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SCHOOL OF SCIENCE AND HUMANITES

DEPARTMENT OF MATHEMAICS

UNIT – I - 3D Analytical Geometry and Vector Calculus – SMT1303

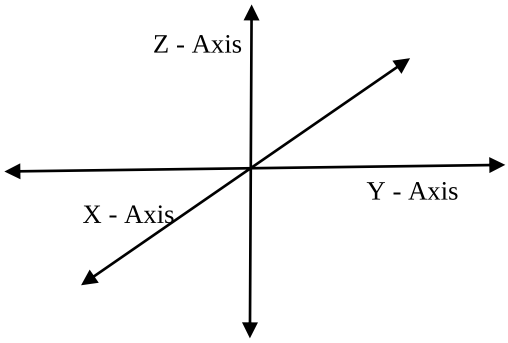
RECTANGULAR CARTESIAN CO- ORDINATES

Direction cosines of a line – Direction ratios of the join of two points - Projection on a line – Angle between the lines -Equation of a plane in different forms - Intercept form- normal form Angle between two planes - Planes bisecting the angle between two planes, bisector planes.

Introduction:

Let X'OX, Y'OY and Z'OZ be three mutually perpendicular lines in space that are concurrent at 0(origin). These three lines, called respectively as x-axis, y-axis and z-axis (and collectively as co-ordinate axes), form the frame of reference, using which the co-ordinates of a point in space are defined.

3D coordinate plane

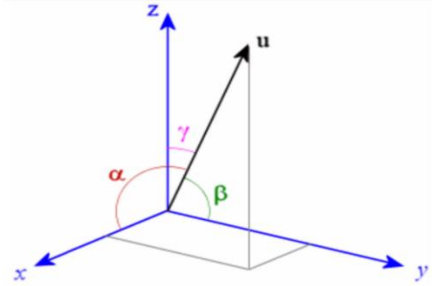


Note:

- The positive parts of the co-ordinate axes, namely OX, OY, OZ should form a right-hand system. The plane XOY determined by the x-axis and y-axis is called xoy plane or xy-plane.
- Similarly the yz plane and zx-plane are defined. These three planes called co-ordinate planes, divide the entire space into 8 parts, called the octants. The octant bounded by OX, OY, OZ is called the positive or the first octant.
- In face, the x co-ordinate of any point in the yz-plane will be zero, the y co-ordinate of any point in the zx-plane will be zero and the z co-ordinate of any point in the xy plane will be zero.
- In other words, the equations of the yz, zx and xy-planes are x = 0, y = 0 and z = 0 respectively. The point A lies on the x-axis and hence in the zx and xy-planes. Hence the co-ordinates of A will be (x, 0, 0), similarly the co-ordinates of B and C will be respectively (0, y, 0) and (0, 0, z).

Definition: Direction Cosines

The cosine of the angles made by a line with the axes X, Y and Z are called directional cosines of the line. (i.e) The triplet $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (D.C.'s) of the line and usually denoted as l, m, n. A set of parallel lines will make the same angles with the co-ordinates axes and hence will have the same D. C.′ s.



Note :

1. If the D.C.'s of line PQ are l, m, n, then the D.C.'s of QP are -l, -m, -n, as the angles made byQP with the co-ordinates axes are $180^0-\alpha, 180^0-\beta, 180^0-\gamma$ when the angles made by PQ with the axes are α, β, γ.

2. The D.C.'s of OX, OY, OZ are respectively 1, 0, 0, 0, 1, 0 and 0, 0, 1.

The **direction cosines of** a line parallel to any coordinate axis are equal to the **direction cosines of** the corresponding axis. The dc's are associated by the **relation** $l^2 + m^2 + n^2 = 1$. If the given line is reversed, then the **direction cosines** will be cos (π − α), cos (π − β), cos (π − γ) or − cos α, − cos β, − cos γ.

Definition: Direction Ratios

The direction ratios are simply a set of three real numbers *a*, *b*, *c* proportional to *l*, *m*, *n*, i.e.

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

From this relation, we can write

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \pm\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}} = \sqrt{a^2+b^2+c^2}$$

$$\Rightarrow \boxed{l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},\quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}}$$

These relations tell us how to find the direction cosines from direction ratios.

Note:

1. $l^2 + m^2 + n^2 = 1$, where as $a^2 + b^2 + c^2 \neq 1$.
2. To specify the direction of a line in space its direction angles, direction cosines or direction ratios must be known.
3. The D.R.'s of two parallel lines are proportional.

Formulae:

1. Direcion Ratios (D.R.'S) of a line joining Two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are **$x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.**

Angle between Two Lines:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of the lines L_1 and L_2, then

$$\boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$$

Corollary 1

If the two lines are perpendicular, then $\theta = 90^0$ or $\cos\theta = 0$

$$\boxed{\text{i-e., } l_1 l_2 + m_1 m_2 + n_1 n_2 = 0}$$

we recall that, if the two lines are parallel, then $\boxed{l_1 = l_2, m_1 = m_2 \text{ and } n_1 = n_2 .}$

Corollary 2

If the D.R.'s of the two lines are a_1, b_1, c and a_2, b_2, c_2 then their D.C.'s are $\left(\frac{a_1}{\sqrt{\Sigma a_1^2}}, \frac{b_1}{\sqrt{\Sigma a_1^2}}, \frac{c_1}{\sqrt{\Sigma a_1^2}}\right)$ and $\left(\frac{a_2}{\sqrt{\Sigma a_2^2}}, \frac{b_2}{\sqrt{\Sigma a_2^2}}, \frac{c_2}{\sqrt{\Sigma a_2^2}}\right)$

If θ is the angle between the two lines, then

$$\boxed{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)\left(a_2^2 + b_2^2 + c_2^2\right)}}}$$

If a_1, b_1, c_1 and a_2, b_2,c_2 are the direction ratios of the lines L_1 and L_2, and if they are perpendicular, then cos θ= a_1 a_2+ b_1 b_2+c_1,c_2=0 or $\theta=90^o$.

we recall that if the two lines are parallel then

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.}$$

Projection of a Line segment on a given Line :

Let AB be a given line and PQ be any line then the the Dr's of the line PQ are **x_2-x_1, y_2-y_1, z_2-z_1** and the Dc's of the given line AB are l, m, and n then

The projection of PQ on $AB = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

THE PLANE

A plane is a surface which is such that the straight line joining any two points on it lies completely on it. This characteristic property of a plane is not true for any other surface.

General Equation of a Plane:

The first degree equation in x, y, z namely ***ax* + *by* +*cz* + *d* = 0** always represents a plane, where a, b, c are not all zero.

Equation of a plane passing through a point:

If $ax + by + cz + d = 0$ is a plane equation and it passes through a given point P(x_1,y_1,z_1) , then the required plane is **a(x- x_1)+b(y - y_1)+c(z - z_1)=0**

Equation of the plane making intercepts a, b, c on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Equation of the plane passing through three points A(x_1,y_1,z_1) , B(x_2,y_2,z_2)and C(x_3,y_3,z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

Equation of a plane in the normal form is $x\cos\alpha + y\cos\beta + z\cos\gamma = \rho$, where ρ is the length of the perpendicular from the origin on it and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the perpendicular line.

Length of the perpendicular from the origin 'O' to the given plane $ax + by + cz + d = 0$ is given by

$$\rho = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

Length of the perpendicular from the point P(x_1,y_1,z_1) **to the plane $ax + by + cz + d = 0$ is given by**

$$\rho = \pm\frac{(ax_1+by_1+cz_1+d)}{\sqrt{a^2+b^2+c^2}}$$

Plane through the Intersection of Two given Planes P_1: $ax + by + cz + d_1 = 0$ and P_2: $ax + by + cz + d_2 = 0$ is $ax + by + cz + d_1 + k(ax + by + cz + d_2) = 0$

Distance between two parallel planes P_1: $ax + by + cz + d_1 = 0$ and P_2: $ax + by + cz + d_2 = 0$ is

$$d = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Problems

1. Find the equation of the plane passing through the point (2,-1,1) and parallel to the plane 3x+7y-10z=5

Solution

Given plane equation is $3x + 7y + 10z - 5 = 0$ ___(1)

Any plane parallel to (1) is of the form $3x + 7y + 10z - 5 + k = 0$ ___(2)

Plane (2) passes through (2, -1, 1)

$\therefore\ 4(2)+2(-4)-7(5)+k=0$

$$\boxed{k=35}$$

$\therefore$ The required plane equation is $4x+2y-7z+35=0$

2. Find the equation of the plane passing through the points (1, -2, 2) and (-3, 1, -2) and Perpendicular to the plane $2x + y - z + 6 = 0$

Solution

Let the required plane equation be $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ___(1)

Plane (1) passes through (1, -2, 2)

$\therefore\ a(x-1)+b(y+2)+c(z-2)=0$ ___(2)

(2) passes through (-3, 1, -2)

$\therefore\ a(-3-1)+b(1+2)+c(-2-2)=0$

$$-4a+3b-4c=0 \qquad \text{\_\_\_(3)}$$

now plane (2) is perpendicular to $2x+y-z+6=0 \Rightarrow 2a+b-c=0$ ___(4)

from (3) & (4), using rule of cross multiplication,

$$\frac{a}{\begin{vmatrix}3 & -4\\ 1 & -1\end{vmatrix}}=\frac{b}{\begin{vmatrix}-4 & -4\\ -1 & 2\end{vmatrix}}=\frac{c}{\begin{vmatrix}-4 & 3\\ 2 & 1\end{vmatrix}}$$

$$\frac{a}{1}=\frac{b}{-12}=\frac{c}{-10}=k$$

Using these values in (2)

$$1(x-1)-12(y+2)-10(z-2)=0$$

$$x-12y-10z-5=0.$$

3. Find the equation of the plane which passes through the points (1, 0, -1) and (2, 1, 1) and parallel to the line joining the points (-2, 1, 3) and (5, 2, 0).

Solution

Equation of a plane passing from a point (x_1, y_1, z_1) is

$$a(x-x_1)+\mathrm{b}(y-y_1)+c(z-z_1)=0$$

since is passes through (1, 0, -1)

$$\Rightarrow a(x-1)+b(y-0)+c(z+1)=0 \qquad \_\_\_\_(1)$$

Plane (1) passes through (2, 1, 1)

$$a(2-1)+b(1-0)+c(1+1)=0$$

$$a+b+2c=0 \qquad \_\_\_\_(2)$$

D.R.'s of the line joining (-2, 1, 3) and (5, 2, 0) are 7, 1, -3

Plane (1) is parallel to this line

$\therefore$ Any normal of plane (1) is $\perp r$ to this line D.R.'s 7, 1, -3

$$\therefore \; a_1a_2+b_1b_2+c_1c_2=0$$

$$7a+b-3c=0 \qquad \_\_\_\_(3)$$

Eliminating a, b, c from (1), (2) and (3)

$$\frac{a}{\begin{vmatrix}1 & 2\\ 1 & -3\end{vmatrix}}=\frac{b}{\begin{vmatrix}2 & 1\\ -3 & 7\end{vmatrix}}=\frac{c}{\begin{vmatrix}1 & 1\\ 7 & 1\end{vmatrix}}$$

$$\frac{a}{-5}=\frac{b}{17}=\frac{c}{-6}=k \text{ (say)}$$

substituting a, b, c in (1)

$$-5(x-1)+17y-6(z+1)=0$$

$$-5x+17y-6z+5-6=0$$

$\boxed{-5x+17y-6z-1=0}$ is the required equation of the plane.

4. Find the equation of the plane through (1, -1, 2) and perpendicular to the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$

Solution

Equation of the plane passing through (1, -1, 2) is $a(x-1)+b(y+1)+c(z-2)=0$ ___(1)

(1) $\perp r$ to $2x+3y-2z=5 \Rightarrow 2a+3b-2c=0$ ____(2)

(I) $\perp r$ to $x+2y-3z=8 \Rightarrow a+2b-3c=0$ ____(3)

Solving (1), (2) and (3) we get

$$\begin{vmatrix} x-1 & y+1 & z-2 \\ 2 & 3 & -2 \\ 1 & 2 & -3 \end{vmatrix} = 0$$

$$(x-1)(-9+4)-(y+1)(-6+2)+(z-2)(4-3)=0$$

$$(x-1)(-5)+4(y+1)+(z-2)=0$$

$\boxed{-5x+4y+z+7=0}$ is the required plane equation.

5. Find the equation of the plane passing through the points (2, 5, -3), (-2, -3, 5) and (5, 3, -3).

Solution

The equation of the plane passing through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$(x-2)(16)-(y-5)(-24)+(z+3)(32)=0$$

$\boxed{2x+3y+4y-7=0}$ is the required plane equation.

6. Show that the fair points (0, -1, -1), (4, 5, 1), (3, 9, 4) and (-4, 4, 4) lie on a plane.

Solution:

The equation of the plane passing through three points (0, -1, -1), (4, 5, 1), (3, 9, 4) is

$$\begin{vmatrix} x-0 & y+1 & z+1 \\ 4 & 6 & 2 \\ 3 & 10 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 5x - 7y + 11z + 4 = 0.$$

To prove (-4, 4, 4) also lies on this plane, we need to prove it satifies the above plane equation $5x - 7y + 11z + 4 = 0$

$$5(-4) - 7(4) + 11(4) + 4 = 0$$

$\therefore$ The given four points lie on $5x - 7y + 11z + 4 = 0$

7. Find the angle between the planes $2x + 4y - 6z = 11$ and $3x + 6y + 5z + 4 = 0$.

Solution

Angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

lie between two planes $2x + 4y - 6z = 11$ and $3x + 6y + 5z + 4 = 0$ is

$$\cos\theta = \pm \frac{3(2) + 4(6) + 5(-6)}{\sqrt{4 + 16 + 36}\sqrt{9 + 36 + 25}}$$

$$\cos\theta = \pm 0$$

$$\boxed{\theta = \pi/2}$$

8. Find the equation of the plane which bisects perpendicularly the join of (2, 3, 5) and (5, -2, 7)

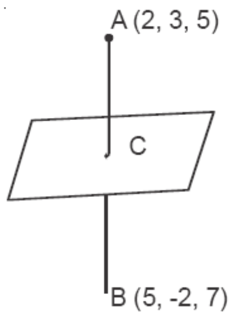
Solution

Let C be the midpoint of the line joining two points A(2, 3, 5) and B(5, -2, 7) then C has coordinates

$$C\left(\frac{2+5}{2}, \frac{3-2}{2}, \frac{5+7}{2}\right)$$

$$i.e.\ C\left(\frac{7}{2}, \frac{1}{2}, 6\right)$$

Equation of plane through $C\left(\frac{7}{2},\frac{1}{2},6\right)$ is



$$a\left(x-\frac{7}{2}\right)+b\left(y-\frac{1}{2}\right)+c(z-6)=0 \qquad \_\_\_\_(1)$$

As AB $\perp r$ to the plane, the DR's of AB are 5 - 2, -2 -3, 7 - 5

i.e. a = 3,b = -5,c = 2)

Substituting in (1)

$$\Rightarrow 3\left(x-\frac{7}{2}\right)-5\left(y-\frac{1}{2}\right)+2(z-6)=0$$

$\boxed{3x-5y+2z-20=0}$ is the required plane equation.

9. Find the distance between the planes x - 2y+ 2z - 8 = 0 and -3x + 6y - 6z = 57

Solution

Distance between two parallel planes

P_1: $ax + by + cz + d_1 = 0$ and P_2: $ax + by + cz + d_2 = 0$ is

$$d=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

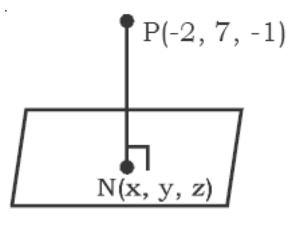
The given planes are x - 2y+ 2z - 8 = 0 and x-2y + 2z + 57/3 =0

$$d=\frac{\left|-8-\frac{57}{3}\right|}{\sqrt{1^2+(-2)^2+2^2}}$$

$$=\frac{|-27|}{\sqrt{1+4+4}}=9$$

10. Find the foot N of the perpendicular drawn from P(-2, 7, -1) to the plane 2x - y + z = 0

Let N be (x_1, y_1, z_1). N lies on $2x-y+z=0$



$\therefore\ 2x_1-y_1+z_1=0$ ____(1)

The D.R.'s of PN are

$$x_1+2,\ y_1-7,\ z_1+1$$

PN is parallel normal to the plane

$$\frac{x_1+2}{2}=\frac{y_1-7}{-1}=\frac{z_1+1}{1}=k \quad \text{(say)}$$

$$x_1 = 2k-2, \quad y_1 = -k+7, \quad z_1 = k-1$$

Substituting in (1), we get

$$2(2k-2)-(-k+7)+(k-1)=0$$

$$\Rightarrow \boxed{k=2}$$

$$\therefore x_1 = 2, \quad y_1 = 5, \quad z_1 = 1$$

Hence the foot of the perpendicular is (2, 5, 1).

11. The foot of the perpendicular from the given point A(1, 2, 3) on a plane is B(-3, 6, Find the plane equation.

Solution

The D.R.'s of AB are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$ i.e., -3-1, 6-2, 1-3 i-e., -4, 4, 2

Since AB is normal to the plane and B (-3, 6, 1) is a point

on the plane, the equation of the plane is $4(x-(-3)) - 4\text{ y} - 6)2(\text{z}-1) = 0$

$\boxed{2x-2y+z+17=0}$ is the required plane equation.

12. Find the image or reflection of the point (5, 3, 2) in the plane $x + y - z = 5$.
Let A be (5, 3, 2)

Solution

Let the image of A be $B(x_1, y_1, z_1)$

The mid-point of AB is $L\left(\frac{x_1+5}{2}, \frac{y_1+3}{2}, \frac{z_1+2}{2}\right)$

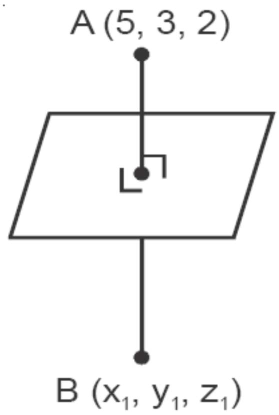
L lies on the plane $x + y - z = 5$ ____ (1)

$$\therefore \frac{x_1+5}{2}+\frac{y_1+3}{2}-\frac{z_1+2}{2}=5$$

$x_1 + y_1 - z_1 = 4$ ____(2)

D.R.'s of AB are $x_1 - 5, y_1 - 3, z_1 - 2$

D.R.'s normal to the plane are 1, 1, -1.
AB is parallel to normal to the plane



$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{x_1 - 5}{1} = \frac{y_1 - 3}{1} = \frac{z_1 - 2}{-1} = \text{k} \text{ (say)}$$

$x_1 = k + 5, \quad y_1 = k + 3, \quad z_1 = -k + 2.$

Substituting in (2)

$(k+5) + (k+3) - (-k+2) = 4$

$$\boxed{k = \frac{-2}{3}}$$

$$\therefore \text{ The image B is } \left(\frac{-2}{3} + 5, \frac{-2}{3} + 3, \frac{-2}{3} + 2 \right)$$

$$\text{i.e., B}\left(\frac{13}{3}, \frac{7}{3}, \frac{8}{3} \right)$$

13. Find the equation of the plane through the line of intersection of $x + y + z = 1$ and $2x + 3y + 4z = 5$ and

(i) Perpendicular to to $x - y + z = 0$

(ii) passing through (1, 2, 3)

Solution:

The equation of the plane passing through the line of intersection of

$x + y + z = 1$ ___(1) and $2x + 3y + 4z - 5 = 0$ ___(2)

(i.e) $(x + y + z - 1) + k(2x + 3y + 4z - 5) = 0$ ___(3)

$(1+ 2k)x + (1+ 3k) y + (1+ 4k)z - (1+ 5k) = 0$ ___(4)

(i) (4) $\perp r$ to $x - y + z = 0$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$(1+2k)\cdot 1 + (1+3k)(-1) + (1+4k)\cdot 1 = 0$$

$$1 + 3k = 0 \Rightarrow \boxed{k = \frac{-1}{3}}$$

substituting in (4)

$$\left(1-\frac{2}{3}\right)x+\left(1-\frac{3}{3}\right)y+\left(1-\frac{4}{3}\right)z-\left(1-\frac{5}{3}\right)=0$$

$$\frac{x}{3}-\frac{z}{3}+\frac{2}{3}=0$$

$\boxed{x-z+2=0}$ is the required plane equation.

(ii) (3) passes through the point (1, 2, 3)

$$(1+2+3-1)+k(2+6+12-5)=0$$

$$5+15k=0$$

$$\boxed{k=\frac{-1}{3}}$$

Substituting in equation (4)

$\boxed{x - z + 2 = 0}$ is the required plane equation

14. Find the equation of the plane passing through the line of intersection of the planes $2x + 5y + z = 3$ and $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$.

Solution:

The given planes are $2x + 5y + z = 3$ ____(1)

$x + y + 4z = 5$ ____(2)

$x + 3y + 6z = 1$____(3)

The required plane equation is is of the form

$(2x + 5y + z - 3) + k(x + y + 4z - 5) = 0$

$(2 + k)x + (k + 5)y + (1+ 4k)z - (3 + 5k) = 0$____(4)

(4) is parallel to (3)

$$\therefore \frac{2+k}{1}=\frac{k-5}{3}=\frac{4k+1}{6}$$

$$\frac{2+k}{1}=\frac{k-5}{3}\Rightarrow k=\frac{-11}{2}$$

Substituting in (4)

$$\left(2-\frac{11}{2}\right)x+\left(\frac{-11}{2}-5\right)y+\left(1+4\left(\frac{-11}{2}\right)\right)z=3+5\left(\frac{-11}{2}\right)$$

$x + 3y + 6z - 7 = 0$ is the required plane equation

15. Find the equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ parallel to y-axis.

Solution:

The given planes are $x + y + z = 1$ ____(1)

$2x + 3y - z + 4 = 0$ ____(2)

Let the required plane equation be

$(x + y + z - 1) + k(2x + 3y - z + 4) = 0$ ____(3)

$(1+ 2k)x + (1+ 3k)y + (1- k)z - 1 + 4k = 0$

Nomal to plane (3) is perpendicular *to* y-axis whose D.R.'s are 0, 1, 0

$$\therefore\ (1+2k)(0)+(1+3k)(1)+(1-k)(0)=0$$

$$\Rightarrow \boxed{k=\frac{-1}{3}}$$

Substituting in (3)

$$\therefore\ (x+y+z-1)\frac{-1}{3}(2x+3y-z+4)=0$$

$\boxed{x+4z-7=0}$ is the required plane equation.

SCHOOL OF SCIENCE AND HUMANITES

DEPARTMENT OF MATHEMAICS

UNIT – II - 3D Analytical Geometry and Vector Calculus – SMT1303

PLANE AND STRAIGHT LINE

Introduction

In two dimensional geometry, an equation in two variables say x, y represents a curve. But in three dimensional geometry, an equation in three variables say x, y, z represents a surface and a curve will be considered as the intersection of two surfaces. Hence a curve equation in three dimensions is represented by two surface equations taken simultaneously.

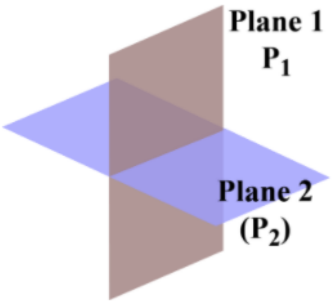
Straight Line

Intersection of the two planes will be a straight line.

Consider the two planes P_1: $a_1x+b_1y+c_1z+d_1=0$ $\longrightarrow$ (1)

and P_2: $a_2x+b_2y+c_2z+d_2=0$ $\longrightarrow$ (2)

The following figure shows the intersection of these two planes will form a straight line



$\therefore$ Equations (1) and (2) taken together represents a straight line and is called the general form of a straight line.

Note :

The x-axis is the line of intersection of xoy and xoz planes whose equations are z = 0, y = 0.

$\therefore$ The equation of the x-axis are y = 0, z = 0. Similarly the equation of the y-axis are x = 0, y = 0. And the equation of the z-axis are x = 0, y = 0.

Symmetrical Form of A Straight Line

1. Equation of a line passing through a point (x_1, y_1, z_1) with direction cosines of the line as l, m, n. is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

Note: Any point on the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$

(i.e) $x=lr + x_1$, $y = mr + y_1$, $z = nr + z_1$

Hence any point on the given line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ is $(lr + x_1, mr + y_1, nr + z_1)$

2. Equation of a straight line passing through (x_1, y_1, z_1) with direction ratios of the line as a, b, c

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

3. Equation of a straight line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Problems:

1. Find the equation of the straight line which passes through the point (2, 3,4) and making angles 60°, 60°, 45° with positive direction of axes.

Solution: Equation of a straight line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \longrightarrow$ (1)

Here $x_1=2, y_1=3, z_1=4$.

$$l=\cos 60=\frac{1}{2}$$

$$m=\cos 60=\frac{1}{2}$$

$$n=\cos 45=\frac{1}{\sqrt{2}}$$

Substituting the values of l, m, n in the straight line equation, we get

$$\frac{x-2}{\frac{1}{2}}=\frac{y-3}{\frac{1}{2}}=\frac{z-4}{\frac{1}{\sqrt{2}}}$$

2. Find the equation of the straight line passing through (2, -1, 1) and parallel to the line joining the points (1, 2, 3) and (-1, 1, 2).

Solution:

The direction ratios of the line joining the points (1, 2, 3) and (-1, 1, 2) are -1-1, 1-2, 2-3
i-e., -2, -1, -1

Equation of a straight line passing through the point (x_1, y_1, z_1) with direction ratios a,b,c is

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ --------(1)

Here $x_1 = 2, y_1 = -1, z_1 = 1$

$a = -2, b = -1, c = -1$

Substituting these values in (1) we get

$$\frac{x-2}{-2} = \frac{y-(-1)}{-1} = \frac{z-1}{-1}$$

(i.e) $\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-1}{1}$ which is the required equation of the line.

3. Find the equation of the line joining the points (1, -1, 2) and (4, 2, 3).

Solution:

The equation of a straight line is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

Here $x_1=1, x_2=-1, x_3=2$

$x_2 = 4, y_2 = 2, z_2 = 3$.

Hence the equation of the required line is

$$\frac{x-1}{4-1} = \frac{y-(-1)}{2-(-1)} = \frac{z-2}{3-2}$$

i.e., $\frac{x-1}{3} = \frac{y+1}{3} = \frac{z-2}{1}$

4. Prove that the points (3, 2, 4) (4, 5, 2) and (5, 8, 0) are collinear.

Solution:

Equation of a straight line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Equation of the line passing through (3, 2, 4) and (4, 5, 2) is

$$\frac{x-3}{4-3} = \frac{y-2}{5-2} = \frac{z-4}{2-4}$$

i.e., $\frac{x-3}{1} = \frac{y-2}{3} = \frac{z-4}{-2}$ ____(1)

If the above two points are collinear with (5, 8, 0) then the point (5, 8, 0) must satisfy equation (1)

Substituting $x = 5$, $y = 8$, $z = 0$ in (1), we get

$$\frac{5-3}{1}=\frac{8-2}{3}=\frac{0-4}{-2}$$

$$\Rightarrow \frac{2}{1}=\frac{6}{3}=\frac{-4}{-2}\Rightarrow \frac{2}{1}=\frac{2}{1}=\frac{2}{1}$$

Hence the point (5, 8, 0) satisfies equation (1)

$\therefore$ The three given points are collinear.

5. Find the angle between the lines

$$\frac{x+1}{2}=\frac{y+3}{2}=\frac{z-4}{-1} \text{ and } \frac{x-4}{1}=\frac{y+4}{2}=\frac{z+1}{2}$$

Solution:

Direction ratios of the first line are 2, 2, -1

Direction cosines of first line are $\frac{2}{3},\frac{2}{3},\frac{-1}{3}$

Direction ratios of the second line are 1, 2, 2.

Direction cosines of second line are $\frac{1}{3},\frac{2}{3},\frac{2}{3}$

Let be the angle between the lines (1) and (2), then

$$\cos\theta=\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right)=\frac{4}{9} \quad [\because \cos\theta=l_1l_2+m_1m_2+n_1n_2]$$

$$\theta=\cos^{-1}\left(\frac{4}{9}\right)$$

Problem for practice

6. Find the equations of the straight line through (a, b, c) which are (i) perpendicular to z-axis (ii) Parallel to z-axis.

Transform of a general form of a straight line into symmetrical form

To express the equation of a line in symmetrical form, we need

(i) The coordinates of a point on the line.

(ii) The direction ratios of the straight line

Method of find a point on the given line

The general form of a straight line is $a_1x+b_1y+c_1z+d_1=0 = a_2x+b_2y+c_2z+d_2=0$

Let us find the coordinates of the point, where this line meets XOY plane. Then z = 0.

Equations of planes are $a_1x+b_1y+ d_1=0$; $a_2x+b_2y +d_2=0$

Solving these equations, we get

$$\frac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$\therefore$ Co-ordinates of a point on the line is $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right) = (x_1, y_1, 0)$ (say).

Note:
To find a point on the line, we can also take x = 0 or y = 0.

Method of find the direction ratios.
Let (l, m, n) be the direction ratios of the required line.

The required line is the intersection of the planes $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 = 0$

It is perpendicular to these planes whose direction ratios of the normal are a_1, b_1, c_1 and a_2, b_2, c_2.

By condition of perpendicularity of two lines we get

$$a_1l + b_1m + c_1n = 0$$
$$a_2l + b_2m + c_2n = 0$$

Using the rule of cross multiplication, we get

$$\frac{l}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{n}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Therefore, the equation of a straight line passing through a point $(x_1, y_1, z_1)$with direction ratios a,b,c is

$$\frac{x - x_1}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y - y_1}{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z - 0}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

7. Find the symmetrical form the equations of the line $3x + 2y - z - 4 = 0$ and $4x + y - 2z + 3 = 0$ and Find its direction cosines.

Solution:
Equation of the given line is

$$\left.\begin{array}{l} 3x + 2y - z - 4 = 0 \\ 4x + y - 2z + 3 = 0 \end{array}\right\} \longrightarrow (1)$$

Let l, m, n be the D.R.'s of line (1).

Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence we have

$$3l+2m-n=0,$$
$$4l+m-2n=0$$

Solving these, we get

$$\frac{l}{-4+1}=\frac{m}{-4+6}=\frac{n}{3-8}$$
$$\Rightarrow \frac{l}{-3}=\frac{m}{2}=\frac{n}{-5}$$

Therefore The D.R.'s of the line (1) ae -3, 2, -5.

$$l=\frac{-3}{\sqrt{38}}, m=\frac{2}{\sqrt{38}}, n=\frac{-5}{\sqrt{38}}$$

Now, to find the co-ordinates of a point on the line given by (1),

Let us find the point where it meets the plane z = 0.

Put z = 0 in the equations given by (1)

we have $\left.\begin{aligned} 3x + 2y &= 4 \\ 4x + y &= -3 \end{aligned}\right\} \longrightarrow$ (2)

Solving these two equations, we get

$$\frac{x}{6+4}=\frac{y}{-16-9}=\frac{1}{3-8}$$
$$\Rightarrow \frac{x}{10}=\frac{y}{-25}=\frac{1}{-5}$$
$$\Rightarrow x=-2, y=5$$

The line meets the plane z = 0 at the point (-2, 5, 0) and has direction ratios -3, 2, -5.

Therefore the equations of the given line in symmetrical form are

$$\frac{x+2}{-3}=\frac{y-5}{2}=\frac{z-0}{-5}.$$

Problem for Practice

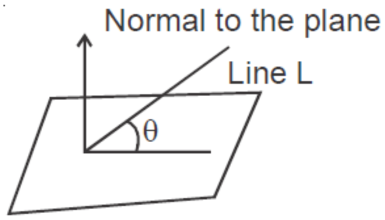
8. Find the symmetrical form of the equation of the straight line $2x - 3y + 3z = 4$, $x + 2y - z = -3$

9. Find the symmetrical form, the equations of the line formed by planes x + y + z + 1 = 0, 4x + y − 2z + 2 = 0 and find its direction-cosines.

The Plane and the Straight Line

Angle between a Line and Plane

Angle between a line $L : \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and a plane $U : ax+by+cz+d=0$



If θ is the angle between the line L and the plane U, then angle between line L & normal to the plane is $90-\theta$.

Direction ratios of lline L are l, m, n

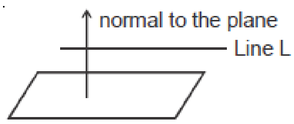
Direction ratios of the normal to the plane U are a, b, c

$$\therefore \cos(90-\theta) = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

$$\text{(i.e)} \quad \theta = \sin^{-1}\left[\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}\right]$$

Consider the line L: $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane P: ax+ by + cz + d = 0

(i) Perpendicular Condition



Line L is perpendicular to the plane U.

$\Rightarrow$ Line L and normal to the plane are parallel. So their direction ratios are proportional.

Direction ratios of line L : l, m, n

Direction ratios of normal to the plane : a, b, c

Hence

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

(ii) Line L is perpendicular to the plane U.

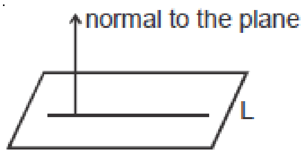
Line L and normal to the plane are parallel. So their direction ratios are proportional.

Hence al + bm + cn = 0

(iii) Line L lies on the plane U.

$\Rightarrow$ Every point of line L lies on the plane $ax+by+cz+d=0 \quad \rightarrow (1)$

$\therefore$ the obvious point (x_1, y_1, z_1) lies on the plane (1)



$\therefore ax_1+by_1+cz_1+d=0$ ________ (2)

(1) - (2) gives $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ________ (2)

Also line L and normal to the plane are perpendicular.

$\therefore al+bm+cn=0 \quad \rightarrow (4)$

Hence if a line L lies on a plane U, then the condition is given by (2) and (4).

And equation of any plane which passes through the given line L is given by (3) and (4).

Problems

10. Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $3x+y+z=7$

Solution:

The angle between a line and a plane is

$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Here $l=2, \quad m=3, \quad n=6$

$a=3, \quad b=1, \quad c=1$

$$\sin\theta=\frac{2(3)+1(3)+1(6)}{\sqrt{3^2+1^2+1^2}\sqrt{2^2+3^2+6^2}}$$

$$=\frac{6+3+6}{\sqrt{11}\sqrt{49}}$$

$$\sin\theta=\frac{15}{7\sqrt{11}}$$

$$\theta=\sin^{-1}\left(\frac{15}{7\sqrt{11}}\right)$$

11. Find the equation of the plane which contains the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z=12$.

Solution:

Let a, b, c be the direction ratios of the normal to the required plane

Equation of a plane which contains line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ is given by

$a(x-1)+b(y+1)+c(z-3)=0$----------(1)

Since line L and normal to the plane are perpendicular $2a-b+4c=0$--------(2)

Also given the required plane is perpendicular to the plane $x + 2y + z = 12$

Hence their normals are perpendicular

Therefore $a + 2b + c = 0$---------(3)

Eliminating a, b, c from (1), (2) & (3), we get the required plane equation.

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

(i.e) $9x - 2y - 5z + 4 = 0$ is the required equation of the plane.

12. Find the image of the line $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$ in the plane $2x - y + z + 3 = 0$.

Solution:

The image of the line is the line joining the images of any two points on the line.

It is an advantage to select one of the points as the point of intersection of the line.

$$L: \frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2} \text{ and the plane } 2x - y + z + 3 = 0$$

Any point on the line L is $(3r + 1, 5r + 3, 2r + 4)$

If this point is taken as A, then it lies on the plane, then$2(3r + 1) - (5r + 3) + (2k + 4) + 3 = 0$

(i.e) $r = -2$.

Substituting $k = -2$, we get the co-ordinate of the point of intersection of the line and the plane A(-5,-7,0).

Let us consider another point on the line L.

Let us choose the obvious point on the line i.e., P(1, 3, 4).

Let the image of the point P (1, 3, 4) on the plane $2x - y + z + 3 = 0$ be P'.

By definition of the image, the midpoint M of PP' lies on the plane and line PP' is normal to the plane.

Let the direction ratios of the line PP' be (, m, n D.R.'s of the normal to the plane are 2, -1, 1

As line PP' and normal to the plane are parallel their direction ratios are proportional.

Then the image of P is P' (-3, 5, 2).

$$\therefore \frac{l}{2} = \frac{m}{-1} = \frac{n}{1} = k \text{ (say)}$$

Equation of line PP', which passes through (1, 3, 4) with direction ratios (2, -1, 1) is given by $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{-1}$

Any point on this line is given by $(2k + 1, -k + 3, -k + 4)$

Suppose this point is M which meets the plane, then it has to satisfy the equation of the plane.

$$\therefore 2(2k+1)-(-k+3)+(k+4)+3=0$$

i.e., $\boxed{k=-1}$

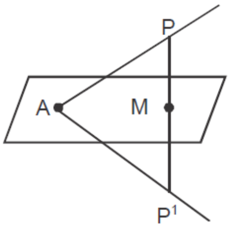
Substituting for k, we get the co-ordinates of M(-1, 4,3)

$\because$ M is the mid point of PP[1]

$$\frac{1+x_1}{2}=-1,\ \frac{1+y_1}{2}-4,\ \frac{1+z_1}{2}=3$$

$$x_1=-3, y_1=5, z_1=2$$

$\because$ the image of is P[1] (-3, 5, 2).



Equation of the image line is given by

$$\frac{x+5}{-3+5}=\frac{y+7}{5+7}=\frac{z-0}{2-0}$$

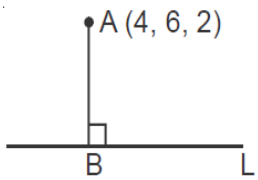
i.e., $$\frac{x+5}{2}=\frac{y+7}{12}=\frac{z}{2}$$

i.e., $$\frac{x+5}{1}=\frac{y+7}{6}=\frac{z}{1}$$

13. Find the foot of the perpendicular from a point (4, 6, 2) to the line $\frac{x-2}{3}=\frac{y-2}{2}=\frac{z-2}{1}$. Also find the length and the equation of the perpendicular.

Solution:

Let B be the foot of the perpendicular drawn from a point A(4, 6, 2) to the line L: $\frac{x-2}{3}=\frac{y-2}{2}=\frac{z-2}{1}=$ k



Then B has coordinates of the form (3*k* + 2, 2*k* + 2, *k* + 2)

Direction ratios of line AB: (3*k* - 2, 2*k* - 4, *k*)

Direction ratios of the line L : (3, 2,1)

Line AB is perpendicular to line L.

Hence 3(3*k* - 2) + 2(2*k* - 4) = *k* = 0

Therefore *k* =1

Hence the foot of the perpendicular is N (5, 4, 3)

Equation of the perpendicular is the equation of line joining the points (4, 6, 2) and (5, 4, 3) is

$$\frac{x-4}{1}=\frac{y-6}{-2}=\frac{z-2}{1}$$

Length of the perpendicular $= AB = \sqrt{(5-4)^2+(4-6)^2+(3-2)^2}$

$$= \sqrt{6} \text{ units.}$$

Condition for Co planarity of the lines

Condition for lines $L_1 : \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $L_2 : \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_1}{n_2}$ to be coplanar is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Equation of the plane containing the coplanar lines L_1 and L_2 is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Problems

14. Show that the lines L1: $\frac{x-7}{2}=\frac{y-10}{3}=\frac{z-13}{4}$ and L2: $\frac{x-3}{1}=\frac{y-5}{2}=\frac{z-7}{3}$ are coplanar. Find the equation of the plane of co planarity and the coordinates of the point of intersection of the lines.

Solution:

Consider the lines

$L_1 : \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $L_2 : \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_1}{n_2}$

Condition for co planarity of two lines is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

L1: $\frac{x-7}{2}=\frac{y-10}{3}=\frac{z-13}{4}$

Here $(x_1,y_1,z_1)=(7,10,13)$

$(x_2,y_2,z_2)=(3,5,7)$

$l_1,m_1,n_1=2,3,4$

$l_2,m_2,n_2=1,2,3$

$$\begin{vmatrix} 3-7 & 5-10 & 7-13 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -4 & -5 & -6 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -4[9-8]+5[6-4]-6[4-3]=0$$

Therefore the lines are coplanar.

Equation of the plane containing the coplanar lines $L1$ and $L2$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

i.e., $$\begin{vmatrix} x-7 & y-10 & z-13 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

i.e., $(x-7)(9-8)-(y-10)(6-4)+(z-13)(4-3)=0$

i.e., $x-2y+z=0$

To find the point of intersection of lines L1 & L2 :

Any point on the line L_1: $\frac{x-7}{2} = \frac{y-10}{3} = \frac{z-13}{4} = k$ is $A(2k + 7,+10, 4k +13)$

Any point on the line L_2: $\frac{x-3}{1} = \frac{y-5}{2} = \frac{z-7}{3} = r$ is $B(r + 3, 2r + 5,3r + 7)$

If L_1 and L_2 intersect, then for some value of r and k, the coordinates A and B are the same.

$$\begin{aligned} \therefore \quad & 2k+7=r+3 && \text{i.e., } 2k-r=-4 \\ & 3k+10=2r+5 && \text{i.e., } 3k-2r=-5 \\ & 4k+13=3r+7 && \text{i.e., } 4k-3r=-5 \end{aligned}$$

Solving any two equations, we get k = -3 and r = -2.

Hence the common point of intersection of the lines L_1 and L_2 is (1, 1, 1).

Problem for practice

15. Show that the lines joining the points (0, 2, -4) & (-1, 1, -2) and (-2, 3, 3) & (-3, -2, 1) are coplanar. Find their point of intersection. Also find the equation of the plane containing them.

SHORTEST DISTANCE BETWEEN TWO SKEW LINES

Two straight lines which do not lie in the same plane are called non-planar or skew lines. Skew lines are neither parallel nor intersecting. Such lines have a common perpendicular. The length of the segment of this common perpendicular line intercepted between the skew lines is called the shortest distance between them. The common perpendicular line itself is called the shortest distance line.

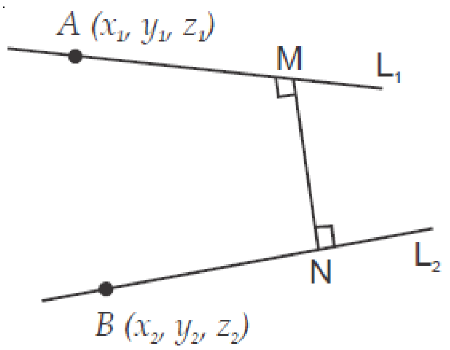
Let us now find the shortest distance and the equations of the shortest distance line between the skew lines.

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \text{\_\_\_\_(1)}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \qquad \text{\_\_\_\_(2)}$$

Let L_1, L_2 be the skew lines passing through A (x_1, y_1, z_1) and B (x_2, y_2, z_2) respectively.

Let MN be the S.D.between L_1 & L_2 and let l, m, n be the D.R.'s of the S.D.Line.



The point M may be taken as $(x_1+l_1r_1, y_1+m_1r_1, z_1+n_1r_1)$ and N may be taken as $(x_2+l_2r_2, y_2+m_2r_2, z_2+n_2r_2)$ Then the D.R.'s of MN are found.

Using the fact that MN is perpendicular to both L_1 and L_2 obtain two equations in r_1 and r_2, solving which we obtain the values of r_1 and r_2.

Substituting there values, we know the coordinates of M and N. Then the length and equations of MN can be found.

Problems

16. Find the length and equations of the shortest distance between the lines L1: $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$

L2: $\frac{x+3}{-3}=\frac{y\mp 7}{2}=\frac{z-6}{4}$

Solution :

Let the S.D. line cut the first line at P and the second line at Q.

$$L_1 : \frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=r \quad \text{(say)}$$

$$L_2 : \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=s \quad \text{(say)}$$

The point P on L_1 has coordinates $(3r+3, -r+8, r+3)$

The point Q on L_2 has coordinates $(-3s-3, 2s-7, 4s+6)$

Hence DR's of PQ are $\quad -3x-3s-6,\ 2s+r-15,\ 4s-r+3$

PQ is perpendicular to L_1

$$\therefore\ 3(-3r-3s-6)-(2s+r-15)+(4s-r+3)=0$$

i.e., $\quad 7s+11r=0 \qquad$ ____(1)

PQ is perpendicular to L_2

$$\therefore -3(-3r-3s-6)+2(2s+r-15)+4(4s-r+3)=0$$

$29s+7r=0 \qquad$ ____(2)

Solving (1) and (2), we get r = s = 0

Using these values of r and s in the co-ordinates of P and Q, we get

P (3, 8, 3) and Q (-3, -7, 6)

Length of S.D is $=\sqrt{270}=3\sqrt{30}$ units

Equation of the S.D. line is

$$\frac{x-3}{-6}=\frac{y-8}{-15}=\frac{z-3}{3} \Rightarrow \frac{x-3}{-2}=\frac{y-8}{-5}=\frac{z-3}{1}$$

Problem for practice

17. Find the length of the shortest distance between the lines

$$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4}; 2x+3y-5z-6=0=3x-2y-z+3$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –III–3D ANALYTICAL GEOMETRY AND VECTOR CALCULUS SMT1304

UNIT – III – SPHERE

Equation of the sphere - general form – plane section of a sphere- tangent line and tangent plane – orthogonal spheres

Introduction

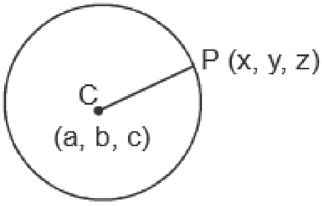
Definition

A sphere is the locus of a point in space which moves in such a way that its distance forms a fixed point is always constant.

The fixed point is called the centre of the sphere and the constant distance the radius of the sphere.

To find the equation of a sphere whose centre and radius are given:

Let r be the radius and (a, b, c) the centre C, and P any point on the sphere whose co-ordinates are (x, y, z).



Then $CP^2 = r^2$

i.e., $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

Corollary

When the centre of the sphere is at the origin and its radius is a, then the equation of the sphere is

$x^2 + y^2 + z^2 = a^2$

Standard Form of the equation of a Sphere

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is called the standard equation of a sphere with centre (-u, -v, -w) and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

Note

1. In the standard equation of the sphere, the coefficient of x^2, y^2, z^2 are all equal
2. The product terms *xy, yz, zx* are absent
3. The equation $a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0$ also representsa sphere with centre $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$ and radius $= \sqrt{\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a}}$

Diameter form of the sphere

Let the extremities of a diameter be A(x_1, y_1, z_1) B(x_2, y_2, z_2). If P(x, y, z) be any point on the sphere, then the equation of the sphere is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0 :$$

Problems

1. Find the equation of the sphere with centre (-1, 2, -3) and radius 3 units.

Solution:

The equation of the sphere is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Here (a, b, c) is (-1, 2, -3) and r = 3

$$(x+1)^2+(y-2)^2+(z+3)^2=3^2$$

(i.e)., $x^2+2x+1+y^2-4y+4+z^2+6z+9=9$

$\therefore$ The equation of the sphere is

$$x^2+y^2+z^2+2x-4y+6z+5=0.$$

2. Find the equation of the sphere with centre at (1, 1, 1) and passing through the point (1, 2, 5)

Solution

Let C (1, 1, 1) be the centre and P (1, 2, 5) be the given point

$$CP = \text{Radius} = \sqrt{(1-1)^2+(2-1)^2+(5-1)^2} = \sqrt{17}$$

The equation of the sphere with centre (a, b, c) and radius r is given by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

The equation of the sphere is

$$(x-1)^2+(y-1)^2+(z-1)^2=17$$

$$\therefore\ x^2-2x+1+y^2-2y+1+z^2-2z+1=17$$

$$\Rightarrow x^2+y^2+z^2-2x-2y-2z+3-17=0$$

The equation of the sphere is

$$x^2+y^2+z^2-2x-2y-2z-14=0$$

3. Find the equation of the sphere described on the line joining the points (2, -1, 4) and (-2, 2, -2) as diameter.

Solution:

Equation of the sphere with (x_1,y_1,z_1) *and* (x_2,y_2,z_2) as the end points of a diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

Here (x_1,y_1,z_1) is (2, -1, 4) and (x_2,y_2,z_2) is (-2, 2, -2)

The required equation of the sphere is

$$(x-2)(x+2)+(y+1)(y-2)+(z-4)(z+2)=0$$
$$x^2-4+y^2-y-2+z^2-2z-8=0$$

$$\Rightarrow \quad x^2+y^2+z^2-y-2z-14=0$$

4. Find the equation of the sphere through the points (2, 0, 1) (1, -5, -1), (0, -2, 3) and (4, -1, 2)

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{\_\_\_\_\_\_\_\_(1)}$$

(1) Passes through the point (2, 0, 1)

$$\therefore 4+0+1+4u+0\cdot v+2w+d=0$$
$$\Rightarrow 4v+2w+d=-5 \qquad \text{\_\_\_\_\_\_\_\_(2)}$$

(1) Passes through the point (1, -5, -1)

$$\therefore 1+25+1+2u-10v-2w+d=0$$
$$\Rightarrow 2u-10v-2w+d=-27 \qquad \text{\_\_\_\_\_\_\_\_(3)}$$

(1) Passes through (0, -2, 3)

$$\therefore 0+4+9+0\cdot u-4v+6w+d=0$$
$$\Rightarrow -4u+6w+d=-13 \qquad \text{\_\_\_\_\_\_\_\_(4)}$$

(1) Passes through (4, -1, 2)

$$\therefore 16+1+4+8u-2v+4w+d=0$$
$$\Rightarrow 8u-2v+4w+d=-21 \qquad \text{\_\_\_\_\_\_\_\_(5)}$$

(2) - (3) gives

$$2u+10v+4w=22$$
$$\text{(i.e)}\ \ u+5v+2w=11 \qquad \text{\_\_\_\_\_\_\_\_(6)}$$

(3) - (4) gives

$$2u-6v-8w=-14$$
$$\text{(i.e)}\ \ u-3v-4w=-7 \qquad \text{\_\_\_\_\_\_\_\_(7)}$$

(4) - (5) gives

$$-8u-2v+2w=8$$
$$\text{(i.e)}\ \ 4u+v-w=-4 \qquad \text{\_\_\_\_\_\_\_\_(8)}$$

(6) - (7) gives

$$8v+6w=18$$
$$\text{(i.e)}\ \ 4u+3w=9 \qquad \text{\_\_\_\_\_\_\_\_(9)}$$

(8) - (7) x 4 gives

$$13v+15w=24 \qquad \text{\_\_\_\_\_\_}(10)$$

Solving (9) and (10)

$$(10)-(9)\times 5 \text{ gives} -7v=-21 \Rightarrow v=3$$

Substituting v in (9), we get

$$4(3)+3w=9$$

$$12+3w=9 \Rightarrow w=\frac{9-12}{3}=-1$$

$$\Rightarrow w=-1$$

Substituting v and w in (8)

$$4u+3+1=-4$$

$$4u=-4-4 \Rightarrow u=-2$$

Substituting u and w in (2)

$$4(-2)+2(-1)+d=-5$$

$$-8-2+d=-5$$

$$d=-5+10 \ \Rightarrow d=5$$

The required equation of the sphere is

$$x^2+y^2+z^2-4x+6y-2z+5=0.$$

To find the point of contact if the two given spheres touch internally or externally

Let S_1 be the sphere with centre $C_1(-u_1,-v_1,-w_1)$ and radius r_1

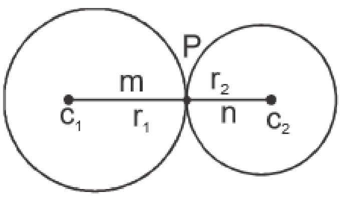
Let S_2 be the sphere with centre $C_2(-u_2,-v_2,-w_2)$ and radius r_2

Let P be the point of contact.

Case I

Two spheres Touch externally

The point of contact is the point which divides the line joining the two points C_1 and C_2 in the ratio m : n internally.



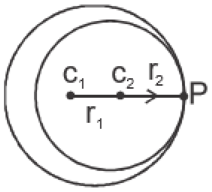
Hence the coordinates of the point of contact are

$$P\left(\frac{m(-u_2)+n(-u_1)}{m+n},\frac{m(-v_2)+n(-v_1)}{m+n},\frac{m(-w_2)+n(-w_1)}{m+n}\right)$$

Case II

Two spheres touch internally

The point of contact is the point which divides the line joining the two points C_1 and C_2 in the ratio m: n externally.



Hence the co-ordinates of the point of contact are

$$P\left(\frac{m(-u_2)+n(-u_1)}{m-n}, \frac{m(-v_2)+n(-v_1)}{m-n}, \frac{m(-w_2)+n(-w_1)}{m-n}\right)$$

Note:

(1) Two spheres S_1 and S_2 whose radii are r_1 and r_2 touch externally if the distance between their centres is equal to the sum of their radii (ie) $d = r_1 + r_2$.

(2) (2) Two spheres S_1 and S_2 whose radii are r_1 and r_2 touch internally if the distance between their centres is equal to the difference of the radii.

Problems

5. Prove that the two spheres

$$x^2+y^2+z^2-2x+4y-4z=0 \text{ and } \quad x^2+y^2+z^2+10x++2z+10=0$$

touch each other and find the coordinates of the point of contact**.**

Solution:

Let $S_1 : x^2+y^2+z^2-2x+4y-4z=0,\ \ S_2 : x^2+y^2+z^2+10x++2z+10=0$

The centre of S_1 is C_1 (1, -2, 2) ; The centre of S_2 is C_2 (-5, 0, -1)

Radius of S_1 is $r_1 = \sqrt{1+4+4} = 3$

Radius of S_2 is $r_2 = \sqrt{25+1-10} = 4$

Distance between C_1 and C_2 is $d = \sqrt{(1+5)^2+(-2+0)^2+(2+1)^2} = 7$

$\therefore\ r_1 + r_2 = 3 + 4 = 7 = d.$

The two spheres touch externally.

To find their point of contact:

The point of contact is the point which divides the line joining the two points C_1(1, -2, 2) and C_2 (-5, 0, -1) in the ratio 3 : 4 internally. Hence the coordinates of the point of contact are

$$P\left(\frac{3(-5)+4(1)}{3+4}, \frac{3(0)+4(-2)}{3+4}, \frac{3(-1)+4(2)}{3+4}\right)$$

(ie) $P\left(\frac{-11}{7}, \frac{-8}{7}, \frac{5}{7}\right)$

PLANE SECTION OF A SPHERE

Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be a sphere with centre at 'C' and radius R. Also let $P \equiv ax + by + cz + d^1 = 0$ be a plane that intersects the sphere S as shown in the figure.

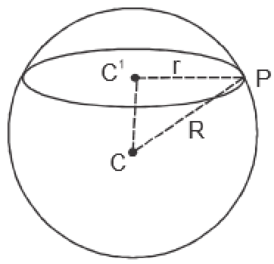
Clearly the curve of intersection of the sphere S and the plane P is a circle. This intersecting portion of the sphere is called as plane section of the sphere.

Therefore in three dimensional spaces any circle can be represented as a plane section of a sphere and a plane. i.e intersecting portion of a sphere and a plane. Its equation can be jointly represented by the equation of the sphere and the plane.

i.e., $$\left.\begin{array}{l} S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \\ P \equiv ax + by + cz + d^1 = 0 \end{array}\right\}$$

Represents a circle in three dimensional space.

The centre of the above circle is the foot of the perpendicular drawn from centre of the sphere S on the plane P and radius of the circle is given by



$$r = \sqrt{R^2 - \overline{CC'}^2}$$

Note

If the intersecting plane passes through the centre of the sphere then such a circle is called a GREAT CIRCLE of the sphere.

For any sphere there are infinitely many great circles that can be identified on its boundary.

Equation of sphere through a given circle

Equation of a sphere that passes through a given circle represented by

$$\left.\begin{array}{l} S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \\ P \equiv ax + by + cz + d^1 = 0 \end{array}\right\} \text{ is of the form}$$

$$\boxed{S + \lambda P = 0}$$

i.e, $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d + \lambda(ax + by + (cz + d')) = 0$

By applying any given additional information about the sphere in the above equation, the value of λ can be found and hence the equation of the sphere can also be found.

Problems

1. Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$ and $2x + 3y + 4z = 8$ is a great circle .

Solution:

The given circle is

$S \equiv x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$ ______(1)

$P \equiv 2x + 3y + 4z - 8 = 0$ ______(2)

Sphere through the above circle is of the form

$$S+\lambda P=0$$

$$x^2+y^2+z^2+7y-2z+2+\lambda(2x+3y+4z-8)=0$$

$$x^2+y^2+z^2+2\lambda x+(7+3\lambda)y+(-2+4\lambda)z+2-8\lambda=0 \quad \_\_\_\_(3)$$

Centre for (3) is given by

$$(-u,-v,-w)$$

$$i.e.,\left(-\left(\frac{2\lambda}{2}\right),-\left(\frac{7+3\lambda}{2}\right),-\left(\frac{-2+4\lambda}{2}\right)\right)$$

$$i.e.,\left(-\lambda,\frac{-7-3\lambda}{2},1-2\lambda\right)$$

If circle (1), (2) is a great circle for (3), then centre of (3) should lie on the plane (2)

$$\Rightarrow 2(-\lambda)+3\left(\frac{-7-3\lambda}{2}\right)+4(1-2\lambda)-8=0$$

$$-2\lambda-\frac{21}{2}-\frac{9\lambda}{2}+4-8\lambda-8=0$$

$$-\frac{29\lambda}{2}=\frac{29}{2}\Rightarrow\boxed{\lambda=-1}$$

Subsituting λ in (3), $x^2+y^2+z^2-2x+4y-6z+10=0$ is the required sphere.

2. Find the centre and radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2y + 4z - 11 = 0$ is cut by the plane $x + 2y + 2z + 15 = 0$.

Solution:

The centre and radius of the sphere

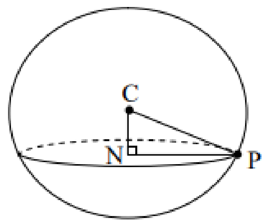
$$x^2+y^2+z^2+2y+4z-11=0 \qquad (1)$$

are C(0, -1, -2) and CP = $\sqrt{0^2+1^2+2^2+11} = 4$

Let N be the foot of the perpendicular from C on the plane

$$x+2y+2z+15=0 \qquad (2)$$

Then $CN=\dfrac{0+2(-1)+2(-2)+15}{\sqrt{1^2+2^2+2^2}}=3$



$\therefore$ The radius of the circle = $\sqrt{CP^2-CN^2}$

$= \sqrt{4^2-3^2}=\sqrt{7}$

The centre of the circle is N

Now CN is perpendicular to plane (2).

$\therefore$ Its D.R.'s are (1, 2, 2). Also CN passes through C (0, -1, -2).

$\therefore$ The equation of line CN is $\dfrac{x-0}{1}=\dfrac{y+1}{2}=\dfrac{z+2}{2}$

Any point on this line is (r, 2r - 1, 2r - 2). If this point is N, it satisfies plane (2).

$\therefore \quad r + 2(2r - 1) + 2(2r - 2) + 15 = 0$

i.e., $\quad 9r + 9 = 0$

i.e., $\quad r = -1.$

$\therefore$ The coordinates of N are (-1, 2(-1)-1, 2(-1)-2)

i.e., $\quad$ (-1, -3, -4)

Thus the centre and radius of the circle are (-1, -3, -4) and $\sqrt{7}$

3. A sphere touches the plane x - 2y - 2z - 7 = 0 in the point (3, -1, -1) and passes through the point (1,1,-3). Find its equation.

Solution:

The equation of the point sphere with centre at (3, -1,-1) is

$$(x - 3)^2 + (y + 1)^2 + (z + 1)^2 = 0$$

i.e., $\quad x^2 + y^2 + z^2 - 6x + 2y + 2z + 11 = 0 \qquad (1)$

The required sphere touches the plane

$$x - 2y - 2z - 7 = 0 \qquad (2)$$

at (3, -1, -1)

Therefore It contains the point circle of intersection of sphere (1) and plane (2).

Hence the equation of the required sphere is of the form.

$$x^2 + y^2 + z^2 - 6x + 2y + 2z + 11 + \lambda(x - 2y - 2z - 7) = 0$$

and passes through (1, 1, -3)

$\therefore \quad 1^2 + 1^2 + (-3)^2 - 6.1 + 2.1 + 2(-3) + 11 + \lambda(1 - 2.1 - 2(-3) - 7) = 0.$

$$12 + \lambda(-2) = 0$$

$$\lambda = 6$$

$\therefore$ The required sphere is

$$x^2 + y^2 + z^2 - 6x + 2y + 2z + 11 + 6(x - 2y - 2z - 7) = 0$$

i.e., $\quad x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$

TANGENT LINE AND TANGENT PLANE

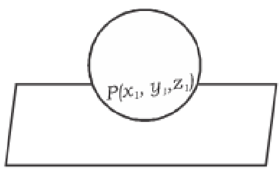
When a straight line intersects a sphere at exactly one point or when it touches a sphere at apoint P, the line is called tangent line of the sphere at P and is perpendicular to the radius of the sphere through P. There are many tangent lines through P which are perpendicular to the radius of the sphere. All these tangent lines lie on a plane through P, which is perpendicular to the radius through P. This plane is called the tangent plane of the sphere at P.

Equation of the tangent plane to a sphere at a given point.

Let the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad$ _____(1)

have a point $P(x_1, y_1, z_1)$ on it. The centre of (1) is (-u, -v, -w). As P lies on (1),

Equation of the tangent plane to (1) at the point (x_1 ,y_1 ,z_1) is



$$\boxed{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

Orthogonal spheres

Two spheres are said to cut each other orthogonally if the tangent planes at a point of intersection are at right angles.

If two spheres cut orthogonally at P, their radii through P, being perpendicular to the tangent planes at P, will also be at right angles.

Consider the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad \_\_\_\_(1)$$

and $$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad \_\_\_\_(2)$$

Let (1) and (2) cut orthogonally at P.

For (1), centre C_1 is $(-u_1,-v_1,-w_1)$

and radius $r_1=\sqrt{u_1^2+v_1^2+w_1^2-d_1}$ ____(3)

For (2), centre C_2 is $(-u_2,-v_2,-w_2)$

and radius $r_2=\sqrt{u_2^2+v_2^2+w_2^2-d_2}$ ____(4)

$$C_1C_2=d=\sqrt{(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2}$$

The condition for the two spheres to cut orthogonally is

$$\boxed{2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2}$$

4. Find the equation of the sphere that passes through the circle $x^2+y^2+z^2+x-3y+2z-1=0$, $2x+5y-z+7=0$ and cuts orthogonally the sphere whose equation $x^2+y^2+z^2-3x+5y-7z-6=0$.

Solution:

The equation of any sphere passing through the given circle is

$$x^2+y^2+z^2+x-3y+2z-1+\lambda(2x+5y-z+7)=0$$

i.e., $$x^2+y^2+z^2+(1+2\lambda)x+(-3+5\lambda)y+(2-\lambda)z-1+7\lambda=0 \qquad (1)$$

Sphere (1) cuts the sphere $x^2+y^2+z^2-3x+5y-7z-6=0$ orthogonally.

$\therefore$ $2uu_1+2vv_1+2ww_1=d+d_1$

i.e., $$2\left(\frac{1+2\lambda}{2}\right)\left(\frac{-3}{2}\right)+2\left(\frac{-3+5\lambda}{2}\right)\left(\frac{5}{2}\right)+2\left(\frac{2-\lambda}{2}\right)\left(\frac{-7}{2}\right)=-1+7\lambda-6$$

i.e., $-3-6\lambda-15+25\lambda-14+7\lambda=-14+14\lambda$

i.e., $12\lambda-18=0$ i.e., $\lambda=3/2$

$\therefore$ Equation of the required sphere is

$$x^2+y^2+z^2+x-3y+2z-1+(3/2)(2x+5y-z+7)=0$$

i.e., $$2(x^2+y^2+z^2)+8x+9y+z+19=0$$

5. Write the equation of the tangent plane at (1, 5, 7) to the sphere $(x-2)^2+(y-3)^2+(z-4)^2=14$

Solution:

The equation of the tangent plane to the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at (x_1, y_1, z_1) is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0 \qquad (1)$$

Given : $(x-2)^2+(y-3)^2+(z-4)^2=14$

$(x^2-4x+4)+(y^2-6y+9)+(z^2-8z+16)=14$

$x^2+y^2+z^2-4x-6y-8z+29-14=0$

Here 2u = -4, 2v = -6, 2w = -8, d = 15

$x_1=1$, $y_1=5$, $z_1=7$

$(1)\Rightarrow x(1)+y(5)+z(7)+(-2)(x+1)+(-3)(y+5)+(-4)(z+7)+15=0$

$x+5y+7z-2x-2-3y-15-4z-28+15=0$

$-x+2y+3z-30=0$

i.e., $x-2y-3z+30=0$

6. Test whether the plane x = 3 touches the sphere $x^2+y^2+z^2=9$

Solution:

The condition that the plane lx+ my + nz =p to touch the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}}=\sqrt{u^2+v^2+w^2-d}$$

i.e., $(lu+mv+nw+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d) \qquad (1)$

$u=0, v=0, w=0, l=1, m=0, n=0, p=3, d=-4$

Hence $(1)\Rightarrow(0+0+3)^2=(1+0+0)(0+0+0+9)$

i.e., $3^2=9$

The plane x=3 touches the sphere $x^2+y^2+z^2=9$.

7. Find the equation of the sphere which has its centre at (-1, 2, 3) and touches the plane 2x-y+2z = 6

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

Given: $-u=-1,\quad -v=2,\quad -w=3$

$u=1,\quad v=-2,\quad w=-3$

$$\therefore(1)\Rightarrow x^2+y^2+z^2+2x-4y-6z+d=0 \qquad (2)$$

To find d: Since the plane 2x-y+2z = 6 touches the sphere whose centre is (-1, 2, 3). The radius of the sphere is equal to the length of the perpendicular drawn from the centre (1, 2, 3) to the plane 2x-y+2z = 6

Length of the perpendicular

$$=\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{(2)(-1)+(-1)(2)+(2)(3)-6}{\sqrt{4+1+4}}$$

$$=\frac{-2-2+6-6}{\sqrt{9}}=\frac{-4}{3}=r$$

We know that $r=\sqrt{u^2+v^2+w^2-d}$

$$r^2=u^2+v^2+w^2-d$$

$$d=u^2+v^2+w^2-r^2$$

$$=(-1)^2+(2)^2+(3)^2-\left(\frac{-4}{3}\right)^2$$

$$=1+4+9-\frac{16}{9}=14-\frac{16}{9}=\frac{110}{9}$$

$$(2)\Rightarrow x^2+y^2+z^2+2x-4y-6z+\frac{110}{9}=0$$

$$9\left(x^2+y^2+z^2\right)+18x-36y-54z+110=0$$

8. Find the equation of the sphere with centre at (2, 3, 5), which touches the XOY plane.

Solution: Let $(x_1,y_1,z_1)=(2,3,5)$

Formula: Radius = perpendicular distance from (x_1,y_1,z_1) to the plane ax + by + cz +d = 0

$$=\pm\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

Radius = perpendicular distance from (2, 3, 5) to the plane z = 0

$$=\pm\frac{5}{\sqrt{0^2+0^2+1^2}}=\pm 5$$

The required sphere is $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$

$$(x-2)^2+(y-3)^2+(z-5)^2=5^2$$

$$x^2-4x+4+y^2-6y+9+z^2-10z+25=25$$

$$x^2+y^2+z^2-4x-6y-10z+13=0$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT-IV-3DANALYTICAL GEOMETRY AND VECTOR CALCULUS
SMT1303

UNIT – IV- VECTOR DIFFERENTIATION

Limit of a vector function – Continuity of vector functions – Derivative of a vector function –– Scalar and vector point functions – Gradient of a scalar point function – Directional derivative of a scalar point function – Divergence and curl of a vector point function – Solenoidal vector – Irrotational vector – Vector identities

Definitions:

Scalars

The quantities which have only magnitude and are not related to any direction in space are called scalars. Examples of scalars are (i) mass of a particle (ii) pressure in the atmosphere (iii) temperature of a heated body (iv) speed of a train.

Vectors

The quantities which have both magnitude and direction are called vectors.

Examples of vectors are (i) the gravitational force on a particle in space (ii) the velocity at any point in a moving fluid.

Scalar point function

If to each point p(x, y, z) of a region R in space there corresponds a unique scalar f(p) then f is called a scalar point function.

Example

Temperature distribution of a heated body, density of a body and potential due to gravity.

Vector point function

If to each point p(x, y, and z) of a region R in space there corresponds a unique vector $\vec{f}(p)$ then $\vec{f}$ is called a vector point function.

Example

The velocity of a moving fluid, gravitational force.

Scalar and vector fields

When a point function is defined at every point of space or a portion of space, then we say that a field is defined. The field is termed as a scalar field or vector field as the point function is a scalar point function or a vector point function respectively.

Vector Differential Operator (∇)

The vector differential operator Del, denoted by ∇ is defined as

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Gradient of a scalar point function

Let $\phi(x, y, z)$ be a scalar point function defined in a region R of space. Then the vector point function given by $\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$

$= \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$ is defined as the gradient of ϕ and denoted by grad ϕ

Directional Derivative (D.D)

The directional derivative of a scalar point function ϕ at point (x,y,z) in the direction of a vector $\vec{a}$ is given by D.D = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$ (or) D.D = $\nabla\phi\,.\,\hat{a}$

The unit normal vector

The unit vector normal to the surface $\phi(x, y, z)$ = c is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Angle between two surfaces

Angle between the surfaces $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ is given by $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Problems

1. Find $\nabla\phi$ if $\phi(x, y, z) = xy - y^2 z$ at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(xy - y^2 z)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(xy - y^2 z) + \vec{j}\,\frac{\partial}{\partial y}(xy - y^2 z) + \vec{k}\,\frac{\partial}{\partial z}(xy - y^2 z)$$

$= y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k} \quad \therefore \nabla\phi = y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k}$.

At (1, 1, 1), $\nabla\phi = \vec{i}(1) + \vec{j}(1 - (2)(1)(1)) - \vec{k}(1)^2 = \vec{i} - \vec{j} - \vec{k}$

2. Find $\nabla\phi$ if $\phi(x, y, z) = x^2 y + 2xz^2 - 8$ at the point (1, 0, 1)

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^2 y + 2xz^2 - 8)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 y + 2xz^2 - 8) + \vec{j}\frac{\partial}{\partial y}(x^2 y + 2xz^2 - 8) + \vec{k}\frac{\partial}{\partial z}(x^2 y + 2xz^2 - 8)$$

$$= (2xy + 2z^2)\vec{i} + (x^2)\vec{j} + 4xz\vec{k}$$

At (1, 0, 1), $\nabla\phi = \vec{i}(2(1)(0) + 2(1^2)) + \vec{j}(1^2) + \vec{k}4(1)(1)$

$$= 2\vec{i} + \vec{j} + 4\vec{k}$$

3. Find the unit normal vector to the surface $\phi(x, y, z) = x^2 yz^3$ at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^2 yz^3) = \vec{i}\frac{\partial}{\partial x}(x^2 yz^3) + \vec{j}\frac{\partial}{\partial y}(x^2 yz^3) + \vec{k}\frac{\partial}{\partial z}(x^2 yz^3)$$

$$= 2xyz^3\vec{i} + x^2 z^3\vec{j} + 3x^2 yz^2\vec{k}$$

At (1,1,1), $\nabla\phi = \vec{i}\,2(1)(1)(1) + \vec{j}(1^2)(1^3) + \vec{k}3(1^2)(1)(1^2)$

$$= 2\vec{i} + \vec{j} + 3\vec{k}$$

$$|\nabla\phi| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{14}}$$

4. Find the unit normal vector to the surface $\phi(x,y,z) = x^2 + y^2 - z$ at the point (1,-1,-2)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(x^2 + y^2 - z)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^2 + y^2 - z) + \vec{j}\,\frac{\partial}{\partial y}(x^2 + y^2 - z) + \vec{k}\,\frac{\partial}{\partial z}(x^2 + y^2 - z)$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

At (1,-1,-2), $\nabla\phi = \vec{i}\,2(1) + \vec{j}\,2(-1) - \vec{k}$

$$= 2\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} - 2\vec{j} - \vec{k}}{3}$$

5. Find the angle between the surfaces xyz and x^3yz at the point (1,1,-2)

Solution:

Given the surface $\phi_1(x,y,z) = xyz$

$$\nabla\phi_1 = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi_1$$

$$\nabla\phi_1 = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(xyz)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(xyz) + \vec{j}\,\frac{\partial}{\partial y}(xyz) + \vec{k}\,\frac{\partial}{\partial z}(xyz)$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

At(1,1,-2), $\nabla\phi_1 = \vec{i}\,(1)(-2) + \vec{j}\,(1)(-2) + (1)(1)\vec{k}$

$$= -2\vec{i} - 2\vec{j} + \vec{k}$$

$$|\nabla\phi_1| = \sqrt{(-2)^2 + (-2)^2 + 1^2} \quad = 3$$

Given the surface $\phi_2(x,y,z) = x^3yz$

$$\nabla\phi_2 = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi_2$$

$$\nabla\phi_2 = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(x^3yz)$$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^3yz) + \vec{j}\,\frac{\partial}{\partial y}(x^3yz) + \vec{k}\,\frac{\partial}{\partial z}(x^3yz)$$

$$= 3x^2yz\vec{i} + x^3z\vec{j} + x^3y\vec{k}$$

At (1,1,-2), $\nabla\phi_2 = \vec{i}\,3(1^2)(1)(-2) + \vec{j}(1^3)(-2) + (1^3)(1)\vec{k} \; = -6\vec{i} - 2\vec{j} + \vec{k}$

$$|\nabla\phi_2| = \sqrt{(-6)^2 + (-2)^2 + 1^2} = \sqrt{41}$$

Angle between the surfaces is given by $\cos\theta = \dfrac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(-2\vec{i} - 2\vec{j} + \vec{k}).(-6\vec{i} - 2\vec{j} + \vec{k})}{3\sqrt{41}}$$

$$= \frac{12 + 4 + 1}{3\sqrt{41}} = \frac{17}{3\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{17}{3\sqrt{41}}\right)$$

6. Find the angle between the normal to the surface xy - z^2 at the point (1,4,-2) and (1,2,3)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})(xy - z^2) = \vec{i}\,\frac{\partial}{\partial x}(xy - z^2) + \vec{j}\,\frac{\partial}{\partial y}(xy - z^2) + \vec{k}\,\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\vec{i} + x\vec{j} - 2z\vec{k}$$

At (1,4,-2), $\nabla\phi_1 = \vec{i}\,(4) + \vec{j}(1) - 2(-2)\vec{k} \quad = \; 4\vec{i} + \vec{j} + 4\vec{k}$

$|\nabla\phi| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

At (1,2,3), $\nabla\phi_2 = \vec{i}(2) + \vec{j}(1) - 2(3)\vec{k} = 2\vec{i} + \vec{j} - 6\vec{k}$

$|\nabla\phi| = \sqrt{2^2 + 1^2 + (-6)^2} = \sqrt{41}$

Angle between the surfaces is given by $\cos\theta = \dfrac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(2\vec{i} + \vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{41}}$$

$$= \frac{8 + 1 - 24}{\sqrt{33}\sqrt{41}} = \frac{-15}{\sqrt{33}\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-15}{\sqrt{33}\sqrt{41}}\right)$$

7. Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (2,-1,1) in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xy^2 + yz^3) = \vec{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \vec{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \vec{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

At (2,-1,1), $\nabla\phi = \vec{i}(-1^2) + \vec{j}(2(2)(-1) + 1^3) + 3(-1)(1^2)\vec{k} = \vec{i} - 3\vec{j} - 3\vec{k}$

To find the directional derivative of ϕ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$

find the unit vector along the direction

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (2,-1,1) is $\nabla\phi . \dfrac{\vec{a}}{|\vec{a}|}$

$$=(\vec{i}-3\vec{j}-3\vec{k}).\frac{(\vec{i}+2\vec{j}+2\vec{k})}{3}$$

$$=\frac{1-6-6}{3}=\frac{-11}{3}\text{ units.}$$

8. Find the directional derivative of $\phi(x,y,z)=xyz+yz^2$ at the point (1, 1, 1) in the direction of $\vec{i}+\vec{j}+\vec{k}$

Solution:

$$\nabla\phi=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})(xyz+yz^2)=\vec{i}\frac{\partial}{\partial x}(xyz+yz^2)+\vec{j}\frac{\partial}{\partial y}(xyz+yz^2)+\vec{k}\frac{\partial}{\partial z}(xyz+yz^2)$$

$$=yz\vec{i}+(xz+z^2)\vec{j}+(xy+2yz)\vec{k}$$

At (1, 1, 1) $\nabla\phi=\vec{i}(1)(1)+\vec{j}((1)(1)+1^2)+((1)(1)+2(1)(1))\vec{k}\quad=\vec{i}+2\vec{j}+3\vec{k}$

To find the directional derivative of ϕ in the direction of the vector $\vec{i}+\vec{j}+\vec{k}$ find the unit vector along the direction

$$\vec{a}=\vec{i}+\vec{j}+\vec{k}\Rightarrow|\vec{a}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (1,1,1) = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

$$=(\vec{i}+2\vec{j}+3\vec{k}).\frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}}=\frac{1+2+3}{\sqrt{3}}=\frac{6}{\sqrt{3}}\text{ units.}$$

Divergence of a differentiable vector point function $\vec{F}$

The divergence of a differentiable vector point function $\vec{F}$ is denoted by div $\vec{F}$ and is defined by

$$\text{Div}\,\vec{F}=\nabla\cdot\vec{F}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot\vec{F}$$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$$

$$=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Curl of a vector point function

The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and is defined by Curl $\vec{F} = \nabla \times \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \times \vec{F}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then Curl $\vec{F}$ = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Vector Identities

Let ϕ be a scalar point function and $\vec{U}$ and $\vec{V}$ be vector point functions. Then

(1) $\nabla \cdot (\vec{U} \pm \vec{V}) = \nabla \cdot \vec{U} \pm \nabla \cdot \vec{V}$
(2) $\nabla \times (\vec{U} \pm \vec{V}) = \nabla \times \vec{U} \pm \nabla \times \vec{V}$
(3) $\nabla \cdot (\phi\vec{U}) = \nabla\phi \cdot \vec{U} + \phi\nabla \cdot \vec{U}$
(4) $\nabla \times (\phi\vec{U}) = \nabla\phi \times \vec{U} + \phi\nabla \times \vec{U}$
(5) $\nabla \cdot (\vec{U} \times \vec{V}) = \vec{V} \cdot (\nabla \times \vec{U}) - \vec{U} \cdot (\nabla \times \vec{V})$
(6) $\nabla \times (\vec{U} \times \vec{V}) = (\nabla \cdot \vec{V})\vec{U} - (\nabla \cdot \vec{U})\vec{V} + \vec{U}(\vec{V} \cdot \nabla) - \vec{V}(\vec{U} \cdot \nabla)$
(7) $\nabla(\vec{U} \cdot \vec{V}) = (\nabla \cdot \vec{V})\vec{U} + (\nabla \cdot \vec{U})\vec{V} + \vec{U} \times (\nabla \times \vec{V}) - (\nabla \times \vec{U}) \times \vec{V}$

Solenoidal and Irrotational vectors
A vector point function is solenoidal if div $\vec{F}$ = 0 and it is irrotational if curl $\vec{F}$ = 0.

Note:
If $\vec{F}$ is irrotational, then there exists a scalar function called Scalar Potential ϕ such that $\vec{F}$ = $\nabla\phi$

Problems
9. Find div $\vec{r}$ and curl $\vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
Solution:

div $\vec{r}$ = $\nabla \cdot \vec{r} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{r}$

$$= (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$$\text{Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = 0.$$

10. Find the divergence and curl of the vector $\vec{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (1,-1, 1)

Solution:

Given $\vec{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$

$$\text{Div}\vec{V} = \nabla \cdot \vec{V} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot \vec{V}$$

$$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k})$$

$$= \frac{\partial(xyz)}{\partial x} + \frac{\partial(3xy^2)}{\partial y} + \frac{\partial(xz^2 - y^2z)}{\partial z} = \text{yz+6xy+2xz-y}^2$$

At (1,-1, 1), $\nabla \cdot \vec{V} = (-1).1 + 6(1)(-1) + 2(1)(1) - (-1)^2$

$$= -1-6+2-1 = -6.$$

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \times \vec{V}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy^2 & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}(\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3xy^2)) - \vec{j}(\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)) + \vec{k}(\frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial y}(xyz)).$$

$$= \vec{i}(-2yz) - \vec{j}(z^2 - yx) + \vec{k}(3y^2 - xz).$$

At (1,-1, 1), $\nabla \times \vec{V} = \vec{i}(-2(-1)(1)) - \vec{j}(1^2 - (-1)(1)) + \vec{k}((3(-1)^2 - 1(1)) = 2\vec{i} - 2\vec{j} + 2\vec{k}$

11. Find the constants a, b, c so that given vector field is irrotational, where $\vec{F} = (x+2y+ax)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$

Solution:

Given $\nabla \times \vec{F} = 0$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \vec{i}\,(\dfrac{\partial}{\partial y}(4x+cy+2z) - \dfrac{\partial}{\partial z}(bx-3y-z)) - \vec{j}\,(\dfrac{\partial}{\partial x}(4x+cy+2z) - \dfrac{\partial}{\partial z}(x+2y+az) + \\ \vec{k}\,(\dfrac{\partial}{\partial x}(bx-3y-z) - \dfrac{\partial}{\partial y}(x+2y+az) \end{bmatrix} = 0\,.$$

$$\Rightarrow \vec{i}\,(c+1) - \vec{j}\,(4-a) + \vec{k}\,(b-2) = 0\,.$$

c+1 = 0, 4-a = 0, b-2 = 0

Hence c = -1, a = 4, b = 2.

12. Prove that $\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ is both solenoidal and irrotational**.**

Solution:

$$\nabla \cdot \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{V}$$

$$= (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot ((2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k})$$

$$= \frac{\partial(2x+yz)}{\partial x} + \frac{\partial(4y+zx)}{\partial y} - \frac{\partial(6z-xy)}{\partial z}$$ = 2+4-6 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

Now, $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -(6z-xy) \end{vmatrix}$

$$= \begin{bmatrix} \vec{i}\,(\frac{\partial}{\partial y}(-(6z-xy)) - \frac{\partial}{\partial z}(4y+zx)) - \vec{j}(\frac{\partial}{\partial x}(-(6z-xy)) - \frac{\partial}{\partial z}(2x+yz) + \\ \vec{k}(\frac{\partial}{\partial x}(4y+zx) - \frac{\partial}{\partial y}(2x+yz) \end{bmatrix}$$

$\Rightarrow \vec{i}\,(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = 0$ for all points (x, y, z)

$\therefore \vec{F}$ is irrotational vector.

13. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and irrotational and find its scalar potential.

Solution:

$$\nabla \cdot \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{F}$$

$$= (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot ((y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k})$$

$$= \frac{\partial(y^2 - z^2 + 3yz - 2x)}{\partial x} + \frac{\partial(3xz + 2xy)}{\partial y} + \frac{\partial(3xy - 2xz + 2z)}{\partial z}$$

= -2+2x-2x+2 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= \begin{bmatrix} \vec{i}\,(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)) - \vec{j}(\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x) + \\ \vec{k}(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x) \end{bmatrix}$$

$\Rightarrow \vec{i}\,(3x - 3x) - \vec{j}(3y - 2z + 2z - 3y) + \vec{k}(3z + 2y - 2y - 3z) = 0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$\Rightarrow (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k} = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial \phi}{\partial x} = y^2 - z^2 + 3yz - 2x \quad \text{……………………………………………………..…..…(1)}$$

$$\frac{\partial \phi}{\partial y} = 3xz + 2xy \quad \text{………………………………………………………………..……...(2)}$$

$$\frac{\partial \phi}{\partial z} = 3xy - 2xz + 2z \quad \text{…………………………………………………………..……(3)}$$

Integrating (1) with respect to ‘x’ treating ‘y’ and ‘z’ as constants, we get

$$\phi = xy^2 - xz^2 + 3xyz - 2\frac{x^2}{2} + f(y,z) \quad \text{……………………………………………………(4)}$$

Integrating (2) with respect to ‘y’ treating ‘x’ and ‘z’ as constants, we get

$$\phi = 3xyz + 2\frac{xy^2}{2} + f(x,z) \quad \text{……………………………………………………….………(5)}$$

Integrating (3) with respect to ‘z’ treating ‘x’ and ‘y’ as constants, we get

$$\phi = 3xyz - 2x\frac{z^2}{2} + 2\frac{z^2}{2} + f(x,y) \quad \text{…………………………………………………………..(6)}$$

Hence from equations (4), (5), (6), we get

14. Prove that $\vec{F} = 3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k}$ is irrotational and find its scalar potential.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2y^2 & 2x^3y + \cos z & -y\sin z \end{vmatrix}$$

$$= \begin{bmatrix} \vec{i}\left(\frac{\partial}{\partial y}(-y\sin z) - \frac{\partial}{\partial z}(2x^3y + \cos z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-y\sin z) - \frac{\partial}{\partial z}(3x^2y^2)\right) \\ \vec{k}\left(\frac{\partial}{\partial x}(2x^3y + \cos z) - \frac{\partial}{\partial y}(3x^2y^2)\right) \end{bmatrix}$$

$$\Rightarrow \vec{i}(-\sin z - (-\sin z)) - \vec{j}(0-0) + \vec{k}(6x^2y - 6x^2y) = 0 \text{ for all points (x,y,z)}$$

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k} = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = 3x^2y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial\phi}{\partial y} = 2x^3y + \cos z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial\phi}{\partial z} = -y\sin z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Integrating (1) with respect to ‘x’ treating ‘y’ and ‘z’ as constants, we get

$$\phi = 3\frac{x^3y^2}{3} + f(y,z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Integrating (2) with respect to ‘y’ treating ‘x’ and ‘z’ as constants, we get

$$\phi = 2\frac{x^3y^2}{2} + y\cos z + f(x,z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Integrating (3) with respect to ‘z’ treating ‘x’ and ‘y’ as constants, we get

$$\phi = y\cos z + f(x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Hence from equations (4), (5), (6), we get

$\phi = x^3y^2 + y\cos z + c$

15. Prove that = div(grad ϕ)= $\nabla^2\phi$

Solution:

$$\begin{aligned}
div(grad\,\phi) &= \nabla.\nabla\phi \\
&= \nabla\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) \\
&= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi \\
&= \nabla^2\phi\,.
\end{aligned}$$

16. Prove that div (curl$\vec{F}$) =0

Solution

$$\nabla.(\nabla\times \mathrm{A}) = \nabla\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\cdot\left[\vec{i}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z}\right) - \vec{j}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z}\right)\right] + \vec{k}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y}\right)$$

$$=\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}\right)$$

$$\therefore div\left(curl\vec{A}\right)=0$$

17. If $\vec{A}$ and $\vec{B}$ are irrotational show that $\vec{A}$ x $\vec{B}$ is solenoidal.

Solution:

Given $\vec{A}$ is irrotational i.e., $\nabla\times\vec{A}=\vec{0}$

$\vec{B}$ is irrotational i.e., $\nabla\times\vec{B}=\vec{0}$

$$\nabla\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}.\left(\nabla\times\vec{B}\right)$$

$$=\vec{B}\cdot\vec{0}-\vec{A}\cdot\vec{0}=\vec{0}$$

$\therefore$ $\vec{A}\times\vec{B}$ is solenoidal.

18. If r= $|\vec{r}|$, where $\vec{r}$ is the position vector of the point P(x, y, z),then prove that $\nabla^2(r^n)=n(n+1)\cdot r^{n-2}$

Solution:

$$\nabla^2(r^n)=\nabla\cdot(\nabla r^n)$$

$$\nabla r^n=\vec{i}\frac{\partial}{\partial x}(r^n)+\vec{j}\frac{\partial}{\partial y}(r^n)+\vec{k}\frac{\partial}{\partial z}(r^n)$$

$$=\vec{i}\left[nr^{n-1}\frac{\partial r}{\partial x}\right]+\vec{j}\left[nr^{n-1}\frac{\partial r}{\partial y}\right]+\vec{k}\left[nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$=\vec{i}\left[nr^{n-1}\cdot\frac{x}{r}\right]+\vec{j}\left[nr^{n-1}\cdot\frac{y}{r}\right]+\vec{k}\left[nr^{n-1}\cdot\frac{z}{r}\right]$$

$$\because r^2=x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x}=2x$$

$$=nr^{n-2}[x\vec{i}+y\vec{j}+z\vec{k}]$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}$$

$$=nr^{n-2}\overline{r}$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

Now

$$\nabla\cdot\nabla r^n=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\left(nr^{n-2}(x\vec{i}+y\vec{j}+z\vec{k})\right)$$

$$=\frac{\partial}{\partial x}(nr^{n-2}x)+\frac{\partial}{\partial y}(nr^{n-2}y)+\frac{\partial}{\partial z}(nr^{n-2}y)$$

$$= n\left[r^{n-2} + x.(n-2)r^{n-3}\left(\frac{x}{r}\right)\right] + n\left[r^{n-2} + y.(n-2)r^{n-3}\left(\frac{y}{r}\right)\right] + n\left[r^{n-2} + z.(n-2)r^{n-3}\left(\frac{z}{r}\right)\right]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4} \cdot r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= nr^{n-2}[3 + n - 2] = n(n+1)r^{n-2}$$

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT –V–3D ANALYTICAL GEOMETRY AND VECTOR CALCULUS SMT1303

UNIT – V – VECTOR INTEGRATION

Vector integration – Line integral – Application of line integral. Surface and Volume integrals – Applications - Gauss Divergence theorem. Stoke's theorem – Green's theorem.

Introduction:

Line Integrals

A line integral (sometimes called a path integral) is the integral of some function along a curve.

(i.e) an integral which is to be evaluated along a curve is called a line integral. One can integrate a scalar-valued function along a curve, obtaining for example, the mass of a wire from its density. One can also integrate a certain type of vector-valued functions along a curve.

Let F (x, y, z) be a vector point function defined at all points in some region of space and let C be a curve in that region. The integral $\int_C \vec{F} . d\vec{r}$ is defined as the line integral of $\vec{F}$ along the curve C.

Note:

(1) Physically, $\int_C \bar{F}.d\bar{r}$ denotes the total work done by the force $\bar{F}$ in displacing a particle from A to B along the curve C.

(2) $\int_A^B \bar{F}.d\bar{r}$ depends not only on the curve C but also on the terminal points A and B.

(3) If the path of integration C is a closed curve, the line integral is denoted as $\oint_C \bar{F}.d\bar{r}$.

(4) If the value of $\int_A^B \bar{F}.d\bar{r}$ does not depend on the curve C, but only on the terminal points A and B, than $\bar{F}$ is called a conservative vector or conservative force.

(5) If $\bar{F}$ is irrotational (conservative) and C is a closed curve then $\int_C \bar{F}.d\bar{r} = 0$.

(6) If $\int_C \bar{F}.d\bar{r}$ is independent of the path C then curl $\bar{F} = 0$.

Problems:

1. If $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$, evaluate $\int_C \vec{F} . d\vec{r}$, where C is the arc of the parabola $y=2x^2$ from (0, 0) to (1, 2).

Solution:

Given $\bar{F} = 3xy\vec{i} - y\vec{j}$

$$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\bar{k}$$

$$\overline{F}.d\overline{r} = 3xydx - y^2dy$$

Given $y = 2x^2$

$$dy = 4xdx$$

$$\therefore\ \overline{F}.d\overline{r} = 3x(2x^2)dx - (2x^2)^2(4xdx)$$

$$= (6x^3 - 16x^5)dx$$

$$\int_C \overline{F}.d\overline{r} = \int_0^1 (6x^3 - 16x^5)dx$$

$$= 6\left[\frac{x^4}{4}\right] - 16\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{6}{4}\ \frac{-16}{6}$$

$$= \frac{-7}{6}$$

2. If $\vec{F} = x^2\vec{\imath} + y^2\vec{\jmath}$, evaluate $\int_C\ \vec{F}\,.\,d\vec{r}$, along the straight liine y=x from (0, 0) to (1, 1).

Solution:

$$\vec{F}.d\vec{r} = \left(x^2\vec{i} + y^2\vec{j}\right)\cdot\left(dx\vec{i} + dy\right)$$
$$= x^2dx + y^2dy$$

Given $y = x$

$dy = dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(x^2dx + y^2dy\right)$$

$$= \int_0^1 x^2dx + x^2dx = 2\int_0^1 x^2dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

3. Find $\int_C\ \vec{F}\,.\,d\vec{r}$ for $\vec{F} = (x^2 + y^2)\vec{\imath} - 2xy\vec{\jmath}$ where C is the rectangle in the xoy plane bounded by x=0,y=0, x=a, y=b.

Solution:

Given $\vec{F}\left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}d\vec{r} = (x^2 + y^2)dx - 2xy\,dy$$

C is the rectangle OABC and C consists of four different paths.

OA (y = 0)

AB (x = a)

BC (y = b)

CO (x = 0)

$$\therefore \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along

$$\begin{array}{lll} OA, & y=0, & dy=0 \\ AB, & x=a, & dx=0 \\ BC, & y=b, & dy=0 \\ CO, & x=0, & dx=0 \end{array}$$

$$\therefore C\int_C \vec{F}.d\vec{r} = \int_{OA} x^2 dx \int_{AB} -2aydy + \int_{BC} \left(x^2+b^2\right)dx + \int_{CO} 0$$

$$= \int_0^a x^2 dx - 2a\int_0^b ydy + \int_a^0 \left(x^2+b^2\right)dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_o^b + \left[\frac{x^3}{3}+b^2x\right]_a^o$$

$$= \left(\frac{a^3}{3}-0\right) - 2a\left(\frac{b^2}{2}-0\right) + \left((0+0)-\left(\frac{a^3}{3}+ab^2\right)\right) = -2ab^2.$$

4. If $\vec{F} = (4xy - 3x^2z^2)\vec{\imath} + 2x^2\vec{\jmath} - 2x^3z\vec{k}$ check whether the integral $\int_C \vec{F}.d\vec{r}$ independent of the path C.

Solution:

Given:

$$\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\int_C \vec{F}.d\vec{r} = \int_C \left(4xy - 3x^2z^2\right)dx + \int_C 2x^2dy - \int_C 2x^3zdz$$

This integral is independent of path of integration if

$$\vec{F} = \nabla\phi \Rightarrow \nabla\times\vec{F} = 0$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy-3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left(0,0\right) - j\left(-6x^2z + 6x^2z\right) + \vec{k}\left(4x-4x\right)$$

$$= 0\vec{i} - 0\vec{i} - 0\vec{j} + 0\vec{k} = 0.$$

Hence the line integral is independent of path.

5. Find the work done in moving a particle in the force field $\vec{F} = (3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$ from t = 0 to t = 1 along the curve $x=2t^2$, $y=t$, $z= 4t^3$

Solution:

Work done = $\int_C \vec{F}\,.\,d\vec{r}$,

Given $\vec{F} = (3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$;

$dr = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$

$$\overline{F} \cdot d\overline{r} = 3x^2dx + (2xz - y)dy - zdz$$

Given $\quad x = 2t^2 \qquad y = t \qquad z = 4t^3$

$\qquad dx = 4tdt \qquad dy = dt \qquad dz = 12t^2dt$

$$\int_C \overline{F}.d\overline{r} = \int_0^1 \left[48t^5 + (16t^5 - t) - 48t^5\right]dt$$

$$= \int_0^1 (16t^5 - t)dt \quad = \left[16 \cdot \frac{t^6}{6} - \frac{t^2}{2}\right]_0^1 \quad = \frac{13}{6} \text{ Units}$$

6. Find the work done by the force $\vec{F} = y(3x^2y - z^2)\vec{\imath} + x(2x^2y - z^2)\vec{\jmath} - 2xyz\vec{k}$ when it moves a particle around a closed curve C.

Solution:

To evaluate the work done by a force, the equation of the path C and the terminal points must be given.

Since C is a closed curve and the particle moves around this curve completely, any point

(x_0, y_0, z_0) can be taken as the initial as well as the final point.

But the equation of C is not given. Hence we verify when the given force $\vec{F}$ is conservative,

i.e. irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\vec{i} - (-2yz + 2yz)\vec{j} + (6x^2y - 6x^2y + z^2 - z^2)\vec{k} \;=0$$

Since $\nabla \times \vec{F} = 0$

$\Rightarrow \vec{F}$ is irrotational

$$\Rightarrow \oint_C \bar{F} \cdot d\bar{r} = 0$$

SURFACE INTEGRAL

Introduction A surface integral is a definite integral taken over a surface. It can be thought of as the double integral analogue of the line integrand. Given the surface, one may integrate over its scalar field (i.e., functions which return scalars as value) and vector field ((i.e.) functions which return vectors as value). Surface integrals have applications in physics, particularly with the classical theory of electromagnetism. Various useful results for surface integrals can be derived using differential geometry and vector calculus, such as the divergence theorem and its generalization stokes theorem.

Consider any surface (planar, curved, closed or open) and let $\vec{F} = \vec{F}(x, y, z)$ be a vector point function, defined and continuous on a region S of the surface. Then $\iint_S \vec{F} \cdot d\vec{s}$ where ds denotes an element of the surface S is called the surface intgral of $\vec{F}$ over S.

Note:

(i) If S is a closed surface, the outer surface is usually chosen as the positive side

(ii) $\int_S \phi d\vec{s}$ and $\int_S \vec{F} \times d\vec{s}$, where ϕ is a scalar point function, are also surface integrals.

(iii) To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int_S \vec{F} \cdot d\vec{s}$ is also denoted as $\iint_S \vec{F} \cdot d\vec{s}$.

(iv) The area of the region S is $\iint_S ds$.

Problems:

7. Obtain $\int_S \vec{F}.\hat{n}d\vec{S}$, where $\vec{F} = (x^2 + y^2)\vec{\imath} - 2x\vec{\jmath} + 2yz\vec{k}$ over the surface of the plane

2x + y + 2z = 6 in the first octant.

Solution:

Let the given surface be $\phi = 2x + y + 2z - 6$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

Let S' be the projection of S in the XOY plane

$$\int_S \vec{F}.\hat{n}ds = \iint_{S'} \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}\cdot\vec{k}|}$$

$$\vec{F}.\hat{n} = ((x^2+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k})\cdot\frac{2\hat{i}+\hat{j}+2\hat{k}}{3} \qquad = \frac{2(x^2+y^2)-2x+4yz}{3}$$

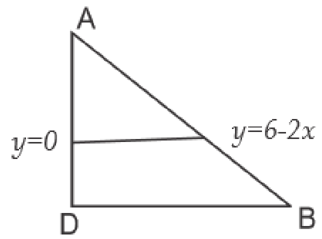
Since $z = \dfrac{6-2x-y}{2}$

$$= \frac{2}{3}\left(x^2+y^2-x+2y\left(\frac{6-2x-y}{2}\right)\right)$$

$$= \frac{2}{3}(x^2+y^2-x+6y-2xy-y^2)$$

$$= \frac{2}{3}(x^2-2xy-x+6y)$$

Since the equation of the line AB is 2x+y=6 (or) y=6-2x. In the region S' as x varies from 0 to 3, y varies from 0 to 6-2x.



$$\therefore \int_S \vec{F}\cdot\hat{n}d\vec{s} = \iint_{S'} \frac{2}{3}(x^2-2xy-x+6y)\frac{dxdy}{\frac{2}{3}}$$

$$= \int_0^3\int_0^{6-2x}(x^2-2xy-x+6y)dxdy$$

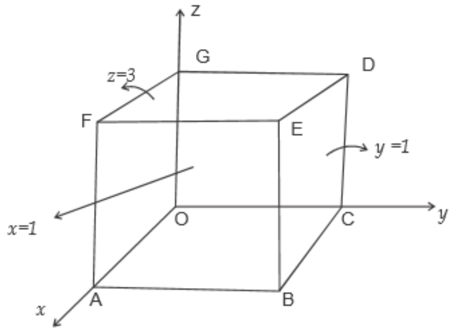
$$= \int_0^3 (x^2y-xy^2-xy+3y^2)_{y=0}^{y=6-2x}\ dx$$

$$= \int_0^3 (108-114x+44x^2-6x^3)dx$$

$$= \frac{1197}{2}$$

8. If $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the rectangle parallelepiped bounded x = 0, y = 0, z = 0, x = 1, y = 2, z = 3 calculate $\iint_S \vec{F}.\hat{n}d\vec{S}$

Solution:



There are six faces of the parallelepiped and we calculate the integral over each of these faces. We denote the values of $\vec{F}$ on these faces by $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_6$

Face	$\hat{n}$	Equation	$\overrightarrow{ds}$	$\vec{F}$
ABEF	$\hat{i}$	$x=1$	dydz	$\vec{F}_1 = 2y\hat{i} + yz^2\hat{j} + z\hat{k}$
COGD	$-\hat{i}$	$x=0$	dydz	$\vec{F}_2 = yz^2\hat{j}$
BCDE	$\hat{j}$	$y=2$	dzdx	$\vec{F}_3 = 4x\hat{i} + 2z^2\hat{j} + xz\hat{k}$
GOAE	$-\hat{j}$	$y=0$	dzdx	$\vec{F}_4 = xz\hat{k}$
EDGE	$\hat{k}$	$z=3$	dxdy	$\vec{F}_5 = 2xy\hat{i} + 9y\hat{j} + 3x\hat{k}$
AOCB	$-\hat{k}$	$z=0$	dxdy	$\vec{F}_6 = 2xy\hat{i}$

$$\therefore \int \vec{F}\cdot ds = \iint_{ABEF} \vec{F}_1.\hat{n}ds + \iint_{COGD} \vec{F}_2.\hat{n}ds + \iint_{BCDE} \vec{F}_3.\hat{n}ds$$

$$+ \iint_{GOAE} \vec{F}_4.\hat{n}ds + \iint_{EDGF} \vec{F}_5.\hat{n}ds + \iint_{AOCB} \vec{F}_6.\hat{n}ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \qquad \text{(say)} \qquad \_\_\_\_\_\_\_\_(1)$$

Consider

$$I_1 = \iint_{ABEF} \vec{F}_1.\hat{n}ds$$

$$\vec{F}_1 \cdot \hat{n} = (2y\hat{i} + yz^2\hat{j} + z\hat{k})\cdot\hat{i} \quad \mathbf{=2y}$$

On the surface ABEF, z varies from 0 to 3 and y varies from 0 to 2

$$\therefore I_1 = \int_{y=0}^{2}\int_{z=0}^{3} 2y\,dy\,dz \qquad = 2\int_0^2 y\,dy\,(z)_0^3$$

$$= 2\times 3\int_0^2 y\,dy \quad = 6\left(\frac{y^2}{2}\right)_0^2 \qquad = 12$$

$$\text{Consider} \therefore I_2 = \iint_{COGD} \vec{F}_2.\hat{n}ds$$

$$\vec{F}_2 \cdot \hat{n} = yz^2\vec{j} \cdot (-\vec{i})$$
$$= 0$$

On the surface COGD, z varies from 0 to 3 and y varies from 0 to 2.

$$\therefore I_2 = \int_{y=0}^{2} \int_{z=0}^{3} 0 \ dydz$$
$$= 0$$

$$\text{Consider} \therefore I_3 = \iint_{BCDE} \vec{F}_3.\hat{n}ds$$

$$\vec{F}_3 \cdot \hat{n} = (4x\hat{i} + 2z^2\hat{j} + xz\hat{k}) \cdot \hat{j}$$
$$= 2z^2$$

On the surface BCDE z varies from 0 to 3 and x varies from 0 to 1

$$\therefore I_3 = \int_{x=0}^{1} \int_{z=0}^{3} 2x^2 dxdz$$

$$= \int_{x=0}^{1} \left(\frac{2z^3}{3} \right)_0^3 dx$$

$$= \frac{2}{3} \int_{x=0}^{1} (z^3)_0^3 dx$$

$$= 18\int_0^1 dx$$

$$= 18$$

$$\text{Consider} \qquad I_4 = \iint_{GOAE} \vec{F}_4.\hat{n}ds$$

$$\vec{F}_4 \cdot \hat{n} = xz\hat{k} \cdot (-\hat{j})$$
$$= 0$$

On the surface GOAE, z varies from 0 to 3, x varies from 0 to 1.

$$\therefore I_4 = \int_{x=0}^{1} \int_{z=0}^{3} 0 \cdot dxdz$$
$$= 0$$

Consider $I_5 = \iint\limits_{EDGF} \vec{F}_5 . \hat{n} ds$

$$\vec{F}_5 \cdot \hat{n} = (2xy\hat{i} + 9y\hat{j} + 3x\hat{k}) \cdot \hat{k}$$
$$= 3x$$

On the surface EDGE, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_5 = \int_{x=0}^{1} \int_{z=0}^{3} 3x dx dy$$

$$= 3\int_0^1 (y)_0^2 x dx$$

$$= 3 \times 2 \int_0^1 x dx$$

$$= 6\left(\frac{x^2}{2}\right)_0^1$$

$$= 3$$

Consider $I_6 = \iint\limits_{AOCB} \vec{F}_6 . \hat{n} ds$

$$\vec{F}_6 \cdot \hat{n} = (2xy\hat{i}) \cdot (-\hat{k})$$
$$= 0$$

On the surface AOCB, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_6 = \int_{x=0}^{1} \int_{y=0}^{2} 0 dx dy$$
$$= 0$$

$$\therefore (1) \Rightarrow \int_S \vec{F} \cdot d\vec{s} = 12 + 0 + 18 + 0 + 3 + 0$$
$$= 33$$

Volume Integral

In multivariable calculus, a volume integral refers to an integral over a 3-dimensional domain. Let V denote the volume enclosed by some closed surfaces and $\overrightarrow{F}$, a vector function defined throughout V. Then $\iiint_V \vec{F} . d\vec{V}$ where d$\vec{V}$ denotes an element of the volume V, is called

the volume integral $\vec{F}$ over V.

Remark

A volume integral is a triple integral of the constant function 1 which gives the volume of the region D (ie) the integral $Vol(D) = \iiint_V dxdydz$A triple integral within a region D in R^3 of a function f(x ,y , z) is usually written as$\iiint_D f(x,y,z)dxdydz$

Problems

1. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_D \vec{F} \cdot dV$ where V is the region bounded by the surfaces $x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2$

Solution:

$$\iiint \vec{F} \cdot dV = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})dxdydz$$

$$= \iiint \vec{F} \cdot dV = \int_{y=0}^{4} \int_{x=0}^{2} \int_{z=x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})dzdydx$$

$$= \int_0^2 \int_0^4 (z^2\hat{i} - xz\hat{j} + yz\hat{k})_{x^2}^{2} dydx$$

$$= \int_0^2 \int_0^4 (4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k})dydx$$

$$= \int_0^2 \left(4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k} \right)_0^4 dx$$

$$= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right)dx$$

$$= \left(16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k} \right)_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} \quad = \frac{32}{5}\hat{i} + \frac{32\hat{k}}{3} \quad = \frac{32}{15}(3\hat{i} + 5\hat{j})$$

2) Evaluate $\iiint_V (\nabla\cdot\vec{F})dV$ if $\vec{F} = x^2\hat{i} + y^2\hat{i} + z^2\hat{i}$ and if V is the volume of the region enclosed by the cube $\theta \le x, y, z \le 1$

Solution

$$\iiint_V (\nabla\cdot\vec{F})d\vec{V} = 2\iiint_V (x+y+z)dV$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x+y+z)dzdydx$$

$$= 2\int_0^1\int_0^1\left(xz + yz + \frac{z^2}{2}\right)_0^1 dydx$$

$$= 2\int_0^1\int_0^1\left(x + y + \frac{1}{2}\right)dydx \qquad = 2\int_0^1\left(xy + \frac{y^2}{2} + \frac{y}{2}\right)_0^1 dx$$

$$= 2\int_0^1\left(xy + \frac{1}{2} + \frac{1}{2}\right)_0^1 dx = 2\left(\frac{x^2}{2} + \frac{1}{2}x + \frac{1}{2}x\right)_0^1$$

$$= 2\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)$$

$$= 3$$

Problem 3 If S is any closed surface enclosing a volume V and $\vec{r}$ is the position vector of a point, prove $\iint_S (\vec{r}.\hat{n})ds = 3V$

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

By Gauss divergence theorem

$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V \nabla\cdot\vec{F}dV$ Here $\vec{F} = \nabla\vec{r}$

$\iint_S \vec{r}\cdot\hat{n}ds = \iiint_V \nabla\cdot\vec{r}dV$

$$= \iiint_V\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x\vec{i} + y\vec{j} + z\vec{k}\right)dV$$

$$= \iiint(1+1+1)dV$$

$\iint_S \vec{r}\cdot\hat{n}ds = 3V.$

Gauss Divergence Theorem

If $\vec{F}$ be a vector point function having continuous partial derivation in the region bounded by a closed surface S, then $\iiint_V (\nabla\cdot\vec{F})dV = \iint_S \vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit outward normal at any point of the surface.

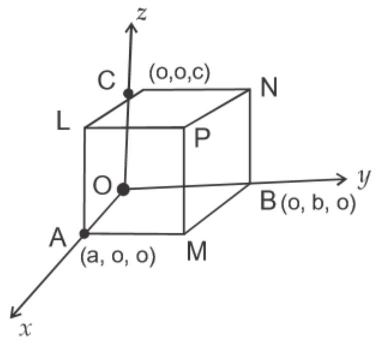
Problems

1. Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{\imath} - (y^2 - xz)\vec{\jmath} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solution:
For verification of divergence theorem, we shall evaluate the volume and surface separately and show that they are equal.
Given $\vec{F} = (x^2 - yz)\vec{i} - (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot F = divF = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \vec{F}$$

$= 2x + 2y + 2z = 2(x + y + z)$
$dV = dxdydz$ or $dV = dzdydx$
x varies from 0 to a, y varies from 0 to b, z varies from 0 to c



$$\therefore \iiint_V (\Delta \cdot \vec{F}) dV = \int_0^a \int_0^b \int_0^c 2(x + y + z) dzdydx$$

$$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right] dydx$$

$$= 2c\int_0^a \int_0^b \left[x + y + \frac{c}{2} \right] dydx$$

$$= 2c\int_0^a \left[xy + \frac{y^2}{2} + \frac{c}{2}y \right]_0^b dx$$

$$= 2bc\int_0^a \left[x + \frac{b}{2} + \frac{c}{2} \right] dx$$

$$= 2bc\left[\frac{x^2}{2} + \frac{bx}{2} + \frac{cx}{2} \right]_0^a = abc[a + b + c]$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallopiped into 6 parts.
S_1 = face OAMB; S_2 = face CLPN; S_3 = face OBNC;
S_4 = face AMPL; S_5 = face OALC; S_6 = face BNPM

$$\therefore \iint_C \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds$$

$$+ \iint_{S_4} \vec{F}.\hat{n}ds + \iint_{S_5} \vec{F}.\hat{n}ds + \iint_{S_6} \vec{F}.\hat{n}ds$$

Face S_1 : z = 0; ds = dxdy; $\hat{n} = -\vec{k}$

$$\vec{F} = x^2\vec{i} + y^2\vec{j} - xy\vec{k}$$

$$\vec{F}\cdot\hat{n} = xy$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}ds = \int_0^a\int_0^b xydxdy = \left[\frac{x^2}{2}\right]_0^a\left[\frac{x^2}{2}\right]_0^b \qquad = \frac{1}{4}a^2b^2$$

Face S_2 : $z = c; \hat{n} = \vec{k}; ds = dxdy$

$$\vec{F} = (x^2 - yc)\vec{i} + (y^2 - (x))\vec{j} + (c^2 - xy)\vec{k}$$

$$\vec{F}\cdot\hat{n} = (c^2 - xy)$$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}ds = \int_0^a\int_0^b (c^2 - xy)dydx$$

$$= \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx$$

$$= \int_0^a \left[c^2b - \frac{xb^2}{2}\right]dx$$

$$= \int_0^a \left[c^2 - \frac{xb}{2}\right]dx$$

$$= b\int_0^a \left[c^2x - \frac{b}{4}x^2\right]_0^b = ab\left[c^2 - \frac{ab}{4}\right]$$

Face S_3 : $\hat{n} = -\vec{i}; ds = dydz; x = 0$

$$\vec{F} = -yz\vec{i} + y^2\vec{j} + z^3\vec{k} \qquad \vec{F}\cdot\hat{n} = yz$$

$$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}ds = \int_0^b\int_0^c yzdydz$$

$$= \left[\frac{y^2}{2}\right]_0^b\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{1}{4}b^2c^2$$

Face S_4 : $x = 0; \hat{n} = \vec{i}; ds = dydz$

$$\vec{F} = (a^2 - yz)\vec{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$$

$$\vec{F}\cdot\hat{n} = a^2 - yz$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}ds = \int_0^c\int_0^b (a^2 - yz)dydz$$

$$= \int_0^c \left[a^2 y - \frac{y^2}{2} z \right]_0^b dz$$

$$= \int_0^c \left[a^2 b - \frac{b^2}{2} z \right] dz$$

$$= \left[a^2 bz - \frac{b^2}{4} z^2 \right]_0^c$$

$$= bc\left[a^2 - \frac{1}{4} bc \right]$$

Face S_5 : $y = 0; \hat{n} = -\vec{j}; ds = dzdx$

$$\vec{F} = x^2\vec{i} - yz\vec{j} + z^2\vec{k} \qquad \vec{F}\cdot\hat{n} = zx$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}ds = \int_0^a\int_0^c zxdzdx = \int_0^a xdx \cdot \int_0^c zdz$$

$$= \frac{1}{2}a^2 \cdot \frac{1}{2}c^2 = \frac{1}{4}a^2c^2$$

Face S_6 : $y = b; \hat{n} = \vec{j}; ds = dzdx$

$$\vec{F} = (x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - bx)\vec{k}$$

$$\vec{F}\cdot\hat{n} = b^2 - zx$$

$$= \int_0^a \left(b^2 z - \frac{z^2}{2} x \right)_0^c dx$$

$$= \int_0^a \left(b^2 c - \frac{c^2}{2} x \right) dx$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}ds = \int_0^a\int_0^c (b^2 - zx)dzdx$$

$$= \left(b^2 cx - \frac{c^2}{2} x^2 \right)_0^a$$

$$= ac\left(b^2 - \frac{1}{4} ac \right)$$

$$\iint_S \vec{F}\cdot\hat{n}ds = abc^2 + ab^2c + a^2bc$$

$$= abc(a+b+c)$$

$$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V dV$$

Hence Gauss divergence theorem is verified.

2. Using divergence theorem evaluate $\iint_S \nabla r^2.\hat{n}ds$ where S in a closed surface.

Solution:

Let $\vec{F} = \nabla r^2$, where $\vec{r}$ = x$\vec{\imath}$ + y$\vec{\jmath}$ + z$\vec{k}$ & $r = |r| = \sqrt{x^2+y^2+z^2}$

By Gauss Divergence Theorem,

$$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V (\nabla\cdot\vec{F})dV \quad = \iiint_V \nabla\cdot(\nabla r^2)dV$$

$$= \iiint_V \nabla^2 r^2 dV = \iiint_V \left(\sum\frac{\partial^2}{\partial x^2}\right)(x^2+y^2+z^2)dV$$

$$= \iiint_V (2+2+2)dV$$

$$= 6\iiint_V dV$$

= 6 x volume of closed surfaces.

Stoke's Theorem

If S be an open surface bounded by a closed curve C and $\vec{F}$ be a continuous and differentiable vector function then $\int_C \vec{F}.d\vec{r} = \iint_S Curl\vec{F}.\hat{n}ds$, where $\hat{n}$ is the unit outward normal at any point of the surfaces.

Problems:

3. Verify stoke's theorem for the vector field $\overrightarrow{F} = (x^2-y^2)\vec{\imath} + 2xy\vec{\jmath}$, in the rectangular region in the xy plane bounded by the lines x=0, x=a, y=0, y=b.

Solution:

By stoke's theorem, $\int_C \vec{F}.d\vec{r} = \iint_S Curl\vec{F}.\hat{n}ds$

To find $\int_C \vec{F}.d\vec{r}$

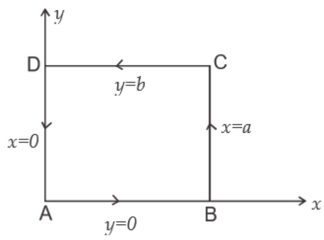
$$\int_C \vec{F}\cdot dr = \int_{AB}\vec{F}\cdot\overrightarrow{dr} + \int_{BC}\vec{F}\cdot\overrightarrow{dr} + \int_{CD}\vec{F}\cdot\overrightarrow{dr} + \int_{DA}\vec{F}\cdot\overrightarrow{dr}$$

Now $\quad \vec{F}\cdot dr = [(x^2-y^2)\vec{i} + 2xy\vec{j}]\cdot[dx\vec{i} + dy\vec{j} + dz\vec{k}]$

$$= (x^2-y^2)dx + 2xy\,dy$$

Along AB: y = 0, d y= 0;

$$\int_{AB} \vec{F}\cdot\overrightarrow{dr} = \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$



Along BC: x = a, dx = 0;

$$\int_{BC} \vec{F}\cdot\overrightarrow{dr} = \int_0^b 2ay dy = 2a\left(\frac{y^2}{2}\right)_0^b = ab^2$$

Along CD: y=a, dx = 0;

$$\int_{CD} \vec{F}\cdot\overrightarrow{dr} = \int_b^0 (x^2-b^2)dx$$

$$= \left(\frac{x^3}{3} - xb^2\right)_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$= \frac{-a^3}{3} + ab^2$$

Along DA: x=0, dx = 0;

$$\int_{DA} \vec{F}\cdot\overrightarrow{dr} = 0$$

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_(1)}$$

To find $\iint_S \ Curl\vec{F}.\hat{n}ds$

$$\text{Now, } curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}$$

Surface S is the rectangle ABCD in xy plane.

$$\hat{n} = \vec{k} \text{ and } \quad ds = \frac{dxdy}{|\hat{n}\cdot\vec{k}|} = \frac{dxdy}{|\vec{k}\cdot\vec{k}|} = dxdy$$

$$\iint_S curl\vec{F}\cdot\hat{n}ds = \iint_S 4y\vec{k}\cdot\vec{k}\ dxdy$$

$$= \int_0^a\int_0^b 4y\ dxdy$$

$$= \int_0^a 4\left(\frac{y^2}{2}\right)_0^b dx$$

$$= 2b^2(x)_0^a$$

$$= 2ab^2 \qquad \text{\_\_\_\_\_\_\_\_\_(2)}$$

From equation (1) and (2).

$$\iint_S curl\vec{F}\cdot\hat{n}ds = \int_C \vec{F}\cdot\overrightarrow{dr}$$

Hence stoke's theorem is verified.

4. Verify Stoke's theorem for F$\vec{F} = (y-z)\vec{\imath} + yz\vec{\jmath} - xz\vec{k}$, where S is the surface bounded by the planes and 1 x=0, x = 1, y = 0, y = 1, z = 0, z = 1 above the XOY plane.

Solution:

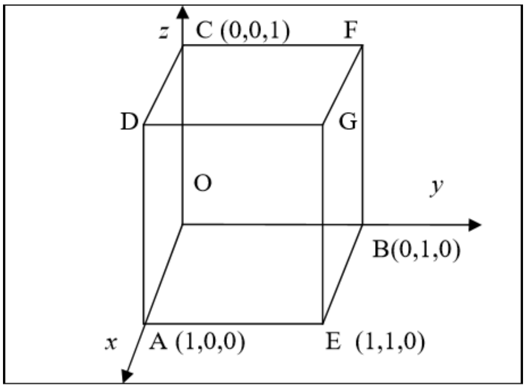
Stoke's theorem is $\int_C \vec{F}.d\vec{r} = \iint_S \nabla\times\vec{F}.\hat{n}ds$

$$\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

$$\iint_S \nabla\times\vec{F}.\hat{n}\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$



$\iint_{S_6}$ is not applicable, since the given condition is above the XOY plane.

$$\iint_{S_1} = \iint_{AEGD}\left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right].\vec{i}dydz$$

$$= \iint_{AEGD} -y\,dy\,dz$$

$$= \int_0^1 \int_0^1 -y\,dy\,dz = \int_0^1 \left[-\frac{y^2}{2}\right]_0^1 dz$$

$$= -\frac{1}{2}(z)_0^1 = -\frac{1}{2}$$

$$\iint_{S_3} = \iint_{EBFG} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right]\vec{j}dxdz$$

$$= \int_0^1 \int_0^1 (z-1)\ dx\,dz = \int_0^1 (xz - x)_0^1\,dz$$

$$= \left(\frac{z^2}{2} - z\right)_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\iint_{S_4} = \iint_{OADC} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right]\left(-\vec{j}\ \right)dxdz$$

$$= \int_0^1 \int_0^1 (-z+1)\ dx\,dz$$

$$= \int_0^1 (-xz + x)_0^1 = \int_0^1 (-z+1)\,dz$$

$$= \left(\frac{-z^2}{2} + z\right)_0^1 = \frac{-1}{2} + 1 = \frac{1}{2}$$

$$\iint_{S_5} = \iint_{DGFC} \left(-y\vec{i} + (z-1)\vec{j} - \vec{k}\right).\vec{k}dxdy$$

$$= \int_0^1 \int_0^1 (-1)dxdy = \int_0^1 (-x)_0^1\,dy$$

$$= \int_0^1 (-1)\ dy = (-y)_0^1 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

$$= -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - 1 = -1$$

$$L.H.S = \int_C \overrightarrow{F}.\overrightarrow{dr} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$\int_{OA} = \int_{OA} (y-z)\,dx + yzdy - xzdz$$

$$= \int_{OA} 0 = 0 \qquad [\because y=0,\ z=0,\ dy=0,\ dz=0]$$

$$\int_{AE} = \int_{AE} (y-z)\,dx + yzdy - xzdz$$

$$= \int_{AE} 0 = 0 \qquad [\because x=1,\ z=0,\ dx=0,\ dz=0]$$

$$\int_{EB} = \int_{EB} (y-z)\,dx + yzdy - xzdz$$

$$= \int_1^0 1\,dx \qquad (y=1,\ z=0,)$$

$$= [x]_1^0 = 0 - 1 = -1$$

$$\int_{BO} = \int_{BO} (y-z)\,dx + yzdy - xzdz$$

$$= \int_{BO} 0 = 0 \qquad [x=0,\ z=0]$$

$$= 0$$

$$\therefore \int_C = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$= 0 + 0 - 1 + 0 = -1$$

Therefore L.HS = R.HS. Hence Stoke's theorem is verified.

5. Evaluate by stoke's theorem $\int (e^x\,dx + 2ydy - dz)$, where C is the curve $x^2 + y^2 = 4$, $z = 2$

Solution:

By Stoke's Theorem $\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

Here $\vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

$$\therefore \int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$$

$$\int_C (e^x dx + 2y dy - dz) = 0$$

Green's Theorem

If C is a regular closed curve in the xy-plane and R be the region bounded by C, then

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

Where $F_1(x, y)$ and $F_2(x, y)$ are continuously differentiable functions inside and on C.

Problems

6. Verify Green's Theorem in a plane for $\int_C (x^2(1+y)dx + (y^3+x^3)dy)$ where C is the square bounded by $x = \pm a, y = \pm a$

Solution:

Let $P = x^2(1+y)$

$$\frac{\partial P}{\partial y} = x^2$$

$$Q = y^3 + x^3$$

$$\frac{\partial Q}{\partial x} = 3x^2$$

By green's theorem in a plane

$$\int_C (Pdx + Qdy) = \int_C\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Now $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

$$= \int_{-a}^{a}\int_{-a}^{a} \left(3x^2 - x^2\right) dx\, dy$$

$$= (y)_{-a}^{a} \left(\frac{2x^3}{3} \right)_{-a}^{a}$$

$$= (a+a)\frac{2}{3}\left(a^3 + a^3\right) \qquad = \frac{8a^4}{3} - (1)$$

Now $\int_C (Pdx + Qdy) = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

Along AB, $y = -a$, $dy = 0$

X varies from $-a$ to a

$$\int_{AB} (Pdx + Qdy) = \int_{-a}^{a} \left(x^2 (1+y) dx + \left(x^3 + y^3 \right) dy \right)$$

$$= \int_{-a}^{a} x^2 (1-a) dx + 0$$

$$= (1-a) \left[\frac{x^3}{3} \right]_{-a}^{a}$$

$$= \left(\frac{1-a}{3} \right) \left(a^3 + a^3 \right) = \frac{2a^3}{3} - \frac{2a^4}{3}$$

Along BC

$x = a$, $dx = 0$

Y varies from $= -a$ to a

$$\int_{BC} (Pdx + Qdy) = \int_{-a}^{a} \left(x^2 (1+y) dx + \left(x^3 + y^3 \right) dy \right)$$

$$= \int_{-a}^{a} (a^3 + y^3) dy$$

$$= \left[a^3 y + \frac{y^4}{4} \right]_{-a}^{a}$$

$$= \left(a^4 + \frac{a^4}{4} \right) - \left(-a^4 + \frac{a^4}{4} \right) = 2a^4$$

Along CD

$y = a$, $dy = 0$

X varies from a to $-a$

$$\int_{CD} (Pdx + Qdy) = \int_{a}^{-a} \left(x^2 (1+y) dx + \left(x^3 + y^3 \right) dy \right)$$

$$= \int_{a}^{-a} x^2 (1+a) dx$$

$$=(1+a)\left(\frac{x^3}{3}\right)_a^{-a} dx$$

$$=(1+a)\left[\frac{-a^3-a^3}{3}\right]$$

$$=-\frac{2a^3}{3}-\frac{2a^4}{3}$$

Along DA,

$x=-a,\ dx=0$

Y Varies from a to $-a$

$$\int_{DA}(Pdx+Qdy)=\int_a^{-a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{+a}^{-a}\left(a^2(1+y)dx+\left(y^3-a^3\right)dy\right)$$

$$=\left[\frac{y^4}{4}-a^3y\right]_a^a$$

$$=\left(\frac{a^4}{4}+a^4\right)-\left(\frac{a^4}{4}-a^4\right)=2a^4$$

$$\int_C(Pdx+Qdy)=\frac{2a^3}{3}-\frac{2a^4}{3}+2a^4-\frac{2a^3}{3}-\frac{2a^4}{3}+2a^4$$

$$=4a^4-\frac{4}{3}a^4$$

$$=\frac{8a^4}{3}\ ......(2)$$

From (1) and (2),

$$\int_C(Pdx+Qdy)=\int\int_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\frac{8a^4}{3}.$$

Hence Green’s Theorem is verified.

7. By the use of Green’s theorem, show that area bounded by a simple closed curve C is given by$\frac{1}{2}\int\ \ xdy-ydy$. Hence find the area of an ellipse.

Solution:

By Green’s theorem in planes,

$$\int_C(F_1dx+F_2dy)=\iint_R\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy$$

Put $\quad F_1 = -y$ and $F_2 = x$

$$\frac{\partial F_1}{\partial y} = -1 \text{ and } \frac{\partial F_2}{\partial x} = 1$$

$$\int_C -ydx + xdy = \iint_R (1+1)dxdy$$

$$= 2\iint_R dxdy$$

$$= 2A$$

Where A is the required area.

$$\therefore A = \frac{1}{2}\int_C (xdy - ydx)$$

Any point (x,y) on the ellipse is given by,

$x = a\cos\theta,\qquad y = b\sin\theta\qquad$ where θ is the parameter.

$dx = -a\sin\theta d\theta\qquad dy = b\cos\theta d\theta$

$\therefore$ Area of the ellipse $= \frac{1}{2}\int_C (xdy - ydx)$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta(b\cos\theta d\theta) - b\sin\theta(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)d\theta$$

$$= \frac{1}{2}(ab)\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} d\theta$$

$$= \frac{ab}{2}(\theta)_0^{2\pi} \qquad = \pi ab$$

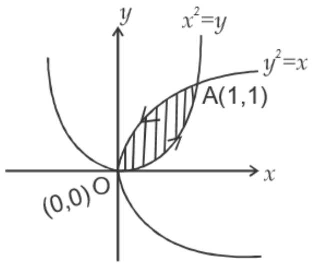
$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

8. Verify Green's theorem in the plane for $\int_C \left((3x^2 - 8y^2)dx + (4y - 6xy)dy\right)$, where C is the boundary of the region defined by $y = \sqrt{x}\,, y = x^2$

Solution: The Green's theorem is

$$\int_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

Here $F_1 = 3x^2 - 8y^2$ $\qquad$ $F_2 = 4y - 6xy$



Given C is $y = \sqrt{x}$, $\qquad y = x^2$

(i.e) $y^2 = x$, $\qquad y = x^2$

$$\therefore \int_C F_1 dx + F_2 dy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= I_1 + I_2 \qquad \text{\_\_\_\_\_\_\_\_(1)}$$

$$I_1 = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along OA, $\quad y = x^2$

$dy = 2xdx$

x varies from 0 to 1

$$\therefore I_1 = \int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)(2xdx)$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)dx$$

$$= (x^3 + 2x^4 - 4x^5)_0^1$$

$$= 1 + 2 - 4$$

$$\therefore I_1 = -1$$

Along AO, $\quad x = y^2$

$dx = 2ydy$

y varies from 1 to 0

$$I_2 = \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= \left[6\left(\frac{y^6}{6}\right) - 22\left(\frac{y^4}{4}\right) + 4\left(\frac{y^2}{2}\right)\right]_1^0 \qquad = -1 + \frac{11}{2} - 2$$

$$\therefore I_2 = \frac{5}{2}$$

$\therefore$ from (1),

$$\int_C F_1 dx + F_2 dy = I_1 + I_2$$

$$= -1 + \frac{5}{2}$$

$$= \frac{3}{2} \qquad \text{\_\_\_\_\_\_\_\_(2)}$$

Now, $\qquad F_1 = 3x^2 - 8y^2$, $F_2 = 4y - 6xy$

$$\frac{\partial F_1}{\partial y} = -16y, \quad \frac{\partial F_2}{\partial x} = -6y$$

$$\therefore \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} (-6y + 16y) dxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} 10y\, dxdy$$

$$= 10 \int_{y=0}^{1} y(x)_{x=y^2}^{\sqrt{y}} dy$$

$$= 10 \int_{y=0}^{1} y\left(\sqrt{y} - y^2\right) dy$$

$$= 10 \int_{y=0}^{1} \left(y^{\frac{3}{2}} - y^3\right) dy$$

$$=10\left[\frac{y^{5/2}}{\frac{5}{2}}-\frac{y^4}{4}\right]_{y=0}^{1}$$

$$=10\left[\frac{2}{5}-\frac{1}{4}\right]$$

$$=10\left(\frac{3}{20}\right)=\frac{3}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_(3)}$$

From (2) and (3), we see that

$$\int_C F_1dx+F_2dy=\iint_R\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy$$

(i.e) Green's theorem is verified.