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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – REAL ANALYSIS – SMT1502

THE REAL AND COMPLEX NUMBER SYSTEMS

INTRODUCTION

A satisfactory discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept. We shall not, however, enter into any discussion of the axioms that govern the arithmetic of the integers, but assume familiarity with the rational numbers (i.e., the numbers of the form m/n, where m and n are integers and $n \neq 0$).

The rational number system is inadequate for many purposes, both as a field and as an ordered set. (These terms will be defined in Secs. 1.6 and 1.12.) For instance, there is no rational p such that $p^2 = 2$. (We shall prove this presently.) This leads to the introduction of so-called "irrational numbers" which are often written as infinite decimal expansions and are considered to be "approximated" by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

"tends to $\sqrt{2}$." But unless the irrational number $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that this sequence "tends to"?

This sort of question can be answered as soon as the so-called "real number system" is constructed.

1.1 Example We now show that the equation

$$p^2 = 2 \tag{1}$$

is not satisfied by any rational p. If there were such a p, we could write $p = m/n$ where m and n are integers that are not both even. Let us assume this is done. Then (1) implies

$$m^2 = 2n^2, \tag{2}$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of (2) is divisible by 4, so that n^2 is even, which implies that n is even.

The assumption that (1) holds thus leads to the conclusion that both m and n are even, contrary to our choice of m and n. Hence (1) is impossible for rational p.

We now examine this situation a little more closely. Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. We shall show that *A contains no largest number and B contains no smallest.*

More explicitly, for every p in A we can find a rational q in A such that $p < q$, and for every p in B we can find a rational q in B such that $q < p$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{3}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{4}$$

If p is in A then $p^2 - 2 < 0$, (3) shows that $q > p$, and (4) shows that $q^2 < 2$. Thus q is in A.

If p is in B then $p^2 - 2 > 0$, (3) shows that $0 < q < p$, and (4) shows that $q^2 > 2$. Thus q is in B.

1.2 Remark The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another: If $r < s$ then $r < (r + s)/2 < s$. The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

In order to elucidate its structure, as well as that of the complex numbers, we start with a brief discussion of the general concepts of *ordered set* and *field.*

Here is some of the standard set-theoretic terminology that will be used throughout this book.

1.3 Definitions If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A.

If x is not a member of A, we write: $x \notin A$.

The set which contains no element will be called the *empty set*. If a set has at least one element, it is called *nonempty*.

If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$, or $B \supset A$. If, in addition, there is an element of B which is not in A, then A is said to be a *proper* subset of B. Note that $A \subset A$ for every set A.

If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise $A \neq B$.

1.4 Definition Throughout Chap. 1, the set of all rational numbers will be denoted by Q.

ORDERED SETS

1.5 Definition Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x precedes y".

It is often convenient to write $y > x$ in place of $x < y$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold. In other words, $x \leq y$ is the negation of $x > y$.

1.6 Definition An *ordered set* is a set S in which an order is defined.

For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E.

Lower bounds are defined in the same way (with $\geq$ in place of $\leq$).

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.
(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of E* [that there is at most one such α is clear from (ii)] or the *supremum of E*, and we write

$$\alpha = \sup E.$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

1.9 Examples

(*a*) Consider the sets A and B of Example 1.1 as subsets of the ordered set Q. The set A is bounded above. In fact, the upper bounds of A are exactly the members of B. Since B contains no smallest member, *A has no least upper bound in Q*.

Similarly, B is bounded below: The set of all lower bounds of B consists of A and of all $r \in Q$ with $r \leq 0$. Since A has no lasgest member, *B has no greatest lower bound in Q*.

(*b*) If $\alpha = \sup E$ exists, then α may or may not be a member of E. For instance, let E_1 be the set of all $r \in Q$ with $r < 0$. Let E_2 be the set of all $r \in Q$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0,$$

and $0 \notin E_1$, $0 \in E_2$.

(*c*) Let E consist of all numbers $1/n$, where $n = 1, 2, 3, \ldots$. Then $\sup E = 1$, which is in E, and $\inf E = 0$, which is not in E.

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Example 1.9(*a*) shows that Q does not have the least-upper-bound property.

We shall now show that there is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least-upper-bound property also has the greatest-lower-bound property.

1.11 Theorem *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ *exists in S.*

Proof Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$, we see that *every $x \in B$ is an upper bound of L.* Thus L is bounded above. Our hypothesis about S implies therefore that L has a supremum in S; call it α.

If $\gamma < \alpha$ then (see Definition 1.8) γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.

FIELDS

1.12 Definition A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
(A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

(M1) If $x \in F$ and $y \in F$, then their product xy is in F.
(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
(M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
(M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y+z) = xy + xz$$

holds for all $x, y, z \in F$.

1.13 Remarks

(*a*) One usually writes (in any field)

$$x - y, \frac{x}{y}, x + y + z, xyz, x^2, x^3, 2x, 3x, \ldots$$

in place of

$$x + (-y), x \cdot \left(\frac{1}{y}\right), (x + y) + z, (xy)z, xx, xxx, x + x, x + x + x, \ldots.$$

(*b*) The field axioms clearly hold in Q, the set of all rational numbers, if addition and multiplication have their customary meaning. Thus Q is a field.

(*c*) Although it is not our purpose to study fields (or any other algebraic structures) in detail, it is worthwhile to prove that some familiar properties of Q are consequences of the field axioms; once we do this, we will not need to do it again for the real numbers and for the complex numbers.

1.14 Proposition *The axioms for addition imply the following statements.*

(*a*) *If* $x + y = x + z$ *then* $y = z$.
(*b*) *If* $x + y = x$ *then* $y = 0$.
(*c*) *If* $x + y = 0$ *then* $y = -x$.
(*d*) $-(-x) = x$.

Statement (*a*) is a cancellation law. Note that (*b*) asserts the uniqueness of the element whose existence is assumed in (A4), and that (*c*) does the same for (A5).

Proof If $x + y = x + z$, the axioms (A) give

$$\begin{aligned} y = 0 + y = (-x + x) + y &= -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (*a*). Take $z = 0$ in (*a*) to obtain (*b*). Take $z = -x$ in (*a*) to obtain (*c*).
Since $-x + x = 0$, (*c*) (with $-x$ in place of x) gives (*d*).

1.15 Proposition *The axioms for multiplication imply the following statements.*

(*a*) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = 1/x$.
(*d*) *If* $x \neq 0$ *then* $1/(1/x) = x$.

The proof is so similar to that of Proposition 1.14 that we omit it.

1.16 Proposition *The field axioms imply the following statements, for any* x, y, $z \in F$.

(*a*) $0x = 0$.
(*b*) *If* $x \neq 0$ *and* $y \neq 0$ *then* $xy \neq 0$.
(*c*) $(-x)y = -(xy) = x(-y)$.
(*d*) $(-x)(-y) = xy$.

Proof $0x + 0x = (0 + 0)x = 0x$. Hence 1.14(*b*) implies that $0x = 0$, and (*a*) holds.

Next, assume $x \neq 0$, $y \neq 0$, but $xy = 0$. Then (*a*) gives

$$1 = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0 = 0,$$

a contradiction. Thus (*b*) holds.

The first equality in (*c*) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with 1.14(*c*); the other half of (*c*) is proved in the same way. Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (*c*) and 1.14(*d*).

1.17 Definition An *ordered field* is a *field F* which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
(ii) $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*; if $x < 0$, x is *negative*.

For example, Q is an ordered field.

All the familiar rules for working with inequalities apply in every ordered field: Multiplication by positive [negative] quantities preserves [reverses] inequalities, no square is negative, etc. The following proposition lists some of these.

1.18 Proposition *The following statements are true in every ordered field.*

(*a*) *If* $x > 0$ *then* $-x < 0$, *and vice versa.*
(*b*) *If* $x > 0$ *and* $y < z$ *then* $xy < xz$.
(*c*) *If* $x < 0$ *and* $y < z$ *then* $xy > xz$.
(*d*) *If* $x \neq 0$ *then* $x^2 > 0$. *In particular*, $1 > 0$.
(*e*) *If* $0 < x < y$ *then* $0 < 1/y < 1/x$.

Proof
(*a*) If $x > 0$ then $0 = -x + x > -x + 0$, so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$, so that $-x > 0$. This proves (*a*).
(*b*) Since $z > y$, we have $z - y > y - y = 0$, hence $x(z - y) > 0$, and therefore

$$xz = x(z - y) + xy > 0 + xy = xy.$$

(*c*) By (*a*), (*b*), and Proposition 1.16(*c*),

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that $x(z - y) < 0$, hence $xz < xy$.
(*d*) If $x > 0$, part (ii) of Definition 1.17 gives $x^2 > 0$. If $x < 0$, then $-x > 0$, hence $(-x)^2 > 0$. But $x^2 = (-x)^2$, by Proposition 1.16(*d*). Since $1 = 1^2$, $1 > 0$.
(*e*) If $y > 0$ and $v \leq 0$, then $yv \leq 0$. But $y \cdot (1/y) = 1 > 0$. Hence $1/y > 0$. Likewise, $1/x > 0$. If we multiply both sides of the inequality $x < y$ by the positive quantity $(1/x)(1/y)$, we obtain $1/y < 1/x$.

THE REAL FIELD

We now state the *existence theorem* which is the core of this chapter.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*
Moreover, R contains Q as a subfield.

The second statement means that $Q \subset R$ and that the operations of addition and multiplication in R, when applied to members of Q, coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of R.

The members of R are called *real numbers.*

The proof of Theorem 1.19 is rather long and a bit tedious and is therefore presented in an Appendix to Chap. 1. The proof actually constructs R from Q.

The next theorem could be extracted from this construction with very little extra effort. However, we prefer to derive it from Theorem 1.19 since this provides a good illustration of what one can do with the least-upper-bound property.

1.20 Theorem

(*a*) *If* $x \in R$, $y \in R$, *and* $x > 0$, *then there is a positive integer* n *such that*

$$nx > y.$$

(*b*) *If* $x \in R$, $y \in R$, *and* $x < y$, *then there exists a* $p \in Q$ *such that* $x < p < y$.

Part (*a*) is usually referred to as the *archimedean property* of R. Part (*b*) may be stated by saying that Q is *dense* in R: Between any two real numbers there is a rational one.

Proof

(*a*) Let A be the set of all nx, where n runs through the positive integers. If (*a*) were false, then y would be an upper bound of A. But then A has a *least* upper bound in R. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m + 1)x \in A$, which is impossible, since α is an upper bound of A.

(*b*) Since $x < y$, we have $y - x > 0$, and (*a*) furnishes a positive integer n such that

$$n(y - x) > 1.$$

Apply (*a*) again, to obtain positive integers m_1 and m_2 such that $m_1 > nx$, $m_2 > -nx$. Then

$$-m_2 < nx < m_1.$$

Hence there is an integer m (with $-m_2 \le m \le m_1$) such that

$$m - 1 \le nx < m.$$

If we combine these inequalities, we obtain

$$nx < m \le 1 + nx < ny.$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

This proves (*b*), with $p = m/n$.

We shall now prove the existence of nth roots of positive reals. This proof will show how the difficulty pointed out in the Introduction (irrationality of $\sqrt{2}$) can be handled in R.

1.21 Theorem *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof That there is at most one such y is clear, since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

Let E be the set consisting of all positive real numbers t such that $t^n < x$.

If $t = x/(1 + x)$ then $0 \leq t < 1$. Hence $t^n \leq t < x$. Thus $t \in E$, and E is not empty.

If $t > 1 + x$ then $t^n \geq t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of E.

Hence Theorem 1.19 implies the existence of

$$y = \sup E.$$

To prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}$$

when $0 < a < b$.

Assume $y^n < x$. Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$

Put $a = y$, $b = y + h$. Then

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

Thus $(y + h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E.

Assume $y^n > x$. Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E.

But $y - k < y$, which contradicts the fact that y is the *least* upper bound of E.

Hence $y^n = x$, and the proof is complete.

Corollary *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

Proof Put $\alpha = a^{1/n}$, $\beta = b^{1/n}$. Then

$$ab = \alpha^n\beta^n = (\alpha\beta)^n,$$

since multiplication is commutative. [Axiom (M2) in Definition 1.12.] The uniqueness assertion of Theorem 1.21 shows therefore that

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}.$$

1.22 Decimals We conclude this section by pointing out the relation between real numbers and decimals.

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \le x$. (Note that the existence of n_0 depends on the archimedean property of R.) Having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x.$$

Let E be the set of these numbers

(5) $$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \qquad (k = 0, 1, 2, \ldots).$$

Then $x = \sup E$. The decimal expansion of x is

(6) $$n_0 \cdot n_1 n_2 n_3 \cdots.$$

Conversely, for any infinite decimal (6) the set E of numbers (5) is bounded above, and (6) is the decimal expansion of $\sup E$.

Since we shall never use decimals, we do not enter into a detailed discussion.

THE EXTENDED REAL NUMBER SYSTEM

1.23 Definition The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$.

It is then clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in R, then $\sup E = +\infty$ in the extended real number system.

Exactly the same remarks apply to lower bounds.

The extended real number system does not form a field, but it is customary to make the following conventions:

(*a*) If x is real then

$$x + \infty = +\infty, \qquad x - \infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(*b*) If $x > 0$ then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.

(*c*) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

When it is desired to make the distinction between real numbers on the one hand and the symbols $+\infty$ and $-\infty$ on the other quite explicit, the former are called *finite*.

THE COMPLEX FIELD

1.24 Definition A *complex number* is an ordered pair (a, b) of real numbers. "Ordered" means that (a, b) and (b, a) are regarded as distinct if $a \neq b$.

Let $x = (a, b)$, $y = (c, d)$ be two complex numbers. We write $x = y$ if and only if $a = c$ and $b = d$. (Note that this definition is not entirely superfluous; think of equality of rational numbers, represented as quotients of integers.) We define

$$x + y = (a + c, b + d),$$
$$xy = (ac - bd, ad + bc).$$

1.25 Theorem *These definitions of addition and multiplication turn the set of all complex numbers into a field, with* $(0, 0)$ *and* $(1, 0)$ *in the role of* 0 *and* 1.

Proof We simply verify the field axioms, as listed in Definition 1.12. (Of course, we use the field structure of R.)

Let $x = (a, b)$, $y = (c, d)$, $z = (e, f)$.

(A1) is clear.

(A2) $x + y = (a + c, b + d) = (c + a, d + b) = y + x$.

(A3) $(x + y) + z = (a + c, b + d) + (e, f)$
$= (a + c + e, b + d + f)$
$= (a, b) + (c + e, d + f) = x + (y + z).$

(A4) $x + 0 = (a, b) + (0, 0) = (a, b) = x.$

(A5) Put $-x = (-a, -b)$. Then $x + (-x) = (0, 0) = 0.$

(M1) is clear.

(M2) $xy = (ac - bd, ad + bc) = (ca - db, da + cb) = yx.$

(M3) $(xy)z = (ac - bd, ad + bc)(e, f)$
$= (ace - bde - adf - bcf, acf - bdf + ade + bce)$
$= (a, b)(ce - df, cf + de) = x(yz).$

(M4) $1x = (1, 0)(a, b) = (a, b) = x.$

(M5) If $x \neq 0$ then $(a, b) \neq (0, 0)$, which means that at least one of the real numbers a, b is different from 0. Hence $a^2 + b^2 > 0$, by Proposition 1.18(d), and we can define

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

Then

$$x \cdot \frac{1}{x} = (a, b)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = (1, 0) = 1.$$

(D) $x(y + z) = (a, b)(c + e, d + f)$

$$= (ac + ae - bd - bf, ad + af + bc + be)$$

$$= (ac - bd, ad + bc) + (ae - bf, af + be)$$

$$= xy + xz.$$

1.26 Theorem *For any real numbers a and b we have*

$$(a, 0) + (b, 0) = (a + b, 0), \qquad (a, 0)(b, 0) = (ab, 0).$$

The proof is trivial.

Theorem 1.26 shows that the complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a. We can therefore identify $(a, 0)$ with a. This identification gives us the real field as a subfield of the complex field.

The reader may have noticed that we have defined the complex numbers without any reference to the mysterious square root of -1. We now show that the notation (a, b) is equivalent to the more customary $a + bi$.

1.27 Definition $i = (0, 1)$.

1.28 Theorem $i^2 = -1$.

Proof $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$.

1.29 Theorem *If a and b are real, then $(a, b) = a + bi$.*

Proof

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) = (a, b). \end{aligned}$$

1.30 Definition If a, b are real and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is called the *conjugate* of z. The numbers a and b are the *real part* and the *imaginary part* of z, respectively.

We shall occasionally write

$$a = \mathrm{Re}(z), \qquad b = \mathrm{Im}(z).$$

1.31 Theorem *If z and w are complex, then*

(*a*) $\overline{z + w} = \bar{z} + \bar{w}$,
(*b*) $\overline{zw} = \bar{z} \cdot \bar{w}$,
(*c*) $z + \bar{z} = 2\,\mathrm{Re}(z)$, $z - \bar{z} = 2i\,\mathrm{Im}(z)$,
(*d*) *$z\bar{z}$ is real and positive (except when $z = 0$).*

Proof (*a*), (*b*), and (*c*) are quite trivial. To prove (*d*), write $z = a + bi$, and note that $z\bar{z} = a^2 + b^2$.

1.32 Definition If z is a complex number, its *absolute value* $|z|$ is the non-negative square root of $z\bar{z}$; that is, $|z| = (z\bar{z})^{1/2}$.

The existence (and uniqueness) of $|z|$ follows from Theorem 1.21 and part (*d*) of Theorem 1.31.

Note that when x is real, then $\bar{x} = x$, hence $|x| = \sqrt{x^2}$. Thus $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$.

1.33 Theorem *Let z and w be complex numbers. Then*

(*a*) $|z| > 0$ *unless* $z = 0$, $|0| = 0$,
(*b*) $|\bar{z}| = |z|$,
(*c*) $|zw| = |z|\,|w|$,
(*d*) $|\mathrm{Re}\, z| \leq |z|$,
(*e*) $|z + w| \leq |z| + |w|$.

Proof (*a*) and (*b*) are trivial. Put $z = a + bi$, $w = c + di$, with a, b, c, d real. Then

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2$$

or $|zw|^2 = (|z|\,|w|)^2$. Now (*c*) follows from the uniqueness assertion of Theorem 1.21.

To prove (*d*), note that $a^2 \le a^2 + b^2$, hence

$$|a| = \sqrt{a^2} \le \sqrt{a^2 + b^2}.$$

To prove (*e*), note that $\bar{z}w$ is the conjugate of $z\bar{w}$, so that $z\bar{w} + \bar{z}w = 2\,\mathrm{Re}\,(z\bar{w})$. Hence

$$\begin{aligned}|z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}\\ &= |z|^2 + 2\,\mathrm{Re}\,(z\bar{w}) + |w|^2\\ &\le |z|^2 + 2|z\bar{w}| + |w|^2\\ &= |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2.\end{aligned}$$

Now (*e*) follows by taking square roots.

1.34 Notation If $x_1, \ldots, x_n$ are complex numbers, we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^{n} x_j.$$

We conclude this section with an important inequality, usually known as the *Schwarz inequality.*

1.35 Theorem *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof Put $A = \Sigma|a_j|^2$, $B = \Sigma|b_j|^2$, $C = \Sigma a_j\bar{b}_j$ (in all sums in this proof, j runs over the values $1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 1.31 we have

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j})\\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j\bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2\\ &= B^2A - B|C|^2\\ &= B(AB - |C|^2).\end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality.

EUCLIDEAN SPACES

1.36 Definitions For each positive integer k, let R^k be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_k),$$

where $x_1, \ldots, x_k$ are real numbers, called the *coordinates* of $\mathbf{x}$. The elements of R^k are called points, or vectors, especially when $k > 1$. We shall denote vectors by boldfaced letters. If $\mathbf{y} = (y_1, \ldots, y_k)$ and if α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_k + y_k),$$
$$\alpha\mathbf{x} = (\alpha x_1, \ldots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y} \in R^k$ and $\alpha\mathbf{x} \in R^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws (the proof is trivial, in view of the analogous laws for the real numbers) and make R^k into a *vector space over the real field*. The zero element of R^k (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordinates are 0.

We also define the so-called "inner product" (or scalar product) of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_1^k x_i^2\right)^{1/2}.$$

The structure now defined (the vector space R^k with the above inner product and norm) is called euclidean k-space.

1.37 Theorem *Suppose* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, *and* α *is real. Then*

(*a*) $|\mathbf{x}| \geq 0$;
(*b*) $|\mathbf{x}| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$;
(*c*) $|\alpha\mathbf{x}| = |\alpha|\,|\mathbf{x}|$;
(*d*) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$;
(*e*) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
(*f*) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Proof (*a*), (*b*), and (*c*) are obvious, and (*d*) is an immediate consequence of the Schwarz inequality. By (*d*) we have

$$\begin{aligned}|\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\\ &= \mathbf{x}\cdot\mathbf{x}+2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}\\ &\le |\mathbf{x}|^2+2|\mathbf{x}|\,|\mathbf{y}|+|\mathbf{y}|^2\\ &= (|\mathbf{x}|+|\mathbf{y}|)^2,\end{aligned}$$

so that (*e*) is proved. Finally, (*f*) follows from (*e*) if we replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y}-\mathbf{z}$.

1.38 Remarks Theorem 1.37 (*a*), (*b*), and (*f*) will allow us (see Chap. 2) to regard R^k as a metric space.

R^1 (the set of all real numbers) is usually called the line, or the real line. Likewise, R^2 is called the plane, or the complex plane (compare Definitions 1.24 and 1.36). In these two cases the norm is just the absolute value of the corresponding real or complex number.

APPENDIX

Theorem 1.19 will be proved in this appendix by constructing R from Q. We shall divide the construction into several steps.

Step 1 The members of R will be certain subsets of Q, called *cuts*. A cut is, by definition, any set $\alpha \subset Q$ with the following three properties.

(I) α is not empty, and $\alpha \neq Q$.
(II) If $p \in \alpha$, $q \in Q$, and $q < p$, then $q \in \alpha$.
(III) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

The letters $p, q, r, \ldots$ will always denote rational numbers, and $\alpha, \beta, \gamma, \ldots$ will denote cuts.

Note that (III) simply says that α has no largest member; (II) implies two facts which will be used freely:

If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

Step 2 Define "$\alpha < \beta$" to mean: α is a proper subset of β.

Let us check that this meets the requirements of Definition 1.5.

If $\alpha < \beta$ and $\beta < \gamma$ it is clear that $\alpha < \gamma$. (A proper subset of a proper subset is a proper subset.) It is also clear that at most one of the three relations

$$\alpha < \beta, \qquad \alpha = \beta, \qquad \beta < \alpha$$

can hold for any pair α, β. To show that at least one holds, assume that the first two fail. Then α is not a subset of β. Hence there is a $p \in \alpha$ with $p \notin \beta$. If $q \in \beta$, it follows that $q < p$ (since $p \notin \beta$), hence $q \in \alpha$, by (II). Thus $\beta \subset \alpha$. Since $\beta \neq \alpha$, we conclude: $\beta < \alpha$.

Thus R is now an ordered set.

Step 3 *The ordered set R has the least-upper-bound property.*

To prove this, let A be a nonempty subset of R, and assume that $\beta \in R$ is an upper bound of A. Define γ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We shall prove that $\gamma \in R$ and that $\gamma = \sup A$.

Since A is not empty, there exists an $\alpha_0 \in A$. This α_0 is not empty. Since $\alpha_0 \subset \gamma$, γ is not empty. Next, $\gamma \subset \beta$ (since $\alpha \subset \beta$ for every $\alpha \in A$), and therefore $\gamma \neq Q$. Thus γ satisfies property (I). To prove (II) and (III), pick $p \in \gamma$. Then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$; this proves (II). If $r \in \alpha_1$ is so chosen that $r > p$, we see that $r \in \gamma$ (since $\alpha_1 \subset \gamma$), and therefore γ satisfies (III).

Thus $\gamma \in R$.

It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$.

Suppose $\delta < \gamma$. Then there is an $s \in \gamma$ and that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in A$. Hence $\delta < \alpha$, and δ is not an upper bound of A.

This gives the desired result: $\gamma = \sup A$.

Step 4 If $\alpha \in R$ and $\beta \in R$ we define $\alpha + \beta$ to be the set of all sums $r + s$, where $r \in \alpha$ and $s \in \beta$.

We define 0^* to be the set of all negative rational numbers. It is clear that 0^* is a cut. *We verify that the axioms for addition* (see Definition 1.12) *hold in R, with 0^* playing the role of* 0.

(A1) We have to show that $\alpha + \beta$ is a cut. It is clear that $\alpha + \beta$ is a nonempty subset of Q. Take $r' \notin \alpha$, $s' \notin \beta$. Then $r' + s' > r + s$ for all choices of $r \in \alpha$, $s \in \beta$. Thus $r' + s' \notin \alpha + \beta$. It follows that $\alpha + \beta$ has property (I).

Pick $p \in \alpha + \beta$. Then $p = r + s$, with $r \in \alpha$, $s \in \beta$. If $q < p$, then $q - s < r$, so $q - s \in \alpha$, and $q = (q - s) + s \in \alpha + \beta$. Thus (II) holds. Choose $t \in \alpha$ so that $t > r$. Then $p < t + s$ and $t + s \in \alpha + \beta$. Thus (III) holds.

(A2) $\alpha + \beta$ is the set of all $r + s$, with $r \in \alpha$, $s \in \beta$. By the same definition, $\beta + \alpha$ is the set of all $s + r$. Since $r + s = s + r$ for all $r \in Q$, $s \in Q$, we have $\alpha + \beta = \beta + \alpha$.

(A3) As above, this follows from the associative law in Q.

(A4) If $r \in \alpha$ and $s \in 0^*$, then $r + s < r$, hence $r + s \in \alpha$. Thus $\alpha + 0^* \subset \alpha$. To obtain the opposite inclusion, pick $p \in \alpha$, and pick $r \in \alpha$, $r > p$. Then

$p - r \in 0^*$, and $p = r + (p - r) \in \alpha + 0^*$. Thus $\alpha \subset \alpha + 0^*$. We conclude that $\alpha + 0^* = \alpha$.

(A5) Fix $\alpha \in R$. Let β be the set of all p with the following property:

There exists $r > 0$ *such that* $-p - r \notin \alpha$.

In other words, some rational number smaller than $-p$ fails to be in α.

We show that $\beta \in R$ *and that* $\alpha + \beta = 0^*$.

If $s \notin \alpha$ and $p = -s - 1$, then $-p - 1 \notin \alpha$, hence $p \in \beta$. So β is not empty. If $q \in \alpha$, then $-q \notin \beta$. So $\beta \neq Q$. Hence β satisfies (I).

Pick $p \in \beta$, and pick $r > 0$, so that $-p - r \notin \alpha$. If $q < p$, then $-q - r > -p - r$, hence $-q - r \notin \alpha$. Thus $q \in \beta$, and (II) holds. Put $t = p + (r/2)$. Then $t > p$, and $-t - (r/2) = -p - r \notin \alpha$, so that $t \in \beta$. Hence β satisfies (III).

We have proved that $\beta \in R$.

If $r \in \alpha$ and $s \in \beta$, then $-s \notin \alpha$, hence $r < -s$, $r + s < 0$. Thus $\alpha + \beta \subset 0^*$.

To prove the opposite inclusion, pick $v \in 0^*$, put $w = -v/2$. Then $w > 0$, and there is an integer n such that $nw \in \alpha$ but $(n + 1)w \notin \alpha$. (Note that this depends on the fact that Q has the archimedean property!) Put $p = -(n + 2)w$. Then $p \in \beta$, since $-p - w \notin \alpha$, and

$$v = nw + p \in \alpha + \beta.$$

Thus $0^* \subset \alpha + \beta$.

We conclude that $\alpha + \beta = 0^*$.

This β will of course be denoted by $-\alpha$.

Step 5 Having proved that the addition defined in Step 4 satisfies Axioms (A) of Definition 1.12, it follows that Proposition 1.14 is valid in R, and we can prove one of the requirements of Definition 1.17:

If $\alpha, \beta, \gamma \in R$ *and* $\beta < \gamma$, *then* $\alpha + \beta < \alpha + \gamma$.

Indeed, it is obvious from the definition of $+$ in R that $\alpha + \beta \subset \alpha + \gamma$; if we had $\alpha + \beta = \alpha + \gamma$, the cancellation law (Proposition 1.14) would imply $\beta = \gamma$.

It also follows that $\alpha > 0^*$ if and only if $-\alpha < 0^*$.

Step 6 Multiplication is a little more bothersome than addition in the present context, since products of negative rationals are positive. For this reason we confine ourselves first to R^+, the set of all $\alpha \in R$ with $\alpha > 0^*$.

If $\alpha \in R^+$ and $\beta \in R^+$, we define $\alpha\beta$ to be the set of all p such that $p \leq rs$ for some choice of $r \in \alpha$, $s \in \beta$, $r > 0$, $s > 0$.

We define 1^* to be the set of all $q < 1$.

Then the axioms (M) and (D) of Definition 1.12 hold, with R^+ in place of F, and with 1^* in the role of 1.

The proofs are so similar to the ones given in detail in Step 4 that we omit them.

Note, in particular, that the second requirement of Definition 1.17 holds: If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta > 0^*$.

Step 7 We complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*, \beta > 0^*, \\ -[\alpha \cdot (-\beta)] & \text{if } \alpha > 0^*, \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6.

Having proved (in Step 6) that the axioms (M) hold in R^+, it is now perfectly simple to prove them in R, by repeated application of the identity $\gamma = -(-\gamma)$ which is part of Proposition 1.14. (See Step 5.)

The proof of the distributive law

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

breaks into cases. For instance, suppose $\alpha > 0^*$, $\beta < 0^*$, $\beta + \gamma > 0^*$. Then $\gamma = (\beta + \gamma) + (-\beta)$, and (since we already know that the distributive law holds in R^+)

$$\alpha\gamma = \alpha(\beta + \gamma) + \alpha \cdot (-\beta).$$

But $\alpha \cdot (-\beta) = -(\alpha\beta)$. Thus

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma).$$

The other cases are handled in the same way.

We have now completed the proof that R is an ordered field with the least-upper-bound property.

Step 8 We associate with each $r \in Q$ the set r^* which consists of all $p \in Q$ such that $p < r$. It is clear that each r^* is a cut; that is, $r^* \in R$. These cuts satisfy the following relations:

(*a*) $r^* + s^* = (r + s)^*$,

(*b*) $r^* s^* = (rs)^*$,

(*c*) $r^* < s^*$ *if and only if* $r < s$.

To prove (*a*), choose $p \in r^* + s^*$. Then $p = u + v$, where $u < r$, $v < s$. Hence $p < r + s$, which says that $p \in (r + s)^*$.

Conversely, suppose $p \in (r + s)^*$. Then $p < r + s$. Choose t so that $2t = r + s - p$, put

$$r' = r - t, s' = s - t.$$

Then $r' \in r^*$, $s' \in s^*$, and $p = r' + s'$, so that $p \in r^* + s^*$.

This proves (*a*). The proof of (*b*) is similar.

If $r < s$ then $r \in s^*$, but $r \notin r^*$; hence $r^* < s^*$.

If $r^* < s^*$, then there is a $p \in s^*$ such that $p \notin r^*$. Hence $r \leq p < s$, so that $r < s$.

This proves (*c*).

Step 9 We saw in Step 8 that the replacement of the rational numbers r by the corresponding "rational cuts" $r^* \in R$ preserves sums, products, and order. This fact may be expressed by saying that the ordered field Q is *isomorphic* to the ordered field Q^* whose elements are the rational cuts. Of course, r^* is by no means the same as r, but the properties we are concerned with (arithmetic and order) are the same in the two fields.

It is this identification of Q with Q^* which allows us to regard Q as a subfield of R.

The second part of Theorem 1.19 is to be understood in terms of this identification. Note that the same phenomenon occurs when the real numbers are regarded as a subfield of the complex field, and it also occurs at a much more elementary level, when the integers are identified with a certain subset of Q.

It is a fact, which we will not prove here, that *any two ordered fields with the least-upper-bound property are isomorphic.* The first part of Theorem 1.19 therefore characterizes the real field R completely.

The books by Landau and Thurston cited in the Bibliography are entirely devoted to number systems. Chapter 1 of Knopp's book contains a more leisurely description of how R can be obtained from Q. Another construction, in which each real number is defined to be an equivalence class of Cauchy sequences of rational numbers (see Chap. 3), is carried out in Sec. 5 of the book by Hewitt and Stromberg.

The cuts in Q which we used here were invented by Dedekind. The construction of R from Q by means of Cauchy sequences is due to Cantor. Both Cantor and Dedekind published their constructions in 1872.

EXERCISES

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

2. Prove that there is no rational number whose square is 12.

3. Prove Proposition 1.15.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

6. Fix $b > 1$.

(*a*) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(*b*) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(*c*) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(*d*) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b.*)

(*a*) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(*b*) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(*c*) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(*d*) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (*c*) with $t = y \cdot b^{-w}$.

(*e*) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(*f*) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(*g*) Prove that this x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

15. Under what conditions does equality hold in the Schwarz inequality?

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically, as a statement about parallelograms.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.

(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – REAL ANALYSIS – SMT1502

BASIC TOPOLOGY

FINITE, COUNTABLE, AND UNCOUNTABLE SETS

We begin this section with a definition of the function concept.

2.1 Definition Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

2.2 Definition Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$

such that $f(x) = y$. If, for each $y \in B, f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

2.3 Definition If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.
It is symmetric: If $A \sim B$, then $B \sim A$.
It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

2.4 Definition For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(*a*) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
(*b*) A is *infinite* if A is not finite.
(*c*) A is *countable* if $A \sim J$.
(*d*) A is *uncountable* if A is neither finite nor countable.
(*e*) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

2.5 Example Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J:

$$\begin{array}{ll} A: & 0, 1, -1, 2, -2, 3, -3, \ldots \\ J: & 1, 2, 3, 4, 5, 6, 7, \ldots \end{array}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \dfrac{n}{2} & (n \text{ even}), \\ -\dfrac{n-1}{2} & (n \text{ odd}). \end{cases}$$

2.6 Remark A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.5, in which J is a proper subset of A.

In fact, we could replace Definition 2.4(b) by the statement: A is infinite if A is equivalent to one of its proper subsets.

2.7 Definition By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A*, or a *sequence of elements of A*.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

2.8 Theorem *Every infinite subset of a countable set A is countable.*

Proof Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots)$, we obtain a 1-1 correspondence between E and J.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

2.9 Definition Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{1}$$

If A consists of the integers $1, 2, \ldots, n$, one usually writes

$$S = \bigcup_{m=1}^{n} E_m \tag{2}$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{3}$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{4}$$

The symbol ∞ in (4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition 1.23.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha, \tag{5}$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

2.10 Examples

(*a*) Suppose E_1 consists of $1, 2, 3$ and E_2 consists of $2, 3, 4$. Then $E_1 \cup E_2$ consists of $1, 2, 3, 4$, whereas $E_1 \cap E_2$ consists of $2, 3$.

(*b*) Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Then

(i) $$E_x \subset E_z \text{ if and only if } 0 < x \leq z \leq 1;$$

(ii) $$\bigcup_{x \in A} E_x = E_1;$$

(iii) $$\bigcap_{x \in A} E_x \text{ is empty};$$

(i) and (ii) are clear. To prove (iii), we note that for every $y > 0$, $y \notin E_x$ if $x < y$. Hence $y \notin \bigcap_{x \in A} E_x$.

2.11 Remarks Many properties of unions and intersections are quite similar to those of sums and products; in fact, the words sum and product were sometimes used in this connection, and the symbols Σ and Π were written in place of $\bigcup$ and $\bigcap$.

The commutative and associative laws are trivial:

(8) $$A \cup B = B \cup A; \qquad A \cap B = B \cap A.$$

(9) $$(A \cup B) \cup C = A \cup (B \cup C); \qquad (A \cap B) \cap C = A \cap (B \cap C).$$

Thus the omission of parentheses in (3) and (6) is justified.

The distributive law also holds:

(10) $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

To prove this, let the left and right members of (10) be denoted by E and F, respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

(11) $$A \subset A \cup B,$$

(12) $$A \cap B \subset A.$$

If 0 denotes the empty set, then

(13) $$A \cup 0 = A, \qquad A \cap 0 = 0.$$

If $A \subset B$, then

(14) $$A \cup B = B, \qquad A \cap B = A.$$

2.12 Theorem *Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets, and put*

(15) $$S = \bigcup_{n=1}^{\infty} E_n .$$

Then S is countable.

Proof Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \ldots$, and consider the infinite array

(16) $$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

(17) $$x_{11};\ x_{21}, x_{12};\ x_{31}, x_{22}, x_{13};\ x_{41}, x_{32}, x_{23}, x_{14};\ \ldots$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable.

Corollary *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha .$$

Then T is at most countable.

For T is equivalent to a subset of (15).

2.13 Theorem *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

Proof That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable $(n = 2, 3, 4, \ldots)$. The elements of B_n are of the form

(18) $$(b, a) \qquad (b \in B_{n-1}, a \in A).$$

For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets. By Theorem 2.12, B_n is countable.

The theorem follows by induction.

Corollary *The set of all rational numbers is countable.*

Proof We apply Theorem 2.13, with $n = 2$, noting that every rational r is of the form b/a, where a and b are integers. The set of pairs (a, b), and therefore the set of fractions b/a, is countable.

In fact, even the set of all algebraic numbers is countable (see Exercise 2).

That not all infinite sets are, however, countable, is shown by the next theorem.

2.14 Theorem *Let A be the set of all sequences whose elements are the digits* 0 *and* 1. *This set A is uncountable.*

The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$.

Proof Let E be a countable subset of A, and let E consist of the sequences $s_1, s_2, s_3, \ldots$. We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A.

We have shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd).

The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences $s_1, s_2, s_3, \ldots$ are placed in an array like (16), it is the elements on the diagonal which are involved in the construction of the new sequence.

Readers who are familiar with the binary representation of the real numbers (base 2 instead of 10) will notice that Theorem 2.14 implies that the set of all real numbers is uncountable. We shall give a second proof of this fact in Theorem 2.43.

METRIC SPACES

2.15 Definition A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(*a*) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
(*b*) $d(p, q) = d(q, p)$;
(*c*) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.16 Examples The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k, especially R^1 (the real line) and R^2 (the complex plane); the distance in R^k is defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \qquad (\mathbf{x}, \mathbf{y} \in R^k). \tag{19}$$

By Theorem 1.37, the conditions of Definition 2.15 are satisfied by (19).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (a) to (c) of Definition 2.15 hold for $p, q, r \in X$, they also hold if we restrict p, q, r to lie in Y.

Thus every subset of a euclidean space is a metric space. Other examples are the spaces $\mathscr{C}(K)$ and $\mathscr{L}^2(\mu)$, which are discussed in Chaps. 7 and 11, respectively.

2.17 Definition By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consists of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a *k-cell*. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball B* with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

We call a set $E \subset R^k$ *convex* if

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

For example, *balls are convex*. For if $|\mathbf{y} - \mathbf{x}| < r$, $|\mathbf{z} - \mathbf{x}| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda \mathbf{y} + (1 - \lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda |\mathbf{y} - \mathbf{x}| + (1 - \lambda)|\mathbf{z} - \mathbf{x}| < \lambda r + (1 - \lambda) r \\ &= r. \end{aligned}$$

The same proof applies to closed balls. It is also easy to see that k-cells are convex.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(*a*) *A neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(*b*) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(*c*) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(*d*) E is *closed* if every limit point of E is a point of E.
(*e*) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.
(*f*) E is *open* if every point of E is an interior point of E.
(*g*) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(*h*) E is *perfect* if E is closed and if every point of E is a limit point of E.
(*i*) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(*j*) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

Let us note that in R^1 neighborhoods are segments, whereas in R^2 neighborhoods are interiors of circles.

2.19 Theorem *Every neighborhood is an open set.*

Proof Consider a neighborhood $E = N_r(p)$, and let q be any point of E. Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E.

2.20 Theorem *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

[we use this notation to denote the smallest of the numbers $d(p, q_1), \ldots, d(p, q_n)$]. The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This contradiction establishes the theorem.

Corollary *A finite point set has no limit points.*

2.21 Examples Let us consider the following subsets of R^2:

(*a*) The set of all complex z such that $|z| < 1$.
(*b*) The set of all complex z such that $|z| \leq 1$.
(*c*) A nonempty finite set.
(*d*) The set of all integers.
(*e*) The set consisting of the numbers $1/n$ $(n = 1, 2, 3, \ldots)$. Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E; we wish to stress the difference between having a limit point and containing one.
(*f*) The set of all complex numbers (that is, R^2).
(*g*) The segment (a, b).

Let us note that (*d*), (*e*), (*g*) can be regarded also as subsets of R^1.
Some properties of these sets are tabulated below:

	Closed	*Open*	*Perfect*	*Bounded*
(*a*)	No	Yes	No	Yes
(*b*)	Yes	No	Yes	Yes
(*c*)	Yes	No	No	Yes
(*d*)	Yes	No	No	No
(*e*)	No	No	No	Yes
(*f*)	Yes	Yes	Yes	No
(*g*)	No		No	Yes

In (*g*), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2, but it is an open subset of R^1.

2.22 Theorem *Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c). \tag{20}$$

Proof Let A and B be the left and right members of (20). If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for any α, hence $x \in E_\alpha^c$ for every α, so that $x \in \bigcap E_\alpha^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_\alpha^c$ for every α, hence $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in (\bigcup_\alpha E_\alpha)^c$. Thus $B \subset A$.

It follows that $A = B$.

2.23 Theorem *A set E is open if and only if its complement is closed.*

Proof First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E, and E is open.

Next, suppose E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary *A set F is closed if and only if its complement is open.*

2.24 Theorem

(*a*) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
(*b*) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
(*c*) *For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.*
(*d*) *For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.*

Proof Put $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α. Since x is an interior point of G_α, x is also an interior point of G, and G is open. This proves (*a*).

By Theorem 2.22,

(21) $$\Big(\bigcap_\alpha F_\alpha\Big)^c = \bigcup_\alpha (F_\alpha^c),$$

and F_α^c is open, by Theorem 2.23. Hence (*a*) implies that (21) is open so that $\bigcap_\alpha F_\alpha$ is closed.

Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x, with radii r_i, such that $N_i \subset G_i$ $(i = 1, \ldots, n)$. Put

$$r = \min(r_1, \ldots, r_n),$$

and let N be the neighborhood of x of radius r. Then $N \subset G_i$ for $i = 1, \ldots, n$, so that $N \subset H$, and H is open.

By taking complements, (*d*) follows from (*c*):

$$\Big(\bigcup_{i=1}^n F_i\Big)^c = \bigcap_{i=1}^n (F_i^c).$$

2.25 Examples In parts (*c*) and (*d*) of the preceding theorem, the finiteness of the collections is essential. For let G_n be the segment $\left(-\frac{1}{n}, \frac{1}{n}\right)$ $(n = 1, 2, 3, \ldots)$. Then G_n is an open subset of R^1. Put $G = \bigcap_{n=1}^{\infty} G_n$. Then G consists of a single point (namely, $x = 0$) and is therefore not an open subset of R^1.

Thus the intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem *If X is a metric space and $E \subset X$, then*

(*a*) *$\bar{E}$ is closed,*
(*b*) *$E = \bar{E}$ if and only if E is closed,*
(*c*) *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

By (*a*) and (*c*), $\bar{E}$ is the *smallest* closed subset of X that contains E.

Proof
(*a*) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.
(*b*) If $E = \bar{E}$, (*a*) implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions 2.18(*d*) and 2.26], hence $\bar{E} = E$.
(*c*) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

2.28 Theorem *Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

Compare this with the examples in Sec. 1.9.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in \bar{E}$.

2.29 Remark Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p, q) < r$, $q \in X$ imply that $q \in E$. But we have already observed (Sec. 2.16) that Y is also a metric space, so that our definitions may equally well be made within Y. To be quite explicit, let us say that E is *open relative to* Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Example 2.21(*g*) showed that a set may be open relative to Y without being an open subset of X. However, there is a simple relation between these concepts, which we now state.

2.30 Theorem *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof Suppose E is open relative to Y. To each $p \in E$ there is a positive number r_p such that the conditions $d(p, q) < r_p$, $q \in Y$ imply that $q \in E$. Let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$, and define

$$G = \bigcup_{p \in E} V_p .$$

Then G is an open subset of X, by Theorems 2.19 and 2.24.

Since $p \in V_p$ for all $p \in E$, it is clear that $E \subset G \cap Y$.

By our choice of V_p, we have $V_p \cap Y \subset E$ for every $p \in E$, so that $G \cap Y \subset E$. Thus $E = G \cap Y$, and one half of the theorem is proved.

Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subset G$. Then $V_p \cap Y \subset E$, so that E is open relative to Y.

COMPACT SETS

2.31 Definition By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} .$$

The notion of compactness is of great importance in analysis, especially in connection with continuity (Chap. 4).

It is clear that every finite set is compact. The existence of a large class of infinite compact sets in R^k will follow from Theorem 2.41.

We observed earlier (in Sec. 2.29) that if $E \subset Y \subset X$, then E may be open relative to Y without being open relative to X. The property of being open thus depends on the space in which E is embedded. The same is true of the property of being closed.

Compactness, however, behaves better, as we shall now see. To formulate the next theorem, let us say, temporarily, that K is compact relative to X if the requirements of Definition 2.32 are met.

2.33 Theorem *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

By virtue of this theorem we are able, in many situations, to regard compact sets as metric spaces in their own right, without paying any attention to any embedding space. In particular, although it makes little sense to talk of *open* spaces, or of *closed* spaces (every metric space X is an open subset of itself, and is a closed subset of itself), it does make sense to talk of *compact* metric spaces.

Proof Suppose K is compact relative to X, and let $\{V_\alpha\}$ be a collection of sets, open relative to Y, such that $K \subset \bigcup_\alpha V_\alpha$. By theorem 2.30, there are sets G_α, open relative to X, such that $V_\alpha = Y \cap G_\alpha$, for all α; and since K is compact relative to X, we have

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \tag{22}$$

for some choice of finitely many indices $\alpha_1, \ldots, \alpha_n$. Since $K \subset Y$, (22) implies

$$K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}. \tag{23}$$

This proves that K is compact relative to Y.

Conversely, suppose K is compact relative to Y, let $\{G_\alpha\}$ be a collection of open subsets of X which covers K, and put $V_\alpha = Y \cap G_\alpha$. Then (23) will hold for some choice of $\alpha_1, \ldots, \alpha_n$; and since $V_\alpha \subset G_\alpha$, (23) implies (22).

This completes the proof.

2.34 Theorem *Compact subsets of metric spaces are closed.*

Proof Let K be a compact subset of a metric space X. We shall prove that the complement of K is an open subset of X.

Suppose $p \in X$, $p \notin K$. If $q \in K$, let V_q and W_q be neighborhoods of p and q, respectively, of radius less than $\frac{1}{2}d(p, q)$ [see Definition 2.18(*a*)]. Since K is compact, there are finitely many points $q_1, \ldots, q_n$ in K such that

$$K \subset W_{q_1} \cup \cdots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \cdots \cap V_{q_n}$, then V is a neighborhood of p which does not intersect W. Hence $V \subset K^c$, so that p is an interior point of K^c. The theorem follows.

2.35 Theorem *Closed subsets of compact sets are compact.*

Proof Suppose $F \subset K \subset X$, F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F.

Corollary *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof Theorems 2.24(*b*) and 2.34 show that $F \cap K$ is closed; since $F \cap K \subset K$, Theorem 2.35 shows that $F \cap K$ is compact.

2.36 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Proof Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α. Then the sets G_α form an open cover of K_1; and since K_1 is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. But this means that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$$

is empty, in contradiction to our hypothesis.

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty K_n$ is not empty.*

2.37 Theorem *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q, if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.

2.38 Theorem *If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty I_n$ is not empty.*

Proof If $I_n = [a_n, b_n]$, let E be the set of all a_n. Then E is nonempty and bounded above (by b_1). Let x be the sup of E. If m and n are positive integers, then

$$a_n \le a_{m+n} \le b_{m+n} \le b_m,$$

so that $x \le b_m$ for each m. Since it is obvious that $a_m \le x$, we see that $x \in I_m$ for $m = 1, 2, 3, \ldots$.

2.39 Theorem *Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty I_n$ is not empty.*

Proof Let I_n consist of all points $\mathbf{x} = (x_1, \ldots, x_k)$ such that

$$a_{n,j} \le x_j \le b_{n,j} \qquad (1 \le j \le k; n = 1, 2, 3, \ldots),$$

and put $I_{n,j} = [a_{n,j}, b_{n,j}]$. For each j, the sequence $\{I_{n,j}\}$ satisfies the hypotheses of Theorem 2.38. Hence there are real numbers $x_j^*(1 \le j \le k)$ such that

$$a_{n,j} \le x_j^* \le b_{n,j} \qquad (1 \le j \le k; n = 1, 2, 3, \ldots).$$

Setting $\mathbf{x}^* = (x_1^*, \ldots, x_k^*)$, we see that $\mathbf{x}^* \in I_n$ for $n = 1, 2, 3, \ldots$. The theorem follows.

2.40 Theorem *Every k-cell is compact.*

Proof Let I be a k-cell, consisting of all points $\mathbf{x} = (x_1, \ldots, x_k)$ such that $a_j \le x_j \le b_j$ $(1 \le j \le k)$. Put

$$\delta = \left\{\sum_1^k (b_j - a_j)^2\right\}^{1/2}.$$

Then $|\mathbf{x} - \mathbf{y}| \le \delta$, if $\mathbf{x} \in I$, $\mathbf{y} \in I$.

Suppose, to get a contradiction, that there exists an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I. Put $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ then determine 2^k k-cells Q_i whose union is I. At least one of these sets Q_i, call it I_1, cannot be covered by any finite subcollection of $\{G_\alpha\}$ (otherwise I could be so covered). We next subdivide I_1 and continue the process. We obtain a sequence $\{I_n\}$ with the following properties:

(*a*) $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$;
(*b*) I_n is not covered by any finite subcollection of $\{G_\alpha\}$;
(*c*) if $\mathbf{x} \in I_n$ and $\mathbf{y} \in I_n$, then $|\mathbf{x} - \mathbf{y}| \le 2^{-n}\delta$.

By (*a*) and Theorem 2.39, there is a point $\mathbf{x}^*$ which lies in every I_n. For some α, $\mathbf{x}^* \in G_\alpha$. Since G_α is open, there exists $r > 0$ such that $|\mathbf{y} - \mathbf{x}^*| < r$ implies that $\mathbf{y} \in G_\alpha$. If n is so large that $2^{-n}\delta < r$ (there is such an n, for otherwise $2^n \le \delta/r$ for all positive integers n, which is absurd since R is archimedean), then (*c*) implies that $I_n \subset G_\alpha$, which contradicts (*b*).

This completes the proof.

The equivalence of (*a*) and (*b*) in the next theorem is known as the Heine-Borel theorem.

2.41 Theorem *If a set E in R^k has one of the following three properties, then it has the other two:*

(*a*) *E is closed and bounded.*
(*b*) *E is compact.*
(*c*) *Every infinite subset of E has a limit point in E.*

Proof If (*a*) holds, then $E \subset I$ for some k-cell I, and (*b*) follows from Theorems 2.40 and 2.35. Theorem 2.37 shows that (*b*) implies (*c*). It remains to be shown that (*c*) implies (*a*).

If E is not bounded, then E contains points $\mathbf{x}_n$ with

$$|\mathbf{x}_n| > n \qquad (n = 1, 2, 3, \ldots).$$

The set S consisting of these points $\mathbf{x}_n$ is infinite and clearly has no limit point in R^k, hence has none in E. Thus (*c*) implies that E is bounded.

If E is not closed, then there is a point $\mathbf{x}_0 \in R^k$ which is a limit point of E but not a point of E. For $n = 1, 2, 3, \ldots,$ there are points $\mathbf{x}_n \in E$ such that $|\mathbf{x}_n - \mathbf{x}_0| < 1/n$. Let S be the set of these points $\mathbf{x}_n$. Then S is infinite (otherwise $|\mathbf{x}_n - \mathbf{x}_0|$ would have a constant positive value, for infinitely many n), S has $\mathbf{x}_0$ as a limit point, and S has no other limit point in R^k. For if $\mathbf{y} \in R^k$, $\mathbf{y} \neq \mathbf{x}_0$, then

$$\begin{aligned} |\mathbf{x}_n - \mathbf{y}| &\geq |\mathbf{x}_0 - \mathbf{y}| - |\mathbf{x}_n - \mathbf{x}_0| \\ &\geq |\mathbf{x}_0 - \mathbf{y}| - \frac{1}{n} \geq \frac{1}{2} |\mathbf{x}_0 - \mathbf{y}| \end{aligned}$$

for all but finitely many n; this shows that $\mathbf{y}$ is not a limit point of S (Theorem 2.20).

Thus S has no limit point in E; hence E must be closed if (*c*) holds.

We should remark, at this point, that (*b*) and (*c*) are equivalent in any metric space (Exercise 26) but that (*a*) does not, in general, imply (*b*) and (*c*). Examples are furnished by Exercise 16 and by the space $\mathscr{L}^2$, which is discussed in Chap. 11.

2.42 Theorem (Weierstrass) *Every bounded infinite subset of R^k has a limit point in R^k.*

Proof Being bounded, the set E in question is a subset of a k-cell $I \subset R^k$. By Theorem 2.40, I is compact, and so E has a limit point in I, by Theorem 2.37.

PERFECT SETS

2.43 Theorem *Let P be a nonempty perfect set in R^k. Then P is uncountable.*

Proof Since P has limit points, P must be infinite. Suppose P is countable, and denote the points of P by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$. We shall construct a sequence $\{V_n\}$ of neighborhoods, as follows.

Let V_1 be any neighborhood of $\mathbf{x}_1$. If V_1 consists of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}_1| < r$, the closure $\bar{V}_1$ of V_1 is the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}_1| \le r$.

Suppose V_n has been constructed, so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P, there is a neighborhood V_{n+1} such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\mathbf{x}_n \notin \bar{V}_{n+1}$, (iii) $V_{n+1} \cap P$ is not empty. By (iii), V_{n+1} satisfies our induction hypothesis, and the construction can proceed.

Put $K_n = \bar{V}_n \cap P$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Since $\mathbf{x}_n \notin K_{n+1}$, no point of P lies in $\bigcap_1^\infty K_n$. Since $K_n \subset P$, this implies that $\bigcap_1^\infty K_n$ is empty. But each K_n is nonempty, by (iii), and $K_n \supset K_{n+1}$, by (i); this contradicts the Corollary to Theorem 2.36.

Corollary *Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.*

2.44 The Cantor set The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.

Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$[0, \tfrac{1}{3}]\ [\tfrac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \tfrac{1}{9}], [\tfrac{2}{9}, \tfrac{3}{9}], [\tfrac{6}{9}, \tfrac{7}{9}], [\tfrac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(*a*) $E_1 \supset E_2 \supset E_3 \supset \cdots$;
(*b*) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and Theorem 2.36 shows that P is not empty.

No segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \tag{24}$$

where k and m are positive integers, has a point in common with P. Since every segment (α, β) contains a segment of the form (24), if

$$3^{-m} < \frac{\beta - \alpha}{6},$$

P contains no segment.

To show that P is perfect, it is enough to show that P contains no isolated point. Let $x \in P$, and let S be any segment containing x. Let I_n be that interval of E_n which contains x. Choose n large enough, so that $I_n \subset S$. Let x_n be an endpoint of I_n, such that $x_n \neq x$.

It follows from the construction of P that $x_n \in P$. Hence x is a limit point of P, and P is perfect.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero (the concept of measure will be discussed in Chap. 11).

CONNECTED SETS

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ *are* separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem *A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.*

Proof If there exist $x \in E$, $y \in E$, and some $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where

$$A_z = E \cap (-\infty, z), \qquad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence E is not connected.

To prove the converse, suppose E is not connected. Then there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup (A \cap [x, y]).$$

By Theorem 2.28, $z \in \bar{A}$; hence $z \notin B$. In particular, $x \leq z < y$.

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \notin \bar{B}$, hence there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

EXERCISES

1. Prove that the empty set is a subset of every set.
2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that
$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$
Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with
$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$
3. Prove that there exist real numbers which are not algebraic.
4. Is the set of all irrational real numbers countable?
5. Construct a bounded set of real numbers with exactly three limit points.
6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?
7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
(*a*) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.
(*b*) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.
Show, by an example, that this inclusion can be proper.
8. Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.
9. Let E° denote the set of all interior points of a set E. [See Definition 2.18(*e*); E° is called the *interior* of E.]
(*a*) Prove that E° is always open.
(*b*) Prove that E is open if and only if $E^\circ = E$.
(*c*) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
(*d*) Prove that the complement of E° is the closure of the complement of E.
(*e*) Do E and $\bar{E}$ always have the same interiors?
(*f*) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in R^1$ and $y \in R^1$, define

$$d_1(x, y) = (x-y)^2,$$
$$d_2(x, y) = \sqrt{|x-y|},$$
$$d_3(x, y) = |x^2 - y^2|,$$
$$d_4(x, y) = |x - 2y|,$$
$$d_5(x, y) = \frac{|x-y|}{1+|x-y|}.$$

Determine, for each of these, whether it is a metric or not.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).
13. Construct a compact set of real numbers whose limit points form a countable set.
14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.
15. Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."
16. Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?
17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?
18. Is there a nonempty perfect set in R^1 which contains no rational number?
19. (*a*) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

    (*b*) Prove the same for disjoint open sets.

    (*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

    (*d*) Prove that every connected metric space with at least two points is uncountable. *Hint*: Use (*c*).
20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2.)
21. Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(*a*) Prove that A_0 and B_0 are separated subsets of R^1.

(*b*) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(*c*) Prove that every convex subset of R^k is connected.

22. A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ ($n = 1, 2, 3, \ldots$), and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint*: Use Exercise 27.

29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots,$ then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – REAL ANALYSIS – SMT1502

NUMERICAL SEQUENCES AND SERIES

As the title indicates, this chapter will deal primarily with sequences and series of complex numbers. The basic facts about convergence, however, are just as easily explained in a more general setting. The first three sections will therefore be concerned with sequences in euclidean spaces, or even in metric spaces.

CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here d denotes the distance in X.)

In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ [see Theorem 3.2(b)], and we write $p_n \to p$, or

$$\lim_{n \to \infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge*.

It might be well to point out that our definition of "convergent sequence" depends not only on $\{p_n\}$ but also on X; for instance, the sequence $\{1/n\}$ converges in R^1 (to 0), but fails to converge in the set of all positive real numbers [with $d(x, y) = |x - y|$]. In cases of possible ambiguity, we can be more precise and specify "convergent in X" rather than "convergent."

We recall that the set of all points p_n ($n = 1, 2, 3, \ldots$) is the *range* of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

As examples, consider the following sequences of complex numbers (that is, $X = R^2$):

(*a*) If $s_n = 1/n$, then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.
(*b*) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.
(*c*) If $s_n = 1 + [(-1)^n/n]$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.
(*d*) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.
(*e*) If $s_n = 1$ ($n = 1, 2, 3, \ldots$), then $\{s_n\}$ converges to 1, is bounded, and has finite range.

We now summarize some important properties of convergent sequences in metric spaces.

3.2 Theorem *Let $\{p_n\}$ be a sequence in a metric space X.*

(*a*) *$\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n* for *all but finitely many n.*
(*b*) *If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.*
(*c*) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*
(*d*) *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Proof (*a*) Suppose $p_n \to p$ and let V be a neighborhood of p. For some $\varepsilon > 0$, the conditions $d(q, p) < \varepsilon, q \in X$ imply $q \in V$. Corresponding to this ε, there exists N such that $n \geq N$ implies $d(p_n, p) < \varepsilon$. Thus $n \geq N$ implies $p_n \in V$.

Conversely, suppose every neighborhood of p contains all but finitely many of the p_n. Fix $\varepsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \varepsilon$. By assumption, there exists N (corresponding to this V) such that $p_n \in V$ if $n \geq N$. Thus $d(p_n, p) < \varepsilon$ if $n \geq N$; hence $p_n \to p$.

(*b*) Let $\varepsilon > 0$ be given. There exist integers N, N' such that

$$n \geq N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2},$$

$$n \geq N' \quad \text{implies} \quad d(p_n, p') < \frac{\varepsilon}{2}.$$

Hence if $n \geq \max(N, N')$, we have

$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon.$$

Since ε was arbitrary, we conclude that $d(p, p') = 0$.

(*c*) Suppose $p_n \to p$. There is an integer N such that $n > N$ implies $d(p_n, p) < 1$. Put

$$r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}.$$

Then $d(p_n, p) \leq r$ for $n = 1, 2, 3, \ldots$.

(*d*) For each positive integer n, there is a point $p_n \in E$ such that $d(p_n, p) < 1/n$. Given $\varepsilon > 0$, choose N so that $N\varepsilon > 1$. If $n > N$, it follows that $d(p_n, p) < \varepsilon$. Hence $p_n \to p$.

This completes the proof.

For sequences in R^k we can study the relation between convergence, on the one hand, and the algebraic operations on the other. We first consider sequences of complex numbers.

3.3 Theorem *Suppose* $\{s_n\}$, $\{t_n\}$ *are complex sequences, and* $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. *Then*

(*a*) $\lim_{n\to\infty} (s_n + t_n) = s + t$;

(*b*) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$, *for any number* c;

(*c*) $\lim_{n\to\infty} s_n t_n = st$;

(*d*) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, *provided* $s_n \neq 0$ $(n = 1, 2, 3, \ldots)$, and $s \neq 0$.

Proof

(*a*) Given $\varepsilon > 0$, there exist integers N_1, N_2 such that

$$n \geq N_1 \quad \text{implies} \quad |s_n - s| < \frac{\varepsilon}{2},$$

$$n \geq N_2 \quad \text{implies} \quad |t_n - t| < \frac{\varepsilon}{2}.$$

If $N = \max(N_1, N_2)$, then $n \geq N$ implies

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon.$$

This proves (*a*). The proof of (*b*) is trivial.

(*c*) We use the identity

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s). \tag{1}$$

Given $\varepsilon > 0$, there are integers N_1, N_2 such that

$$n \geq N_1 \quad \text{implies} \quad |s_n - s| < \sqrt{\varepsilon},$$
$$n \geq N_2 \quad \text{implies} \quad |t_n - t| < \sqrt{\varepsilon}.$$

If we take $N = \max(N_1, N_2)$, $n \geq N$ implies

$$|(s_n - s)(t_n - t)| < \varepsilon,$$

so that

$$\lim_{n \to \infty} (s_n - s)(t_n - t) = 0.$$

We now apply (*a*) and (*b*) to (1), and conclude that

$$\lim_{n \to \infty} (s_n t_n - st) = 0.$$

(*d*) Choosing m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \tfrac{1}{2}|s| \qquad (n \geq m).$$

Given $\varepsilon > 0$, there is an integer $N > m$ such that $n \geq N$ implies

$$|s_n - s| < \tfrac{1}{2}|s|^2 \varepsilon.$$

Hence, for $n \geq N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \varepsilon.$$

3.4 Theorem

(*a*) *Suppose* $\mathbf{x}_n \in R^k$ $(n = 1, 2, 3, \ldots)$ *and*

$$\mathbf{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n}).$$

Then $\{\mathbf{x}_n\}$ *converges to* $\mathbf{x} = (\alpha_1, \ldots, \alpha_k)$ *if and only if*

$$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j \qquad (1 \leq j \leq k). \tag{2}$$

(*b*) *Suppose* $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ *are sequences in* R^k, $\{\beta_n\}$ *is a sequence of real numbers, and* $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. *Then*

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \qquad \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \qquad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta\mathbf{x}.$$

Proof

(*a*) If $\mathbf{x}_n \to \mathbf{x}$, the inequalities

$$|\alpha_{j,n} - \alpha_j| \le |\mathbf{x}_n - \mathbf{x}|,$$

which follow immediately from the definition of the norm in R^k, show that (2) holds.

Conversely, if (2) holds, then to each $\varepsilon > 0$ there corresponds an integer N such that $n \ge N$ implies

$$|\alpha_{j,n} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}} \qquad (1 \le j \le k).$$

Hence $n \ge N$ implies

$$|\mathbf{x}_n - \mathbf{x}| = \left\{\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\right\}^{1/2} < \varepsilon,$$

so that $\mathbf{x}_n \to \mathbf{x}$. This proves (*a*).

Part (*b*) follows from (*a*) and Theorem 3.3.

SUBSEQUENCES

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p. We leave the details of the proof to the reader.

3.6 Theorem

(*a*) *If* $\{p_n\}$ *is a sequence in a compact metric space* X, *then some subsequence of* $\{p_n\}$ *converges to a point of* X.

(*b*) *Every bounded sequence in* R^k *contains a convergent subsequence.*

Proof

(*a*) Let E be the range of $\{p_n\}$. If E is finite then there is a $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \cdots$, such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The subsequence $\{p_{n_i}\}$ so obtained converges evidently to p.

If E is infinite, Theorem 2.37 shows that E has a limit point $p \in X$. Choose n_1 so that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{i-1}$, we see from Theorem 2.20 that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p.

(*b*) This follows from (*a*), since Theorem 2.41 implies that every bounded subset of R^k lies in a compact subset of R^k.

3.7 Theorem *The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.*

Proof Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^*. We have to show that $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$. (If no such n_1 exists, then E^* has only one point, and there is nothing to prove.) Put $\delta = d(q, p_{n_1})$. Suppose $n_1, \ldots, n_{i-1}$ are chosen. Since q is a limit point of E^*, there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$. Since $x \in E^*$, there is an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Thus

$$d(q, p_{n_i}) \leq 2^{1-i}\delta$$

for $i = 1, 2, 3, \ldots$. This says that $\{p_{n_i}\}$ converges to q. Hence $q \in E^*$.

CAUCHY SEQUENCES

3.8 Definition A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

In our discussion of Cauchy sequences, as well as in other situations which will arise later, the following geometric concept will be useful.

3.9 Definition Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the *diameter* of E.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \ldots,$ it is clear from the two preceding definitions that *$\{p_n\}$ is a Cauchy sequence if and only if*

$$\lim_{N\to\infty} \operatorname{diam} E_N = 0.$$

3.10 Theorem

(*a*) *If $\bar{E}$ is the closure of a set E in a metric space X, then*

$$\operatorname{diam} \bar{E} = \operatorname{diam} E.$$

(*b*) *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$ and if*

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Proof

(*a*) Since $E \subset \bar{E}$, it is clear that

$$\operatorname{diam} E \le \operatorname{diam} \bar{E}.$$

Fix $\varepsilon > 0$, and choose $p \in \bar{E}, q \in \bar{E}$. By the definition of $\bar{E}$, there are points p', q', in E such that $d(p, p') < \varepsilon$, $d(q, q') < \varepsilon$. Hence

$$\begin{aligned} d(p, q) &\le d(p, p') + d(p' q') + d(q', q) \\ &< 2\varepsilon + d(p', q') \le 2\varepsilon + \operatorname{diam} E. \end{aligned}$$

It follows that

$$\operatorname{diam} \bar{E} \le 2\varepsilon + \operatorname{diam} E,$$

and since ε was arbitrary, (*a*) is proved.
(*b*) Put $K = \bigcap_1^\infty K_n$. By Theorem 2.36, K is not empty. If K contains more than one point, then $\operatorname{diam} K > 0$. But for each n, $K_n \supset K$, so that $\operatorname{diam} K_n \ge \operatorname{diam} K$. This contradicts the assumption that $\operatorname{diam} K_n \to 0$.

3.11 Theorem

(*a*) *In any metric space X, every convergent sequence is a Cauchy sequence.*
(*b*) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.*
(*c*) *In R^k, every Cauchy sequence converges.*

Note: The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Thus Theorem 3.11(*b*) may enable us to decide whether or not a given sequence converges without knowledge of the limit to which it may converge.

The fact (contained in Theorem 3.11) that a sequence converges in R^k if and only if it is a Cauchy sequence is usually called the ***Cauchy criterion*** for convergence.

Proof

(*a*) If $p_n \to p$ and if $\varepsilon > 0$, there is an integer N such that $d(p, p_n) < \varepsilon$ for all $n \geq N$. Hence

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\varepsilon$$

as soon as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

(*b*) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N = 1, 2, 3, \ldots$, let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \ldots$. Then

$$\lim_{N\to\infty} \operatorname{diam} \bar{E}_N = 0, \tag{3}$$

by Definition 3.9 and Theorem 3.10(*a*). Being a closed subset of the compact space X, each $\bar{E}_N$ is compact (Theorem 2.35). Also $E_N \supset E_{N+1}$, so that $\bar{E}_N \supset \bar{E}_{N+1}$.

Theorem 3.10(*b*) shows now that there is a unique $p \in X$ which lies in every $\bar{E}_N$.

Let $\varepsilon > 0$ be given. By (3) there is an integer N_0 such that $\operatorname{diam} \bar{E}_N < \varepsilon$ if $N \geq N_0$. Since $p \in \bar{E}_N$, it follows that $d(p, q) < \varepsilon$ for every $q \in \bar{E}_N$, hence for every $q \in E_N$. In other words, $d(p, p_n) < \varepsilon$ if $n \geq N_0$. This says precisely that $p_n \to p$.

(*c*) Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in R^k. Define E_N as in (*b*), with $\mathbf{x}_i$ in place of p_i. For some N, $\operatorname{diam} E_N < 1$. The range of $\{\mathbf{x}_n\}$ is the union of E_N and the finite set $\{\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$. Hence $\{\mathbf{x}_n\}$ is bounded. Since every bounded subset of R^k has compact closure in R^k (Theorem 2.41), (*c*) follows from (*b*).

3.12 Definition A metric space in which every Cauchy sequence converges is said to be *complete*.

Thus Theorem 3.11 says that *all compact metric spaces and all Euclidean spaces are complete*. Theorem 3.11 implies also that *every closed subset E of a complete metric space X is complete*. (Every Cauchy sequence in E is a Cauchy sequence in X, hence it converges to some $p \in X$, and actually $p \in E$ since E is closed.) An example of a metric space which is not complete is the space of all rational numbers, with $d(x, y) = |x - y|$.

Theorem 3.2(*c*) and example (*d*) of Definition 3.1 show that convergent sequences are bounded, but that bounded sequences in R^k need not converge. However, there is one important case in which convergence *is* equivalent to boundedness; this happens for monotonic sequences in R^1.

3.13 Definition A sequence $\{s_n\}$ of real numbers is said to be

(*a*) *monotonically increasing* if $s_n \le s_{n+1}$ $(n = 1, 2, 3, \ldots)$;
(*b*) *monotonically decreasing* if $s_n \ge s_{n+1}$ $(n = 1, 2, 3, \ldots)$.

The class of monotonic sequences consists of the increasing and the decreasing sequences.

3.14 Theorem *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof Suppose $s_n \le s_{n+1}$ (the proof is analogous in the other case). Let E be the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let s be the least upper bound of E. Then

$$s_n \le s \qquad (n = 1, 2, 3, \ldots).$$

For every $\varepsilon > 0$, there is an integer N such that

$$s - \varepsilon < s_N \le s,$$

for otherwise $s - \varepsilon$ would be an upper bound of E. Since $\{s_n\}$ increases, $n \ge N$ therefore implies

$$s - \varepsilon < s_n \le s,$$

which shows that $\{s_n\}$ converges (to s).

The converse follows from Theorem 3.2(*c*).

UPPER AND LOWER LIMITS

3.15 Definition Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \ge N$ implies $s_n \ge M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \ge N$ implies $s_n \le M$, we write

$$s_n \to -\infty.$$

It should be noted that we now use the symbol $\rightarrow$ (introduced in Definition 3.1) for certain types of divergent sequences, as well as for convergent sequences, but that the definitions of convergence and of limit, given in Definition 3.1, are in no way changed.

3.16 Definition Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \rightarrow x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits as defined in Definition 3.5, plus possibly the numbers $+\infty$, $-\infty$.

We now recall Definitions 1.8 and 1.23 and put

$$s^* = \sup E,$$

$$s_* = \inf E.$$

The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\limsup_{n \to \infty} s_n = s^*, \qquad \liminf_{n \to \infty} s_n = s_*.$$

3.17 Theorem *Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in Definition* 3.16. *Then s^* has the following two properties:*

(*a*) $s^* \in E$.

(*b*) *If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.*

Moreover, s^* is the only number with the properties (*a*) *and* (*b*).

Of course, an analogous result is true for s_*.

Proof

(*a*) If $s^* = +\infty$, then E is not bounded above; hence $\{s_n\}$ is not bounded above, and there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \rightarrow +\infty$.

If s^* is real, then E is bounded above, and at least one subsequential limit exists, so that (*a*) follows from Theorems 3.7 and 2.28.

If $s^* = -\infty$, then E contains only one element, namely $-\infty$, and there is no subsequential limit. Hence, for any real M, $s_n > M$ for at most a finite number of values of n, so that $s_n \rightarrow -\infty$.

This establishes (*a*) in all cases.

(*b*) Suppose there is a number $x > s^*$ such that $s_n \geq x$ for infinitely many values of n. In that case, there is a number $y \in E$ such that $y \geq x > s^*$, contradicting the definition of s^*.

Thus s^* satisfies (*a*) and (*b*).

To show the uniqueness, suppose there are two numbers, p and q, which satisfy (*a*) and (*b*), and suppose $p < q$. Choose x such that $p < x < q$. Since p satisfies (*b*), we have $s_n < x$ for $n \geq N$. But then q cannot satisfy (*a*).

3.18 Examples

(*a*) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n\to\infty} s_n = +\infty, \qquad \liminf_{n\to\infty} s_n = -\infty.$$

(*b*) Let $s_n = (-1^n)/[1 + (1/n)]$. Then

$$\limsup_{n\to\infty} s_n = 1, \qquad \liminf_{n\to\infty} s_n = -1.$$

(*c*) For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

We close this section with a theorem which is useful, and whose proof is quite trivial:

3.19 Theorem *If $s_n \le t_n$ for $n \ge N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n,$$

$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n.$$

SOME SPECIAL SEQUENCES

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \le x_n \le s_n$ for $n \ge N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

3.20 Theorem

(*a*) *If $p > 0$, then* $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(*b*) *If $p > 0$, then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(*c*) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(*d*) *If $p > 0$ and α is real, then* $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(*e*) *If $|x| < 1$, then* $\lim_{n\to\infty} x^n = 0$.

Proof

(*a*) Take $n > (1/\varepsilon)^{1/p}$. (Note that the archimedean property of the real number system is used here.)
(*b*) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem,

$$1 + nx_n \le (1 + x_n)^n = p,$$

so that

$$0 < x_n \le \frac{p-1}{n}.$$

Hence $x_n \to 0$. If $p = 1$, (*b*) is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals.
(*c*) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \ge 0$, and, by the binomial theorem,

$$n = (1 + x_n)^n \ge \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \le x_n \le \sqrt{\frac{2}{n-1}} \qquad (n \ge 2).$$

(*d*) Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1 + p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \qquad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha - k} \to 0$, by (*a*).
(*e*) Take $\alpha = 0$ in (*d*).

SERIES

In the remainder of this chapter, all sequences and series under consideration will be complex-valued, unless the contrary is explicitly stated. Extensions of some of the theorems which follow, to series with terms in R^k, are mentioned in Exercise 15.

3.21 Definition Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n \qquad (p \le q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n. \tag{4}$$

The symbol (4) we call an *infinite series*, or just a *series*. The numbers s_n are called the *partial sums* of the series. If $\{s_n\}$ converges to s, we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that *s is the limit of a sequence of sums*, and is not obtained simply by addition.

If $\{s_n\}$ diverges, the series is said to diverge.

Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} a_n. \tag{5}$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write Σa_n in place of (4) or (5).

It is clear that every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.

The Cauchy criterion (Theorem 3.11) can be restated in the following form:

3.22 Theorem *Σa_n converges if and only if for every $\varepsilon > 0$ there is an integer N such that*

$$\left| \sum_{k=n}^{m} a_k \right| \le \varepsilon \tag{6}$$

if $m \ge n \ge N$.

In particular, by taking $m = n$, (6) becomes

$$|a_n| \le \varepsilon \qquad (n \ge N).$$

In other words:

3.23 Theorem *If* Σa_n *converges, then* $\lim_{n\to\infty} a_n = 0$.

The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of Σa_n. For instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges; for the proof we refer to Theorem 3.28.

Theorem 3.14, concerning monotonic sequences, also has an immediate counterpart for series.

3.24 Theorem *A series of nonnegative*[1] *terms converges if and only if its partial sums form a bounded sequence.*

We now turn to a convergence test of a different nature, the so-called "comparison test."

3.25 Theorem

(*a*) *If* $|a_n| \le c_n$ *for* $n \ge N_0$, *where* N_0 *is some fixed integer, and if* Σc_n *converges, then* Σa_n *converges.*
(*b*) *If* $a_n \ge d_n \ge 0$ *for* $n \ge N_0$, *and if* Σd_n *diverges, then* Σa_n *diverges.*

Note that (*b*) applies only to series of nonnegative terms a_n.

Proof Given $\varepsilon > 0$, there exists $N \ge N_0$ such that $m \ge n \ge N$ implies

$$\sum_{k=n}^{m} c_k \le \varepsilon,$$

by the Cauchy criterion. Hence

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k \le \varepsilon,$$

and (*a*) follows.

Next, (*b*) follows from (*a*), for if Σa_n converges, so must Σd_n [note that (*b*) also follows from Theorem 3.24].

[1] The expression "nonnegative" always refers to *real* numbers.

The comparison test is a very useful one; to use it efficiently, we have to become familiar with a number of series of nonnegative terms whose convergence or divergence is known.

SERIES OF NONNEGATIVE TERMS

The simplest of all is perhaps the geometric series.

3.26 Theorem *If* $0 \le x < 1$, *then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \ge 1$, *the series diverges.*

Proof If $x \ne 1$,

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x}.$$

The result follows if we let $n \to \infty$. For $x = 1$, we get

$$1 + 1 + 1 + \cdots,$$

which evidently diverges.

In many cases which occur in applications, the terms of the series decrease monotonically. The following theorem of Cauchy is therefore of particular interest. The striking feature of the theorem is that a rather "thin" subsequence of $\{a_n\}$ determines the convergence or divergence of Σa_n.

3.27 Theorem *Suppose* $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \tag{7}$$

converges.

Proof By Theorem 3.24, it suffices to consider boundedness of the partial sums. Let

$$s_n = a_1 + a_2 + \cdots + a_n,$$

$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}.$$

For $n < 2^k$,

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k, \end{aligned}$$

so that

$$s_n \le t_k. \tag{8}$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \tfrac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} \\ &= \tfrac{1}{2}t_k, \end{aligned}$$

so that

$$2s_n \ge t_k. \tag{9}$$

By (8) and (9), the sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. This completes the proof.

3.28 Theorem $\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$.

Proof If $p \le 0$, divergence follows from Theorem 3.23. If $p > 0$, Theorem 3.27 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now, $2^{1-p} < 1$ if and only if $1 - p < 0$, and the result follows by comparison with the geometric series (take $x = 2^{1-p}$ in Theorem 3.26).

As a further application of Theorem 3.27, we prove:

3.29 Theorem *If* $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \tag{10}$$

converges; if $p \le 1$, *the series diverges.*

Remark "log n" denotes the logarithm of n to the base e (compare Exercise 7, Chap. 1); the number e will be defined in a moment (see Definition 3.30). We let the series start with $n = 2$, since $\log 1 = 0$.

Proof The monotonicity of the logarithmic function (which will be discussed in more detail in Chap. 8) implies that $\{\log n\}$ increases. Hence $\{1/n \log n\}$ decreases, and we can apply Theorem 3.27 to (10); this leads us to the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}, \tag{11}$$

and Theorem 3.29 follows from Theorem 3.28.

This procedure may evidently be continued. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n} \tag{12}$$

diverges, whereas

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2} \tag{13}$$

converges.

We may now observe that the terms of the series (12) differ very little from those of (13). Still, one diverges, the other converges. If we continue the process which led us from Theorem 3.28 to Theorem 3.29, and then to (12) and (13), we get pairs of convergent and divergent series whose terms differ even less than those of (12) and (13). One might thus be led to the conjecture that there is a limiting situation of some sort, a "boundary" with all convergent series on one side, all divergent series on the other side—at least as far as series with monotonic coefficients are concerned. This notion of "boundary" is of course quite vague. The point we wish to make is this: No matter how we make this notion precise, the conjecture is false. Exercises 11(*b*) and 12(*b*) may serve as illustrations.

We do not wish to go any deeper into this aspect of convergence theory, and refer the reader to Knopp's "Theory and Application of Infinite Series," Chap. IX, particularly Sec. 41.

THE NUMBER *e*

3.30 Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Since

$$s_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3,$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits:

3.31 Theorem $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Proof Let

$$s_n = \sum_{k=0}^{n}\frac{1}{k!}, \qquad t_n = \left(1+\frac{1}{n}\right)^n.$$

By the binomial theorem,

$$t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

Hence $t_n \le s_n$, so that

(14) $$\limsup_{n\to\infty} t_n \le e,$$

by Theorem 3.19. Next, if $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

Let $n \to \infty$, keeping m fixed. We get

$$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that

$$s_m \le \liminf_{n\to\infty} t_n.$$

Letting $m \to \infty$, we finally get

(15) $$e \le \liminf_{n\to\infty} t_n.$$

The theorem follows from (14) and (15).

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows: If s_n has the same meaning as above, we have

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right\} = \frac{1}{n!n}$$

so that

$$0 < e - s_n < \frac{1}{n!n}. \tag{16}$$

Thus s_{10}, for instance, approximates e with an error less than 10^{-7}. The inequality (16) is of theoretical interest as well, since it enables us to prove the irrationality of e very easily.

3.32 Theorem *e is irrational.*

Proof Suppose e is rational. Then $e = p/q$, where p and q are positive integers. By (16),

$$0 < q!(e - s_q) < \frac{1}{q}. \tag{17}$$

By our assumption, $q!e$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

is an integer, we see that $q!(e - s_q)$ is an integer.

Since $q \geq 1$, (17) implies the existence of an integer between 0 and 1. We have thus reached a contradiction.

Actually, e is not even an algebraic number. For a simple proof of this, see page 25 of Niven's book, or page 176 of Herstein's, cited in the Bibliography.

THE ROOT AND RATIO TESTS

3.33 Theorem (Root Test) *Given Σa_n, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.*

Then

(*a*) *if $\alpha < 1$, Σa_n converges;*
(*b*) *if $\alpha > 1$, Σa_n diverges;*
(*c*) *if $\alpha = 1$, the test gives no information.*

Proof If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$ [by Theorem 3.17(*b*)]. That is, $n \geq N$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\Sigma\beta^n$ converges. Convergence of Σa_n follows now from the comparison test.

If $\alpha > 1$, then, again by Theorem 3.17, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \to \alpha.$$

Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for convergence of Σa_n, does not hold (Theorem 3.23).

To prove (*c*), we consider the series

$$\sum\frac{1}{n}, \sum\frac{1}{n^2}.$$

For each of these series $\alpha = 1$, but the first diverges, the second converges.

3.34 Theorem (Ratio Test) *The series* Σa_n

(*a*) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(*b*) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

Proof If condition (*a*) holds, we can find $\beta < 1$, and an integer N, such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta$$

for $n \geq N$. In particular,

$$\begin{aligned}
&|a_{N+1}| < \beta|a_N|,\\
&|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|,\\
&\dots\dots\dots\dots\dots\dots\\
&|a_{N+p}| < \beta^p|a_N|.
\end{aligned}$$

That is,

$$|a_n| < |a_N|\beta^{-N} \cdot \beta^n$$

for $n \geq N$, and (*a*) follows from the comparison test, since $\Sigma\beta^n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (*b*) follows.

Note: The knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of Σa_n. The series $\Sigma 1/n$ and $\Sigma 1/n^2$ demonstrate this.

3.35 Examples

(*a*) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots,$$

for which

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0,$$

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}},$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = +\infty.$$

The root test indicates convergence; the ratio test does not apply.

(*b*) The same is true for the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots,$$

where

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8},$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2,$$

but

$$\lim \sqrt[n]{a_n} = \tfrac{1}{2}.$$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

or

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

(18) $$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta,$$

by Theorem 3.20(*b*). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

POWER SERIES

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{19}$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle, the circle of convergence, such that (19) converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

3.39 Theorem *Given the power series* $\Sigma c_n z^n$, *put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, $R = +\infty$; *if* $\alpha = +\infty$, $R = 0$.) *Then* $\Sigma c_n z^n$ *converges if* $|z| < R$, *and diverges if* $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

3.40 Examples

(*a*) The series $\Sigma n^n z^n$ has $R = 0$.

(*b*) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)

(c) The series Σz^n has $R = 1$. If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \to \infty$.

(d) The series $\sum \frac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$. It converges for all other z with $|z| = 1$. (The last assertion will be proved in Theorem 3.44.)

(e) The series $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, by the comparison test, since $|z^n/n^2| = 1/n^2$.

SUMMATION BY PARTS

3.41 Theorem *Given two sequences $\{a_n\}$, $\{b_n\}$, put*

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \tag{20}$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1},$$

and the last expression on the right is clearly equal to the right side of (20).

Formula (20), the so-called "partial summation formula," is useful in the investigation of series of the form $\Sigma a_n b_n$, particularly when $\{b_n\}$ is monotonic. We shall now give applications.

3.42 Theorem *Suppose*

(a) *the partial sums A_n of Σa_n form a bounded sequence;*
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n \to \infty} b_n = 0$.

Then $\Sigma a_n b_n$ converges.

Proof Choose M such that $|A_n| \le M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $b_N \le (\varepsilon/2M)$. For $N \le p \le q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\le M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2Mb_p \le 2Mb_N \le \varepsilon.\end{aligned}$$

Convergence now follows from the Cauchy criterion. We note that the first inequality in the above chain depends of course on the fact that $b_n - b_{n+1} \ge 0$.

3.43 Theorem *Suppose*

(*a*) $|c_1| \ge |c_2| \ge |c_3| \ge \cdots$;
(*b*) $c_{2m-1} \ge 0$, $c_{2m} \le 0$ $\quad (m = 1, 2, 3, \ldots)$;
(*c*) $\lim_{n\to\infty} c_n = 0$.

Then Σc_n *converges.*

Series for which (*b*) holds are called "alternating series"; the theorem was known to Leibnitz.

Proof Apply Theorem 3.42, with $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

3.44 Theorem *Suppose the radius of convergence of* $\Sigma c_n z^n$ *is* 1, *and suppose* $c_0 \ge c_1 \ge c_2 \ge \cdots$, $\lim_{n\to\infty} c_n = 0$. *Then* $\Sigma c_n z^n$ *converges at every point on the circle* $|z| = 1$, *except possibly at* $z = 1$.

Proof Put $a_n = z^n$, $b_n = c_n$. The hypotheses of Theorem 3.42 are then satisfied, since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \le \frac{2}{|1 - z|},$$

if $|z| = 1$, $z \ne 1$.

ABSOLUTE CONVERGENCE

The series Σa_n is said to *converge absolutely* if the series $\Sigma |a_n|$ converges.

3.45 Theorem *If* Σa_n *converges absolutely, then* Σa_n *converges.*

Proof The assertion follows from the inequality

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|,$$

plus the Cauchy criterion.

3.46 Remarks For series of positive terms, absolute convergence is the same as convergence.

If Σa_n converges, but $\Sigma|a_n|$ diverges, we say that Σa_n converges *non-absolutely*. For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges nonabsolutely (Theorem 3.43).

The comparison test, as well as the root and ratio tests, is really a test for absolute convergence, and therefore cannot give any information about non-absolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

3.47 Theorem *If $\Sigma a_n = A$, and $\Sigma b_n = B$, then $\Sigma(a_n + b_n) = A + B$, and $\Sigma c a_n = cA$, for any fixed c.*

Proof Let

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k.$$

Then

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k).$$

Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

The proof of the second assertion is even simpler.

Thus two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series. To begin with, we have to define the product. This can be done in several ways; we shall consider the so-called "Cauchy product."

3.48 Definition Given Σa_n and Σb_n, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots)$$

and call Σc_n the *product* of the two given series.

This definition may be motivated as follows. If we take two power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots\\ &= c_0 + c_1 z + c_2 z^2 + \cdots.\end{aligned}$$

Setting $z = 1$, we arrive at the above definition.

3.49 Example If

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k,$$

and $A_n \to A$, $B_n \to B$, then it is not at all clear that $\{C_n\}$ will converge to AB, since we do not have $C_n = A_n B_n$. The dependence of $\{C_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one (see the proof of Theorem 3.50). We shall now show that the product of two convergent series may actually diverge.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges (Theorem 3.43). We form the product of this series with itself and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\\ - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \cdots,\end{aligned}$$

so that

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2.$$

we have

$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

so that the condition $c_n \to 0$, which is necessary for the convergence of Σc_n, is not satisfied.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

3.50 Theorem *Suppose*

(*a*) $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

(*b*) $\sum_{n=0}^{\infty} a_n = A,$

(*c*) $\sum_{n=0}^{\infty} b_n = B,$

(*d*) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots).$

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Proof Put

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k, \qquad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 .$$

We wish to show that $C_n \to AB$. Since $A_n B \to AB$, it suffices to show that

$$\lim_{n \to \infty} \gamma_n = 0. \tag{21}$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n| .$$

[It is here that we use (*a*).] Let $\varepsilon > 0$ be given. By (*c*), $\beta_n \to 0$. Hence we can choose N such that $|\beta_n| \le \varepsilon$ for $n \ge N$, in which case

$$\begin{aligned} |\gamma_n| &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha. \end{aligned}$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n \to \infty} |\gamma_n| \le \varepsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since ε is arbitrary, (21) follows.

Another question which may be asked is whether the series Σc_n, if convergent, must have the sum AB. Abel showed that the answer is in the affirmative.

3.51 Theorem *If the series* Σa_n, Σb_n, Σc_n *converge to* A, B, C, *and* $c_n = a_0 b_n + \cdots + a_n b_0$, *then* $C = AB$.

Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after Theorem 8.2.

REARRANGEMENTS

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \ldots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J, in the notation of Definition 2.2). Putting

$$a_n' = a_{k_n} \qquad (n = 1, 2, 3, \ldots),$$

we say that $\Sigma a_n'$ is a *rearrangement* of Σa_n.

If $\{s_n\}$, $\{s'_n\}$ are the sequences of partial sums of Σa_n, $\Sigma a'_n$, it is easily seen that, in general, these two sequences consist of entirely different numbers. We are thus led to the problem of determining under what conditions all rearrangements of a convergent series will converge and whether the sums are necessarily the same.

3.53 Example Consider the convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \tag{22}$$

and one of its rearrangements

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots \tag{23}$$

in which two positive terms are always followed by one negative. If s is the sum of (22), then

$$s < 1 - \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}.$$

Since

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$$

for $k \geq 1$, we see that $s'_3 < s'_6 < s'_9 < \cdots$, where s'_n is nth partial sum of (23). Hence

$$\limsup_{n\to\infty} s'_n > s'_3 = \tfrac{5}{6},$$

so that (23) certainly does not converge to s [we leave it to the reader to verify that (23) does, however, converge].

This example illustrates the following theorem, due to Riemann.

3.54 Theorem *Let Σa_n be a series of real numbers which converges, but not absolutely. Suppose*

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement $\Sigma a'_n$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha, \qquad \limsup_{n\to\infty} s'_n = \beta. \tag{24}$$

Proof Let

$$p_n = \frac{|a_n| + a_n}{2}, \qquad q_n = \frac{|a_n| - a_n}{2} \qquad (n = 1, 2, 3, \ldots).$$

Then $p_n - q_n = a_n, p_n + q_n = |a_n|, p_n \geq 0, q_n \geq 0$. The series Σp_n, Σq_n must both diverge.

For if both were convergent, then

$$\Sigma(p_n + q_n) = \Sigma |a_n|$$

would converge, contrary to hypothesis. Since

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n,$$

divergence of Σp_n and convergence of Σq_n (or vice versa) implies divergence of Σa_n, again contrary to hypothesis.

Now let $P_1, P_2, P_3, \ldots$ denote the nonnegative terms of Σa_n, in the order in which they occur, and let $Q_1, Q_2, Q_3, \ldots$ be the absolute values of the negative terms of Σa_n, also in their original order.

The series $\Sigma P_n, \Sigma Q_n$ differ from $\Sigma p_n, \Sigma q_n$ only by zero terms, and are therefore divergent.

We shall construct sequences $\{m_n\}, \{k_n\}$, such that the series

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots, \tag{25}$$

which clearly is a rearrangement of Σa_n, satisfies (24).

Choose real-valued sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n, \beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} > \beta_1,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1;$$

let m_2, k_2 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2;$$

and continue in this way. This is possible since ΣP_n and ΣQ_n diverge.

If x_n, y_n denote the partial sums of (25) whose last terms are P_{m_n}, $-Q_{k_n}$, then

$$|x_n - \beta_n| \leq P_{m_n}, \qquad |y_n - \alpha_n| \leq Q_{k_n}.$$

Since $P_n \to 0$ and $Q_n \to 0$ as $n \to \infty$, we see that $x_n \to \beta$, $y_n \to \alpha$.

Finally, it is clear that no number less than α or greater than β can be a subsequential limit of the partial sums of (25).

3.55 Theorem *If Σa_n is a series of complex numbers which converges absolutely, then every rearrangement of Σa_n converges, and they all converge to the same sum.*

Proof Let $\Sigma a'_n$ be a rearrangement, with partial sums s'_n. Given $\varepsilon > 0$, there exists an integer N such that $m \geq n \geq N$ implies

$$\sum_{i=n}^{m} |a_i| \leq \varepsilon. \tag{26}$$

Now choose p such that the integers $1, 2, \ldots, N$ are all contained in the set $k_1, k_2, \ldots, k_p$ (we use the notation of Definition 3.52). Then if $n > p$, the numbers $a_1, \ldots, a_N$ will cancel in the difference $s_n - s'_n$, so that $|s_n - s'_n| \leq \varepsilon$, by (26). Hence $\{s'_n\}$ converges to the same sum as $\{s_n\}$.

EXERCISES

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
2. Calculate $\lim_{n \to \infty} (\sqrt{n^2 + n} - n)$.
3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

6. Investigate the behavior (convergence or divergence) of Σa_n if

(*a*) $a_n = \sqrt{n+1} - \sqrt{n}$;

(*b*) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;

(*c*) $a_n = (\sqrt[n]{n} - 1)^n$;

(*d*) $a_n = \dfrac{1}{1 + z^n}$, for complex values of z.

7. Prove that the convergence of Σa_n implies the convergence of

$$\Sigma \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

8. If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:

(*a*) $\sum n^3 z^n$, (*b*) $\sum \frac{2^n}{n!} z^n$,

(*c*) $\sum \frac{2^n}{n^2} z^n$, (*d*) $\sum \frac{n^3}{3^n} z^n$.

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(*a*) Prove that $\sum \frac{a_n}{1 + a_n}$ diverges.

(*b*) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(*c*) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(*d*) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

12. Suppose $a_n > 0$ and Σa_n converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(*a*) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(*b*) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(*a*) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
(*b*) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
(*c*) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
(*d*) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim (na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (*a*), but under the additional assumption that $na_n \to 0$.]
(*e*) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k. Absolute convergence is defined as convergence of $\Sigma|\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(*a*), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(*a*) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(*b*) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \ldots).$$

(*c*) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \qquad \varepsilon_6 < 4 \cdot 10^{-32}.$$

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(*a*) Prove that $x_1 > x_3 > x_5 > \cdots$.

(*b*) Prove that $x_2 < x_4 < x_6 < \cdots$.

(*c*) Prove that $\lim x_n = \sqrt{\alpha}$.

(*d*) Compare the rapidity of convergence of this process with the one described in Exercise 16.

18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21. Prove the following analogue of Theorem 3.10(*b*): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n \to \infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\bar{E}_n \subset G_n$, and apply Exercise 21.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n,

$$d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

24. Let X be a metric space.

(*a*) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(*b*) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(*c*) Prove that the resulting metric space X^* is complete.

(*d*) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(*e*) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (*d*), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – REAL ANALYSIS – SMT1502

CONTINUITY

The function concept and some of the related terminology were introduced in Definitions 2.1 and 2.2. Although we shall (in later chapters) be mainly interested in real and complex functions (i.e., in functions whose values are real or complex numbers) we shall also discuss vector-valued functions (i.e., functions with values in R^k) and functions with values in an arbitrary metric space. The theorems we shall discuss in this general setting would not become any easier if we restricted ourselves to real functions, for instance, and it actually simplifies and clarifies the picture to discard unnecessary hypotheses and to state and prove theorems in an appropriately general context.

The domains of definition of our functions will also be metric spaces, suitably specialized in various instances.

LIMITS OF FUNCTIONS

4.1 Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

(1) $$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon \tag{2}$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta. \tag{3}$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some euclidean space R^k, the distances d_X, d_Y are of course replaced by absolute values, or by norms of differences (see Sec. 2.16).

It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

We can recast this definition in terms of limits of sequences:

4.2 Theorem *Let X, Y, E, f, and p be as in Definition* 4.1. *Then*

$$\lim_{x \to p} f(x) = q \tag{4}$$

if and only if

$$\lim_{n \to \infty} f(p_n) = q \tag{5}$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \qquad \lim_{n \to \infty} p_n = p. \tag{6}$$

Proof Suppose (4) holds. Choose $\{p_n\}$ in E satisfying (6). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that (5) holds.

Conversely, suppose (4) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ ($n = 1, 2, 3, \ldots$), we thus find a sequence in E satisfying (6) for which (5) is false.

Corollary *If f has a limit at p, this limit is unique.*

This follows from Theorems 3.2(*b*) and 4.2.

4.3 Definition Suppose we have two complex functions, f and g, both defined on E. By $f+g$ we mean the function which assigns to each point x of E the number $f(x)+g(x)$. Similarly we define the difference $f-g$, the product fg, and the quotient f/g of the two functions, with the understanding that the quotient is defined only at those points x of E at which $g(x) \neq 0$. If f assigns to each point x of E the same number c, then f is said to be a constant function, or simply a constant, and we write $f=c$. If f and g are real functions, and if $f(x) \geq g(x)$ for every $x \in E$, we shall sometimes write $f \geq g$, for brevity.

Similarly, if $\mathbf{f}$ and $\mathbf{g}$ map E into R^k, we define $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ by

$$(\mathbf{f}+\mathbf{g})(x)=\mathbf{f}(x)+\mathbf{g}(x), \qquad (\mathbf{f}\cdot\mathbf{g})(x)=\mathbf{f}(x)\cdot\mathbf{g}(x);$$

and if λ is a real number, $(\lambda\mathbf{f})(x)=\lambda\mathbf{f}(x)$.

4.4 Theorem *Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and*

$$\lim_{x\to p} f(x)=A, \qquad \lim_{x\to p} g(x)=B.$$

Then (*a*) $\lim_{x\to p} (f+g)(x)=A+B$;

(*b*) $\lim_{x\to p} (fg)(x)=AB$;

(*c*) $\lim_{x\to p} \left(\frac{f}{g}\right)(x)=\frac{A}{B}$, *if* $B \neq 0$.

Proof In view of Theorem 4.2, these assertions follow immediately from the analogous properties of sequences (Theorem 3.3).

Remark If $\mathbf{f}$ and $\mathbf{g}$ map E into R^k, then (*a*) remains true, and (*b*) becomes

(*b′*) $\lim_{x\to p} (\mathbf{f}\cdot\mathbf{g})(x)=\mathbf{A}\cdot\mathbf{B}$.

(Compare Theorem 3.4.)

CONTINUOUS FUNCTIONS

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at p* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E, then f is said to be *continuous on E*.

It should be noted that f has to be defined at the point p in order to be continuous at p. (Compare this with the remark following Definition 4.1.)

If p is an isolated point of E, then our definition implies that every function f which has E as its domain of definition is continuous at p. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$; then

$$d_Y(f(x), f(p)) = 0 < \varepsilon.$$

4.6 Theorem *In the situation given in Definition* 4.5, *assume also that p is a limit point of E. Then f is continuous at p if and only if* $\lim_{x\to p} f(x) = f(p)$.

Proof This is clear if we compare Definitions 4.1 and 4.5.

We now turn to compositions of functions. A brief statement of the following theorem is that a continuous function of a continuous function is continuous.

4.7 Theorem *Suppose X, Y, Z are metric spaces,* $E \subset X$, *f maps E into Y, g maps the range of f,* $f(E)$, *into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is continuous at a point $p \in E$ *and if g is continuous at the point* $f(p)$, *then h is continuous at p.*

This function h is called the *composition* or the *composite* of f and g. The notation

$$h = g \circ f$$

is frequently used in this context.

Proof Let $\varepsilon > 0$ be given. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that

$$d_Z(g(y), g(f(p))) < \varepsilon \text{ if } d_Y(y, f(p)) < \eta \text{ and } y \in f(E).$$

Since f is continuous at p, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \eta \text{ if } d_X(x, p) < \delta \text{ and } x \in E.$$

It follows that

$$d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \varepsilon$$

if $d_X(x, p) < \delta$ and $x \in E$. Thus h is continuous at p.

4.8 Theorem *A mapping f of a metric space X into a metric space Y is continuous on X if and only if* $f^{-1}(V)$ *is open in X for every open set V in Y.*

(Inverse images are defined in Definition 2.2.) This is a very useful characterization of continuity.

Proof Suppose f is continuous on X and V is an open set in Y. We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if $d_Y(f(p), y) < \varepsilon$; and since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$ and $\varepsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \varepsilon$. Then V is open; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $x \in f^{-1}(V)$as soon as $d_X(p, x) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$.

This completes the proof.

Corollary *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

This follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.

We now turn to complex-valued and vector-valued functions, and to functions defined on subsets of R^k.

4.9 Theorem *Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X.*

In the last case, we must of course assume that $g(x) \neq 0$, for all $x \in X$.

Proof At isolated points of X there is nothing to prove. At limit points, the statement follows from Theorems 4.4 and 4.6.

4.10 Theorem

(a) Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into R^k defined by

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \qquad (x \in X); \tag{7}$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.
(b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into R^k, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X.

The functions $f_1, \ldots, f_k$ are called the *components* of $\mathbf{f}$. Note that $\mathbf{f} + \mathbf{g}$ is a mapping into R^k, whereas $\mathbf{f} \cdot \mathbf{g}$ is a *real* function on X.

Proof Part (*a*) follows from the inequalities

$$|f_j(x) - f_j(y)| \le |\mathbf{f}(x) - \mathbf{f}(y)| = \left\{ \sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right\}^{\frac{1}{2}},$$

for $j = 1, \ldots, k$. Part (*b*) follows from (*a*) and Theorem 4.9.

4.11 Examples If $x_1, \ldots, x_k$ are the coordinates of the point $\mathbf{x} \in R^k$, the functions ϕ_i defined by

$$\phi_i(\mathbf{x}) = x_i \qquad (\mathbf{x} \in R^k) \tag{8}$$

are continuous on R^k, since the inequality

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \le |\mathbf{x} - \mathbf{y}|$$

shows that we may take $\delta = \varepsilon$ in Definition 4.5. The functions ϕ_i are sometimes called the *coordinate functions*.

Repeated application of Theorem 4.9 then shows that every monomial

$$x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \tag{9}$$

where $n_1, \ldots, n_k$ are nonnegative integers, is continuous on R^k. The same is true of constant multiples of (9), since constants are evidently continuous. It follows that every polynomial P, given by

$$P(\mathbf{x}) = \Sigma c_{n_1 \cdots n_k} x_1^{n_1} \ldots x_k^{n_k} \qquad (\mathbf{x} \in R^k), \tag{10}$$

is continuous on R^k. Here the coefficients $c_{n_1 \cdots n_k}$ are complex numbers, $n_1, \ldots, n_k$ are nonnegative integers, and the sum in (10) has finitely many terms.

Furthermore, every rational function in $x_1, \ldots, x_k$, that is, every quotient of two polynomials of the form (10), is continuous on R^k wherever the denominator is different from zero.

From the triangle inequality one sees easily that

$$\big| |\mathbf{x}| - |\mathbf{y}| \big| \le |\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in R^k). \tag{11}$$

Hence the mapping $\mathbf{x} \to |\mathbf{x}|$ is a continuous real function on R^k.

If now $\mathbf{f}$ is a continuous mapping from a metric space X into R^k, and if ϕ is defined on X by setting $\phi(p) = |\mathbf{f}(p)|$, it follows, by Theorem 4.7, that ϕ is a continuous real function on X.

4.12 Remark We defined the notion of continuity for functions defined on a *subset* E of a metric space X. However, the complement of E in X plays no role whatever in this definition (note that the situation was somewhat different for limits of functions). Accordingly, we lose nothing of interest by discarding the complement of the domain of f. This means that we may just as well talk only about continuous mappings of one metric space into another, rather than of mappings of subsets. This simplifies statements and proofs of some theorems. We have already made use of this principle in Theorems 4.8 to 4.10, and will continue to do so in the following section on compactness.

CONTINUITY AND COMPACTNESS

4.13 Definition A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \le M$ for all $x \in E$.

4.14 Theorem *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, Theorem 4.8 shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

(12) $$X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, (12) implies that

(13) $$f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

This completes the proof.

Note: We have used the relation $f(f^{-1}(E)) \subset E$, valid for $E \subset Y$. If $E \subset X$, then $f^{-1}(f(E)) \supset E$; equality need not hold in either case.

We shall now deduce some consequences of Theorem 4.14.

4.15 Theorem *If $\mathbf{f}$ is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.*

This follows from Theorem 2.41. The result is particularly important when f is real:

4.16 Theorem *Suppose f is a continuous real function on a compact metric space X, and*

(14) $$M = \sup_{p \in X} f(p), \qquad m = \inf_{p \in X} f(p).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

The notation in (14) means that M is the least upper bound of the set of all numbers $f(p)$, where p ranges over X, and that m is the greatest lower bound of this set of numbers.

The conclusion may also be stated as follows: *There exist points p and q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$*; that is, f attains its maximum (at p) and its minimum (at q).

Proof By Theorem 4.15, $f(X)$ is a closed and bounded set of real numbers; hence $f(X)$ contains

$$M = \sup f(X) \qquad \text{and} \qquad m = \inf f(X),$$

by Theorem 2.28.

4.17 Theorem *Suppose f is a continuous* 1-1 *mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x \qquad (x \in X)$$

is a continuous mapping of Y onto X.

Proof Applying Theorem 4.8 to f^{-1} in place of f, we see that it suffices to prove that $f(V)$ is an open set in Y for every open set V in X. Fix such a set V.

The complement V^c of V is closed in X, hence compact (Theorem 2.35); hence $f(V^c)$ is a compact subset of Y (Theorem 4.14) and so is closed in Y (Theorem 2.34). Since f is one-to-one and onto, $f(V)$ is the complement of $f(V^c)$. Hence $f(V)$ is open.

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon \tag{15}$$

for all p and q in X for which $d_X(p, q) < \delta$.

Let us consider the differences between the concepts of continuity and of uniform continuity. First, uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point. To ask whether a given function is uniformly continuous at a certain point is meaningless. Second, if f is continuous on X, then it is possible to find, for each $\varepsilon > 0$ and for each point p of X, a number $\delta > 0$ having the property specified in Definition 4.5. This δ depends on ε *and* on p. If f is, however, uniformly continuous on X, then it is possible, for each $\varepsilon > 0$, to find *one* number $\delta > 0$ which will do for *all* points p of X.

Evidently, every uniformly continuous function is continuous. That the two concepts are equivalent on compact sets follows from the next theorem.

4.19 Theorem *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Proof Let $\varepsilon > 0$ be given. Since f is continuous, we can associate to each point $p \in X$ a positive number $\phi(p)$ such that

$$q \in X,\ d_X(p, q) < \phi(p) \quad \text{implies} \quad d_Y(f(p), f(q)) < \frac{\varepsilon}{2}. \tag{16}$$

Let $J(p)$ be the set of all $q \in X$ for which

$$d_X(p, q) < \tfrac{1}{2}\phi(p). \tag{17}$$

Since $p \in J(p)$, the collection of all sets $J(p)$ is an open cover of X; and since X is compact, there is a finite set of points $p_1, \ldots, p_n$ in X, such that

$$X \subset J(p_1) \cup \cdots \cup J(p_n). \tag{18}$$

We put

$$\delta = \tfrac{1}{2} \min\,[\phi(p_1), \ldots, \phi(p_n)]. \tag{19}$$

Then $\delta > 0$. (This is one point where the finiteness of the covering, inherent in the definition of compactness, is essential. The minimum of a finite set of positive numbers is positive, whereas the inf of an infinite set of positive numbers may very well be 0.)

Now let q and p be points of X, such that $d_X(p, q) < \delta$. By (18), there is an integer m, $1 \leq m \leq n$, such that $p \in J(p_m)$; hence

$$d_X(p, p_m) < \tfrac{1}{2}\phi(p_m), \tag{20}$$

and we also have

$$d_X(q, p_m) \leq d_X(p, q) + d_X(p, p_m) < \delta + \tfrac{1}{2}\phi(p_m) \leq \phi(p_m).$$

Finally, (16) shows that therefore

$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)) < \varepsilon.$$

This completes the proof.

An alternative proof is sketched in Exercise 10.

We now proceed to show that compactness is essential in the hypotheses of Theorems 4.14, 4.15, 4.16, and 4.19.

4.20 Theorem *Let E be a noncompact set in R^1. Then*

(*a*) *there exists a continuous function on E which is not bounded;*
(*b*) *there exists a continuous and bounded function on E which has no maximum.*

If, in addition, E is bounded, then

(*c*) *there exists a continuous function on E which is not uniformly continuous.*

Proof Suppose first that E is bounded, so that there exists a limit point x_0 of E which is not a point of E. Consider

(21)
$$f(x) = \frac{1}{x - x_0} \qquad (x \in E).$$

This is continuous on E (Theorem 4.9), but evidently unbounded. To see that (21) is not uniformly continuous, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking t close enough to x_0, we can then make the difference $|f(t) - f(x)|$ greater than ε, although $|t - x| < \delta$. Since this is true for every $\delta > 0$, f is not uniformly continuous on E.

The function g given by

(22)
$$g(x) = \frac{1}{1 + (x - x_0)^2} \qquad (x \in E)$$

is continuous on E, and is bounded, since $0 < g(x) < 1$. It is clear that

$$\sup_{x \in E} g(x) = 1,$$

whereas $g(x) < 1$ for all $x \in E$. Thus g has no maximum on E.

Having proved the theorem for bounded sets E, let us now suppose that E is unbounded. Then $f(x) = x$ establishes (*a*), whereas

(23)
$$h(x) = \frac{x^2}{1 + x^2} \qquad (x \in E)$$

establishes (*b*), since

$$\sup_{x \in E} h(x) = 1$$

and $h(x) < 1$ for all $x \in E$.

Assertion (*c*) would be false if boundedness were omitted from the hypotheses. For, let E be the set of all integers. Then every function defined on E is uniformly continuous on E. To see this, we need merely take $\delta < 1$ in Definition 4.18.

We conclude this section by showing that compactness is also essential in Theorem 4.17.

4.21 Example Let X be the half-open interval $[0, 2\pi)$ on the real line, and let $\mathbf{f}$ be the mapping of X onto the circle Y consisting of all points whose distance from the origin is 1, given by

$$\mathbf{f}(t) = (\cos t, \sin t) \qquad (0 \le t < 2\pi). \tag{24}$$

The continuity of the trigonometric functions cosine and sine, as well as their periodicity properties, will be established in Chap. 8. These results show that $\mathbf{f}$ is a continuous 1-1 mapping of X onto Y.

However, the inverse mapping (which exists, since $\mathbf{f}$ is one-to-one and onto) fails to be continuous at the point $(1, 0) = \mathbf{f}(0)$. Of course, X is not compact in this example. (It may be of interest to observe that $\mathbf{f}^{-1}$ fails to be continuous in spite of the fact that Y *is* compact!)

CONTINUITY AND CONNECTEDNESS

4.22 Theorem *If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.*

Proof Assume, on the contrary, that $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y. Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \bar{A}$ (the closure of A), we have $G \subset f^{-1}(\bar{A})$; the latter set is closed, since f is continuous; hence $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B$ is empty, we conclude that $\bar{G} \cap H$ is empty.

The same argument shows that $G \cap \bar{H}$ is empty. Thus G and H are separated. This is impossible if E is connected.

4.23 Theorem *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

A similar result holds, of course, if $f(a) > f(b)$. Roughly speaking, the theorem says that a continuous real function assumes all intermediate values on an interval.

Proof By Theorem 2.47, $[a, b]$ is connected; hence Theorem 4.22 shows that $f([a, b])$ is a connected subset of R^1, and the assertion follows if we appeal once more to Theorem 2.47.

4.24 Remark At first glance, it might seem that Theorem 4.23 has a converse. That is, one might think that if for any two points $x_1 < x_2$ and for any number c between $f(x_1)$ and $f(x_2)$ there is a point x in (x_1, x_2) such that $f(x) = c$, then f must be continuous.

That this is not so may be concluded from Example 4.27(*d*).

DISCONTINUITIES

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is *discontinuous* at x, or that f *has a discontinuity* at x. If f is defined on an interval or on a segment, it is customary to divide discontinuities into two types. Before giving this classification, we have to define the *right-hand* and the *left-hand limits* of f at x, which we denote by $f(x+)$ and $f(x-)$, respectively.

4.25 Definition Let f be defined on (a, b). Consider any point x such that $a \le x < b$. We write

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \le b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

It is clear that any point x of (a, b), $\lim_{t \to x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

4.26 Definition Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity*, at x. Otherwise the discontinuity is said to be of the *second kind*.

There are two ways in which a function can have a simple discontinuity: either $f(x+) \ne f(x-)$ [in which case the value $f(x)$ is immaterial], or $f(x+) = f(x-) \ne f(x)$.

4.27 Examples

(*a*) Define

$$f(x) = \begin{cases} 1 & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f has a discontinuity of the second kind at every point x, since neither $f(x+)$ nor $f(x-)$ exists.

(*b*) Define

$$f(x) = \begin{cases} x & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point.

(*c*) Define

$$f(x) = \begin{cases} x + 2 & (-3 < x < -2), \\ -x - 2 & (-2 \leq x < 0), \\ x + 2 & (0 \leq x < 1). \end{cases}$$

Then f has a simple discontinuity at $x = 0$ and is continuous at every other point of $(-3, 1)$.

(*d*) Define

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Since neither $f(0+)$ nor $f(0-)$ exists, f has a discontinuity of the second kind at $x = 0$. We have not yet shown that $\sin x$ is a continuous function. If we assume this result for the moment, Theorem 4.7 implies that f is continuous at every point $x \neq 0$.

MONOTONIC FUNCTIONS

We shall now study those functions which never decrease (or never increase) on a given segment.

4.28 Definition Let f be real on (a, b). Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a *monotonically decreasing* function. The class of monotonic functions consists of both the increasing and the decreasing functions.

4.29 Theorem *Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t). \tag{25}$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-). \tag{26}$$

Analogous results evidently hold for monotonically decreasing functions.

Proof By hypothesis, the set of numbers $f(t)$, where $a < t < x$, is bounded above by the number $f(x)$, and therefore has a least upper bound which we shall denote by A. Evidently $A \leq f(x)$. We have to show that $A = f(x-)$.

Let $\varepsilon > 0$ be given. It follows from the definition of A as a least upper bound that there exists $\delta > 0$ such that $a < x - \delta < x$ and

(27) $$A - \varepsilon < f(x - \delta) \leq A.$$

Since f is monotonic, we have

(28) $$f(x - \delta) \leq f(t) \leq A \qquad (x - \delta < t < x).$$

Combining (27) and (28), we see that

$$|f(t) - A| < \varepsilon \qquad (x - \delta < t < x).$$

Hence $f(x-) = A$.

The second half of (25) is proved in precisely the same way.

Next, if $a < x < y < b$, we see from (25) that

(29) $$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t).$$

The last equality is obtained by applying (25) to (a, y) in place of (a, b). Similarly,

(30) $$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t).$$

Comparison of (29) and (30) gives (26).

Corollary *Monotonic functions have no discontinuities of the second kind.*

This corollary implies that every monotonic function is discontinuous at a countable set of points at most. Instead of appealing to the general theorem whose proof is sketched in Exercise 17, we give here a simple proof which is applicable to monotonic functions.

4.30 Theorem *Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.*

Proof Suppose, for the sake of definiteness, that f is increasing, and let E be the set of points at which f is discontinuous.

With every point x of E we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Since $x_1 < x_2$ implies $f(x_1+) \le f(x_2-)$, we see that $r(x_1) \ne r(x_2)$ if $x_1 \ne x_2$.

We have thus established a 1-1 correspondence between the set E and a subset of the set of rational numbers. The latter, as we know, is countable.

4.31 Remark It should be noted that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of (a, b), which may even be dense, we can construct a function f, monotonic on (a, b), discontinuous at every point of E, and at no other point of (a, b).

To show this, let the points of E be arranged in a sequence $\{x_n\}$, $n = 1, 2, 3, \ldots$. Let $\{c_n\}$ be a sequence of positive numbers such that Σc_n converges. Define

$$f(x) = \sum_{x_n < x} c_n \qquad (a < x < b). \tag{31}$$

The summation is to be understood as follows: Sum over those indices n for which $x_n < x$. If there are no points x_n to the left of x, the sum is empty; following the usual convention, we define it to be zero. Since (31) converges absolutely, the order in which the terms are arranged is immaterial.

We leave the verification of the following properties of f to the reader:

(*a*) f is monotonically increasing on (a, b);
(*b*) f is discontinuous at every point of E; in fact,

$$f(x_n+) - f(x_n-) = c_n.$$

(*c*) f is continuous at every other point of (a, b).

Moreover, it is not hard to see that $f(x-) = f(x)$ at all points of (a, b). If a function satisfies this condition, we say that f is *continuous from the left*. If the summation in (31) were taken over all indices n for which $x_n \le x$, we would have $f(x+) = f(x)$ at every point of (a, b); that is, f would be *continuous from the right*.

Functions of this sort can also be defined by another method; for an example we refer to Theorem 6.16.

INFINITE LIMITS AND LIMITS AT INFINITY

To enable us to operate in the extended real number system, we shall now enlarge the scope of Definition 4.1, by reformulating it in terms of neighborhoods.

For any real number x, we have already defined a neighborhood of x to be any segment $(x - \delta, x + \delta)$.

4.32 Definition For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

4.33 Definition Let f be a real function defined on $E \subset R$. We say that

$$f(t) \to A \text{ as } t \to x,$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

A moment's consideration will show that this coincides with Definition 4.1 when A and x are real.

The analogue of Theorem 4.4 is still true, and the proof offers nothing new. We state it, for the sake of completeness.

4.34 Theorem *Let f and g be defined on $E \subset R$. Suppose*

$$f(t) \to A, \qquad g(t) \to B \qquad \text{as } t \to x.$$

Then

(a) $f(t) \to A'$ *implies* $A' = A$.
(b) $(f + g)(t) \to A + B$,
(c) $(fg)(t) \to AB$,
(d) $(f/g)(t) \to A/B$,

provided the right members of (b), (c), and (d) are defined.

Note that $\infty - \infty$, $0 \cdot \infty$, ∞/∞, $A/0$ are not defined (see Definition 1.23).

EXERCISES

1. Suppose f is a real function defined on R^1 which satisfies

$$\lim_{h \to 0} [f(x + h) - f(x - h)] = 0$$

for every $x \in R^1$. Does this imply that f is continuous?

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\bar{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\bar{E}$ denotes the closure of E.) Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

4. Let f and g be continuous mappings of a metric space X into a metric space Y,

and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

5. If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. *Hint:* Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.

6. If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.

   Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

7. If $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by: $f(0, 0) = g(0, 0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ if $(x, y) \neq (0, 0)$. Prove that f is bounded on R^2, that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!

8. Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

   Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$.

10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous.

    State this more precisely and prove it.

13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X

(see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.) *Hint:* For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \dots$, consists of a single point, say $g(p)$, of R^1. Prove that the function g so defined on X is the desired extension of f.

Could the range space R^1 be replaced by R^k? By any compact metric space? By any complete metric space? By any metric space?

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

15. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

16. Let $[x]$ denote the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

17. Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. *Hint:* Let E be the set on which $f(x-) < f(x+)$. With each point x of E, associate a triple (p, q, r) of rational numbers such that

(*a*) $f(x-) < p < f(x+)$,

(*b*) $a < q < t < x$ implies $f(t) < p$,

(*c*) $x < t < r < b$ implies $f(t) > p$.

The set of all such triples is countable. Show that each triple is associated with at most one point of E. Deal similarly with the other possible types of simple discontinuities.

18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \dfrac{1}{n} & \left(x = \dfrac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

19. Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed.

Prove that f is continuous.

Hint: If $x_n \to x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction. (N. J. Fine, *Amer. Math. Monthly*, vol. 73, 1966, p. 782.)

20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(*a*) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

(*b*) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$, $y \in X$.

Hint: $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, so that

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

21. Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. *Hint:* ρ_F is a continuous positive function on K.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

22. Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}([0, \tfrac{1}{2})), \qquad W = f^{-1}((\tfrac{1}{2}, 1]),$$

show that V and W are open and disjoint, and that $A \subset V$, $B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

23. A real-valued function f defined in (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

24. Assume that f is a continuous real function defined in (a, b) such that

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

25. If $A \subset R^k$ and $B \subset R^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(*a*) If K is compact and C is closed in R^k, prove that $K + C$ is closed.

Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.

(*b*) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1.

26. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – REAL ANALYSIS – SMT1502

DIFFERENTIATION

In this chapter we shall (except in the final section) confine our attention to *real* functions defined on intervals or segments. This is not just a matter of convenience, since genuine differences appear when we pass from real functions to vector-valued ones. Differentiation of functions defined on R^k will be discussed in Chap. 9.

THE DERIVATIVE OF A REAL FUNCTION

5.1 Definition Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

(1) $$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b,\ t \neq x),$$

and define

(2) $$f'(x) = \lim_{t \to x} \phi(t),$$

provided this limit exists in accordance with Definition 4.1.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit (2) exists; f' is called the *derivative* of f.

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

It is possible to consider right-hand and left-hand limits in (2); this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b, the derivative, if it exists, is a right-hand or left-hand derivative, respectively. We shall not, however, discuss one-sided derivatives in any detail.

If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined by (1) and (2), as above. But $f'(a)$ and $f'(b)$ are not defined in this case.

5.2 Theorem *Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Proof As $t \to x$, we have, by Theorem 4.4,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

The converse of this theorem is not true. It is easy to construct continuous functions which fail to be differentiable at isolated points. In Chap. 7 we shall even become acquainted with a function which is continuous on the whole line without being differentiable at any point!

5.3 Theorem *Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and*

(*a*) $(f + g)'(x) = f'(x) + g'(x)$;

(*b*) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;

(*c*) $\displaystyle \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$.

In (*c*), we assume of course that $g(x) \neq 0$.

Proof (*a*) is clear, by Theorem 4.4. Let $h = fg$. Then

$$h(t) - h(x) = f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)].$$

If we divide this by $t - x$ and note that $f(t) \to f(x)$ as $t \to x$ (Theorem 5.2), (b) follows. Next, let $h = f/g$. Then

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)}\left[g(x)\frac{f(t) - f(x)}{t - x} - f(x)\frac{g(t) - g(x)}{t - x}\right].$$

Letting $t \to x$, and applying Theorems 4.4 and 5.2, we obtain (c).

5.4 Examples The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of (b) and (c) then shows that x^n is differentiable, and that its derivative is nx^{n-1}, for any integer n (if $n < 0$, we have to restrict ourselves to $x \neq 0$). Thus every polynomial is differentiable, and so is every rational function, except at the points where the denominator is zero.

The following theorem is known as the "chain rule" for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives. We shall meet more general versions of it in Chap. 9.

5.5 Theorem *Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \qquad (a \le t \le b),$$

then h is differentiable at x, and

(3) $$h'(x) = g'(f(x))f'(x).$$

Proof Let $y = f(x)$. By the definition of the derivative, we have

(4) $$f(t) - f(x) = (t - x)[f'(x) + u(t)],$$

(5) $$g(s) - g(y) = (s - y)[g'(y) + v(s)],$$

where $t \in [a, b]$, $s \in I$, and $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$. Let $s = f(t)$. Using first (5) and then (4), we obtain

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)] \cdot [g'(y) + v(s)] \\ &= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)], \end{aligned}$$

or, if $t \neq x$,

(6) $$\frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)].$$

Letting $t \to x$, we see that $s \to y$, by the continuity of f, so that the right side of (6) tends to $g'(y)f'(x)$, which gives (3).

5.6 Examples

(*a*) Let f be defined by

(7) $$f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Taking for granted that the derivative of $\sin x$ is $\cos x$ (we shall discuss the trigonometric functions in Chap. 8), we can apply Theorems 5.3 and 5.5 whenever $x \neq 0$, and obtain

(8) $$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \qquad (x \neq 0).$$

At $x = 0$, these theorems do not apply any longer, since $1/x$ is not defined there, and we appeal directly to the definition: for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \to 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

(*b*) Let f be defined by

(9) $$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

As above, we obtain

(10) $$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad (x \neq 0).$$

At $x = 0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t| \qquad (t \neq 0);$$

letting $t \to 0$, we see that

(11) $$f'(0) = 0.$$

Thus f is differentiable at all points x, but f' is not a continuous function, since $\cos(1/x)$ in (10) does not tend to a limit as $x \to 0$.

MEAN VALUE THEOREMS

5.7 Definition Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

5.8 Theorem *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

The analogous statement for local minima is of course also true.

Proof Choose δ in accordance with Definition 5.7, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

5.9 Theorem *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \qquad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b). \tag{12}$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$.

If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum

(Theorem 4.16). By (12), $x \in (a, b)$, and Theorem 5.8 shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a *generalized mean value theorem*; the following special case is usually referred to as "the" mean value theorem:

5.10 Theorem *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof Take $g(x) = x$ in Theorem 5.9.

5.11 Theorem *Suppose f is differentiable in (a, b).*

(*a*) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(*b*) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(*c*) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Proof All conclusions can be read off from the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

which is valid, for each pair of numbers x_1, x_2 in (a, b), for *some* x between x_1 and x_2.

THE CONTINUITY OF DERIVATIVES

We have already seen [Example 5.6(*b*)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Theorem 4.23). The precise statement follows.

5.12 Theorem *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

A similar result holds of course if $f'(a) > f'(b)$.

Proof Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attains its minimum on $[a, b]$ (Theorem 4.16) at some point x such that $a < x < b$. By Theorem 5.8, $g'(x) = 0$. Hence $f'(x) = \lambda$.

Corollary *If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.*

But f' may very well have discontinuities of the second kind.

L'HOSPITAL'S RULE

The following theorem is frequently useful in the evaluation of limits.

5.13 Theorem *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

(13) $$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$

If

(14) $$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a,$$

or if

(15) $$g(x) \to +\infty \text{ as } x \to a,$$

then

(16) $$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

The analogous statement is of course also true if $x \to b$, or if $g(x) \to -\infty$ in (15). Let us note that we now use the limit concept in the extended sense of Definition 4.33.

Proof We first consider the case in which $-\infty \leq A < +\infty$. Choose a real number q such that $A < q$, and then choose r such that $A < r < q$. By (13) there is a point $c \in (a, b)$ such that $a < x < c$ implies

(17) $$\frac{f'(x)}{g'(x)} < r.$$

If $a < x < y < c$, then Theorem 5.9 shows that there is a point $t \in (x, y)$ such that

(18) $$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Suppose (14) holds. Letting $x \to a$ in (18), we see that

(19) $$\frac{f(y)}{g(y)} \leq r < q \qquad (a < y < c).$$

Next, suppose (15) holds. Keeping y fixed in (18), we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Multiplying (18) by $[g(x) - g(y)]/g(x)$, we obtain

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \qquad (a < x < c_1). \tag{20}$$

If we let $x \to a$ in (20), (15) shows that there is a point $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q \qquad (a < x < c_2). \tag{21}$$

Summing up, (19) and (21) show that for any q, subject only to the condition $A < q$, there is a point c_2 such that $f(x)/g(x) < q$ if $a < x < c_2$.

In the same manner, if $-\infty < A \le +\infty$, and p is chosen so that $p < A$, we can find a point c_3 such that

$$p < \frac{f(x)}{g(x)} \qquad (a < x < c_3), \tag{22}$$

and (16) follows from these two statements.

DERIVATIVES OF HIGHER ORDER

5.14 Definition If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. Continuing in this manner, we obtain functions

$$f, f', f'', f^{(3)}, \ldots, f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the nth derivative, or the derivative of order n, of f.

In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x. Since $f^{(n-1)}$ must exist in a neighborhood of x, $f^{(n-2)}$ must be differentiable in that neighborhood.

TAYLOR'S THEOREM

5.15 Theorem *Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k. \tag{23}$$

Then there exists a point x between α and β such that

(24) $$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$, and that (24) allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof Let M be the number defined by

(25) $$f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

and put

(26) $$g(t) = f(t) - P(t) - M(t - \alpha)^n \qquad (a \le t \le b).$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. By (23) and (26),

(27) $$g^{(n)}(t) = f^{(n)}(t) - n!M \qquad (a < t < b).$$

Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β.

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \ldots, n - 1$, we have

(28) $$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β, by the mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}, that is, between α and β.

DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

5.16 Remarks Definition 5.1 applies without any change to complex functions f defined on $[a, b]$, and Theorems 5.2 and 5.3, as well as their proofs, remain valid. If f_1 and f_2 are the real and imaginary parts of f, that is, if

$$f(t) = f_1(t) + if_2(t)$$

for $a \le t \le b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

(29) $$f'(x) = f_1'(x) + if_2'(x);$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Passing to vector-valued functions in general, i.e., to functions $\mathbf{f}$ which map $[a, b]$ into some R^k, we may still apply Definition 5.1 to define $\mathbf{f}'(x)$. The term $\phi(t)$ in (1) is now, for each t, a point in R^k, and the limit in (2) is taken with respect to the norm of R^k. In other words, $\mathbf{f}'(x)$ is that point of R^k (if there is one) for which

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0, \tag{30}$$

and $\mathbf{f}'$ is again a function with values in R^k.

If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, as defined in Theorem 4.10, then

$$\mathbf{f}' = (f_1', \ldots, f_k'), \tag{31}$$

and $\mathbf{f}$ is differentiable at a point x if and only if each of the functions $f_1, \ldots, f_k$ is differentiable at x.

Theorem 5.2 is true in this context as well, and so is Theorem 5.3(a) and (b), if fg is replaced by the inner product $\mathbf{f} \cdot \mathbf{g}$ (see Definition 4.3).

When we turn to the mean value theorem, however, and to one of its consequences, namely, L'Hospital's rule, the situation changes. The next two examples will show that each of these results fails to be true for complex-valued functions.

5.17 Example Define, for real x,

$$f(x) = e^{ix} = \cos x + i \sin x. \tag{32}$$

(The last expression may be taken as the definition of the complex exponential e^{ix}; see Chap. 8 for a full discussion of these functions.) Then

$$f(2\pi) - f(0) = 1 - 1 = 0, \tag{33}$$

but

$$f'(x) = ie^{ix}, \tag{34}$$

so that $|f'(x)| = 1$ for all real x.

Thus Theorem 5.10 fails to hold in this case.

5.18 Example On the segment $(0, 1)$, define $f(x) = x$ and

$$g(x) = x + x^2 e^{i/x^2}. \tag{35}$$

Since $|e^{it}| = 1$ for all real t, we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1. \tag{36}$$

Next,

(37) $$g'(x) = 1 + \left\{2x - \frac{2i}{x}\right\} e^{i/x^2} \qquad (0 < x < 1),$$

so that

(38) $$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1.$$

Hence

(39) $$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

and so

(40) $$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0, 1)$, by (38).

However, there *is* a consequence of the mean value theorem which, for purposes of applications, is almost as useful as Theorem 5.10, and which remains true for vector-valued functions: From Theorem 5.10 it follows that

(41) $$|f(b) - f(a)| \leq (b - a) \sup_{a<x<b} |f'(x)|.$$

5.19 Theorem *Suppose* $\mathbf{f}$ *is a continuous mapping of* $[a, b]$ *into* R^k *and* $\mathbf{f}$ *is differentiable in* (a, b). *Then there exists* $x \in (a, b)$ *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|.$$

Proof[1] Put $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$, and define

$$\varphi(t) = \mathbf{z} \cdot \mathbf{f}(t) \qquad (a \leq t \leq b).$$

Then φ is a real-valued continuous function on $[a, b]$ which is differentiable in (a, b). The mean value theorem shows therefore that

$$\varphi(b) - \varphi(a) = (b - a)\varphi'(x) = (b - a)\mathbf{z} \cdot \mathbf{f}'(x)$$

for some $x \in (a, b)$. On the other hand,

$$\varphi(b) - \varphi(a) = \mathbf{z} \cdot \mathbf{f}(b) - \mathbf{z} \cdot \mathbf{f}(a) = \mathbf{z} \cdot \mathbf{z} = |\mathbf{z}|^2.$$

The Schwarz inequality now gives

$$|\mathbf{z}|^2 = (b - a)|\mathbf{z} \cdot \mathbf{f}'(x)| \leq (b - a)|\mathbf{z}|\,|\mathbf{f}'(x)|.$$

Hence $|\mathbf{z}| \leq (b - a)|\mathbf{f}'(x)|$, which is the desired conclusion.

[1] V. P. Havin translated the second edition of this book into Russian and added this proof to the original one.

EXERCISES

1. Let f be defined for all real x, and suppose that

$$|f(x)-f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

3. Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

6. Suppose

(*a*) f is continuous for $x \geq 0$,

(*b*) $f'(x)$ exists for $x > 0$,

(*c*) $f(0) = 0$,

(*d*) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g is monotonically increasing.

7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t-x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

9. Let f be a continuous real function on R^1, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = \left\{\frac{f(x)}{x} - A\right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.

11. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Hint: Use Theorem 5.13.

12. If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

13. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

(*a*) f is continuous if and only if $a > 0$.
(*b*) $f'(0)$ exists if and only if $a > 1$.
(*c*) f' is bounded if and only if $a \ge 1 + c$.
(*d*) f' is continuous if and only if $a > 1 + c$.
(*e*) $f''(0)$ exists if and only if $a > 2 + c$.
(*f*) f'' is bounded if and only if $a \ge 2 + 2c$.
(*g*) f'' is continuous if and only if $a > 2 + 2c$.

14. Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \ge 0$ for all $x \in (a, b)$.

15. Suppose $a \in R^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \le 4M_0 M_2.$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0 M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \dfrac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

Does $M_1^2 \leq 4M_0 M_2$ hold for vector-valued functions too?

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Hint: Let $a \to \infty$ in Exercise 15.

17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.

Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(*a*) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.

(*b*) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.

(*c*) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

21. Let E be a closed subset of R^1. We saw in Exercise 22, Chap. 4, that there is a real continuous function f on R^1 whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on R^1, or one which is n times differentiable, or even one which has derivatives of all orders on R^1?

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(*a*) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(*b*) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(*c*) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

(*d*) Show that the process described in (*c*) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots.$$

23. The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ, where

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2.$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(*a*) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(*b*) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.

(*c*) If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.

Thus β can be located by this method, but α and γ cannot.

24. The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right), \qquad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)

Do the same when $0 < \alpha < 1$.

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

Complete the details in the following outline of *Newton's method* for computing ξ.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \to \infty} x_n = \xi.$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chap. 3.)

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup|f(x)|, \qquad M_1 = \sup|f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, \ldots, y_k), \qquad y_j(a) = c_j \qquad (j = 1, \ldots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, $\mathbf{\Phi}$ is the mapping of a $(k + 1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \ldots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \ldots, c_k)$. Use Exercise 26, for vector-valued functions.

29. Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \qquad (j = 1, \ldots, k - 1),$$

$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x)y_j,$$

where $f, g_1, \ldots, g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \cdots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_2, \qquad \ldots, \qquad y^{(k-1)}(a) = c_k.$$