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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I – Ordinary and Partial Differential Equations – SMT1305

Unit I

Differential Equations of First Order

Topics covered in this unit are: Definition of an ordinary differential equation, Degree and order of a differential equation, Formation of differential equations, Solutions: General, particular, and singular, First order exact equations and integrating factors, Equations in which the variable are separable, Homogeneous equations, Equations of first order and first degree, Linear equations and equations reducible to linear form, First order higher degree equations solvable for x, y, p, Clairaut's form and singular solutions, Orthogonal trajectories.

1 Introduction to Ordinary Differential Equations (ODE)

1.1 Definition

A differential equation is an equation which involves differentials or differential coefficients. Differential equation is an equation involving one or more functions with its derivatives. The derivatives of the function define the rate of change of a function. Differential equations are used to model physical phenomena that involves rate of change.

Examples: (i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = sinx$ (ii) $\frac{dy}{dx} = x^2$.

1.2 Order and Degree of a ODE

Order is the highest derivative present in the differential equation and degree is the exponent of the highest derivative.

Example: In $\left(\frac{d^2y}{dx^2}\right)^3 + 2\left(\frac{dy}{dx}\right)^2 + y = 2$, Order = 2 and Degree = 3.

2 Formation of Ordinary Differential Equations

The following are the steps involved in forming a ODE:

Given the general solution of a ODE in the form $f(x, y, c_1, c_2, \ldots, c_n) = 0$_____(1)

Step 1: Find the number of arbitrary constants 'n' present in equation (1).

Step 2: Differentiate (1) w.r.t. independent variable x, present in (1).

Step 3: Keep differentiating 'n' times, so that (n+1) equations are obtained.

Step 4: Using the (n+1) equations are obtained, eliminate the constants $c_1, c_2, \ldots, c_n$.

Example 1

Construct an ordinary differential equation whose general solution is $y = Ae^{2x} + Be^{-2x}$.

Solution.

Given $y = Ae^{2x} + Be^{-2x}$________________(1)

Since there are two arbitrary constants A and B, we differentiate (1) twice.

$y_1 = 2Ae^{2x} - 2Be^{-2x}$________________(2)

$y_2 = 4Ae^{2x} + 4Be^{-2x}$________________(3)

This implies that $y_2 = 4(Ae^{2x} + Be^{-2x})$.

$\Longrightarrow y_2 = 4y$

Therefore, $\frac{d^2y}{dx^2} - 4y = 0$ is the required ODE.

Example 2

Construct a differential equation whose general solution is $y = Ae^x + Be^{2x} + Ce^{-3x}$.

Solution.

Given $y = Ae^x + Be^{2x} + Ce^{-3x}$________________(1)

Since there are three arbitrary constants A, B and C, we differentiate (1) thrice.

$y_1 = Ae^x + 2Be^{2x} - 3Ce^{-3x}$________________(2)

$y_2 = Ae^x + 4Be^{2x} + 9Ce^{-3x}$________________(3)

$y_3 = Ae^x + 8Be^{2x} - 27Ce^{-3x}$________________(4)

First we eliminate A from (1), (2), (3) and (4).

(2) − (1) $\Rightarrow y_1 - y = Be^{2x} - 4Ce^{-3x}$.

(3) − (2) $\Rightarrow y_2 - y_1 = 2Be^{2x} + 12Ce^{-3x}$.

(2) − (1) $\Rightarrow y_3 - y_2 = 4Be^{2x} - 36Ce^{-3x}$.

We then eliminate Band C, by using the determinant,

$$\begin{vmatrix} y_1 - y & B & -4C \\ y_2 - y_1 & 2B & 12C \\ y_3 - y_2 & 4B & -36C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y_1 - y & 1 & 1C \\ y_2 - y_1 & 2 & 3C \\ y_3 - y_2 & 4 & 9C \end{vmatrix} = 0$$

Expanding the determinant we get,

$7y_1 - 6y - y_3 = 0$

or,
$y_3 - 7y_1 + 6y = 0.$

$\Rightarrow \frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$ is the required ODE.

3 Types of Solution of a ODE

Solution: Any relation connecting the variables of an equation and not involving their derivatives, which satisfies the given differential equation is called a solution.
General Solution: A solution of a differential equation in which the number of arbitrary constants is equal to the number of independent variables in the equation is called a general or complete solution or complete primitive of the equation.
Example: $y = Ax + B$.

Particular Solution: The solution obtained by giving particular values to the arbitrary constants of the general solution, is called a particular solution of the equation.
Example: $y = 3x + 5$.
Singular Solution: A solution of a differential equation in which contains no arbitrary constants is called the singular solution.

4 Exact Linear Differential Equations

A differential equation of the type $Mdx + Ndy = 0$ is called an exact differential equation where M and N are functions of x and y if and only if $M_y = N_x$. The solution of an exact differential equation is of the form $F(x, y) = c$, where c is an arbitrary constant.

The following are the steps involved in solving an exact equation:
Step 1: Test the exactness of the given equation.
Step 2: Write the general solution as $F(x, y) = c$ where $F_x = M$ and $F_y = N$.
Step 3: Integrate F w.r.t. x and y and write the constants in terms of g(y) and h(x).
Step 4: Compare F and find g(y) and h(x).
Step 5: Substitute F in Step 2 which is the general solution.

Example 3

Solve $4xsinydx + 2x^2cosydy = 0$.

Solution.

Let $M = 4xsiny$ and $N = 2x^2cosy$.
$\Rightarrow M_y = 4xcosy$ and $N_x = 4xcosy$
$\Rightarrow M_y = N_x$.
$\Rightarrow$The given equation is exact.
Therefore, the general solution is $F(x, y) = c$_____(1), where $F_x = M$ and $F_y = N$.
$\Rightarrow F_x = 4xsiny$ and $F_y = 2x^2cosy$.
Integrating F w.r.t x and y, we get, $F = 2x^2siny$.
Substituting F in equation (1),
$2x^2siny = c$ is the general solution of the given differential equation.

Example 4

Solve $(3x^2y - 6x)dx + (x^3 + 2y)dy = 0$.

Solution.

Let $M = 3x^2y - 6x$ and $N = x^3 + 2y$.
$\Rightarrow M_y = 3x^2$ and $N_x = 3x^2$
$\Rightarrow M_y = N_x$.
$\Rightarrow$The given equation is exact.
Therefore, the general solution is $F(x, y) = c$_____(1), where $F_x = M$ and $F_y = N$.

$\Longrightarrow F_x = 3x^2 - 6x$ and $F_y = x^3 + 2y$.
Integrating F w.r.t x and y, we get, $F = x^3y - 3x^2 + y^2$.
Substituting F in equation (1),
$x^3y - 3x^2 + y^2 = c$ is the general solution of the given differential equation.

5 Separable Equations

A first order differential equation $y' = f(x, y)$ is called a separable equation if the function $f(x, y)$ can be factorised into two functions $g(y)$ and $h(x)$. The following are the steps involved in solving separable equations:

Step 1: Check whether the given equation is separable.
Step 2: Separate the variables y and dy to the LHS and those of x and dx to the RHS.
Step 3: Integrate on both sides to get the general solution.
Step 4: To find the particular solution, substitute the initial conditions in the general solution.

Example 5

Solve $\frac{dy}{dx} = \frac{2x}{3y^2}$.

Solution.

The given differential equation is separable.
Therefore, $3y^2dy = 2xdx$.
Integrating on both sides, we get,
$y^3 = x^2 + C$.
$\Longrightarrow y = (x^2 + C)^{\frac{1}{3}}$ is the general solution.

Example 6

Solve $y' = y^2 sinx$.

Solution.

Given equation can be written as $\frac{dy}{dx} = y^2 sinx$.
The given differential equation is separable.
Therefore, $\frac{dy}{y^2} = sinxdx$.
Integrating on both sides, we get,
$\frac{1}{y} = cosx + C$.
$\Longrightarrow y = \frac{1}{cosx+C}$ is the general solution.

6 Homogeneous Equations

An expression is said to be homogeneous if the degree of the variables (or the sum of the powers of different variables) in each term is the same.

Examples: $2x + 5y$, $5x^2 - 3xy + 4y^2$.

An homogeneous differential equation is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous expressions in x and y of same degree.

Example: $\frac{dy}{dx} = \frac{x^2+y^2}{x^2-xy+y^2}$ is a homogeneous equation.

Steps involved to solve homogeneous equations:

Step 1: Perform the substitution using $v = \frac{y}{x}$. This implies $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
Step 2: Solve the resulting equation using separation of variables.
Step 3: Substitute for v in terms of x and y.

Example 7

Solve $\frac{dy}{dx} = \frac{3y^2+xy}{x^2}$.

Solution.

Given equation $\frac{dy}{dx} = \frac{3y^2+xy}{x^2}$_________(1) is homogeneous

Therefore, put $y = vx$__________(2)

$\Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ___________(3)

Substituting (2) and (3) in (1) and simplifying we get,

$x\frac{dv}{dx} = 3v^2$.

Separating the variables, we get,

$x\frac{dv}{3v^2} = \frac{dx}{x}$.

Integrating on both sides, we get,

$-\frac{1}{3v} = logx + C$_____(4)

Put $v = \frac{y}{x}$ in (4) we get, $y = \frac{x}{C_1 - 3logx}$ where $C_1 = -3C$.

Example 8

Solve $\frac{dy}{dx} = \frac{x+y}{x}$.

Solution.

Given equation $\frac{dy}{dx} = \frac{x+y}{x}$_________(1) is homogeneous

Therefore, put $y = vx$__________(2)

$\Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ___________(3)

Substituting (2) and (3) in (1) and simplifying we get,

$x\frac{dv}{dx} = 1.$

Separating the variables, we get,

$dv = \frac{dx}{x}.$

Integrating on both sides, we get,

$v = logx + C$______(4)

Put $v = \frac{y}{x}$ in (4) we get, $y = x(logx + C)$ is the general solution.

7 Linear Differential Equations and Equations Reducible to Linear Form

A first order linear differential equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are functions of x or constant. The following are the steps involved in solving a linear ODE:

Step 1: Find the integrating factor $(I.F.) = e^{\int Pdx}$.

Step 2: The solution is given by $y(I.F.) = \int Q(I.F.)dx + C$.

A differential equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$ where $P(x)$ and $Q(x)$ are functions of x, can be reduced to a linear differential equation. The following are the steps involved in solving these equations:

Step 1: Put $f(y) = v$. This implies $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

Step 2: Substitute Step equations in the given differential equation.

Step 3: Step 2 gives a linear equation of the form $\frac{dv}{dx} + P(x)v = Q(x)$

Step 4: The solution is given by $v(I.F.) = \int Q(I.F.)dx + C$.

Example 9

Solve $\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{1}{(x^2+1)^3}$.

Solution.

The given equation a linear equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Here $P = \frac{4x}{x^2+1}$ and $Q = \frac{1}{(x^2+1)^3}$.

$I.F. = e^{\int Pdx} = e^{\int \frac{4xdx}{x^2+1}}$.

Put $t = x^2 + 1$, $dt = 2xdx \Longrightarrow \frac{dt}{2} = xdx$.

Therefore $I.F. = e^{2\int \frac{dt}{t}} = e^{2logt} = e^{2\log(x^2+1)} = e^{\log(x^2+1)^2} = (x^2 + 1)^2$.

The solution is given by $y(I.F.) = \int Q(I.F.)dx + C$.

Therefore, $y(x^2+1)^2 = \int \frac{1}{(x^2+1)^3}(x^2+1)^2 dx + C$.

$\Rightarrow y(x^2+1)^2 = \int \frac{1}{x^2+1} dx + C$.

$\Rightarrow y(x^2+1)^2 = tan^{-1}x + C$.

$\Rightarrow y = \frac{tan^{-1}x+C}{(x^2+1)^2}$ is the general solution.

Example 10

Solve $\frac{dy}{dx} + xsin2y = x^3cos^2y$.

Solution.

Since the RHS of the given equation must contain only x terms, we divide the equation throughout by cos^2y.

$sec^2y\frac{dy}{dx} + \frac{2xsinycosy}{cos^2y} = x^3$.

$\Rightarrow sec^2y\frac{dy}{dx} + 2xtany = x^3$______________(1)

This equation is in the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$.

Here $f'(y) = sec^2y$ and $f(y) = tany$.

To solve this, we put $f(y) = v$ and $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ in (1).

Therefore, we get, $\frac{dv}{dx} + Pv = Q$ where $P = 2x, Q = x^3$.

$I.F. = e^{\int Pdx} = e^{\int 2xdx} = e^{x^2}$.

The solution is given by $v(I.F.) = \int Q(I.F.)dx + C$.

$\Rightarrow\ vx^2 = \int x^3e^{x^2}dx + C$.

Using integration by parts, we get $ve^{x^2} = \frac{1}{2}\left[x^2e^{x^2} - x^2\right] + C$ where $v = tany$.

$tany = \frac{x^2}{2}\left[1 - e^{-x^2}\right] + C$ is the general solution.

8 First order higher degree differential equations solvable for x, y, p

The differential equation which involves $\frac{dy}{dx}$, denoted by p, in higher degree is of the form $f(x,y,p) = 0$ is called a first order higher degree differential equation. These equations can be solved by the following methods:

(i) Equations solvable for x
(ii) Equations solvable for y
(iii) Equations solvable for p

Example 11

Solve $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$.

Solution.

Given $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$_______(1)

Put $\frac{dy}{dx} = p$ in (1), we get, $yp^2 + (x-y)p - x = 0$.

Treating this as quadratic equation in p and solving we get, two equations namely,

$p = 1$ and $p = -\frac{x}{y}$.

$\Longrightarrow \frac{dy}{dx} = 1$ and $\frac{dy}{dx} = -\frac{x}{y}$.

Separating the variables and integrating, we get,

$dy = dx$ and $ydy = -xdx$.

$\Longrightarrow y = x + c$ and $y^2 = x^2 + c_2$.

$\Longrightarrow y - x + c_1 = 0$ and $y^2 - x^2 + c_2 = 0$.

$\Longrightarrow (y - x + c_1 = 0)(y^2 - x^2 + c_2 = 0)$ is the general solution.

Example 12

Solve $y\ = sinp - pcosp$.

Solution.

Given $y\ = sinp - pcosp$ _____(1)

Differentiating the given equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = cosp\frac{dp}{dx} - \left[p\left(-sinp\frac{dp}{dx}\right) + cosp\frac{dp}{dx}\right]$.

$p = psinp\frac{dp}{dx}$.

$1 = sinp\frac{dp}{dx}$.

Separating the variables,

$sinpdp = dx$.

Integrating we get, $-cosp = x + C$.

$\Longrightarrow cosp = C - x$______(2)

$\Longrightarrow p = cos^{-1}(C - x)$_____(3)

$\Rightarrow sinp = \sqrt{1 - cos^2 p} = \sqrt{1 - (C - x)^2}$ ______(4)

Substituting (2), (3) and (4) in (1), we get,

$y = \sqrt{1 - (C - x)^2} - (C - x)cos^{-1}(C - x)$ is the general solution.

9 Clairaut's Equations

The non-linear differential equation of the form $y = px + f(p)$ is called the Clairaut's equation.

To solve Clairaut's equation:

Step 1: Put $p = c$ in the given equation to obtain the general solution.

Step 2: To obtain the singular solution, differentiate the general solution w.r.t 'c'.

Step 3: Eliminate 'c' from equations in Steps 1 and 2.

Example 13

Solve the Clairaut's equation $y = px + p^2$.

Solution.

Given $y = px + p^2$_____(1)

Put $p = c$ in (1), we get, $y = cx + c^2$ _____(2), which is the general solution.

Differentiating (2) w.r.t. 'c' we get,

$0 = x + 2c$.

$\Rightarrow c = -\frac{x}{2}$ ______(3)

Substituting (3) in (2), we get,

$x^2 + 4y = 0$ which is the singular integral.

Example 14

Solve the Clairaut's equation $y = px + \frac{1}{p^2}$.

Solution.

Given $y = px + \frac{1}{p^2}$_____(1)

Put $p = c$ in (1), we get, $y = cx + \frac{1}{c^2}$ _____(2), which is the general solution.

Differentiating (2) w.r.t. 'c' we get,

$0 = x + \frac{2}{c^3}$.

$\Rightarrow c = \left(\frac{2}{x}\right)^{\frac{1}{3}}$ ______(3)

Substituting (3) in (2), we get,

$4y^3 = 27x^2$, which is the singular integral.

10 Orthogonal Trajectories

An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles.

Example:

For the family of concentric circles $x^2 + y^2 = r^2$ with the origin as the center,

each member of the family $y = mx$ of straight lines through the origin is an orthogonal

trajectory. We say that the two families are orthogonal trajectories of each other.

Algorithm to Find Orthogonal Trajectories

Step 1: Find $\frac{dy}{dx}$, eliminate the constant and form a differential equation.

Step 2: Find the slope of the orthogonal trajectory by taking the negative reciprocal of the slope of given family of curves.

Step 3: Separate the variables and integrate to find the orthogonal trajectory.

Example 15

Find the orthogonal trajectories of the family of parabolas $y^2 = cx$

Solution

Given family of curves is: $y^2 = cx \longrightarrow$ (1)

Diff. w.r.t. x $\quad 2y\frac{dy}{dx} = c \longrightarrow$ (2)

To eliminate c, we use eqn. (1)

From (1), $\quad c = \frac{y^2}{x} \longrightarrow$ (3)

Substituting (3) in (2) we get,

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx} = \frac{y}{2x} \longrightarrow (4)$$ (Slope of the given family of curves)

Slope of the orthogonal trajectory is:

$\frac{dy}{dx} = -\frac{2x}{y}$ (Negative reciprocal of (4))

$ydy = -2xdx$ (Separating the variables)

$$\int ydy = -2\int xdx$$

$$\frac{y^2}{2} = -2\cdot\frac{x^2}{2} + c$$

$$\frac{y^2}{2} + x^2 = c$$

$\implies x^2 + \frac{y^2}{2} = c$ (Family of ellipses)

Therefore, the family of parabolas $y^2 = cx$ and family of ellipses $x^2 + \frac{y^2}{2} = c$ are orthogonal trajectories.

Example 16

Obtain the orthogonal trajectories of the family of the circles $x^2 + y^2 = c$.

Solution

Given

$$x^2 + y^2 = c \quad \rightarrow (1)$$

Diff. w.r.to x we get,

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$ydy = -x$$

$\frac{dy}{dx} = -x$ (Slope of the given curves)

Slope of the trajectory is

$$\frac{dy}{dx} = +\frac{y}{x}$$

$$\int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\log y = \log x + \log c$$

$$\log y = \log xc$$

$$y = xc$$

or $y = cx$ (represents family of straight lines).

Example 17

Construct the orthogonal trajectories of the curve $xy = c$.

Solution

Given

$$xy = c$$

$$x \cdot \frac{dy}{dx} + y(1) = 0$$

$$x \cdot \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$\Longrightarrow$ Slope of the orthogonal trajectory,

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$ydy = xdx$$

$$\Longrightarrow \frac{y^2}{2} = \frac{x^2}{2} + c_1$$

$$y^2 - x = 2c_1$$

or $\quad x^2 - y^2 = -2c_1$

$$\Longrightarrow x^2 - y^2 = c \quad \text{where} \quad -2c_1 = c$$

Therefore, the orthogonal trajectory is $x^2 - y^2 = c$, which is the family of hyperbolas.

Exercise

Identify the type of differential equations and solve the following:

1. $\frac{dy}{dx} + ycosx = sinxcosx \quad (Ans\colon y = sinx + c\, e^{-sinx} - 1)$
2. $\frac{dy}{dx} + 2xy = 2x,$ also $y = 3$ when $x = 0$ obtain a particular solution. (Ans:$y = 1 + c\, e^{-x^2}$and P.S. is $y = 1 + 2\, e^{-x^2}$)
3. $\frac{dy}{dx} + ytanx = secx$ (Ans: $y = sinx + ccosx$)
4. $(x + 2y^3)\frac{dy}{dx} = y$ (Ans: $x = y^3 + c\, y$)
5. $(x^4 - 2x\, y^2 + y^4)d\, x - (2x^2y - 4xy^3 + siny)d\, y$
   (Ans:$x^5 - 5x^2y^2 + 5y^4x + 5\, cosy = c$)

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II– Ordinary and Partial Differential Equations – SMT1305

Unit II

DIFFERENTIAL EQUATIONS OF SECOND ORDER

This unit covers the following topics: Linear differential equations of second order, Second order equation with constant coefficient with particular integrals for e^{ax}, x^m, $e^{ax} sinmx$, $e^{ax} cosmx$. Method of variation of parameters, Ordinary simultaneous differential equations, Transformation of the equation by changing — the dependent variable and the independent variable.

1 Introduction

The second order ODE with constant coefficients is of the form $ay'' + by' + cy = f(x)$__(1) where $a \neq 0$, b and c are constants and y is a function of x. The general solution of (1) is given by y = Complementary Function + Particular Integral. That is, $y = C.F. + P.I.$___(2)

2 The following are the rules to find $C.F.$:

(i) Solve the characteristic equation $am^2 + bm + c = 0$____(3) and let the roots of (3) be m_1 and m_2.

(ii) If the roots of (3) are real and distinct then $C.F. = Ae^{m_1 x} + Be^{m_2 x}$.

(iii) If the roots of (3) are real and equal, say $m_1 = m_2 = m$ then $C.F. = (A + Bx)e^{mx}$.

(iv) If the roots of (3) are complex, say, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then $C.F. = e^{\alpha x}(Acos\beta x + Bsin\beta x)$.

3 Rules to find $P.I.$:

Type I: If the RHS of (1) = 0, then $P.I. = 0$.

Type II: If the RHS of (1) = e^{ax} then $P.I. = \frac{1}{f(D)} e^{ax}$.

Replace D by a.

$\Rightarrow P.I. = \frac{1}{f(a)} e^{ax}$.

Note:

If the denominator becomes zero then, multiply the numerator by x and differentiate the denominator w.r.t D. Repeat until the denominator is not zero.

Type III: f the RHS of (1) = $sinax$ or $cosax$ then $P.I. = \frac{1}{f(D)}$ $sinax$ or $cosax$.

Replace D^2 by $-a^2$.

$\Rightarrow P.I. = \frac{1}{f(-a^2)}$ $sinax$ or $cosax$.

Note:

(a) If the denominator becomes zero then, multiply the numerator by x and differentiate the denominator w.r.t D. Repeat until the denominator is not zero and then use integration for $\frac{1}{D}$.

(b) If the denominator contains factors of $f(D)$ then multiply and divide by the conjugate of the factors and then replace D^2 by $-a^2$.

Type IV: If the RHS of (1) is a polynomial say, x^m then $P.I. = \frac{1}{f(D)} x^m$.

Write the denominator as $(1 + \phi(D))^{-1}$ and expand using either $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ or $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ and then operate D on x^m.

Type V: If the RHS of (1) = $e^{ax} sinbx$ or $e^{ax} cosbx$, then $P.I. = \frac{1}{f(D)} e^{ax} sinbx$ or $e^{ax} cosbx$.

Replace D by $D + a$.

$\Rightarrow P.I. = \frac{1}{f(a)} e^{ax} sinbx$ or $e^{ax} cosbx$.

Proceed as in Type III.

4 Method of Variation of Parameters (Type VI)

If the RHS of (1) is $secx$, $tanx$, $cosecx$ etc., then we use method of variation of parameters.

Steps involved in method of variation of parameters:

Step 1: Find $C.F.$ say, $C.F. = Af_1 + Bf_2$.

Step 2: Find f_1' and f_2'.

Step 3: Compute $f_1 f_2' - f_1' f_2$.

Step 4: Find $P = \int \frac{-f_2\, X\, dx}{f_1 f_2' - f_1' f_2}$ where X = RHS of (1).

Step 5: Find $Q = \int \frac{f_1\, X\, dx}{f_1 f_2' - f_1' f_2}$ where X = RHS of (1).

Step 6: Write $P.I. = Pf_1 + Qf_2$

Step 7: The general solution is : $y = C.F. + P.I.$.

Example 1

Solve $y'' - 6y' + 9y = 0$.

Solution.

Given equation can be written as $(D^2 - 6D + 9)y = 0$.

Therefore, the auxiliary equation is $m^2 - 6m + 9 = 0$.

Solving the roots are $m = 3, 3$ (equal roots)

Therefore, $C.F. = (A + Bx)e^{3x}$.

RHS of the given equation is zero.

Therefore, $P.I. = 0$.

$\Rightarrow y = C.F. + P.I. = (A + Bx)e^{3x}$.

Example 2

Solve $y" - 5y' + 6y = 12e^{5x}$.

Solution.

Given equation can be written as $(D^2 - 5D + 6)y = 12e^{5x}$.

Therefore, the auxiliary equation is $m^2 - 5m + 6 = 0$.

Solving the roots are $m = 2, 3$ (distinct roots)

Therefore, $C.F. = Ae^{2x} + Be^{3x}$.

RHS of the given equation is $12e^{5x}$.

Therefore, $P.I. = \frac{1}{D^2-5D+6} 12e^{5x}$.

Replace D by 5.

$P.I. = 12 \frac{1}{5^2-5(5)+6} e^{5x}$.

$P.I. = 12 \frac{1}{25-25+6} e^{5x}$.

$P.I. = 12 \frac{1}{6} e^{5x}$.

$P.I. = 2e^{5x}$.

$\Longrightarrow y = C.F. + P.I.$

$\Longrightarrow y = Ae^{2x} + Be^{3x} + 2e^{5x}$.

Example 3

Solve $y" + y = sin2x$.

Solution.

Given equation can be written as $(D^2 + 1)y = sin2x$.

Therefore, the auxiliary equation is $m^2 + 1 = 0$.

Solving the roots are $m = \pm i$ (complex roots) where $\alpha = 0$ and $\beta = 1$.

Therefore, $C.F. = e^{0x}(Acosx + Bsinx) = Acosx + Bsinx$.

RHS of the given equation is $sin2x$.

Therefore, $P.I. = \frac{1}{D^2+1} sin2x$.

Replace D^2 by $-2^2 = -4$.

$P.I. = \frac{1}{-4+1} sin2x$.

$P.I. = \frac{1}{-3} sin2x$.

$\Rightarrow y = C.F. + P.I.$

$\Rightarrow y = Acosx + Bsinx - \frac{sin2x}{3}.$

Example 4

Solve $y" + y' - 6y = 36x.$

Solution.

Given equation can be written as $(D^2 + D - 6)y = 36x.$

Therefore, the auxiliary equation is $m^2 + m - 6 = 0.$

Solving the roots are $m = 2, -3$ (distinct roots).

Therefore, $C.F. = Ae^{2x} + Be^{-3x}.$

RHS of the given equation is a polynomial $36x.$

Therefore, $P.I. = \frac{1}{D^2+D-6}\ 36x..$

$P.I. = 36\frac{1}{-6+D+D^2}\ x.$

Write the denominator as $(1 + \phi(D))^{-1}$.

$P.I. = \frac{36}{-6}\frac{1}{\left(1-\frac{(D+D^2)}{6}\right)}x.$

$P.I. = -6\left(1 - \frac{(D+D^2)}{6}\right)^{-1} x.$

Expand using $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ and then operate D on the given polynomial.

$P.I. = -6\left(1 + \frac{(D+D^2)}{6}\right)x.$

$P.I. = -6\left(x + \frac{D(x)}{6} + \frac{D^2(x)}{6}\right).$

$P.I. = -6\left(x + \frac{1}{6} + \frac{0}{6}\right).$

$P.I. = -6x - 1$

$\Rightarrow y = C.F. + P.I.$

$\Rightarrow y = Ae^{2x} + Be^{-3x} - 6x - 1.$

Example 5

Solve $(D^2 - 4D + 13)y = e^{4x}sin2x.$

Solution.

Given equation is $(D^2 - 4D + 13)y = e^{4x}sin2x.$

Therefore, the auxiliary equation is $m^2 - 4m + 13 = 0$.

Solving the roots are $m = 2 \pm 3i$ (complex roots) where $\alpha = 2$ and $\beta = 3$.

Therefore, $C.F. = e^{2x}(Acos3x + Bsin3x)$.

RHS of the given equation is $e^{4x}sin2x$.

Therefore, $P.I. = \frac{1}{D^2-4D+13}\ e^{4x}sin2x$.

Replace D by $D + 4$..

$P.I. = e^{4x}\frac{1}{(D+4)^2-4(D+4)+13}\ sin2x$.

$P.I. = e^{4x}\frac{1}{D^2+4D+13}sin2x$.

Replace D^2 by $-2^2 = -4$.

$P.I. = e^{4x}\frac{1}{-4+4D+13}sin2x$.

$$P.I. = e^{4x}\frac{1}{4D + 9}sin2x$$

$P.I. = e^{4x}\frac{4D-9}{(4D+9)(4D-9)}sin2x$.

$P.I. = e^{4x}\frac{(4D-9)}{16D^2-169}sin2x$.

Replace D^2 by $-2^2 = -4$.

$P.I. = e^{4x}\frac{(4D-9)}{16(-4)-169}sin2x$.

$P.I. = e^{4x}\frac{(4Dsin2x-9sin2x)}{16(-4)-169}$.

$P.I. = e^{4x}\frac{(8cos2x-9sin2x)}{-233}$.

$\Rightarrow y = C.F. + P.I.$

$\Rightarrow y = e^{2x}(Acos3x + Bsin3x) - e^{4x}\frac{(8cos2x-9sin2x)}{233}$.

Example 6

Solve $y" + y = sec^2x$.

Solution.

Given equation can be written as $(D^2 + 1)y = sec^2x$.

Therefore, the auxiliary equation is $m^2 + 1 = 0$.

Solving the roots are $m = \pm i$ (complex roots) where $\alpha = 0$ and $\beta = 1$.

Therefore, $C.F. = Acosx + Bsinx$.

RHS of the given equation is sec^2x.

Therefore, $P.I. = Pf_1 + Qf_2$ where $P = \int \frac{-f_2\, X\, dx}{f_1f_2' - f_1'f_2}$ and $Q = \int \frac{f_1\, X\, dx}{f_1f_2' - f_1'f_2}$ and $X = sec^2x$.

Here $f_1 = cosx$ and $f_2 = sinx$.

Therefore, $f_1' = -sinx$ and $f_2' = cosx$.

$\Rightarrow f_1f_2' - f_1'f_2 = 1.$

$P = \int \frac{-f_2\, X\, dx}{f_1f_2' - f_1'f_2}.$

$\Rightarrow\ P = \int \frac{-sinxsec^2x\, dx}{1}.$

$\Rightarrow\ P = \int \frac{-sinx\, dx}{cos^2x}.$

$\Rightarrow\ P = \int -tanxsecx\, dx.$

$\Rightarrow\ P = -secx.$

$Q = \int \frac{f_1\, X\, dx}{f_1f_2' - f_1'f_2}.$

$\Rightarrow\ Q = \int \frac{cosxsec^2x\, dx}{1}.$

$\Rightarrow\ Q = \int \frac{1}{cosx} dx.$

$\Rightarrow\ Q = \int secx\, dx.$

$\Rightarrow\ Q = \log(secx + tanx).$

$P.I. = Pf_1 + Qf_2.$

$\Rightarrow\ P.I. = -secxcosx + \log(secx + tanx)sinx.$

$\Rightarrow\ P.I. = -1 + \log(secx + tanx)sinx.$

$\Rightarrow y = C.F. + P.I.$

$\Rightarrow y =\ Acosx + Bsinx - 1 + \log(secx + tanx)sinx.$

5 Solution of Simultaneous Differential Equations

Example 7

Solve $\frac{dx}{dt} + y = sint; \frac{dy}{dt} + x = cost$ given $x = 2, y = 0$ when $t = 0$.

Solution.

Given equations can be written as $Dx + y = sint$_____(1) and $Dy + x = cost$____(2)

Solving equations (1) and (2) simultaneously, we get, $(D^2 - 1)y = -2sint$.

The characteristic equation is: $m^2 - 1 = 0$.

Solving we get, $m = \pm 1$ (distinct roots).

$\Rightarrow C.F. = Ae^t + Be^{-t}$.

$P.I. = \frac{1}{D^2-1} - 2sint.$

$$\Rightarrow \ P.I. = -2\frac{1}{D^2 - 1} \ sint$$

Replace D^2 by $-1^2 = -1$.

$P.I. = -2\frac{1}{-1+1} \ sint.$

$P.I. = \frac{-2}{-2} sint.$

$P.I. = sint.$

$\Rightarrow y = C.F. + P.I.$

$\Rightarrow y = Ae^t + Be^{-t} + sint$_______(3)

To find x, substitute $y = Ae^t + Be^{-t} + sint$, in equation (2), we get,

$x = cost - D(Ae^t + Be^{-t} + sint).$

$\Rightarrow \ x = cost - Ae^t + Be^{-t} + cost.$

$\Rightarrow \ x = -Ae^t + Be^{-t}$______(4)

To find the constants A and B in (3) and (4), we use the initial conditions given in the problem.

Given: $x = 2$ when $t = 0$ and $y = 0$ when $t = 0$.

Substituting these values in (3) and (4), we get,

$0 = A + B$_____(5)

$2 = -A + B$_____(6)

Solving (5) and (6), we get, $A = 1$ and $B = 1$.

Putting the values of A and B in (3) and (4), we get the solution as:

$y = e^t + e^{-t} + sint$ and $x = -e^t + e^{-t}$.

6 Transformation of Differential Equations

(i) By changing the independent variable x to z

Let the linear differential equation of second order be $y'' + Py' + Qy = R$___(1) where P, Q and R are functions of x.

Procedure to solve equation (1)

Step 1: If the independent variable x is changed to z (z being a function of x) then (1) becomes $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$_____(2)

Step 2: Here $P_1 = \frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$.

Step 3: To compute $\frac{dz}{dx}$, Q_1 is equated to a constant.

Step 4: To find z integrate $\frac{dz}{dx}$ and substitute in R_1.

Step 5: Solve equation (2) by finding $y = C.F.+P.I.$

Step 6: Replace the value of z by a function of x.

(ii) By changing the dependent variable y to v

Procedure to solve equation (1)

Step 1: compare with equation (1) and find P, Q, R.

Step 2: Calculate $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$.

Step 3: If $I =$ constant then the complete solution is $y = uv$, where $u = e^{-\int\frac{1}{2}P\,dx}$.

Step 4: Here v is obtained from $\frac{d^2v}{dx^2} + Iv = Ru$.

Step 5: Solving Step 4 for v, to get $v = C.F.+P.I.$

Step 6: Substitute u and v in Step 3 to get the complete solution.

Example 8

Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$ by changing the independent variable.

Solution.

Given $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$______(1)

Convert the given equation (1) to the standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by dividing by x.

$\Longrightarrow \frac{d^2y}{dz^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin(x^2)$.

Here $P = -\frac{1}{x}$, $Q = -4x^2$ and $R = 8x^2\sin(x^2)$.

By changing the independent variable x to z, we have,

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ ______(2) where $P_1 = \frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$.

Now $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{\left(\frac{dz}{dx}\right)^2}$ =constant.

$\Longrightarrow \frac{-4x^2}{\left(\frac{dz}{dx}\right)^2} = -1$ (say)

$\Longrightarrow \left(\frac{dz}{dx}\right)^2 = 4x^2.$

$\Longrightarrow \frac{dz}{dx} = 2x.$

$\Longrightarrow \frac{d^2z}{dx^2} = 2.$

$\Longrightarrow P_1 = \frac{2-\frac{1}{x}(2x)}{(2x)^2} = 0.$

$\Longrightarrow Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{(2x)^2} = -1.$

$\Longrightarrow R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2\sin(x^2)}{(2x)^2} = 2\sin(x^2).$

From (2), $\frac{d^2y}{dz^2} - y = 2\sin(x^2)$______(3)

Since $\frac{dz}{dx} = 2x$, we get, $z = x^2$.

Therefore equation (3) becomes, $\frac{d^2y}{dz^2} - y = 2\sin z.$

$\Longrightarrow (D^2 - 1)y = 2sinz.$

Solving $m^2 - 1 = 0$, we get, $C.F. = Ae^z - Be^{-z}.$

$\Longrightarrow C.F. = Ae^{x^2} - Be^{-x^2}.$

$P.I. = \frac{1}{D^2-1}2sinz.$

Replace $D^2$by -1.

$P.I. = 2\frac{1}{-1-1}sinz.$

$P.I. = -sinz = \sin(x^2).$

Therefore, $y = C.F. + P.I.$

$\Longrightarrow\ y = Ae^{x^2} - Be^{-x^2} - \sin(x^2).$

Example 9

Solve by changing the dependent variable $\frac{d^2y}{dx^2} - 2tanx\frac{dy}{dx} + 5y = e^x secx.$

Solution.

The standard form is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$

Therefore, from the given equation $P = 2tanx$, $Q = 5$ and $R = e^x secx.$

Now $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$.

$\Longrightarrow\ I = 5 - \frac{1}{2}(-2sec^2x) - \frac{4tan^2x}{4}$.

$\Longrightarrow\ I = 5 + sec^2x - tan^2x = 5 + 1 = 6$ (constant)

Therefore, complete solution is $y = uv$ where $u = e^{-\int\frac{1}{2}P\,dx}$, v is obtained from $\frac{d^2v}{dx^2} + Iv = Ru$.

$\Longrightarrow u = e^{-\int\frac{1}{2}P\,dx} = e^{-\int\frac{1}{2}(-2tanx)dx} = e^{\int tanx\,dx} = e^{logsecx} = secx.$

To find v, we solve $\frac{d^2v}{dx^2} + Iv = Ru.$

$\Longrightarrow \frac{d^2v}{dx^2} + 6v = \frac{e^x secx}{secx}$.

$\Longrightarrow (D^2 + 6)v = e^x$.

Solving $m^2 + 6 = 0$, we get, $m = \pm\sqrt{6}$.

$\Longrightarrow C.F. = Acos\sqrt{6}x + Bsin\sqrt{6}x$.

$P.I. = \frac{1}{D^2+6}e^x$.

Replace D by 1.

$P.I. = \frac{1}{1+6}e^x$.

$P.I. = \frac{1}{7}e^x$.

Therefore, $v = C.F. + P.I.$

$\Longrightarrow\ v = Acos\sqrt{6}x + Bsin\sqrt{6}x + \frac{1}{7}e^x$.

The complete solution is $v = (secx)(Acos\sqrt{6}x + Bsin\sqrt{6}x + \frac{1}{7}e^x)$.

Exercise

Solve the following differential equations:

1. $y'' - 2y' + 10y = 0$ [Ans: $y = e^x[c_1cos\,(3x) + c_2sin(3x)]$
2. $(D^2 + 3D + 2)y = sin\,(3x)\,cos\,(2x)$
   [Ans: $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{884}[10\,cos\,(5x) - 11\,sin\,(5x)] + \frac{1}{20}[sin\,(x) + 2\,cos\,(x)]$
3. $\frac{dy^2}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ [Ans: $y = c_1 + c_2e^{-x} + \frac{x^3}{3} + 4x - 4$]
4. $(D^2 + 16)y = tan\,(4x)$ [Ans: $Acos4x + Bsin4x - \frac{1}{16}[\log(sec4x + tan4x)]cos4x$]
5. Find P. I. of $(D^2 - 4D + 3)y = e^{4x}sin2x$ [Ans: P.I. = $\frac{e^{-4x}}{65}[8\,cos\,(2x) + sin\,(2x)]$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III – Ordinary and Partial Differential Equations – SMT1305

Unit III

PARTIAL DIFFERENTIAL EQUATIONS

This unit covers topics that explain the formation of partial differential equations and the solutions of special types of first order partial differential equations (PDE).

1 Introduction

A partial differential equation (PDE) is one which involves one or more partial derivatives. The order of the highest derivative is called the order of the equation. A partial differential equation contains more than one independent variable. But, here we shall consider partial differential only equation two independent variables x and y so that z = f(x, y). We shall denote

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

A partial differential equation is linear if it is of the first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous, otherwise it is non homogeneous. Partial differential equations are used to formulate and thus aid the solution of problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity.

2 Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of arbitrary constants or by the elimination of arbitrary functions.

(i) By the elimination of arbitrary constants

Let us consider the function f (x, y, z, a, b) = 0 ------------- (1)

where a & b are arbitrary constants

Differentiating equation (1) partially w.r.t x & y, we get

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Eliminating a and b from equations (1), (2) and (3), we get a partial differential equation of the first order of the form f (x, y, z, p, q) = 0.

(ii) By the elimination of arbitrary functions

Let u and v be any two functions which are arbitrary. This relation can be expressed as u = f(v) ______________ (1)

Differentiating (1) partially w.r.t x and y and eliminating the arbitrary functions from these relations, we get a PDE of the first order of the form f(x, y, z, p, q) = 0.

Example 1

Eliminate the arbitrary constants a and b from $z = ax + by + ab$ to construct a the PDE.

Solution. Consider $z = ax + by + ab$ ____________ (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e,} \quad p = a \qquad \_\_\_\_\_\_\_\_\_\_ \quad (2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e,} \quad q = b \qquad \_\_\_\_\_\_\_\_ \quad (3)$$

Using (2) and (3) in (1), we get, $z = px + qy + pq$, which is the required PDE.

Example 2

Construct the partial differential equation by eliminating the arbitrary constants a and b from $z = (x^2 + a^2) (y^2 + b^2)$.

Solution. Given $z = (x^2 + a^2) (y^2 + b^2)$ _______(1)

Differentiating (1) partially w.r.t x and y, we get

$$p = 2x \ (y^2 + b^2)$$

$$q = 2y \ (x^2 + a^2)$$

Substituting the values of p and q in (1), we get, $4xyz = pq$, which is the PDE.

Example 3

Find the partial differential equation of the family of spheres of radius one whose centre lie on the xy – plane.

Solution.

The equation of the sphere is given by $(x-a)^2 + (y-b)^2 + z^2 = 1$ ________ (1)

Differentiating (1) partially w.r.t x & y, we get, $2(x-a) + 2zp = 0$ and $2(y-b) + 2zq = 0$.

From these equations we obtain

$x - a = -zp$ __________ (2)

$y - b = -zq$ __________ (3)

Put (2) and (3) in (1), we get, $z^2p^2 + z^2q^2 + z^2 = 1$ or $z^2(p^2+q^2+1)^2 = 1$

Example 4

Eliminate the arbitrary constants a, b and c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and form the partial differential equation.

Solution.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)$$

Differentiating (1) partially w.r.t x & y, we get

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$$

Therefore we get

$$\frac{x}{a^2} + \frac{zp}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Again differentiating (2) partially w.r.t 'x', we set

$$(1/a^2) + (1/c^2)(zr + p^2) = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (4)}$$

Multiplying (4) by x, we get

$$\frac{x}{a^2} + \frac{xz\,r}{c^2} + \frac{p^2x}{c^2} = 0$$

From (2), we have

$$\frac{-zp}{c^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

or $-zp + xzr + p^2x = 0$

Therefore, $xp^2 - zp + xzr = 0$ is the required PDE.

Example 5

Form the PDE by eliminating f & Φ from $z = f(x+ay) + \Phi(x-ay)$

Consider $z = f(x+ay) + \Phi(x-ay)$ _______________ (1)

Differentiating (1) partially w.r.t x &y , we get

$$p = f'(x+ay) + \Phi'(x-ay) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$q = f'(x+ay).a + \Phi'(x-ay)(-a) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Differentiating (2) & (3) again partially w.r.t x & y, we get

$$r = f''(x+ay) + \Phi''(x-ay)$$
$$t = f''(x+ay).a^2 + \Phi''(x-ay)(-a)^2$$

i.e, $t = a^2\{f''(x+ay) + \Phi''(x-ay)\}$

or $t = a^2r$

Example 6
Form the partial differential equation by eliminating the arbitrary function f from $z = e^y f(x + y)$

Solution. Consider $z = e^y f(x + y)$ ___________(1)

Differentiating (1) partially w.r.t x & y, we get

$p = e^y f'(x + y)$

$q = e^y f'(x + y) + f(x + y).e^y$
Hence, we have, $q = p + z$, which is the required PDE.

Example 7
Obtain the partial differential equation by eliminating f from the equation
$z = (x + y)f(x^2 - y^2)$.
Solution.
Let us now consider the equation $z = (x + y)f(x^2 - y^2)$ _____________ (1)

Differentiating (1) partially w.r.t x & y, we get

$p = (x + y)f'(x^2 - y^2).2x + f(x^2 - y^2)$

$y = (x + y)f'(x^2 - y^2).(-2y) + f(x^2 - y^2)$

$$p - f(x^2 - y^2) = (x + y)f'(x^2 - y^2).2x \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$
$$q - f(x^2 - y^2) = (x + y)f'(x^2 - y^2).(-2y) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$
Hence, we get

$$\frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)} = -\frac{x}{y}$$

$$py - yf(x^2 - y^2) = -qx + xf(x^2 - y^2)$$

$$py + qx = (x + y)f(x^2 - y^2)$$

Therefore, we have by (1), $py + qx = z$, which is the required PDE.

Exercises:

1. Form the partial differential equation by eliminating the arbitrary constants a & b from the following equations.

(i) $z = ax + by$

(ii) $\dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} = 1$

(iii) $z = ax + by + \sqrt{a^2 + b^2}$

(iv) $ax^2 + by^2 + cz^2 = 1$

(v) $z = a^2x + b^2y + ab$

2. Find the PDE of the family of spheres of radius 1 having their centres lie on the xy plane{Hint: $(x-a)^2 + (x-a)^2 + z^2 = 1$}

3. Find the PDE of all spheres whose centre lie on the (i) z axis (ii) x-axis

4. Form the partial differential equations by eliminating the arbitrary functions in the following cases. (i) $z = f(x + y)$

(ii) $z = f(x^2 - y^2)$

(iii) $z = f(x^2 + y^2 - z^2)$

(iv) $f(xyz, x + y + z) = 0$

(v) $F(xy + z^2, x + y + z) = 0.$

3 Solutions of a Partial Differential Equation

A solution or integral of a partial differential equation is a relation connecting the dependent and the independent variables which satisfies the given differential equation. A partial differential equation can result both from elimination of arbitrary constants and from elimination of arbitrary functions. But there is a basic difference in the two forms of solutions. A solution containing as many arbitrary constants as there are independent variables is called a complete integral. Here, the partial differential equations contain only two independent variables so that the complete integral will include two constants. The solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let f (x,y,z,p,q) = 0 ________ (1)

be the partial differential equation whose complete integral is

f (x,y,z,a,b) = 0 ________(2)

where a and b are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain

$$\frac{\partial \phi}{\partial a} = 0 \qquad \text{----------- (3)}$$

and

$$\frac{\partial \phi}{\partial b} = 0 \qquad \text{----------- (4)}$$

The eliminant of a and b from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put b = F(a), we get

f (x,y,z,a, F(a)) = 0 ---------- (5)

Differentiating (2), partially w.r.t. a, we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} F'(a) = 0 \qquad \text{-------- (6)}$$

The eliminant of a between (5) and (6), if it exists, is called the general integral of (1).

4 Lagrange's Linear Equation

Equations of the form Pp + Qq = R ________ (1), where P, Q and R are functions of x, y, z, are known as Lagrange equations. To solve this equation, let us consider the equations u = a and v = b, where a, b are constants and u, v are functions of x, y, z.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Comparing (2) and (3), we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (3)}$$

Similarly,

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (4)}$$

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$$

(or)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (5)$$

Equation (5) represent a pair of simultaneous equations which are of the first order and of first degree. Therefore, the two solutions of (5) are $u = a$ and $v = b$. Thus, $f(u, v) = 0$ is the required solution of (1).

Note:

To solve the Lagrange's equation, we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping or by the method of multipliers.

Example 8
Find the general solution of $px + qy = z$.

Solution.

Here, the subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking the first two ratios, $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log x = \log y + \log c_1$

or $x = c_1 y$ i.e, $c_1 = x / y$

From the last two ratios,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, $\log y = \log z + \log c_2$

or $y = c_2 z$

i.e, $c_2 = y / z$

Hence the required general solution is $\Phi(x/y, y/z) = 0$.

Example 9

Solve $p \tan x + q \tan y = \tan z$

Solution.

The subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two ratios, $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

ie, $\cot x\, dx = \cot y\, dy$

Integrating, $\log \sin x = \log \sin y + \log c_1$

ie, $\sin x = c_1 \sin y$

Therefore, $c_1 = \sin x / \sin y$

Similarly, from the last two ratios, we get

$$\sin y = c_2 \sin z$$

i.e, $c_2 = \sin y / \sin z$

Hence the required general solution is

$$\Phi\left[\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right] = 0$$

.

Example 10

Solve $(y - z)\,p + (z - x)\,q = x - y.$

Solution.

Here the subsidiary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Using multipliers 1,1,1,

$$\text{each ratio} = \frac{dx + dy + dz}{0}$$

Therefore, $dx + dy + dz = 0$.

Integrating, $x + y + z = c_1$ ______________ (1)

Again using multipliers x, y and z,

$$\text{each ratio} = \frac{xdx + ydy + zdz}{0}$$

Therefore, $xdx + ydy + zdz = 0$.

Integrating, $x^2/2 + y^2/2 + z^2/2 = \text{constant}$

or $x^2 + y^2 + z^2 = c_2$ __________ (2)

Hence from (1) and (2), the general solution is

$$\Phi(x + y + z, x^2 + y^2 + z^2) = 0$$

Example 11

Find the general solution of $(mz - ny)p + (nx - lz)q = ly - mx$.

Solution.

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using the multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

` $\therefore$ $xdx + ydy + zdz = 0$, which on integration gives

$x^2/2 + y^2/2 + z^2/2 = \text{constant}$

or $x^2 + y^2 + z^2 = c_1$ __________ (1)

Again using the multipliers l, m and n, we have

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

` $\therefore$ $ldx + mdy + ndz = 0$, which on integration gives

$lx + my + nz = c_2$ __________ (2)

Hence, the required general solution is

$$\Phi(x^2 + y^2 + z^2, lx + my + nz) = 0$$

Example 12

Solve $(x^2 - y^2 - z^2)\,p + 2xy\,q = 2xz.$

The subsidiary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking the last two ratios,

$$\frac{dx}{2xy} = \frac{dz}{2xz}$$

ie, $$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log c_1$

or $y = c_1 z$

i.e, $c_1 = y/z$ __________ (1)

Using multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + y\,dy + zdz}{x(x^2-y^2-z^2) + 2xy^2 + 2xz^2} = \frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)}$$

Comparing with the last ratio, we get

$$\frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$$

i.e, $$\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Integrating, $\log(x^2 + y^2 + z^2) = \log z + \log c_2$

or $x^2 + y^2 + z^2 = c_2 z$

i.e, $$c_2 = \frac{x^2 + y^2 + z^2}{z} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

From (1) and (2), the general solution is $\Phi(c_1, c_2) = 0$.

i.e, $$\Phi\left((y/z), \frac{x^2 + y^2 + z^2}{z}\right) = 0$$

Exercise

Solve the following equations

1. $px^2 + qy^2 = z^2$
2. $pyz + qzx = xy$
3. $xp - yq = y^2 - x^2$
4. $y^2zp + x^2zq = y^2x$
5. $z(x - y) = px^2 - qy^2$
6. $(a - x)p + (b - y)q = c - z$
7. $(y^2z\,p)/x + xzq = y^2$
8. $(y^2 + z^2)p - xyq + xz = 0$
9. $x^2p + y^2q = (x + y)z$
10. $p - q = \log(x+y)$
11. $(xz + yz)p + (xz - yz)q = x^2 + y^2$
12. $(y - z)p - (2x + y)q = 2x + z$

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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – Ordinary and Partial Differential Equations – SMT1305

UNIT – IV

PARTIAL DIFFERENTIAL EQUATIONS (CONTD…)

1 Some Special Types of Equations which can be Solved Easily by Methods other than the General Methods

The first order partial differential equation can be written as f(x, y, z, p, q) = 0, where $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$. In this section, we shall solve some standard forms of equations by special methods.

Type I: f (p, q) = 0. (Equations containing p and q only).
Suppose that z = ax + by +c is a solution of the equation f(p, q) = 0,
where f (a, b) = 0.

Solving this for b, we get b = F (a).
Hence the complete integral is z = ax + F(a) y + c __________(1)
To find the singular integral, differentiate (1) w.r.t. a, we get, 0 = x + yF'(a)______(2)

Now, the singular integral is obtained by eliminating a and c from (1) and (2), we get 0 = 1.

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take c = F(a).

Then, z = ax + F(a) y + F(a) __________ (2)

Differentiating (2) partially w.r.t. a, we get

0 = x + F'(a). y + F '(a) ________ (3)

Eliminating a from (2) and (3), we get the general integral.

Example 1

Solve $pq = 2$

Solution.

The given equation is of the form f (p, q) = 0

The solution is z = ax + by +c, where ab = 2.

Solving, b = 2/a.

The complete integral is $z = ax + 2/a\ y + c$ ---------- (1)

Differentiating (1) partially w.r.t c, we have, $0 = 1$, which is absurd. Hence, there is no singular integral. To find the general integral, put $c = \Phi(a)$ in (1), we get, $z = ax + 2/a\ y + F(a)$

Differentiating partially w.r.t. a, we get,

$0 = x - 2/a^2\ y + F'(a)$

Eliminating a between these equations, gives the general integral.

Example 2

Solve $pq + p + q = 0$

Solution.

The given equation is of the form $f(p, q) = 0$.

The solution is $z = ax + by + c$, where $ab + a + b = 0$.

Solving, we get

$$b = -\frac{a}{1+a}$$

Hence the complete Integral is $z = ax - \left(\frac{a}{1+a}\right) y + c$ ------ (1)

Differentiating (1) partially w.r.t. c, we get, $0 = 1$.

The above equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral, put $c = F(a)$ in (1), we have

$$z = ax - \left(\frac{a}{1+a}\right) y + \Phi(a) \quad \text{------(2)}$$

Differentiating (2) partially w.r.t a, we get

$$0 = x - \frac{1}{(1+a)^2}\, y + \Phi'(a) \qquad \text{----- (3)}$$

Eliminate a from (2) and (3) gives the general integral.

Example 3
Solve $p^2 + q^2 = npq$.

Solution.

The solution of this equation is z = ax + by + c, where $a^2 + b^2 = nab$.

Solving, we get,

$$b = a\left(\frac{n \pm \sqrt{(n^2-4)}}{2}\right)$$

Hence the complete integral is

$$z = ax + a\left(\frac{n \pm \sqrt{n^2-4}}{2}\right) y + c \qquad \text{-----------(1)}$$

Differentiating (1) partially w.r.t c, we get 0 = 1, which is absurd. Therefore, there is no singular integral for the given equation.

To find the general integral, put C = F (a), we get

$$z = ax + a\left(\frac{n + \sqrt{n^2-4}}{2}\right) y + \Phi(a)$$

Differentiating partially w.r.t 'a', we have

$$0 = x + \left(\frac{n \pm \sqrt{n^2-4}}{2}\right) y + \Phi'(a)$$

The eliminant of a between these equations gives the general integral.

Type II: Equations of the form f (x,p,q) = 0, f (y,p,q) = 0 and f (z,p,q) = 0. (One of the variables x, y and z occurs explicitly)

(i) Let us consider the equation f (x,p,q) = 0.
Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $dz = pdx + qdy$

Assume that $q = a$.

Then the given equation takes the form $f\ (x, p, a\ ) = 0$.

Solving, we get p = F(x, a). Therefore, dz = F(x, a) dx + a dy.

(ii) Let us consider the equation f(y, p, q) = 0. Assume that p = a.

Then the equation becomes f (y, a, q) = 0 Solving, we get q = F (y, a).

Therefore, dz = adx + F(y,a) dy.

Integrating, $z = ax + \int F(y,a)\, dy + b$, which is a complete Integral.

(iii) Let us consider the equation f(z, p, q) = 0.

Assume that q = ap.

Then the equation becomes f (z, p, ap) = 0
Solving, we get p = F(z,a). Hence dz = F(z,a) dx + a F(z, a) dy.

$$\text{ie,}\quad \frac{dz}{\Phi\,(z,a)} = dx + a dy.$$

$$\text{Integrating,}\quad \int \frac{dz}{\Phi\,(z,a)} = x + ay + b, \text{ which is a complete Integral.}$$

Example 4

Solve $q = xp + p^2$

Solution.

Given $q = xp + p^2$ -------------(1)

This is of the form f (x, p, q) = 0.

Put q = a in (1), we get

$a = xp + p^2$

i.e, $p^2 + xp - a = 0$.

Therefore,

$$p = \frac{-x + \sqrt{(x^2 + 4a)}}{2}$$

Integrating ,

$$z = \int \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2} \right) dx + ay + b$$

Thus,

$$z = -\frac{x^2}{4} \pm \left\{ \frac{x}{2\sqrt{a}} \sqrt{(4a + x^2)} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{2\sqrt{a}} \right) \right\} + ay + b$$

Example 5

Solve $q = y\,p^2$

Solution.

This is of the form f (y, p, q) = 0

Then, put p = a.

Therefore, the given equation becomes $q = a^2y$.

Since $dz = pdx + qdy$, we have

$dz = adx + a^2y\, dy$

Integrating, we get $z = ax + (a^2 y^2/2) + b$

Example 6

Solve $9 (p^2 z + q^2) = 4$

Solution.

This is of the form $f (z, p, q) = 0$

Then, putting $q = ap$, the given equation becomes $9 (p^2z + a^2p^2) = 4$.

Therefore, $$p = \pm \frac{2}{3 (\sqrt{z + a^2})}$$

and $$q = \pm \frac{2a}{3 (\sqrt{z + a^2})}$$

Since $dz = pdx + qdy$,

$$dz = \pm \frac{2}{3} \frac{1}{\sqrt{z + a^2}} dx \pm \frac{2}{3} a \frac{1}{\sqrt{z + a^2}} dy$$

Multiplying both sides by $\sqrt{z + a^2}$, we get

$$\sqrt{z + a^2}\, dz = \frac{2}{3} dx + \frac{2}{3} a\, dy\,, \text{ which on integration gives,}$$

$$\frac{(z+a^2)^{3/2}}{3/2} = \frac{2}{3} x + \frac{2}{3} ay + b.$$

or $(z + a^2)^{3/2} = x + ay + b$.

Type III: $f1(x, p) = f2(y, q)$. ie, equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that $f(x,p) = g(y,q) = a$ (say).

Solving for p and q, we get $p = F(x, a)$ and $q = G(y, a)$.

$$\text{But} \qquad dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Hence $dz = pdx + qdy = F(x, a)\,dx + G(y, a)\,dy$

Therefore, $z = \int F(x, a)\,dx + \int G(y, a)\,dy + b$, which is the complete integral of the given equation containing two constants a and b. The singular and general integrals are found in the usual way.

Example 7

Solve $pq = xy$

Solution.

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q} = a \text{ (say)}$$

Therefore, $\frac{p}{x} = a$ implies $p = ax$

and $\frac{y}{q} = a$ implies $q = \frac{y}{a}$

Since $dz = pdx + qdy$, we have

$dz = axdx + \frac{y}{a}dy$, which on integration gives.

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Example 8

Solve $p^2 + q^2 = x^2 + y^2$

Solution.

The given equation can be written as $p^2 - \; = y^2 - q^2 = a^2$ (say)

$p^2 - x^2 = a^2$ implies $p = \sqrt{x^2 + a^2}$

and $y^2 - q^2 = a^2$ implies $q = \sqrt{y^2 - a^2}$

But $dz = pdx + qdy$

ie, $dz = \sqrt{a^2 + x^2}\, dx + \sqrt{y^2 - a^2}\, dy$

Integrating, we get

$$z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b$$

Type IV (Clairaut's) form

Equation of the type $z = px + qy + f(p,q)$ ------(1) is known as Clairaut's form. Differentiating (1) partially w.r.t x and y, we get $p = a$ and $q = b$.

Therefore, the complete integral is given by
$z = ax + by + f(a,b)$.

Example 9
Solve $z = px + qy + pq$

Solution.

The given equation is in Clairaut's form

Putting $p = a$ and $q = b$, we have ,
$z = ax + by + ab$ -------- (1)

which is the complete integral.

To find the singular integral, differentiating (1) partially w.r.t a and b, we get

$0 = x + b$

$0 = y + a$

Therefore, we have, $a = -y$ and $b = -x$.

Substituting the values of a & b in (1), we get, $z = -xy - xy + xy$
or $z + xy = 0$, which is the singular integral.

To get the general integral, put $b = F(a)$ in (1).

Then $z = ax + F(a)y + a F(a)$ ---------- (2)

Differentiating (2) partially w.r.t a, we have

$0 = x + F'(a) y + aF'(a) + F(a)$ ---------- (3)

Eliminating a between (2) and (3), we get the general integral.

Example 10

Find the complete and singular solutions of

$$z = px + qy + \sqrt{1 + p^2 + q^2}$$

The complete integral is given by

$$z = ax + by + \sqrt{1 + a^2 + b^2} \qquad \text{-------- (1)}$$

To obtain the singular integral, differentiating (1) partially w.r.t a & b. Then,

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}}$$

Therefore,

$$x = \frac{-a}{\sqrt{(1 + a^2 + b^2)}} \quad \text{----------(2)}$$

and

$$y = \frac{-b}{\sqrt{(1 + a^2 + b^2)}} \quad \text{---------- (3)}$$

Squaring (2) & (3) and adding, we get

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

Now,

$$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

i.e,

$$1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Therefore,

$$\sqrt{(1 + a^2 + b^2)} = \frac{1}{\sqrt{1 - x^2 - y^2}} \quad \text{----------(4)}$$

Using (4) in (2) & (3), we get

$$x = -a\sqrt{1 - x^2 - y^2}$$

and

$$y = -b\sqrt{1 - x^2 - y^2}$$

Hence,

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \quad \text{and} \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Substituting the values of a & b in (1) , we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

which on simplification gives

$$z = \sqrt{1 - x^2 - y^2}$$

or $$x^2 + y^2 + z^2 = 1, \quad \text{which is the singular integral.}$$

Exercises

Solve the following Equations

1. $pq = k$
2. $p + q = pq$
3. $p + q = x + y$
4. $p = y^2q^2$
5. $z = p^2 + q^2$
6. $p + q = x + y$
7. $p^2z^2 + q^2 = 1$
8. $z = px + qy - 2pq$
9. $\{z - (px + qy)\}^2 = c^2 + p^2 + q^2$
10. $z = px + qy + p^2q^2$

2 EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non – linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By changing the variables suitably, we will reduce them into any one of the four standard forms.

Type (I): Equations of the form $F(x^m p, y^n q) = 0$ (or) $F(z, x^m p, y^n q) = 0$.

Case(i): If m ^{1}1 and n ^{1}1, then put $x^{1-m} = X$ and $y^{1-n} = Y$.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)\, x^{-m}$$

$$\text{Therefore, } x^m p = \frac{\partial z}{\partial X}(1-m) = (1-m)P, \text{ where } P = \frac{\partial z}{\partial X}$$

$$\text{Similarly, } y^n q = (1-n)Q, \text{ where } Q = \frac{\partial z}{\partial Y}$$

Hence, the given equation takes the form $F(P, Q) = 0$ (or) $F(z, P, Q) = 0$.

Case(ii) : If m = 1 and n = 1, then put log x = X and log y = Y.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \; \frac{1}{x}$$

$$\text{Therefore, } xp = \frac{\partial z}{\partial X} = P.$$

Similarly, $yq = Q$

Example 11

Solve $x^4p^2 + y^2zq = 2z^2$

Solution.

The given equation can be expressed as $(x^2p)^2 + (y^2q)z = 2z^2$

Here m = 2, n = 2

Put X = x1-m = x -1 and Y = y 1-n = y -1.

We have $x^mp = (1- m) P$ and $y^nq = (1- n)Q$

i.e, $x^2p = -P$ and $y^2q = -Q$.

Hence the given equation becomes

$P^2 – Qz = 2z^2$ ----------(1)

This equation is of the form f (z, P, Q) = 0.

Let us take Q = aP.

Then equation (1) reduces to

$P^2 –aPz =2z^2$

Hence, $$P = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

and $$Q = a\left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

Since $dz = PdX + QdY$, we have

$$dz = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\, dX + a \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\, dY$$

i.e, $$\frac{dz}{z} = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) (dX + a\, dY)$$

Integrating, we get

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)(X + aY) + b$$

Therefore, $$\log z = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) \left(\frac{1}{x} + \frac{a}{y}\right) + b$$ which is the complete solution.

Example 12

Solve $x^2p^2 + y^2q^2 = z^2$

Solution.

The given equation can be written as $(xp)^2 + (yq)^2 = z^2$

Here $m = 1$, $n = 1$.

Put $X = \log x$ and $Y = \log y$.

Then $xp = P$ and $yq = Q$.

Hence the given equation becomes

$P^2 + Q^2 = z^2$ ------------- (1)

This equation is of the form F(z, P, Q) = 0.

Therefore, let us assume that Q = aP.

Now, equation (1) becomes,

$$P^2 + a^2 P^2 = z^2$$

Hence $\quad P = \dfrac{z}{\sqrt{(1+a^2)}}$

and $\quad Q = \dfrac{az}{\sqrt{(1+a^2)}}$

Since dz = PdX + QdY, we have

$$dz = \frac{z}{\sqrt{(1+a^2)}} dX + \frac{az}{\sqrt{(1+a^2)}} dY.$$

i.e, $\sqrt{(1+a^2)}\dfrac{dz}{z} = dX + a\,dY.$

Integrating, we get, $\sqrt{1+a^2}\log z = X + aY + b$.

Therefore, $\sqrt{1+a^2}\log z = \log x + a\log y + b$, which is the complete solution.

Type (II) : Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y, z^k q)$.

Case (i) : If $k \neq -1$, put $Z = z^{k+1}$,

$$\text{Now } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\,\frac{\partial z}{\partial x} = (k+1)z^k.\frac{\partial z}{\partial x} = (k+1)\,z^k p.$$

$$\text{Therefore, } z^k p = \frac{1}{k+1} \frac{\partial Z}{\partial x}$$

$$\text{Similarly, } z^k q = \frac{1}{k+1} \frac{\partial Z}{\partial y}$$

Case (ii) : If k = -1, put $Z = \log z$.

$$\text{Now, } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \frac{\partial z}{\partial x} = \frac{1}{z} p$$

$$\text{Similarly, } \quad \frac{\partial Z}{\partial y} = \frac{1}{z} q.$$

Example 13

Solve $z^4q^2 - z^2p = 1$

Solution.

The given equation can also be written as $(z^2q)^2 - (z^2p) = 1$

Here k = 2. Putting $Z = z^{k+1} = z^3$, we get

$$Z^k p = \frac{1}{k+1} \frac{\partial Z}{\partial x} \quad \text{and} \quad Z^k q = \frac{1}{k+1} \frac{\partial Z}{\partial y}$$

$$\text{i.e, } Z^2 p = \frac{1}{3} \frac{\partial Z}{\partial x} \quad \text{and} \quad Z^2 q = \frac{1}{3} \frac{\partial Z}{\partial y}$$

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \left(\frac{P}{3}\right) = 1$$

i.e, $\quad Q^2 - 3P - 9 = 0,$

i.e, $\quad Q^2 - 3P - 9 = 0$, which is of the form $F(P, Q) = 0$.

Hence its solution is z = ax + by + c, where $b^2 - 3a - 9 = 0$.

Solving for b, we get, $b = \pm \sqrt{3a+9}$

Hence the complete solution is $Z = ax \pm \sqrt{3a+9}\, y + c$

or $z^3 = ax \pm \sqrt{3a+9}\, y + c$

3 Charpit's Method

This is a general method to solve the most general non-linear PDE $f(x, y, z, p, q) = 0$____(1) of order one involving two independent variables. To solve (1), we solve the system of auxiliary equations called Charpit's equations.

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0} \quad \text{\_\_\_\_\_(2).}$$

Working rule of Charpit's Method:

Step 1: Transfer all the terms of the PDE to LHS and denote the entire expression by $f(x, y, z, p, q) = 0$.

Step 2: Write down Charpit's auxiliary equations.

Step 3: Find f_x, f_y, f_z, f_p and f_q. Put them in Step 2 and simplify.

Step 4: Choose two fractions such that the resulting integral is a simplest relation involving p or q or both.

Step 5: Use Step 4 to find p and q and put p and q in the equation $dz = pdx + qdy$, which on integration gives the complete integral.

Example 14

Find the complete integral of the PDE $3p^2 = q$ using Charpit's method.

Solution.

Given $3p^2 = q$.

$\Longrightarrow 3p^2 - q = 0$_______(1)

$\Longrightarrow f = 3p^2 - q$.

$\Longrightarrow f_x = 0$, $f_y = 0$, $f_z = 0$, $f_p = 6p$ and $f_q = -1$.

Charpit's auxiliary equations are:

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-6p^2+q} = \frac{dx}{-6p} = \frac{dy}{1}.$$

Taking the first fraction, $\frac{dp}{0} = k$.

$\Rightarrow dp = 0$.

Integrating, $p = a$.

Substituting $p = a$ in (1), $3a^2 - q = 0$.

$\Rightarrow q = 3a^2$.

Substituting p and q in $dz = pdx + qdy$, we get, $dz = adx + 3a^2dy$.

Integrating, $z = ax + 3a^2y + b$ is the complete integral.

4 Jacobi's Method

This method is used to solve non-linear first order PDE which involves three or more independent variables. Consider a non-linear PDE of order one of the form:

$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$______(1)

involving three independent variables x_1, x_2, x_3 where the dependent variable z do not occur except by its partial derivatives $p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, p_3 = \frac{\partial z}{\partial x_3}$.

To solve (1), we solve the following auxiliary equations:

$$\frac{dp_1}{f_{x_1}} = \frac{d\,x_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}}.$$

Working rule of Jacobi's Method:

Step 1: Transfer all the terms of the PDE to LHS and denote the entire expression by $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ ___(1).

Step 2: Write down Jacobi's auxiliary equations.

Step 3: Find f_{x_1}, f_{x_2}, f_{x_3}, f_{p_1}, f_{p_2} and f_{p_3}. Put them in Step 2 and simplify.

Step 4: Choose two fractions such that we obtain two additional equations as $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = c_1$_____(3) and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = c_2$_____(4) where c_1 and c_2 are arbitrary constants. While obtaining (3) and (4), we try to select simple fractions from (2), so that solution of equations (1), (3) and (4) may be as easy as possible.

Step 5: Verify that the relations (3) and (4) satisfy the condition $\sum_{i=1}^{3}\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_2}{\partial x_i}\frac{\partial F_1}{\partial p_i}\right) = 0.$

Step 6: If Step 5 is satisfied, then solve equations (1), (3) and (4) to find p_1, p_2, p_3.

Step 7: Substitute p_1, p_2, p_3 in the equation $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, which on integration gives the complete integral.

Example 15

Find the complete integral of the PDE $p_1{}^3 + p_2{}^2 + p_3 = 1$ using Jacobi's method.

Solution.

Given $p_1{}^3 + p_2{}^2 + p_3 = 1$.

$\Longrightarrow p_1{}^3 + p_2{}^2 + p_3 - 1 = 0$_______(1)

$\frac{dp_1}{f_{x_1}} = \frac{d\,x_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}}\,.$

$\frac{dp_1}{0} = \frac{d\,x_1}{-\,3p_1{}^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}$ _______(2)

Taking the first and the third fractions of (2), we have $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = c_1$and $p_2 = c_2$, where c_1 and c_2 are arbitrary constants.

$\Longrightarrow p_1 = c_1$_____(3) and $p_2 = c_2$_____(4)

To verify whether $\sum_{i=1}^{3}\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_2}{\partial x_i}\frac{\partial F_1}{\partial p_i}\right) = 0$:

$\sum_{i=1}^{3}\left(\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_2}{\partial x_i}\frac{\partial F_1}{\partial p_i}\right) = [(0)(0) - (1)(0)] + [(0)(1) - (0)(0)] + [(0)(0) - (0)(0)] = 0.$

Since the equation is verified, we substitute $p_1 = c_1$and $p_2 = c_2$ in (1) to find p_3.

$\Longrightarrow p_3 = 1 - c_1{}^3 - c_2{}^2.$

Putting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ we get, $z = c_1 x_1 + c_2 x_2 + (1 - c_1{}^3 - c_2{}^2)x_3 + c_3.$

Exercise

Solve the following equations.

1. $x^2p^2 + y^2p^2 = z^2$
2. $z^2 (p^2+q^2) = x^2 + y^2$
3. $z^2 (p^2x^2 + q^2) = 1$
4. $2x^4p^2 - yzq - 3z^2 = 0$
5. $p^2 + x^2y^2q^2 = x^2 z^2$
6. $x^2p + y^2q = z^2$
7. $x^2/p + y^2/q = z$
8. $z^2 (p^2 - q^2) = 1$
9. $z^2 (p^2/x^2 + q^2/y^2) = 1$
10. $p^2x + q^2y = z$.

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – Ordinary and Partial Differential Equations – SMT1305

Unit V

Second and Higher Order Partial Differential Equations

This unit covers the following topics: Partial differential equations of second and higher order, Classification of linear partial differential equations of second order, Homogeneous and non-homogeneous equations with constant coefficients, Monge's methods.

1 Classification of linear partial differential equations of second order

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \_\_\_\_\_\_\_ (1)$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y, but do not depend on the unknown function u. The classification of second order linear PDEs is given by the following:

The second order linear PDE (1) is called

(i) Hyperbolic, if $B^2 - 4AC > 0$

(ii) Parabolic, if $B^2 - 4AC = 0$

(ii) Elliptic, if $B^2 - 4AC < 0$

Example 1

Determine the regions in the xy −plane where the following equation is hyperbolic, parabolic, or elliptic: $u_{xx} + yu_{yy} + 12u_y$.

Solution.

Given $u_{xx} + yu_{yy} + 12u_y$.

The coefficients of the leading terms in this equation are: $A = 1$, $B = 0$, $C = y$.

The discriminant is then $B^2 - 4AC = -4y$.

Hence the equation is (i) hyperbolic when $y < 0$, (ii) parabolic when $y = 0$, and (iii) elliptic when $y > 0$.

Example 2

Classify the equation $u_{xx} + 2u_{xy} + 2u_{yy} = 0$.

Solution.

Here $A = 1$, $B = 2$, $C = -4$.

The discriminant is then $B^2 - 4AC = 4 - 4(1)(2) = -4 < 0$.

Hence the equation is elliptic.

2 Homogeneous Partial Linear Differential Equations with constant Coefficients.

A homogeneous linear partial differential equation of the n^{th} order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots\dots + c_n \frac{\partial^n z}{\partial y^n} = F(x,y) \qquad \text{---------} (1)$$

where c_0, c_1,---------, c_n are constants and F is a function of 'x' and 'y'. It is

homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$(c_0 D^n + c_1 D^{n-1} D' + \dots\dots + c_n D'^n) z = F(x,y)$$

$$\text{or} \qquad f(D,D') z = F(x,y) \qquad \text{---------}(2),$$

$$\text{where, } \frac{\partial}{\partial x} \equiv D \text{ and } \frac{\partial}{\partial y} \equiv D'.$$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of f (D, D') z = 0-------(3), which must contain *n* arbitrary functions as the degree of the polynomial f(D, D'). The particular integral is the particular solution of equation (2).

Finding the complementary function

Let us now consider the equation f(D, D') z = F (x, y).

The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.

$$\text{i.e, } c_0 m^n + c_1 m^{n-1} + \dots\dots + c_n = 0 \qquad \text{---------}(4)$$

Solving equation (4) for m, we get n roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	Nature of the roots	Complementary function(C.F)
$m_1, m_2, m_3 \ldots\ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x)+\ldots\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = m, m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x)+f_3(y+m_3x)+\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = \ldots\ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx)+x^2f_3(y+mx)+\ldots\ldots+\ldots+x^{n-1}f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D,D')\,z = F(x,y)$.

Now, the P.I is given by $\dfrac{1}{f(D,D')}F(x,y)$

Case (i) : When $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')}e^{ax+by}$$

Replacing D by 'a' and D' by 'b', we have

$$P.I = \frac{1}{f(a,b)}e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

Case (ii) : When $F(x,y) = \sin(ax+by)$ (or) $\cos(ax+by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)}\sin(ax+by) \text{ or } \cos(ax+by)$$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)}\sin(ax+by) \text{ or } \cos(ax+by), \text{ where } f(-a^2, -ab, -b^2) \neq 0.$$

Case (iii) : When $F(x,y) = x^m y^n$,

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv) : When F(x,y) is any function of x and y.

$$P.I = \frac{1}{f(D,D')} F(x,y).$$

$$\text{Resolve} \frac{1}{f(D,D')}$$

into partial fractions considering f (D,D') as a function of D alone.

Then operate each partial fraction on F(x,y) in such a way that

$$\frac{1}{D-mD'} F(x,y) = \int F(x,c-mx)\, dx,$$

where c is replaced by y+mx after integration

Example 3

Solve $(D^3 - 3D^2D' + 4D'^3) z = e^{x+2y}$

Solution.

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$

The roots are : m = -1, 2, 2

Therefore, C.F. is $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$P.I. = \frac{e^{x+2y}}{D^3 - 3D^2D' + 4D'^3} \text{ (Replace D by 1 and } D' \text{ by 2)}$$

$$= \frac{e^{x+2y}}{1-3(1)(2)+4(2)^3}$$

$$= \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I.

$$\text{ie,} \quad z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Example 4

Solve $(D^2 - 4DD' + 4D'^2)\, z = \cos(x - 2y)$

Solution.

The auxiliary equation is $m^2 - 4m + 4 = 0$

Solving, we get, m = 2, 2.

Therefore, C.F is $f_1(y+2x) + xf_2(y+2x)$.

$$\therefore P.I = \frac{1}{D^2 - 4DD' + 4D'^2} \cos(x-2y)$$

Replacing D^2 by -1, DD' by 2 and D'^2 by -4, we have

$$P.I = \frac{1}{(-1) - 4(2) + 4(-4)} \cos(x-2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

$$\therefore \text{Solution is } z = f_1(y+2x) + xf_2(y+2x) - \frac{\cos(x-2y)}{25}.$$

Example 5

Solve $(D^2 - 2DD')\, z = x^3y + e^{5x}$

Solution.

The auxiliary equation is $m^2 - 2m = 0$.

Solving, we get, $m = 0, 2$.

Hence, C.F. is $f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}(x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right)(x^3y)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D}D'(x^3y) + \frac{4}{D^2}D'^2(x^3y) + \ldots\right)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \ldots\right)$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$$P.I_2 = \frac{e^{5x}}{D^2 - 2DD'} \quad \text{(Replace D by 5 and } D' \text{ by 0)}$$

$$= \frac{e^{5x}}{25}$$

$$\therefore \text{Solution is } z = f_1(y) + f_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60} + \frac{e^{5x}}{25}$$

Example 6

Solve $(D^2 + DD' - 6D'^2)\, z = y \cos x$.

Solution.

The auxiliary equation is $m^2 + m - 6 = 0$.

Therefore, $m = -3, 2$.

Hence, C.F. is $f_1(y-3x) + f_2(y + 2x)$.

$$P.I = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y \cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D+3D')} \frac{1}{(D - 2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \cos x \, dx, \quad \text{where } y = c - 2x$$

$$= \frac{1}{(D+3D')} \int (c - 2x)\, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y \sin x - 2 \cos x)]$$

$$= \int [(c + 3x) \sin x - 2 \cos x]\, dx, \quad \text{where } y = c + 3x$$

$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2\sin x$

$= -y\cos x + \sin x$

Hence the complete solution is

$z = f_1(y - 3x) + f_2(y + 2x) - y\cos x + \sin x$

Example 7

Solve $r - 4s + 4t = e^{2x+y}$

Solution.

$$\text{Given equation is } \frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

i.e, $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

The auxiliary equation is $m^2 - 4m + 4 = 0$.

Therefore, $m = 2, 2$

Hence, C.F. is $f_1(y + 2x) + x f_2(y + 2x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD' + 4D'^2 = 0$ for $D = 2$ and $D' = 1$, we have to apply the general rule.

$$\therefore \text{P.I.} = \frac{e^{2x+y}}{(D - 2D')(D - 2D')}$$

$$= \frac{1}{(D - 2D')}\frac{1}{(D - 2D')}e^{2x+y}$$

$$= \frac{1}{(D - 2D')}\int e^{2x+c-2x}\,dx\,, \quad \text{where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \frac{1}{(D-2D')} e^{2x+y}$$

$$= \frac{1}{(D-2D')} \int e^{2x+c-2x}\, dx \,, \quad \text{where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \int e^{c}\, dx$$

$$= \frac{1}{(D-2D')} e^{c}.x$$

$$= \frac{1}{D-2D'} xe^{y+2x}$$

$$= \int xe^{c-2x+2x}\, dx \quad , \quad \text{where } y = c - 2x.$$

$$= \int xe^{c}\, dx$$

$$= e^{c}.\, x^2/2$$

$$= \frac{x^2 e^{y+2x}}{2}$$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y+2x) + \frac{1}{2} x^2 e^{2x+y}$$

3 Non –Homogeneous Linear Equations

Let us consider the partial differential equation

f (D,D') z = F (x,y)------- (1)

If f (D,D') is not homogeneous, then (1) is a non–homogeneous linear partial differential equation. Here also, the complete solution = C.F + P.I.

The methods for finding the Particular Integrals are the same as those for homogeneous linear equations.

But for finding the C.F, we have to factorize f (D, D') into factors of the form D –mD' –c.

Consider now the equation (D –mD' –c) z = 0 ----------- (2).
This equation can be expressed as p –mq = cz ---------(3), which is in Lagrangian form.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \quad \text{-----------(4)}$$

The solutions of (4) are y + mx = a and $z = be^{cx}$.

Taking b = f (a), we get $z = e^{cx} f(y+mx)$ as the solution of (2).

Note:

1. If $(D-m_1D' - C_1)(D - m_2D' - C_2) \ldots (D - m_nD^{n\prime} - C_n) z = 0$ is the partial differential equation, then its complete solution is

   $z = e^{c_1x} f_1(y+m_1x) + e^{c_2x} f_2(y+m_2x) + \ldots + e^{c_nx} f_n(y+m_nx)$.

2. In the case of repeated factors, the equation $(D-mD' - C_n)z = 0$ has a complete

   solution $z = e^{cx} f_1(y+mx) + x\, e^{cx} f_2(y+mx) + \ldots + x_{n-1} e^{cx} f_n(y+mx)$.

Example 8
Solve $(D-D'-1)(D-D'-2)z = e^{2x-y}$

Solution.

Here, $m_1 = 1$, $m_2 = 1$, $c_1 = 1$, $c_2 = 2$.

Therefore, the C.F is $e^{x} f_1(y+x) + e^{2x} f_2(y+x)$.

$$\text{P.I.} = \frac{e^{2x-y}}{(D - D' - 1)(D - D' - 2)} \quad \text{Put } D = 2, D' = -1.$$

$$= \frac{e^{2x-y}}{(2 - (-1) - 1)(2 - (-1) - 2)}$$

$$= \frac{e^{2x-y}}{2}$$

Hence the solution is $z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \dfrac{e^{2x-y}}{2}$.

Example 9

Solve (D2 –DD' + D' –1) z = cos (x + 2y)

Solution.

The given equation can be rewritten as

(D-D'+1) (D-1) z = cos (x + 2y)

Here $m_1 = 1$, $m_2 = 0$, $c_1 = -1$, $c_2 = 1$.

Therefore, C.F. = $e^{-x} f_1(y+x) + e^x f_2(y)$

$$\text{P.I} = \frac{1}{(D^2 - DD' + D' - 1)} \cos(x+2y) \quad [\text{Put } D^2 = -1, DD' = -2, D'^2 = -4]$$

$$= \frac{1}{-1 - (-2) + D' - 1} \cos(x+2y)$$

$$= \frac{1}{D'} \cos(x+2y)$$

$$= \frac{\sin(x+2y)}{2}$$

Hence the solution is $z = e^{-x} f_1(y+x)\, e^x f_2(y) + \dfrac{\sin(x+2y)}{2}$.

Example 10

Solve $[(D + D'-1)(D + 2D'-3)]\, z = e^{x+2y} + 4 + 3x + 6y$

Solution.

Here $m_1 = -1$, $m_2 = -2$, $c_1 = 1$, $c_2 = 3$.

Hence the C.F is $z = e^x f_1(y-x) + e^{3x} f_2(y-2x)$.

Hence the C.F is $z = e^x f_1(y-x) + e^{3x} f_2(y-2x)$.

$$P.I_1 = \frac{e^{x+2y}}{(D+D'-1)(D+2D'-3)} \quad [\text{Put } D = 1, D' = 2]$$

$$= \frac{e^{x+2y}}{(1+2-1)(1+4-3)}$$

$$= \frac{e^{x+2y}}{4}$$

$$P.I_2 = \frac{1}{(D+D'-1)(D+2D'-3)}(4 + 3x + 6y)$$

$$= \frac{1}{3\,[1-(D+D')]\left(1-\dfrac{D+2D'}{3}\right)}(4 + 3x + 6y)$$

$$= \frac{1}{3}\,[1-(D+D')]^{-1}\left(1-\frac{D+2D'}{3}\right)^{-1}(4+3x+6y)$$

$$= \frac{1}{3}[1 + (D+D') + (D+D')^2 + \ldots]\left(1+\frac{D+2D'}{3}+\frac{1}{9}(D+2D')^2 + \ldots\right) . (4 + 3x + 6y)$$

$$= \frac{1}{3}\left(1 + \frac{4}{3}D + \frac{5}{3}D' + \ldots\ldots\right)(4 + 3x + 6y)$$

$$= \frac{1}{3}\left(4 + 3x + 6y + \frac{4}{3}(3) + \frac{5}{3}(6)\right)$$

$$= x + 2y + 6$$

Hence the complete solution is

$$z = e^x f_1(y-x) + e^{3x} f_2(y - 2x) + \frac{e^{x+2y}}{4} + x + 2y + 6.$$

4 Monge's Method

This method is used to solve non-linear second order PDE with the standard form

$$Rr + Ss + Tt = V \qquad (1)$$

Where R, S, T and V are functions of x, y, z, p and q.

Procedure to solve by Monge's method:

Step1: Write the given PDE in the standard form and find R, S, T and V.

Step 2: The auxiliary equations are:

$Rdpdy + Tdqdx - Vdxdy = 0$ ____________(2)

$R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ____________(3)

Step 3: First factorise equation (3) and get the factors in terms of dx and dy.

Step 4: If these factors are equations (4) and (5), use each factor in (2) to get equations (6) and (7).

Step 5: Obtain the values of p and q using equations (6) and (7).

Step 6: The general solution of (1) is $dz = pdx + qdy$.

Example 11

Using Monge's method, solve $r^2 = at$.

Solution.

Given equation is $r^2 = at$ ______________(1)

Here $R = 1, S = 0, T = -a^2, V = 0$.

Monge's auxiliary equations are:

$Rdpdy + Tdqdx - Vdxdy = 0$ ____________(2)

$R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ____________(3)

From (3), we have, $(dy)^2 - a^2(dx)^2 = 0$.

$\Rightarrow (dy - adx) = 0$ and $(dy + adx) = 0$.

$\Rightarrow dy = adx$ and $dy = -adx$.

Integrating both the equations, we get,

$\Rightarrow y = ax + c_1$ and $y = -ax + c_2$.
$\Rightarrow y - ax = c_1$ ______(4) and $y + ax = c_2$______(5)
Case(i) Put (4) in (2).
$\Rightarrow dp(adx) - a^2 dqdx = 0.$
$\Rightarrow dp - adq = 0.$
Integrating, we get,
$\Rightarrow p - aq = \varphi_1(y - ax)$__________(6) [using equation(4)]
Case(ii) Put (5) in (2).
$\Rightarrow dp(adx) + a^2 dqdx = 0.$
$\Rightarrow dp + adq = 0.$
Integrating, we get,
$\Rightarrow p + aq = \varphi_2(y + ax)$__________(7) [using equation(5)]
(6) + (7) $\Rightarrow\ 2p = \varphi_1(y - ax) + \varphi_2(y + ax).$
$\Rightarrow p = \frac{1}{2}[\varphi_1(y - ax) + \varphi_2(y + ax)].$
$(7) - (6) \Rightarrow\ 2aq = \varphi_2(y + ax) - \varphi_1(y - ax).$
$\Rightarrow q = \frac{1}{2a}[\varphi_2(y + ax) - \varphi_1(y - ax)].$
The general solution of (1) is $dz = pdx + qdy$.
Substituting p and q in $dz = pdx + qdy$, we get,
$dz = \frac{1}{2}[\varphi_1(y - ax) + \varphi_2(y + ax)]dx + \frac{1}{2a}[\varphi_2(y + ax) - \varphi_1(y - ax)]dy.$
Simplifying and integrating we get,
$z = \psi_1(y - ax) + \psi_2(y + ax)].$
The arbitrary constant of integration may be considered as absorbed in either of the functions $\psi_1(y - ax)$ or $\psi_2(y + ax)$.
Therefore, the complete integral is $z = \psi_1(y - ax) + \psi_2(y + ax)]$ where ψ_1 and ψ_2 are arbitrary functions.

Exercises

(a) Solve the following homogeneous Equations.

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$

2. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x . \cos 2y$

3. $(D^2 + 3DD' + 2D'^2)\, z = x + y$

4. $(D^2 - DD' + 2D'^2)\, z = xy + e^x . \cosh y$

$\left\{ \text{Hint: } e^x . \cosh y = e^x . \left(\frac{e^y + e^{-y}}{2}\right) = \frac{e^{x+y} + e^{x-y}}{2} \right\}$

5. $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x+2y) + e^{2x+y}$

6. $(D^2 + 4DD' - 5D'^2)\, z = 3e^{2x-y} + \sin(x - 2y)$

7. $(D^2 - DD' - 30D'^2)\, z = xy + e^{6x+y}$

8. $(D^2 - 4D'^2)\, z = \cos 2x.\ \cos 3y$
9. $(D^2 - DD' - 2D'^2)\, z = (y - 1)e^x$

10. $4r + 12s + 9t = e^{3x - 2y}$

(b) Solve the following non –homogeneous equations.

1. $(2DD' + D'^2 - 3D')\, z = 3\cos(3x - 2y)$
2. $(D^2 + DD' + D' - 1)\, z = e^{-x}$
3. $r - s + p = x^2 + y^2$
4. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$
5. $(D^2 - D'^2 - 3D + 3D')\, z = xy + 7$

(c) Solve the following using Monge’s method.

1. Solve $r = t$ using Monge’s method
2. Solve $t - r\sec^4 y = 2q\tan y$ by Monge’s method
3. Solve $r - t\cos^2 x + p\tan x = 0$ using Monge’s method
4. Solve $x^2 r - y^2 t = xy$ using Monge’s method
5. Solve $(r - s)y + (x - t)x + q - p = 0$ by Monge’s method