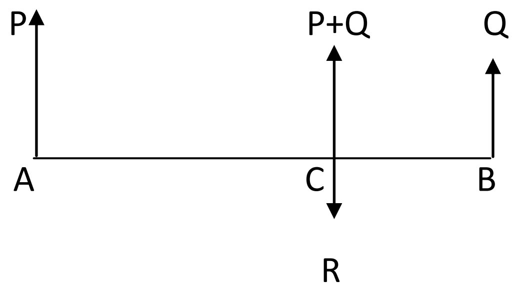
(1) and (2) $\Rightarrow$ $\boxed{\text{P.AC = Q.CB}}$

ie) $\frac{CA}{CB} = \frac{Q}{P}$

ie) ‘C’ divides AB externally.

Note : The effect of two equal and unlike parallel forces can not be replaced by a single force.

The condition of equilibrium of three coplanar parallel forces



Let P, Q, R be the three coplanar parallel forces in equilibrium. Draw a line to meet the forces P, Q, R at the points A, B, C respectively.

Equilibrium is not possible if all the three forces are in the same direction.

Let P + Q be the resultant of P and Q parallel to P. Hence R must be equal and opposite to P + Q.

$\therefore$ R = P + Q (in magnitude, opposite in direction)

$$\therefore P.AC = Q.CB$$

$$\therefore \frac{P}{CB} = \frac{Q}{AC} = \frac{P+Q}{CB+AC} = \frac{R}{AB}$$

Hence, $\boxed{\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}}$

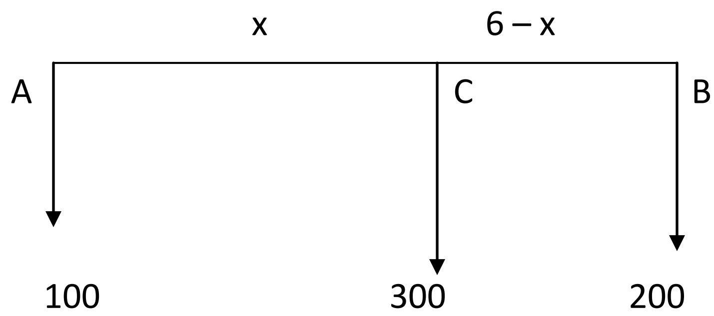
ie) If three parallel forces are in equilibrium then each force is proportional to the distance between the other two.

Note: The centre of two parallel forces is a fixed point through which their resultant always passes.

Problem 11

Two men, one stronger than the other, have to remove a block of stone weighing 300 kgs. with a light pole whose length is 6 metre. The weaker man cannot carry more than 100 kgs. Where the stone be fastened to the pole, so as just to allow him his full share of weight?

Solution:



Let A be the weaker man bearing 100 kgs., B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC = x. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

$\therefore$ 100.AC = 200.BC

i.e. 100x = 200 (6-x) = 1200 – 200x

$\therefore$ 300x = 1200 or x=4

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

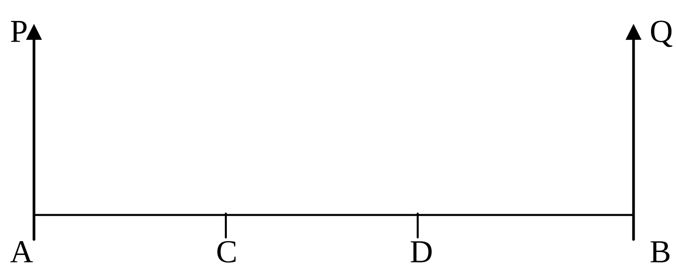
Problem 12

Two like parallel forces P and Q act on a rigid body at A and B respectively.

a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

b) If P and Q be interchanged in position, show that the point of application of the resultant will be displayed along AB through a distance d, where $d = \frac{P-Q}{P+Q}.AB$.

Solution:



Let C – be the centre of the two forces.

Then P. AC = Q.CB ………… (1)

(a) If Q is changed to $\frac{P^2}{Q}$, (P remaining the same), let D be the new centre of parallel forces.

Then P.AD = $\frac{P^2}{Q}$ DB …. …. ….. (2)

Q.AD = P.DB ……………. (3)

Relation (3) shows that D is the centre of two like parallel forces, with Q at A and P at B.

(b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

Let CD = d

From (3), Q. (AC+CD) = P. (CB – CD)

i.e. Q.AC + Q.d = P.CB – P.d

(Q + P).d = P.CB – Q.AC

= P (AB – AC) – Q (AB – CB)

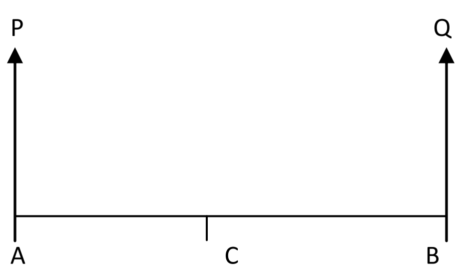
= (P – Q).AB[$\because$P.AC = Q.CB from (1)]

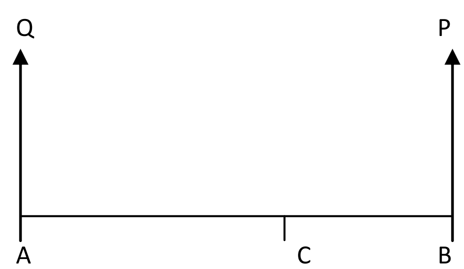
$$\boxed{d = \frac{P-Q}{P+Q}.AB}$$

Problem 13

The position of the resultant of two like parallel forces P and Q is unaltered, when the position of P and Q are interchanged. Show that P and Q are of equal magnitude.

Solution:





Let C be the centre of two like parallel forces P at A and Q at B.

$\therefore$ P.AC = Q.CB ……………… (1)

When P and Q are interchanged, the centre C is not altered (given)

$\therefore$ Q.AC =P.CB ………………. (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{P}{Q} = \frac{Q}{P}$$

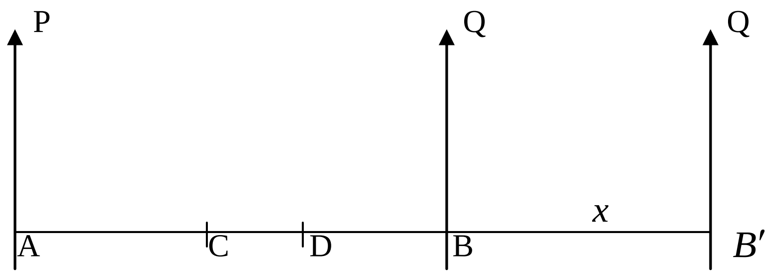
$$\therefore P^2 = Q^2$$

$$\boxed{\therefore P = \pm \ Q}$$

Problem 14

P and Q are like parallel forces. If Q is moved parallel to itself through a distance x, prove that the resultant of P and Q moves through a distance $\dfrac{Qx}{P+Q}$.

Solution:



Let C be the centre of P and Q at A and B.

$\therefore P.AC = Q.CB$ ………… (1)

Let D be the new centre of P at A and Q at B' such that BB' = x

$\therefore P.AD = Q.DB'$ …………………… (2)

ie) $P(AC + CD) = Q[DB + BB']$ = $Q[(CB - CD) + x]$

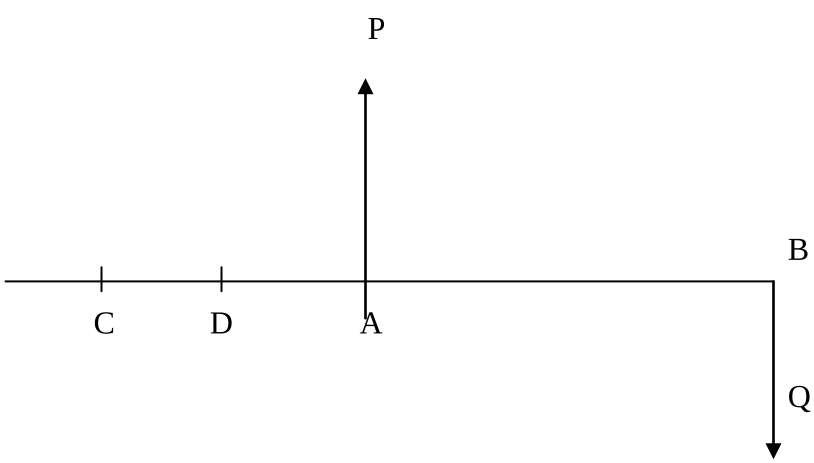
$$\therefore (P+Q)CD = Q.x \text{ using (1)}$$

$$\boxed{\therefore CD = \frac{Qx}{P+Q}}$$

Problem 15

Two unlike parallel forces P and Q (P>Q) acting on a rigid body at A and B respectively be interchanged in position, show that the point application of the resultant in AB will be displayed along AB through a distance $\frac{P+Q}{P-Q}AB$.

Solution:



Let C be the centre of two unlike parallel forces P at A and Q at B.

$$\therefore P.AC = Q.CB \text{ ………………… (1)}$$

Let D be the new centre when P and Q are interchanged in position.

$$\therefore Q.AD = P.DB \text{ ……………….. (2)}$$

i.e.) $Q(AC - CD) = P.(DA + AB)$

i.e.) $Q[(CB - AB) - CD] = P.[(AC - CD) + AB]$

$$Q.CB - Q.AB - Q.CD = P.AC - P.CD + P.AB$$

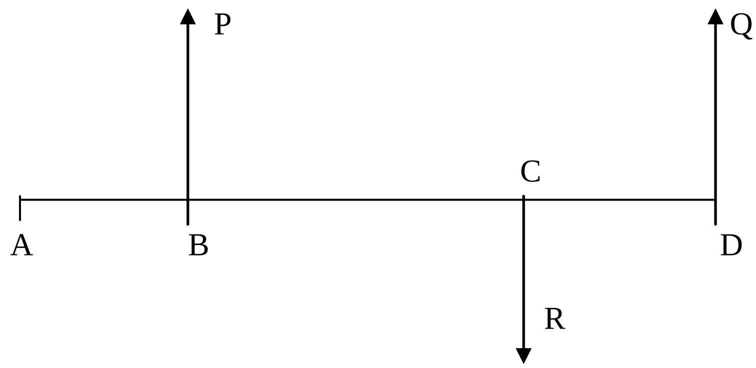
$$\therefore (P-Q).CD = (P+Q).AB \text{ using (1)}$$

$$\boxed{\therefore CD = \frac{P+Q}{P-Q}.AB}$$

Problem 16

A light rod is acted on by three parallel forces P, Q, and R, acting at three points distant 2, 8 and 6 ft. respectively from one end. If the rod is in equilibrium, show that P: Q: R = 1:2:3.

Solution



P, Q, R are parallel forces acting on the rod AD at B, D, C respectively.

Given, AB = 2 ft, AD = 8ft, AC = 6ft.

$\therefore$ BC = 4ft, CD = 2ft, BD = 6ft.

For equilibrium of the rod, each force should be proportional to the distance between the other two.

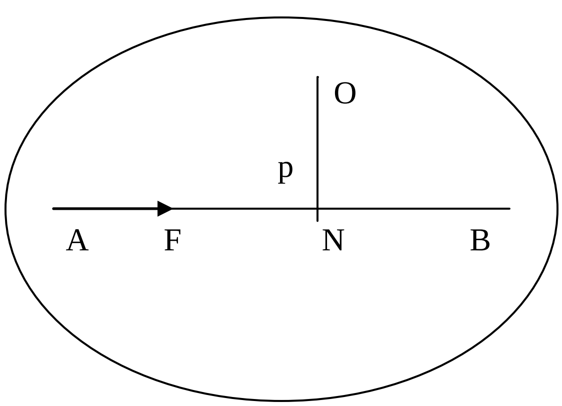
$$\therefore \frac{P}{2} = \frac{Q}{4} = \frac{R}{6} \Rightarrow P:Q:R = 2:4:6$$

$$\boxed{\therefore P:Q:R = 1:2:3}$$

1.5 Moment of a force (or) Turning effect of a force

Definition:

The moment of a force about a point is defined as the product of the force and the perpendicular distance of the point from the line of action of the force.



Moment of F about O = F x ON = F x p.

Note: Moment of F about O is zero if either F = O (or) ON = O.

i.e.) F = 0 (or) AB passes through O.

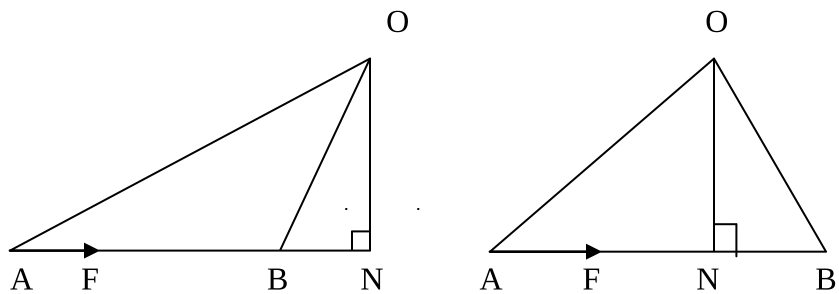
Hence, moment of a force about any point is zero if either

the force itself is zero (or) the force passes through that point.

Physical significance of the moment of a force

It measures the tendency to rotate the body about the fixed point.

Geometrical Representation of a moment



Let AB represent the force F both in magnitude and direction and O be any given point.

$\therefore$ the moment of the force F about O

= F x ON = AB x ON = 2. Δ AOB

= Twice the area of the triangle AOB

Sign of the moment

If the force tends to turn the body in the **anticlockwise** direction, moment is **positive.**

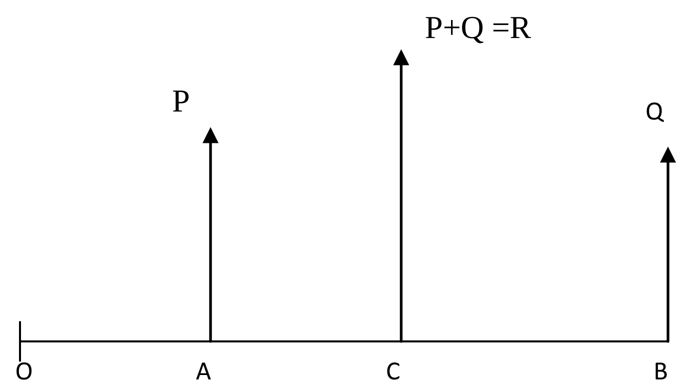
If the force tends to turn the body in the **clockwise** direction, moment is **negative.**

Varignon's Theorem of Moments

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

Proof:

Case 1 Let the forces be parallel and O lies **i) Outside AB**



Let P and Q be the two parallel forces acting at A and B. P + Q be their resultant R acting at C. such that

P.AC = Q.CB ……………... (1)

Algebraic sum of the moments of P and Q about O

= P.OA + Q.OB

= P x (OC – AC) + Q x (OC + CB)

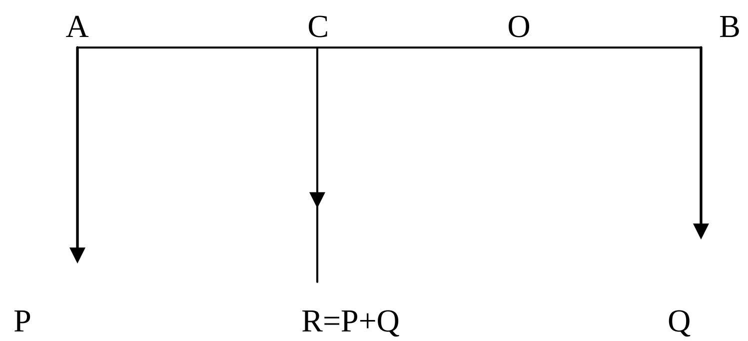
= (P +Q).OC – P.AC +Q.CB

= (P+Q).OC using (1)

= R.OC

= moment of R about O.

ii) P and Q are parallel and O lies within AB



Algebraic sum of the moments of P and Q about O

= P.OA – Q.OB

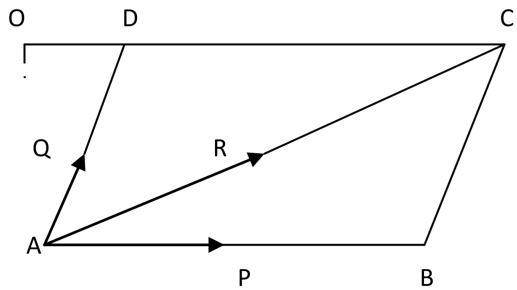
= P. (OC+CA) – Q. (CB – CO)

= (P+Q).OC + P.CA – Q.CB by (1)

= R.OC

= moment of R about O.

Case II iii) P and Q meet at a point and O any point in their plane. O lies outside the angle BAD



Through O, draw a line parallel to the direction of P, to meet the line of action of Q at D. Complete the parallelogram ABCD such that AB, AD represent the magnitude of P and Q and the diagonal AC represents the resultant R of P and Q.

Algebraic sum of the moments of P and Q about O

$= 2. \Delta$ AOB + 2. Δ AOD

$= 2 \Delta$ ACB + 2. Δ AOD [$\because \Delta$ AOB = Δ ACB]

$= 2 \Delta$ ADC + 2 Δ AOD

$= 2 (\Delta$ ADC + Δ AOD)

$= 2. \Delta$ AOC

= Moment of R about O.

iv) O lies inside the angle BAD

Algebraic sum of the moments of P and Q about O:

$= 2 \Delta$ AOB – 2 Δ AOD

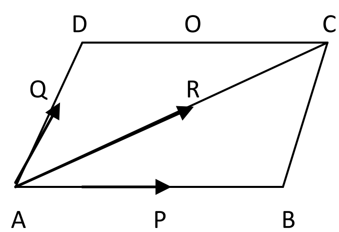
$= 2 \Delta$ ACB – 2 Δ AOD

$= 2 \Delta$ ADC – 2 Δ AOD

$= 2 (\Delta$ ADC – Δ AOD)

$= 2. \Delta$ AOC

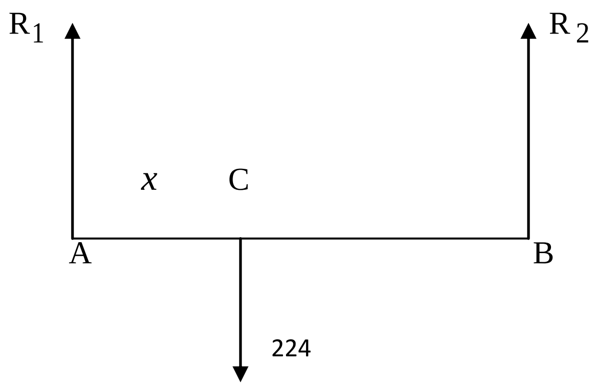
= moment of R about O.



Problem 17

Two men carry a load of 224 kg. wt, which hangs from a light pole of length 8 m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other. What is the pressure on each shoulder?

Solution



AB is the light pole of length 8m. C is the point from which the load of 224 kgs. is hung.

Let AC = x. Then BC = 8 – x. given (8 – x) – x =2

i.e) 8 – 2x = 2 0r 2x = 6.

$\therefore$ x = 3. i.e. AC = 3 and BC = 5.

Let the pressures at A and B be R_1 and R_2 kg. wt. respectively. Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2 and 224 kg. wt. about any point must be equal to zero.

Taking moments about B,

224 CB – R_1.AB = 0

i.e. $224 \times 5 - R_1 \times 8 = 0.$

$$\therefore R_1 = \frac{224 \times 5}{8} = 140.$$

Taking moments about A,

R_2.AB – 224.AC = 0.

i.e. $8R_2 - 224 \times 3 = 0.$

$$\therefore R_2 = \frac{224 \times 3}{8} = 84$$

Problem 18

A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

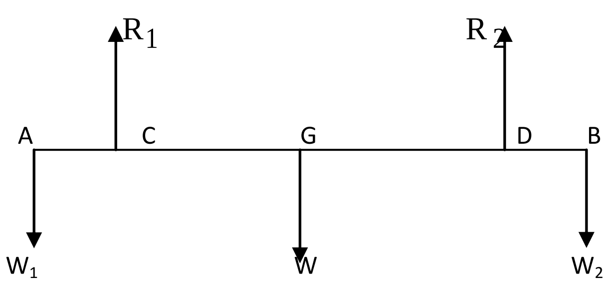
$$\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}.$$

Solution

Let AB be the plank placed upon two vertical props at C and D. CD = b. The weight W of the plank acts at G, the midpoint of AB,

AG = GB = a

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D is zero.



There is upward reaction R_1 at C.

Take moments about C, we have

W_1. AC = W.CG

i.e. W_1 (AG – CG) = W.CG

W_1.AG = (W +W_1).CG

i.e. $W_1.a$ = (W+W_1) CG

CG = $\dfrac{W_1 a}{W + W_1}$ ……………. (1)

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Take moments about D, we get

W.GD = W_2 (GB –GD)

GD (W+W_2) = W_2 .GB = W_2 .a

GD = $\dfrac{W_2 a}{W + W_2}$ ………… (2)

$$\boxed{CG + GD = CD = b}$$

$$\therefore \frac{W_1 a}{W + W_1} + \frac{W_2 a}{W + W_2} = b$$

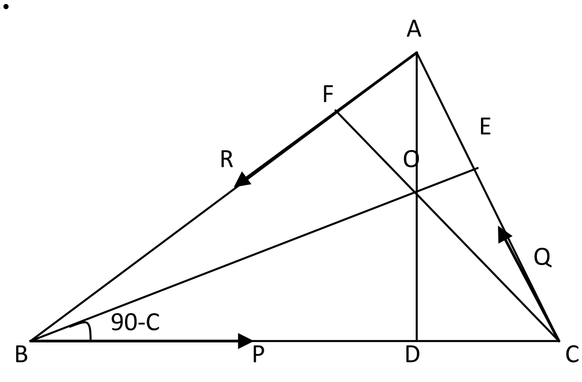
$$\frac{W_1}{W + W_1} + \frac{W_2}{W + W_2} = \frac{b}{a}$$

Problem 19

The resultant of three forces P, Q, R, acting along the sides BC, CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled.

If $\angle A = 120°$, and B = C, show that Q+R = P$\sqrt{3}$.

Solution:



Let AD, BE and CF be the altitudes of the triangle intersecting at O, the orthocentre.

As the resultant passes through O, moment of the resultant about O = O.

$\therefore$ Sum of the moments of P, Q, R about O = O

P.OD+Q.OE+R.OF = 0 ……….. (1)

In rt. $\angle d\Delta BOD, \angle OBD = \angle EBC = 90° - C.$

$$\therefore \tan(90° - C) = \frac{OD}{BD}$$

i.e) cot C = $\frac{OD}{BD}$

OD = BD cot C …………. (2)

From rt. $\angle d\Delta ABD, \cos B = \frac{BD}{AB}$

$$\therefore From(2), OD = c\cos B.\cot C = c\cos B.\frac{\cos C}{\sin C}$$

$$= \frac{c}{\sin C}.\cos B\cos C$$

= $2R'\cos B\cos C(\because \frac{c}{\sin C} = 2R'$, R' is the circumradius of the Δ)

Similarly OE = $2R'\cos C\cos A$

and OF = $2R'\cos A\cos B$

Hence (1) becomes

$$P.2R'\cos B\cos C + Q.2R'\cos C\cos A + R.2R'\cos A\cos B = 0$$

Dividing by $2R'\cos A\cos B\cos C,$

$$\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0 \text{ …… (3)}$$

Now, P, Q, R being magnitudes of the forces, are all positive.

(3) may hold good, if at least one of the terms must be negative.

Hence one of the cosines must be negative.

i.e) the triangle must be obtuse angled.

If A = $120°$ and the other angles equal, then B = C = $30°$

Hence (3) becomes

$$\frac{P}{\cos 120^\circ} + \frac{Q}{\cos 30^\circ} + \frac{R}{\cos 30^\circ} = 0$$

i.e. $\dfrac{P}{\left(-\dfrac{1}{2}\right)} + \dfrac{Q+R}{\left(\dfrac{\sqrt{3}}{2}\right)} = 0$

i.e. P$\sqrt{3} = Q + R$

1.6 Couples: Definition

Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.

Examples of a couple are the forces used in winding a clock or turning tap. Such forces acting upon a rigid body can have only a rotator effect on the body and they can not produce a motion of translation.

The moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them,

The perpendicular distance (p) between the two equal forces P of a couple is called the **arm of the couple.** A couple each of whose forces is P and whose arm is p is usually denoted by (P, p).

A couple is positive when its moment is positive i.e., if the forces of the couple tend to produce rotation in the anti-clockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

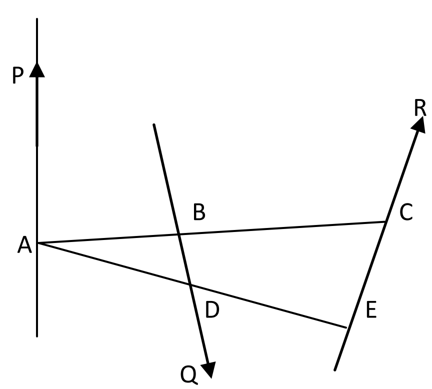
1.7 Equilibrium of three forces acting on a Rigid Body.

In the previous sections we have studied theorems and problems involving parallel forces and forces acting at a point. Here we study three important theorems and solved problems on forces acting on a rigid body and their conditions of equilibrium.

Theorem

If three forces acting on a rigid body are in equilibrium, they must be coplanar.

Proof:



Let the three forces be P,Q,R

Given : They are acting on a rigid body and in equilibrium.

Take 'A' on the force P, and B on the force Q such that AB is not parallel to R.

$\therefore$ Sum of the moments of P, Q, R about AB = 0 [$\therefore$ P,Q, R are in equilibrium]

Now, moment of P and Q about AB = 0 [$\because$ P and Q intersect AB].

$\therefore$ Moment of R about AB = 0, Hence R must intersect AB at a point C

Similarly if D is another point on Q such that AD is not parallel to R, we prove, R must intersect AD at a point E.

Since BC and DE intersect at A, BD, CE, A lie on the same plane. i.e) 'A' lies on the plane formed by Q and R. Since A is an arbitrary point on the force P, every point on the force P lie on the same plane.

ie) P, Q, R lie on the same plane.

Three Coplanar Forces – theorem

If three coplanar forces acting on a rigid body keep it in equilibrium, they must be either concurrent or all parallel.

Proof:

Let P, Q, R be the three forces acting on a rigid body keep it in equilibrium.

$\therefore$ One force must be equal and opposite to the resultant of the other two.

$\therefore$ they must be parallel or intersect.

Case 1: If P and Q are parallel (like or unlike)

Then the resultant of P and Q is also parallel. Hence R must be parallel to P and Q.

Case 2: If P and Q are not parallel: (intersect)

They meet at O. Therefore, by parallelogram law, the third force R must pass through O.

i.e) the three forces are concurrent.

Note: A couple and a single force can not be in equilibrium

Conditions of equilibrium

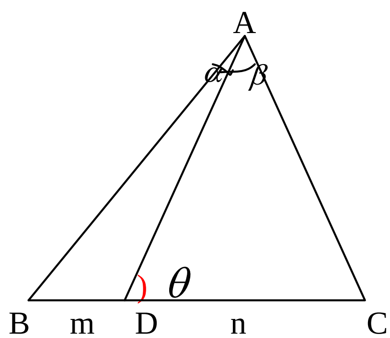
1. If three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two.
2. If three forces in equilibrium are parallel, then each force is proportional to the distance between the other two

Two Trigonometrical theorems

If D is any point on BC of a triangle ABC such that $\frac{BD}{DC}=\frac{m}{n}$ and $\angle ADC=\theta$, $\angle BAD=\alpha, \angle DAC=\beta$ then

1) $(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta$ 2) $(m+n)\cot\theta = n.\cot B - m.\cot C.$

Proof:



1. Given, $\frac{m}{n}=\frac{BD}{DC}=\frac{BD}{DA}.\frac{DA}{DC}$

Using, sine formula in Δ ABD, Δ ADC,

$$\frac{m}{n}=\frac{\sin\angle BAD}{\sin\angle ABD}\times\frac{\sin\angle ACD}{\sin\angle DAC}$$

$$\frac{m}{n}=\frac{\sin\alpha}{\sin(\theta-\alpha)}\times\frac{\sin(\theta+\beta)}{\sin\beta}$$

$$=\frac{\sin\alpha}{\sin\beta}\times\frac{(\sin\theta.\cos\beta+\cos\theta.\sin\beta)}{(\sin\theta\cos\alpha-\cos\theta.\sin\alpha)}$$

Divide by $\sin\alpha.\sin\theta.\sin\beta$

$$\frac{m}{n}=\frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$$

$$\therefore m(\cot\alpha-\cot\theta)=n(\cot\beta+\cot\theta)$$

$$\boxed{(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta}$$

2. $\dfrac{m}{n} = \dfrac{BD}{DA} . \dfrac{DA}{DC}$

$= \dfrac{\sin \angle BAD}{\sin \angle ABD} \times \dfrac{\sin \angle ACD}{\sin \angle DAC}$

$= \dfrac{\sin(\theta - B).\sin C}{\sin B.\sin[180° - (\theta + C)]} = \dfrac{\sin C.\sin(\theta - B)}{\sin B.\sin(\theta + C)}$

$= \dfrac{\sin C \times (\sin \theta.\cos B - \cos \theta \sin B)}{\sin B(\sin C \cos \theta + \cos C \sin \theta)}$

Divide by sin B sin C sinθ

$\dfrac{m}{n} = \dfrac{\cot B - \cot \theta}{\cot \theta + \cot C}$

$\therefore m(\cot \theta + \cot C) = n(\cot B - \cot \theta)$

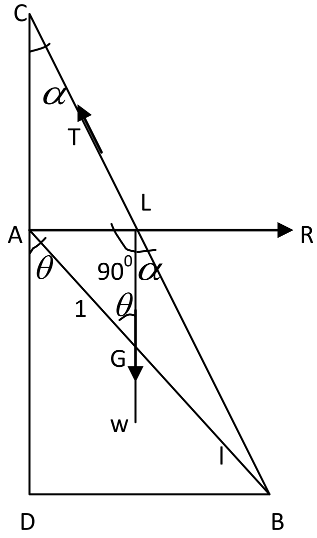
$$\boxed{\therefore (m+n)\cot\theta = n\cot B - m\cot C}$$

Problem 20

A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length *l*, tied to one end of the rod, the other end of the string being attached to a point in the wall: show that the rod can rest inclined to the wall at an angle θ given by $\cos^2 \theta = \dfrac{l^2 - a^2}{3a^2}$.

What are the limits of the ratio of a: *l* in order that equilibrium may be possible?

Solution:



AB is the rod of length a, with G its centre of gravity and BC is the string of length *l*. The forces acting on the rod are:

(i). Its weight W acting vertically downwards through G.

(ii). The reaction R at A which is normal to the wall and therefore horizontal.

iii) The tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L.

Let the string make an angle α with the vertical.

$\therefore \angle ACB = \alpha = \angle GLB.$

$\angle LGB = 180° - \theta \ and \angle ALG = 90°$, AG:GB = 1 :1,

Using the trigonometrical theorem in Δ ALB

$(1+1)\cot(180° - \theta) = 1.\cot 90° - 1.\cot\alpha$

i.e) $-2\cot\theta = -\cot\alpha$

$2\cot\theta = \cot\alpha$ ……………… (1)

Draw BD $\perp$ to CA.

From rt. $\angle d\Delta CDB, BD = BC.\sin\alpha = l.\sin\alpha$

rt. $\angle d\Delta ABD, BD = AB\sin\theta = a\sin\theta$

$\therefore l\sin\alpha = a\sin\theta$ ………… (2)

Eliminate α between (1) and (2).

We know that $\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$ …………………… (3)

(2) $\Rightarrow \sin\alpha = \dfrac{a\sin\theta}{l}$ $\therefore \operatorname{cosec}\alpha = \dfrac{l}{a\sin\theta}$ …………………… (4)

Substitute (4) and (1) in (3)

$$\frac{l^2}{a^2\sin^2\theta} = 1 + 4\cot^2\theta$$

i.e. $\dfrac{l^2}{a^2} = \sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$

$$\therefore 3\cos^2\theta = \frac{l^2}{a^2} - 1 = \frac{l^2 - a^2}{a^2}$$

$$\therefore \cos^2\theta = \frac{l^2 - a^2}{3a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Equilibrium position is possible, if $\cos^2\theta$ positive and less than 1

$\therefore l^2 - a^2 > 0$ i.e. $l^2 > a^2 or a^2 < l^2$ ……………….. (6)

Also $\frac{l^2 - a^2}{3a^2} < 1$ i.e. $l^2 - a^2 < 3a^2 or \; l^2 < 4a^2$

i.e. $a^2 > \frac{l^2}{4}$ …………………… (7)

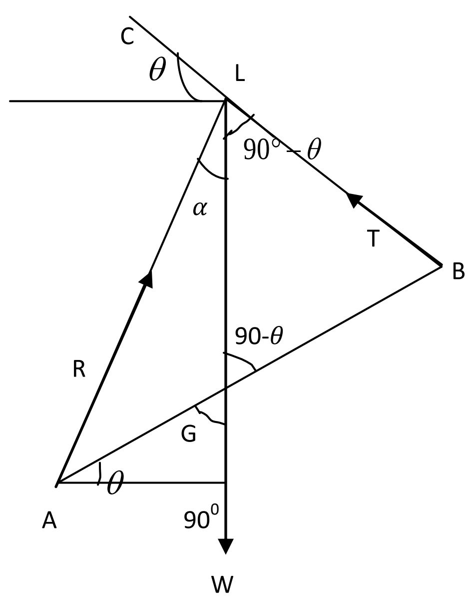
$$\frac{l^2}{4} < a^2 < l^2$$

[By (6) & (7)] $\frac{1}{4} < \frac{a^2}{l^2} < 1 = \frac{1}{2} < \frac{a}{l} < 1.$

Problem 21

A beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that the reaction at the hinge is $\frac{W}{4}\sqrt{8 + \cos ec^2\theta}$

Solution:



Let AB be the beam of weight W and G its centre of gravity.

BC is the string

The force acting on the beam are:

i) Its wt. W acting vertically down wards at G

ii) the tension T along BC

iii) the reaction R at the hinge A.

For equilibrium (i) , (ii) and (iii) must meet at L.

BC and AB make the same angle θ with the horizon.

$\therefore$ They make $90° - \theta$ with the vertical LG,

i.e. $\angle BLG = 90° - \theta = \angle LGB$

Let $\angle ALG = \alpha$

Using trigonometrical theorem in Δ ALB, AG:GB = 1:1

$(1+1)\cot(90° - \theta) = 1.\cot\alpha - 1.\cot(90° - \theta)$

i.e. 2 $\tan\theta = \cot\alpha - \tan\theta$

3 $\tan\theta = \cot\alpha$ ………………. (1)

Applying Lami's theorem at L,

$$\frac{R}{\sin(90° - \theta)} = \frac{W}{\sin(90° - \theta + \alpha)}$$

i.e. $\dfrac{R}{\cos\theta} = \dfrac{W}{\sin\left(90° - \overline{\theta - \alpha}\right)} = \dfrac{W}{\cos(\theta - \alpha)}$

$$\therefore R = \frac{W\cos\theta}{\cos(\theta - \alpha)} = \frac{W\cos\theta}{\cos\theta\cos\alpha + \sin\theta\sin\alpha}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta\cot\alpha + \sin\theta)}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta.3\tan\theta + \sin\theta)} \quad \text{[By (1)]}$$

$$= \frac{W\cos\theta\,\mathrm{cosec}\,\alpha}{3\sin\theta + \sin\theta} = \frac{W\cot\theta}{4}.\mathrm{cosec}\,\alpha = \frac{W}{4}\cot\theta\sqrt{1 + \cot^2\alpha}$$

$$= \frac{W}{4}.\cot\theta\sqrt{1 + 9\tan^2\theta}$$

$$= \frac{W}{4}\sqrt{\cot^2\theta + 9} = \frac{W}{4}\sqrt{\cot^2\theta + 1 + 8}$$

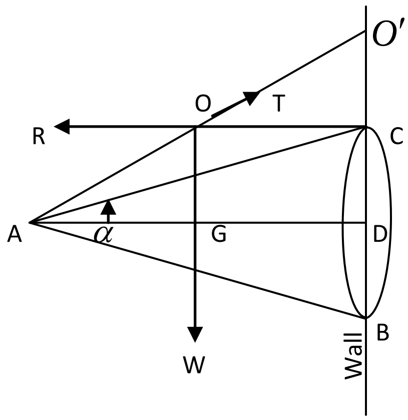
$$= \frac{W}{4}\sqrt{\mathrm{cosec}^2\theta + 8}$$

Problem 22

A solid cone of height h and semi-vertical angle α is placed with its base flatly against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$.

(The centre of gravity of a solid cone lies on its axis and divides it in the ratio 3 : 1 from the vertex.)

Solution:



Let A be the vertex, & height AD = h.

Semi-vertical angle $D\hat{A}C = \alpha$.

G divides AD in the ratio 3: 1

Length AO' is greatest, when the cone is just in the point of turning about C.

At that time, normal reaction R must be perpendicular to the wall.

Since, the cone is in equilibrium, the three forces T, W, R must be concurrent at O. $\Delta AOG \,\&\, \Delta AO'D$ are similar.

$$\therefore \frac{AO'}{AO} = \frac{AD}{AG} = \frac{h}{\left(\frac{3}{4}h\right)} = \frac{4}{3} \qquad \boxed{\therefore AO' = \frac{4}{3}AO} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, OG = CD.

From $\Delta ACD, \tan\alpha = \frac{CD}{AD} = \frac{CD}{h} \quad \therefore CD = h\tan\alpha$

$$\boxed{\therefore OG = h.\tan\alpha}$$

From $\Delta AOG, AO^2 = AG^2 + GO^2$

$$= \left(\frac{3}{4}h\right)^2 + (h.\tan\alpha)^2$$

$$= \frac{9h^2}{16} + h^2.\tan^2\alpha$$

$$= \frac{9h^2 + 16h^2\tan^2\alpha}{16}$$

$$AO^2 = h^2\left(\frac{9}{16} + \tan^2\alpha\right)$$

$$\therefore AO = h.\sqrt{\frac{9}{16} + \tan^2\alpha}$$

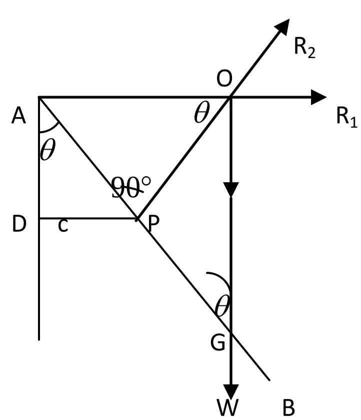
$$(1) \Rightarrow AO' = \frac{4}{3} \times h \times \sqrt{\frac{9}{16} + \tan^2\alpha}$$

$$\boxed{AO' = h.\sqrt{1 + \frac{16}{9}\tan^2\alpha}}$$

Problem 23

A heavy uniform rod of length 2a lies over a smooth peg with one end resting on a smooth vertical wall. If c is the distance of the peg from the wall and θ the inclination of the rod to the wall, show that c = a $\sin^3\theta$

Solution:



Forces acting on the rod AB are

i) Weight W at G $(\downarrow)$

ii) Reaction R_1 at A ($\perp$ to the wall)

iii) Reaction R_2 at the peg P ($\perp$ to the rod)

For equilibrium, W, R_1,R_2 must be concurrent at O.

From rightangled triangle ADP (DP = c)

$$\sin\theta = \frac{c}{AP} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From $\Delta AOP, \sin\theta = \dfrac{AP}{AO}$ …………………….. (2)

From $\Delta OGA, \sin\theta = \dfrac{OA}{AG}$ …………………….. (3)

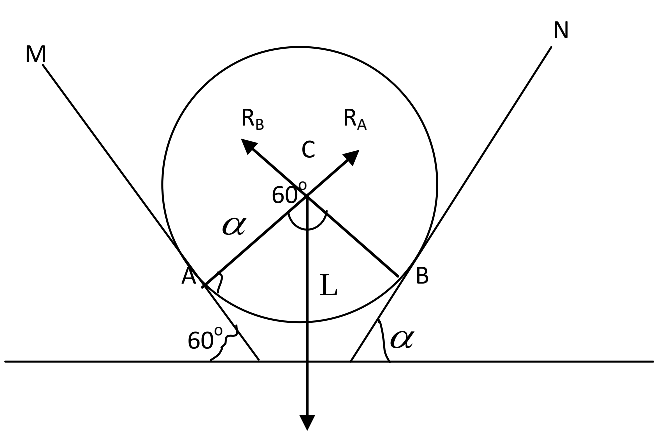
$$(1)\times(2)\times(3) \Rightarrow \sin^3\theta = \frac{c}{AP}\times\frac{AP}{AO}\times\frac{OA}{AG} = \frac{c}{AG} = \frac{c}{a}$$

$\therefore$ $\boxed{c = a\sin^3\theta}$

Problem 24

A heavy uniform sphere rests touching two smooth inclined planes one of which is inclined at $60°$ to the horizontal. If the pressure on this plane is one-half of the weight of the sphere, prove that the inclination of the other plane to the horizontal is $30°$

Solution:



Let the sphere centre C rest on the inclined planes AM and BN. MA makes 60° with the horizontal and let NB make an angle α with the horizon.

The forces acting are

i) Reaction R_A at A perpendicular to the inclined plane AM and to the sphere and hence passing through C.

ii) Reaction R_B at B which is normal to the inclined plane BN and to the sphere and hence passing through C.

iii) W, the weight of the sphere acting vertically downwards at C along CL.

Clearly the above three forces meet at C.

Also $\angle ACL = 60^\circ and \ \angle BCL = \alpha$

Applying Lami's theorem,

$$\frac{R_A}{\sin\alpha} = \frac{W}{\sin(60+\alpha)}$$

$$\therefore R_A = \frac{W\sin\alpha}{\sin(60^\circ+\alpha)} \text{ ………………….. (1)}$$

But $R_A = \dfrac{W}{2}$ ……………… (2)

From (1) and (2), we have

$$\frac{W\sin\alpha}{\sin(60^\circ+\alpha)} = \frac{W}{2}$$

i.e. 2 sin $\alpha = \sin(60^\circ+\alpha) = \sin 60^\circ\cos\alpha + \cos 60^\circ \sin\alpha$

i.e. 2 sin $\alpha = \dfrac{\sqrt{3}}{2}\cos\alpha + \dfrac{1}{2}\sin\alpha$ or $4\sin\alpha = \sqrt{3}\cos\alpha + \sin\alpha$

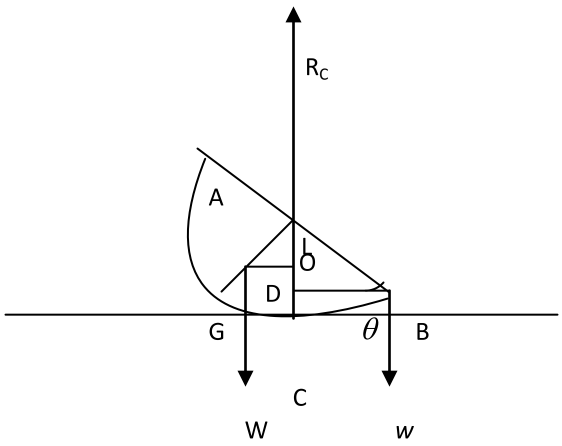
i.e. 3 sin $\alpha = \sqrt{3}\cos\alpha$ or $\dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$

i.e. tan $\alpha = \dfrac{1}{\sqrt{3}}$ or $\alpha = 30^\circ$

Problem 25

A uniform solid hemisphere of weight W rests with its curved surface on a smooth horizontal plane. A weight w is suspended from a point on the rim of the hemisphere. If the plane base of the rim is inclined to the horizontal at an angle θ, prove that $\tan\theta = \frac{8w}{3W}$

Solution:



Draw GL perpendicular to OC and BD perpendicular to OC. Base AB is inclined at an angle θ with the horizontal BD. Forces acting are i) Reaction R_c ii) Weight W at G iii) Weight w at B.

Since these three forces are parallel, and in equilibrium each force is proportional to the distance between the other two.

$\therefore \frac{W}{BD} = \frac{w}{GL}$ ………………… (1)

Now, $\Delta OBD \Rightarrow BD = OB\cos\theta = r\cos\theta$

Here, OG = $\frac{3r}{8}$, r – radius

GL = OG. sin $\theta = \frac{3r}{8}\sin\theta$

$$\therefore (1) \Rightarrow \frac{W}{r\cos\theta} = \frac{w}{\left(\frac{3r}{8}\sin\theta\right)}$$

$$\boxed{\therefore \tan\theta = \frac{8w}{3W}}$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

III YEAR B.SC MATHEMATICS

SMT5103 MECHANICS 1

Unit 2- Principles of Momentum – SMT1503

UNIT II

2.1 Friction

In the previous sections we have studied problems on equilibrium of smooth bodies. Practically no bodies are perfectly smooth. All bodies are rough to a certain extent. Friction is the force that opposes the motion of an object. Only because of this friction we are able to travel along the road by walking or by vehicles. So friction helps motion. It is a tangential force acting at the point on contact of two bodies. To stop a moving object a force must act in the opposite direction to the direction of motion. Such force is called a frictional force. For example if you push your book across your desk, the book will move. The force of the push moves the book. As the books slides across the desk, it slows down and stops moving. When you ride a bicycle the contact between the wheel and the road is an example of dynamic friction.

Definition

If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called ***friction;*** the force exerted is called the ***force of friction***.

Types of Friction

There are three types of friction

1) Statical Friction 2) Limiting Friction 3) Dynamical friction.

1. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium is called ***statical friction***.

2. When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called ***limiting friction***; the equilibrium is said to be limiting equilibrium.

3. When motion ensues by one body sliding over another, the friction exerted is called ***dynamical friction***.

2.2 Laws of Friction

Friction is not a mathematical concept; it is a physical reality.

Law 1 When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

Law 2 When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law 3 The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law 4 The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.

Law 5 (Law of dynamical Friction)

When motion ensues by one body sliding over the other the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

Friction is a passive force: Explain

1) Friction is only a resisting force.
2) It appears only when necessary to prevent or oppose the motion of the point of contact.
3) It can not produce motion of a body by itself, but maintains relative equilibrium.
4) It is a self-adjusting force.
5) It assumes magnitude and direction to balance other forces acting on the body.

Hence, friction is purely a passive force.

Co-efficient of friction

The ratio of the limiting friction to the normal reaction is called the co-efficient of friction. It is denoted by μ

i.e.) $\boxed{\frac{F}{R} = \mu}$ $\Rightarrow$ $\boxed{F = \mu R}$

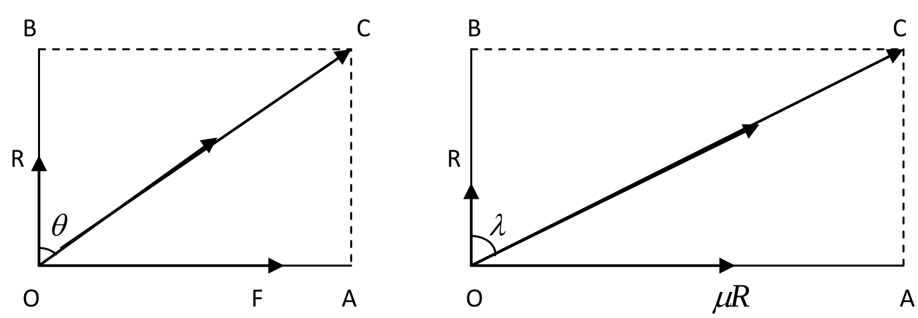
Note: 1) μ depends on the nature of the materials in contact.

2) Friction is maximum when it is limiting. μR is the maximum value of friction.

3) When equilibrium is non-limiting, $F < \mu R$ i.e.) $\frac{F}{R} < \mu$

4) Friction 'F' takes any value from zero upto μR.

Angle of Friction



Let OA = F(Friction), $\overrightarrow{OB} = R$ (Normal reaction) & $\overrightarrow{OC}$ be the resultant of F and R.

If $B\hat{O}C = \theta,\ \tan\theta = \dfrac{BC}{OB} = \dfrac{OA}{OB} = \dfrac{F}{R}$ ……….. (1)

As F increases, θ - increases until F reaches its maximum value μR. In this case, equilibrium is limiting.

Definition

"When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called **the angle of friction** and is denoted by λ"

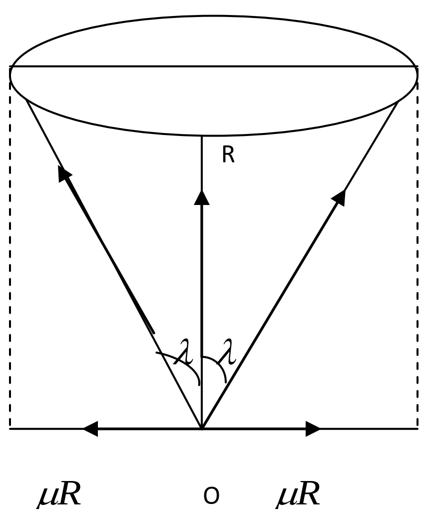
In the limiting equilibrium, $B\hat{O}C = \lambda$ = angle of friction.

$\therefore \tan\lambda = \dfrac{BC}{OB} = \dfrac{OA}{OB} = \dfrac{\mu R}{R} = \mu$

$$\boxed{\mu = \tan\lambda}$$

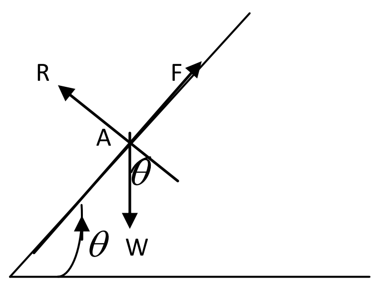
i.e.) The co-efficient of friction is equal to the tangent of the angle of friction.

Cone of Friction



We know, the greatest angle made by the resultant reaction with the normal is λ (angle of friction) where $\lambda = \tan^{-1}(\mu)$. Consider the motion of a body at O (its point of contact) with another. When two bodies are in contact, consider a cone drawn with O as vertex, common normal as the axis of the cone, λ - be the semi-vertical angle of the cone. Now, the resultant reaction of R and μR will have a direction which lies within the surface or on the surface of the cone. It can not fall outside the cone. This cone generated by the resultant reaction is called the ***cone of friction***.

2.3 Equilibrium of a particle on a rough inclined plane.



Let θ - be the inclination of the rough inclined plane, on which a particle of weight W, is placed at A. Forces acting on the particle are,

1) Weight W vertically downwards
2) Normal reaction R, $\perp$r to the plane.
3) Frictional force F, along the plane upwards (Since the body tries to slip down).

Resolving the forces along and perpendicular to the plane,

$$F = W\sin\theta, \quad R = W\cos\theta$$

$$\therefore \frac{F}{R} = \tan\theta$$

But $\frac{F}{R} < \mu \;\therefore \tan\theta < \mu$

i.e) $\tan\theta < \tan\lambda$

$\therefore \theta < \lambda$

When $\theta = \lambda, \frac{F}{R} = \tan\lambda = \mu$

Hence, it is clear that "***when a body is placed on a rough inclined plane and is on the point of sliding down the plane, the angle of inclination of the plane is equal to the angle of friction***." Now λ is called as the ***angle of repose.***

Thus the angle of repose of a rough inclined plane is equal to the angle friction when there is no external force act on the body.

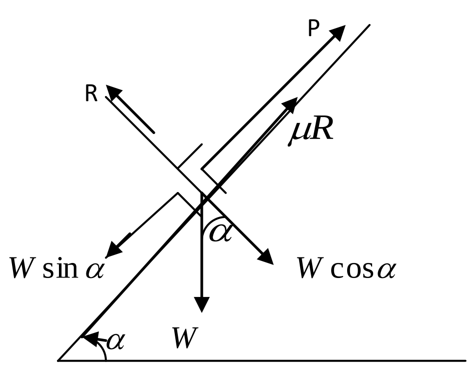
2.4 Equilibrium of a body on a rough inclined plane under a force parallel to the plane.

A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted on by a force parallel to the plane. Find the limits between which the force must lie.

Proof:

Let α be the inclination of the plane, W be the weight of the body& R be the normal reaction.

Case 1: Let the body be on the point of slipping down. Therefore μR acts upwards along the plane.



Let P be the force applied to keep the body at rest.

Resolving the forces along and perpendicular to the plane,

$P + \mu R = W \sin \alpha$ ……………….. (1)

$R = W.\cos \alpha$ ………………… (2)

$\therefore P = W.\sin \alpha - \mu.W \cos\alpha$

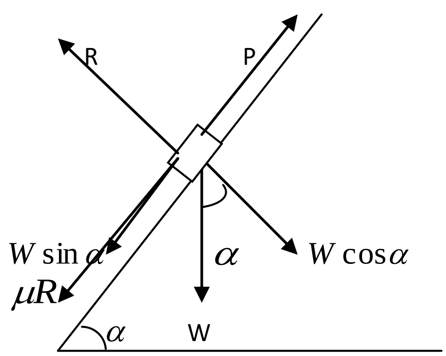
$= W[\sin \alpha - \tan \lambda.\cos \alpha]$

$= \dfrac{W}{\cos \lambda}[\sin \alpha.\cos \lambda - \cos \alpha \sin \lambda]$

$= \dfrac{W}{\cos \lambda}.\sin(\alpha - \lambda)$

Let $P_1 = \dfrac{W.\sin(\alpha - \lambda)}{\cos \lambda}$

Case ii Let the body be on the point of moving up. Therefore limiting frictional force μR acts downward along the plane.



Let P be the external force applied to keep the body at rest.

Resolving the force,

$R = W \cos \alpha;\ P = \mu R + W \sin \alpha$

$\therefore P = \mu.W \cos \alpha + W \sin \alpha$

$= \dfrac{W}{\cos \lambda}[\sin \lambda \cos \alpha + \cos \lambda.\sin \alpha]$

$$= \frac{W}{\cos\lambda}.\sin(\alpha+\lambda)$$

Let $P_2 = \frac{W}{\cos\lambda}.\sin(\alpha+\lambda)$

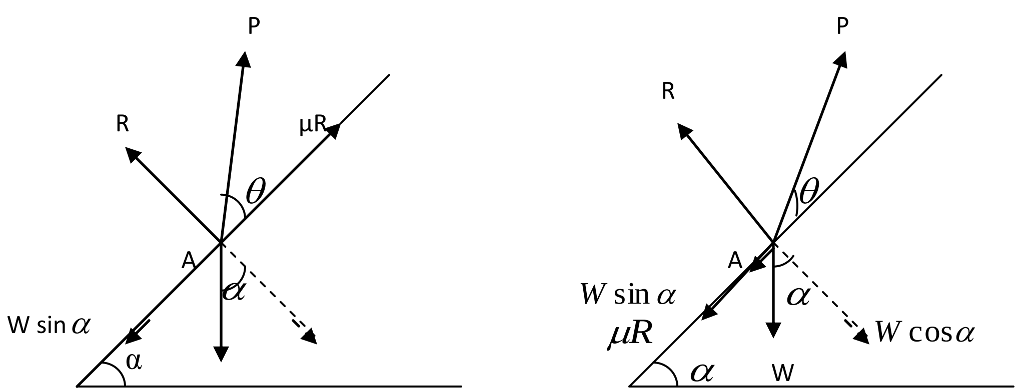
If $P < P_1$, body will move down the plane. If $P > P_2$, body will move up the plane.

$\therefore$ For equilibrium P must lie between P_1 and P_2.

i.e.) $\boxed{P_1 > P > P_2}$

2.5 Equilibrium of a body on a rough inclined plane under any force.

Theorem: A body is at rest on a rough inclined plane of inclination α to the horizon, being acted on by a force making an angle θ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.



Let α be the inclination of the plane, W be the weight of the body, P – be the force acting at an angle θ with the inclined plane and R – be the normal reaction.

Case i: The body is just on the point of slipping down. Therefore the limiting friction μR acts upwards.

Resolving the forces along and $\perp r$ to the inclined plane,

$P\cos\theta + \mu R = W\sin\alpha$ ……………….. (1)

$P\sin\theta + R = W\cos\alpha$ ………………….. (2)

$$\therefore R = W\cos\alpha - P\sin\theta$$

$$\therefore (1) \Rightarrow P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$$

$$P(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha)$$

$$\therefore P = \frac{W(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

We have $\mu = \tan\lambda$

$$\therefore P = \frac{W(\sin\alpha - \tan\lambda.\cos\alpha)}{\cos\theta - \tan\lambda.\sin\theta}$$

$$= W\frac{(\sin\alpha\cos\lambda - \cos\alpha.\sin\lambda)}{\cos\theta.\cos\lambda - \sin\theta.\sin\lambda}$$

$$= W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$$

Let $P_1 = W.\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$

Case ii: The body is just on the point of moving up the plane. Therefore μR acts downwards.

Resolving the forces along and $\perp r$ to the plane.

$P\cos\theta - \mu R = W.\sin\alpha$ …………………. (3)

$P\sin\theta + R = W.\cos\alpha$ …………………. (4)

$$R = W\cos\alpha - P\sin\theta$$

$$(3) \Rightarrow P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W.\sin\alpha$$

$$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$$

$$\therefore P = \frac{W(\sin\alpha + \tan\lambda.\cos\alpha)}{(\cos\theta + \tan\lambda.\sin\theta)}$$

$$= \frac{W(\sin\alpha.\cos\lambda + \sin\lambda.\cos\alpha)}{(\cos\theta\cos\lambda + \sin\theta.\sin\lambda)}$$

$$= \frac{W.\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Let $P_2 = \dfrac{W.\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

To keep the body in equilibrium, P_1 and P_2 are the limiting values of P.

Find the least force required to drag the body up the inclined plane

We have, P = $W.\dfrac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

P is least when $\cos(\theta-\lambda)$ is greatest.

i.e.) When $\cos(\theta-\lambda)=1$

i.e.) When $\theta-\lambda=0$

i.e.) When $\theta=\lambda$

$$\boxed{\therefore Least\ \ value\ \ of\ \ P = W.\sin(\alpha+\lambda)}$$

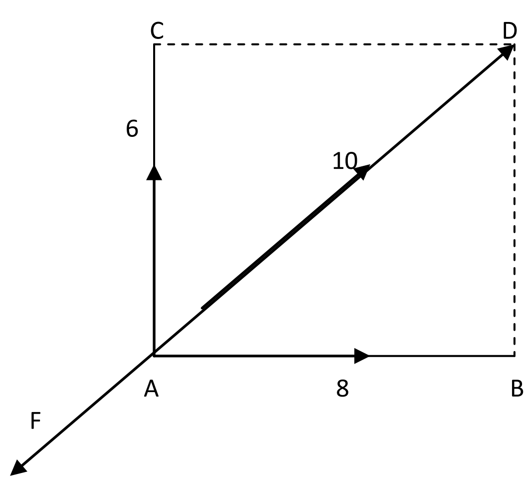
Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

i.e.) ***"The best angle of traction up a rough inclined plane is the angle of friction"***

Problem 1

A particle of weight 30 kgs. resting on a rough horizontal plane is just on the point motion when acted on by horizontal forces of 6kg wt. and 8kg. wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.

Solution:



Let AB = 8 and AC = 6 represent the directions of the forces, A being the particle.

The resultant force = $\sqrt{8^2+6^2}$ = 10kg. wt. and this acts along AD, making an angle $\cos^{-1}\left(\frac{4}{5}\right)$ with the 8kg force.

Let F be the frictional force. As motion just begins, magnitude of F is equal to that of the resultant force.

$\therefore F = 10$ ……………… (1)

If R is the normal reaction on the particle,

R = 30 ………………….. (2)

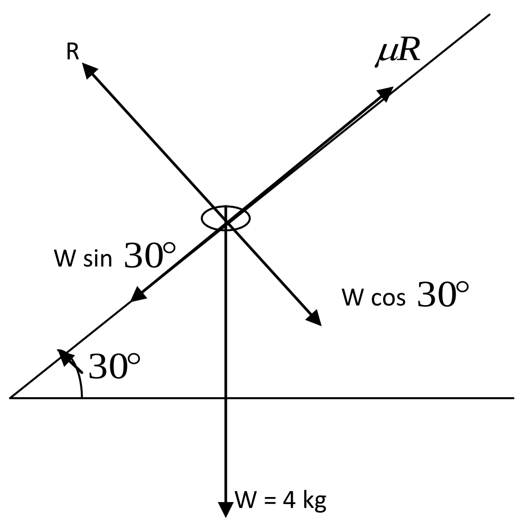
If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$

10 = $\mu.30$ $\therefore \mu = \frac{10}{30} = \frac{1}{3}$.

Problem 2

A body of weight 4 kgs. rests in limiting equilibrium on an inclined plane whose inclination is 30°. Find the coefficient of friction and the normal reaction.

Solution:



Since the body is in limiting equilibrium on the inclined plane, it tries to move in the downward direction along the inclined plane.

$\therefore$ Frictional force μR acts in the upward direction along the inclined plane. Resolving along and $\perp r$ to the plane,

$\mu R = W \sin 30°$ ………………… (1)

$= 4.\dfrac{\sqrt{3}}{2} = 2\sqrt{3}$

R = $W.\cos 30°$ …………………. (2)

$= 4\dfrac{1}{2} = 2$

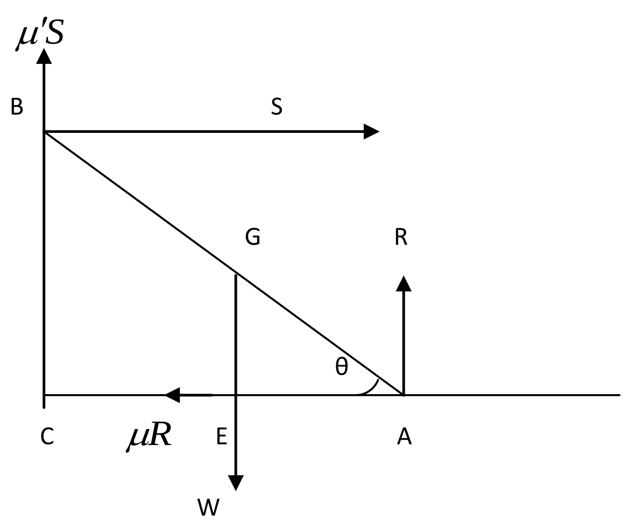
$\dfrac{(1)}{(2)} \Rightarrow \mu = \dfrac{1}{\sqrt{3}}$

$\tan\lambda = \dfrac{1}{\sqrt{3}}, \therefore \quad \boxed{\lambda = 30°}$

Problem 3

A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to the horizon is given by $\tan\theta = \dfrac{1-\mu\mu'}{2\mu}$. Find also the reactions at the wall and ground.

Solution:



AB is the uniform ladder, whose weight W is acting at G such that AG = GB. Forces acting are,

1. Weight W
2. Normal reaction R at A
3. Normal reaction S at B
4. μR
5. $\mu' S$

When the ladder is on the point of slipping at both ends, frictional forces $\mu' S, \mu R$ act along CB, AC respectively.

Since the ladder is in equilibrium resultant is zero.

$\therefore$ Resolving horizontally and vertically,

$S = \mu R$ ……….. (1)

$R + \mu' S = W$ …………. (2)

$\therefore R + \mu'(\mu R) = W$

$R(1 + \mu\mu') = W \Rightarrow$ $\boxed{R = \dfrac{W}{1 + \mu\mu'}}$ $\boxed{\therefore S = \dfrac{\mu W}{1 + \mu\mu'}}$

By Varigon's theorem on moments, taking moments about A

$$S.BC + \mu' S.AC = W.AE$$

$$S.AB\sin\theta + \mu' S.AB\cos\theta = W.AG.\cos\theta$$

$$S.\sin\theta + \mu' S.\cos\theta = W.\frac{1}{2}.\cos\theta \quad \left[\because AG = \frac{AB}{2}\right]$$

$$\therefore S.\sin\theta = \left[\frac{W}{2} - \mu' S\right].\cos\theta$$

$$\therefore \tan\theta = \frac{W}{2S} - \mu' = \frac{W}{2\left[\dfrac{\mu W}{1 + \mu\mu^1}\right]} - \mu^1 = \frac{1 + \mu\mu'}{2\mu} - \mu'$$

$$= \frac{1 + \mu\mu' - 2\mu\mu'}{2\mu} \quad \boxed{\tan\theta = \frac{1 - \mu\mu'}{2\mu}}$$

Problem 4

In the previous problem, when $\mu = \mu'$ show that $\theta = 90° - 2\lambda,$ where λ is the angle of friction.

Solution:

In the previous problem, we have proved $\tan\theta = \frac{1-\mu\mu'}{2\mu}$

Put $\mu = \mu'$, we get

$$\tan\theta = \frac{1-\mu^2}{2\mu} = \frac{1-\tan^2\lambda}{2\tan\lambda}; [\because \mu = \tan\lambda]$$

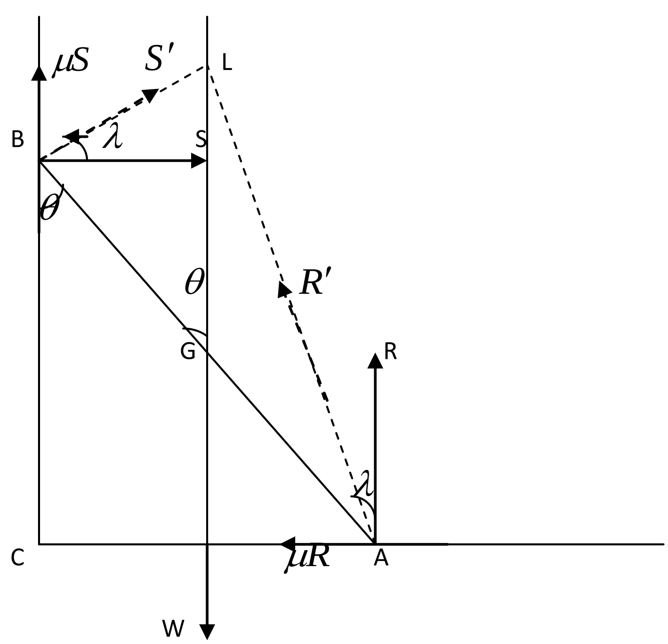
$$= \frac{1}{\tan 2\lambda} = \cot 2\lambda = \tan(90° - 2\lambda)$$

i.e.) $\tan\theta = \tan(90° - 2\lambda)$ $\therefore$ $\boxed{\theta = 90° - 2\lambda}$

Problem 5

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against an equally rough vertical wall. If θ be the inclination of the ladder to the vertical, prove that $\tan\theta = \frac{2\mu}{1-\mu^2}$ where μ is the coefficient of friction.

Solution:



When the ladder AB is in limiting equilibrium, five forces are acting as marked in the figure.

1) Weight of the ladder W
2) Normal reaction R at A
3) Normal reaction S at B
4) Frictional force μR
5) frictional force μS

Let R', S' be the resultant reactions of R, μR and S, μS respectively.

$\therefore$ We have 3 forces R', S', W. For equilibrium, they must be concurrent at L.

In $\Delta LAB, L\hat{G}A = 180^\circ - \theta; A\hat{L}G = \lambda$

$B\hat{L}G = 90 - \lambda, AG : GB = 1:1$

$\therefore$ By trigonometrical theorem in Δ LBA,

(1+1) $\cot(180^\circ - \theta) = 1.\cot(90^\circ - \lambda) - 1.\cot\lambda$

$$-2.\cot\theta = \tan\lambda - \cot\lambda \;= \frac{\tan^2\lambda - 1}{\tan\lambda}$$

$$\therefore \cot\theta = \frac{1 - \tan^2\lambda}{2\tan\lambda}$$

i.e.) $\dfrac{1}{\tan\theta} = \dfrac{1-\mu^2}{2\mu}$ $\quad \boxed{\therefore \tan\theta = \dfrac{2\mu}{1-\mu^2}}$

Problem 6

A uniform ladder rests with its lower end on a rough horizontal ground its upper end against a rough vertical wall, the ground and the wall being equally rough and the angle of friction being λ. Show that the greatest inclination of the ladder to the vertical is 2λ.

Solution

In the previous problem, we have proved, $\tan\theta = \dfrac{2\mu}{1-\mu^2}$ But $\mu = \tan\lambda$

$$\therefore \tan\theta = \frac{2\tan\lambda}{1-\tan^2\lambda} = \tan 2\lambda \Rightarrow \boxed{\therefore \theta = 2\lambda}$$

Problem 7

A ladder which stands on a horizontal ground, leaning against a vertical wall, is so loaded that its C. G. is at a distance a and b from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium, its inclination θ to the horizontal is given by $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$ where μ, μ' are the coefficients of friction between the ladder and the ground and the wall respectively.

Solution:

As in problem 5, five forces are acting on the ladder

Here, AG : GB = a: b

$\therefore$ By Trigonometrical theorem in ΔLBA,

$(b+a).\cot(90+\theta) = b.\cot(90-\lambda') - a.\cot\lambda$

i.e.) $(a+b)(-\tan\theta) = b.\tan\lambda^1 - a.\cot\lambda$

$$\therefore \tan\theta = \frac{\left(\dfrac{a}{\mu}\right) - b.\mu'}{a+b} = \frac{a - b.\mu\mu'}{(a+b)\mu}$$

Problem 8

A ladder AB rests with A on a rough horizontal ground and B against an equally rough vertical wall. The centre of gravity of the ladder divides AB in the ratio a: b. If the ladder is on the point of slipping, show that the inclination θ of the ladder to the ground is given by $\tan\theta = \dfrac{a - b\mu^2}{\mu(a+b)}$ where μ is the coefficient of friction.

Solution:

In the previous problem,

Put $\mu = \mu'$ in $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$

$$\boxed{\therefore \tan\theta = \frac{a - b\mu^2}{\mu(a+b)}}$$

Problem 9

A ladder AB rests with A resting on the ground and B against a vertical wall, the coefficients of friction of the ground and the wall being μ and μ' respectively. The centre of gravity G of the ladder divides AB in the ratio 1: n. If the ladder is on the point of slipping at both ends, show that its inclination to the ground is given by $\tan\theta = \dfrac{1-n\mu\mu'}{(n+1)\mu}$.

Solution:

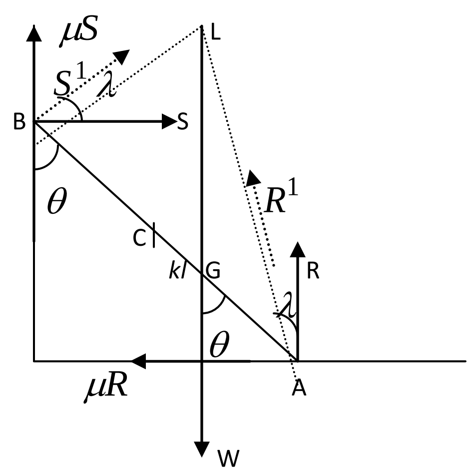
Put a : b = 1 : n in problem7.

$$\therefore \tan\theta = \frac{1-n\mu\mu'}{(1+n)\mu}$$

Problem 10

A ladder of length $2l$ is in contact with a vertical wall and a horizontal floor, the angle of friction being λ at each contact. If the weight of the ladder acts at a point distant kl below the middle point, prove that its limiting inclination θ to the vertical is given by $\cot\theta = \cot 2\lambda - k\,\mathrm{cosec}\,2\lambda.$

Solution:



Forces are acting as marked in the figure. For equilibrium, the three forces R', S', W must be concurrent at L, where W – be the weight of the ladder.

In $\Delta LAB, BC = CA = l; CG = kl.$

$$\therefore BG = BC + CG = l + kl = (1+k)l$$

$B\hat{L}G = 90° - \lambda, L\hat{G}A = 180° - \theta$

$A\hat{L}G = \lambda; GA = CA - CG = l - kl = (1-k)l.$

$BG : GA = (1+k) : (1-k)$

$\therefore$ By Trigonometrical theorem in $\Delta LBA,$

$[(1+k)+(1-k)].\cot(180° - \theta) = (1+k).\cot(90° - \lambda) - (1-k).\cot\lambda.$

$2(-\cot\theta) = (1+k).\tan\lambda - (1-k).\cot\lambda$

$\therefore 2\cot\theta = (1-k)\cot\lambda - (1+k)\tan\lambda$

$$= \frac{(1-k).\cot^2\lambda - (1+k)}{\cot\lambda}$$

$$= \frac{(\cot^2\lambda - 1) - k(\cot^2\lambda + 1)}{\cot\lambda}$$

$$\cot\theta = \frac{(\cot^2\lambda - 1) - k.\operatorname{cosec}^2\lambda}{2.\cot\lambda}$$

$$= \frac{1-\tan^2\lambda}{2\cot\lambda.\tan^2\lambda} - k\left[\frac{1+\cot^2\lambda}{2.\cot\lambda}\right]$$

$$= \frac{1}{\left(\dfrac{2\tan\lambda}{1-\tan^2\lambda}\right)} - k\left[\frac{1+\tan^2\lambda}{2.\tan^2\lambda.\cot\lambda}\right]$$

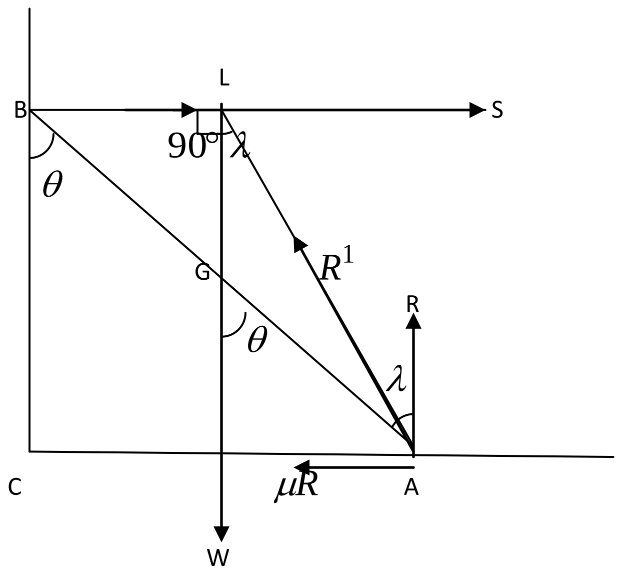
$$= \frac{1}{\tan 2\lambda} - k.\frac{1}{\sin 2\lambda}$$

ie) $\cot\theta = \cot 2\lambda - k.\operatorname{cosec} 2\lambda$

Problem 11

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and with the upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, prove that, $\tan\theta = 2\mu,$ where μ is the coefficient of friction.

Solution:



Since the wall is smooth, there is no frictional force. Forces acting on the ladder are i) its weight W, ii) Frictional force μR iii) R at A iv) S at B. For equilibrium, the three forces W, R', S must be concurrent at L. where R^1 is the resultant of R and μR. In triangle LAB,

$L\hat{G}A = 180^\circ - \theta, A\hat{L}G = \lambda, B\hat{L}G = 90^\circ; BG : GA = 1:1. A\hat{B}C = \theta$

By Trigonometrical theorem in ΔLAB,

$(1+1)\cot(180^\circ - \theta) = 1.\cot 90^\circ - 1.\cot\lambda$

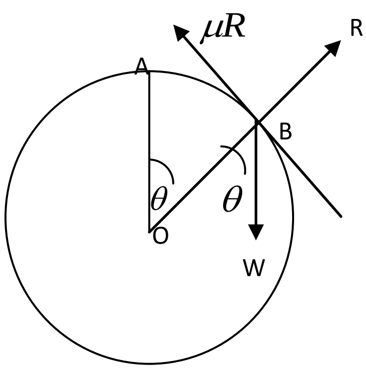
$-2.\cot\theta = 0 - \cot\lambda$

$\therefore \dfrac{2}{\tan\theta} = \dfrac{1}{\tan\lambda}$ $\therefore \tan\theta = 2\tan\lambda$ i.e) $\boxed{\tan\theta = 2\mu}$

Problem 12

A particle is placed on the outside of a rough sphere whose coefficient of friction is μ. Show that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1}\mu$ with the vertical.

Solution:



Let O be the centre, A the highest point of the sphere and B the position of the particle which is just on the point of motion. Let $\angle AOB = \theta$

The forces acting at B are:

1) the normal reaction R
2) limiting friction μR
3) Its weight W,

Since the particle at B is in limiting equilibrium,

Resolving along the normal OB,

$R = W\cos\theta$ …………………. (1)

Resolving along the tangent at B,

$\mu R = W\sin\theta$ ……………….. (2)

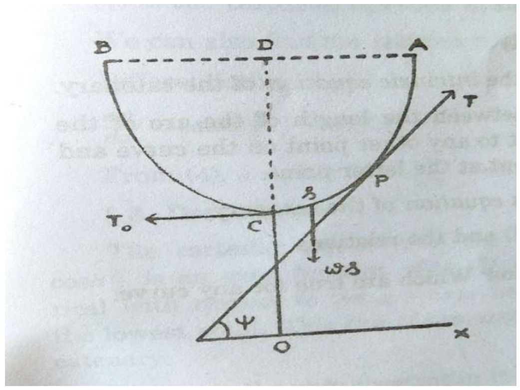
$\frac{(2)}{(1)} \Rightarrow \mu = \tan\theta \Rightarrow$ $\boxed{\theta = \tan^{-1}\mu}$

2.6 Equilibrium of Strings

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a *catenary*. If the weight per unit length of the chain or string is constant, the catenary is called the *uniform or common catenary*.

2.7 Equation of the common catenary:

A uniform heavy inextensible string hangs freely under the action of gravity; to find the equation of the curve which it forms.



Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest, of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the are CP = s

Let ω be the weight per unit length of the chain.

Consider the equilibrium of the portion CP of the chain.

The forces acting on it are:

(i) Tension T_0 acting along the tangent at C and which is therefore horizontal.

(ii) Tension T acting at P along the tangent at P making an angle Ψ with OX.

(iii) Its weight **ws** acting vertically downwards through the C.G. of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight **ws** must pass through the point of the intersection of T and T_o.

Resolving horizontally and vertically, we have

$$T\cos \Psi = T_o \; \ldots \; \ldots \qquad (1)$$

$$\text{and } T\sin \Psi = \mathbf{ws} \; \ldots \; \ldots \; (2)$$

Dividing (2) by (1), $\tan \Psi = \frac{\mathbf{ws}}{T_0}$

Now it will be convenient to write the value of T_0 the tension at the lowest point, as $T_o = wc$ … … (3) where c is a constant. This means that we assume T_o, to be equal to the weight of an unknown length c of the cable.

Then tan $\Psi = \frac{\boldsymbol{ws}}{\boldsymbol{wc}} = \frac{\boldsymbol{s}}{\boldsymbol{c}}$

$\therefore$ S = ctanΨ … … … (4)

Equation (4) is called the *intrinsic* equation of the catenary.

It gives the relation between the length of the area of the curve from the lowest point to any other point on the curve and the inclination of the tangent at the latter point.

To obtain the *certesian equation* of the catenary,

We use the equation (4) and the relations

$\frac{dy}{ds}$ = sin Ψ and $\frac{dy}{dx}$ = tan Ψ which are true for any curve.

Now $\frac{dy}{d\Psi} = \frac{dy}{ds} . \frac{ds}{d\Psi}$

$= \sin \Psi \frac{d}{d\Psi} c \tan \Psi$

$= \sin c\sec^2\Psi = c\sec \Psi \tan \Psi$

$\therefore y = \int c\sec \Psi \tan \Psi \, d\Psi + A$

$= c\sec \Psi + S$

If $y = c$ when $\Psi = 0$, then c = $c$sec0 + A

$\therefore$ A = 0

Hence y = csec Ψ … … … (5)

$\therefore y^2 = c^2 \sec \Psi = c^2 (1 + \tan^2 \Psi)$

$= c^{\,2} + s^{\,2}$ … … (6)

$\frac{dy}{dx} = \tan \Psi = \frac{s}{c} = \frac{\sqrt{y^2 - c^2}}{\mathrm{c}}$

$\therefore \ \frac{\mathrm{dy}}{\sqrt{y^2 - c^2}} = \frac{\mathrm{dx}}{\mathrm{c}}$

Integrating, $\cos h^{-1} \left(\frac{y}{c}\right) = \frac{x}{c} + B$

When x = 0, y = c

i.e. $\cos h^{-1} 1 = 0 + B$ or B = 0

$\therefore \cos h^{-1} \left(\frac{y}{c}\right) = \frac{x}{c}$

i.e. $y = c\cos h \left(\frac{\mathrm{x}}{c}\right)$ … … (7)

(7) is the Cartesian equation to the catenary.

We can also find the relation connecting s and x.

Differentiating (7).

$\frac{dy}{dx}$ = csinh $\frac{x}{c}$. $\frac{1}{c}$ = sinh $\frac{x}{c}$

From (4), s = ctan Ψ = c. $\frac{dy}{dx}$ = csinh $\frac{x}{c}$ … (8)

Definitions:

The Cartesian equation to the catenary is y = ccosh $\frac{x}{c}$. cosh $\frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the axis of the catenary.

Since c is the only constant, in the equation, it is called the *parameter* of the catenary and it determines the size of the curve.

The lowest point *C* is called the vertex of the catenary. The horizontal line at the depth *c* below the vertex (which is taken by us the x – axis) is called the directrix of the catenary.

If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the span and the distance CD (i.e. the depth of the lowest point C below AB) is called the sag.

2.8 Tension at any point:

We have derived the equations

T cos Ψ = T_0 … … … … (1)

And T sin Ψ = *ws* … … … … (2)

We have also put T_0 = *wc* … … … (3)

Equation (3) shows that the tension at the lowest point is a constant and is equal to the weight of a portion of the string whose length is equal to the parameter of the catenary. From the equation (1), we find that the horizontal component of the tension at any point on the curve is equal to the tension at the lowest point and hence is a constant.

From equation (2), we deduce that the vertical component of the tension at any point is equal to ws i.e. equal to the weight of the portion of the string lying between the vertex and the point. (∴ s = are CP)

Squaring (1) and (2) and then adding,

$T^2 = T^2_0 + w^2s^2$

$=w^2c^2+w^2a^2$

$=w^2(c^2+s^2)$

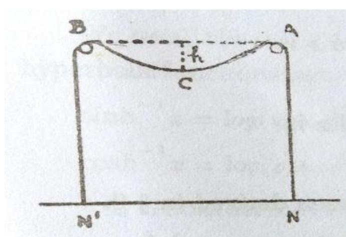
$=w^2y^2$ using equation (6) of page 377

$\therefore T = wy$ ... … … (4)

Thus the tension at any point is proportional to the height of the point above the origin. It is equal to the weight of a portion of the string whose length is equal to the height of the point above the directrix.

Important Corollary:

Suppose a long chains is thrown over two smooth pegs A and B and is in equilibrium with the portions AN and B*N*' hanging vertically. The potion BCA of the chain will from a catenary.



The tension of the chain is unaltered by passing overt the smooth peg A. The tension at A can be calculated by two methods.

On one side (i.e. from the catenary portion), Tension at A = $w.y$ where y is the height of A above the directrix.

On the other side, tension at A = weight of the free part AN hanging down

= w. AN

$\therefore y$=AN

In other words, N is on the directrix of the catenary.

Similarly N' is on the directrix.

Hence if a long chain is thrown over two smooth pegs and is in equilibrium, the free ends must reach the directrix of the catenary formed by it.

Important Formulae:

The Cartesian coordinates of a point P on the catenary are (x, y) and its intrinsic coordinates are (s, Ψ). Hence there are four variable quantities we can have a relation connecting any two of them. There will be $_4C_2 = 6$ such relations, most of them having been already derived. We shall derive the remaining. It is worthwhile to collect these results for ready reference.

(i) The relation connecting x and y is

$y = c\cosh\frac{x}{c}$ … … (1)

and this is the Cartesian equation to the catenary.

(ii) The relation connecting s and Ψ is

$s = c\tan\Psi$ … … (2)

(iii) The relation connecting y and Ψ is

$y = c\sec\Psi$ … … … (3)

(iv) The relation connecting y and s is

$y^2 = c^2+s^2$ …. … … (4)

(v) The relation connecting s and x is

$s = c\sinh\frac{x}{c}$

(vi) We have $y = c\cosh\frac{x}{c}$ and $y = c\sec\Psi$,

$\therefore \sec\Psi = \cosh\frac{x}{c}$

$\therefore \frac{x}{c} = \cosh{-1}(\sec\Psi)$

$= \log(sec\Psi + \sqrt{sec^2\Psi - 1}$

$= \log(sec\Psi + \tan\Psi)$

$\therefore x = clog\ (sec\Psi + tan\Psi)$ … … (6)

This relation can also be obtained thus:

$\frac{dx}{d\Psi} = \frac{dx}{ds}.\frac{ds}{d\Psi}$

$= \cos\Psi.\frac{d}{d\Psi}(c\tan\Psi)$ since $\frac{dx}{ds} = \cos\Psi$ for any curve

$= \cos\Psi.\ C\sec2\Psi - c\sec\Psi$

Integrating, x = $\int c\sec\Psi\, d\Psi + D$

= clog (secΨ + ranΨ) + D

At the lowest point, $\Psi = 0$ and $x = 0$

$\therefore$ 0 = clog (sec0+tan0 + D

i.e. 0 = D

$\therefore$ x= clog (secΨ + tan Ψ)

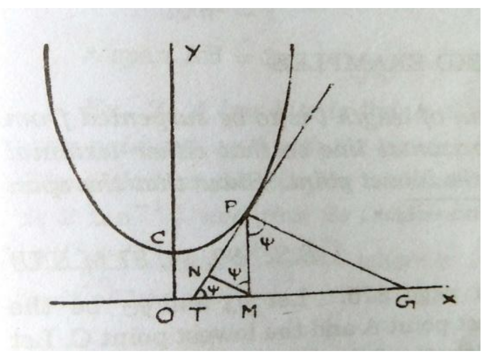
(vii) The tension at any point = wy … … (7), where y is the distance of the point from the directrix.

(viii) The tension at the lowest point = wc … … (8)

$\sinh^{-1} x = \log(x+\sqrt{x^2+1})$

$\cosh^{-1} x = \log(x+\sqrt{x^2-1})$

2.9 Geometrical Properties of the Common catenary:



Let P be any point on the catenary y = ccosh $\frac{x}{c}$.

PT is the tangent meeting the directrix (i.e. the x axis) at T.

angle PTX = Ψ

PM (=y) is the ordinate of P and PG is the normal at P.

Draw MN $\perp$ to PT.

From ΔPMN. MN = PMcosΨ

=$y$cosΨ

$=c\sec\Psi \cos \Psi$

$=c$=constant

i.e. *The length of the perpendicular from the foot of the ordinate on the tangent at any point of the catenary is constant.*

Again $\tan \Psi = \frac{PN}{MN} = \frac{PN}{C}$

$\therefore$ PN = $_C\tan \Psi = S$ arc CP

$PM^2 = NM^2 + PN^2$

$\therefore y^2 = c^2+s^2$, a relation already obtained.

If is the radius of curvature of the catenary at P,

$P= \frac{ds}{d\Psi} = \frac{d}{d\Psi}(c\tan \Psi) = c\sec^2\Psi$

Let the normal at P cut the x axis at G.

Then PG. $\cos \Psi$ = PM = y

$\therefore$ PG = $\frac{y}{\cos\Psi} = c\sec\Psi.\sec\Psi = c\sec^2\Psi$

$\therefore \rho =$ PG

Hence the radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.

Problem 13

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1}$

Solution:

Tension at A = wy_A

And tension at C = $w.y_C$ since T = wy at any point

Now $w.y_A = n.w.y_C$

$\therefore y_A = ny_C = nc$

But $y_A = c\cosh\frac{x_A}{c} = nc$

$\therefore \cosh\frac{x_A}{c} = n$

or $\frac{x_A}{c}$ = $\cosh^{-1} n$ = $\log(n+\sqrt{n^2-1})$

$\therefore x_A = c\log(n+\sqrt{n^2-1})$ … … … (1)

We have to find c.

$y^2_A = c^2+s^2_A$, s_A denoting the length of CA.

$= c^2+\frac{l^2}{4}$ (as total length = l)

i.e. $n^2c^2 = c^2+\frac{l^2}{4}$

or $c^2 = \frac{l^2}{4(n^2-1)}$

$\therefore c = \frac{l^2}{2\sqrt{n^2-1}}$ … … (2)

Substituting (2) in (1),

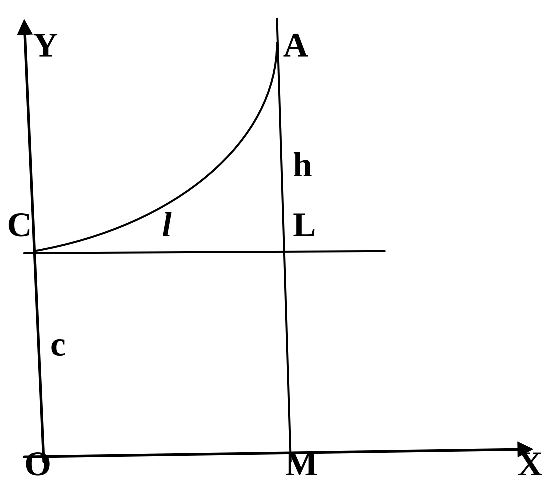
$$x_A = \frac{l^2}{2\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$$

$\therefore$ span AB $= 2x_A = \frac{l}{\sqrt{n^2-1})}\log(n+\sqrt{n^2-1})$

Problem 14

A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite, the inclination of the wire to the ground is $2\tan^{-1}\frac{h}{l}$ and that its tensions there and at the ground are $\frac{w(l^2+h^2)}{2h}$ and $\frac{w(l^2-h^2)}{2h}$ where w is the weight of the wire per unit of length.

Solution:



C is the vertex of the catenary CA, A being the kite. The origin O is taken at a depth c below C.

Then $y_A = c + h$ and s_A = arc CA = l

Since $y^2 = c^2 + s^2$, we have $(c+h)^2 = c^2 + l^2$

i.e. $h^2+2ch = l^2$

or $c = \frac{l^2-h^2}{2h}$ … … … (1)

We know that $s = c \tan \Psi$ … … …. …. …. (2)

Applying (2) at the point A, we have

$l = c. \tan \Psi_A$

$\therefore \operatorname{Tan} \Psi_A = \frac{l}{c} = \frac{2hl}{l^2-h^2}$ substituting for c from (1)

$= \frac{2(\frac{h}{l})}{1-(\frac{h}{l})2}$ … … (3)

But $\tan\Psi = \frac{2tan\frac{\Psi}{2}}{1-\tan^2\frac{\Psi}{2}}$ … … (4)

Comparing (3) and (4), we find that

$tan\frac{\Psi}{2}$ at A = $\frac{h}{l}$

$\therefore \frac{\Psi}{2} = \tan^{-1}\frac{h}{l}$

or Ψ at A = $2\tan^{-1}\frac{h}{l}$

The tension at A = $w.y_A$

$= w.(c + h)$

$= w\left(\frac{l^2-h^2}{2h} + h\right) = \frac{w(l^2+h^2)}{2h}$

Problem 15

A uniform chain of length l is to have its extremities fixed at two points in the same horizontal line. Show that the span must be $\frac{l}{\sqrt{8}}\log(3+\sqrt{8})$ in order that the tension at each support shall be three times that at the lowest point.

Solution:

Put n = 3 in problem number 13.

Problem 16

A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension A is twice that at the lowest point, show that the span AB is $\frac{l}{\sqrt{3}}\log(2+\sqrt{3})$

Solution:

Put n = 2 in problem number 13.

Problem 17

A uniform chain of length $2l$ hangs between two points A and B on the same level. The tension both at A and B is five times that at the lowest point. Show that the horizontal distance between A and B is $\frac{l}{\sqrt{6}}\log(5+2\sqrt{3})$

Solution:

Put n = 5 and length = $2l$ in problem number 13.

Problem 18

If T is the tension at any point P and T_0 is the tension at the lowest point C then prove that $T^2 - T_0^2 = W^2$ where W is the weight of the arc CP of the string.

Solution:

Given T is the tension at P. Let w be the weight per unit length and y is the ordinate of P.

Then $T = wy$.

Also $T_0 = wc$

$$\therefore T^2 - T_0^2 = w^2y^2 - w^2c^2$$

$$= w^2(y^2 - c^2)$$

$$= w^2s^2$$

$$= W^2$$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

III YEAR B.SC MATHEMATICS

SMT5103 MECHANICS 1

Unit 3- Kinematics – SMT1503

2 Kinematics

An object (eg. ball, planet,...) is idealized as a point particle (zero size) with a quantity of matter called *mass*.
Good approx if size of object $\ll$ trajectory, and rotation not important.
A point particle has position vector $\mathbf{r}(t)$ at time t, given a chosen origin $\mathcal{O}$.
Write down the equation of motion for $\mathbf{r}(t)$ (ODE) and solve it to find the trajectory $\mathbf{r}(t)$ ie. a curve in space.

2.1 Definitions

Definitions of some quantities.

velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

is tangent to the trajectory.
speed, $v = |\mathbf{v}| \geq 0$, magnitude of the velocity.
momentum, $\mathbf{p} = m\mathbf{v}$.
acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}$$

kinetic energy, $T = \frac{1}{2}mv^2$.

2.2 Cartesian coordinates

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed orthogonal unit vectors ie $\mathbf{i} \cdot \mathbf{i} = 1, \mathbf{i} \cdot \mathbf{j} = 0$, etc, eg. $\mathbf{i} = (1, 0, 0)$.
In mechanics do not write just the components, $\mathbf{r} = (x, y, z)$, but include the basis vectors ie. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

This is because sometimes the basis vectors are not constant, and then we would get the wrong answer for the velocity if we just differentiated the components.
If $\mathbf{r} = \alpha\mathbf{e}$, then $\dot{\mathbf{r}} = \dot{\alpha}\mathbf{e} + \alpha\dot{\mathbf{e}} \neq \dot{\alpha}\mathbf{e}$ if $\dot{\mathbf{e}} \neq 0$.

eg. $\mathbf{r} = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$ with $m = 2$.
$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{i} + 2t\mathbf{k}$, $\quad v = |\mathbf{v}| = \sqrt{1+4t^2}$, $\quad \mathbf{p} = m\mathbf{v} = 2\mathbf{v} = 2\mathbf{i} + 4t\mathbf{k}$, $\mathbf{a} = \dot{\mathbf{v}} = 2\mathbf{k}$, $\quad T = \frac{1}{2}mv^2 = (1+4t^2)$.

Note: Acceleration can be non-zero even if the speed is constant, since the direction of the velocity might not be constant.

Given the acceleration at all times and initial position and velocity, the position can be found by integration.
eg. $\mathbf{a} = 2\mathbf{k}$, $\quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, $\quad \mathbf{v}(0) = \mathbf{i}$.
$\mathbf{v} = 2t\mathbf{k} + \mathbf{c}$, but $\mathbf{v}(0) = \mathbf{c} = \mathbf{i}$ therefore $\mathbf{v} = 2t\mathbf{k} + \mathbf{i}$.
$\mathbf{r} = t^2\mathbf{k} + t\mathbf{i} + \mathbf{d}$, but $\mathbf{r}(0) = \mathbf{d} = \mathbf{j} + \mathbf{k}$, therefore $\mathbf{r} = (t^2+1)\mathbf{k} + t\mathbf{i} + \mathbf{j}$.

2.3 Polar coordinates and vectors

Consider motion in a plane, using polar coordinates r, θ, where $x = r\cos\theta$ and $y = r\sin\theta$.
$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$.
The **radial unit vector** $\mathbf{e}_r$ is a vector in the direction of $\mathbf{r}$,
$\mathbf{e}_r = \frac{\mathbf{r}}{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.
The **tangential unit vector** $\mathbf{e}_\theta$ is a vector perpendicular to $\mathbf{e}_r$, and is
$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, $\quad$ (increasing θ is anti-clockwise).

If the particle is moving then r and θ can depend on time.
$\dot{\mathbf{e}}_r = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_r = \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\mathbf{e}_\theta$.
$\dot{\mathbf{e}}_\theta = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_\theta = \dot{\theta}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = -\dot{\theta}\mathbf{e}_r$.

Note: $\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1$ and $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ for all time.

$\mathbf{r} = r\mathbf{e}_r$ therefore $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$.

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.$$

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta, \qquad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.}$$

Eg. Motion in a circle with constant speed
$r = \rho$ with ρ constant.
$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = \rho\dot{\theta}\mathbf{e}_\theta$.
Note that for circular motion $\mathbf{v} \cdot \mathbf{r} = 0$ since $\mathbf{e}_\theta \cdot \mathbf{e}_r = 0$.
$v = |\mathbf{v}| = |\rho\dot{\theta}||\mathbf{e}_\theta| = |\rho\dot{\theta}|$.
Hence for constant speed $\dot{\theta} = \omega$ with ω constant (choose $\omega > 0$.)
$v = \omega\rho$ hence $\omega = v/\rho$.
Since $\dot{\theta}$ is constant then $\ddot{\theta} = 0$, so
$\mathbf{a} = -\rho\omega^2\mathbf{e}_r$. Hence $a = |\mathbf{a}| = \rho\omega^2 = v^2/\rho$.
The acceleration is directed radially **inwards**.
This is called centripetal (centre-seeking) acceleration.
Warning: do not confuse with centrifugal (centre-fleeing) – see later.

2.4 Units and dimensions

Generally use SI units (often drop units altogether).
Mass kg, length m, time s. Remember to convert eg. mins to seconds.

Dimensions are similar to units but more significant.

Quantity	Dimension
Mass	M
Length	L
Time	T

Write [mass] = M etc
$[\text{velocity}] = [\frac{\text{length}}{\text{time}}] = \text{LT}^{-1}, \quad [\text{acceleration}] = [\frac{\text{velocity}}{\text{time}}] = \text{LT}^{-2}.$

Correct equations must have the same dimensions on each side.
Can check consistency using dimensional analysis.

Eg. Period of a pendulum
Pendulum of length l and mass m swings under gravity (acceleration due to gravity g). Its period is $2\pi\sqrt{l/g}$. Check this has the correct dimensions

$$[2\pi\sqrt{\frac{l}{g}}] = \sqrt{\frac{[l]}{[g]}} = \sqrt{\frac{L}{LT^{-2}}} = T$$

.
An expression like mg/l is obviously wrong, since

$$[mg/l] = MLT^{-2}L^{-1} = MT^{-2} \neq T.$$

Can calculate the dimensions of constants in expressions.
Eg. suppose a force is given by κA, where A is the surface area of an object.

$$[\text{force}] = [\text{mass} \times \text{acceleration}] = MLT^{-2} = [\kappa A] = [\kappa]L^2$$

hence $[\kappa] = ML^{-1}T^{-2}$, so could be given in units of $kg/m/s^2$.

2.5 Relative motion

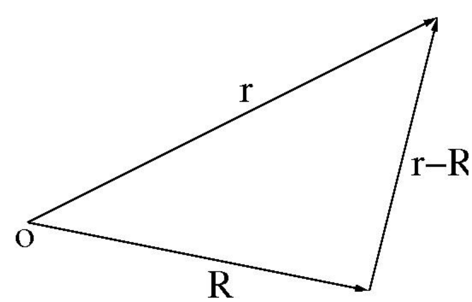


Figure 1: Relative position

r is the position of an object with respect to a fixed origin $\mathcal{O}$. Let an observer (possibly moving) have position **R**. Then the **relative position** of the object to the observer is

$$\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{R}.$$

Relative velocity $\dot{\tilde{\mathbf{r}}} = \dot{\mathbf{r}} - \dot{\mathbf{R}}$, relative acceleration $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}}$, *Eg. Bart is going north (direction* $\mathbf{j}$*) on his skateboard at 10mph and feels a headwind of 25mph. What is the windspeed (velocity) relative to the ground?*

$\mathbf{R}$ is Bart's position, and $\mathbf{r}$ is the position of an air particle.
(Units are miles and hours).
$\dot{\mathbf{R}} = 10\mathbf{j}, \quad \dot{\tilde{r}} = -25\mathbf{j} = \dot{\mathbf{r}} - \dot{\mathbf{R}}$
$\dot{\mathbf{r}} = -25\mathbf{j} + 10\mathbf{j} = -15\mathbf{j}$
Windspeed relative to the ground is 15mph southward.

If Bart now goes east at 15mph what wind does he feel?
$\dot{\mathbf{R}} = 15\mathbf{i}, \quad \dot{\mathbf{r}} = -15\mathbf{j}$, so $\dot{\tilde{\mathbf{r}}} = \dot{\mathbf{r}} - \dot{\mathbf{R}} = -15\mathbf{j} - 15\mathbf{i}$
$|\dot{\tilde{\mathbf{r}}}| = 15\sqrt{2}$, so feels a wind of $15\sqrt{2}$mph in the direction $-(\mathbf{i} + \mathbf{j})/\sqrt{2}$ ie. southwest.

Centrifugal acceleration
This is a result of viewing centripetal acceleration in rotating coordinates.
Let $\mathbf{R}$ be the position of an observer moving in circular motion with radius ρ and constant speed v eg. child on a roundabout.
$\ddot{\mathbf{R}} = -\frac{v^2}{\rho}\mathbf{e}_R$ hence $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}} = \ddot{\mathbf{r}} + \frac{v^2}{\rho}\mathbf{e}_R$,
so even for an object with no forces acting in this plane $\ddot{\mathbf{r}} = \mathbf{0}$,
eg. ball released by the child, then $\ddot{\tilde{\mathbf{r}}} = \frac{v^2}{\rho}\mathbf{e}_R$, so observer sees a relative acceleration directed radially **outwards**.
This is **centrifugal** (centre-fleeing) acceleration.
eg. Child sees the ball flying outwards.

2.6 Inertial frames

The above example appears to contradict **N1** – no forces, no acceleration.
In fact **N1** defines the type of observer (or better reference frame) for which **N2** holds.

An **inertial frame** is one which is not accelerating ie. $\ddot{\mathbf{R}} = \mathbf{0}$,
then $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}} = \ddot{\mathbf{r}}$ so see the 'true' acceleration.

Velocity of a body is defined as the time rate of displacement, where as acceleration is defined as the time rate of change of velocity. Acceleration is a vector quantity. The motion may be uniformly accelerated motion or it may be non-uniformly accelerated, depending on how the velocity changes with time.

Uniform Acceleration

The acceleration of a body is said to be uniform if its velocity changes by equal amounts in equal intervals.

Non-Uniform Acceleration

The acceleration of a body is said to be non-uniform if its velocity changes by unequal amounts in equal intervals of time.

Average velocity

$$lt; v_{avg} gt; = \frac{\int_0^t v dt}{\int_0^t dt}$$

Average acceleration

$$lt; a_{avg} gt; = \frac{\int_0^t a dt}{\int_0^t dt}$$

Illustration:

A particle moves with a velocity $v(t) = (1/2)kt^2$ along a straight line. Find the average speed of the particle in time T.

Solution:

$$lt; v_{avg} gt; = \frac{1}{T}\int_0^T v(t)dt = \frac{1}{2T}\int_0^T kt^2 dt = \frac{1}{6}kT^2$$

Illustration:

A particle having initial velocity is moving with a constant acceleration 'a' for a time t.

(a)Find the displacement of the particle in the last 1 second.

(b)Evaluate it for u = 2 m/s, a = 1 m/s^2 and t = 5 sec.

Solution:

(a) The displacement of a particle at time t is given $s = ut + 1/2at^2$

At time (t - 1), the displacement of a particle is given by

$S' = u (t-1) + 1/2a(t-1)^2$

So, Displacement in the last 1 second is,

$S_t = S - S'$

$= ut + 1/2\ at^2 – [u(t-1)+1/2\ a(t-1)^2\ ]$

$= ut + 1/2at^2 - ut + u - 1/2a(t - 1)^2$

$= 1/2at^2 + u - 1/2\ a\ (t+1-2t) = 1/2at^2 + u - 1/2at^2 - a/2 + at$

$S = u + a/2(2t - 1)$

(b) Putting the values of $u = 2$ m/s, $a = 1$ m/s^2 and $t = 5$ sec, we get

$S = 2 + 1/2(2 \times 5 - 1) = 2 + 1/2 \times 9$

$= 2 + 4.5 = 6.5$ m

Illustration:

Position of a particle moving along x-axis is given by $x = 3t - 4t^2 + t^3$, where x is in meters and t in seconds.

(a)Find the position of the particle at t = 2 s.

(b)Find the displacement of the particle in the time interval from t = 0 to t = 4 s.

(c)Find the average velocity of the particle in the time interval from t = 2s to t=4s.

(d)Find the velocity of the particle at t = 2 s.

Solution:

(a) $x_{(t)} = 3t - 4t^2 + t^3$

=> $x_{(2)} = 3 \times 2 - 4 \times (2)^2 + (2)^3 = 6 - 4 \times 4 + 8 = -2$m.

(b) $x_{(o)} = 0$

$X_{(4)} = 3 \times 4 - 4 \times (4)^2 + (4)^3 = 12$ m.

Displacement = $x_{(4)} - x_{(0)} = 12$ m.

(c) $< v > = X_{(4)}X_{(2)/(4-2)} = (12-(-2))/2$ m/s = 7 m/s

(d) $dx/dt = 3 - 8t + 3t^2$

$v_{(2)}\ (dx/dt)_2 = 3 - 8 \times 2 + 3 \times (2)^2 = -1$m/s

Illustration:

Two trains take 3 sec to pass one another when going in the opposite direction but only 2.5 sec if the speed of the one is increased by 50%. The time one would take to pass the other when going in the same direction at their original speed is

(a) 10 sec (b) 12 sec

(c) 15 sec (d) 18 sec

Solution:

Using the equation,

$t = d/v_r$

We have,

$3 = d/v_1+v_2$

$2.5 = d/1.5v_1+v_2$

Solving we get,

$v_1 = 2d/15$ and $v_2 = d/5$

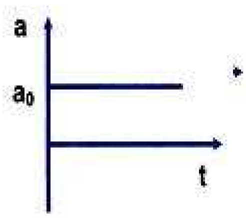
When they are going in same direction,

$v_r = v_2 – v_1 = d/15$

Thus, $t = d/v_r = d/(d/15) = 15$ s

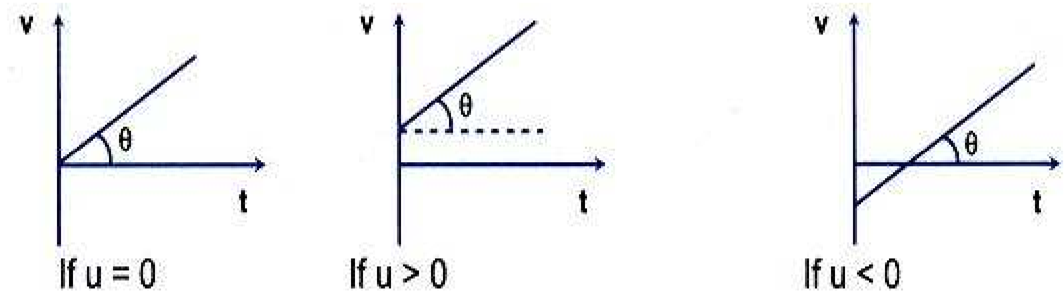
From the above observation we conclude that, option (c) is correct.

Analysis of Uniformly Accelerated Motion



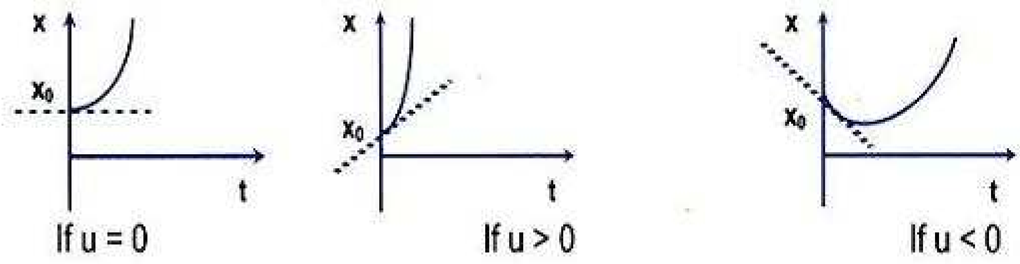
Case-I:
For uniformly accelerated motion with initial velocity u and initial position x_0.

Velocity Time Graph



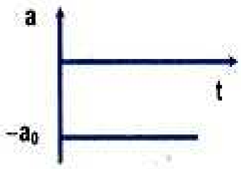
In every case $\tan\theta = a_0$

Position Time Graph



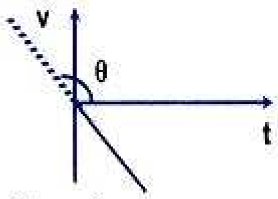
Initial position x of the body in every case is x_0 (> 0)

Case II:

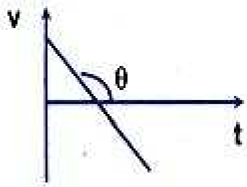


For uniformly retarded motion with initial velocity u and initial position x_0.

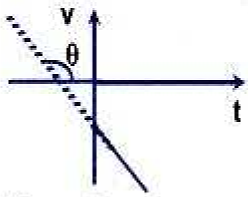
Velocity Time Graph



If u = 0



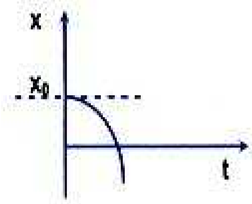
If u > 0



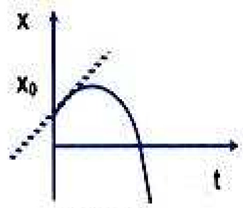
If u < 0

In every case $\tan\theta = -a_0$

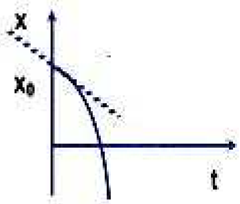
Position Time Graph



If u = 0



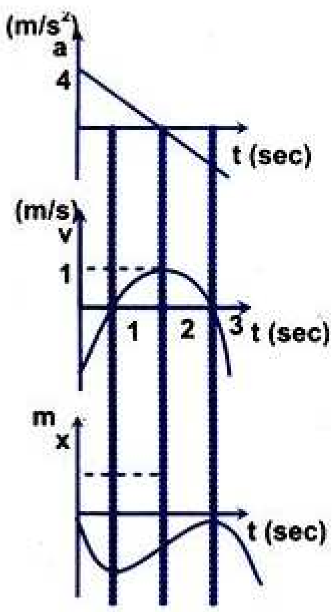
If u > 0



If u < 0

Initial position x of the body in every case is x_0 (> 0)

Illustration:



A particle is moving rectilinearly with a time varying acceleration a = 4 - 2t, where a is in m/s^2 and t is in sec. If the particle is starting its motion with a velocity of -3 m/s from x = 0. Draw a-t, v-t and x-t curve for the particle.

Solution:

a = 4-2t

$$\int_{-3}^{v} dv = \int_{0}^{t} a dt$$

v = 4t-t^2-3

$$\int_{0}^{x} dx = \int_{0}^{t} v dt$$

x = 2t^2 – t^3/3 – 3t

Acceleration

Acceleration is the rate of change of velocity with time. The concept of acceleration is understood in non-uniform motion. It is a vector quantity.

Average acceleration is the change in velocity per unit time over an interval of time.

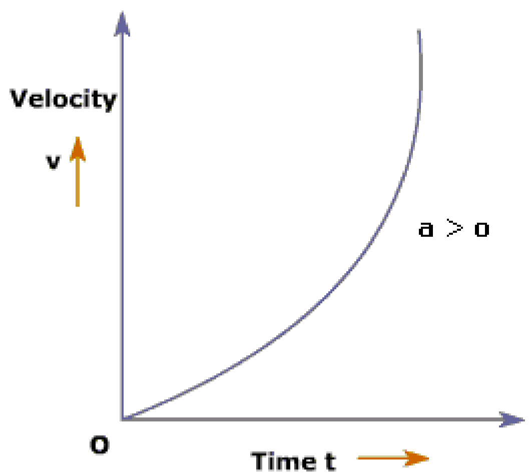
$$a_{\vec{avg}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v_2} - \vec{v_1}}{\vec{t_2} - \vec{t_1}}$$

Instantaneous acceleration is defined as

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\Rightarrow \vec{a}\frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Acceleration Vector in Non Uniform Motion



Suppose that at the instant t_1 a particle as in figure above, has velocity $\vec{v1}$ and at t_2, velocity is $\vec{v_2}$. The average acceleration $lt; \vec{a} gt;$ during the motion is defined as

$$lt; \vec{a} gt; = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_2} - \vec{v_1}}{\vec{t_2} - \vec{t_1}}$$

Variable Acceleration

The acceleration at any instant is obtained from the average acceleration by shrinking the time interval closer zero. As Δt tends to zero average acceleration approaching a limiting value, which is the acceleration at that instant called instantaneous acceleration which is vector quantity.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

i.e. the instantaneous acceleration is the derivative of velocity.

Hence instantaneous acceleration of a particle at any instant is the rate at which its velocity is changing at that instant. Instantaneous acceleration at any point is the slope of the curve v (t) at that point as shown in figure above.

Equations of Motion

The relationship among different parameter like displacement velocity, acceleration can be derived using the concept of average acceleration and concept of average acceleration and instantaneous acceleration.

When acceleration is constant, a distinction between average acceleration and instantaneous acceleration loses its meaning, so we can write

$$\vec{a} = \frac{\vec{v} - \vec{v_0}}{t - t_0} = \frac{d\vec{v}}{dt}$$

where $\vec{v_0}$ is the velocity at t = 0 and $\vec{v}$ is the velocity at some time t

Now,

$$\vec{a}t = \vec{v} - \vec{v_0}$$

Hence,

$$\vec{v} - \vec{v_0} = \vec{a}t \qquad \text{...... (2)}$$

This is the first useful equation of motion.

Similarly for displacement

$$\vec{x} = \vec{x_0} + lt; \vec{v} gt; t \qquad \text{...... (3)}$$

in which $\vec{x_0}$ is the position of the particle at t_0 and $lt; \vec{v} gt;$ is the average velocity between t_0 and later time t. If at t_0 and t the velocity of particle is

$$lt;\vec{v}gt; = \frac{1}{2}(\vec{v_0} + \vec{v}) = \frac{1}{2}[\vec{v_0} + \vec{v_0} + \vec{a}t]$$

$$lt;\vec{v}gt; = \vec{v_0} + \vec{a}\frac{t}{2}$$ **…... (4)**

From equation (3) and (4), we get,

$$\vec{x} - \vec{x_0} = \vec{v_0}t + \frac{1}{2}\vec{a}t^2$$ **…... (5)**

This is the second important equation of motion.

Now from equation (2), square both side of this equation we get

$$v_2 = v_0^2 + a^2t^2 + 2v_0\vec{a}t = v_0^2 + 2\vec{a}t + [v_0 + \frac{\vec{a}t}{2}]$$

$$v_2 = v_0^2 + 2\vec{a}tlt;\, vgt;$$ **[Using equation (4)]**

Using equation (3), we get,

$$\vec{v}^2 = \vec{v_0}^2 + 2\vec{a}(\vec{x} - \vec{x_0})$$ **…... (6)**

This is another important equation of motion.

Caution:The equation of motion derived above are possible only in uniformly accelerated motion i.e. the motion in which the acceleration is constant.

Refer this Simulation for Motion in a Straight Line

Illustration:

The nucleus of helium atom (alpha-particle) travels inside a straight hollow tube of length 2.0 meters long which forms part of a particle accelerator. (a) If one assumes uniform acceleration, how long is the particle in the tube if it enters at a speed of 1000 meter/sec and leaves at 9000 meter/sec? (b) What is its acceleration during this interval?

Solution:

(a) We choose x-axis parallel to the tube, its positive direction being that in which the particle is moving and its origin at the tube entrance. We are given x and v_x and we seek t. The acceleration ax is not involved. Hence we use equation 3, $x = x_0 + <v> t$.

We get

$x = v_0 + ½ (v_{x0}) + v_x)$ t, with $x_0 = 0$ or

$t = 2x/(v_{x0}+v_{x}),$

t = ((2)(2.0 meters))/((1000+9000)meters/sec) = $4.0/10^{-4}$ sec Ans.

(b) The acceleration follows from equation 2, $v_x = v_{x0} + a_xt$

=> ax = $(v_0-v_{x0})/t$ = ((9000-1000)meters/sec)/($4.0×10^{(-4)}$ sec)

= 2.0 × 10^7 meter/sec^2 Ans.

Pause: The above equations of motion are, however, universal and can be derived by using differential calculus as given below:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\Rightarrow d\vec{v} = \vec{a}dt$$

$$\Rightarrow \int d\vec{v} = \vec{a}\int dt$$

Or,

Let at t = 0, $\vec{v} = \vec{v_0}$

then, $C = \vec{v_0}$

Or, $\vec{v} = \vec{a}t + \vec{v_0}$

Further we know that,

$$\frac{d\vec{x}}{dt} = \vec{v}$$

or $d\vec{x} = \vec{v}dt$

Integrating,

$$\int d\vec{x} = \int \vec{v}dt + c'$$

Or,

$$\vec{x} = \int (\vec{v_0} + \vec{a}t)dt + c' = \vec{v_0}t + \frac{1}{2}\vec{a}t^2 + c'$$

At, t = 0, x = x_0 then c' = x_0

Hence,

$$\vec{x} = \vec{x_0} + \vec{v_0}t + \frac{1}{2}\vec{a}t^2$$

Thus, we have derived the same equation of motion using calculus.

To understand the use of calculus in solving the kinematics problems we can look into the following illustrations.

Illustration:

The displacement x of a particle moving in one dimension, under the action of a constant force is related to the time t by the equation t = √x + 3 where x is in meter and t is in seconds. Find the displacement of the particle when its velocity is zero.

Solution:

Here t = √x + 3 => √x = t - 3

Squaring both sides, we get $x = t^2 - 6t + 9$,

As we know velocity, v = dx/dt

Hence we get v = dx/dt = 2t - 6

Put v = 0, we get, 2t - 6 = 0

So, t = 3s

When t = 3s, $x = t^2 - 6t + 9 = 9 - 6(3) + 9 = 0$

Hence the displacement of the particle is zero when its velocity is zero.

Illustration:

A particle starts from a point whose initial velocity is v_1 and it reaches with final velocity v^2, at point B which is at a distance 'd' from point A. The path is straight line. If acceleration is proportional to velocity, find the time taken by particle from A to B.

Solution:

Here acceleration a is proportional to velocity v.

Hence a α v

=> a = kv, where k is constant

=> dv/dt = kv .... (1)

=> (dv/ds)(ds/dt) = kv => (dv/ds) v = kv

$$\int_{v_1}^{v_2} dv = k \int_0^d ds$$

From equation (1),

dv/v = kdt

or, $$\int_{v_1}^{v_2} \frac{dv}{v} = k \int_0^t dt$$

Or, $\ln (v_2/v_1) = kt$

Or, $t = \ln (v_2/v_1) /k$

= $[d \ln (v_2/v_1)/(v_2-v_1)]$

- **The displacement remains unaffected due to shifting of origin from one point to the other.**
- **The displacement can have positive, negative or zero value.**
- **The displacement is never greater than the actual distance travelled.**
- **The displacement has unit of length.**
- **Velocity can be considered to be a combination of speed and direction.**
- **A change in either speed or direction of motion results in a change in velocity.**
- **It is not possible for a particle to possess zero speed with a non-zero velocity.**
- **A particle which completes one revolution, along a circular path, with uniform speed is said to possesss zero velocity and non-zero speed.**
- **In case a body moves with uniform velocity, along a straight line, its average speed is equal to its instantaneous speed**

Revision Notes on Kinematics

- Inertial frame of reference:- Reference frame in which Newtonian mechanics holds are called inertial reference frames or inertial frames. Reference frame in which Newtonian mechanics does not hold are called non-inertial reference frames or non-inertial frames.

- The average speed v_{av} and average velocity $\vec{V}_{av}$ of a body during a time interval ?*t* is defined as,

v_{av}= average speed

= ?*s*/?*t*

$\vec{V}_{av}$ = average velocity

$$= \frac{\Delta \vec{r}}{\Delta t}$$

- Instantaneous speed and velocity are defined at a particular instant and are given by

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \text{ and } \vec{V} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Note:

(a) A change in either speed or direction of motion results in a change in velocity

(b) A particle which completes one revolution, along a circular path, with uniform speed is said to possess zero velocity and non-zero speed.

(c) It is not possible for a particle to possess zero speed with a non-zero velocity.

- Average acceleration is defined as the change in velocity $\Delta \vec{V}$ over a time interval ?*t*.

$$\vec{a}_{av} = \frac{\Delta \vec{V}}{\Delta t}$$

The instantaneous acceleration of a particle is the rate at which its velocity is changing at that instant.

$$\vec{a}_{ins} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t} = \frac{d\vec{V}}{dt}$$

- The three equations of motion for an object with constant acceleration are given below.

(a) $v = u+at$

(b) $s = ut+1/2\ at^2$

(c) $v^2=u^2+2as$

Here *u* is the initial velocity, *v* is the final velocity, *a* is the acceleration , *s* is the displacement travelled by the body and *t* is the time.

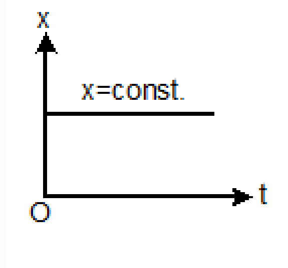
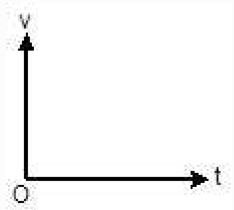
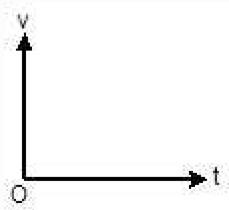
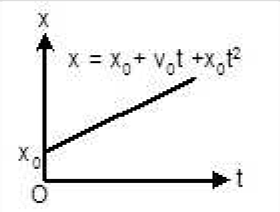
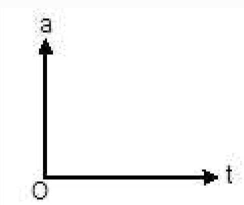
Note: Take '+ve' sign for a when the body accelerates and takes '–ve' sign when the body decelerates.

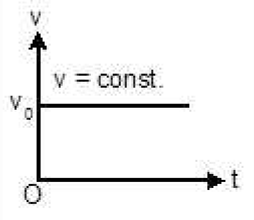
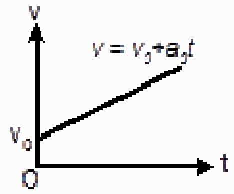
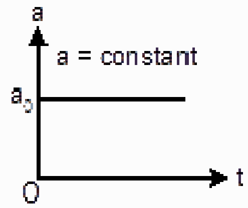
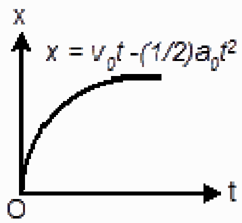
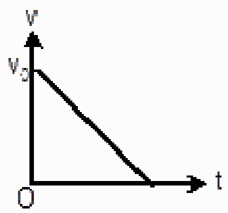
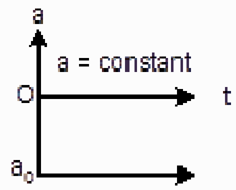
- The displacement by the body in n^{th} second is given by,

$s_n = u + a/2\ (2n-1)$

- Position-time (*x* vs *t*), velocity-time (*v* vs *t*) and acceleration-time (*a* vs *t*) graph for motion in one-dimension:

(i) Variation of displacement (x), velocity (v) and acceleration (a) with respect to time for different types of motion.

	Displacement(x)	Velocity(v)	Acceleration (a)
(a) At rest			
(b) Motion with constant velocity			

			
(c) Motion with constant acceleration			
(d) Motion with constant deceleration			

- **Scalar Quantities:-** Scalar quantities are those quantities which require only magnitude for their complete specification.(e.g-mass, length, volume, density)

- **Vector Quantities:-** Vector quantities are those quantities which require magnitude as well as direction for their complete specification. (e.g-displacement, velocity, acceleration, force)

- **Null Vector (Zero Vectors):-** It is a vector having zero magnitude and an arbitrary direction.

When a null vector is added or subtracted from a given vector the resultant vector is same as the given vector.

Dot product of a null vector with any arbitrary is always zero. Cross product of a null vector with any other vector is also a null vector.

- **Collinear vector:-** Vectors having a common line of action are called collinear vector. There are two types.

Parallel vector (ϑ=0°):- Two vectors acting along same direction are called parallel vectors.

Anti parallel vector (ϑ=180°):-Two vectors which are directed in opposite directions are called anti-parallel vectors.

- **Co-planar vectors-** Vectors situated in one plane, irrespective of their directions, are known as co-planar vectors.

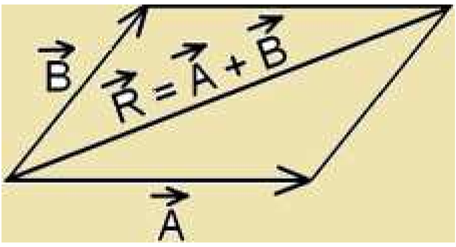
- **Vector addition:-**

Vector addition is commutative- $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

Vector addition is associative- $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$

Vector addition is distributive- $m\vec{A}+m\vec{B}=m(\vec{A}+\vec{B})$

- **Triangles Law of Vector addition:-** If two vectors are represented by two sides of a triangle, taken in the same order, then their resultant in represented by the third side of the triangle taken in opposite order.



$\vec{R}=\vec{A}+\vec{B}$

Magnitude of resultant vector $\vec{R}$:-

$R=\sqrt{(A^2+B^2+2AB\cos\vartheta)}$

Here ϑ is the angle between $\vec{A}$ and $\vec{B}$.

If β is the angle between $\vec{R}$ and $\vec{A}$,

then,

$$\beta = \tan^{-1}\{\frac{B\sin\theta}{A+B\cos\theta}\}$$

- If three vectors acting simultaneously on a particle can be represented by the three sides of a triangle taken in the same order, then the particle will remain in equilibrium.

So, $\vec{A}+\vec{B}+\vec{C}=0$

- **Parallelogram law of vector addition:-**

$\vec{R}=\vec{A}+\vec{B}$

$R=\sqrt{(A^2+B^2+2AB\cos\vartheta)}$,

$$\beta = \tan^{-1}\{\frac{B\sin\theta}{A+B\cos\theta}\}$$

Cases 1:- When, $\vartheta=0°$, then,

$R= A+B$ (maximum), $\beta=0°$

Cases 2:- When, $\vartheta=180°$, then,

$R= A-B$ (minimum), $\beta=0°$

Cases 3:- When, $\vartheta=90°$, then,

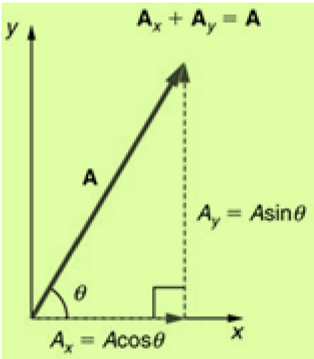
$R=\sqrt{(A^2+B^2)}$, $\beta = \tan^{-1} (B/A)$

- The process of subtracting one vector from another is equivalent to adding, vectorially, the negative of the vector to be subtracted.

So,

$$\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$$

- **Resolution of vector in a plane:-**



- **Product of two vectors:-**

(a) Dot product or scalar product:-

$$\vec{A}\vec{B} = AB\cos\theta\ ,$$

$$\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z z$$

Here A is the magnitude of $\vec{A}$, B is the magnitude of $\vec{B}$ and ϑ is the angle between $\vec{A}$ and $\vec{B}$.

(i) Perpendicular vector:-

$$\vec{A}.\vec{B} = 0$$

(ii) Collinear vector:-

When, Parallel vector (θ=0°), $\vec{A}.\vec{B} = AB$

When, Anti parallel vector (θ=180°), $\vec{A}.\vec{B} = -AB$

(b) Cross product or Vector product:-

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

Or,

$$\vec{A}\times\vec{B} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$$

Here A is the magnitude of $\vec{A}$, B is the magnitude of $\vec{B}$, ϑ is the angle between $\vec{A}$ and $\vec{B}$ and $\hat{n}$ is the unit vector in a direction perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

(i) Perpendicular vector (θ=90°):-

$$\vec{A}\times\vec{B} = AB$$

(ii) Collinear vector:-

When, Parallel vector (θ=0°), (null vector)

When, θ=180°, (null vector)

- **Unit Vector:-** Unit vector of any vector is a vector having a unit magnitude, drawn in the direction of the given vector.

In three dimension,

$$\hat{A} = \frac{\vec{A}}{A}$$
$$= \frac{A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

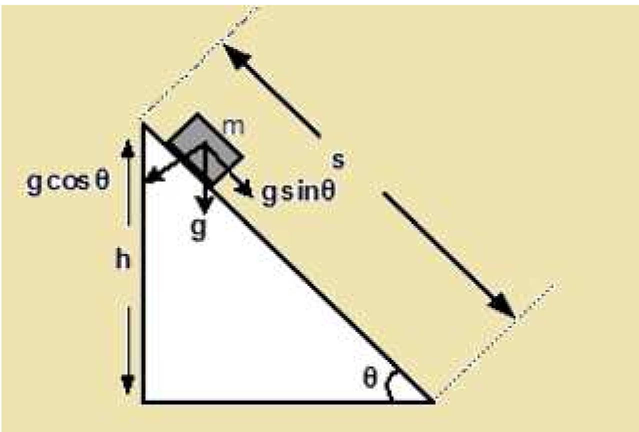
- **Area:-**

Area of triangle:- $A = \frac{1}{2}\left|\vec{A}\times\vec{B}\right|$

Area of parallelogram:- $A = \left|\vec{A}\times\vec{B}\right|$

Volume of parallelepiped:- $\vec{V} = \vec{A}.\left(\vec{B}\times\vec{C}\right)$

- **Equation of Motion in an Inclined Plane:**



(i) Perpendicular vector :- At the top of the inclined plane (t = 0, u = 0 and a = g sin$q$), the equation of motion will be,

(a) $v = (g \sin\theta)t$

(b) $s = ½ (g \sin\theta) t^2$

(c) $v^2 = 2(g \sin\theta)s$

(ii) If time taken by the body to reach the bottom is t, then $s = ½ (g \sin\vartheta) t^2$

$t = \sqrt{(2s/g \sin\vartheta)}$

But $\sin\vartheta = h/s$ or $s = h/\sin\vartheta$

So, $t = (1/\sin\vartheta) \sqrt{(2h/g)}$

(iii) The velocity of the body at the bottom

$v = g(\sin\vartheta)t$

$= \sqrt{2gh}$

- The relative velocity of object A with respect to object B is given by

$V_{AB} = V_A - V_B$

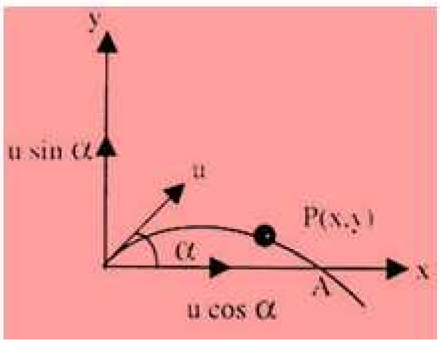
Here, V_B is called reference object velocity.

- **Variation of mass:-** In accordance to Einstein's mass-variation formula, the relativistic mass of body is defined as,

$m = m_0/\sqrt{(1-v^2/c^2)}$

Here, m_0 is the rest mass of the body, v is the speed of the body and c is the speed of light.

- **Projectile motion in a plane:-** If a particle having initial speed u is projected at an angle ϑ (angle of projection) with x-axis, then,



Time of Flight, $T = (2u \sin\alpha)/g$

Horizontal Range, $R = u^2\sin2\alpha/g$

Maximum Height, $H = u^2\sin^2\alpha/2g$

Equation of trajectory, $y = x\tan\alpha-(gx^2/2u^2\cos^2\alpha)$

- **Motion of a ball:-**

(a) When dropped:- Time period, $t=\sqrt{(2h/g)}$ and speed, $v=\sqrt{(2gh}$

(b) When thrown up:- Time period, $t=u/g$ and height, $h = u^2/2g$

- **Condition of equilibrium:-**

(a) $\vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$

(b) $|F_1+F_2| \geq |F_3| \geq | F_1-F_2|$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

III YEAR B.SC MATHEMATICS

SMT5103 MECHANICS 1

Unit 4- Newton’s Law of Motion – SMT1503

UNIT IV

NEWTON’S LAW OF MOTION

https://cnx.org/contents/UYPplaH7@29.25:OR0Da3iU@12/Vertical-motion-under-gravity

Newton's laws of motion are of fundamental importance in classical physics. Newton gave three laws connected with motion and are, popularly, known as Newton’s laws of motion.

Newton’s First Law of motion

To study Newton’s first law of motion, the concept of equilibrium should be clear to us. Whenever a number of forces act on a body and they neutralize each other’s effect, the body is said to be in equilibrium. In such a case there is no change in the state of rest or of motion. If however, the system of forces have a resultant, the state of rest or that of motion undergoes a change. This is explained by Newton’s first law of motion.

It states that,” **Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by some external force to change that state. Therefore, every object persists in its natural state of motion i.e. continues to be at rest or moves in a straight line with uniform (constant) velocity, in the absence of a net external force acting (impressed) on it.**

Law of Inertia

Inertia is the property of all bodies by virtue of which they are unable to change their state of rest or of uniform motion in a straight line without the help of an external force. In other words inertia can also be termed as a resistance to change the state of motion of a body.

Inertia can be classified into following three categories.

(a) Inertia of Rest:-

It is the property of a body by virtue of which it is unable to change its state of rest without the help of an external force.

(b) Inertia of Motion:-

It is the property of a body by virtue of which it is not able to change its speed without the help of an external force.

(c) Inertia of Direction:-

It is the property of a body by virtue of which it is unable to change its direction of motion without the help of an external force.

Qualitative definition of force from first law:-

Newton’s first law states that there cannot be any change in the state of rest or that of motion of a body unless some external force acts upon it. In other words force is an agent which is capable of producing any change in state of rest or that of motion (including direction). This provides a qualitative definition of force.

Some Conceptual Questions

Question 1:-

A car moving at constant speed is suddenly braked. The occupants, all wearing seat belts, are thrown forward. The instant the car stops, however, the occupants are all jerked backward. Why? Is it possible to stop an automobile without this ‘jerk’?

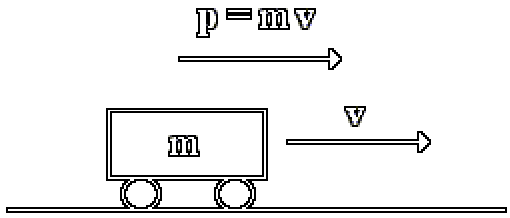
Solution:-

Newton’s first law states that, without any external force, if a body is at rest, it will remain at rest and if the body is moving with constant velocity, it will continue to do so. When the car is suddenly braked, due to the inertia, the occupants in the car will tend to move in the forward direction of car. When the car stops the sit belt in the car will produce backward momentum on the occupants. Since the all the occupants wearing seat belts, therefore the occupants are all jerked backward.

Yes, it is possible to stop an automobile without this jerk. This can be done by slowing down the car a little longer time.

Newton’s second law of motion

Momentum



Momentum of a body is defined as the amount of motion contained in a body.

Quantity of motion or the momentum of the body depends upon,

(a) mass of the body.

(b) velocity of the body.

Therefore momentum of a body of mass ‘m’ and velocity ‘v’ will be,

$\vec{p} = m\vec{v}$

Definition of Quantitative

Momentum of a body is equal to the product of its mass and velocity. Momentum is a vector quantity and possesses the direction of velocity.

Units:-

S.I:- kg m s^{-1}

C.G.S:- g cm s^{-1}

Momentum can be put into following two categories.

Dimension:-

$[MLT^{-1}]$

(a) Non-Relativistic Momentum

According to classical physics (or non-relativistic physics) which is based upon the concepts of Newton’s laws of motion, mass of a body is considered to be a constant quantity, independent of the velocity of body. In that case momentum $\vec{P}$ is given by,

$\vec{p} = m\vec{v}$.

Thus, momentum of a body is a linear function of its velocity.

(b) Relativistic Momentum

In accordance to Einstein’s special theory of relativity, mass of a body depends upon the relative velocity ‘v’ of the body with respect to the observer. If ‘m_0’ is the mass of body observed by an observer at rest with respect to body, its relativistic mass ‘m’ is given by,

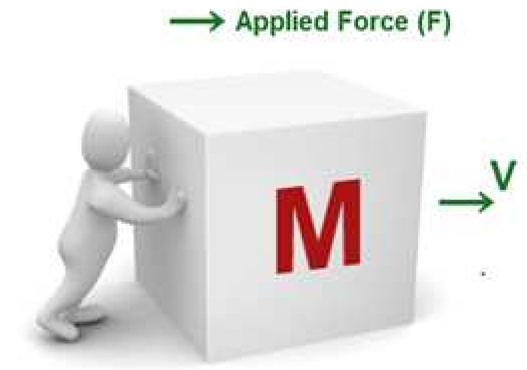
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, momentum of a body according to the concepts of theory of relativity is given by,

$$\vec{p} = \frac{m_0\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus, relativistic momentum is not a linear function of v.

Newton’s Second Law



The rate of change of momentum of a body is directly proportional to the impressed force and takes place in the direction of the force.

Newton's first law provides a qualitative definition of the force while second law provides a quantitative definition of the force.

Let $\vec{v}$ be the instantaneous velocity of the body. Momentum $\vec{P}$ of the body is given by,

$$\vec{p} = m\vec{v}$$

According to second law,

$\vec{F} \propto$ (rate of change of momentum)

Or,

$$\vec{F} \alpha \frac{d\vec{p}}{dt}$$

Or,

$$\vec{F} \alpha \frac{d(m\vec{v})}{dt}$$

Or,

$$\vec{F} = k\frac{d(m\vec{v})}{dt}$$

Here 'k' is the constant of proportionality. Mass 'm' of a body is considered to be a constant quantity.

$$\vec{F} = km\frac{d(\vec{v})}{dt}$$

or,

$$\vec{F} = km\vec{a}$$

The units of force are also selected that 'k' becomes one.

Thus, if a unit force is chosen to be the force which produces a unit acceleration in a unit mass, i.e., F = 1, m = 1 and a = 1.

Then, k = 1

So, Newton's second law can be written , in mathematical form, as

$$\vec{F} = m\vec{a}$$

i.e., Force = (mass) (acceleration)

This provides us a measure of the force.

Here, if F = 0 then we find a = 0. This reminds us of first law of motion. That is, if net external force is absent, then there will be no change in state of motion, that means its acceleration is zero.

Further we can extend second law of motion, (in fact its decomposition) to three mutually perpendicular directions as per our coordinate system.

If components in x, y and z direction are F_x, F_y & F_z respectively, the three acceleration produced when F_x, F_y & F_z act simultaneously) in the body are, Now,

$$\vec{F}_x = m\vec{a}_x, \quad \vec{F}_y = m\vec{a}_y, \quad \vec{F}_z = m\vec{a}_z$$

If we add three forces then resultant is called net external force.

Similarly,

$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

is called net acceleration produced in the body.

Unit of Force:-

S.I:- Newton [kg.m/sec^2]

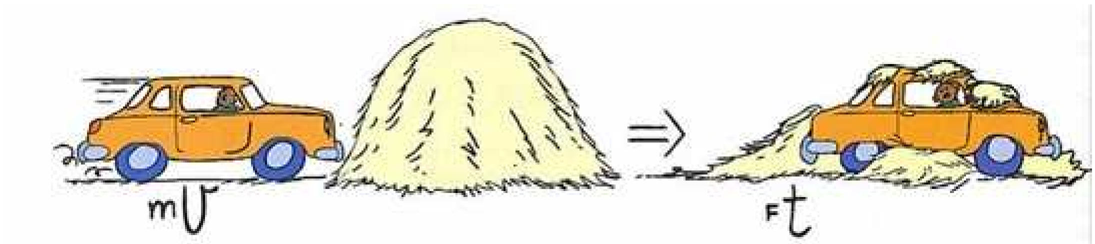
C.G.S:- Dyne [g.cm/sec^2]

Dimension:-

[MLT^{-2}]

Impulse

Impulse of a force is defined as the change in momentum produced by the force and it is equal to the product of force and the time for which it acts. Therefore, a large force acting for a short time to produce a finite change in momentum which is called impulse of this force and the force acted is called impulsive force or force of impulse.



According to Newton's second law of motion,

$$\vec{F} = m\vec{a} = m\frac{(\vec{v}-\vec{u})}{t}$$

or,

$$\vec{F}t = m\vec{v} - m\vec{u} = \vec{J}$$

So, Impulse of a force = change in momentum.

If the force acts for a small duration of time, the force is called impulsive force.

As force is a variable quantity, thus impulse will be,

$$J = \int_{t_1}^{t_2} F\,dt$$

The area under F - t curve gives the magnitude of impulse.

Impulse is a vector quantity and its direction is same as the direction of $\vec{F}$.

Unit of Impulse:- The unit in S.I. system is kgm/sec or newton -second.

Dimension:- MLT^1

Problem 1:-

The Sun yacht Diana, designed to negative in the solar system using the pressure of the sunlight, has a sail area of 3.1 km^2 and a mass of 930 kg. Near Earth's orbit, the sun could exert a radiation force of 29 N on its sail. (a) What acceleration would such a force impart to the craft? (b) A small acceleration can produce large effects if it acts steadily for a long enough time. Starting from rest then, how far would the craft have moved after 1 day under these conditions? (c) What would then be its speed? (See “The Wind from the Sun,” a fascinating science fiction account by Arthur C.Clarke of a Sun yacht race.)

Solution:-

(a)

Given Data:-

Mass of the yacht Diana, m = 930 kg

Force exerted by the sun light, F = 29 N

Force acting on the body (F) is equal to the product of mas of the body (m) and acceleration of the body (a).

So, $F = ma$ …… (1)

From equation (1), the acceleration (a) of the body would be,

$a = F/m$ …… (2)

Putting the value of m and a in equation (2), the acceleration such force impart to the craft would be,

$a = F/m$

= 29 N /930 kg

= $(3.1\times10^{-2}$ N/kg) (1 kg. m/s^2 /1 N)

= 3.1×10^{-2} m/s^2 …… (3)

Thus acceleration such force impart to the craft would be, 3.1×10^{-2} m/s^2.

(b)

Given Data:-

Time, t = 1 day

= (1day) (24 h/1 day) (60 min/1 h) (60 s/1 min)

= 86400 s

Initial velocity, $v_i = 0$

Acceleration, a = 3.1×10^{-2} m/s^2

From equation of motion, we know that,

Distance travelled by the body (x) = $v_i + ½\, at^2$

So, $x = v_i t + ½\, at^2$ …… (4)

Putting the value of v_i, a and t in equation (4), the distance travelled by the craft will be,

$x = v_i t + ½\, at^2$

= 0+½ (3.1×10^{-2} m/s^2) (86400 s)^2 (Since, $a = 3.1\times10^{-2}$ m/s^2 and t = 86400 s)

=1.1571×10^8 …… (5)

Rounding off to two significant figures, the distance will be 1.2×10^8 m.

Thus from the above observation we conclude that, the craft have moved1.2×10^8 m after 1 day under these conditions.

(c)

Given data:

Acceleration, $a = 3.1\times10^{-2}$ m/s^2

Time, t = 86400 s

Acceleration of an object is equal to the velocity of the object divided by time.

$a = v/t$

So, $v = at$ ……(6)

Putting the value of a and t in equation (6), velocity would be,

$v = at$

$= (3.1\times10^{-2}\ \text{m/s}^2)\ (86400\ \text{s})$

$= 2678.4\ \text{m/s}$

Rounding off to two significant figures, speed will be 2700 m/s.

Thus from the above observation we conclude that, speed will be 2700 m/s.

Problem 2:-

A car travelling at 53 km/h hits a bridge abutment. A passenger in the car moves forward a distance of 65 cm (with respect to the road) while being brought to rest by an inflated air bag. What force (assumed constant) acts on the passenger's upper torso, which has a mass of 39 kg?

Concept:-

Force acting (F) on the body is equal to the mass of the body (m) times deceleration of the body (a).

$F = ma$ …… (1)

Solution:-

First we have to find out the deceleration (a) of the car.

If v_0 is the initial speed of car and v is the final speed of the car, then the average speed (v_{av}) of the car will be,

$v_{av,} = ½\ (v+ v_0)$ …… (2)

To obtain the average speed (v_{av}) while the car is decelerating, substitute 53 km/h for v_0 and 0 m/s for v in the equation $v_{av} = ½\ (v+ v_0)$,

$v_{av} = ½\ (v+ v_0)$

$= ½\ ((53\ \text{km/h})+ (0\ \text{m/s}))$

$= (½ \times 53\ \text{km/h})\ (1{,}000\ \text{m}/1\ \text{km})\ (1\ \text{h}/60\ \text{min})\ (1\ \text{min}/60\ \text{s})$

$= 7.4\ \text{m/s}$ …… (3)

But average speed (v_{av}) is equal to the rate of change of displacement (x).

$v_{av} = x/\ t$

So, $t = x/\ v_{av}$ …… (4)

To obtain the time of deceleration t, substitute 0.65 m for x and 7.4 m/s for v_{av} in the equation $t = x/\ v_{av}$,

$t = x/ v_{av}$

$= 0.65 \text{ m} /7.4 \text{ m/s}$

$= 8.8\times10^{-2}$ s …… (5)

Deceleration (a) is equal to rate of change of velocity.

So, $a = \Delta v /t$

$= ((0) - (53 \text{ km/h}))/ 8.8\times10^{-2}$ s

$= (-53 \text{ km/h})/ 8.8\times10^{-2}$ s

$= ((-53 \text{ km/h}) (1{,}000 \text{ m}/1 \text{ km}) (1 \text{ h}/60 \text{ min}) (1 \text{ min}/60 \text{ s}))/ 8.8\times10^{-2}$ s

$= (-14.7 \text{ m/s})/ (8.8\times10^{-2} \text{ s})$

$= -1.7\times10^{2} \text{ m/s}^2$ ……(6)

To obtain the force (F) acting on the passengers upper torso having mass 39 kg, substitute 39 kg for mass m and -1.7×10^2 m/s^2 for deceleration a in the equation, $F = ma$,

$F = ma$

$= (39 \text{ kg}) (-1.7\times10^2 \text{ m/s}^2)$

$= -6630 \text{ kg. m/s}^2$

$= -(6630 \text{ kg. m/s}^2) (1 \text{ N}/1 \text{ kg. m/s}^2)$

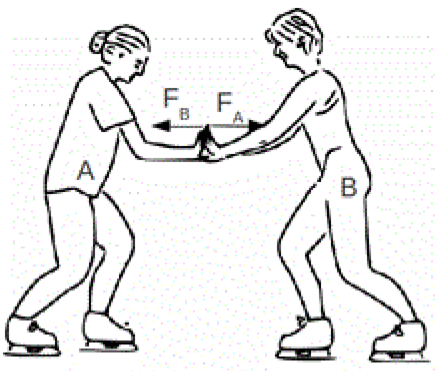
$= -6630$ N …… (7)

Rounding off to two significant figures, the magnitude of the force will be 6600 N.

Newton’s third law of motion

It states that,

“To every action there is an equal and opposite reaction”.



Whenever one force acts on a body, it gives rise to another force called **reaction**. A single isolated force is an impossibility. The two forces involved in any interaction between two bodies are called “**<u>action”</u>** and “**<u>reaction</u>**”. But this does not imply any difference in their nature, or that one force is the ‘cause‘ and the other is the ‘effect’. Either force may be considered as ‘action’ and the other ‘reaction’ to it.

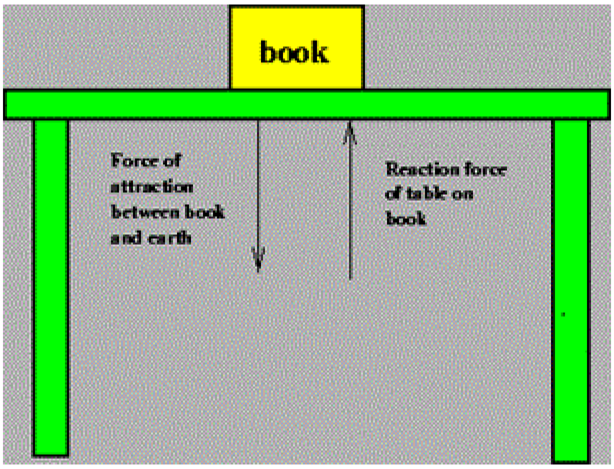
It may be noted that action and reaction never act on same body.

Note: The most important fact to notice here is that these oppositely directed equal action and reaction can never balance or cancel each other because they always act, on two different point (broadly on two different objects) For balancing any two forces the first requirement is that they should act one and the same object. (or point, if object can be treated as a point mass, which is a common practice)

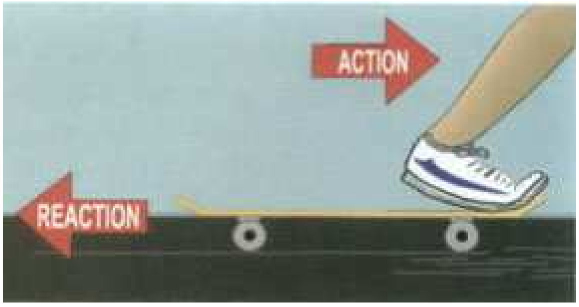
Few Examples on Newton’s third Law of Motion

(a) Book Kept on a Table

A book lying on a table exerts a force on the table which is equal to the weight of the book. This is the force of action. The table supports the book, by exerting an equal force on the book. This is the force of reaction, as shown in the below figure. As the system is at rest, net force on it is zero. Therefore, forces of action and reaction must be equal and opposite.



(b) Walking on the ground:-



While walking a person presses the ground in the backward direction (action) by his feet. The ground pushes the person in forward direction with an equal force (reaction). The component of reaction in the horizontal direction makes the person move forward.

(c) Process of Swimming:-

A swimmer pushes the water backwards (action). The water pushed the swimmer forward (reaction) with the same force. Hence the swimmer swims.

(d) Firing from a gun:-

When a gun is fired, the bullet moves forward (action). The gun recoils backwards (reaction).

(e) Fight of jet planes and rockets:-

The burnt fuel which appears in the form of hot and highly compressed gases escapes through the nozzle (action) in the backward direction. The escaping gases push the jet plane or rocket forward (reaction) with the same force, hence, the jet or rocket moves.

(f) Rubber ball re-bounds from a wall:-

When a rubber ball is struck against a wall or floor it exerts a force on a wall (action). The ball rebounds with an equal force (reaction) exerted by the wall or floor on the ball.

(g) It is difficult to walk on sand or ice:-

This is because on pushing, sand gets displaced and reaction from sandy ground is very little. In case of ice, force of reaction is again small because friction between feet and ice is very small.

(h) Driving a nail in to a wooden block without holding the block is difficult:-

This is because when the wooden block is not resting against a support, the block and nails both move forward on being hit with a hammer. However, when the block is held firmly against a support, and the nail is hit, an equal reaction of the support drives the nail into the block.

(i) A tea cup breaks on falling on the ground:-

Tea cup exerts certain force (action) on ground while the ground exerts an equal and opposite reaction on the cup. Ground is able to withstand the action of cup, but the cup being relatively more delicate breaks due to reaction.

Problem 1:-

Two blocks, with masses m_1 = 4.6 kg and m_2 = 3.8 kg, are connected by a light spring on a horizontal frictionless table. At a certain instant, when m_2 has an acceleration a_2 = 2.6 m/s^2, (a) what is the force on m_2 and (b) what is the acceleration of m_1?

Concept:-

Force acting on the body (F) is equal to the product of mas of the body (m) and acceleration of the body (a).

So, $F = ma$

From equation $F = ma$, the acceleration (a) of the body would be,

$a = F/m$

Solution:-

(a) The net force $\sum F_x$ on the second box having mass m_2 will be,

$\sum F_x = m_2 a_{2x}$

Here a_{2x} is the acceleration of the second block.

To obtain the net force $\sum F_x$ on the second box having mass m_2, substitute 3.8 kg mass m_2 and 2.6 m/s^2 for a_{2x} in the equation $\sum F_x = m_2 a_{2x}$,

$\sum F_x = m_2 a_{2x}$

= (3.8 kg) (2.6 m/s^2)= 9.9 kg .m/s^2

= (9.9 kg .m/s^2) (1 N/ 1 kg .m/s^2)= 9.9 N

From the above observation we conclude that, the net force $\sum F_x$ on the second box having mass m_2 would be 9.9 N. There is only one (relevant) force on the block, the force of block 1 on block 2.

(b) There is only one (relevant) force on block 1, the force of block 2 on block 1. By Newton's third law this force has a magnitude of 9.9 N.

So the Newton's second law gives,

$\sum F_x = m_1 a_{1x} = -9.9$ N

But, $m_1 a_{1x} = (4.6 \text{ kg})(a_{1x})$ (Since, $m_1 = 4.6$ kg)

$(4.6 \text{ kg})(a_{1x}) = -9.9$ N

So, $a_{1x} = -9.9$ N/4.6 kg

$= (-2.2 \text{ N/kg})(1 \text{ kg.m/s}^2 / 1 \text{ N}) = -2.2 \text{ m/s}^2$

From the above observation we conclude that, the acceleration of m_1 will be -2.2 m/s^2.

Problem 2:-

A meteor of mass 0.25 kg is falling vertically through Earth's atmosphere with an acceleration of 9.2 m/s^2. In addition to gravity, a vertical retarding force (due to frictional drag of the atmosphere) acts on the meteor as shown in the below figure. What is the magnitude of this retarding force?

Solution:-

Given Data:

Mass of the meteor, $m = 0.25$ kg

Acceleration of the meteor, $a = 9.2$ m/s^2

The net force exerted (F_{net}) on the meteor will be,

$F_{net} = ma$

$= (0.25 \text{ kg})(9.2 \text{ m/s}^2) = (2.30 \text{ kg. m/s}^2)(1 \text{ N}/ 1 \text{ kg. m/s}^2) = 2.30$ N …… (1)

If g ($g = 9.80$ m/s^2) is the free fall acceleration of meteor, then the weight of the meteor (W) will be,

$W = mg = (0.25 \text{ kg})(9.80 \text{ m/s}^2)$

$= (2.45 \text{ kg. m/s}^2)(1 \text{ N}/ 1 \text{ kg. m/s}^2) = 2.45$ N …… (2)

The vertical retarding force would be equal to the net force exerted on the meteor (F_{net}) minus weight of the meteor (W).

So, vertical retarding force = F_{net} –W …… (3)

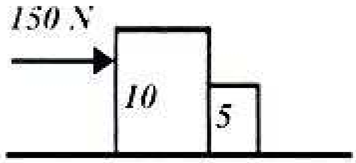
Putting the value of F_{net} and W in equation (3), the vertical retarding force will be,

Vertical retarding force = F_{net} –W = 2.30 N -2.45 N = -0.15 N ……. (4)

From equation (4) we observed that, magnitude of the vertical retarding force would be, -0.15 N.

Problem 3:-

Suppose in figure shown above we put one more block of 5 kg mass adjacent to 10 kg and a force of 150 N acts as shown in the figure below, then find the forces acting on the interface.

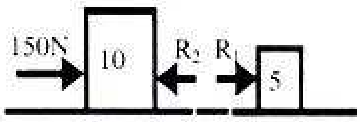


Solution:-

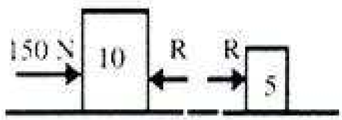
The combined acceleration of the two bodies when treated as one is
 $a = F/((10+5))=150/15=10/sec^2$

 So each one moves with $a = 10m/sec^2$ keeping their contact established.

Here you can feel that due to 150N force the body of 5 kg feels as if it is being pushed by the 10 kg mass. There is force acting on 5kg called R_1, to oppose it by third law this body exerts a force R_2 on 10kg. The interface is as shown in Figure given below.



Also, third law tells us that $R_1 = R_2$ in magnitude and is opposite in direction.



$R_1 = R_2 = R$

Here since 150 N force acts on the 10kg mass and only r acts on the 5kg mass. For motion in 5kg only R is responsible. We can write the initial equation as:

$F = 150 = (10 + 5) a$

$150 = 10a + 5a$

Here 10a is force experienced by 10kg mass. And 5a is experienced by 5kg mass.

$R = 5a \quad a = 10m/sec^2$

So,R = 50N

Thus,Net force experienced by 10kg block is (150-R) = 10a 150-R = 1010 = 100 N

Therefore, R = 50

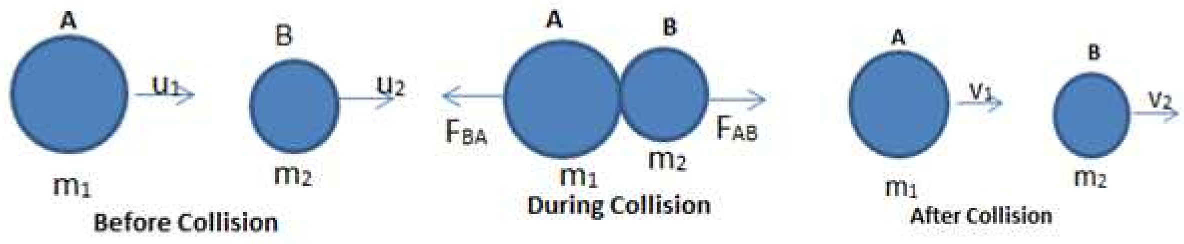
Therefore we get R = 50N for both blocks. Hence we find "action and reaction are equal and opposite". Now net force on the body of 10kg mass is 100N & Net force on the body of 5kg mass is 50N and on the interface action and reaction are both equal and also are equal to force experienced by second body.

It states that,

“In an isolated system (no external force), the algebraic some of the momenta of bodies, along any straight line, remains constant and is not changed due to their mutual action and reaction on each other”.

This can be verified by a following simple experiment.

Consider a body ‘A’ of mass ‘m_1’ moving with a velocity $\vec{u_1}$ strike against another body ‘B’ of mass m_2,moving with velocity in same direction as shown in the below figure. Two bodies remain in contact with each other for small time '?t'. They get separated and move with velocities$\vec{v_1}$ and $\vec{v_2}$ after collision.



Let $\vec{F_{AB}}$ be the force exerted by 'A' upon 'B' and $\vec{F_{BA}}$ be its reaction. Since the system is isolated, i.e., no external force is there,

$$\vec{F_{AB}} + \vec{F_{BA}} = 0$$

So,

$$\vec{F_{AB}} = -\vec{F_{BA}} \qquad \text{...... (1)}$$

This is in accordance with Newton's third law of motion that 'action and reaction are equal and opposite'.

Considering the momenta of the bodies before and after collision.

Body A	**Body B**
Momentum of A before collision = $m_1\vec{u_1}$ Momentum of A after collision = $m_1\vec{v_1}$ Change in momentum of A = $m_1\vec{v_1} - m_1\vec{u_1}$ Time taken for the change of momentum =?t Rate of change of momentum of A (=Force on A) = $\dfrac{m_1\vec{v_1} - m_1\vec{u_1}}{\Delta t}$ So,	Momentum of A before collision =$m_2\vec{u_2}$ Momentum of B after collision = $m_2\vec{v_2}$ Change in momentum of B = $m_2\vec{v_2} - m_2\vec{u_2}$ Time taken for the change of momentum =?t Rate of change of momentum of B (=Force on B)= $\dfrac{m_2\vec{v_2} - m_2\vec{u_2}}{\Delta t}$ So, $\vec{F_{AB}} = \dfrac{m_2\vec{v_2} - m_2\vec{u_2}}{\Delta t}$

Substituting for $\vec{F_{AB}}$ and $\vec{F_{BA}}$ in equation (1),

$$\frac{m_2\vec{v_2} - m_2\vec{u_2}}{\Delta t} = -\frac{m_1\vec{v_1} - m_1\vec{u_1}}{\Delta t}$$

Or, $m_1\vec{u_1} + m_2\vec{u_2} = m_1\vec{v_1} + m_2\vec{v_2}$

Thus, the total momentum of the system before collision is equal to the total momentum of the system after collision.

This verifies the law of conservation of momentum.

It may be noted that the conservation of momentum is closely connected with the validity of Newton's third law of motion, since we have used equation (1) [which is nothing but third law] to prove it.

Alternative Method

According to Newton's second law of motion,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v})$$

Since $m\vec{v} = \vec{p}$ (momentum of body),

$$\vec{F} = \frac{d}{dt}\vec{p}$$

Incase of an isolated system,

$$\vec{F} = 0$$

Thus,

$$\frac{d\vec{p}}{dt} = 0$$

or, $\vec{p} = constant$

Therefore, momentum (in vector form) of an isolated system remains constant. This is in accordance with the law of conservation of momentum.

IMPORTANT NOTE:-

While applying law of conservation of momentum to a system following consideration must be kept in mind:

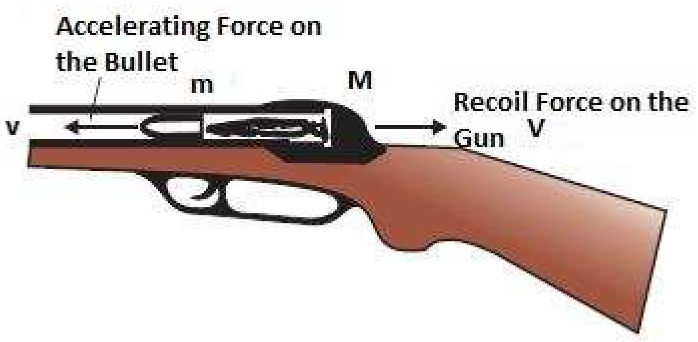
(a) **The system must be isolated.**

(b) **While finding the algebraic sum of momenta it must be ensured that all of them are along a particular straight line.**

Applications of Conservation of Momentum

Following few examples with illustrate the law of conservation of momentum.

(a) Recoil of Gun



A gun and a bullet constitute one isolated system. On firing the gun, bullet moves out with a very high velocity $\vec{v}$. The gun experiences a recoil. It moves in the opposite direction as shown in the below figure. Velocity '$\vec{v}$' of the recoil gun can be calculated by the application of law of conservation of momentum.

Before Firing	After Firing
Momentum of bullet = 0 Momentum of gun = 0 Total momentum of the system = 0	Momentum of bullet = $m\vec{v}$ Momentum of gun = $m\vec{v}$ Total momentum of the system = $M\vec{V} + m\vec{v}$

Here 'm' and 'M' are the masses of bullet and gun respectively. According to the law of conservation of momentum, momentum before collision and after collision must be same.

$$M\vec{V} + m\vec{v} = 0$$

or, $M\vec{V} = -m\vec{v}$

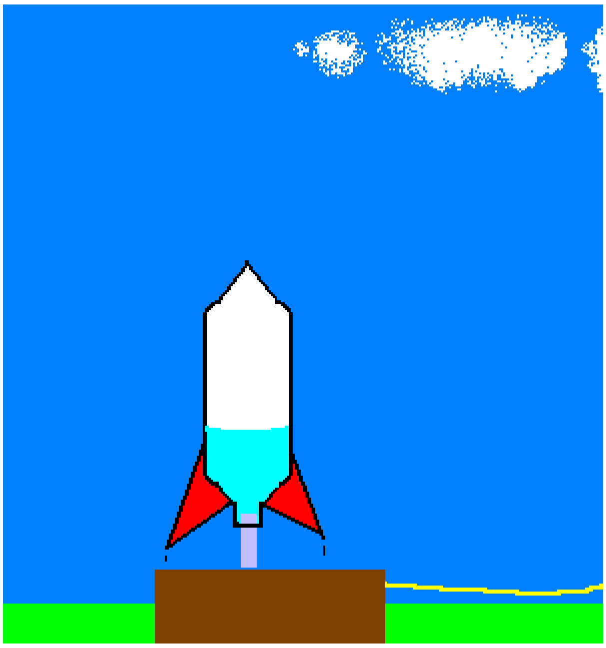
or,

$$\vec{V} = -\frac{m\vec{v}}{M}$$

Negative sign indicates that direction of motion of gun is in opposite direction.

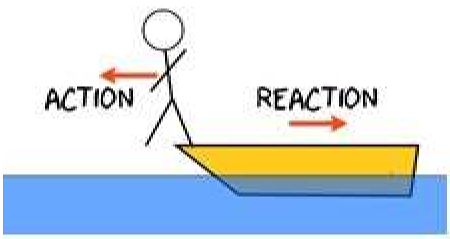
(b) Rocket and Jet Plane

Fuel and oxygen is burnt in the ignition chamber. As hot gases escape from a rear opening, with some momentum, the rocket moves in the forward direction with the same momentum.



(c) Explosion of a Bomb

Momentum of a bomb before explosion is zero. After explosion different fragments fly in various directions. It will be observed that their momenta, when represented by the slide of a polygon, from a closed polygon, indicating that net momentum after explosion is also zero. Thus, if the bomb exploded into two fragments, they must move in opposite directions.



(d) A man Jumping from a Boat

When a man jumps from the boat to the shore, the boat is pushed backward. It can, exactly, be explained as in the case of recoil of gun.

Some Conceptual Questions

Question 1:-

Figure below shows a popular carnival device, in which the contestant tries to see how high a weighted marker can be raised by hitting a target with a sledge hammer. What physical quantity does the device measure? Is it the average force, the maximum force, the work done, the impulse, the energy transferred, the momentum transferred, or something else? Discuss your answer.

Answer:-

The device will measure impulse. The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval. Since the contestant is hitting the target with a sledge hammer the change in momentum is large and the time of collision is small, therefore it signifies that the average impulsive force will relatively large. Suppose two persons bring the harmer from the same height, but they are hitting with different forces. The person who hits with greater force for the short time interval the impulse will be more and this results the height of the mark will be more. Thus the device will measure impulse.

Question 2:-

Can the impulse of a force be zero, even if the force is not zero? Explain why or why not?

Answer:-

Yes, the impulse of a force can be zero, even if the force is not zero.

Impulse of a force is defined as the change in momentum produced by the force and it is equal to the product of force and the time for which it acts. The impulse of a force can be zero, if the net force acting on the particle during that time interval is constant. Since the force is constant (both magnitude and direction), so change in momentum produced by the force will be zero. Therefore impulse of the force will be zero.

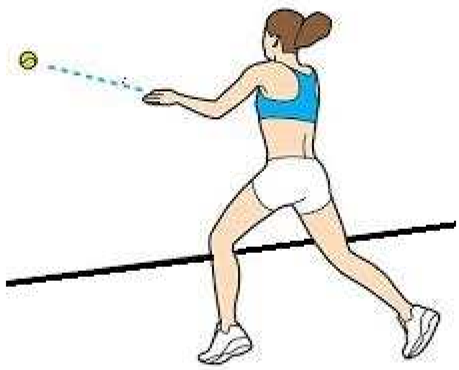
From the above observation we conclude that, impulse of a force can be zero, even if the force is not zero.

Question 3:-

Explain how conservation of momentum applies to a handball bouncing off a wall.

Answer:-

Law of conservation of linear momentum states that, in an isolated system (no external force), the algebraic sum of momenta of bodies, along any straight line, remains constant and is not changed due to their mutual action and reaction on each other.



The momentum of particle (p) is equal to the mass of particle (m) times the velocity of particle (v).

So,$p = mv$ …… (1)

Let us consider m is the mass of the ball and v is the velocity of the ball when the ball is collides with wall.

So using equation (1), the momentum of the ball before collision (p_1) will be,

$p_1 = mv$ …… (2)

After collision, when the ball re bounces, the velocity of the ball will be, $-v$.

So again using equation (1), the momentum of the ball after collision (p_2) will be,

$p_2 = -mv$ …… (3)

Conservation of linear momentum states that, the algebraic sum of momenta of bodies, along any straight line, remains constant and is not changed due to their mutual action and reaction on each other.

$p_1 + p_2 = 0$

So, $mv + (-mv) = 0$ …… (4)

From equation (4) we observed that, linear momentum of the hand ball is conserved.

Question 4:-

Give a plausible explanation for the breaking of wooden boards or bricks by a karate punch. (See “Karate Strikes.” by Jearl D. Walker, American Journal of Physics, October 1975, p.845.)

Answer:-

In the process, breaking of wooden boards or bricks by a karate punch, the collision between the hand and brick is only for a few milliseconds. Because the applied external force is large and the time of collision is small therefore the average impulsive force is relatively large. Thus when you break a wooden board or bricks by a karate punch you have to apply large force for the minimum time which is impulse. Therefore the impact force on the brick or wooden boards will be high.

Some Solved Problems

Problem 1:-

A 75.2-kg man is riding on a 38.6-kg cart travelling at a speed of 2.33 m/s. He jumps off in such a way as to land on the ground with zero horizontal speed. Find the resulting change in the speed of the cart.

Concept:-

Momentum of the body p is equal to the mass of the body m times velocity of the body v.

So, $p = mv$

In accordance to the principle of conservation of energy, the final momentum of the system is equal to the initial momentum of the system.

Consider the initial momentum of the man is $p_{i,m}$, initial momentum of the cart is $p_{i,c}$, final momentum of the man is $p_{f,m}$ and final momentum of the cart is $p_{f,c}$.

We define, $p_{f,m} = m_m v_{f,m}$

$p_{f,c} = m_c v_{f,c}$

$p_{i,m} = m_m v_{i,m}$

$p_{i,c} = m_c v_{i,c}$

Here, mass of the man is m_m, mass of the cart is m_c, initial velocity of the man is $v_{i,m}$ and cart is $v_{i,c}$, and final velocity of the man is $v_{f,m}$ and cart is $v_{f,c}$.

Solution:-

So applying conservation of momentum to this system, the sum of the initial momentum of the man and cart will be equal to the sum of the final momentum of the man and cart.

$p_{f,m} + p_{f,c} = p_{i,m} + p_{i,c}$

Substitute, $m_m v_{f,m}$ for $p_{f,m}$, $m_c v_{f,c}$ for $p_{f,c}$, $m_m v_{i,m}$ for $p_{i,m}$ and $m_c v_{i,c}$ for $p_{i,c}$ ijn the equation $p_{f,m} + p_{f,c} = p_{i,m} + p_{i,c}$,

$p_{f,m} + p_{f,c} = p_{i,m} + p_{i,c}$

$m_m v_{f,m} + m_c v_{f,c} = m_m v_{i,m} + m_c v_{i,c}$

$v_{f,c} - v_{i,c} = (m_m v_{i,m} - m_m v_{f,m}) / m_c$

$\Delta v_c = (m_m v_{i,m} - m_m v_{f,m}) / m_c$

To obtain the resulting change in the speed of the cart Δv_c, substitute 75.2 kg for m_m, 2.33 m/s for $v_{i,m}$ and 0 m/s for $v_{f,m}$ in the equation $\Delta v_c = (m_m v_{i,m} - m_m v_{f,m}) / m_c$,

$\Delta v_c = (m_m v_{i,m} - m_m v_{f,m}) / m_c$

$= (75.2 \text{ kg})(2.33 \text{ m/s}) - (75.2 \text{ kg})(0 \text{ m/s})/(38.6 \text{ kg})$

$= 4.54 \text{ m/s}$

As the sign of the change in the speed of the cart Δv_c is positive, this signifies that, the cart speed increases.

From the above observation we conclude that, the resulting change in the speed of the cart Δv_c would be 4.54 m/s.

Collision of Elastic Bodies

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yields slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the force of restitution. Also, the properly which causes a solid body to recover its shape is called elasticity. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be inelastic. When a body completely regains its shape after a collision, it is said to be perfectly elastic. If it does not come to its original shape, it is said to be perfectly inelastic.

Definitions:

Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.

Two bodies are said to impinge obliquely, if the direction of motion of either body or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the line of impact. Thus, in the cause of two spheres, the line of impact is the line joining their centres.

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3.8. Fundamental Laws of Impact:

1. Newton"s Experimental Law (NEL):

When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the *coefficient (or modulus) of elasticity (or restitution or resilience).*

This law can be put symbolically as follows: If u1, u2 are the components of the velocities of two impinging bodies along their common normal before impact and v1, v2 their component velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution, then

$v2-v1=-e(u2-u1)$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which e = 0 are said to be inelastic. For perfectly elastic bodies, e=1. Probably, there are no bodies in nature coming strictly under wither of these headings. Newton‟s law is purely empirical and is true only approximately, like many experimental laws

2. Motion of two smooth bodies perpendicular to the line of Impact:

When two smooth bodies impinge, the only force between them at the time of impact is the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence the velocity of either body resolved in a direction perpendicular to the line of impact is not altered by impact.

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3. Principle of Conservation of Momentum (PCM) :

We can apply the law of conservation of momentum in the case of two impinging bodies.

The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum of their moments before impact, all momenta being measured along the common normal.

3.9. Impact of a smooth sphere on a fixed smooth plane:

A smooth sphere, or particle whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.

Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure.

Since the plane and the sphere are smooth, the only force acting during impact is the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere, resolved in a direction parallel to the plane is unaltered by the impact.

$v\sin\theta = u\sin\alpha$

By Newton‟s experimental law, the relative velocity of the sphere along the common normal after impact is (-e) time its relative velocity along the common normal before impact. Hence

$$v \cos\theta - 0 = -e(-u\cos\alpha - 0)$$

i.e. $v\cos\theta = eu\cos\alpha$ ...(2)

Squaring (1) and (2), and adding, we have

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

i.e. $v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$ ... (3)

Dividing (2) by (1), we have $\cot\theta = e\cot\alpha$ ... (4)

Hence the (3) and (4) give the velocity and direction of motion after impact.

Corollary 1: If e = 1, we find that from (3) v = u and from (4) $\theta = \alpha$. Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Cor. 2: If e = 0, then from (2), $v\cos\theta = 0$ and from (3), $v = u\sin\alpha$. Hence $\cos\theta = 0$ i.e. $\theta = 90^\circ$. Hence the inelastic sphere slides along the plane with velocity $u\sin\alpha$

Cor. 3: If the impact is direct we have $\alpha = 0$. Then $\theta = 0$ and from (3) v=cu. Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.

Cor. 4: The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$$I = mv\cos\theta - (-mu\cos\alpha)$$
$$= m(v\cos\theta + u\cos\alpha)$$
$$= m(cu\cos\alpha + u\cos\alpha)$$
$$= mu\cos\alpha(1+e)$$

Cor. 5: Loss of kinetic energy due to the impact

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}mu^2(\sin^2\alpha + e^2\cos^2\alpha)$$
$$= \frac{1}{2}mu^2(1 - \sin^2\alpha + e^2\cos^2\alpha)$$
$$= \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$
$$= \frac{1}{2}(1-e^2)mu^2\cos^2\alpha$$

If the sphere is perfectly elastic, e = 1 and the loss of kinetic energy is zero.

Problems

1. A particle falls from a height h upon a fixed horizontal plane: if e be the coefficient of restitution, show that the whole distance described before the particle has

finished rebounding is $h\left(\frac{1+e^2}{1-e^2}\right)$. Show also that the whole time taken is $\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$.

Solution

Let u the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$. After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity eu. After the second impact, it rebounds with a velocity c^2u and the process is repeated a number of times. The velocities after the third, fourth etc. impacts are e^3u e^4u etc.

The height ascended after the first impact with velocity eu is $\frac{(\text{velocity})^2}{2g}$

$$= \frac{e^2u^2}{2g}$$

The height ascended after the second impact with velocity e^2u is $e^4u^2/2g$ and so on.

$\therefore$ Total distance travelled before the particle stops rebounding

$$= h + 2\left(\frac{e^2u^2}{2g} + \frac{e^4u^2}{2g} + \frac{e^6u^2}{2g} + \dots\dots\dots\right)$$

$$= h + \frac{2 \cdot e^2u^2}{2g}\left(1 + e^2 + e^4 + \dots\dots\dots \text{ to } \infty\right)$$

$$= h + \frac{e^2u^2}{g} \cdot \frac{1}{1-e^2}$$

$$= h + \frac{e^2 .2gh}{g} \cdot \frac{1}{1-e^2}$$

$$= h\left(1 + \frac{2e^2}{1-e^2}\right)$$

$$= h. \frac{(1+e^2)}{(1-e^2)}$$

Considering the motion before the first impact, we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken, u = 0 + gt.

$$\therefore t = \frac{u}{g} = \frac{\text{velocity}}{g}$$

Time interval between the first and second impacts is

= 2 x time taken for gravity to reduce the velocitiy to 0.

= 2. velocity / g

= 2 e^u / g.

Similarly time interval between the second and third impacts

= 2 e^2 u/g and so on.

So total time taken

$$= \frac{u}{g} + 2\left(\frac{eu}{g} + \frac{e^{2}u}{g} + \frac{e^{3}u}{g} + \dots \dots \infty\right)$$

$$= \frac{u}{g} + \frac{2eu}{g}\left(1 + e + e^{2} + \dots \dots \dots \text{ to } \infty\right)$$

$$= \frac{u}{g} + \frac{2eu}{g} \cdot \frac{1}{1-e} = \frac{u}{g}\left[1 + \frac{2e}{1-e}\right]$$

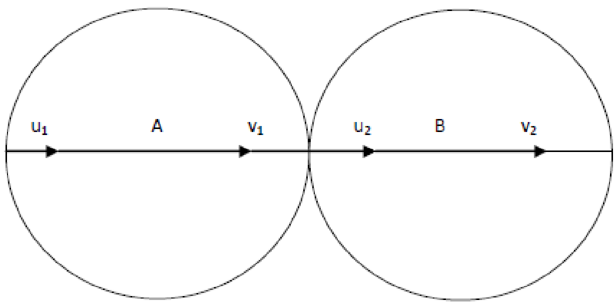
$$= \frac{u}{g} + \left(\frac{1+e}{1-e}\right)$$

$$= \frac{\sqrt{2gh}}{g}\left(\frac{1+e}{1-e}\right) = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}.$$

Direct impact of two smooth spheres:

A smooth sphere of mass m1 impinges directly with velocity u1 on another smooth sphere

of mass m2, moving in the same direction with velocity u2. If the coefficient of restitution is e, to find their velocities after the impact:



AB is the line of impact, i.e. the common normal. Due to the impact there is no tangential

force and hence, for either sphere the velocity along the tangent is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere tangential velocity after impact = its tangent velocity before impact = 0. So, after impact, the spheres will move only in the direction AB. Let their velocities be v1 and v2.

By Newton‟s experimental law, the relative velocity of m2 with respect to m1 after impact

is (-e) times the corresponding relative velocity before impact.

∴ $v2 – v1 = -e (u2 – u1)$ …….(1)

By the principle of conservation of momentum, the total momentum along the common

normal after impact is equal to the total momentum in the same direction before impact.

∴ $m1 v1 + m2 v2 = m1 u1 + m2 u2$ …….(2)

(2) – (1) x m2 gives

$v1 (m1 + m2) = m1 u1 + m2 u2 + em2 (u2 – u1)$

$= m2 u2 (1 + e) + (m1 – em2) u1$

$$\therefore v_1 = \frac{m_2 u_2 (1 + e) + (m_1 - em_2) u_1}{m_1 + m_2} \quad \ldots (3)$$

(1) x m_1 + (2) gives

$$v_2 (m_1 + m_2) = - em_1 (u_2 - u_1) + m_1u_1 + m_2u_2$$

$$= m_1u_1 (1 + e) + (m_2 - em_1) u_2$$

$$\therefore v_2 = \frac{m_1 u_1 (1 + e) + (m_2 - em_1) u_2}{m_1 + m_2} \quad \ldots (4)$$

Equations (3) and (4) give the velocities of the spheres after impact.

Note: If one sphere say m2 is moving originally in a direction opposite to that of m1, the

sign of u2 will be negative. Also it is most important that the directions of v1 and v2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v1 and v2 are in this direction. If either of them is actually in the opposite direction, the value obtained for it will turn to be negative.

In writing equation (1) corresponding to Newton‟s law, the velocities must be subtracted

in the same order on both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and m1

= m2. Then, from equations (3) and (4), we have

$$v_1 = \frac{m_1 u_2 . 2 + 0}{2m_1} = u_2 \text{ and } v_2 = \frac{m_1 u_1 . 2 + 0}{2m_1} = u_1.$$

i.e. *If two equal perfectly elastic spheres impinge directly, they interchange their velocities.*

Cor: 2. The impulse of the blow on the sphere A of mass m1 = change of momentum of

A = m1 (v1 – u1).

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 - em_2)u_1}{m_1 + m_2} - u_1\right]$$

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 u_1 - em_2 u_1 - m_1 u_1 - m_2 u_1}{m_1 + m_2}\right]$$

$$= \frac{m_1 [m_2 u_2 (1+e) - m_2 u_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 m_2 (1+e)(u_2 - u_1)}{m_1 + m_2}$$

The impulsive blow on m2 will be equal and opposite to the impulsive blow on m1.

Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinge directly; to show that there is

a loss of kinetic energy and to find the amount:

Let m1 m2 be the masses of the spheres, u1 and u2, v1 and v2 be their velocities before and

after impact and e the coefficient of restitution.

By Newton‟s law, v2 – v1 = -e (u2 – u1) … (1)

By the principle of conservation of momentum,

m1v1 + m2v2 = m1u1 + m2u2 ….(2)

Total kinetic energy before impact

$$= \frac{1}{2} m_1 u_1^{\ 2} + \frac{1}{2} m_2 u_2^{\ 2}$$

and total kinetic energy after impact

$$= \frac{1}{2} m_1 v_1^{\ 2} + \frac{1}{2} m_2 v_2^{\ 2}$$

Change in K.E. = initial K.E. – final K.E.

$$= \frac{1}{2} m_1 u_1^{\ 2} + \frac{1}{2} m_2 u_2^{\ 2} - \frac{1}{2} m_1 v_1^{\ 2} - \frac{1}{2} m_2 v_2^{\ 2}$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_2 (u_2 - v_2)(u_2 + v_2)$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_1 (v_1 - u_1)(u_2 + v_2)$$

$[\because \ m_2 (u_2 - v_2) = m_1 (v_1 - u_1)$ from (2)]

$$= \frac{1}{2} m_1 (u_1 - v_1)[u_1 - u_2 - (v_2 - v_1)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)[u_1 - u_2 + e \ (u_2 - u_1)] \quad \text{using (1)}$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 - u_2)(1 - e) \qquad \ldots..(3)$$

Now, from (2), $m_1 (u_1 - v_1) = m_2 (v_2 + u_2)$

$$\therefore \frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1} \text{ and each} = \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$$

$$\text{i.e. each} = \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2}$$

$$= \frac{(u_1 - u_2) - e\,(u_2 - u_1)}{m_1 + m_2} \quad \text{using (1)}$$

$$= \frac{(u_1 - u_2)\ (1 + e)}{m_1 + m_2}$$

$\therefore u_1 - v_1 = \dfrac{m_2 (u_1 - u_2)\ (1 + e)}{m_1 + m_2}$ and substituting this in (3),

$$\text{Change in K.E.} = \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)(1 + e)(u_1 - u_2)\ (1 - e)}{m_1 + m_2}$$

$$= \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)^2 (1 - e^2)}{m_1 + m_2} \qquad \ldots(4)$$

As $e < 1$, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the

case, when e=1, i.e. only when the bodies are perfectly elastic, the expression (4) becomes zero
and hence the total K.E. is unchanged by impact.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

III YEAR B.SC MATHEMATICS

SMT5103 MECHANICS 1

Unit 5- Projectiles – SMT1503

UNIT V

5.1 Projectiles.

Definitions:

i. A particle projected into the air in any direction with any velocity is called a **projectile**.

ii. **The angle of projection** is the angle made by the initial velocity with the horizontal plane through the point of projection.

iii. **The velocity of projection** is the velocity with which the particle is projected.

iv. **The trajectory** is the path described by the projectile.

v. The **range on a plane** through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.

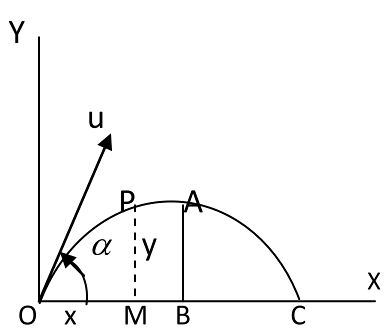
vi. **The time of flight** is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

Two fundamental principles

i. The horizontal velocity remains constant throughout the motion.

ii. The vertical component of the velocity will be subjected to retardation *g*.

5.2 Equation of the path of the projectile



Let a particle be projected from O, with initial velocity u and α be the angle of projection. Take OX and OY as x and y axes respectively. Let P (x,y) be the position of the particle in time t secs. Now u can be divided into two components as u $\cos\alpha$ in the horizontal direction and $u\sin\alpha$ in the vertical direction.

Now, horizontal velocity $u\cos\alpha$ is constant throughout the motion.

$$\therefore x = (u\cos\alpha)\ t \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Vertical velocity is subjected to retardation 'g'

$$\therefore y = (u\sin\alpha)\ t - \frac{1}{2}gt^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eliminate 't' using (1) and (2)

$$(1) \Rightarrow t = \frac{x}{u\cos\alpha}$$

$$(2) \Rightarrow y = u\sin\alpha\frac{x}{u\cos\alpha} - \frac{1}{2}g.\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$= \frac{x\tan\alpha.2u^2\cos^2\alpha - gx^2}{2u^2\cos^2\alpha}$$

$$2u^2\cos^2\alpha.y = x.2u^2\sin\alpha\cos\alpha - gx^2$$

$$\therefore gx^2 - 2u^2\sin\alpha\cos\alpha.x = -2u^2\cos^2\alpha.y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = \frac{-2u^2\cos^2\alpha}{g}y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} - \frac{2u^2\cos^2\alpha}{g}.y$$

ie) $\left(x - \dfrac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\dfrac{2u^2\cos^2\alpha}{g}\left(y - \dfrac{u^2\sin^2\alpha}{2g}\right) \ldots\ldots\ldots (4)$

Shifting the origin to $\left(\dfrac{u^2\sin\alpha\cos\alpha}{g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$

$$X^2 = -\frac{2u^2\cos^2\alpha}{g}.Y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

(5) is the equation of a parabola of the form $X^2 = -4aY$,

whose latus-rectum is $\dfrac{2u^2\cos^2\alpha}{g} = \dfrac{2}{g}(u\cos\alpha)^2$

$= \dfrac{2}{g}(horizontal\ \ velocity)^2$

Vertex is $\left(\dfrac{u^2\sin\alpha.\cos\alpha}{g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$

5.3 Characteristics of the motion of the projectile

1. Greatest height attained by a projectile.

2. Time taken to reach the greatest height.

3. Time of flight.

4. The range on the horizontal plane through the point of projection.

Derive formula for the characteristics

5.3.1 Greatest height h

When the particle reaches the highest point at A, its direction is horizontal.

$\therefore$ At A, vertical velocity = 0

Let AB = h.

Consider the vertical motion and using the formula "$v^2 = u^2 + 2aS$"

$$O = (u\sin\alpha)^2 - 2g.h \qquad \therefore h = \frac{u^2\sin^2\alpha}{2g}$$

- Highest point of the path is the vertex of the parabola.

5.3.2 Time taken to reach the greatest height T

Let T be the time taken to travel from O to reach the greatest height at A.

At A final vertical velocity is zero

At O initial vertical velocity is $u\sin\alpha$

Using the formula "$v = u + at$"

$$O = u\sin\alpha - gT \quad \therefore \boxed{T = \frac{u\sin\alpha}{g}}$$

5.3.3 Time of flight t

Let t be the time taken to travel from O to C along its path. At C, vertical distance traveled is zero. Consider the vertical motion and by the formula $S = ut + \frac{1}{2}at^2$,

$$O = u\sin\alpha \;.t - \frac{1}{2}gt^2$$

ie) $t\left(u\sin\alpha - \frac{1}{2}gt\right) = 0$

$\therefore t = 0$ or $u\sin\alpha - \frac{1}{2}gt = 0$

ie) $t = 0$ or $t = \frac{2u\sin\alpha}{g} = 2\left(\frac{u\sin\alpha}{g}\right) = 2T$

t = 0 gives the time of projection.

$$\boxed{\therefore \text{ Time of flight } \; t = \frac{2u\sin\alpha}{g}}$$

- ❖ Time of flight = 2 x time taken to reach the greatest height.

5.3.4 The range on the horizontal plane through the point of projection R

Range R = OC = horizontal distance traveled during the time of flight.

= horizontal velocity x time of flight

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

- ❖ Horizontal range R = $\frac{2(u\cos\alpha)(u\sin\alpha)}{g} = \frac{2UV}{g}$

Where U – initial horizontal velocity, V – initial vertical velocity.

Problem 1

A body is projected with a velocity of 98 metres per sec. in a direction making an angle $\tan^{-1} 3$ with the horizon; show that it rises to a vertical height of 441 metres and that its time of flight is about 19 sec. Find also horizontal range through the point of projection (g=9.8 metres / sec^2)

Solution:

Given u = 98; α = tan $^{-1}$3 i.e tan α = 3

$$\therefore \sin\alpha = \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{3}{\sqrt{10}}$$

$$\cos\alpha = \frac{\sin\alpha}{\tan\alpha} = \frac{1}{\sqrt{10}}$$

Greatest height = $\dfrac{u^2\sin^2\alpha}{2g} = \dfrac{98\times 98\times 9}{10\times 2\times 9.8}$ = 441 metres

Time of flight = $\dfrac{2u\sin\alpha}{g} = \dfrac{2\times 98\times 3}{\sqrt{10}\times 9.8} = 6\sqrt{10}$

= $6\times 3.162 = 18.972 = 19$ secs. nearly

Horizontal range = $\dfrac{2u^2\sin\alpha\cos\alpha}{g}$

= $\dfrac{2\times 98\times 98}{9.8}\times\dfrac{3}{\sqrt{10}}\times\dfrac{1}{\sqrt{10}}$ = 588 metres

Problem 2

If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection

Solution

Let u be the initial velocity and α the angle of projection

Greatest height = $\dfrac{u^2\sin^2\alpha}{2g}$

Horizontal range = $\dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

Given $\dfrac{u^2 \sin^2\alpha}{2g} = \dfrac{1}{4} \times \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

i.e $\dfrac{u^2 \sin^2\alpha}{2g} = \dfrac{u^2 \sin\alpha \cos\alpha}{2g}$

i.e $\sin\alpha = \cos\alpha \Rightarrow \tan\alpha = 1 \quad \therefore \alpha = 45^0$

Problem 3

A particle is projected so as to graze the tops of two parallel walls, the first of height ‘a’ at a distance b from the point of projection and the second of height b at a distant ‘a’ from the point of projection. If the path of particle lies in a plane perpendicular to both the walls, find the range on the horizontal plane and show that the angle of projection exceeds $\tan^{-1}3$.

Solution:

Let u be the initial velocity, α be the angle of projection.

Equation to the path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

i.e $y = xt - \dfrac{gx^2}{2u^2}\left(1+t^2\right)$ where $t = \tan\alpha$ ……. (1)

The tops of the two walls are (b, a) and (a, b) lie on (1)

$\therefore\ a = bt - \dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ……… (2)

$b = at - \dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (3)

From (2), $a - bt = -\dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ………. (4)

From (3), $b - at = -\dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (5)

Dividing (4) by (5), $\frac{a-bt}{b-at}=\frac{b^2}{a^2}$

i.e $b^3 - ab^2 t = a^3 - a^2bt \quad \Rightarrow \quad t(a^2b - ab^2) = a^3 - b^3$

$$\therefore t=\frac{a^3-b^3}{a^2b-ab^2}=\frac{(a-b)\left(a^2+ab+b^2\right)}{ab(a-b)}=\frac{a^2+ab+b^2}{ab}$$

$$\therefore \tan\alpha=\frac{a^2+ab+b^2}{ab}=\frac{\left(a^2-2ab+b^2\right)+3ab}{ab}=\frac{(a-b)^2}{ab}+3 \text{.....(6)}$$

$$(6) \Rightarrow \tan\alpha>3 \;\; or \;\; \alpha>\tan^{-1}3$$

From (4), $\frac{g\left(1+t^2\right)}{2u^2}=\frac{a-bt}{-b^2}=\frac{bt-a}{b^2}$

$$=\frac{\frac{b\left(a^2+ab+b^2\right)}{ab}-a}{b^2}=\frac{a^2+ab+b^2-a^2}{ab^2}$$

$$=\frac{b(a+b)}{ab^2}=\frac{a+b}{ab} \qquad \text{........... (7)}$$

Horizontal range = $\frac{u^2\sin 2\alpha}{g}=\frac{2u^2t}{g\left(1+t^2\right)} \quad \because \sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}$

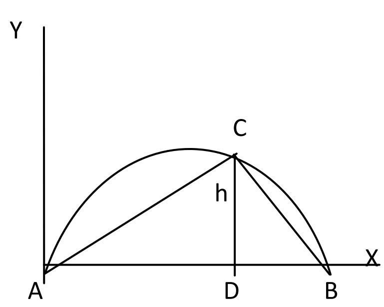
$$= t.\frac{ab}{a+b} \text{ from (7)}$$

$$=\frac{\left(a^2+ab+b^2\right)}{ab}\cdot\frac{ab}{a+b}=\frac{a^2+ab+b^2}{a+b}$$

Problem 4

A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, show that $\tan\alpha$ = tan A + tan B

Solution:



Let u be the velocity of projection and α the angle of projection and let t secs be the time taken from A to C. Draw CD $\perp$ AB and let CD = h.

Consider the vertical motion, h = vertical distance described in time t

$$= u \sin \alpha \cdot t - \frac{1}{2} g t^2$$

AD = horizontal distance described in time t = u cos $\alpha \cdot$t

From ΔCAD, $\tan A = \frac{CD}{AD} = \frac{h}{AD} = \frac{u \sin \alpha \cdot t - \frac{1}{2} g t^2}{u \cos \alpha \cdot t}$

$$= \tan \alpha - \frac{gt}{2u \cos \alpha} \qquad \ldots\ldots (1)$$

AB = horizontal range = $\frac{2u^2 \sin \alpha \cos \alpha}{g}$

$\therefore$ DB = AB – AD = $\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t$

From ΔCDB, $\tan B = \frac{CD}{DB} = \frac{h}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t\right)}$

$$= \frac{u \sin \alpha . t - \frac{1}{2} g t^2}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha . t\right)}$$

$$= \frac{gt\left(u \sin \alpha - \frac{1}{2} gt\right)}{u \cos \alpha (2u \sin \alpha - gt)}$$

$$= \frac{gt(2u \sin \alpha - gt)}{2u \cos \alpha (2u \sin \alpha - gt)} = \frac{gt}{2u \cos \alpha} \ldots\ldots\ldots.. (2)$$

(1) + (2) $\Rightarrow$ tanA + tanB = tanα

Problem 5

Show that the greatest height which a particle with initial velocity v can reach on a vertical wall at a distance 'a' from the point of projection is $\frac{v^2}{2g} - \frac{ga^2}{2v^2}$ Prove also that the greatest height above the point of projection attained by the particle in its fight is $v^6 / 2g\left(v^4 + g^2a^2\right)$

Solution:

Equation to the path is $y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$ …….. (1)

Put x = a in (1), $y = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$

$y = at - \frac{ga^2}{2v^2}\left(1+t^2\right)$ where t = tan α …….. (2)

y is a function of t. $\therefore$ y is maximum when $\frac{dy}{dt} = 0$ and $\frac{d^2y}{dt^2}$ is negative.

Differentiating (2) with respect to t,

$$\frac{dy}{dt} = a - \frac{ga^2}{2v^2}\cdot 2t = a - \frac{ga^2t}{v^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{v^2} = \text{negative}$$

So y is maximum when $a - \frac{ga^2t}{v^2} = 0$ or $t = \frac{v^2}{ga}$ ……(3)

Put $t = \frac{v^2}{ga}$ in (2)

Max value of $y = a\cdot\frac{v^2}{ga} - \frac{ga^2}{2v^2}\left(1 + \frac{v^4}{g^2a^2}\right)$

$$= \frac{v^2}{g} - \frac{ga^2}{2v^2} - \frac{v^2}{2g} = \frac{v^2}{2g} - \frac{ga^2}{2v^2}$$

Greatest height during the flight

$$= \frac{v^2\sin^2\alpha}{2g} = \frac{v^2}{2g}\cdot\frac{1}{\operatorname{cosec}^2\alpha} = \frac{v^2}{2g\left(1+\cot^2\alpha\right)}$$

$$= \frac{v^2}{2g\left(1+\dfrac{g^2a^2}{v^4}\right)} \text{ from (3)}$$

$$= \frac{v^6}{2g\left(v^4+g^2a^2\right)}$$

Problem 6

a. A projectile is thrown with a velocity of 20 m/sec. at an elevation 30^0. Find the greatest height attained and the horizontal range.

b. A particle is projected with a velocity of 9.6 metres at an angle of 30^0. Find

i. The time of flight

ii. the greatest height of the particle.

Solution:

Given u = 20m/sec; $\alpha = 30^0$

Greatest height = $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{20^2\left(\sin 30^0\right)^2}{2\times 9.8} = 5.1m$

Horizontal range = $\dfrac{u^2 \sin 2\alpha}{g} = \dfrac{20^2 \cdot \sin 60^0}{9.8} = 35.35m$

Problem 7

(a) A particle is projected under gravity in a vertical plane with a velocity u at an angle α to the horizontal. If the range on the horizontal be R and the greatest height attained by h, show that $\dfrac{u^2}{2g} = h + \dfrac{R^2}{16h}$ and $\tan\alpha = \dfrac{4h}{R}$.

(b) A particle is projected so that on its upward path, it passes through a point x feet horizontally and y feet vertically from the point of projection. Show that, if R be the horizontal range, the angle of projection is $\tan^{-1}\left(\dfrac{y}{x}\cdot\dfrac{r}{R-x}\right)$.

Solution:

a) $$h+\frac{R^2}{16h}=\frac{u^2\sin^2\alpha}{2g}+\frac{\left(\frac{2u^2\sin\alpha\cos\alpha}{g}\right)^2}{16\left(\frac{u^2\sin^2\alpha}{2g}\right)}$$

$$=\frac{u^2\sin^2\alpha}{2g}+\frac{u^2\cos^2\alpha}{2g}=\frac{u^2}{2g}$$

b) Equation of the path is, $y=x\tan\alpha-\dfrac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore x\tan\alpha=y+\frac{gx^2}{2u^2\cos^2\alpha}$$

$$\therefore \tan\alpha=\frac{y}{x}+\frac{gx}{2u^2\cos^2\alpha} \qquad \text{...........}\ (1)$$

We have $R=\dfrac{2u^2\sin\alpha\cdot\cos\alpha}{g}\Rightarrow g=\dfrac{2u^2\sin\alpha\cos\alpha}{R}$

$$\therefore (1)\Rightarrow \tan\alpha=\frac{y}{x}+\frac{x}{2u^2\cos^2\alpha}\times\frac{2u^2\cdot\sin\alpha\cos\alpha}{R}=\frac{y}{x}+\frac{x\tan\alpha}{R}$$

$$\therefore \tan\alpha\left(1-\frac{x}{R}\right)=\frac{y}{x}$$

ie $\tan\alpha\left(\dfrac{R-x}{R}\right)=\dfrac{y}{x}$ or $\tan\alpha=\dfrac{y}{x}.\dfrac{R}{R-x}$

$$\therefore \alpha=\tan^{-1}\left(\frac{y}{x}\cdot\frac{R}{R-x}\right)$$

Problem 8

If the time of flight of a shot is T seconds over a range of x metres, show that the elevation is $\tan^{-1}\left(\dfrac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.

Solution:

Given, horizontal range R = x metres

Time of flight $T = \dfrac{2u\sin\alpha}{g}$ .... (1)

where α -is the angle of projection

$$\therefore x = \frac{2u^2 \sin\alpha\cos\alpha}{g}$$

$\therefore$ (1) $\Rightarrow$ gT = 2u sinα . $\Rightarrow$ $\boxed{u = \dfrac{gT}{2\sin\alpha}}$

$$\therefore x = \frac{2\cdot g^2T^2\cdot\sin\alpha\cos\alpha}{4\sin^2\alpha\cdot g} = \frac{1}{2}gT^2\cdot\cot\alpha$$

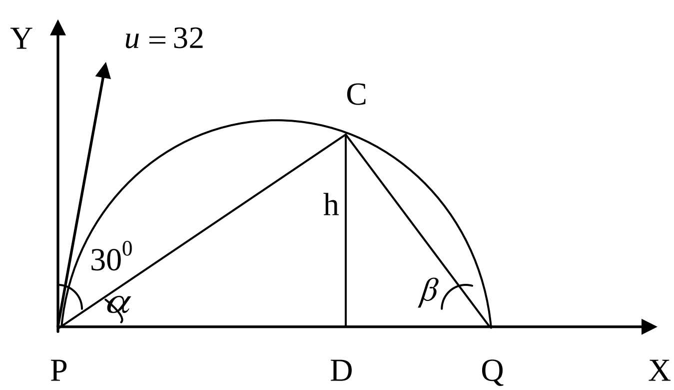
$$\therefore \tan\alpha = \frac{gT^2}{2x} \Rightarrow \boxed{\alpha = \tan^{-1}\left(\frac{gT^2}{2x}\right)}$$

Maximum height = $\dfrac{u^2\sin^2\alpha}{2g} = \dfrac{g^2T^2}{4\sin^2\alpha}\cdot\dfrac{\sin^2\alpha}{2g} = \dfrac{gT^2}{8}$

Problem 9

A particle is projected from a point P with a velocity of 32m per second at an angle of 30^0 with the horizontal. If PQ be its horizontal range and if the angles of elevation from P and Q at any instant of its flight be α and β respectively, show that $\tan\alpha + \tan\beta = \dfrac{1}{\sqrt{3}}$

Solution:



Given, initial velocity u = 32 m/sec, 30^0 is the angle of projection. P-be the point of projection. 't' – be the time taken from P to C.

Let CD = h = $u\sin\alpha.t - \frac{1}{2}gt^2$

$h = (32.\sin 30^0)\ t - \frac{1}{2}gt^2 =$ vertical distance described in t secs

$= 16t - \frac{1}{2}gt^2$

PD = horizontal distance described in t secs = $u\cos\alpha.t$

$= \left(32\cos 30^0\right)\ t \quad = 32\cdot\frac{\sqrt{3}}{2}t \quad = 16\sqrt{3}t$.

From Δ PCD, $\tan\alpha = \frac{h}{PD} = \frac{h}{16\sqrt{3}t}$ .... (1)

From Δ QCD, $\tan\beta = \frac{h}{DQ} = \frac{h}{PQ-PD}$, PQ = range

ie $\tan\beta = \dfrac{h}{\left(\dfrac{2(32)^2\sin 30^0\cdot\cos 30^0}{g}\right) - 16\sqrt{3}t}$

$= \dfrac{hg}{512\sqrt{3} - 16\sqrt{3}gt}$ ....(2)

$\therefore (1)+(2) \Rightarrow \tan\alpha + \tan\beta = \frac{h}{16\sqrt{3}}\left[\frac{1}{t} + \frac{g}{32-gt}\right]$

$= \dfrac{\left(16t - \frac{1}{2}gt^2\right)}{16\sqrt{3}}\left[\dfrac{32-gt+gt}{t(32-gt)}\right]$

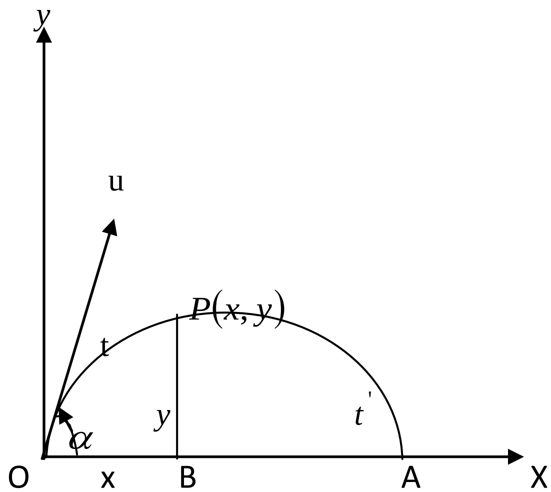
$= \dfrac{t\ (32-gt)}{32\sqrt{3}} \times \dfrac{32}{t(32-gt)} = \dfrac{1}{\sqrt{3}}$

$\therefore \quad \boxed{\tan\alpha + \tan\beta = \dfrac{1}{\sqrt{3}}}$

Problem 10

A particle is projected and after time t reaches a point P. If t is the lime it takes to move from P to the horizontal plane through the point of projection, prove that the height of P above the plane is $\frac{1}{2} gt\ t^{'}$

Solution:



Let u be the velocity of projection, α be the angle of projection, P be the position of the particle after t secs. Let $t^{'}$ be the time taken to travel from P to A

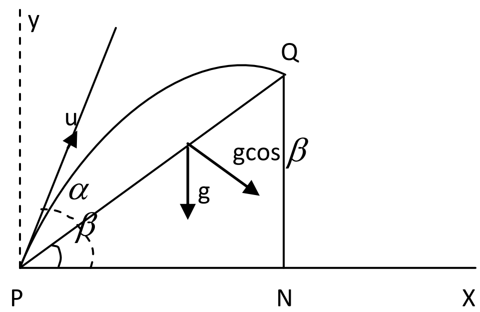
$\therefore$ We have $t + t^{'}$ = time of flight = $\dfrac{2u\sin\alpha}{g}$ $\therefore u\sin\alpha = \dfrac{g(t+t^{'})}{2}$

Now, y = vertical distance described in t secs $= (u\sin\alpha)\ t - \frac{1}{2}gt^2$

$$= \frac{g(t+t^{'})\ t}{2} - \frac{1}{2}gt^2 \qquad = \frac{gtt^{'}}{2}$$

$\therefore$ Height of P above the plane = $= \dfrac{gtt^{'}}{2}$

5.4 Range on an inclined Plane:



Let P be the point of projection on a plane of inclination β, u be the velocity of projection at an angle α with the horizontal. The particle strikes the inclined plane at Q. Then PQ = r is the range on the inclined plane. Take PX and PY as x and y axes.

Draw $QN \perp PX$.

From $\Delta PQN,\ PN = r\cos\beta,\ QN = r\sin\beta$

$Q(r\cos\beta, r\sin\beta)$ lies on the path. $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore r\sin\beta = r\cos\beta.\tan\alpha - \frac{g(r\cos\beta)^2}{2u^2\cos^2\alpha}$$

Dividing by r we get $\dfrac{gr\cos^2\beta}{2u^2\cos^2\alpha} = \cos\beta.\dfrac{\sin\alpha}{\cos\alpha} - \sin\beta$

$$\therefore r = \frac{2u^2\cos^2\alpha}{g\cos^2\beta}\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}\right]$$

$$\boxed{r = \frac{2u^2\cos\alpha}{g\cos^2\beta}\sin(\alpha-\beta)}$$

5.5 Maximum range on the inclined plane, given u the velocity of projection and β the inclination of the plane:

Range *r* on the inclined plane is

$$r = \frac{2u^2 \cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta)-\sin\beta\right] \quad \ldots\ldots (1)$$

Now u and β are given, g constant.

So *r* is maximum when $\left[\sin(2\alpha-\beta)-\sin\beta\right]$ is maximum.

i.e. when sin $(2\alpha-\beta)$ is maximum.

i.e.when. $2\alpha-\beta=\frac{\pi}{2}$

$\therefore$ $\boxed{\alpha=\frac{\pi}{4}+\frac{\beta}{2}}$ for maximum range.

From (1), maximum range on the inclined plane

$$= \frac{u^2}{g\cos^2\beta}(1-\sin\beta) = \frac{u^2}{g(1+\sin\beta)}$$

3.5.1 Time of flight T (up an inclined plane):

From the figure in 6.11, the time taken to travel from P to Q is the time of flight. Consider the motion perpendicular to the inclined plane. At the end of time T, the distance travelled perpendicular to the inclined plane S = 0, component of g perpendicular to the inclined plane is $g\cos\beta$, initial velocity perpendicular to the inclined plane is $u\sin(\alpha-\beta)$.

$$0 = u\sin(\alpha-\beta)T - \frac{1}{2}g\cos\beta.T^2 \qquad \text{using } "S = ut + \frac{1}{2}at^2"$$

$$\boxed{\therefore T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}}$$

5.5.2 Greatest distance S of the projectile from the inclined plane and show that it is attained in half the total time of flight:

Consider the motion perpendicular to the inclined plane. The initial velocity perpendicular to the plane is u sin (α - β) and this is subjected to an acceleration gcos β in the same direction but acting downwards. Let S be the greatest distance travelled by the particle perpendicular to the inclined plane. At the greatest distance the velocity becomes parallel to the inclined plane and hence the velocity perpendicular to the plane is zero.

Using the formula "$v^2 = u^2 + 2as$"

$$0 = [u \sin(\alpha - \beta)]^2 - 2g \cos \beta . S$$

$$\boxed{S = \frac{u^2 . \sin^2(\alpha - \beta)}{2.g \cos \beta}}$$

5.5.3 Time taken to reach the greatest distance t :

When the particle is at the greatest distance from the inclined plane, its velocity becomes parallel to the inclined plane and the velocity perpendicular to the inclined plane is zero. So, if t is the time taken to reach the greatest distance, using the formula

"$v = u + at$"

$$\therefore 0 = [u \sin(\alpha - \beta)] - g \cos \beta \cdot t$$

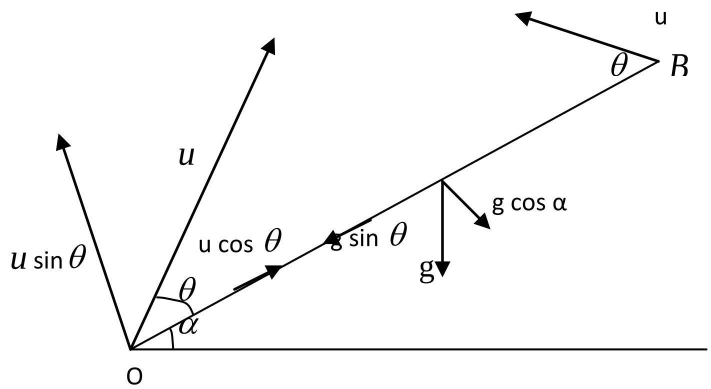
i.e. $\boxed{t = \dfrac{u \sin(\alpha - \beta)}{g \cos \beta}}$

Note : Time of flight T = $\dfrac{2u \sin(\alpha - \beta)}{g \cos \beta}$ = 2.t = 2 $\times$ time taken to reach the greatest distance.

Problem 11

Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\dfrac{1 + \sin \alpha}{1 - \sin \alpha}$

Solution



Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane. u has two components u$\cos\theta$ along the upward inclined plane and u$\sin\theta$ perpendicular to the inclined plane. g has two components, g sin α along the downward inclined plane and g$\cos\alpha$ perpendicular to the inclined plane and downwards.

Consider the motion perpendicular to the inclined plane. Let T be the time of flight. Distance travelled perpendicular to the inclined plane in time T = 0

$$\therefore 0 = u\sin\theta\cdot T - \frac{1}{2}g\cos\alpha\cdot T^2 \qquad \left(\because S = ut + \frac{1}{2}at^2\right)$$

i.e. $T = \dfrac{2u\sin\theta}{g\cos\alpha}$

Range up the plane = R_1

R_1= distance travelled along the plane in time T

$$= u\cos\theta\cdot T - \frac{1}{2}g\sin\alpha\cdot T^2$$

$$= u\cos\theta\cdot\frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2}g\sin\alpha\cdot\frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha) = \frac{u^2}{g\cos^2\alpha}\cdot 2\cos(\theta+\alpha)\sin\theta$$

$$= \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha)-\sin\alpha\right]$$

R_1 is maximum, when $\sin(2\theta+\alpha)=1$

$\therefore$ Maximum range up the plane

$$= \frac{u^2}{g\cos^2\alpha}(1-\sin\alpha) = \frac{u^2}{g(1+\sin\alpha)} \quad \text{………… (1)}$$

When the particle is projected down the plane from B at the same angle θ to the plane, the time of flight T has the same value $\frac{2u\sin\theta}{g\cos\alpha}$. The component of the initial velocity along the inclined plane is u $\cos\theta$ downwards and the component of acceleration g sin α is also downwards.

Range down the plane = R_2

R_2 = distance travelled along the plane in time T

$$= u\cos\theta\cdot T + \frac{1}{2}g\sin\alpha\cdot T^2$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha) = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha)+\sin\alpha\right]$$

R_2 is maximum, when sin $(2\theta - \alpha) = 1$.

Maximum range down the plane

$$= \frac{u^2}{g\cos^2\alpha}(1+\sin\alpha) = \frac{u^2}{g(1-\sin\alpha)} \quad \text{………….. (2)}$$

$$\therefore \frac{Max\cdot range\ down\ the\ plane}{Max\cdot range\ up\ the\ plane} = \frac{u^2}{g(1-\sin\alpha)}\cdot\frac{g(1+\sin\alpha)}{u^2} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

Problem 12

A particle is projected at an angle α with a velocity u and it strikes up an inclined plane of inclination β at right angles to the plane. Prove that (i) cotβ = 2tan(α - β) (ii) cotβ = tan $\alpha - 2\tan\beta$. If the plane is struck horizontally, show that $\tan\alpha = 2\tan\beta$.

Solution:

The initial velocity and acceleration are split into components along the plane and perpendicular to the plane.

The time of flight is $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ ….. (1)

Since the particle strikes the inclined plane normally, its velocity parallel to the inclined plane at the end of time T is = 0.

i.e. 0 = u cos (α - β) – g sin $\beta \cdot T$

$$T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{….. (2)}$$

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \text{ from (1) and (2)}$$

i.e. cot β = 2 tan ($\alpha - \beta$) …… (i)

i.e. $\cot\beta = \dfrac{2(\tan\alpha - \tan\beta)}{1+\tan\alpha\tan\beta}$, Simplifying we get

cot β + tan α = 2 tan $\alpha - 2$ tan β

cot β = tan $\alpha - 2$ tan β ….. (ii)

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T = 0. Initial vertical velocity = u sinα, and acceleration in this direction = g (downwards).

Vertical velocity in time T = u sin $\alpha -$ gT

$\therefore$ u sin $\alpha - $gT = 0 or T = $\dfrac{u\sin\alpha}{g}$ ….. (3)

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\sin\alpha}{g}$$ from (1) and (3)

Simplifying we get

2 sin ($\alpha - \beta$) = sin α cos β

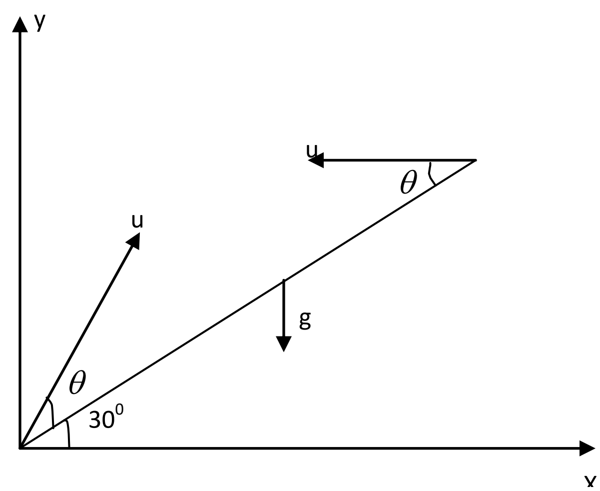
2(sinα cos β – cosα sinβ) = sinα cos β.

sinα cos β = 2cosα sinβ or tanα = 2tanβ

Problem 13

The greatest range with a given velocity of projection on a horizontal plane is 3000 metres. Find the greatest ranges up and down a plane inclined at 30^0 to the horizon.

Solution:



Let u be the velocity of projection, θ be the inclination of direction of projection with the plane. Given $\dfrac{u^2}{g} = 3000\,m \Rightarrow u^2 = 3000\times g$

At the end of time t, distance travelled perpendicular to the inclined plane is zero.

$$\therefore 0 = u\sin\theta\cdot T - \frac{1}{2}g\cos 30^0\cdot T^2$$

$$0 = u\sin\theta \cdot T - \frac{1}{2} g \cdot \frac{\sqrt{3}}{2} \cdot T^2$$

$$\boxed{\therefore T = \frac{4u\sin\theta}{g\sqrt{3}}}$$

Range up the inclined plane, S = $u\cos\theta \cdot T - \frac{1}{2} g \cdot \sin 30^0 \cdot T^2$

$$= u\cos\theta \cdot \frac{4u\sin\theta}{g\sqrt{3}} - \frac{1}{4} \cdot g \cdot \times \frac{16u^2 \sin^2\theta}{3g^2}$$

$$= \frac{4u^2 \sin\theta\cos\theta}{g\sqrt{3}} - \frac{4u^2\sin^2\theta}{3g}$$

$$S = \frac{4u^2\sin\theta}{3g}\left[\sqrt{3}\cos\theta - \sin\theta\right]$$

Max. range is got when $\sin(2\theta + 30^0) = 1$

i.e. $2\theta + 30^0 = 90^0 \therefore \theta = 30^0$

Max. range up the inclined plane

$$= S_{\max} = \frac{4u^2 \sin 30^0}{3g}\left[\sqrt{3}\cos 30^0 - \sin 30^0\right]$$

$$= \frac{4u^2 \times \frac{1}{2}}{3g}\left[\sqrt{3} \times \frac{\sqrt{3}}{2} - \frac{1}{2}\right] = \frac{2}{3} \times 3000 \quad S_{\max} = 2000m$$

$\therefore$ Range down the inclined plane = $\frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta - \alpha) + \sin\alpha\right]$

Max. range down the inclined plane

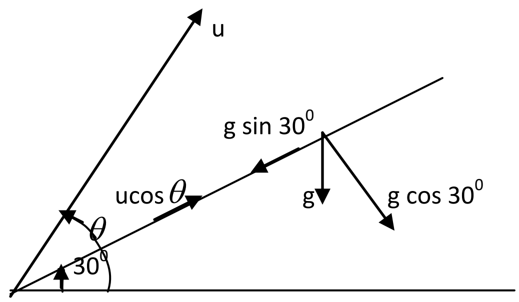
$$= \frac{u^2}{g \cdot \cos^2 30^0}\left[1 + \sin 30^0\right] = \frac{4u^2}{3g}\left[1 + \tfrac{1}{2}\right]$$

$$= \frac{2u^2}{g} = 2 \times 3000 = 6000m$$

Problem 14

An inclined plane is inclined at an angle of 30^0 to the horizon. Show that, for a given velocity of projection, the maximum range up the plane is 1/3 of the maximum range down the plane.

Solution:



Max. range up the plane = $\dfrac{u^2}{g \cdot \cos^2 30^0}\left[1 - \sin 30^0\right] = \dfrac{2u^2}{3g}$

Max. range down the plane = $\dfrac{u^2}{g \cdot \cos^2 30^0}\left[1 + \sin 30^0\right]$

$$= \frac{4u^2}{3g} \times \frac{3}{2} = \frac{2u^2}{g}$$

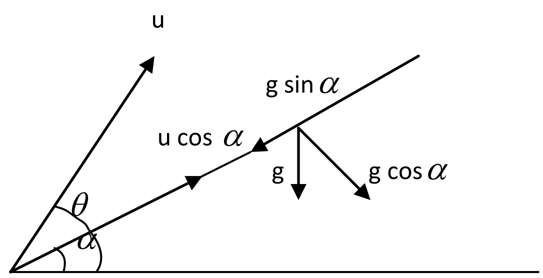
Max. range up the plane = $\dfrac{1}{3} \times \dfrac{2u^2}{g}$

$$= \frac{1}{3} \times \max \cdot \textit{range down the plane}$$

Problem 15

If the greatest range down an inclined plane is three times its greatest range up the plane then show that the plane is inclined at 30^0 to the horizon..

Solution



Greatest range down the inclined plane R_1

$$R_1 = \frac{u^2}{g\cos^2\alpha}[1+\sin\alpha]$$

Greatest range down the inclined plane R_2

$$R_2 = \frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

Given, $R_1 = 3R_2$

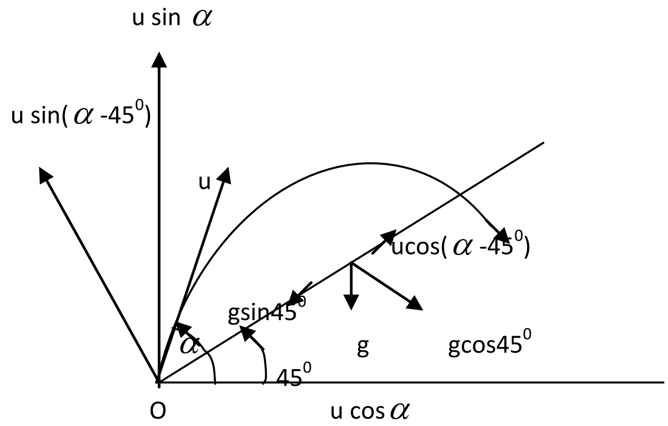
i.e. $$\frac{u^2}{g\cos^2\alpha}[1+\sin\alpha] = 3\cdot\frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

$$\sin\alpha = \frac{1}{2} \quad \therefore \alpha = 30^0$$

Problem 16

A particle is projected in a vertical plane at an angle α to the horizontal from the foot of a plane whose inclination to the horizon is 45^0. Show that the particle will strike the plane at right angles if $\tan\alpha = 3$.

Solution:



When the particle strikes the plane at right angles, velocity parallel to the plane is zero.

$$\therefore O = u\cos(\alpha - 45^0) - g \cdot \sin 45^0 \cdot T$$

$$\therefore T = \frac{u\cos(\alpha - 45^0)}{g\sin 45^0} = \frac{u\cos(\alpha - 45^0)}{g \cdot \frac{1}{\sqrt{2}}} \qquad \ldots\ldots. (1)$$

Also, time of flight, $T = \dfrac{2u \cdot \sin\left(\alpha - 45^0\right)}{g \cdot \cos 45^0}$ ……. (2)

$$(1) \,\&\, (2) \Rightarrow \frac{u\cos\left(\alpha - 45^0\right)}{g \cdot \frac{1}{\sqrt{2}}} = \frac{2u \cdot \sin\left(\alpha - 45^0\right)}{g . \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \cos\left(\alpha - 45^0\right) = 2 \cdot \sin(\alpha - 45^0) \;\Rightarrow 2 \cdot \tan\left(\alpha - 45^0\right) = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - \tan 45^0}{1 + \tan\alpha \cdot \tan 45^0}\right] = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - 1}{1 + \tan\alpha}\right] = 1$$

i.e. $2(\tan\alpha - 1) = 1 + \tan\alpha$

$\therefore \tan\alpha = 3$