SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – I –THEORY OF EQUATIONS- SMT1302

THEORY OF EQUATIONS

1.0 Introduction

In this module, we will study about polynomial functions and various methods to find out the roots of polynomial equations. 'Solving equations' was an important problem from the beginning of study of Mathematics itself. The notion of complex numbers was first introduced because equations like $x^2 + 1 = 0$ has no solution in the set of real numbers. The "fundamental theorem of algebra" which states that every polynomial of degree ≥ 1 has at least one zero was first proved by the famous German Mathematician Karl Fredrich Gauss. We shall look at polynomials in detail and will discuss various methods for solving polynomial equations.

1.1. Polynomial Functions

Definition:

A function defined by

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, *where* $a_o \neq 0$, n is a non negative integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of **degree** n in x. Then numbers $a_o, a_1, ..., a_n$ are called the **coefficients** of f.

If α is a complex number such that f(α)= 0, then α is called **zero** of the polynomial.

1.1.1 Theorem (Fundamental Theorem of Algebra)

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark:

Fundamental theorem of algebra says that, if f(x) = $a_0x^n + a_1x^{n-1} + ... + a_n$,

where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result.

1.1.2 Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, *where* $a_o \neq 0$, be a polynomial of degree n $\geq$ 1.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of f(x).

Therefore, we can write:

f(x) = $(x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree n - 1.

If n -1 $\geq$ 1, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.

Therefore, $f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$ *where* $Q_2(x)$ is a polynomial function of degree n - 2.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2).....(x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function of degree n - n = 0, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x - \alpha_1)(x - \alpha_2)....(x - \alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, ...\alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2 ...., \alpha_n$.

Note:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n; a_o \neq 0$ be an n^{th} degree polynomial in x.

Then, $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

Hence every polynomial equation of degree n has n and only n roots.

Solved Problems

1. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given $2 + i\sqrt{3}$ is a root.

Solution :

Given that $2 + i\sqrt{3}$ is a root of $x^4 - 4x^2 + 8x + 35 = 0;$ since complex roots occurs in conjugate pairs $2 - i\sqrt{3}$ is also a root of it.

$\Rightarrow [x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = (x - 2)^2 + 3 = x^2 - 4x + 7$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 + 4x + 5$.

The roots of $x^2 + 4x + 5 = 0$ are given by $\frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$.

Hence the roots of the given polynomial are $2 + i\sqrt{3},\ 2 - i\sqrt{3},\ -2 + i$ *and* $-2 - i$.

2. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that one of the roots is $1 - \sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1 + \sqrt{5}$ is also a root of the given polynomial.

$\Rightarrow [x - (1 - \sqrt{5})][x - (1 + \sqrt{5})] = (x - 1)^2 - 5 = x^2 - 2x - 4$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2$.

Also, $x^2 - 3x + 2 = (x - 2)(x - 1)$

Thus the roots of the given polynomial equation are $1 + \sqrt{5}, 1 - \sqrt{5}, 1, 2$.

3. Find a polynomial equation of the lowest degree with rational coefficients having $\sqrt{3}$ and $1 - 2i$ as two of its roots.

Solution:

Since quadratic surds occur in pairs as roots, $-\sqrt{3}$ is also a root.

Since complex roots occur in conjugate pairs, $1 + 2i$ is also a root of the required polynomial equation. Therefore the desired equation is given by

$$(x - \sqrt{3})(x + \sqrt{3})(x - (1 - 2i)\big(x - (1 + 2i)\big) = 0$$

$$\text{i.e.,}\quad x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$$

4. Solve $4x^5 + x^3 + x^2 - 3x + 1 = 0$, given that it has rational roots.

Solution:

Let $f(x) = 4x^5 + x^3 + x^2 - 3x + 1$.

By theorem (1.1.5.), any rational root p/q (in its lowest terms) must satisfy the condition that, p is divisor of 1 and q is positive divisor of 4.

So the possible rational roots are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$.

Note that $f(-1) = 0$, $f(\frac{1}{2}) = 0$. But $f(1) \neq 0$, $f(-\frac{1}{2}) \neq 0$, $f(\frac{1}{4}) \neq 0$ and $f(-1/4) \neq 0$.

Since $f(-1) = 0$ and $f(\frac{1}{2}) = 0$, we see that $(x + 1)$ and $(x - \frac{1}{2})$ are factors of the given polynomial. Also by factorizing , we find that

$$f(x) = (x - \tfrac{1}{2})(x + 1)(4x^3 - 2x^2 + 4x - 2)$$

Note that $x = \frac{1}{2}$ is a root of the third factor, if we divide $4x^3 - 2x^2 + 4x - 2$ by $x - \frac{1}{2}$, we obtain

$$f(x) = (x - \tfrac{1}{2})^2 (x + 1)(4x^2 + 4)$$
$$= 4(x - \tfrac{1}{2})^2 (x + 1)(x^2 + 1)$$

Hence the roots of $f(x) = 0$, are $\frac{1}{2}, \frac{1}{2}, -1, \pm i$.

5. Solve $x^3 - x^2 - 8x + 12 = 0$, given that has a double root.

Solution:

Let $f(x) = x^3 - x^2 - 8x + 12$

Differentiating, we obtain:

$$f^1(x) = 3x^2 - 2x - 8.$$

Since the multiple roots of $f(x) = 0$ are also the roots of $f^1(x) = 0$, the product of the factors corresponding to these roots will be the g.c.d of $f(x)$ and $f^1(x)$. Let us find the g.c.d of $f(x)$ and $f^1(x)$.

$3x$	$3x^2 - 2x - 8$		$x^3 - x^2 - 8x + 12$	
	$3x^2 - 6x$		3	
4	$4x - 8$		$3x^3 - 3x^2 - 24x + 36$	x
	$4x - 8$		$3x^3 - 2x^2 - 8x$	
0	0		$-x^2 - 16x + 36$	
			3	
			$-3x^2 - 48x + 108$	-1
			$-3x^2 + 2x + 8$	
		-50	$-50x + 100$	
			$x - 2$	

Therefore, g.c.d $= (x - 2)$

$\Rightarrow f(x)$ has a factor $(x - 2)^2$.

Also, $f(x) = (x - 2)^2 (x + 3)$

Thus the roots are 2, 2, - 3 .

6. Show that the equation $x^3 + qx + r = 0$ has two equal roots if $27r^2 + 4q^3 = 0$.

Solution:

Let $f(x) = x^3 + qx + r$ -------------- (1)

Differentiating, we obtain: $f^1(x) = 3x^2 + q$ ------------------ (2)

Given that $f(x) = 0$ has two equal roots, i.e., it has a double root, say α.

Then α is a root of both $f(x) = 0$ and $f^1(x) = 0$.

From the 2nd equation, we obtain $\alpha^2 = -q/3$

Now the first equation can be written as: $\alpha(\alpha^2 + q) + r = 0$

i.e., $\alpha(-q/3 + q) + r = 0 \quad \Rightarrow \quad \alpha = \dfrac{-3r}{2q}$

Squaring and simplifying, we obtain: $27r^2 + 4q^3 = 0$

Relation between the Roots and Coefficients of a Polynomial Equation

Consider the polynomial function $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n,\ a_0 \neq 0$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then we can write $f(x) = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Equating the two expressions for $f(x)$, we obtain:

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Dividing both sides by a_0,

$$x^n + \left(\frac{a_1}{a_o}\right)x^{n-1} + \ldots + \left(\frac{a_n}{a_o}\right) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n S_n$$

where S_r stands for the sum of the products of the roots $\alpha_1, \ldots, \alpha_n$ taken r at a time.

Comparing the coefficients on both sides , we see that

$$S_1 = \frac{-a_1}{a_o},\quad S_2 = \frac{a_2}{a_o}, \ldots \quad S_n = (-1)^n \frac{a_n}{a_o}.$$

Special Cases

If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

If α and β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then $\alpha + \beta + \gamma = \frac{-b}{a}$,

and $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = \frac{-d}{a}$.

Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a = \frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow \left(-p/3\right)^3 + p.\left(-p/3\right)^2 + q.\left(-p/3\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\frac{a}{r}, a, ar$

Then , $\frac{a}{r}.a.ar = a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$

Since $a = \frac{2}{3}$ is a root, $\left(x - \frac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\frac{-60 \pm \sqrt{60^2 - 4 \times 27 \times 12}}{2 \times 27} = \frac{-2}{9}$ or -2

Hence the roots of the given polynomial eqution are $\frac{-2}{9}$, -2, $\frac{2}{3}$.

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $^1/_a$, $^1/_b$, $^1/_c$ are in arithmetic progression and hence $b = \frac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{9}{15}$ ……….. (1)

$$\alpha\beta\gamma = \frac{1}{15} \qquad \text{………. (2)}$$

Since α, β, γ are in harmonic progression, $\beta = \frac{2\alpha\gamma}{\alpha + \gamma}$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \frac{.9}{15} \Rightarrow 3\alpha\gamma = \frac{9}{15}$

$$\Rightarrow \alpha\gamma = \frac{3}{15}.$$

Substitute in (2), we obtain $\frac{3}{15}\beta = \frac{1}{15}$

$\Rightarrow \beta = \frac{1}{3}$ is a root of the given polynomial.

Proceeding as in the above problem, we find that the roots are $\frac{1}{3}$, 1, $\frac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\frac{k}{r}$, k, kr

Then $\frac{k}{r}.k.kr = \frac{-d}{a}$

i.e., $k^3 = \frac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$$ak^3 + bk^2 + ck + d = 0$$

$$a\left(\frac{-d}{a}\right) + bk^2 + ck + d = 0$$

$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow \; bk^2 = -ck$

$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \;\; b^3k^6 = -c^3k^3$

$$\Rightarrow b^3 \frac{d^2}{a^2} = -c^3\left(\frac{-d}{a}\right)$$

$$\Rightarrow \quad \frac{b^3d}{a} = c^3 \quad \Rightarrow \quad b^3d = c^3a .$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio 3: 2.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9$ ………………. (1)

$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$

i.e., $6\alpha^2 + 5\alpha\beta = 14$ ……………… (2)

and $3\alpha.2\alpha.\beta = 6\alpha^2\beta = -24$

$\Rightarrow \alpha^2\beta = -4$ ……………… (3)

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$

i.e., $19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\alpha = 2 \;\; or \;\; \frac{7}{19}$.

When $\alpha = \frac{7}{19}$, from (1), we get $\beta = \frac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.

Symmetric Functions of the Roots

Consider the expressions like $\alpha^2 + \beta^2 + \gamma^2, (\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$, $(\beta + \gamma)(\gamma + \alpha)(\alpha - \beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions**.

Generally, a function $f(\alpha_1, \alpha_2, ..., \alpha_n)$ is said to be a symmetric function of $\alpha_1, \alpha_2, ..., \alpha_n$ if it remains unchanged by interchanging any two of $\alpha_1, \alpha_2, ..., \alpha_n$.

Remark:

The expressions $S_1, S_2, ..., S_n$ where S_r is the sum of the products of $\alpha_1, \alpha_2, ..., \alpha_n$ taken r at a time, are symmetric functions. These are called **elementary symmetric functions**.

Now we discuss some results about the sums of powers of the roots of a given polynomial equation.

1.3.1. Theorem

The sum of the r^{th} powers of the roots of the equation f(x) = 0 is the coefficient of x^{-r} in the expansion of $\frac{xf'(x)}{f(x)}$ in descending powers of x.

Proof:

Let f(x) = 0 be the given n^{th} degree equation and let its roots be $\alpha_1, \alpha_2, ..., \alpha_n$ then, f(x) = $a_0(x-\alpha_1)(x-\alpha_2)....(x-\alpha_n)$ where a_0 is some constant.

Taking logarithm, we obtain

$$\log f(x) = \log a_0 + \log(x-\alpha_1) + ... + \log(x-\alpha_n)$$

Differentiating w.r.t. x, we have:

$$\frac{f^1(x)}{f(x)} = \frac{1}{x-\alpha_1} + ... + \frac{1}{x-\alpha_n}$$

Multiplying by x,

$$\frac{x f^1(x)}{f(x)} = \frac{x}{x-\alpha_1} + ... + \frac{x}{x-\alpha_n}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1} + ... + \left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= \left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}+....\right) + ... + \left(1+\frac{\alpha_n}{x}+\frac{\alpha_n^2}{x^2}+....\right)$$

$$= n + (\sum \alpha_i)x^{-1} + \left(\sum \alpha_i^2\right)x^{-2} + ... + ...$$

Therefore $\sum \alpha_i^r$ is the coefficient of x^{-r} in the expansion of $\frac{xf'(x)}{f(x)}$ in descending powers of x.

Theorem (Newton's Theorem on the Sum of the Powers of the Roots)

If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the equation $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = 0$,

and $S_r = \alpha_1^r + ... + \alpha_n^r$. Then, $S_r + S_{r-1}P_1 + ... + S_1 P_{r-1} + rP_r = 0, \quad if \quad r \le n.$

and $S_r + S_{r-1}P_1 + S_{r-2}P_2 + ... + S_{r-n}P_n = 0$ if r > n .

Proof:

We have $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = (x - \alpha_1)(x - \alpha_2).....(x - \alpha_n)$

Put $x = \dfrac{1}{y}$

$$\Rightarrow \quad \frac{1}{y^n} + \frac{P_1}{y^{n-1}} + \frac{P_2}{y^{n-2}} + ... + P_n = (\frac{1}{y} - \alpha_1)(\frac{1}{y} - \alpha_2).....(\frac{1}{y} - \alpha_n),$$

and then multiplying by y^n, we obtain:

$$1 + P_1 y + P_2 y^2 + ... + P_n y^n = (1 - \alpha_1 y)(1 - \alpha_2 y).....(1 - \alpha_n y)$$

Taking logarithm and differentiating w.r.t y, we get

$$\frac{P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1}}{1 + P_1 y + P_2 y^2 + ... + P_n y^n} = \frac{-\alpha_1}{1 - \alpha_1 y} + \frac{-\alpha_2}{1 - \alpha_2 y} + ... + \frac{-\alpha_n}{1 - \alpha_n y}$$

$$=$$

$$-\alpha_1 (1 - \alpha_1 y)^{-1} - \alpha_2 (1 - \alpha_2 y)^{-1} - ...... - \alpha_n (1 - \alpha_n y)^{-1}$$

$$=$$

$$-\alpha_1 (1 + \alpha_1 y + \alpha^2 y^2 + ...) - \alpha_2 (1 + \alpha_2 y + \alpha_2^2 y^2 + ...) -$$

$$...... - \alpha_n (1 + \alpha_n y + \alpha_n^2 y^2 + ...)$$

$$= -S_1 - S_2 y - S_3 y^2 - ..... - S_{r+1} y^r - ...$$

Cross – multiplying, we get

$$P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1} = -(1 + P_1 y + P_2 y^2 + ... + P_n y^n)$$

$$[S_1 + S_2 y + ... + S_{r+1} y^r + ...]$$

Equating coefficients of like powers of y, we see that

$P_1 = -S_1 \quad \Rightarrow \quad S_1 + 1.P_1 = 0$

$2P_2 = -S_2 - S_1 P_1 \quad \Rightarrow \quad S_2 + S_1 P_1 + 2P_2 = 0$

$3p_3 = -S_3 - S_2 P_1 - S_1 P_2 \quad \Rightarrow \quad S_3 + S_2 P_1 + S_1 P_2 + 3P_3 = 0$, and so on .

If $r \le n$, equating coefficients of y^{r-1} on both sides,

$$rP_r = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - ...... - S_1 P_{r-1}$$

$$\Rightarrow \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_1P_{r-1} + rP_r = 0$$

If $r > n$, then $r-1 > n-1$.

Equating coefficients of y^{r-1} on both sides,

$$0 = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - \ldots\ldots - S_{r-n}P_n$$

$$\text{i.e.,} \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_{r-n}P_n = 0$$

Remark:

To find the sum of the negative powers of the roots of $f(x) = 0$, put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the new equation

Examples

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of the following in terms of the coefficients.

(i) $\sum \frac{1}{\beta\gamma}$ (ii) $\sum \frac{1}{\alpha}$ (iii) $\sum \alpha^2\beta$

Solution:

Here $\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \alpha\gamma = q$, $\alpha\beta\gamma = -r$

(i) $\sum \frac{1}{\beta\gamma} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$

(ii) $\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$

(iii) $\sum \alpha^2\beta = \alpha^2\beta + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\gamma + \beta^2\gamma$

$= (\alpha\beta + \beta\gamma + \alpha\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = (q \,.\, -p) - 3(-r) = 3r - pq$.

2. If α is an imaginary root of the equation $x^7 - 1 = 0$ form the equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4$.

Solution:

Let $a = \alpha + \alpha^6 \quad b = \alpha^2 + \alpha^5 \quad c = \alpha^3 + \alpha^4$

The required equation is $(x - a)(x - b)(x - c) = 0$

i.e., $x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc = 0$ …………… (1)

$$a + b + c = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = \frac{\alpha(\alpha^6 - 1)}{\alpha - 1} = \frac{\alpha^7 - \alpha}{\alpha - 1} = \frac{1 - \alpha}{\alpha - 1} = -1$$

(Since α is a root of $x^7 - 1 = 0$, we have $\alpha^7 = 1$)

Similarly we can find that $ab + bc + ac = -2$, $abc = 1$.

Thus from (1), the required equation is

$x^3 + x^2 - 2x - 1 = 0$

3. If α, β, γ are the roots of $x^3 + 3x^2 + 2x + 1 = 0$, find $\sum \alpha^3$ and $\sum \overline{\alpha}^2$.

Solution:

Here $\alpha + \beta + \gamma = -3$, $\alpha\beta + \beta\gamma + \alpha\gamma = 2$, $\alpha\beta\gamma = -1$

Using the identity $a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ac)$, we find that

$$\sum \alpha^3 = (\alpha+\beta+\gamma)\left[\alpha^2+\beta^2+\gamma^2 - (\alpha\beta+\beta\gamma+\alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= (\alpha+\beta+\gamma)\left[[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)] - (\alpha\beta+\beta\gamma+\alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= -3[(9-4)-2] - 3$$

$$= -9 - 3 = -12$$

Also, $$\sum \alpha^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{(\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2\sum\alpha^2\beta\gamma}{\alpha^2\beta^2\gamma^2} \quad \text{……………… (1)}$$

We have:

$$\sum \alpha^2\beta\gamma = (\alpha+\beta+\gamma)\alpha\beta\gamma = -3 . -1 = 3$$

$$(1) \Rightarrow \sum \overline{\alpha}^2 = \frac{4 - 2.3}{1} = -2$$

4. Find the sum of the 4th powers of the roots of the equation $x^4 - 5x^3 + x - 1 = 0$.

Solution:

Let $f(x) = x^4 - 5x^3 + x - 1 = 0$

Then $f^1(x) = 4x^3 - 15x^2 + 1$

Now, $\dfrac{xf^1(x)}{f(x)}$ can be evaluated as follows :

$$
\begin{array}{l}
\qquad\qquad\quad \underline{4+5+25+122}+609+\ldots.. \\
1-5+0+1-1\,\big)\,4-15+0+1+0 \\
\qquad\qquad\quad \underline{4-20+0+4-4} \\
\qquad\qquad\qquad 5+0-3+4 \\
\qquad\qquad\qquad \underline{5-25+0+5-5} \\
\qquad\qquad\qquad\quad 25-3-1+5 \\
\qquad\qquad\qquad\quad \underline{25-125+0+25-25} \\
\qquad\qquad\qquad\qquad 122-1-20+25 \\
\qquad\qquad\qquad\qquad \underline{122-610+0+122-122} \\
\qquad\qquad\qquad\qquad\qquad 609-20-97+122 \\
\qquad\qquad\qquad\qquad\qquad \underline{609-3045+0+609-609} \\
\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Therefore,

$$\frac{xf^{|}(x)}{f(x)} = 4 + \frac{5}{x} + \frac{25}{x^2} + \frac{122}{x^3} + \frac{609}{x^4} + \ldots\ldots$$

Sum of the fourth powers of the roots = coefficient of x^{-4}.

= 609.

5. If $\alpha+\beta+\gamma=1$, $\alpha^2+\beta^2+\gamma^2=2$, $\alpha^3+\beta^3+\gamma^3=3$. Find $\alpha^4+\beta^4+\gamma^4$.

Solution:

Let $x^3 + P_1x^2 + P_2x + P_3 = 0$ be the equation whose roots are α, β, γ, then

$$\alpha+\beta+\gamma = -P_1 \Rightarrow P_1 = -1$$

By Newton's theorem,

$$S_2 + S_1P_1 + 2P_2 = 0$$

i.e., $2 + 1.(-1) + 2P_2 = 0 \Rightarrow P_2 = {}^{-1}/_2$

Again, by Newton's theorem

$$S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$$

i.e., $3 + 2.-1 + 1.{}^{-1}/_2 + 3.P_3 = 0$

$\Rightarrow P_3 = {}^{-1}/_6$

Also $S_4 + S_3P_1 + S_2P_2 + S_1P_3 = 0$ (By Newton's theorem for the case $r < n$)

Substituting and simplifying, we obtain $S_4 = {}^{25}/_6$

Thus $\alpha^4 + \beta^4 + \gamma^4 = \dfrac{25}{6}$

6. Calculate the sum of the cubes of the roots of $x^4 + 2x + 3 = 0$

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – RECIPROCAL EQUATIONS-SMT1302

RECIPROCAL EQUATIONS

Let α be a solution of the equation.

$2x^6 - 3x^5 + \sqrt{2}x^4 + 7x^3 + \sqrt{2}x^2 - 3x + 2 = 0$. ... (1)

Then $\alpha \neq 0$ (why?) and

$2\alpha^6 - 3\alpha^5 + \sqrt{2}\alpha^4 + 7\alpha^3 + \sqrt{2}\alpha^2 - 3\alpha + 2 = 0$.

Substituting $1/\alpha$ for x in the left side of (1), we get

$$2\left(\frac{1}{\alpha}\right)^6 - 3\left(\frac{1}{\alpha}\right)^5 + \sqrt{2}\left(\frac{1}{\alpha}\right)^4 + 7\left(\frac{1}{\alpha}\right)^3 + \sqrt{2}\left(\frac{1}{\alpha}\right)^2 - 3\left(\frac{1}{\alpha}\right) + 2$$

$$= \frac{2 - 3\alpha + \sqrt{2}\alpha^2 + 7\alpha^3 + \sqrt{2}\alpha^4 - 3\alpha^5 + 2\alpha^6}{\alpha^6} = \frac{0}{\alpha^6} = 0.$$

Thus $1/\alpha$ is also a solution of (1). Similarly we can see that if α is a solution of the equation

$x^5 + 3x^4 - 4x^3 + 4x^2 - 3x - 2 = 0$ ... (2)

then $1/\alpha$ is also a solution of (2).

Equations (1) and (2) have a common property that, if we replace *x* by 1/x in the equation and write it as a polynomial equation, then we get back the same equation. The immediate question that flares up in our mind is “Can we identify whether a given equation has this property or not just by seeing it?” Theorem 3.6 below answers this question.

Definition 3.1

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial if one of the following conditions is true:

(i) $P(x) = x^n P\left[\frac{1}{x}\right]$ $\quad$ $(ii) P(x) = -x^n P\left[\frac{1}{x}\right]$

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial of Type I if $P(x) = x^n P\left(\frac{1}{x}\right)$ called a reciprocal equation of Type I.

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial of Type II if $P(x) = - x^n P\left(\frac{1}{x}\right)$ called a reciprocal equation of Type II.

Theorem

A polynomial equation $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$, $(a_n \neq 0)$ is a reciprocal equation if, and only if, one of the following two statements is true:

(i) $a_n = a_0$, $a_{n-1} = a_1$, $\quad a_{n-2} = a_2 \ldots.$

(ii) $a_n = - a_0$, $a_{n-1} = - a_1$, $a_{n-2} = - a_2$, …

Proof

Consider the polynomial equation

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$. … (1)

Replacing x by 1/x in (1), we get

$$P\left(\frac{1}{x}\right) = \frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \frac{a_{n-2}}{x^{n-2}} + \cdots + \frac{a_2}{x^2} + \frac{a_1}{x} + a_0 = 0. \qquad \ldots (2)$$

Multiplying both sides of (2) by x^n, we get

$$x^n P\left(\frac{1}{x}\right) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n = 0. \qquad \ldots (3)$$

Now, (1) is a reciprocal equation $\Leftrightarrow$ $P(x) = \pm x^n P(1/x)$ $\Leftrightarrow$ (1) and (3) are same .

This is possible $\Leftrightarrow \dfrac{a_n}{a_0} = \dfrac{a_{n-1}}{a_1} = \dfrac{a_{n-2}}{a_2} = \cdots = \dfrac{a_2}{a_{n-2}} = \dfrac{a_1}{a_{n-1}} = \dfrac{a_0}{a_n}$.

Let the proportion be equal to λ. Then, we get $a_n/a_0 = \lambda$ and $a_0/a_n = \lambda$. Multiplying these equations, we get $\lambda^2 = 1$. So, we get two cases $\lambda = 1$ and $\lambda = -1$.

Case (i) :

$\lambda = 1$ In this case, we have $a_n = a_0$, $a_{n-1} = a_1$, $a_{n-2} = a_2$, …..

That is, the coefficients of (1) from the beginning are equal to the coefficients from the end.

Case (ii) :

$\lambda = -1$ In this case, we have $a_n = -a_0$, $a_{n-1} = -a_1$, $a_{n-2} = -a_2$, …..

That is, the coefficients of (1) from the beginning are equal in magnitude to the coefficients from the end, but opposite in sign.

Note

Reciprocal equations of Type I correspond to those in which the coefficients from the beginning are equal to the coefficients from the end.

For instance, the equation $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ is of type I.

Reciprocal equations of Type II correspond to those in which the coefficients from the beginning are equal in magnitude to the coefficients from the end, but opposite in sign.

For instance, the equation $6x^5 - 41x^4 + 97x^3 - 97x2 + 41x - 6 = 0$ is of Type II.

Remark

(i) A reciprocal equation cannot have 0 as a solution.

(ii) The coefficients and the solutions are not restricted to be real.

(iii) The statement "If $P(x) = 0$ is a polynomial equation such that whenever α is a root, $1/\alpha$ is also a root, then the polynomial equation $P(x) = 0$ must be a reciprocal equation" is not true. For instance $2x^3 - 9x^2 + 12x - 4 = 0$ is a polynomial equation whose roots are 2, 2,1/2.

Note that x^3 P($1/x$) $\neq \pm$ P(x) and hence it is not a reciprocal equation. Reciprocal equations are classified as Type I and Type II according to $a_{n-r} = a_r$ or $a_{n-r} = -a_r$, r = 0, 1, 2,...n. We state some results without proof :

· For an odd degree reciprocal equation of Type I, $x = -1$ must be a solution.

· For an odd degree reciprocal equation of Type II, $x = 1$ must be a solution.

· For an even degree reciprocal equation of Type II, the middle term must be 0 Further $x = 1$ and $x = -1$ are solutions.

• For an even degree reciprocal equation, by taking x + (1/x) or x – (1/x) as y , we can obtain a polynomial equation of degree one half of the degree of the given equation ; solving this polynomial equation, we can get the roots of the given polynomial equation.

As an illustration, let us consider the polynomial equation

$6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$

which is an even degree reciprocal equation of Type II. So 1 and -1 are two solutions of the equation and hence $x^2 -1$ is a factor of the polynomial. Dividing the polynomial by the factor $x^2 -1$, we get $6x^4 - 35x^3 + 62x^2 - 35x + 6$ as a factor. Dividing this factor by x2 and rearranging the terms we get $6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62$. Setting u = ($x + 1/x$) it becomes a quadratic polynomial as $6(u^2 - 2) - 35u + 62$ which reduces to $6u^2 - 35u + 50$. Solving we obtain u = 10/3 , 5/2 . Taking u = 10/3 gives x = 3, 1/3 and taking u = 5/2 gives x = 2, 1/2. So the required solutions are +1, -1, 2, 1/2 , 3, 1/3 .

Example

Solve the equation $7x^3 - 43x^2 = 43x - 7$.

Solution

The given equation can be written as $7x^3 - 43x^2 - 43x + 7 = 0$.

This is an odd degree reciprocal equation of Type I. Thus -1 is a solution and hence x +1 is a factor.

Dividing the polynomial $7x^3 - 43x^2 - 43x + 7$ by the factor x +1,we get $7x^2 - 50x + 7$ as a quotient.

Solving this we get 7 and 1/7 as roots. Thus -1, 1/7 , 7 are the solutions of the given equation.

Example

Solve the following equation: $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

Solution

This equation is Type I even degree reciprocal equation. Hence it can be rewritten as

$$x^2\left[\left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26\right]=0 \text{ Since } x\neq 0,$$

$$\text{we get } \left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26=0$$

Let y = x + [1/x] . Then, we get

$(y^2 - 2) - 10y + 26 = 0 \Rightarrow y^2 - 10y + 24 = 0 \Rightarrow (y - 6)(y - 4) = 0 \Rightarrow y = 6$ or $y = 4$

Case (i)

$y = 6 \Rightarrow x + (1/x) = 6 \Rightarrow x = 3 + 2\sqrt{2}, x = 3 - \sqrt{2}$.

Case (ii)

$y = 4 \Rightarrow x + (1/x) = 4 \Rightarrow x = 2 + \sqrt{3}, x = 2 - \sqrt{3}$

Hence, the roots are $3 \pm 2\sqrt{2}, 2 \pm \sqrt{3}$

Solve the equations

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

(ii) $x^4 + 3x^3 - 3x - 1 = 0$

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – III –TRANSFORMATION OF EQUATIONS- SMT1302

UNIT-III

Transformations of Equations

Let f(x) = 0 be a polynomial equation. Without explicitly knowing the roots of f(x) = 0, we can often transform the given equation into another equation whose roots are related to the roots of the first equation in some way. Now we discuss some important such transformations.

To form an equation whose roots are k-times the roots of a given equation.

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n$ ----------- (1)

Suppose that $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of f(x) = 0

Then $f(x) = a_o (x-\alpha_1)(x-\alpha_2)........(x-\alpha_n)$ …………….. (2)

Put y = kx in (2), we obtain:

$$f\left(\frac{y}{k}\right) = a_o\left(\frac{y}{k}-\alpha_1\right)\left(\frac{y}{k}-\alpha_2\right)......\left(\frac{y}{k}-\alpha_n\right)$$

Thus the roots of $f(y/k) = 0, \;\; are \;\; k\alpha_1, ..., k\alpha_n$

Therefore the required equation is

$$f\left(\frac{y}{k}\right) = a_o\left(\frac{y}{k}\right)^n + a_1\left(\frac{y}{k}\right)^{n-1} + ... + a_n = 0$$

i.e., $a_o y^n + k a_1 y^{n-1} + k^2 a_2 y^{n-2} + ... + k^n a_n = 0$

Thus; to obtain the equation whose roots are k times the roots of a given equation, we have to multiply the coefficients of $x^n, x^{n-1}, ..., x$ and the constant term by 1, k, k^2,……k^{n-1} and k^n respectively.

Remark:

To form an equation whose roots are the negatives of the roots of a given equation of degree n, multiply the coefficients of x^n, x^{n-1}, ... by 1, -1, 1, -1, … respectively.

To form an equation whose roots are the reciprocals of the roots of a given equation.

Consider, $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ ………….. (1)

Let $\alpha_1, \alpha_1, ..., \alpha_n$ be the roots of the equation. Then,

$$f(x) = a_o (x - \alpha_1)(x - \alpha_2)......(x - \alpha_n) \qquad \text{………….. (2)}$$

In (1), put $y = \frac{1}{x}$ *i.e.,* $x = \frac{1}{y}$

Then $f\left(\frac{1}{y}\right) = a_o\left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)........\left(\frac{1}{y} - \alpha_n\right)$

The roots of this equation are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ...\frac{1}{\alpha_n}$

But from (1) , $f\left(\frac{1}{y}\right) = a_o\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + ... + a_n = 0$

i.e., $a_o + a_1 y + a_o y^2 + ... + a_n y^n = 0$

Therefore, the required equation is $a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y + a_o = 0$

To form an equation whose roots are less by 'h' then the roots of a given equation. (i.e., Diminishing the roots by h)

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ …………. (1)

Suppose that $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of $f(x) = 0$

Therefore, $f(x) = a_o (x - \alpha_1)(x - \alpha_2).......(x - \alpha_n)$ …………. (2)

Put $y = x - h$ so that $x = y + h$

From (2), $f(y + h) = a_o (y + h - \alpha_1)(y + h - \alpha_2).......(y + h - \alpha_n)$

$$= a_o (y - (\alpha_1 - h))(y - (\alpha_2 - h)).......(y - (\alpha_n - h))$$

The roots of $f(y + h) = 0$ are $\alpha_1 - h, ..., \alpha_n - h$.

By (1), we obtain,

$$a_o (y + h)^n + a_1 (y + h)^{n-1} + ... + a_n = 0$$

Expanding using binomial theorem and combining like terms, we get an equation of the form

$$b_o y^n + b_1 y^{n-1} + ... + b_n = 0 \qquad .... (3)$$

Replacing y = x – h, we get

$$b_o (x-h)^n + b_1 (x-h)^{n-1} + ... + b_n = 0 \qquad .... (4)$$

Now, equation (1) and (4) represents the same equation.

Dividing equation (4) continuously by (x – h), we obtain the remainders as

$b_n, b_{n-1}, ..., b_o$

Substituting these in (3), we obtain the required equation.

Remark:

Increasing the roots by h is equivalent to decreasing the roots by –h.

To form an equation in which certain specified terms of the given equation are absent.

Consider the equation $a_o x^n + a_1 x^{n-1} + ... + a_n = 0 \qquad .... (1)$

Suppose it is required to remove the second term of the equation (1). Diminish the roots of the given equation by h.

For this, put $y = x - h$ i.e., x = y + h in (1), we obtain the new equation as

$$a_o (y+h)^n + a_1 (y+h)^{n-1} + ... + a_n = 0$$

$$\text{ie} \quad a_o y^n + (na_o h + a_1) y^{n-1} + ... + a_n = 0$$

Now to remove the second term of the equation (1), we must have $na_o h + a_1 = 0$

i.e., we must have $h = {-a_1}/{na_o}$.

Thus to remove the second term of the equation (1), we have to diminish its roots by

$h = {a_1}/{na_o}$

Remarks:

If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the polynomial equation f(x) = 0. Formation of an equation whose roots are $\phi(\alpha_1), \phi(\alpha_2), ..., \phi(\alpha_n)$ is known as a **general transformation** of the given equation.

In this case, the relation between a root x of f(x) = 0 and a root y of the transformed equation is that $y = \phi(x)$. Also, to obtain this new equation we have to eliminate x between f(x) = 0 and $y = \phi(x)$.

Solved Problems

1. Form an equation whose roots are three times those of the equation

$$x^3 - x^2 + x + 1 = 0.$$

Solution:

To obtain the required equation, we have to multiply the coefficients of x^3, x^2, x, and 1 by 1, 3, 3^2, and 3^3 respectively.

Thus $x^3 - 3x^2 + 9x + 27 = 0$ is the desired equation.

2. Form an equation whose roots are the negatives of the roots of the equation

Solution:

By multiplying the coefficients successively by 1, -1, 1, -1 we obtain the required equation as $x^3 + 6x^2 + 8x + 9 = 0$.

3. Form an equation whose roots are the reciprocals of the roots of

$$x^4 - 5x^3 + 7x^2 - 4x + 5 = 0.$$

Solution:

We obtain the required equation, by replacing the coefficients in the reverse order, as $5x^4 - 4x^3 + 7x^2 - 5x + 1 = 0$

4. Find the equation whose roots are less by 2, than the roots of the equation $x^5 - 3x^4 - 2x^3 + 15x^2 + 20x + 15 = 0$.

Solution:

To find the desired equation, divide the given equation successively by x – 2.

2	1	-3	-2	+15	+20	+15
		2	-2	-8	+14	68
	1	-1	-4	+7	+34	83
		2	+2	-4	+6	
	1	+1	-2	+3	+40	
		+2	+6	+8		
	1	+3	+4	+11		
		2	+10			
	1	+5	+14			
		+2				
	1	+7				
	1					

Thus the required equation is

$$x^5 + 7x^4 + 14x^3 + 11x^2 + 40x + 83 = 0$$

5. Solve the equation $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ by removing its second term.

Solution:

To remove the second term, we have to diminish the roots of the given equation by $h = \frac{-a_1}{na_o} = \frac{8}{4.1} = 2$.

Dividing the given equation successively by x – 2, we obtain the new equation as

$x^4 - 25x^2 + 144 = 0$

On solving, we get x = - 4 , 4, - 3 , 3.

Thus the roots of the original equation are –2, 6, -1 and 5.

6. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$. Form the equation whose roots are $\alpha\beta, \beta\gamma, \gamma\alpha$.

Solution:

Note that $\alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = \frac{-c}{\gamma}$

Put $y = \frac{-c}{x} \Rightarrow x = \frac{-c}{y}$

Hence the given equation becomes

$$\left(\frac{-c}{y}\right)^3 + a\left(\frac{-c}{y}\right)^2 + b\left(\frac{-c}{y}\right) + c = 0$$

i.e., $y^3 - by^2 + acy - c^2 = 0$, which is the required equation.

7. If α, β, γ are the roots of $x^3 - x + 1 = 0$, show that $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = 1$

Solution:

We have to form the equation whose roots are $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$.

For this, put $y = \frac{1+x}{1-x}$ i.e., $x = \frac{y-1}{y+1}$

Therefore the required equation is $\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) + 1 = 0$

On simplifying, we obtain $y^3 - y^2 + 7y + 1 = 0$

The sum of the roots of this equation is 1. i.e., $\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}=1$

Reciprocal Equations

Let f(x) = 0 be an equation with roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

If $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots\ldots, \frac{1}{\alpha_n}$ are also roots of the same equation, then such equations are called **reciprocal equations**.

Suppose that $a_o x^n + a_1 x^{n-1} + \ldots\ldots + a_n = 0$ …... (1) is a reciprocal equation with roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

Then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots\ldots, \frac{1}{\alpha_n}$ are also roots of the same equation. The equation with roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots\ldots, \frac{1}{\alpha_n}$ is : $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_o = 0$ …… (2)

Since (1) and (2) represents the same equation, we must have $\frac{a_o}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_n}{a_o} = k$

Taking the first and last terms in the above equality, we obtain $k^2 = 1$ i.e., $k = \pm 1$

when k = 1, we have $a_o = a_n$, $a_1 = a_{n-1}$ . . . .

Such equations are called reciprocal equations of **first type**.

When k = –1, we have $a_o = -a_n$, $a_1 = -a_{n-1}$, …… These type of equations are called reciprocal equations of **second type**.

A reciprocal equation of first type and even degree is called a **standard reciprocal equation**.

Note:

1. If f(x) = 0 is a reciprocal equation of first type and odd degree, the x = –1 is always a root. If we remove the factor x + 1 corresponding to this root, we obtain a standard reciprocal equation.
2. If f(x) = 0 is a reciprocal equation of second type and odd degree, then x = 1 is always a roots. If we remove the factor x –1 corresponding to this root, we obtain a standard reciprocal equation.

3. If f(x) = 0 is a reciprocal equation of second type and even degree, then x = 1 and x = –1 are roots. If we remove the factor $x^2 - 1$ corresponding to these roots, we obtain a standard reciprocal equation.

Solved Problems:

1. Solve the equation $60x^4 - 736x^3 + a433x^2 - 736x + 60 = 0$

Solution:

The given equation is a standard reciprocal equation. Dividing throughout by x^2, we obtain,

$$60x^2 - 736x + 1433 - \frac{736}{x} + \frac{60}{x^2} = 0$$

$$60\left(x^2 + \frac{1}{x^2}\right) - 736\left(x + \frac{1}{x}\right) + 1433 = 0$$

Putting $y = x + 1/x$ and simplifying, we obtain

$$60y^2 - 736y + 1313 = 0$$

On solving, we get $y = \frac{101}{10} \text{ or } \frac{13}{6}$

When $y = \frac{101}{10}, x + \frac{1}{x} = \frac{101}{10} \Rightarrow 10x^2 - 101x + 10 = 0$

i.e., $x = 10, \frac{1}{10}$

Similarly when $y = \frac{13}{6}$, we get $x = \frac{3}{2}, 2/3$

Thus the roots of the given equation are $10, \frac{1}{10}, \frac{3}{2}, \frac{2}{3}$

2. Solve :

$$x^2 - 5x^2 + 9x3 - 9x^2 + 5x - 1 = 0$$

Solution:

This is a second type reciprocal equation of odd degree. So x = 1 is a root. On division by the corresponding factor x – 1, we obtain the other factor as $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$, which is a standard reciprocal equation.

Proceeding exactly as in the above problem, we may find that

$$x = \frac{1 \pm i\sqrt{3}}{2} \text{ or } x = \frac{3 \pm \sqrt{5}}{2}$$

Hence the roots of the given equation are $1, \frac{1 \pm i\sqrt{3}}{2}, \frac{3 \pm \sqrt{5}}{2}$

3. Show that on diminishing the roots of the equation

$$6x^4 - 43x^3 + 76x^2 + 25x - 100 = 0$$

by 2, it becomes a reciprocal equation and hence solve it.

Solution:

To diminish the roots of the given equation by 2, divide it successively by (x – 2), we obtain:

2	6	-43	+76	+25	-100
		+12	-62	+28	+106
	6	-31	+14	+53	+6
		+12	-38	-48	
	6	-19	-24	+5	
		+12	-14		
	6	-7	-38		
		+12			
	6	+5			
	6				

$\Rightarrow 6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ is the required equation, which is a standard reciprocal equation.

It can be written as

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$$

Putting $x + \frac{1}{x} = y$ and solving for y, we get $y = \frac{-10}{3} \text{ or } \frac{5}{2}$

When $y = \frac{5}{2}$, we have $x + \frac{1}{x} = \frac{5}{2}$. On solving we get: $x = 2, \frac{1}{2}$

When $y = \frac{-10}{3}$, we have $3x^2 + 10x + 3 = 0$ or $x = -3 \text{ or } -\frac{1}{3}$

Thus the roots of the original equation are $4, \frac{5}{2}, -1, \frac{5}{3}$ (by adding 2 to each of the above roots)

Descartes's Rule of Signs:

Nature of Roots - Descarte's Rule of Signs

To determine the nature of some of the roots of a polynomial equation it is not always necessary to solve it; for instance, the truth of the following statements will be readily admitted.

1. If the coefficients of a polynomial equation are all positive, the equation has no positive root; for example, the equation

$$x^4 + 3x^2 + 3 = 0$$

cannot have a positive root.

2. If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the opposite sign, the equation has no negative root; thus for example, the equation

$$-x^8 + x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 3 = 0$$

cannot have a negative root.

3. If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus for example, the equation

$$-x^8 - 2x^4 - 3x^2 - 3 = 0$$

cannot have a real root.

4. If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$; thus the equation

$$x^7 + x^5 + 3x^3 + 8x = 0$$

has no real root except $x = 0$.

Suppose that the signs of the terms in a polynomial are $++--+---+-+-$; here the number of changes of sign is 7. We shall show that if this polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+-$, there will be at least one more change of sign in the product than in the original polynomial.

Writing down only the signs of the terms in the multiplication, we have the following:

$$\begin{array}{l} ++--+---+-+- \\ +- \\ \hline ++--+---+-+- \\ \quad --++-+++-+-+ \\ \hline +\pm-\pm+-\pm\pm+-+-+ \end{array}$$

Here in the last line the ambiguous sign $\pm$ is placed wherever there are two different signs to be added.

Here we see that in the product

(i) an ambiguity replaces each continuation of sign in the original polynomial;

(ii) the signs before and after an ambiguity or set of ambiguities are unlike;

(iii) a change of sign is introduced at the end.

Let us take the most unfavourable case (i.e., the case where the number of changes of sign is less) and suppose that all the ambiguities are replaced by continuations; then the sign of the terms become

$$++--+---+-+-+,$$

and the number of changes of sign is 8.

We conclude that if a polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+-$, there will be at least one more change of sign in the product than in the original polynomial.

If then we suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x-a$ corresponding to a positive root introduces at least one change of sign; therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x)=0$ are equal to those of $f(x)=0$ but opposite to them in sign; therefore the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$; but the number of these positive roots cannot exceed the number of changes of sign in $f(-x)$; that is, the number of negative roots of $f(x)=0$ cannot exceed the number of changes in sign in $f(-x)$.

All the above observations are included in the following result, known as **Descarte's Rule of Signs.**

In any polynomial equation f(x) = 0, the number of real positive roots cannot exceed the number of changes in the signs of the coefficients of the terms in f(x), and the number of real negative roots cannot exceed the number of changes in the signs of the coefficients of f(–x).

Example:

Consider the equation $f(x) = x^4 + 3x - 1 = 0$

This a polynomial equations of degree 4, and hence must have four roots.

The signs of the coefficients of f(x) are + + –

Therefore, the number of changes in signs = 1

By Descarte's rule of signs, number of real positive roots $\leq$ 1.

Now $f(-x) = x^4 - 3x - 1 = 0$

The signs of the coefficients of f(–x) are + – –

Therefore, the number of changes in signs = 1.

Hence the number of real negative roots of f(x) = 0 is $\leq$ 1.

Therefore, the maximum number of real roots is 2.

If the equation has two real roots, then the other two roots must be complex roots.

Since complex roots occur in conjugate pairs, the possibility of one real root and three complex roots is not admissible.

Also $f(0) < 0$, and $f(1) > 0$, so f(x) = 0 has a real roots between 0 and 1.

Therefore, the given equation must have two real roots and two complex roots.

Problem.

Discuss the nature of roots of the equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0.$

Solution.

With $f(x) = x^9 + 5x^8 - x^3 + 7x + 2,$ there are two changes of sign in $f(x) = 0$, and therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2,$ and there are three changes of sign, therefore the given equation has at most three negative roots.

Obviously 0 is not a root of the given equation.

Hence the given equation has at most 2 + 3 + 0 = 5 real roots. Thus the given equation has at least four imaginary roots.

Exercises

1. Solve the equation $x^4 + x^3 - x^2 - 2x - 2 = 0$ given that one root is $\sqrt{2}$.
2. Form a rational quartic whose roots are 1, –1, $2 + \sqrt{3}$
3. Solve $x^5 - x^3 + 4x^2 - 3x + 2 = 0$ given that it has multiple roots.
4. Solve the equation $x^4 - 2x^3 - 21x^2 + 2^2 x + 40 = 0$ whose roots are in A.P.
5. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.
6. Find the condition that the roots of the equation $x^3 + px^2 + qx + r = 0$ may be in geometric progression.
7. Find the condition that the roots of the equation $x^3 - lx^2 + mx - n = 0$ may be in arithmetic progression.
8. If α, β, γ are the roots of $x^3 + px + 1 = 0$, prove that $\frac{1}{5}\sum \alpha^5 = \frac{1}{6}\sum \alpha^3 \cdot \sum \alpha^2$.
9. If α, β, γ are the roots of $x^3 + qx + r = 0$, then find the values of $\sum \frac{1}{\beta + \gamma}$ and $\sum (\beta - \gamma)^2$.
10. Prove that the sum of the ninth powers of the roots of $x^3 + 3x + 9 = 0$ is zero.
11. If α, β, γ are the roots of $x^3 - 7x + 7 = 0$, find the value of $\alpha^{-4} + \beta^{-4} + \gamma^{-4}$.
12. Find the equation whose roots are the roots of the equation

    $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$, each increased by 7.
13. Remove the second term of the equation $x^3 - 6x^2 + 4x - 7 = 0$.
14. Solve the equation $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$ by removing its second term.
15. If α, β, γ are the roots of $x^3 + px + q = 0$, form the equation whose roots are $\alpha^2 + \beta\gamma, \beta^2 + \gamma\alpha, \gamma^2 + \alpha\beta$.
16. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, find the equation whose roots are $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.
17. Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.
18. Solve $x^5 - 5x^3 + 5x^2 - 1 = 0$.
19. Solve $x^3 - 9x - 12 = 0$ using Cardan's method.

20. Solve $2x^3 + 3x^2 + 3x + 1 = 0$ using Cardan's method.

21. Solve $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ using Ferrari's method.

22. Solve $x^4 + 6x^3 + 4x^2 - 32 = 0$ using Ferrari's method.

23. Find the greatest possible number of real roots of the equation

$$x^5 - 6x^2 - 4x + 5 = 0$$

24. Find the number of real roots of $x^7 - x^5 - x^4 - 6x^2 + 7 = 0$.

25. Show that $x^5 - 2x^2 + 7 = 0$ has at least two imaginary roots.

26. Determine the nature of the roots of the equation $x^4 + 3x^2 + 2x - 7 = 0.$

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – ELEMENTARY CONCEPTS-SMT1302

Matrices

When some numbers are arranged in rows and columns and are surrounded on both sides by square brackets, we call it as a matrix. Matrix or matrices have very important applications in mathematics. In this chapter, we will learn about matrices, their types and various operations on them. .

Definition: Matrix refers to an ordered rectangular arrangement of numbers which are either real or complex or functions. We enclose Matrix by [ ] or ( ).

The different types of Matrix are Row Matrix, Square Matrix, Column Matrix, Rectangle Matrix, Diagonal Matrix, Scalar Matrix, Zero or Null Matrix, Unit or Identity Matrix, Upper Triangular Matrix and Lower Triangular Matrix Symmetric and skew-symmetric matrices, Hermitian and skew-Hermitian matrices, Orthogonal and unitary matrices and diagonal matrices

Symmetric Matrix and Skew-Symmetric matrix

A symmetric matrix and skew-symmetric matrix both are square matrices. But the difference between them is, the symmetric matrix is equal to its transpose whereas skew-symmetric matrix is a matrix whose transpose is equal to its negative.

If A is a symmetric matrix, then $A = A^T$ and if A is a skew-symmetric matrix then $A^T = -A$.

To understand if a matrix is a symmetric matrix, it is very important to know about transpose of a matrix and how to find it. If we interchange rows and columns of an $m \times n$ matrix to get an $n \times m$ matrix, the new matrix is called the transpose of the given matrix. There are two possibilities for the number of rows (m) and columns (n) of a given matrix:

- If $m = n$, the matrix is square
- If $m \neq n$, the matrix is rectangular
- If $m \neq n$, the matrix is rectangular

Properties of Symmetric Matrix

Addition and difference of two symmetric matrices results in symmetric matrix.

- If A and B are two symmetric matrices and they follow the commutative property, i.e. AB =BA, then the product of A and B is symmetric.
- If matrix A is symmetric then A^n is also symmetric, where n is an integer.
- If A is a symmetrix matrix then A^{-1} is also symmetric.

Properties of Skew Symmetric Matrix

- When we add two skew-symmetric matrices then the resultant matrix is also skew-symmetric.
- Scalar product of skew-symmetric matrix is also a skew-symmetric matrix.
- The diagonal of skew symmetric matrix consists of zero elements and therefore the sum of elements in the main diagonals is equal to zero.
- When identity matrix is added to skew symmetric matrix then the resultant matrix is invertible.
- The determinant of skew symmetric matrix is non-negative

. Example

A= $\begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}$ is a symmetric matrix. When transforming rows and Columns in to columns and rows it results the same matrix. ie $A^T = A$

A = $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ is a skew- Symmetric matrix When the row elements and the column elements are transformed to column and row respectively we get –A. ie $A^T = -A$

Hermitian matrix, Skew-Hermitian matrix**.**

Hermitian matrix**:** A square matrix such that a_{ij} is the complex conjugate of a_{ji} for all elements a_{ij} of the matrix i.e. a matrix in which corresponding elements with respect to the diagonal are conjugates of each other. The diagonal elements are always real numbers.

Example.

Let A = $\begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$

A Hermitian matrix can also be defined as a square matrix A in which the transpose of the conjugate of A is equal to A i.e. where

$\left(\overline{A}\right)^T = A$

Skew-Hermitian matrix**.** A square matrix such that $a_{ij} = -\overline{a_{ji}}$
for all elements a_{ij} of the matrix. The diagonal elements are either zeros or pure imaginaries.

Example. LetA= $\begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$ A Skew-Hermitian matrix can also be defined as a square matrix A in which $\left(\overline{A}\right)^T = -A$

Rank of a matrix .

Let A be any matrix of order mxn.The determinants of the sub square matrices of A are called the minors of A. If all the minors of order **(r+1)** are zero but there is at least one non zero minor of order **r,** then **r** is called the rank of A and is written as R(A). For an mxn matrix,

- If m is less than n then the maximum rank of the matrix is m
- If m is greater than n then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

Example

1. Find the rank of A = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$|A|$ = 1(20-12) -2(5-4) +3(6-8) = 0

Hence R(A) < 3.

Let the second order minor $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ =2 $\neq 0$

R(A)=2.

2. Find the Rank of B = $\begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

= $\begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$ $R_2 = R_2 - 2R_1$, $R_3 = R_3 - 3R_1$, $R_4 = R_4 - 6R_1$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix} \quad R_2 = 1/5R_2,\ R_3 = R_3,\ R_4 = R_4$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{pmatrix} \quad R_3 = R_3 - 4R_2,\ R_4 = R_4 - 9R_2$$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_4 = R_4 - R_3$$

The number of Nonzero Rows is 3.Hence R(B)=3.

3. Find the Rank of the Matrix A = $\begin{pmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{pmatrix} \quad R_1 = R_2,\ R_2 = R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{pmatrix} \quad R_1 = R_2,\ R_2 = R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{pmatrix} \quad R_2 = R_2 - 2R_1,\ R_3 = R_3 - 4R_1$$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{pmatrix} \quad R_3 = R_3 - R_2$$

The number of Nonzero Rows is 3. Hence R(A)=3.

Unitary Matrices: Recall that a real matrix A is orthogonal if and only if In the complex system, matrices having the property that * are more useful and such matrices are called unitary.

A complex matrix A is unitary if $A^{-1} = (\overline{A})^T$ in other words $A(\overline{A})^T = I$

EXAMPLE :

Show that the matrix A = $\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ is unitary.

Solution :

$$A(\overline{A})^T = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

you can conclude that $A^{-1} = (\overline{A})^T$ So, A is a unitary matrix.

Orthogonal matrix:

Definition. An $n \times n$ matrix is *orthogonal* if $A^t A = \mathbb{I}_n$.

Recall the basic property of the transpose (for *any* A):

$$Av \cdot w = v \cdot A^t w, \quad \forall v, w \in \mathbb{R}^n.$$

It implies that requiring A to have the property:

$$Av \cdot Aw = v \cdot w, \quad \forall v, w \in \mathbb{R}^n.$$

is the same as requiring:

$$v \cdot A^t Aw = v \cdot w, \quad \forall v, w \in \mathbb{R}^n.$$

This is certainly true for orthogonal matrices; thus the action of an orthogonal matrices on vectors in $\mathbb{R}^n$ preserves lengths and angles.

(2) and (3) (plus the fact that the identity is orthogonal) can be summarized by saying the $n \times n$ orthogonal matrices form a matrix group, the *orthogonal group* $\mathbb{O}_n$.

(4)The 2×2 rotation matrices R_θ are orthogonal. Recall:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(R_θ rotates vectors by θ radians, counterclockwise.)

(5)The determinant of an orthogonal matrix is equal to 1 or -1. The reason is that, since $det(A) = det(A^t)$ for any A, and the determinant of the product is the product of the determinants, we have, for A orthogonal:

$$1 = det(I_n) = det(A^tA) = det(A(^t)det(A) = (detA)^2.$$

(6) Any real eigenvalue of an orthogonal matrix has absolute value 1. To see this, consider that $|Rv| = |v|$ for any v, if R is orthogonal. But if $v \neq 0$ is an eigenvector with eigenvalue λ:

$$Rv = \lambda v \quad \Rightarrow |v| = |Rv| = |\lambda||v|,$$

hence $|\lambda| = 1$. (Actually, it is also true that each complex eigenvalue must have modulus 1, and the argument is similar).

Basic properties. (1) A matrix is orthogonal exactly when its column vectors have length one, and are pairwise orthogonal; likewise for the row vectors. In short, the columns (or the rows) of an orthogonal matrix are an *orthonormal basis* of $\mathbb{R}^n$, and any orthonormal basis gives rise to a number of orthogonal matrices.

(2) Any orthogonal matrix is invertible, with $A^{-1} = A^t$. If A is orthogonal, so are A^T and A^{-1}.

(3) The product of orthogonal matrices is orthogonal: if $A^tA = I_n$ and $B^tB = I_n$,

$$(AB)^t(AB) = (B^tA^t)AB = B^t(A^tA)B = B^tB = \mathbb{I}_n.$$

Example:

The matrix A = $\begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal Science

$$A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Echelan fom of a matrix

Reduced row echelon form and elementary row operation

In above motivating example, the key to solve a system of linear equations is to transform the original augmented matrix to some matrix with some properties via a few elementary row operations. As a matter of fact, we can solve any system of linear equations by transforming the associate augmented matrix to a matrix in some form. The form is referred to as the reduced row echelon form.

A matrix in reduced row echelon form has the following properties:

1. **All rows consisting entirely of 0 are at the bottom of the matrix.**
2. **For each nonzero row, the first entry is 1. The first entry is called a leading 1.**
3. **For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.**
4. **If a column contains a leading 1, then all other entries in that column are 0.**

Note: **a matrix is in row echelon form as the matrix has the first 3 properties.**

Example:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are the matrices in reduced row echelon form.

The matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in reduced row echelon form but in row echelon form since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not 0. The matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are not in row echelon form (also not in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not 0.

Definition of elementary row operation:

There are 3 elementary row operations:

1. **Interchange two rows**
2. **Multiply a row by some nonzero constant**
3. **Add a multiple of a row to another row.**

Example:

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}.$$

Interchange rows 1 and 3 of A

$$\Rightarrow \quad \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Multiply the third row of A by $\frac{1}{3}$

$$\Rightarrow \qquad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

Multiply the second row of A by -2, then add to the third row of A

$$\Rightarrow \qquad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Important result:

Every nonzero m n matrix can be transformed to a unique matrix in reduced row echelon form via elementary row operations.
If the augmented matrix $[A \vdots b]$ can be transformed to the matrix in reduced row echelon form $[C \vdots d]$ via elementary row operations, then the solutions for the linear system corresponding to C d is exactly the same as the one corresponding to $[A \vdots b]$.

Example:

$$\begin{array}{rl} 2x_2 + 3x_3 - 4x_4 = 1 & (1) \\ 2x_3 + 3x_4 = 4 & (2) \\ 2x_1 + 2x_2 - 5x_3 + 2x_4 = 4 & (3) \\ 2x_1 \qquad - 6x_3 + 9x_4 = 7 & (4) \end{array} \Leftrightarrow \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\xrightarrow{(1)\leftrightarrow(3)}$$

$$\begin{array}{rl} 2x_1 + 2x_2 - 5x_3 + 2x_4 = 4 & (1) \\ 2x_3 + 3x_4 = 4 & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ 2x_1 \qquad - 6x_3 + 9x_4 = 7 & (4) \end{array} \Leftrightarrow \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\xrightarrow{(1)=\frac{(1)}{2}}$$

$$\begin{cases} x_1 + x_2 - \frac{5}{2}x_3 + x_4 = 2 & (1) \\ 2x_3 + 3x_4 = 4 & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ 2x_1 - 6x_3 + 9x_4 = 7 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\xrightarrow{(4)=(4)-2*(1)}$$

$$\begin{cases} x_1 + x_2 - \frac{5}{2}x_3 + x_4 = 2 & (1) \\ 2x_3 + 3x_4 = 4 & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ -2x_2 - x_3 + 7x_4 = 3 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\xrightarrow{(2)\leftrightarrow(4)}$$

$$\begin{cases} x_1 + x_2 - \frac{5}{2}x_3 + x_4 = 2 & (1) \\ -2x_2 - x_3 + 7x_4 = 3 & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ 2x_3 + 3x_4 = 4 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & -2 & -1 & 7 & 3 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{(2)=\frac{(2)}{-2}}$$

$$\begin{cases} x_1 + x_2 - \frac{5}{2}x_3 + x_4 = 2 & (1) \\ x_2 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = \frac{-3}{2} & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ 2x_3 + 3x_4 = 4 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 1/2 & -7/2 & -3/2 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{(1)=(1)-(2)}$$

$$\begin{cases} x_1 + \quad -3x_3 + \frac{9}{2}x_4 = \frac{7}{2} & (1) \\ x_2 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = \frac{-3}{2} & (2) \\ 2x_2 + 3x_3 - 4x_4 = 1 & (3) \\ 2x_3 + 3x_4 = 4 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 & 9/2 & 7/2 \\ 0 & 1 & 1/2 & -7/2 & -3/2 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{(3)=(3)-2*(2)}$$

$$\begin{cases} x_1 \quad -3x_3 + \frac{9}{2}x_4 = \frac{7}{2} & (1) \\ x_2 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = \frac{-3}{2} & (2) \\ 2x_3 + 3x_4 = 4 & (3) \\ 2x_3 + 3x_4 = 4 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 & 9/2 & 7/2 \\ 0 & 1 & 1/2 & -7/2 & -3/2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{(3)=\frac{(3)}{2}}$$

$$\begin{cases} x_1 \quad -3x_3 + \frac{9}{2}x_4 = \frac{7}{2} & (1) \\ x_2 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = \frac{-3}{2} & (2) \\ x_3 + \frac{3}{2}x_4 = 2 & (3) \\ 2x_3 + 3x_4 = 4 & (4) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 & 9/2 & 7/2 \\ 0 & 1 & 1/2 & -7/2 & -3/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{(4)=(4)-2*(3)}$$

$$\begin{aligned} x_1 \qquad -3x_3+\frac{9}{2}x_4 &= \frac{7}{2} \quad (1) \\ x_2+\frac{1}{2}x_3-\frac{7}{2}x_4 &= \frac{-3}{2} \quad (2) \\ x_3+\frac{3}{2}x_4 &= 2 \quad (3) \\ 0x_3+0x_4 &= 0 \quad (4) \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 & 9/2 & 7/2 \\ 0 & 1 & 1/2 & -7/2 & -3/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(2)=(2)-\frac{1}{2}*(3)}$$

$$\begin{aligned} x_1 \qquad -3x_3+\frac{9}{2}x_4 &= \frac{7}{2} \quad (1) \\ x_2 \qquad -\frac{17}{4}x_4 &= \frac{-5}{2} \quad (2) \\ x_3+\frac{3}{2}x_4 &= 2 \quad (3) \\ 0x_3+0x_4 &= 0 \quad (4) \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 0 & -3 & 9/2 & 7/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(1)=(1)+3*(3)}$$

$$\begin{aligned} x_1 \qquad +9x_4 &= \frac{19}{2} \quad (1) \\ x_2 \qquad -\frac{17}{4}x_4 &= \frac{-5}{2} \quad (2) \\ x_3+\frac{3}{2}x_4 &= 2 \quad (3) \\ 0x_3+0x_4 &= 0 \quad (4) \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

can be transformed to the unique matrix in reduce row echelon form,

$$\begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

via elementary row operations.

The linear system

$$\begin{aligned} 2x_2 + 3x_3 - 4x_4 &= 1 \quad (1) \\ 2x_3 + 3x_4 &= 4 \quad (2) \\ 2x_1 + 2x_2 - 5x_3 + 2x_4 &= 4 \quad (3) \\ 2x_1 \qquad - 6x_3 + 9x_4 &= 7 \quad (4) \end{aligned}$$

has the exactly the same solution as the linear system

$$\begin{aligned} x_1 \qquad + 9x_4 &= \frac{19}{2} \quad (1) \\ x_2 \qquad - \frac{17}{4}x_4 &= \frac{-5}{2} \quad (2) \\ x_3 + \frac{3}{2}x_4 &= 2 \quad (3) \\ 0x_3 + 0x_4 &= 0 \quad (4) \end{aligned}$$

The solution for the linear system corresponding to the augmented matrix in reduced row echelon form is

$$x_1 = \frac{19}{2} - 9t,\ x_2 = \frac{-5}{2} + \frac{17}{4}t,\ x_3 = 2 - \frac{3}{2}t,\ x_4 = t,\ t \in R$$

$$\Leftrightarrow \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (19/2) - 9t \\ (-5/2) + (17/4)t \\ 2 - (3/2)t \\ t \end{bmatrix} = \begin{bmatrix} 19/2 \\ -5/2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 17/4 \\ -3/2 \\ 1 \end{bmatrix} t$$

The above solutions are also the solutions for the original linear system.

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
(DEEMED TO BE UNIVERSITY)
Accredited "A" Grade by NAAC | 12B Status by UGC | Approved by AICTE
www.sathyabama.ac.in

SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – V – MATRICES - SMT1302

MATRICES

CHARACTERISTIC EQUATION:

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A

Working rule to find characteristic equation:

<u>For a 3 x 3 matrix:</u>

<u>Method 1:</u>

The characteristic equation is $|A - \lambda I| = 0$

<u>Method 2:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$,

$S_3 = Determinant of A = \ |A|$

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of A = \ |A|$

Problems:

1. **Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 = sumofthemaindiagonalelements = 1 + 2 = 3,$

$S_2 = Determinant of A = \ |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the characteristic equation of $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$**

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where $S_1 = sumofthemaindiagonalelements$ = 8 + 7 + 3 = 18,

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$, $S_3 = Determinant of A = \ |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

3. **Find the characteristic polynomial of $\begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic polynomial of A is $\lambda^2 - S_1\lambda + S_2$ where $S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic polynomial is $\lambda^2 - 5\lambda + 7$

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

2. **If A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$ $\boldsymbol{write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem}$**

Solution:Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 6$

$S_2 = |A| = 5$

Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

i.e., $A^2 = 6A - 5I$

3. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \ \Rightarrow \ A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - \ 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by $A^{-1}, A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4. **Verify that A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4**

Solution:Given A =$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = \text{Sum of the main diagonal elements} = 0$

$S_2 = |A| = -1 - 4 = -5$

Therefore, the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find A^4:

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

5. **Find A^{-1} if A = $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$, using Cayley-Hamilton theorem**

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 - 1 = 2$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (-2 + 1) + (-1 - 8) + (2 + 3)$
$= -1 - 9 + 5 = -5$

$S_3 = |A| = 1(-2 + 1) + 1(-3 + 2) + 4(3 - 4) = -1 - 1 - 4 = -6$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ------------ (1)

To find A^{-1}:

Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I)$ ------------- (2)

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

6. If A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$, $\boldsymbol{find\ A^n\ in\ terms\ of\ A}$

Solution:The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 = 3$

$S_2 = |A| = 2 - 0 = 2$

The characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$ i.e., $\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm 1}{2} = 2,1$

To find A^n:

When λ^n is divided by $\lambda^2 - 3\lambda + 2$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + b$

$\lambda^n = (\lambda^2 - 3\lambda + 2)Q(\lambda) + a\lambda + b$ ---------- (1)

When $\lambda = 1, 1^n = a + b$

When $\lambda = 2, 2^n = 2a + b$

$2a + b = 2^n$ ------------ (2)

$a + b = 1^n$ ------------ (3)

Solving (2) and (3), we get, (2) - (3)⇒ $a = 2^n - 1^n$

(2) – 2 x (3) ⇒ $b = -2^n + 2(1)^n$

i.e., $a = 2^n - 1^n$

$b = 2(1)^n - 2^n$

Since $A^2 - 3A + 2I = 0$ by Cayley-Hamilton theorem, (1) ⇒ $A^n = aA + bI$

$$A^n = (2^n - 1^n)\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} + [2(1)^n - 2^n]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

7. **Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ to express as a linear polynomial in A (i) $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ (ii) $A^4 - 4A^3 - 5A^2 + A + 2I$**

Solution: Given A = $\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

By Cayley-Hamilton theorem, we get, $A^2 - 4A - 5I = 0$ ------------ (1)

$\lambda^3 - 2\lambda + 3$

$\lambda^2 - 4\lambda - 5 \,|\, \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

$\lambda^5 - 4\lambda^4 - 5\lambda^3$

$-2\lambda^3 + 11\lambda^2 - \lambda$

(-)$- 2\lambda^3 + 8\lambda^2 + 10\lambda$

$3\lambda^2 - 11\lambda - 10$

$(-)\ 3\lambda^2 - 12\lambda - 15$

$\lambda + 5$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I = 0 + A + 5I$$

$= A + 5I$ (by (1)) which is a linear polynomial in A

(i)

λ^2

$\lambda^2 - 4\lambda - 5 \,|\, \lambda^4 - 4\lambda^3 - 5\lambda^2 + \lambda + 2$

$\lambda^4 - 4\lambda^3 - 5\lambda^2$

(-) $\lambda + 2$

$A^4 - 4A^3 - 5A^2 + A + 2I$ **=**$A^2(A^2 - 4A - 5I) + A + 2I$ = 0 + A + 2I = A +2I (by (1)) which is a linear polynomial in A

8. **Using Cayley-Hamilton theorem, find A^{-1} when A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$**

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 + 1 = 3$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (1 - 1) + (1 - 3) + (1 - 0)$
$= 0 - 2 + 1 = -1$

$S_3 = |A| = 1(1 - 1) + 0(2 + 1) + 3(-2 - 1) = 1(0) + 0 - 9 = -9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

Pre-multiplying by A^{-1}, we get, $A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-A^2 = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix};\ 3A = \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix};\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

9. **Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$**

Solution: Given A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements = 1+2+1 = 4

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (2 - 6) + (1 - 7) + (2 - 12)$
$= \quad -4 - 6 - 10 = -20$

$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3 + 42 = 35$

The characteristic equation is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

To prove that: $A^3 - 4A^2 - 20A - 35I = 0$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - \begin{bmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{bmatrix} - \begin{bmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Therefore, Cayley-Hamilton theorem is verified.

10. Verify Cayley-Hamilton theorem for the matrix (i) A = $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{5} \end{bmatrix}$ $(\boldsymbol{ii}) \boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$

Solution:(i) Given A = $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 3 + 5 = 8$

$S_2 = |A| = 15 - 1 = 14$

The characteristic equation is $\lambda^2 - 8\lambda + 14 = 0$

To prove that: $A^2 - 8A + 14I = 0$

$$A^2 = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 9+1 & -3-5 \\ -3-5 & 1+25 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$8A = 8\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix}$

$14I = 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$A^2 - 8A + 14I = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} - \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

(ii) Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

To prove that: $\boldsymbol{A^2 - 4A - 5I = 0}$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$4A = 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix}; 5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors
2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.
   If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

Symmetric matrix:

If a square matrix A is symmetric, then A = A^T

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors
2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors
   If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

Problems:

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = sumofthemaindiagonalelements = 1 - 1 = 0,$

$S_2 = DeterminantofA = \ |A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

A is a non-symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$**--------------- (1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$

$$3x_1 + x_2 = 0$$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$$

i.e., we get only one equation $x_1 - x_2 = 0$

$$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. **Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$**

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 3 + 2 = 7$,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 =$

11,

$S_3 = Determinant of A =$ $|A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

Therefore, the eigen values are 1, 1, and 5

A is a non-symmetric matrix with repeated eigen values

<u>To find the eigen vectors:</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + 2x_2 - 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

2 1 -3 2

-2 1 1 -2

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 . Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 1 = 2,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$

$-4 - 4 + 4 = -4,$

$S_3 = Determinant of A = |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8

$0 \qquad 2 \qquad 0 \qquad -8$

1 0 -4 0

$$(\lambda-2)(\lambda^2-4)=0 \Rightarrow \lambda=2, \qquad \lambda = 2,-2$$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$$[A-\lambda I]X=0$$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\boldsymbol{\lambda=-2}$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1-2x_2+2x_3=0$ --------- (1)

$x_1+3x_2+x_3=0$ ------------- (2)

$x_1+3x_2+x_3=0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

-1 1 2 -1

3 1 1 3

$$\Rightarrow \frac{x_1}{-4}=\frac{x_2}{-1}=\frac{x_3}{7} \Rightarrow \frac{x_1}{4}=\frac{x_2}{1}=\frac{x_3}{-7}$$

Therefore, $X_1=\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = 2}$**,** $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

-2	2	0	-2
-1	1	1	-1

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

4. **Find the eigen values and eigen vectors of** $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 -$

$8 + 4 = 0,$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = - 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

A is a symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 ⤨ 3 ⤨ 3 ⤨ 1

7   1   1   7

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{4} = \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If$\lambda = 3$, $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 ⤨ 3 ⤨ -2 ⤨ 1

2   1   1   2

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If$\lambda = 6$, $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 -5 1

-1 1 1 -1

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

5. **Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Determine the algebraic and geometric multiplicity**

Solution: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 - 1 - 1 = -3,$

$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

$$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

A is a symmetric matrix with repeated eigen values. The algebraic multiplicity of $\lambda = -1$ is 2

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\boldsymbol{\lambda = 2}$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1	1	-2	1
-2	1	1	-2

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = -1}$**,** $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 is orthogonal to X_1 *and* X_2 .

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

	l		m		n	
1		1		1		1
1		-1		0		1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1,$ there corresponds two linearly independent eigen vectors $X_2\ and\ X_3$. So, the geometric multiplicity of eigen value $\lambda = -1$ is 2

Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$

   Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

   Product of the eigen values = |A| = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Product of two eigen values of the matrix A =** $\begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$ **is 16. Find the third eigen value**

   Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

   Given $\lambda_1 \lambda_2 = 16$

   We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

   $\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

   $$There\ fore, \lambda_1 \lambda_2 \lambda_3 = 32 \Rightarrow\ 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

3. **Find the sum and product of the eigen values of the matrix A =** $\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ **without finding the roots of the characteristic equation**

   Solution:We know that the sum of the eigen values = Trace of A = a + d

   Product of the eigen values = |A| = ad – bc

4. **If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find │A│, without expanding the determinant**

Solution:Given $\lambda_1 = 3\ and\ \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \ \lambda_3 = 0$$

$We\ know\ that\ the\ product\ of\ the\ eigen\ values = \ |A|$

$\Rightarrow (3)(15)(0) = \ |A|$

$\Rightarrow \ |A| \ = 0$

5. **If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T**

Solution:By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of $A^T\ are\ 2{,}2{,}3$

6. **Find the eigen values of A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$**

Solution:Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

7. **Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}**

Solution:Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 ⇒ $k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. Find the eigen values of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ $and - 1$

Solution: Let A = $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =$ 5 and $S_2 = |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ $and - 1$ is A^{-1}

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

|A| = 4 -10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

9. Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

Solution:We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. Find the eigen values of A^3 given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of A^2 and A^{-1}?

Solution:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 *and* 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ and $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ and $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

= $3(2)^3 + 5(2)^2 - 6(2) + 2 = 24 + 20 - 12 + 2 = 34$

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

= $3(5)^3 + 5(5)^2 - 6(5) + 2$

= 375+ 125-30 + 2 = 472

14. Find the eigen values of adj A if A = $\begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{4} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

<u>Solution</u>:Given A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\ adj\ A$

Adj A = |A| A^{-1}

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

|A| =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

<u>Note:</u> A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times the third. Find them

<u>Solution</u>:Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1,1, 5

16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a*

<u>Solution</u>:Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 $\Rightarrow k = 1$

We know that product of the eigen values = $|A|$

$\Rightarrow$2(3)(k) = $|A|$

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

17. Prove that the eigen vectors of the real symmetric matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

Solution:The characteristic equation of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$where $S_1 = sum\ of\ the\ main\ diagonal\ elements = 7$;

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements\ = 4 + (-8) + 4 = 0$

$$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4) - 1(-2) + 3(-14) = -36$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

3	1-7	0	36	
	0	3	-12	-36
	1	-4	-12	0

Therefore, $\lambda = 3, \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda = 3, \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} = \frac{4 \pm 8}{2} = 6, -2$

Therefore, the eigen values of A are -2, 3, 6

To find the eigen vectors:

$$(A - \lambda I)X = 0$$

Case 1: When $\lambda = -2, \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 7x_2 + x_3 = 0$ ---------- (2)

$3x_1 + x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 3 1

7 1 1 7

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2:When $\lambda = 3$, $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 2x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 2x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -2 1

2 1 1 2

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case 3:When $\lambda = 6$, $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 - x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 5x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -5 1

-1 1 1 -1

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To prove that: $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0$

$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$X_2^T X_3 = [1 - 1 \quad 1]\begin{bmatrix}1\\2\\1\end{bmatrix} = 1 - 2 + 1 = 0$

$X_3^T X_1 = [1 \quad 2 \quad 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = -1 + 0 + 1 = 0$

Hence, the eigen vectors are orthogonal in pairs

18. Find the sum and product of all the eigen values of the matrix A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\\mathbf{2} & \mathbf{2} & \mathbf{4}\\\mathbf{1} & \mathbf{2} & \mathbf{7}\end{bmatrix}$**.Is the matrix singular?**

Solution:Sum of the eigen values = Sum of the main diagonal elements =Trace of the matrix

Therefore, the sum of the eigen values = 1+2+7=10

Product of the eigen values = ⎪A⎪ = 1(14 - 8) -2(14 - 4) + 3(4 - 2) = 6-20+ 6= - 8

⎪A⎪≠0. Hence the matrix is non-singular.

19. Find the product of the eigen values of A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-2}\\\mathbf{1} & \mathbf{0} & \mathbf{3}\\\mathbf{-2} & \mathbf{-1} & \mathbf{-3}\end{bmatrix}$

Solution:Product of the eigen values of A = ⎪A⎪= $\begin{vmatrix}1 & 2 & -2\\1 & 0 & 3\\-2 & -1 & -3\end{vmatrix} = 1(3) - 2(3) - 2(-1) =$

$3 - 6 + 2 = -1$

ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TODIAGONAL FORM:

Orthogonal matrices:

A square matrix A (with real elements) is said to be orthogonal if $AA^T = A^T A = I$ or $A^T = A^{-1}$

Problems:

1. **Check whether the matrix B is orthogonal. Justify. B =** $\begin{bmatrix}\boldsymbol{\cos\theta} & \boldsymbol{\sin\theta} & \mathbf{0}\\\boldsymbol{-\sin\theta} & \boldsymbol{\cos\theta} & \mathbf{0}\\\mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}$

Solution: Condition for orthogonality is $AA^T = A^T A = I$

To prove that: $\boldsymbol{BB^T = B^T B = I}$

B = $\begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$; $B^T = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$

$$BB^T = \begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & -sin\theta cos\theta + sin\theta cos\theta & 0 \\ -sin\theta cos\theta + \sin\theta\cos\theta + 0 & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$B^T B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta cos\theta - \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, B is an orthogonal matrix

2. **Show that the matrix P = $\begin{bmatrix} \boldsymbol{cos\,\theta} & \boldsymbol{sin\,\theta} \\ \boldsymbol{-sin\,\theta} & \boldsymbol{cos\,\theta} \end{bmatrix}$ is orthogonal**

<u>Solution</u>:To prove that: $PP^T = P^T P = I$

$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$; $P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$PP^T = \begin{bmatrix} cos^2\theta + sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Similarly, P^T P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, P is an orthogonal matrix

WORKING RULE FOR DIAGONALIZATION

[ORTHOGONAL TRANSFORMATION]:

Step 1: To find the characteristic equation

Step 2: To solve the characteristic equation

Step 3:To find the eigen vectors

Step 4: If the eigen vectors are orthogonal, then form a normalized matrix N

Step 5: Find N^T

Step 6: Calculate AN

Step 7: Calculate D = $N^T AN$

Problems:

1. **Diagonalize the matrix** $\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{5} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

<u>Solution</u>: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 3 + 5 + 3 = 11$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (15 - 1) + (9 - 1) + (15 - 1)$
$= 14 + 8 + 14 = 36$

$S_3 = |A| = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5) = 3(14) - 2 - 4 = 42 - 6 = 36$

Therefore, the characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 2, \lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = 6,3$

Hence, the eigen values of A are 2, 3, 6

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda} = \mathbf{2}$, $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 3x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

-1 1 1 -1

3 -1 -1 3

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\boldsymbol{\lambda = 3}$, $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 2x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

-1 1 0 -1

2 -1 -1 2

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: When $\boldsymbol{\lambda = 6}$, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 - x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 - 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$x_1 x_2 x_3$$

-1 1 -3 -1

-1 -1 -1 -1

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = [1 \quad 1 \quad 1] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = [1 \quad -2 \quad 1] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N $= \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN $= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix}$

$$N^T A N = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{6}} & \frac{9}{3} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{12}} & \frac{0}{\sqrt{18}} & \frac{36}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

i.e., $D = N^T A N = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

The diagonal elements are the eigen values of A

2. Diagonalize the matrix $\begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$\mathrm{S}_1 = \text{Sum of the main diagonal elements} = 8 + 7 + 3 = 18$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (21 - 16) + (24 - 4) + (56 - 36)$
$= 5 + 20 + 20 = 45$

$S_3 = |A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 8(5) - 60 + 20 = 0$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda = 0, \lambda = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(45)}}{2(1)} = \frac{18 \pm \sqrt{324 - 180}}{2} = \frac{18 \pm 12}{2}$
$= 15, 3$

Hence, the eigen values of A are 0, 3, 15

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda = 0}$, $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ -------- (2)

$2x_1 - 4x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 8 -6

7 -4 -6 7

$$\frac{x_1}{24-14}=\frac{x_2}{-12+32}=\frac{x_3}{56-36}\Rightarrow\frac{x_1}{10}=\frac{x_2}{20}=\frac{x_3}{20}\ \Rightarrow\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

$$X_1=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Case 2: When $\boldsymbol{\lambda=3}$, $\begin{bmatrix}5 & -6 & 2\\-6 & 4 & -4\\2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$5x_1-6x_2+2x_3=0$ -------- (1)

$-6x_1+4x_2-4x_3=0$ ------- (2)

$2x_1-4x_2+0x_3=0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 5 -6

4 -4 -6 4

$$\frac{x_1}{24-8}=\frac{x_2}{-12+20}=\frac{x_3}{20-36}\Rightarrow\frac{x_1}{16}=\frac{x_2}{8}=\frac{x_3}{-16}\ \Rightarrow\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$X_2=\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Case 3: When $\boldsymbol{\lambda=15}$, $\begin{bmatrix}-7 & -6 & 2\\-6 & -8 & -4\\2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-7x_1-6x_2+2x_3=0$ ---------- (1)

$-6x_1-8x_2-4x_3=0$ ---------- (2)

$2x_1-4x_2-12x_3=0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 -7 -6

-8 -4 -6 -8

$$\frac{x_1}{24+16}=\frac{x_2}{-12-28}=\frac{x_3}{56-36}\Rightarrow\frac{x_1}{40}=\frac{x_2}{-40}=\frac{x_3}{20}\ \Rightarrow\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}$$

$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$X_1^T X_2 = [1 \quad 2 \quad 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 2 - 4 = 0$

$X_2^T X_3 = [2 \quad 1 \quad -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ =4-2-2 = 0

$X_3^T X_1 = [2 \quad -2 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 - 4 + 2 = 0$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

AN =

$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$

$\frac{1}{3} \begin{bmatrix} 8 - 12 + 4 & 16 - 6 - 4 & 16 + 12 + 2 \\ -6 + 14 - 8 & -12 + 7 + 8 & -12 - 14 - 4 \\ 2 - 8 + 6 & 4 - 4 - 6 & 4 + 8 + 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$

$$N^T AN = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 + 0 + 0 & 2 + 2 - 4 & 10 - 20 + 10 \\ 0 + 0 + 0 & 4 + 1 + 4 & 20 - 10 - 10 \\ 0 + 0 + 0 & 4 - 2 - 2 & 20 + 20 + 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

The diagonal elements are the eigen values of A

QUADRATIC FORM- REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION:

Quadratic form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form

Example: $2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$ is a quadratic form in three variables

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$$

Problems:

1. **Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$**

Solution:Q = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

Here $x_2x_1 = x_1x_2$; $x_3x_1 = x_1x_3$; $x_2x_3 = x_3x_2$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

2. **Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$**

Solution: Q = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

3. **Write down the quadratic form corresponding to the following symmetric matrix $\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} \end{bmatrix}$**

Solution:Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

The required quadratic form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$$

$= 0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$

CONSISTENCY OF LINEAR ALGEBRAIC EQUATION

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

Denoting the matrices by A, X, and C, the system of equation is, AX = C where A is called the coefficient matrix, C is called the right hand side vector and X is called the solution vector. Sometimes AX=C systems of equations are written in the augmented form. That is

The number of Nonzero Rows is 3. Hence R(A)=3.

5. Find the Rank of the Matrix B = $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

A possible minor of least order is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{pmatrix}$ whose determinant is non zero.

Hence it is possible to find a nonzero minor of order 3.

Hence R(B)=3.

CONSISTENCY OF LINEAR ALGEBRAIC EQUATION

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots.. + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

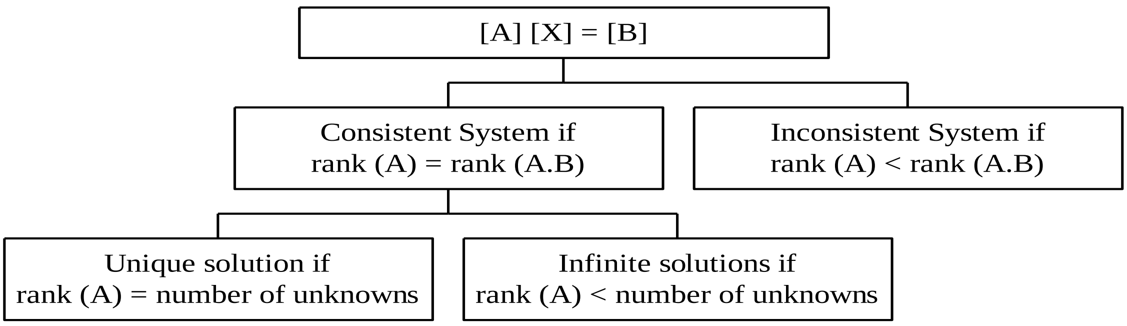
Denoting the matrices by A, X, and C, the system of equation is, AX = C where A is called the coefficient matrix, C is called the right hand side vector and X is called the solution vector. Sometimes AX=C systems of equations are written in the augmented form. That is

$$[\mathrm{A} : \mathrm{C}] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots c_m \end{bmatrix}$$

Rouche'sTheorem

1. A system of equations AX =C is **consistent** if the rank of A is equal to the rank of the augmented matrix $(\mathrm{A} : \mathrm{C})$. If in addition, the rank of the coefficient matrix A is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix A is less than the number of unknowns, then infinite solutions exist.

2. A system of equations AX=C is **inconsistent** if the rank of A is not equal to the rank of the augmented matrix $(\mathrm{A} : \mathrm{C})$.



Problems

1. **Check whether the following system of equations**

$25x_1 + 5x_2 + x_3 = 106.8$

$64x_1 + 8x_2 + x_3 = 177.2$

$89x_1 + 13x_2 + 2x_3 = 280$ is consistent or inconsistent.

Solution

The augmented matrix is

$$[A : B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :280.0 \end{bmatrix}$$

To find the rank of the augmented matrix consider a square sub matrix of order 3×3 as

$\begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$ whose determinant is 12. Hence R $[A : B]$ is 3.

So the rank of the augmented matrix is 3 but the rank of the coefficient matrix $[A]$ is 2

as the Determinant of A is zero. Hence R$[A:B] \neq$ R [A].Hence the system is inconsistent.

2. Check the consistency of the system of linear equations and discuss the nature of the solution?

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Solution

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_2 = R_2 - 3R_1$, $R_3 = R_3 - 4R_1$, $R_4 = R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_2 = R_2 / -5$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_3 = R_3 + 11R_1$

Here R$[A:B]$=R[A] =3.Hence the system is consistent. Also R[A] is equal to the number of unknowns. Hence the system has an unique solution.

3. Check whether the following system of equations is a consistent system of equations. Is the solution unique or does it have infinite solutions

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10$$

Solution

The given system has the augmented matrix given by

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix} \quad R_2 = R_2 - R_1,\ R_3 = R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2$$

A and $[A:B]$ are each of rank r = 3, the given system is consistent but R[A] is not equal to the number of unknowns. Hence the system does not has a unique solution.

4. Check whether the following system of equations

$3x - 2y + 3z = 8$

$x + 3y + 6z = -3$

$2x + 6y + 12z = -6$

is a consistent system of equations and hence solve them.

Solution

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 3 & -2 & 3 & 8 \\ 1 & 3 & 6 & -3 \\ 2 & 6 & 12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 3 & -2 & 3 & 8 \\ 2 & 6 & 12 & -6 \end{bmatrix} \quad R_1 = R_2 \ \ R_2 = R_1$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 0 & 11 & 15 & -17 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 2R_1$$

R[A:B] = R[A] = 2**.**Therefore the system is consistent and posses solution but rank is not equal to the number of unknowns which is 3.Hence the system has infinite solution. From the upper triangular matrix we have the reduced system of equations given by

x +3 y+6z = −3 ; 11y+15z = -17 .

By assuming a value for y we have one set of values for z and x.For example when y=3, z = $-10/3$ and x = 8.Similarly by choosing a value for z the corresponding y and x can be calculated. Hence the system has infinite number of solutions.

5. Check whether the following system of equations

X+y + z = 6
3 x −2 y+4z = 9
x − y −z = 0

Is a consistent system of equations and hence solve them.

<u>Solution</u>

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & 4 & 9 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & 1 & -9 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2=R_2- 3R_1 ,\ R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2 = R_2/-5$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & -12/5 & -12/5 \end{bmatrix} \quad R_3 = R_3+2 R_2$$

Hence R[A B] =R[A] =3 which is equal to the number of unknowns. Hence the system is consistent with unique solution. Now the system of equations takes the form

x+y + z = 6**;** y-z/5 =9/5; $-12/5$z = $-12/5$.

Hence z =1. Substituting z = 1 in y-z/5 =9/5 we have y-1/5 = 9/5 or y = 1/5+9/5 = 10/5.

Hence y =2. Substituting the values of y,z in x+y + z = 6 we have x= 3. Hence the system has the unique solution as x= 3 , y =2 , z =1.

<u>CHARACTERISTIC EQUATION</u>

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X