SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I

DIRECT AND ITERATIVE METHODS

Gauss Elimination method

To solve the system of equation represented by matrix form AX=B, A is a square matrix of order 'n' and X and B are column matrices with n elements. The coefficient matrix is reduced to upper triangular matrix by means linear transformation thereby the values of the variable are found one after other by back substitution methods.

We consider the system of n linear equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

There are two steps in the solution viz., the elimination of unknowns and back substitution

Problem 1. Solve the following system of equations using Gaussian elimination.

$$x_1 + 3x_2 - 5x_3 = 2$$
$$3x_1 + 11x_2 - 9x_3 = 4$$
$$-x_1 + x_2 + 6x_3 = 5$$

Solution :

An augmented matrix is given by

$$\left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right].$$

We use the boxed element to eliminate any non-zeros below it.

This involves the following row operations

$$\left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right] \begin{array}{l} R2 - 3 \times R1 \\ R3 + R1 \end{array} \Rightarrow \left[\begin{array}{rrr|r} \boxed{1} & 3 & -5 & 2 \\ 0 & 2 & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right].$$

And the next step is to use the 2 to eliminate the non-zero below it. This requires the final row operation

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right] \; R3 - 2 \times R2 \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 0 & -11 & 11 \end{array}\right].$$

This is the augmented form for an upper triangular system, writing the system in extended form we

$$
\begin{aligned}
x_1 + 3x_2 - 5x_3 &= 2 \\
2x_2 + 6x_3 &= -2 \\
-11x_3 &= 11
\end{aligned}
$$

This gives $x_3 = -1$; $x_2 = 2$; $x_1 = -9$.

Problem 2: Solve the system of equation $2x + 4y + 6z = 22$
$3x + 8y + 5x = 27$
$-x + y + 2z = 2$

Solution

$$\begin{bmatrix} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_1}' = \mathbf{1/2}\boldsymbol{R}_1$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_2}' = \boldsymbol{R_2} - \mathbf{3}\boldsymbol{R}_1$; $\boldsymbol{R_3}' = \boldsymbol{R_3} + \boldsymbol{R}_1$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$$

$\boldsymbol{R_2}' = \mathbf{1/2}\boldsymbol{R}_2$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \mathbf{2}\boldsymbol{R}_2$; $\boldsymbol{R_3}' = \boldsymbol{R_3} - \mathbf{3}\boldsymbol{R_2}$

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$\boldsymbol{R_3}' = \mathbf{1/11}\boldsymbol{R}_1$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \mathbf{7}\boldsymbol{R}_3$; $\boldsymbol{R_1}' = \boldsymbol{R_1} - \mathbf{7}\boldsymbol{R}_3$; $\boldsymbol{R_2}' = \boldsymbol{R_2} + \mathbf{2}\boldsymbol{R}_3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus the solution to the system is $x = 3$, $y = 1$, $z = 2$.

Problem 3. Using Gauss-Elimination method solve $2x + y + 4z = 12$, $8x - 3y + 2z = 20$, $4x + 11y - z = 33$.

Solution: Given system of equations in Matrix form AX = B

$$\begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$$

Consider the Augmented Matrix

$$[A \quad B] \sim \begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{bmatrix} \quad R_3 \leftrightarrow R_3 - 2R_1$$

$$R_2 \leftrightarrow R_2 - 4R_1$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & -7 & -14 & -28\\ 0 & 0 & -27 & -27\end{bmatrix} \quad R_3 \leftrightarrow R_3 + (\frac{9}{7})R_2$$

-27Z=-27, Z=1, -7Y-14Z=-28, Y = 2, 2X + Y + 4Z =12, X=3

X = 3, Y = 2, Z = 1

Problem 4. Solve 2x + y + 4z = 4, x – 3y – z = -5, 3x – 2y + 2z = -1 by Gauss Elimination method

Solution: Given system of equations in Matrix form AX = B

$$\begin{pmatrix}2 & 1 & 4\\ 1 & -3 & -1\\ 3 & -2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}4\\ -5\\ -1\end{pmatrix}$$

Consider the Augmented Matrix

$$[A \quad B]\sim\begin{bmatrix}2 & 1 & 4 & 4\\ 1 & -3 & -1 & -5\\ 3 & -2 & 2 & -1\end{bmatrix}$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & -7 & -6 & -14\\ 0 & -7 & -8 & -14\end{bmatrix} R_3 \leftrightarrow 2R_3 - 3R_1, R_2 \leftrightarrow 2R_2 - R_1$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & -7 & -6 & -14\\ 0 & 0 & -2 & 0\end{bmatrix} R_3 \leftrightarrow R_3 - R_2$$

By back substitution methods -2z = 0 , z = 0, -7y-6z=-14, y = 2, 2x + y + 4z = 12 , x=1

X = 1, Y = 2, Z = 0

Problem 5. Solve x + 2y + 3z = 6, 2x +4y + z = 7, 3x + 2y + 9z = 14 by Gauss Elimination method

Solution: Given system of equations in Matrix form AX = B

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 1\\ 3 & 2 & 9\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}6\\ 7\\ 14\end{pmatrix}$$

Consider the Augmented Matrix

$$[A \quad B]\sim\begin{bmatrix}1 & 2 & 3 & 6\\ 2 & 4 & 1 & 7\\ 3 & 2 & 9 & 14\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 3 & 6\\ 0 & 0 & -5 & -5\\ 0 & -4 & 0 & -4\end{bmatrix} R_3 \leftrightarrow R_3 - 3R_1, R_2 \leftrightarrow R_2 - 2R_1$$

$$\sim\begin{bmatrix}1 & 2 & 3 & 6\\ 0 & -4 & 0 & -4\\ 0 & 0 & -5 & -5\end{bmatrix} R_3 \leftrightarrow R_2$$

By back substitution methods -5z = -5 , z = 1, -4y= -4, y = 1, x +2 y + 3z = 6 , x=1

X = 1, Y = 1, Z = 1

ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

As a numerical technique, Gaussian elimination is rather unusual because it is direct. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error. Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

Gauss-Jacobi method

The Jacobi Method The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal. If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal. A matrix A is diagonally dominated if, in each row, the absolute value of the entry on the diagonal is greater than the sum of the absolute values of the other entries. More compactly, A is diagonally dominated if

$$|A_{ii}| > \sum_{j, j \neq i} |A_{ij}| \quad \text{for all } i$$

To begin the Jacobi method, solve the first equation for the second equation for and so on, as follows

$$
\begin{aligned}
x_1 &= 1/a_{11}[b_1 - a_{12}x_2 - \dots - a_{1n}x_n] \\
x_2 &= 1/a_{22}[b_2 - a_{21}x_1 - \dots - a_{2n}x_n] \\
&\vdots \\
x_n &= 1/a_{nn}[b_n - a_{n1}x_1 - a_{n2}x_2 - \dots]
\end{aligned}
$$

Then make an initial approximation of the solution, Initial approximation and substitute these values of into the right-hand side of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often converges to the actual solution.

Problem 1:Use the Jacobi method to approximate the solution of the following system of linear equations.

$$
\begin{aligned}
5x_1 - 2x_2 + 3x_3 &= -1 \\
-3x_1 + 9x_2 + x_3 &= 2 \\
2x_1 - x_2 - 7x_3 &= 3
\end{aligned}
$$

Solution

To begin, write the system in the form

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3$$
$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3$$
$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2.$$

Let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$
as a convenient initial approximation. So, the first approximation is

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$
$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222$$
$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429.$$

Continuing this procedure, you obtain the sequence of approximations shown in Table

n	0	1	2	3	4	5	6	7
x_1	0.000	-0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Because the last two columns in the above table are identical, you can conclude that to three significant digits the solution is $x_1 = 0.186$, $x_2 = 0.331$, $x_3 = -0.423$.

Problem 2. Solve the system of equation using Gauss-Jacobi method 4x-10y+3z= - 3, x+6y+10z= - 3, 10x-5y-2z = 3

Sol. Given equation can be rearranged such that they are diagonally dominant as follows.

10x-5y-2z = 3 →x = 1/10[3+5y+2z]
4x-10y+3z= - 3 →y= -1/10[3+4x+3z]
x+6y+10z= - 3 →z= -1/10[3x+x+6y]

By iteration process, the values are tabulated as follows

Iteration	x = 1/10[3+5y+2z]	y= -1/10[3+4x+3z]	z= -1/10[3x+x+6y]
0	0	0	0
1	0.3	0.3	-0.3
2	0.39	0.33	-0.51
3	0.363	0.303	-0.537
4	0.344	0.284	-0.518
5	0.338	0.282	-0.505

6	0.34	0.284	-0.503
7	0.341	0.285	-0.504
8	0.342	0.285	-0.505
9	0.342	0.285	-0.505

Solution is x = 0.342, y = 0.285, z = -0.505

GAUSS SEIDEL METHOD

Intuitively, the Gauss-Seidel method seems more natural than the Jacobi method. If the solution is converging and updated information is available for some of the variables, surely it makes sense to use that information! From a programming point of view, the Gauss-Seidel method is definitely more convenient, since the old value of a variable can be overwritten as soon as a new value becomes available. With the Jacobi method, the values of all variables from the previous iteration need to be retained throughout the current iteration, which means that twice as much as storage is needed. On the other hand, the Jacobi method is perfectly suited to parallel computation, whereas the Gauss-Seidel method is not. Because the Jacobi method updates or 'displaces' all of the variables at the same time (at the end of each iteration) it is often called the method of simultaneous displacements. The Gauss-Seidel method updates the variables one by one (during each iteration) so its corresponding name is the method of successive displacements.

Problem 1

Solve the following system of equations by Gauss – Seidel method

28x +4y –z = 32

x + 3y + 10z = 24

2x + 17y + 4z = 35

Solution: Since the diagonal element in given system are not dominant, we rearrange the equation as follows

28x +4y – z = 32

2x + 17y + 4z = 35

x + 3y + 10z = 24

Hence

x =1/28[32 – 4y +z]

y = 1/17[35-2x -4z]

z = 1/10[24 –x – 3y]

Setting y = 0 and z = 0, we get,

First iteration

$x^{(1)}$ = 1/28 [32- 4(0) +(0)] = 1.1429

$y^{(1)}$ = 1/17 [35 – 2(1.1429) -4(0)] = 1.9244

$z^{(1)}$ = 1/10 [24 – 1.1429 – 3(1.9244)] = 1.8084

Second Iteration

$x^{(2)}$ = 1/28 [32- 4(1.9244) +(1.8084)] = 0.9325

$y^{(2)} = 1/17\ [\ 35 – 2(0.9325)\ -4(1.8084)\ ] = 1.5236$
$z^{(2)} = 1/10\ [\ 24 – 0.9325 – 3(1.5236)] = 1.8497$
Third Iteration
$x^{(3)} = 1/28\ [\ 32- 4(1.5236) +(1.8497)] = 0.9913$
$y^{(3)} = 1/17\ [\ 35 – 2(0.9913)\ -4(1.8497)] = 1.5070$
$z^{(3)} = 1/10\ [\ 24 –0.9913– 3(1.5070)] = 1.8488$
Fourth Iteration
$x^{(4)} = 1/28\ [\ 32- 4(1.5070\ ) +(1.8488)] = 0.9936$
$y^{(4)} = 1/17\ [\ 35 – 2(0.9936)\ -4(1.8488)] = 1.5069$
$z^{(4)} = 1/10\ [\ 24 – 0.9936 – 3(1.5069)] = 1.8486$
Fifth Iteration
$x^{(5)} = 1/28\ [\ 32- 4(1.5069) +(1.8486)] = 0.9936$
$y^{(5)} = 1/17\ [\ 35 – 2(0.9936)\ -4(1.8486)] = 1.5069$
$z^{(5)} = 1/10\ [\ 24 – 0.9936 – 3(1.5069)] =1.8486$
Since the values of x, y, z are same in the 4^{th} and 5^{th} Iteration, we stop the procedure here.
Hence x = 0.9936, y = 1.5069, z = 1.8486.

Problem 2. Solve the following system of equation by Gauss-Seidel method 4x+2y+z=14, x+5y-z=10, x+y+8z=20

Here the diagonal elements are dominant. Hence we apply Gauss-Seidel method.

Iteration	X=1/4(14-2y-z)	Y=1/5(10-x+z)	Z=1/8(20-x-y)
0	-	0	0
1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.05625	1.98175	1.99525
4	2.0103125	1.9970	1.9991
5	2.001734	1.99947	1.9998
6	2.00030	1.99991	1.99997

The values of solution correct to 4 decimal places are x=2.0000, y = 1.9999, z= 1.9999

1. Solve the system of equations using Gauss-Jacobi method, 8x-3y+2z=20, 4x+11y-z=33, 6x+3y+12z=35

Ans: x=3.0168, y=1.9858, z=0.9117

2. Solve by Gauss-Seidel method x+y+54z=110, 27x+6y-z=85,6x+15y+2z=72
   Ans: x=2.425, y = 3.573, z= 1.926

Power method: To find the numerically largest eigenvalue called dominant eigenvalue andthe corresponding eigen vector of a square matrix A.

1. Find the numerically largest eigenvalue of A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ by power method.

Soln.: Let $X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then $Y_1 = AX_0 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} = 7\begin{pmatrix} 0.714 \\ 1 \\ 0.714 \end{pmatrix} = 7X1$

$Y_2 = AX_1 = \begin{pmatrix} 3.856 \\ 6.428 \\ 3.856 \end{pmatrix} = 6.428\begin{pmatrix} 0.6 \\ 1 \\ 0.6 \end{pmatrix} = 6.428X2$

$Y_3 = AX_2 = \begin{pmatrix} 3.4 \\ 6.2 \\ 3.4 \end{pmatrix} = 6.2\begin{pmatrix} 0.548 \\ 1 \\ 0.548 \end{pmatrix} = 6.2X3$

$Y_4 = AX_3 = \begin{pmatrix} 3.192 \\ 6.096 \\ 3.192 \end{pmatrix} = 6.096\begin{pmatrix} 0.524 \\ 1 \\ 0.524 \end{pmatrix} = 6.096X4$

$Y_5 = AX_4 = \begin{pmatrix} 3.096 \\ 6.048 \\ 3.096 \end{pmatrix} = 6.048\begin{pmatrix} 0.512 \\ 1 \\ 0.512 \end{pmatrix} = 6.048X5$

$Y_6 = AX_5 = \begin{pmatrix} 3.048 \\ 6.024 \\ 3.048 \end{pmatrix} = 6.024\begin{pmatrix} 0.506 \\ 1 \\ 0.506 \end{pmatrix} = 6.024X6$

$Y_7 = AX_6 = \begin{pmatrix} 3.024 \\ 6.012 \\ 3.024 \end{pmatrix} = 6.012\begin{pmatrix} 0.503 \\ 1 \\ 0.503 \end{pmatrix} = 6.012X7$

$Y_8 = AX_7 = \begin{pmatrix} 3.012 \\ 6.006 \\ 3.012 \end{pmatrix} = 6.006\begin{pmatrix} 0.501 \\ 1 \\ 0.501 \end{pmatrix} = 6.006X8$

$Y_9 = AX_8 = \begin{pmatrix} 3.004 \\ 6.002 \\ 3.004 \end{pmatrix} = 6.002\begin{pmatrix} 0.5 \\ 1 \\ 0.5 \end{pmatrix} = 6.002X9$

$Y_{10} = AX_9 = \begin{pmatrix} 3.0 \\ 6.0 \\ 3.0 \end{pmatrix} = 6.0\begin{pmatrix} 0.5 \\ 1 \\ 0.5 \end{pmatrix} = 6.0X10$

$Y_{11} = AX_{10} = \begin{pmatrix} 3.0 \\ 6.0 \\ 3.0 \end{pmatrix} = 6.0\begin{pmatrix} 0.5 \\ 1 \\ 0.5 \end{pmatrix} = 6.0X11$

Convergence has occurred. The dominant eigen value is 6 and the corresponding eigen vector $\begin{pmatrix} 0.5 \\ 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT II

NUMERICAL DIFFERENTIATION AND INTEGRATION

2. **Interpolation**: Interpolation is the process of finding the intermediate values of a function (which is not explicitly known) from a set of values at specific points given in a tabulated form. The process of computing y corresponding to x where $x_i < x < x_{i+1}$, I = 0, 1, 2, ….. (n-1), is interpolation.

Extrapolation: If $x < x_0$ or $x > x_n$ then the process is called extrapolation

: Lagrange's Interpolation Formula for Unequal intervals

If $y_0, y_1, y_2, \ldots\ldots Y_n$ are the values of a function y=f(x) corresponding to the arguments $x_0, x_1, x_2, \ldots\ldots Xn$ which are not necessarily equally spaced then

$$Y = f(x) = \frac{(x-x1)(x-x2)\ldots\ldots(x-xn)}{(x0-x1)(x0-x2)\ldots\ldots(x0-xn)}y_0 + \ldots.+ \frac{(x-x0)(x-x1)\ldots\ldots(x-xn-1)}{(xn-x0)(xn-x1)\ldots\ldots(xn-xn-1)}\, y_n$$

Problem 1. Determine by Lagrange's method the percentage number of patients over 40 years using the following data

Age over x years	30 x_0	35 x_1	45 x_2	55 x_3
% number y of patients	148 y_0	96 y_1	68 y_2	34 y_3

Soln. By Lagrange's polynomial

$$Y = f(x) = \frac{(x-x1)(x-x2)(x-x3)}{(x0-x1)(x0-x2)(x0-x3)}\, y_0 + \ldots\ldots\ldots..$$

$$Y = \frac{(x-35)(x-45)(x-55)}{(-5)(-15)(-25)}148 + \frac{(x-30)(x-45)(x-55)}{(5)(-10)(-20)}96 + \frac{(x-30)(x-35)(x-55)}{(15)(10)(-10)}68 + \frac{(x-30)(x-35)(x-45)}{(25)(20)(10)}34$$

$$Y = \frac{-148}{5} + \frac{3}{4}\, X\, 96 + \frac{68}{2} - \frac{34}{20} = 74.7$$

2. Apply Lagrange's interpolation formula to find f(x) if f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16

and f(7) =128. Hence find f(5) and f(6).

Soln. Given data

X	1 x_0	2 x_1	3 x_2	4 x_3	7 x_4
Y= f(x)	2 y_0	4 y_1	8 y_2	16 y_3	128 y_4

By Lagrange's polynomial $Y = f(x) = \frac{(x-x1)(x-x2)(x-x3)(x-x4)}{(x0-x1)(x0-x2)(x0-x3)(x0-x4)} y_0 + \ldots\ldots\ldots\ldots$

f(x) =

$$\frac{(x-2)(x-3)(x-4)(x-7)}{(-1)(-2)(-3)(-6)}2+\frac{(x-1)(x-3)(x-4)(x-7)}{(-1)(-2)(-1)(-5)}4+\frac{(x-1)(x-2)(x-4)(x-7)}{(-1)(2)(-4)(1)}8+\frac{(x-1)(x-2)(x-3)(x-7)}{(3)(2)(-3)(1)}16+\frac{(x-1)(x-2)(x-3)(x-4)}{(5)(4)(3)(6)}128$$

f(x) = 1/90 [$11x^4 - 80x^3 + 295x^2 - 310x + 264$]

f(5) = 32.93 and f(6) = 66.67

Problem 3: Determine the value of $y(1)$ from the following data using Lagrange's Interpolation

x	-1	0	2	3
y	-8	3	1	12

Solution: given

x	$x_0=-1$	$x_1=0$	$x_2=3$	$x_n=3$
y	$y_0=-8$	$y_1=3$	$y_2=1$	$y_n=12$

Since the intervals ere not uniform we cannot apply Newton's interpolation.

Hence by Lagrange's interpolation for unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}y_2+\frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}y_n$$

$$y(x)=\frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)----(1)$$

To compute $y(1)$ put $x = 1$ in (1), we get

$$y(x=1)=\frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(1+1)(1-2)(1-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(1+1)(1-0)(1-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(1+1)(1-0)(1-2)}{(3+1)(3-0)(3-2)}(12)$$

$$\Rightarrow y(x=1)=2$$

To find polynomial $y(x)$, from (1) we get

$$y(x)=\frac{2}{3}(x^3-5x^2+6x)+\frac{1}{2}(x^3-4x^2+x+6)$$
$$-\frac{1}{6}(x^3-2x^2-3x)+\frac{1}{1}(x^3-x^2-2x)----(1)$$

$$y(x)=x^3\left(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1\right)+x^2\left(\frac{-10}{3}+\frac{-4}{2}+\frac{2}{6}-1\right)+x\left(\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2\right)+\left(\frac{6}{2}\right)$$

$$\Rightarrow y(x)=2x^3-6x^2+3x+3-----(2)$$

To compute $y(1)$ put $x = 1$ in (2), we get $y(x=1)=2-6+3+3=2$

Inverse interpolation

For a given set of values of x and y, the process of finding $x(dependent)$ given $y(independent)$ is called Inverse interpolation

$$x(y)=\frac{(y-y_1)(y-y_2)--(y-y_n)}{(y_0-y_1)(y_0-y_2)--(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)--(y-y_n)}{(y_1-y_0)(y_1-y_2)--(y_1-y_n)}x_1$$
$$+\frac{(y-y_0)(y-y_1)--(y-y_n)}{(y_2-y_0)(y_2-y_1)--(y_2-y_n)}x_2+---+\frac{(y-y_0)(y-y_1)--(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)--(y_n-y_{n-1})}x_n$$

Problem 4: Estimate the value of x given $y=100$ from the following data, $y(3)=6$ $y(5)=24$, $y(7)=58$, $y(9)=108$, $y(11)=174$

Solution: given

x	$x_0=3$	$x_1=5$	$x_2=7$	$x_3=9$	$x_n=11$
y	$y_0=6$	$y_1=24$	$y_2=58$	$y_3=108$	$y_n=174$

By applying Lagrange's inverse interpolation

$$x(y)=\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_n)}{(y_0-x_1)(y_0-y_2)(y_0-y_3)(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_n)}x_1$$
$$+\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_n)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_n)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_n)}x_3$$
$$+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)(y_n-y_{n-1})}x_n$$
$$\Rightarrow x(100)=\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3)+\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$
$$+\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7)+\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$
$$+\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$
$$\Rightarrow x(100)=0.35344-1.51547+2.88703+7.06759-0.13686=8.65573$$

Gregory Newton's Interpolation:

Newton's Forward Interpolation for equal intervals:

If y_0, y_1, y_2,................Y_n are the values of a function y=f(x) corresponding to the arguments x_0, x_1, x_2,............Xn which are equally spaced where $x_i - x_{i-1} = h$, for i = 1 to n then

$$y(x)=y_0+\frac{(u)}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2(y_0)+\frac{u(u-1)(u-2)}{3!}\Delta^3(y_0)+\cdots \text{ where } u=\frac{x-x_0}{h}$$

Newton's Backward Interpolation for equal intervals:

If y_0, y_1, y_2, ... Y_n are the values of a function y=f(x) corresponding to the arguments x_0, x_1, x_2,............ Xn which are equally spaced where $x_i - x_{i-1} = h$, for i = 1 to n then

$y(x) = y_n + \frac{(v)}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2(y_n) + \frac{v(v+1)(v+2)}{3!}\nabla^3(y_n) + \cdots$ where $v = \frac{x-x_n}{h}$

Remark:

(i) The process of finding the values of $y(x_i)$ outside the interval (x_0, x_n) is called *extrapolation*

(ii) The *interpolating polynomial* is a function $p_n(x)$ through the data points $y_i = f(x_i) = P_n(x_i)$ i=0,12,..n

(iii) Gregory-Newton's forward interpolation formula (a) can be applicable if the interval difference h is constant and used to interpolate the value of $y(x_i)$ nearer to beginning value x_0 of the data set

(iv) If $y = f(x)$ is the exact curve and $y = p_n(x)$ is the interpolating polynomial then the *Error in polynomial interpolation* is $y(x) - p_n(x)$ given by

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)--(x-x_n): x_0 < x < x_n, x_0 < c < x_n ---(c)$$

(v) *Error in Newton's forward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}u(u-1)(u-2)--(u-n): x_0 < x < x_n, x_0 < c < x_n ----(d)$$

(vi) *Error in Newton's backward interpolation is*

$$Error = \frac{h^{n+1}y^{(n+1)}(c)}{(n+1)!}v(v+1)(v+2)--(v+n): x_0 < x < x_n, x_0 < c < x_n ----(e)$$

Problem 1: If y(10) = 35.3, y(15) = 32.4, y(20) = 29.2, y(25) = 26.1, y(30) = 23.2 and y(35) = 20.5, find y(12) using Newton's forward interpolation formula.

Solution: The difference table is

X	Y	Δ	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	35.3	- 2.9				
15	32.4	- 3.2	- 0.3			
20	29.2	- 3.1	0.1	0.4	- 0.3	
25	26.1	- 2.9	0.2	0.1	- 0.1	0.2
30	23.2	- 2.7	0.2	0.1		
35	20.5					

By Newton's forward interpolation formula

$y(x) = y_0 + \frac{(u)}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2(y_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3(y_0) + \cdots$ where $u = \frac{x-x_0}{h} = \frac{12-10}{5} = 0.4$

$$y(12) = 35.3 + \frac{(0.4)}{1!}(-2.9) + \frac{(o.4)(-0.6)}{2!}(-0.3) + \frac{(0.4)(-0.6)(-1.6)}{3!}(0.4) + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{4!}(-0.3) + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-3.6)}{5!}(0.2)$$

$$y(12) = 35.3 - 1.16 + 0.036 + 0.0256 + 0.01248 + 0.0059904$$

$$y(12) = 34.22$$

Problem 2. The population of a town in the census is as given in the data. Estimate the population in the year 1996 using Newton's (i) forward interpolation (ii) backward interpolation formula.

Year(x)	1961	1971	1981	1991	2001
Population (in 1000's)	46	66	81	93	101

Solution: The difference table is

X	Y	Δ	$\Delta^2 y$	Δ^3y	$\Delta^4 y$
1961	46	20			
1971	66	15	-5		
1981	81	12	-3	2	-3
1991	93	8	-4	-1	
2001	101				

By Newton's forward interpolation formula

$$y(x) = y_0 + \frac{(u)}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2(y_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3(y_0) + \cdots$$

where u = $\frac{x-x_0}{h} = \frac{1996-1961}{10}$ = 3.5

$$y(x=1996) = 46 + \frac{(3.5)}{1!}20 + \frac{(3.5)(2.5)}{2!}(-5) + \frac{(3.5)(2.5)(1.5)}{3!}2 + \frac{(3.5)(2.5)(1.5)(0.5)}{4!}(-3)$$

$$y(x=1996) = 46 + 70 - 21.875 + 4.375 - 0.8203125$$

$$y(x=1996) = 97.6796875$$

By Newton's backward interpolation formula

$$y(x) = y_n + \frac{(v)}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2(y_n) + \frac{v(v+1)(v+2)}{3!}\nabla^3(y_n) + \cdots$$

where v = $\frac{x-x_n}{h} = \frac{1996-2001}{10} = -0.5$

$$y(x=1996) = 101 + \frac{(-0.5)}{1!}8 + \frac{(-0.5)(0.5)}{2!}(-4) + \frac{(-0.5)(0.5)(1.5)}{3!}(-1) + \frac{(-0.5)(2.5)(1.5)(0.5)}{4!}(-3)$$

$y(x=1996)=101-4+0.5+0.06250.1171875$
$y(x=1996)=97.6796$

Problem 3. Find the interpolating polynomial for y from the following data using both Newton's forward and backward formula

X	4	6	8	10
Y	1	3	8	16

Solution: The difference table is

X	Y	Δy	$\Delta^2 y$	Δ^3y
4	1			
6	3	2		
8	8	5	3	0
10	16	8	3	

By Newton's forward interpolation formula

$$y(x)=y_0+\frac{(u)}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2(y_0)+\frac{u(u-1)(u-2)}{3!}\Delta^3(y_0)+\cdots \text{ where u } =\frac{x-x_0}{h}=\frac{x-4}{2}$$

$$y(x)=1+\frac{\left(\frac{x-4}{2}\right)}{1!}2+\frac{\frac{x-4}{2}\left(\frac{x-4}{2}-1\right)}{2!}3+\frac{\frac{x-4}{2}\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!}0$$

$$y(x)=1+x-4+\frac{(3)}{8}(x-4)(x-6)$$

$y(x)=\frac{(3x^2-22x+48)}{8}$ which is the required interpolating polynomial for y.

By Newton's backward interpolation formula

$$y(x)=y_n+\frac{(v)}{1!}\nabla y_n+\frac{v(v+1)}{2!}\nabla^2(y_n)+\frac{v(v+1)(v+2)}{3!}\nabla^3(y_n)+\cdots \text{ where v } =\frac{x-x_n}{h}=\frac{x-10}{2}$$

$y(x)=16+\frac{\frac{x-10}{2}}{1!}8+\frac{\frac{x-10}{2}\left(\frac{x-10}{2}+1\right)}{2!}3+0$, $y(x)=\frac{(3x^2-22x+48)}{8}$ which is the required interpolating polynomial for y.

Problem4: Estimate the number of students whose weight is between 60 lbs and 70 lbs from the following data

Weight(lbs)	0-40	40-60	60-80	80-100	100-120
No.Students	250	120	100	70	50

Solution: let x-Weight less than 40 lbs, y-Number of Students, $\Rightarrow x_0=40, x_1=60, x_2=80, x_3=100, x_n=120$, Here all the intervals are equal with h=x_1-x_0=20 we apply Newton interpolation

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$250 = y_0$	$y_1 - y_0 = 120 = \Delta y_0$			
60	$370 = y_1$	$y_2 - y_1 = 100 = \Delta y_1$	$-20 = \Delta^2 y_0$	$-10 = \Delta^3 y_0$	
80	$470 = y_2$	$y_3 - y_2 = 70 = \Delta y_2$	$-30 = \Delta^2 y_1$	$10 = \nabla^2 y_n$	$20 = \Delta^4 y_0 = \nabla^4 y_n$
100	$540 = y_3$	$y_n - y_{n-1} = 50 = \nabla y_n$	$-20 = \nabla^2 y_n$		
120	$590 = y_n$				

Case (i): to find the number of students y whose weight less than 60 lbs $(x = 60)$

From the difference table the number of students y whose weight less than 60 lbs $(x = 60) = 370$

Case (ii): to find the number of students y whose weight less than 70 lbs $(x = 70)$

Since $x = 70$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ----(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{20}(70-40) = \frac{3}{2} \Rightarrow u-1 = \frac{3}{2}, u-2 = \frac{2}{2}, u-2 = \frac{-1}{2}, u-3 = \frac{-3}{2} -----(2)$$

Substituting (2) in (1), we get

$$y(x=70) = 250 + \frac{120}{1}\left(\frac{3}{2}\right) + \frac{-20}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + \frac{-10}{6}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) + \frac{20}{24}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right) = 423.59$$

The number of students y whose weight less than 70 lbs $(x = 70)$ =424

Number of students whose weight is between 60 lbs and 70 lbs $=$

$$\left\{\begin{matrix}\text{The number of students } y \\ \text{whose weight less than 70 lbs}\end{matrix}\right\} - \left\{\begin{matrix}\text{The number of students } y \\ \text{whose weight less than 60 lbs}\end{matrix}\right\} = 424\text{-}370 = 54$$

NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i), I = 0,1,12,….,n of a function. The process of comuputing the derivative of the function y at a particular value of x from the given set of values is called Numerical Differentiation. This maybe done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired. Numerical diffentiation can be done for equal and unequal intervals.

Gregory Newton's Forward Difference Formula for Derivatives:

$$y'^{(x)} = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2(y_0) + \frac{3u^2-6u+2}{6}\Delta^3(y_0) + \frac{[4u^3-18u^2+22u-6]}{24}\Delta^4(y_0)\cdots\right]$$

$$y''^{(x)} = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3(y_0) + \frac{6u^2-18u+11}{12}\Delta^4(y_0) + \cdots\right]$$

$$y'''(x) = \frac{d^3y}{dx^3} = \frac{1}{h^3}[\Delta^3(y_0) + \frac{12u-18}{12}\Delta^4(y_0) + \cdots]$$

And so on where u = $\frac{x-x_0}{h}$, x is the value at which the derivative needs to be found. X_0 is the first value of x, h is the common difference in x values.
At particular case, x = X_0, u = 0, then the derivative formula reduced to

$$\left(\frac{dy}{dx}\right)(x = X0) = \frac{1}{h}[\Delta y_0 - \frac{\Delta^2(y_0)}{2} + \frac{\Delta^3(y_0)}{3} - \frac{\Delta^4(y_0)}{4} + \cdots]$$
$$\left(\frac{d^2y}{dx^2}\right)(x = X0) = \frac{1}{h^2}[\Delta^2 y_0 - \Delta^3(y_0) + \frac{11}{12}\Delta^4(y_0) + \cdots]$$
$$\left(\frac{d^3y}{dx^3}\right)(x = X0) = \frac{1}{h^3}[\Delta^3(y_0) - \frac{3}{2}\Delta^4(y_0) + \cdots]$$

2.3.2. Gregory Newton's Backward Difference Formula for Derivatives:

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}[\nabla y_n + \frac{2v+1}{2}\nabla^2(y_n) + \frac{3u^2-6u+2}{6}\nabla^3(y_n) + \frac{[4u^3-18u^2+22u-6]}{24}\nabla^4(y_n)\cdots]$$
$$y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}[\nabla^2 y_n + (u-1)\nabla^3(y_n) + \frac{6u^2-18u+11}{12}\nabla^4(y_n) + \cdots]$$
$$y'''(x) = \frac{d^3y}{dx^3} = \frac{1}{h^3}[\nabla^3(y_n) + \frac{12u-18}{12}\nabla^4(y_n) + \cdots]$$

And so on where u = $\frac{x-x_0}{h}$, x is the value at which the derivative needs to be found. X_0 is the first value of x, h is the common difference in x values.
At particular case, x = X_0, u = 0, then the derivative formula reduced to

$$\left(\frac{dy}{dx}\right)(x = Xn) = \frac{1}{h}[\nabla y_n - \frac{\nabla^2(y_n)}{2} + \frac{\nabla^3(y_n)}{3} - \frac{\nabla^4(y_n)}{4} + \cdots]$$
$$\left(\frac{d^2y}{dx^2}\right)(x = Xn) = \frac{1}{h^2}[\nabla^2 y_n - \nabla^3(y_n) + \frac{11}{12}\nabla^4(y_n) + \cdots]$$
$$\left(\frac{d^3y}{dx^3}\right)(x = Xn) = \frac{1}{h^3}[\nabla^3(n) - \frac{3}{2}\nabla^4(y_n) + \cdots]$$

Problem 5: Find the rate of growth of population in the year 1941&1961 from the following table

Year	1931	1941	1951	1961	1971
Population	40.62	60.80	79.95	103.56	132.65

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation
Difference Table: let x-year,y-Population

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	$40.62 = y_0$	$y_1 - y_0 = 20.18 = \Delta y_0$			
1941	$60.80 = y_1$	$y_2 - y_1 = 19.15 = \Delta y_1$	$-1.03 = \Delta^2 y_0$	$5.49 = \Delta^3 y_0$	
1951	$79.95 = y_2$	$y_3 - y_2 = 23.61 = \Delta y_2$	$4.46 = \Delta^2 y_1$	$1.02 = \nabla^2 y_n$	$-4.47 = \Delta^4 y_0 = \nabla^4 y_n$
1196	$103.56 = y_3$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$5.48 = \nabla^2 y_n$		
1971	$132.65 = y_n$				

Case (i): to find rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$

Since $x = 1941$ is nearer to x_0 we apply Newton's forwarded formula for derivative

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{\Delta^2 y_0}{2}(2u-1) + \frac{\Delta^3 y_0}{6}(3u^2 - 6u + 2) + \frac{\Delta^4 y_0}{24}(4u^3 - 18u^2 + 22u - 6) + --- \right\}$$

$$where\, u = \frac{1}{h}(x - x_0) = \frac{1}{10}(1941-1931) = 1$$

$$\Rightarrow y'(x = 1941) = \frac{dy}{dx} = \frac{1}{10}\left\{20.18 + \frac{-1.03}{2}(2-1) + \frac{5.49}{6}(3-6+2) + \frac{-4.47}{24}(4-18+22-6) + --- \right\}$$

The rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$ $y'(1941) = 2.36425$

Case (ii): to find rate of growth of population $(\frac{dy}{dx})$in the year $(x = 1961)$

Since $x = 1961$ is nearer to x_n we apply Newton's backward formula for derivative

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2}(2v+1) + \frac{\nabla^3 y_n}{6}(3v^2 + 6v + 2) + \frac{\nabla^4 y_n}{24}(4v^3 + 18v^2 + 22v + 6) + --- \right\}$$

$$v = \frac{1}{h}(x - x_n) = \frac{1}{10}(1961-1971) = -1$$

$$\Rightarrow y'(x = 1961) = \frac{dy}{dx} = \frac{1}{10}\left\{29.09 + \frac{5.48}{2}(-2+1) + \frac{1.02}{6}(3-6+2) + \frac{-4.47}{24}(-4+18-22+6) + --- \right\}$$

The rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1961)$= $y'(1961) = 2.65525$

Problem 6 A rod is rotating in a plane, estimate the angular velocity and angular acceleration of the rod at time 6 secs from the following table

Time-t(sec)	0	0.2	0.4	0.6	0.8	1.0
Angle-θ(radians)	0	0.12	0.49	1.12	2.02	3.20

Solution: Here all the intervals are equal with $h=x_1-x_0=0.2$ we apply Newton interpolation Difference Table: let x- time (sec), y-Angle (radians)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$0 = y_0$	$y_1 - y_0 = 0.12 = \Delta y_0$			
	$0.12 = y_1$	$y_2 - y_1 = 0.37 = \Delta y_1$	$0.25 = \Delta^2 y_0$	$0.01 = \Delta^3 y_0$	
	$0.49 = y_2$	$y_3 - y_2 = 0.63 = \Delta y_2$	$0.26 = \Delta^2 y_1$	$0.01 = \Delta^3 y_1$	$0 = \Delta^4 y_0$
	$1.12 = y_3$	$y_4 - y_3 = 0.90 = \Delta y_3$	$0.27 = \Delta^2 y_2$	$0.01 = \nabla^2 y_n$	$0 = \nabla^4 y_n$
	$2.02 = y_4$	$y_n - y_{n-1} = 1.18 = \nabla y_n$	$0.28 = \nabla^2 y_n$		
	$3.20 = y_n$				

Case (i): to find Angular velocity $\left(\frac{dy}{dx}\right)$ in time $(x = 0.6\ sec)$

Since $x = 0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2}(2v+1) + \frac{\nabla^3 y_n}{6}(3v^2+6v+2) + \frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6) + --- \right\}$$

$$v = \frac{1}{h}(x - x_n) = \frac{1}{0.2}(0.6-1.0) = -2$$

$$y'(x=0.6) = \frac{dy}{dx} = \frac{1}{0.2}\left\{1.18 + \frac{0.28}{2}(-4+1) + \frac{0.01}{6}(12-12+2) + \frac{0}{24}(4v^3+18v^2+22v+6) + --- \right\}$$

$\Rightarrow$ *The angular velocity* $y'(x=0.6) = 3.81665\ radian/\sec$

Case (ii): to find Angular acceleration $\left(\frac{d^2y}{dx^2}\right)$ in time $(x = 0.6\ sec)$

Since $x = 0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n (v+1) + \frac{\nabla^4 y_n}{24}(12v^2 + 36v + 22) + --- \right\}$$

$$where\ v = \frac{1}{h}(x - x_n) = \frac{1}{0.2}(0.6-1.0) = -2$$

$$\Rightarrow y''(x=0.6) = \frac{1}{0.2^2}\left\{0.28 + \frac{0.01}{1}(-2+1) + 0\right\}$$

$y''(0.6) = 6.75\ radian/\sec^2$

Numerical Integration

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + --)\} \text{ where } h = \frac{1}{n}(x_n - x_0), n - \text{number of intervals}$$

Simpson's $^1/_3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + y_6 + -) + 4(y_1 + y_3 + y_5 + --)\}$$

$$\text{where } h = \frac{1}{n}(x_n - x_0), n - \text{number of intervals}$$

Simpson's $^3/_8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0 + y_n) + 2(y_3 + y_6 + y_9 + -) + 3(y_1 + y_2 + y_4 + y_5 + --)\}$$

$$\text{where } h = \frac{1}{n}(x_n - x_0), n - \text{number of intervals}$$

Remarks:

1) Geometrical interpretation of $\int_{x_0}^{x_n} y(x)dx$ is approximated by the sum of area of the trapezium
2) Simpson's $^1/_3$ rule is applicable when number of intervals are multiples of 2 and Simpson's $^3/_8$ rule is applicable when number of intervals are multiples of 3
3) The error in trapezoidal rule is $\frac{b-a}{12}h^2 M$ where $M = max\{y_0'', y_1'', \ldots\}$ which is of order h^2
4) The error in Simpson's $^1/_3$ rule rule is $\frac{b-a}{180}h^4 M$ where $M = max\{y_0'''', y_2'''', \ldots\}$ which is of order h^4

Problem7: Evaluate $\int_1^6 \frac{1}{1+x^2}dx$ using (i) Trapezoidal rule (ii) Simpson's $^1/_3$ rule (iii) Simpson's $^3/_8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^6 \frac{1}{1+x^2}dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x) = \frac{1}{1+x^2}, x_0 = 0, x_0 + nh = 6 - - - - (1)$

Choose the number of interval (n)=6 so that we can apply all rules

x	$x_0 = 0$	$x_1 = x_0 + h = 1$	$x_2 = x_1 + h = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$	$x_n = 6$
$y(x) = \frac{1}{1+x^2}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

case(i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+y_3+y_4+--)$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\left\{(1+\frac{1}{37})+2(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26})\right\}=1.410799$$

Case (ii) Simpson's $1/3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\left\{(y_0+y_n)+2(y_2+y_4+y_6+-)+4(y_1+y_3+y_5+--)\right\}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{1}{3}\left\{(1+\frac{1}{37})+2(\frac{1}{5}+\frac{1}{17})+4(\frac{1}{2}+\frac{1}{10}+\frac{1}{26})\right\}=1.36617$$

Case(iii) Simpson's $3/8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\left\{(y_0+y_n)+2(y_3+y_6+y_9+-)+3(y_1+y_2+y_4+y_5+--)\right\}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3}{8}\left\{(1+\frac{1}{37})+2(\frac{1}{10})+3(\frac{1}{2}+\frac{1}{5}+\frac{1}{17}+\frac{1}{26})\right\}=1.35708$$

Comparison

Exact value $\int_0^6 \frac{1}{1+x^2}dx = \left[\tan^{-1}(x)\right]_{x=0}^{x=6} = \tan^{-1}(6)-\tan^{-1}(0) = 1.40565$

Hence trapezoidal rule gives better approximation than Simpson's rule.

Problem 8: By dividing the range into 10 equal part Determine the value of $\int_0^{\pi} \sin x\, dx$ using (i) Trapezoidal rule (ii) Simpson's $1/3$ rule (iii) Simpson's $3/8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^{\pi} \sin x\, dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x)=\sin x, x_0=0, x_0+nh=\pi \text{ and } n=10----(1)$

$given\,number\,of\,\text{int}ervals(n)=10,(1)\Rightarrow h=\frac{1}{n}(x_n-x_0)=\frac{1}{10}(\pi-0)=\frac{\pi}{10}$

x	$x_0=0$	$x_1=x_0+h=\frac{\pi}{10}$	$x_2=x_1+h=\frac{2\pi}{10}$	$x_3=\frac{3\pi}{10}$	$x_4=\frac{4\pi}{10}$	$x_5=\frac{5\pi}{10}$
$y(x)=\sin(x)$	$\sin(0)$ $=0$	$\sin(\frac{\pi}{10})$ $=0.30901$	$\sin(\frac{2\pi}{10})$ $=0.58779$	$\sin(\frac{3\pi}{10})$ $=0.80901$	$\sin(\frac{4\pi}{10})$ $=0.95106$	$\sin(\frac{5\pi}{10})$ $=1.0$

x		$x_6 = \frac{6\pi}{10}$	$x_7 = \frac{7\pi}{10}$	$x_8 = \frac{8\pi}{10}$	$x_9 = \frac{9\pi}{10}$	$x_n = \pi$
$y(x) = \sin(x)$		$\sin(\frac{6\pi}{10}) = 0.95106$	$\sin(\frac{7\pi}{10}) = 0.80902$	$\sin(\frac{8\pi}{10}) = 0.58779$	$\sin(\frac{9\pi}{10}) = 0.30902$	$\sin(\frac{10\pi}{10}) = 0$

Case (i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + \text{---})$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\{(0+0) + 2(0.30901 + 0.58779 + 0.80901 + 0.95106 + 1.0 + 0.95106 + 0.80901 + 0.58779 + 0.30901)\ \}$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = 1.98352$$

Case (ii) Simpson's $^1/_3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \text{-}) + 4(y_1 + y_3 + y_5 + \text{---})\ \}$$

$$\Rightarrow \int_0^6 \sin(x)dx = \frac{\pi}{30}\{(0+0) + 2(0.58779 + 0.95106 + 0.95106 + 0.58779) + 4(0.30901 + 0.80901 + 1.0 + 0.80901 + 0.30901\}$$

$$\Rightarrow \int_0^6 \sin(x)dx = 2.00010$$

Case (iii) Simpson's $^3/_8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \text{-}) + 3(y_1 + y_2 + y_4 + y_5 + \text{---})\ \}$$

This rule cannot be applied since n is not a multiple of 3

Comparison

Exact value $\int_0^\pi \sin(x)dx = \left[-\cos(x)\right]_{x=0}^{x=\pi} = -\left[\cos(\pi) - \cos(0)\right] = 2.0$

Hence, Simpson's $^1/_3$ rule gives better approximation than trapezoidal rule

2.4.4 Gausian Quadrature Formula: $\int_a^b f(x)dx$,

I = $\left(\frac{b-a}{2}\right)\int_{-1}^{1} \emptyset(u)du$, $x = \left(\frac{b-a}{2}\right)u + \left(\frac{b+a}{2}\right)$

1. Evaluate the integral $\int_0^1 \frac{dx}{1+x^2}$ using 2 point and 3 point Gaussian Quadrature Formula

Solution: Putting $x = \left(\frac{1}{2}\right)u + \left(\frac{1}{2}\right)$, we get $I = \int_{-1}^{1} \frac{\frac{du}{2}}{1+\left(\frac{u+1}{2}\right)^2} = \int_{-1}^{1} \emptyset(u)\,du, \emptyset(u) = \frac{2}{u^2+2u+5}$

By Gaussian 2-point formula $I = \emptyset\left(-\frac{1}{\sqrt{3}}\right) + \emptyset\left(\frac{1}{\sqrt{3}}\right) =$

$$= \frac{2}{\frac{1}{3} - \frac{2}{\sqrt{3}} + 5} + \frac{2}{\frac{1}{3} + \frac{2}{\sqrt{3}} + 5}$$

=0.4786+0.3083

I= 0.7869

By Gaussian 3-point formula

$$I = \frac{5}{9}\phi(u_1) + \frac{8}{9}\phi(u_2) + \frac{5}{9}\phi(u_3)$$

$$I = \frac{5}{9}\phi\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\phi(0) + \frac{5}{9}\phi\left(\sqrt{\frac{3}{5}}\right)$$

I = 5/9(0.4937288) + 8/9(0.4) + 5/9(0.2797518) = 0.785267

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –III

POLYNOMIAL APPROXIMATIONS

LEAST SQUARES APPROXIMATION

To find an approximate function to the given set of values is called least squares regression. The approximating function is called least squares approximation.
The sum of the squares of the residuals of the plotted points be assumed to be the least. This is the principle of least squares, $S = \sum_{r=1}^{n}\{y_r - (ax_r + b)\}^2$ is least.

Fitting the straight line y = ax + b

The normal equations are $\sum y = a\sum x + nb --- (1)$

$$\sum xy = a\sum x^2 + b\sum x --- (2)$$

Fitting a second degree parabola y = a$x^2 + bx + c$

By the principle of least squares, $S = \sum_{r=1}^{n}\{y_r - (ax_r^2 + bx_r + c)\}^2$ is least.
The normal equations are $\sum y = a\sum x^2 + b\sum x + nc --- (1)$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x --- (2)$$

$$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2 --- (3)$$

Problem 1. Fit a straight line to the following data by the method of least squares:

x	3.4	4.3	5.4	6.7	8.7	10.6
y	4.5	5.8	6.8	8.1	10.5	12.7

Soln. Let the equation of the best fitting straight line be y = ax + b--------- (1)
The normal equations are $\sum y = a\sum x + nb --- (2)$

$$\sum xy = a\sum x^2 + b\sum x --- (3)$$

X	Y	x^2	xy
3.4	4.5	11.56	15.30

4.3	5.8	18.49	24.94
5.4	6.8	29.16	36.72
6.7	8.1	44.89	54.27
8.7	10.5	75.69	91.35
10.6	12.7	112.36	132.62
Total 39.1	48.4	292.15	357.20

Substituting all the values in eqn (2) and (3) 39.1a+6b=48.4 ---------------------- (4)

292.15a+39.1b=357.2 -------------- (5)

By solving (4) and (5) we get a= 1.119, b = 0.7744, the required equation is y =1.119 x + 0.7744

Problem 2. Fit a second degree parabola to the following data by the method of least squares:

x	0.5	1	2	3	5
y	3.1	6	11.2	14.8	20

Solution: Fitting a second degree parabola y = a$+bx+cx^2$ --------------------(1)

The normal equations are $\sum y = na + b\sum x + c\sum x^2$---------(2)

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 --- (3)$$
$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 --- (4)$$

X	Y	x^2	x^3	x^4	xy	x^2y
0.5	3.1	0.25	0.125	0.0625	1.55	0.775
1	6	1	1	1	6	6
2	11.2	8	8	16	22.4	44.8
3	14.8	27	27	81	44.4	133.2
5	20	125	125	625	100	500
Total 11.5	55.1	39.25	161.125	723.0625	174.35	684.775

Substituting all the values in eqn (2), (3) and (4)

5a + 11.5b + 39.25c = 55.1----------------- (5)

11.5a + 39.25b + 161.125c = 174.35----- (6)

39.25a + 161.125b + 723.0625c = 684.775 ----(7)

By solving these equations a= 0.0882, b=6.4523, c= -0.4979

The best fitting parabola y = y = 0.0882 $+6.4523\,x - 0.4979x^2$

Chebyshev polynomial:

$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} - \frac{1}{2}\frac{[f(x_r)]^2 f''(x_r)}{[f'(x_r)]^3}$ is the iterative formula of Chebyshev method.

Problem 1: Find the positive root of the equation x^3- 4x + 1=0, correct to 4 places of decimals, using Chebyshev method.

Soln.: Let f(x) = x^3- 4x + 1. Then f '(x) = $3x^2$-4, f ''(x) = 6x

f(0) = 1>0, f(1) = - 4 < 0, The positive lies between 0 and 1

Taking x_0=0, f_0=f(x_0)=1, f'_0=f ' (x_0)=- 4, f''_0=f''(x_0)=0

Chebyshev iterative formula $x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} - \frac{1}{2}\frac{[f(x_r)]^2 f''(x_r)}{[f'(x_r)]^3}$

$x_1 = x_0 - \frac{f_1}{f'_1} - \frac{1}{2}\frac{f_0^2}{f_0'^3} f''_0$= 0 – [1/(-4)] = 0.25, $f_1 = 0.05613;\ f'_1 = -3.8125; f''_1 = 1.5$

$x_2 = x_1 - \frac{f_1}{f'_1} - \frac{1}{2}\frac{f_1^2}{f_1'^3} f''_1$=0.25+$\frac{0.05613}{3.8125} + \frac{1}{2}\frac{(0.05613)^2}{(3.8125)^3} X(1.5) = 0.25410$,

$f_2 = 0.00001;\ f'_2 = -3.8063; f''_1 = 1.5246$

$x_3 = x_2 - \frac{f_2}{f'_2} - \frac{1}{2}\frac{f_2^2}{f_2'^3} f''_2$=0.25410+$\frac{0.00001}{3.8063} + \frac{1}{2}\frac{(0.00001)^2}{(3.8063)^3} X(1.5246) = 0.25410$

Since $x_2 = x_3$ = 0.25410, the required root.

Problem 2: Find the root of the equation $2x^3 - 3x + 6 = 0$, that lies between -2 and -1, correct to 4 places of decimals, using Chebyshev method

Soln.: Let $f(x) = 2x^3 - 3x + 6$. Then $f'(x) = 6x^2 - 3$, $f''(x) = 12x$

$x_0 = -2$, $f_0 = f(x_0) = -4$, $f'_0 = f'(x_0) = 21$, $f''_0 = f''(x_0) = -24$

By Chebyshev iterative formula $x_1 = x_0 - \frac{f_1}{f_1'} - \frac{1}{2}\frac{{f_0}^2}{{f_0'}^3} f_0'' = -2 + \frac{4}{21} + \frac{1}{2}\frac{16}{21^3} 24 = -1.78879$

$f_1 = -0.08106;\ f_1' = 16.19862;\ f_1'' = -21.46548$

$x_2 = x_1 - \frac{f_1}{f_1'} - \frac{1}{2}\frac{{f_1}^2}{{f_1'}^3} f_1'' =$ -1.78377,

$f_2 = -0.00002;\ f_2' = 16.09101;\ f_2'' = -21.40524$

$$x_3 = x_2 - \frac{f_2}{f_2'} - \frac{1}{2}\frac{{f_2}^2}{{f_2'}^3} f_2'' = -1.78377$$

Since $x_2 = x_3 = -1.78377$ the required root.

Piecewise Linear and Cubic Spine Interpolation:

Piecewise Linear Interpolation: $P_i(x) = \left(\frac{x - x_i}{x_{i-1} - x_i} y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i\right)$ for $x_{i-1} \le x \le x_i$ and i = 1,2,3,….n. Then the interpolation polynomial is given by $P(x) = \sum_{i=1}^{n} P_i(x)$

Problem 1: Find the piecewise linear interpolating polynomial

X	0	1	2
Y	1	3	35
Y^I	1	6	81

Solution: The piecewise linear interpolating polynomial

$Y = \left(\frac{x - x_i}{x_{i-1} - x_i} y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i\right)$ for $x_{i-1} \le x \le x_i$.

Thus if for $0 \le x \le 1$, $Y = \left(\frac{x-1}{0-1} 1 + \frac{x-0}{1-0} 3\right) = 2x + 1$

if for $1 \le x \le 2$, $Y = \left(\frac{x-2}{1-2} 3 + \frac{x-1}{2-1} 35\right) = 32x - 29$

Hence the linear interpolating polynomial $Y = \begin{cases} 2x + 1, & for\ \ 0 \le x \le 1, \\ 32x - 29, & for\ 1 \le x \le 2, \end{cases}$

Cubic Spine Interpolation: A cubic spline function S(x) with respect to $x_0, x_1, x_2 \dots \dots x_n$ is a polynomial of degree three in each interval (x_{i-1}, x_i): I = 1,2,….n such that S(x), S'(x) and S''(x) are continuous in (x_0, x_n)

S(x)=

$$\frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6} M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6} M_i\right)$$

Where $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$, i=1,2,…….(n-1) and $M_0 = M_n = 0$

Problem 1. Obtain the cubic spline approximation for the function y = f(x) from the following data, given that $y_0'' = y_3'' = 0$.

X: -1 0 1 2, y: -1 1 3 35

Solution: Since the values of x are equally spaced with h = 1, we have

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$, -----(1) i = 1,2 and

$M_0 = y_0'' = 0, M_3 = y_3'' = 0$

Put i = 1 in (1) $M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$= 6(-1-2+3)=0

Put i = 2 in (1) $M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$= 6(1-6+35)=180

i.e. $4M_1 + M_2 = 0, M_1 + 4M_2 = 180$, since $M_0 = 0, M_3 = 0$

i.e. $M_1 = -12, M_2 = 48$

The Cubic Spine in $x_{i-1} \le x \le x_i$. Is given by

y=

$$\frac{1}{6}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{1}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{1}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right)$$

---(2)

putting i = 1 in (2) for -1$\le x \le 0$, y = $\frac{1}{6}(x+1)^3(-12) + (0-x)(-1) + (x+1)\left\{1 - \frac{(-12)}{6}\right\}$= -2x^3-6x^2-2x+1

putting i = 2 in (2) for 0$\le x \le 1$, y = 10x^3-6x^2-2x+1

putting i = 3 in (2) for 1$\le x \le 2$, y = -8x^3+48x^2-56x+19

Hence the required cubic spline approximation $y = \begin{cases} -2x3 - 6x2 - 2x + 1, & -1 \le x \le 0 \\ 10x3 - 6x2 - 2x + 1, & 0 \le x \le 1, \\ -8x3 + 48x2 - 56x + 19, & 1 \le x \le 2 \end{cases}$

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SCHOOL OF SCIENCE AND HUMANITIES
DEPARTMENT OF MATHEMATICS
Unit – IV

Numerical Solution of Ordinary Differential equations

Numerical Solution to Ordinary Differential Equation

Introduction

An ordinary differential equation of order *n* in of the form $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$, where $y^{(n)} = \frac{d^n y}{dx^n}$.

We will discuss the Numerical solution to first order linear ordinary differential equations by Taylor series method, Euler method and Runge - Kutta method, given the initial condition $y(x_0) = y_0$.

Taylor Series method

Consider the first order differential equation of the form $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$.
The solution of the above initial value problem is obtained in two types

- Power series solution
- Point wise solution

(i) Power series solution

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

(ii) Point wise solution

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

Problems:

1. Using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = y + 1, y(0) = 1$.

Solution:

Given $\frac{dy}{dx} = y + 1$ and $x_0 = 0$, $y_0 = 1, h = 0.1$

Taylor series formula for $y(0.1)$ is

$$y(x) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots$$

$y'(x) = y + 1$	$y'(0) = y(0) + 1 = 1 + 1 = 2$
$y''(x) = y'$	$y''(0) = y'(0) = 2$
$y'''(x) = y''$	$y'''(0) = y''(0) = 2$

Substituting in the Taylor's series expansion:

$$y(0.1) = y(0) + hy'(0) + \frac{h^2}{2!}y''(0) + \cdots$$

$$= 1 + 0.1 \times 2 + \frac{0.01}{2} \times 2 + \frac{0.001}{6} \times 2 + \cdots$$

$$y(0.1) = 1.2103$$

2. Find the Taylor series solution with three terms for the initial value problem $\frac{dy}{dx} = x^2 + y, y(1) = 1$

Solution:

Given $\frac{dy}{dx} = x^2 + y, x_0 = 1, y_0 = 1$

$y'(x) = x^2 + y$	$y'(1) = 1 + 1 = 2$
$y''(x) = 2x + y'$	$y''(1) = 2 + 2 = 4$
$y'''(x) = 2 + y''$	$y'''(1) = 2 + 4 = 6$
$y^{iv}(x) = y'''$	$y^{iv}(1) = 6$

The Taylor's series expansion about a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Hence at $x = 1$

$$y(x) = y(1) + \frac{(x - 1)}{1!}y'(1) + \frac{(x - 1)^2}{2!}y''(1) + \frac{(x - 1)^3}{3!}y'''(1) + \cdots$$

$$y(x) = 1 + 2\frac{(x - 1)}{1!} + 4\frac{(x - 1)^2}{2!} + 6\frac{(x - 1)^3}{3!} + \cdots$$

Runge-Kutta method

Runge-kutta methods of solving initial value problem do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formula possesses the advantage of requiring only the function values at some selected points. These methods agree with Taylor series solutions up to the term in h^r where r is called the order of that method.

Fourth-order Runge-Kutta method

Let $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ be given.

Working rule to find $y(x_1)$

The value of $y_n = y(x_n)$ where $x_n = x_{n-1} + h$ where *h* is the incremental value for *x* is obtained as below:

Compute the auxiliary values

$k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

Compute the incremental value for *y*

$\Delta y = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

The iterative formula to compute successive value of *y* is $y_{n+1} = y_n + \Delta y$

Problems

1. Find the value of *y* at $x = 0.2$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1$, using R-K method of order IV.

Sol:

Here $f(x,y) = x^2 + y, y(0) = 1$, Choosing $h = 0.1, x_0 = 0, y_0 = 1$

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0$$
$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.00525$$
$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.00525$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.0110050$$

$y(0.1) = 1.0053$

To find $y(0.2)$ given $x_2 = x_1 + h = 0.2, y_1 = 1.0053$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.0110$$
$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.01727$$
$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.01728$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.02409$$

y(0.2)=1.0227

Euler's Method:

Let $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ be given

- The simple Euler's formula y(x+h) = y(x) + h f(x,y)
- The improved Euler's formula y(x+h) = y(x) + h/2 [f(x,y) + f{x+h, y+hf(x,y)}]
- The modified Euler's formula y(x+h) = y(x) + h f [x+h/2 , y + h/2f(x,y)]

1. Given that $5x\frac{dy}{dx} + y^2 - 2 = 0, y(4) = 1, find\ y(4.1) and\ y(4.2)\ by\ Euler's\ method.$

Soln.: The given equation is $\frac{dy}{dx} = \frac{2-y^2}{5x} = f(x,y) - - - (1)$
Given $x_0 = 4, y_0 = 1$

The simple Euler's formula y(x+h) = y(x) + h f(x,y) = y(x) + h $\left[\frac{2-y^2}{5x}\right] - - - (2)$

By taking h=0.1, $x_1 = x_0 + h = 4 + 0.1 = 4.1,$

$x_2 = x_1 + h = 4.1 + 0.1 = 4.2$

$$y(x_1) = y(x_0) + 0.1\left[\frac{2-{y_0}^2}{5x_0}\right]$$

y(4.1) = 1 + 0.1$\left[\frac{2-1^2}{5(4)}\right]$= 1.005, $x_1 = 4.1, y_1 = 1.005$

y(x_2) = y(x_1)+0.1$\left[\frac{2-{y_1}^2}{5x_1}\right]$

y(4.2) = 1.005 + 0.1$\left[\frac{2-(1.005)^2}{5(4.1)}\right]$= 1.0098 , $x_2 = 4.2, y_2 = 1.0098$

Predictor corrector method:

To solve $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$, by knowing 4 consecutive values of y namely $y_{n-3}, y_{n-2}, y_{n-1,}$ and y_n

Milne's Predictor formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y_{n-2}' - y_{n-1}' + 2y_n']$$

$$when\, n = 3, y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

Milne's Corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}']$$

$$when\, n = 3, y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

Problem 1: Find y(2) if $\frac{dy}{dx} = \frac{(x+y)}{2}$ given y(0) = 2, y(0.5)= 2.636, y(1) = 3.595 and y(1.5)= 4.968

Solution:Here

$x_0 = 0, x_1 = 0.5, x_2 = 1, x_{3=} 1.5, x_4 = 2, h = 0.5, y_0 = 2, y_1 = 2.636, y_2 = 3.595, y_{3=} 4.968, f(x,y) = y' = \frac{(x+y)}{2}$ ....(1)

Milne's Predictor formula $y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] - - - -(2)$

From (1) $y_1' = \frac{(x_1+y_1)}{2} = \frac{(0.5+2.636)}{2} = 1.5680$, $y_2' = \frac{(x_2+y_2)}{2}$= 2.2975, $y_3' = \frac{(x_3+y_3)}{2}$= 3.2340

$$y_{4,p} = 2 + \frac{4(0.5)}{3}[2(1.568) - 2.2975 + 2(3.2340)] = 6.8710$$

$y_4' = \frac{(x_4+y_4)}{2}$=4.4355, Milne's Corrector formula $y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$$y_{4,c} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.234) + 4.4355] = 6.8732$$

Note: Suppose y_1, y_2, y_3 *values are not given then by using Taylor's series method, Euler's method and*

R.K method to get all initial values.

Problem 2: Determine the value of y(0.4) using Milne's method given that y' = xy + y², y(0)=1, use Taylor's method find y(0.1), y(0.2), y(0.3)

Solution: Given y(0)=1, $x_0 = 0, y_0 = 1$

$y' = xy + y^2$	$y_0' = x_0 y_0 + y_0^2 = 1$	$y_1' = x_1 y_1 + y_1^2 = 1.3587$	$y_2' = x_2 y_2 + y_2^2 = 1.8853$
$y'' = xy' + y + 2yy'$	$y_0'' = x_0 y_0' + y_0 + 2y_0 y_0' = 3$	$y_1'' = x_1 y_1' + y_1 + 2y_1 y_1' = 4.2871$	$y_2'' = x_2 y_2' + y_2 + 2y_2 y_2' = 6.4677$
$y''' = xy'' + 2y' + 2yy'' + 2y'^2$	$y_0''' = x_0 y_0'' + 2y_0' + 2y_0 y_0'' + 2{y_0'}^2 = 10$	$y_1''' = x_1 y_1'' + 2y_1' + 2y_1 y_1'' + 2{y_1'}^2 = 16.4131$	$y_2''' = x_2 y_2'' + 2y_2' + 2y_2 y_2'' + 2{y_2'}^2 = 28.6875$

$$y_1 = (y_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots = 1 + (0.1) + \frac{(0.01)}{2}3 + \frac{(0.001)}{6}10$$

$$= 1 + 0.1 + 0.015 + 0.001666 = 1.1167$$

$$y_2 = (y_1) + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$= 1.1167 + (0.1)1.3587 + \frac{(0.01)}{2}4.2871 + \frac{(0.001)}{6}16.4131$$

$$y_2 = y(0.2) = 1.2767$$

$$y_3 = (y_2) + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

$$= 1.2767 + (0.1)1.8853 + \frac{(0.01)}{2}6.4677 + \frac{(0.001)}{6}28.6875$$

$$y_3 = y(0.3) = 1.5023$$

$x_0 = 0$	$y_0 = 1$	
$x_1 = 0.1$	$y_1 = y(0.1) = 1.1167$	$y_1' = 1.3587$
$x_2 = 0.2$	$y_2 = y(0.2) = 1.2767$	$y_2' = 1.8853$
.3	$y_3 = y(0.3) = 1.5023$	$y_3' = 2.7076$

		$y_4{}' = 4.09296$

Milne's Predictor formula $y_{4,p} = y_0 + \frac{4h}{3}[2y_1{}' - y_2{}' + 2y_3{}']$ = 1.83297

Milne's Corrector formula $y_{4,c} = y_2 + \frac{h}{3}[y_2{}' + 4y_3{}' + y_4{}']$ = 1.83698

Adam's Bashforth Predictor Corrector formula:

Adam's Predictor formula

$$y_{n+1,p} = y_n + \frac{h}{24}[55y_n{}' - 59y_{n-1}{}' + 37y_{n-2}{}' - 9y_{n-3}{}']$$

$$when\ n = 3, y_{4,p} = y_3 + \frac{h}{24}[55y_3{}' - 59y_2{}' + 37y_1{}' - 9y_0{}']$$

Adam's Corrector formula

$$y_{n+1,c} = y_n + \frac{h}{24}[9y_{n+1}{}' + 19y_n{}' - 5y_{n-1}{}' + y_{n-2}{}']$$

$$when\ n = 3, y_{4,c} = y_3 + \frac{h}{24}[9y_4{}' + 19y_3{}' - 5y_2{}' + y_1{}']$$

Problem 3: : Find y(2) by Adam's method if $\frac{dy}{dx} = \frac{(x+y)}{2}$ given y(0) = 2, y(0.5)= 2.636, y(1) = 3.595 and y(1.5)= 4.968

Solution: Here $x_0 = 0, x_1 = 0.5, x_2 = 1, x_{3=} 1.5, x_4 = 2, h = 0.5, y_0 = 2, y_1 = 2.636, y_2 = 3.595, y_{3=} 4.968, f(x,y) = y' = \frac{(x+y)}{2}$ ....(1)

From (1) $y_1{}' = \frac{(x_1+y_1)}{2} = \frac{(0.5+2.636)}{2} = 1.5680$, $y_2{}' = \frac{(x_2+y_2)}{2}$= 2.2975, $y_3{}' = \frac{(x_3+y_3)}{2}$= 3.2340

$$y_{4,p} = y_3 + \frac{h}{24}[55y_3{}' - 59y_2{}' + 37y_1{}' - 9y_0{}']$$
$$= 4.968 + \frac{(0.5)}{24}[55(3.2340) - 59(2.2975) + 37(1.5680) - 9(1)] = 6.8708$$

$y_4{}' = \frac{(x_4+y_4)}{2}$=4.4354, $y_{4,c} = y_3 + \frac{h}{24}[9y_4{}' + 19y_3{}' - 5y_2{}' + y_1{}']$

$y(2) = y_4 = 6.8731$

Boundary value problems – by Finite Difference Method:

When the differential equation is to be solved satisfying the conditions specified at the end points of an interval, the problem is called a boundary value problem.

Problem 1. Solve, by finite difference method, the boundary value problem y''(x)-y(x)=0,where y(0)=0 and y(1)=1,taking h=1/4

Solution: The finite difference approximation of the given differential equation is

$\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)-y_i=0$, i.e. $y_{i-1}-\left(2+\frac{1}{16}\right)y_i+y_{i+1}=0$,

$i.e., y_{i-1}-2.0625y_i+y_{i+1}=0$, i=1,2,3,…(1)

The boundary conditions are $y_0= y(0) = 0$, and $y_4= y(1) =1$----- (2)
From (1) and (2) we have
$0 – 2.0625\ y_1 + y_2 = 0$ --------- (3)
$y_1 – 2.0625\ y_2 + y_3 = 0$ --------- (4)
$y_2 – 2.0625\ y_3 + I = 0$----------(5)
Solving the equations (3), (4) and (5),
we get $y_1= y(0.25) = 0.2151$, $y_2 = y(0.5) = 0.4437$, $y_2= y(0.75)=0.7$

Problem 2. Solve the equation $y''(x) – [14/x]\ y'(x) + x^3\ y(x) = 2x^3$, for y(1/3) and y(2/3), given that y(0) = 2 and y(1) = 0.
Soln.: The finite difference approximation of the given differential equation is

$\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)-\frac{14}{x_i}\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)+x_i^3y_i=2x_i^3$, -----(1)
Putting h = 1/3, we get
$\left(1+\frac{7}{3x_i}\right)y_{i-1}-\left(2-\frac{1}{9}\right)y_i+\left(1+\frac{7}{3x_i}\right)y_{i+1}=\frac{2}{9}x_i^3$ ---(2)
Putting i = 1, 2 and using $x_1 = 1/3$ and $x_2 = 2/3$ in (2) , we have
$8y_0 - \frac{485}{243}y_1 - 6y_2 = \frac{2}{243}$ ----------(3)
$\frac{9}{2}y_1 - \frac{478}{243}y_2 - \frac{5}{2}y_3 = \frac{18}{243}$..........(4)
Using $y_0= y(0) = 2$, and $y_3= y(1) =0$ in (3) and (4), we have
$485\ y_1 + 1458y_2 = 3886$ ------ (5)
$2187\ y_1 -956\ y_2 = 36$----------(6)
Solving (5) and (6) we get $y_1= y(1/3)=1.0315$ and $y_2= y(2/3)=2.3222$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

Unit – V

Numerical Solution of Partial Differential equations

Solution of Laplace Equation and Poisson equation

Partial differential equations with boundary conditions can be solved in a region by replacing the partial derivative by their finite difference approximations. The finite difference approximations to partial derivatives at a point (x_i, y_i) are given below:

$$u_x(x_i, y_i) = \frac{u(x_{i+1}, y_i) - u(x_i, y_i)}{h}$$

$$u_y(x_i, y_i) = \frac{u(x_i, y_{i+1}) - u(x_i, y_i)}{k}$$

$$u_{xx}(x_i, y_i) = \frac{u_x(x_{i+1}, y_i) - u_x(x_i, y_i)}{h} = \frac{u(x_{i+1}, y_i) - 2u(x_i, y_i) + u(x_{i-1}, y_i)}{h^2}$$

$$u_{yy}(x_i, y_i) = \frac{u_y(x_i, y_{i+1}) - u_y(x_i, y_i)}{k} = \frac{u(x_i, y_{i+1}) - 2u(x_i, y_i) + u(x_i, y_{i-1})}{k^2}$$

Graphical Representation

The xy plane is divided into small rectangles of length h and breadth k by drawing the lines $x = ih$ and $y = ik$, parallel to the coordinate axes. The points of intersection of these lines are called grid points or mesh points or lattice points. The grid points (x_i, y_j) is denoted by (i, j) and is surrounded by the neighbouring grid points $(i-1, j)$ to the left, $(i+1, j)$ to the right, $(i, j+1)$ above and $(i, j-1)$ below.

Note

The most general linear P.D.E of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x, y)$$

where A,B,C,D,E,F are in general functions of x and y.

A partial differential equation in the above form is said to be

- Elliptic if $B^2 - 4AC < 0$
- Parabolic if $B^2 - 4AC = 0$
- Hyperbolic if $B^2 - 4AC > 0$

Standard Five Point Formula (SFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

Diagonal Five Point Formula (DFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right]$$

Solution of Laplace equation $u_{xx}+u_{yy}=0$

Leibmann's Iteration Process

We compute the initial values of $u_1, u_2, \ldots, u_9$ by using standard five point formula and diagonal five point formula .First we compute u_5 by standard five point formula (SFPF).

$$u_5 = \frac{1}{4}[b_7 + b_{15} + b_{11} + b_3]$$

We compute $u_1, u_3, u_7 . u_9$ by using diagonal five point formula (DFPF)

$$u_1 = \frac{1}{4}[b_1 + u_5 + b_3 + b_{15}]$$

$$u_3 = \frac{1}{4}[u_5 + b_5 + b_3 + b_7]$$

$$u_7 = \frac{1}{4}[b_{13} + u_5 + b_{15} + b_{11}]$$

$$u_9 = \frac{1}{4}[b_7 + b_{11} + b_9 + u_5]$$

Finally we compute u_2, u_4, u_6, u_8 by using standard five point formula.

$$u_2 = \frac{1}{4}[u_5 + b_3 + u_1 + u_3]$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + b_{15} + u_7]$$

$$u_6 = \frac{1}{4}[u_3 + u_9 + u_5 + b_7]$$

$$u_8 = \frac{1}{4}[u_7 + b_{11} + u_9 + u_5]$$

Solve the system of simultaneous equations obtained by finite difference method to get the value at the interior mesh points. This process is called *Leibmann's method.*

Problems

1. Classify the PDE $u_{xx}+4u_{xy}+(x^2+4y^2)u_{yy}=0$

Solution: Here A=1, B=4, C = x^2+4y^2, $B^2-4AC=16-4(x^2+4y^2)$,
The equation is elliptic, if

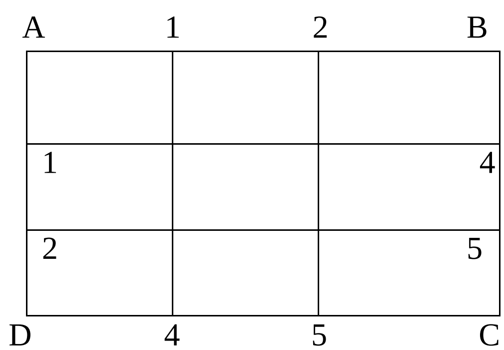
$4-x^2-4y^2<0,\ x^2+4y^2>4,\ \frac{x^2}{4}+\frac{y^2}{1}>1.$

It is elliptic in the region outside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1.$

It is Hyperbolic inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1.$

It is parabolic on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1.$

2. Solve $u_{xx}+u_{yy}=0$ for the following square mesh with boundary values as shown in the figure below.



Solution: The boundary values are symmetrical about the diagonal AC but not about BD. Let the values at the interval grid points be $u_1, u_2, u_3, u_4.$

By Symmetry, $u_2=u_3;\ u_1\neq u_4.$

Assume $u_2=3\ \left(Since\ \ u_2=2+\frac{1}{3}(5-2)=3\right).$

Rough values $u_1=\frac{1}{4}(1+1+2u_2)$ =2. (SFPF).

$u_2=3,\ u_4=\frac{1}{4}(5+5+2u_2)=\frac{1}{2}(5+u_2)=4$

$First\ Iteration: u_1=\frac{1}{2}(1+u_2)=2,\ u_2=\frac{1}{4}(6+u_1+u_4)=\frac{1}{4}(6+2+4)=3,\ \ u_4=\ \frac{1}{2}(5+u_2)=4.$

$Result\ \ u_1=2,\ u_2=3,\ u_4=4.$

3. Solve $U_{xx}+U_{yy}=0$ over the square mesh of side 4 unit satisfying the boundary conditions:

$U(0,y)=0$ for $0\le y\le 4,\ u(4,y)=12+y\ for\ 0\le y\le 4,\ u(x,0)=3x\ for\ 0\le x\le 4,\ u(x,4)=x^2$ for $0\le x\le 4.$

Solution:

We divide the square mesh into 16 sub-squares of side 1 unit and calculate the

numerical values of u on the boundary
using given analytical expressions.

0	1	4	9	16
	U1	U2	U 3 15	
0	U4	U5	U 6 14	
0	U7	U8	U 9 13	
0	3	6	9 12	

Let the internal grid points be $u_1, u_2, u_3, \ldots\ldots\ldots, u_9$.

Rough values: $U_5 = 1/4(4+6+14+0) = 6$ (SFPF)

$U_1 = 1/4\ (0+6+4+0) = 2.5$ (DFPF)

$U_3 = 1/4\ (16+6+14+4) = 10$ (DFPF)

$U_7 = 1/4\ (0+6+6+0) = 3$ (DFPF)

$U_9 = 1/4\ (6+6+14+12) = 9.5$ (DFPF)

We use SFPF to get the other values of u.

$U_2 = 1/4\ (2.5+6+4+10) = 5.625$ (SFPF)

$U_4 = 1/4\ (0+6+2.5+3) = 3.125$ (SFPF)

$U_6 = 1/4\ (10+6+14+9.5) = 9.875$ (SFPF)

$U_8 = 1/4\ (6+6+3+9.5) = 6.125$ (SFPF)

Now we proceed for iteration using always SFPF.

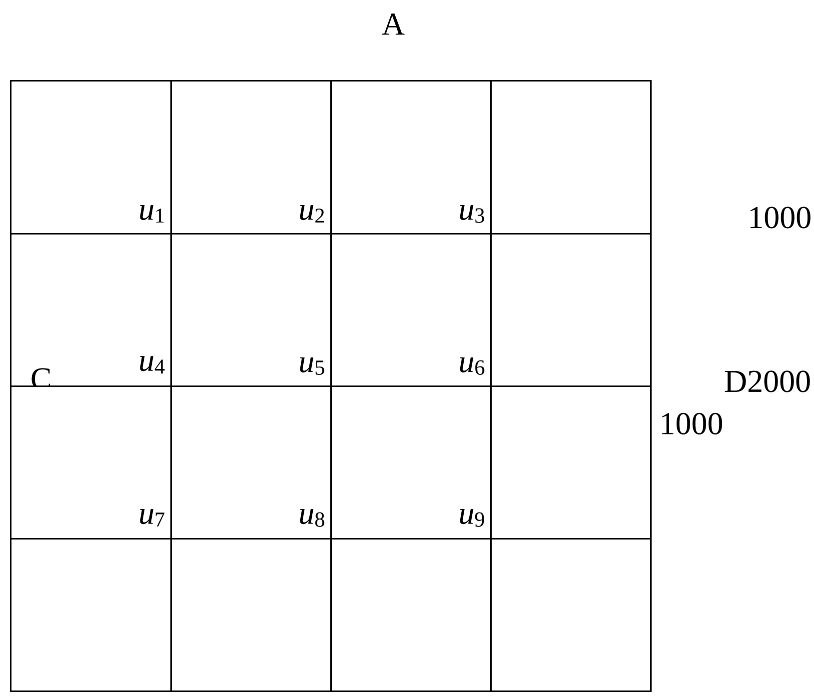
U1	U2	U3
2.4375	5.6094	9.8711
2.3672	5.5888	9.8652

1000

U4 2.8594 2.8698	U5 6.1172 6.1209	U6 9.8721 9.8731
U7 2.9948 3.0057	U8 6.153 6.1582	U9 9.5063 9.5078

Repeating one more iteration, we conclude, correct to 2 decimals,
$u_1 = 2.37, u_2 = 5.59, u_3 = 9.87, u_4 = 2.88, u_5 = 6.13, u_6 = 9.88, u_7 = 3.01, u_8 = 6.16, u_9 = 9.51.$

4. **Solve the equation $\nabla^2 u = 0$ for the following mesh, with boundary values as shown using Leibmann's iteration process.**



Sol:

Let $u_1, u_2 \ldots\ldots u_9$ be the values of u at the interior mesh points of the given region. By symmetry about the vertical lines AB and the horizontal line CD, we observe
$u_1 = u_3 = u_9 = u_7; u_2 = u_8; u_4 = u_6$

Hence it is enough to find $u_1, u_2, u_4,$
Using SFPF $u_5 = 1500$
Using DFPF $u_1 = 1125$ $u_2 = 1187.5$ $u_4 = 1437.5$

Solution of Poisson equation

An equation of the type $\nabla^2 u = f(x,y)$ i.e., is called Poisson's equation where $f(x,y)$ is a function of x and y. Substituting the finite difference approximations to the partial differential coefficients, we get $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

Problem: 1

Solve the poisson equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides

Applying the formula below at the interior point of the mesh we get a system of simultaneous equations $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

2. Solve $\nabla^2 u = 8X^2Y^2$ for square mesh given u=0 on the 4 boundaries dividing the square into 16 sub-squares of length 1 unit.

Solution:

	U1	U2	U3
	U4	U5	U6
	U7	U8	U9

Take the coordinate system with origin at the center of the square.

Since the boundary conditions are symmetrical about the x, y axes and x=y, we have

U1= U3= U7= U9, U2= U4= U6= U8

At i=-1, j=-1, we have,

At i=0, j=1, we have,

At i=0, j=0, we have,

Solution of One dimensional heat equation

In this chapter, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit and implicit method

Explicit Method (Bender-Schmidt method)

Consider the one dimensional heat equation .This equation is an example of parabolic equation.

Implicit method (Crank-Nicholson method)

This expression is called Crank-Nicholson's implicit scheme. We note that Crank Nicholson's scheme converges for all values of λ

When $\lambda=1$, i.e., $k=ah^2$ the simplest form of the formula is given by
]
4
The use of the above simplest scheme is given below.

The value of u at A=Average of the values of u at B, C, D, E

Note

In this scheme, the values of u at a time step are obtained by solving a system of linear equations in the unknowns u_i.

Solved Examples

1.Solve $u_{xx} = 2u_t$ **when u(0,t)=0,u(4,t)=0 and with initial condition u(x,0)=x(4-x) .** Assume h=1.Find the values of u up to t=5 by Bender-Schmidt recurrence equation.

Solution: Here a=2. By Bender-Schmidt recurrence relation , Step -size in time =k=1. The values of are tabulated below

i j	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

subject to the conditions u(0,t)=u(5,t)=0 and

$u(x,0)$ **taking h=1 and k=1/2,tabulate the values of u upto t=4 sec.**

Sol:

Here a=1,h=1

For λ=1/2,we must choose $k=ah^2/2$
K=1/2

The values of u upto 4 sec are tabulated as follows

j\i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0

3. Using Crank-Nicholson scheme, solve U_{XX} - 16 U_t =0, given u(x,0)=0, u(0,t)=0,u(1,t)=100t. Compute

for one step in t direction taking h = 1/4

Solution: Here a=16, h=1/4, $k=ah^2$,   16(1/16)=1.

4. Solve U_{XX} - 32 U_t =0 taking h=0.25 for t>0,0<x<1 and u(x,0)=0, u(0,t)=0, u(1,t)=t using Bender –

Schmidt method.
Solution: The range of x is (0, 1); h=0.25

J i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5

Solution of One dimensional wave equation

Introduction

The one dimensional wave equation is of hyperbolic type. In this chapter, we discuss the finite difference solution of the one dimensional wave equation u_{tt} $a^2 u_{xx}$.

Explicit method to solve u

Problems

1.Solve numerically ,$4u_{xx}$ U_{tt} **with the boundary conditions u(0,t)=0,u(4,t)=0 and the initial conditions** $u_t(x,0)= 0$ & $u(x,0)$ $x(4 -x)$, **taking h=1.Compute u upto t=3sec.**

Sol:

Here $a^2=4$

A=2 and h=1

We choose k=h/ak=1/2

The values of u for steps t=1,1.5,2,2.5,3 are calculated using (1) and tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	3	-2	0
4	0	-3	-4	3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0

$x(4 \quad x)$

2.Solve $u_{xx} = 1/4u_{tt}$ **Given u(0,t)=0,u(4,t)=0,u(x,0)=**$u(x,0)$ & $u_t(x,0)=0$.**Take h=1.Find the solution upto 5 steps in t-direction.**

Sol:

Here $a^2=4$

A=2 and h=1

We choose k=h/ak=1/2

The values of u upto t=5 are tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	1.5	2	1.5	0
1	0	1	1.5	1	0
2	0	0	0	0	0
3	0	-1	-1.5	-1	0
4	0	-1.5	-2	-1.5	0
5	0	-1	-1.5	-1	0

3. Solve , $25U_{xx} = U_{tt}$ for u at the pivotal points, given u (0, t)= u(5,t)=0, $U_t(x,0)=0$ and u(x,0)= 2x for $0<x<2.5$,=10-2x, for $2.5<x<5$ for one half

period of vibration.

Solution: a^2 = =25, a=5

Period of vibration=2l/a= 2 seconds. Half period=1 second. We want values up to t=1 second. Taking h=1, k= =1/5. Step-size in t-direction=1/5.

The explicit scheme is,

we have u(0,0)=0,u(1, 0)=2, u(2,0)=4, u(3,0)=4, u(4,0)=2, u(5,0)=0

T x	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	-1	-1	-1	-1	0
4/5	0	-2	-3	-3	-2	0
1	0	-2	-4	-4	-2	0

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SATHYABAMA INSTITUTE OF SCIENCE AND TECHNOLOGY

Course and Branch: B. Sc Maths

Course Name: NUMERICAL METHODS

Course Code: SMT1405

QUESTION BANK

UNIT 1 DIRECT AND ITERATIVE METHODS

PART – A

1. Explaintwo indirect methods to solve simultaneous linear Algebraic equations. (1)
2. Determine the principle used in Gauss Seidel Method. (1)
3. Solve the system of equations $x - 2y = 0$ and $2x + y = 5$ by Gauss Elimination method. (1)
4. Using Gauss Jacobi method solve x – 3y = 1, 3x + y = 4 . (1)
5. Evaluate the dominant eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method. (1)
6. By Gauss Elimination method 4x – 3y = 11, 3x + 2y = 4 estimate the value of x and y (1)
7. Write any two direct methods in linear Algebraic equations (1)
8. Define Diagonally dominant (1)
9. Discuss power method to calculate dominant eigen value. (1)
10. 4x + 2y + z = 14, x + 5y – z = 10, x + y + 8z = 20 check the given system of equation is diagonally dominant. (1)

PART - B

1. Estimate the following system of equations by Gauss Seidel method
   28x+4y-z=32, x+3y+10z=24, 2x+17y+4z=35. (1)
2. Calculate following system of equation by using Gauss – Seidel Jacobi method
   $$x + 17y - 2z = 48$$
   $$30x - 2y + 3z = 75$$
   $$2x + 2y + 18z = 30$$
   (1).
3. Evaluate the following equation using Gauss-Jocobi method
   10x + y + z = 12, 2x+10y+z = 13, x+y+5z = 7 (1)
4. By Gauss Elimination method solve
   $x + 3y + 8z = 4;$
   $x + 4y + 3z = -2;$ (1)
   $x + 3y + 4z = 1.$
5. Examine the system of equations using Gauss Seidel method.
   $$8x - y + z = 18$$
   $$2x + 5y - 2z = 3$$
   $$x + y - 3z = -16$$
   (1)
6. Solve the following system of equations by using Gauss-Jocobi method and Gauss-Seidel method:
   $3x_1 - x_2 - x_3 = 1,$
   $3x_1 + 6x_2 + 2x_3 = 0,$ (1)
   $3x_1 + 3x_2 + 7x_3 = 4$

7. Find the dominant Eigen value and Eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by Power method. (1)

8. Manipulate the following system of equation by Gauss – Seidel method:

   10x – 5y – 2z = 3
   4x – 10y + 3z = -3 (1)
   x + 6y + 10z = -3

9. Recall the system of equation by Gaussian elimination method $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -3x_1 + 2x_2 - 3x_3 &= -6 \\ 2x_1 - 5x_2 + 4x_3 &= 5 \end{aligned}$ (1)

10. By Gaussian elimination method evaluate: $\begin{aligned} x + 5y + z &= 14 \\ 2x + y + 3z &= 13 \\ 3x + y + 4z &= 17 \end{aligned}$. (1)

UNIT 2 NUMERICAL DIFFERENTIATION AND INTEGRATION

PART – A

1. State Lagrange's interpolation formula. (2)
2. Write Gregory-Newton forward interpolation formula. (2)
3. Listthe first and second derivative of Newton's backward formula. (2)
4. Form the difference table for the points (0,-1), (1,1), (2,1) and (3,-2). (2)
5. From the following table, find the rate of growth of the population in 1931 (2)

Year x:	1931	1941	1951	1961	1971
Population in thousands y:	40.62	60.80	79.95	103.56	132.65

6. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h = 0.2 (2)
7. Evaluate $\int_{0.2}^{1.4} \left(\sin x - \log x + e^x\right)dx$ by Simpson's $\frac{1}{3}$ rule. (2)
8. A river is 80 meters wide. The depth "d" in meters at a distance "x" meters from one bank is given by the following table.Calculate the area of cross-section of the river using Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule. (2)

x:	0	10	20	30	40	50	60	70	80
d:	0	4	7	9	12	15	14	8	3

9. Write down Trapezoidal rule to evaluate $\int_1^6 f(x)dx$ with h = 0.5 (2)
10. Recall the errors in trapezoidal and Simpson's rule of numerical integration. (2)
11. Explain the order of errors in trapezoidal and Simpson's rule of numerical integration. (2)

PART – B

1. Using Lagrange's formula calculate $f(3)$ from the following table. (2)

X	0	1	2	4	5	6
f(x)	1	14	15	5	6	19

2. Estimate y(9.5) using Lagrange's formula of interpolation (2)

X	7	8	9	10
Y	3	1	1	9

3. Infer the number of student whose weight is between 80 and 90 (2)

Weight:	0-40	40-60	60-80	80-100	100-120
Number of students:	250	120	100	70	50

4. Discuss the age corresponding to the annuity value $y = 13.6$ given the table (2)

Age (x):	30	35	40	45	50
Annuity Value (y):	15.9	14.9	14.1	13.3	12.5

5. Form the parabola of the form $y = ax^2 + bx + c$ passing through the points (0,0), (1,1), (2,20). (2)
6. The following data are taken from the steam table.
   Temperature ^oC: 140 150 160 170 180
   Pressure kgf/cm^2: 3.685 4.854 6.302 8.076 10.225
   Calculate the pressure at temperature $t=142^o$ and $t=175^o$. (2)
7. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x dx$ by Simpson's 1/3 th rule and Trapezoidal rule. (2)
8. Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal parts using (a) Trapezoidal rule, (b) Simpson's rule. (2)
9. Dividing the range into 10 equal parts, find the approximate value of $\int_4^{5.2} \log_e x dx$ by
   (a) Trapezoidal rule (b) Simpson's rule. (2)
10. From the following table , Compute Θ at x = 43 and x = 84 (2)
    x ; 40 50 60 70 80 90
    Θ ; 184 204 226 250 276 304
11. Estimate y'(10), y''(5), y'''(11) from the following data: (2)

X	5	6	9	11
Y	12	13	14	16

UNIT 3 POLYNOMIAL APPROXIMATION

PART - A

1. Define curve fitting (3)
2. State two categories of fitting a curve to a given set of data points. (3)
3. Explain least-squares polynomials. (3)
4. Recollect piecewise polynomials (3)
5. Discuss the term 'knots' or 'nodes' (3)
6. The conditions which satisfy the spline function s(x)---? (3)
7. The contribution of Russian mathematician Chebyshev in minimizing the truncation error in interpolation is --? (3)

8. Define the natural cubic spline (3)
9. Elaborate the term Chebyshev points. (3)
10. Various approaches for fitting a "best" line through the line. (3)

PART – B

1. Examine whether the following piecewise polynomials are spline or not

$$f(x)=\begin{cases} x+1, -1 \le x \le 0 \\ 2x+1, 0 \le x \le 1 \\ 4-x, 1 \le x \le 2 \end{cases}$$ (3)

2. Check the polynomials are spline

$$f(x)=\begin{cases} x^2-3x+1, 0 \le x \le 1 \\ x^3+x^2-3, 1 \le x \le 2 \\ x^3+5x-9, 2 \le x \le 3 \end{cases}$$ (3)

3. Develop cubic splines for the data given below and predict f(1.5) (3)

X	0	1	2	3
f(x)	1	-1	-1	0

4. Fit a straight line using method of least squares to the following data: (3)

x	1	2	3	4	5
y	14	27	40	55	68

5. Given the data points

i	0	1	2	3
x_i	1	3	4	7
$f(x_i)$	1.5	4.5	9	25.5

Estimate the function value at x = 1.5 using cubic splines. (3)

6. Frame a straight line to the following set of data (3)

x	1	2	3	4	5
y	3	4	5	6	8

7. Fit a second order polynomial to the data in the table below: (3)

x	1	2	3	4
y	6	11	18	27

8. Given the data points (3)

i	0	1	2
x_i	4	9	16
$f(x_i)$	2	3	4

Estimate the function value f at x = 7 using cubic splines.

9. The velocity distribution of a fluid near a flat surface is given below: (3)

x	0.1	0.3	0.5	0.7	0.9
y	0.72	1.81	2.73	3.47	3.98

X is the distance from the surface (cm) and v is the velocity (cm/sec). Using a suitable interpolation formula obtain the velocity at x = 0.2, 0.4, 0.6 and 0.8

10. Identify a second degree parabola by the method of least square to the following also estimate y at x = 3.5. (3)

X	1	2	3	4	5
Y	5	12	26	60	97

UNIT 4NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

PART - A

1. Using Euler's method, calculate $y(0.2)$ *if* $\frac{dy}{dx} = y - x^2, y(0) = 1.$ (4)
2. Compute y(0.1) given that $y' = 1 - y$, y(0)=0 by Taylor's method (4)
3. Define Fourth order Runge-Kutta Method. (4)
4. Solve numerically $y' = y + e^x$, *y(0)=0* for *x=0.2, 0.4* by Euler's Method. (4)
5. Find y(0.1) given $y' = \frac{1}{2}(x+y), y(0) = 1$ by Modified Euler's Method. (4)
6. State Taylor's series formula for $y(x_1)$ in solving $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$. (4)
7. Outline Adam's Bashforth predictor corrector formula. (4)
8. Explain Milne's predictor corrector formula. (4)
9. Write down the recurrence formula for Euler method. (4)
10. Discuss single step and multi-step methods. (4)

PART – B

1. Apply Taylor's series method, find *y* when *x = 1.1* from (4 decimal places) $\frac{dy}{dx} = xy^{\frac{1}{3}}, y(1) = 1$ (4)
2. Estimate y (0.2) from y' = y – x, y(0)=2 taking h = 0.1 by the fourth order Runge-Kutta method (4)
3. By Milne's Predictor and Corrector Method, evaluate y (4.4) given $5xy' + y^2 - 2 = 0$. Given y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097 and y(4.3) = 1.0143. (4)
4. Discuss the equation $\frac{dy}{dx} = 1 - y$, given *y(0)=0* using Euler's Method and tabulated the solutions at *x=0.1, 0.2* and *0.3*. Compute your results with the Exact Solutions. (4)
5. Generate y(0.8) by solving $y' = \frac{1}{x+y}$,y(0)=2 using Milne's predictor-corrector given y(0.2)=2.0933,y(0.4)=2.1755,y(0.6)=2.2493. (4)
6. Estimate $\frac{dy}{dx} = y^2 + x^2$ with y(0) = 1 (4)

(a) Use Taylor series at x = 0.2 and x = 0.4 and

(b) Use Runge-Kutta method of order 4 at x = 0.6.

7. Substitute Runge-Kutta method of 4th order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given *y (0) = 1 at x = 0.2, 0.4*. Take *h = 0.2*. (4)

8. Given that $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ obtain y for $x = 0.1, 0.2$ *and* 0.3 by Taylor's series method and find the solution for $y(0.4)$ by Milne's Predictor and Corrector Method. (4)

9. Using Adams Bashforth method calculate y(0.4) given $y' = \frac{xy}{2}$, y(0.1)=1.01, Y(0.2)=1.022,y(0.3)=1.023,y(0)=1. (4)

10. The differential equation $\frac{dy}{dx} = y - x^2$ is satisfied by y(0) = 1, y(0.2) = 1.012186, y(0.4)=1.46820, y(0.6) = 1.7379. Compute the value of y(0.8) by Milne's Predictor-Corrector formula. (4)

UNIT 5 NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS.

PART – A

1. Write down the Crank-Nicholson difference formula. (5)
2. Explain the Explicit scheme to solve one dimensional wave Equation. (5)
3. Define Poisson's Equation. (6)
4. Discuss the Bender-Schmidt recurrence equation. (5)
5. Classify the Partial differential equation $xu_{xx} + u_{yy} = 0$ when (i) x > 0 (ii) x < 0 (iii) x = 0. (5)
6. State Diagonal five point formula for Laplace equation. (5)
7. Discuss standard five point formula (5)
8. Define period of oscillation (5)
9. What are Lattice points (5)
10. Classify the partial differential equation $xf_{xx} + yf_{yy} = 0, x < 0, y < 0$. (5)

PART – B

1. Analyze $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following boundary conditions.

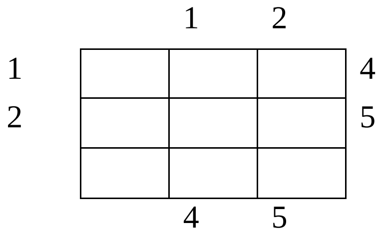
   (i) u(0,y)=0 for 0≤y≤4 (iii) u(x,0)=3x for 0≤x≤4

   (ii) u(4,y)=12+y for 0≤y≤4 (iv) u(x,4)= x^2 for 0≤x≤4 (5)

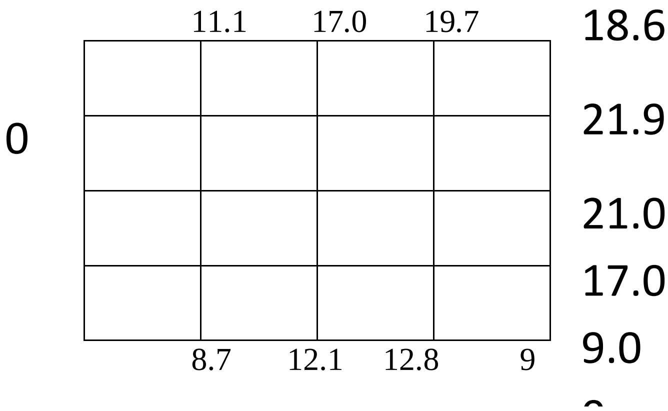
2. Estimate $25u_{xx} - u_{tt} = 0$ for u at the pivotal points given u(0,t)=u(5,t)=0, $u_t(x,0) = 0$ and

   u(x,0)= $\begin{cases} 2x, & 0 \le x \le 2.5 \\ 10 - 2x, & 2.5 \le x \le 5 \end{cases}$ for one half period of vibration. (5)

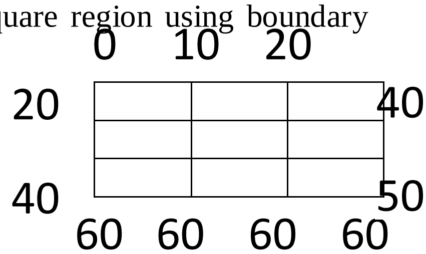
3. Using Leibmann method, solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the following square mesh with boundary values as shown in given figure. Iterate until the maximum difference between successive values at any point is less than 0.001. (5)



4. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions *u(x,0)=0, u(0,t)=0, u(1,t)=t*. Compute u for t = 1/8 in two steps, using Crank-Nicolson formula. (5)
5. Evaluate using Crank-Nicholson's scheme, solve $u_{xx} = 16\, u_t$, $0 < x < 1$, $t > 0$ given $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 100\,t$. Compute u for one step in t direction taking h =1/4. (5)
6. ApplyLiebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown in the following figure. (5)



7. Determine$\nabla^2 u$ = 0at all node point for the following square region using boundary conditions. (5)



8. Solve $u_t = u_{xx}$ given $u(0,t) = u(4,t) = 0$ $u(x, 0) = \frac{1}{2} x\,(4 - x)$,

   $u_t(x, 0) = 0$ Taking h = 1, find the solutions upto 5 steps in t – direction. (5)

9. Calculate the Poisson equation $U_{xx} + U_{xx} = -10(x^2+y^2+10)$ over the square mesh with sides x = 0, y = 0, x = 3, y = 3 with u = 0 on the boundary and mesh length 1unit, correct to one place of decimal. (5)
10. Interpret the pivotal value of the equation $U_{tt} = U_{xx}$ for x =0 (1) 4 and t = 0 (1) 4, given that u(0,t)=0, u(4,t)=0, u(x,0) = 0, $U_t(x,0) = x(4-x)/10$ (6)

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